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Refined curve counting on complex surfaces

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We define refined invariants which "count" nodal curves in sufficiently ample linear systems on surfaces, conjecture that their generating function is multiplicative, and conjecture explicit formulas in the case of $K3$ and abelian surfaces. We also give a refinement of the Caporaso–Harris recursion, and conjecture that it produces the same invariants in the sufficiently ample setting. The refined recursion specializes at $y=-1$ to the Itenberg–Kharlamov–Shustin recursion for Welschinger invariants. We find similar interactions between refined invariants of individual curves and real invariants of their versal families.

14C05, 14H20; 14N10, 14N35

In memory of Friedrich Hirzebruch

1 Introduction

Given a general elliptic fibration $K3 \to \mathbb{P}^1$, we learn by computing $\chi(K3) = 24$ that it must have 24 nodal fibers. For more general irreducible curve classes on a $K3$, Yau and Zaslow [73] argued that taking the Euler characteristic of the relative compactified Jacobian would again yield the number of maximally degenerate fibers; their arguments were clarified by Beauville [5] and by Fantechi, van Straten and Göttsche [22].

For more general families of curves, similar arguments may be made in terms of the relative Hilbert schemes of points. Recall that for a smooth projective curve C of genus g, the "Macdonald formula" asserts[1]

$$\sum_{n=0}^{\infty} q^{n-g+1}\chi(C^{[n]}) = \left(\frac{q}{(1-q)^2}\right)^{1-g}.$$

[1] We recall the derivation of this formula from Macdonald [48]. For any reasonable topological space we have $H^*(X^{(n)}) = H^*(X^n/S_n) = H^*(X^n)^{S_n} = (H^*(X)^{\otimes n})^{S_n}$. From this it follows that taking the generating function of Euler characteristics gives $\sum_{n=0}^{\infty} q^n \chi(X^{(n)}) = (\sum_{n=0}^{\infty} q^n)^{\chi(X)}$. Taking X to be a smooth curve C and recalling that for these $C^{(n)} = C^{[n]}$ gives the formula stated.

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Let $\mathcal{C} \to B$ be a family of reduced planar curves of arithmetic genus g, and let $\mathcal{C}^{[n]} \to B$ be the relative Hilbert schemes. Certain string-theoretic ideas of Gopakumar, Katz, Klemm and Vafa [25; 26; 37] motivate the consideration of the following series, and the following change of variables:[2]

$$\sum_{n=0}^{\infty} q^{n+1-g} \chi(\mathcal{C}^{[n]}) = \sum_{i=0}^{\infty} n^i_{\mathcal{C}/B} \cdot \left(\frac{q}{(1-q)^2}\right)^{i+1-g}.$$

If in fact B were a union of points and $\mathcal{C}$ a disjoint union of smooth curves, the numbers n_i would just count the number of curves of cogenus i. In general we view the n_i as the "virtual" number of curves of cogenus i in the family $\mathcal{C} \to B$.

This makes sense even when $\mathcal{C} \to B$ is a single curve $C \to B = \mathrm{pt}$; in this case we write simply n^i_C. The Macdonald formula is equivalent to the assertion that when C is smooth, $n^0_C = 1$ and $n^i_C = 0$ for all $i > 0$. More generally, Pandharipande and Thomas [62] prove that $n^i_C = 0$ when C has cogenus $\delta(C) < i$. Whenever the singularities of C are unions of smooth branches, the last nonvanishing term is calculated either from [5] or [22] to be $n^i_C = 1$.

In the relative situation, it is helpful to view n^i as a constructible function $n^i \colon B \to \mathbb{Z}$ given by $b \mapsto n^i_{\mathcal{C}_b}$; evidently $n^i_{\mathcal{C}B/B} = \int_B n^i(b) d\chi(b)$.

In addition to the naive or virtual interpretation above, in good cases the n^i carry actual enumerative meaning. Suppose that i is the maximum cogenus of any curve in the family, that there are finitely many curves of cogenus i, and that all these curves are immersed. Then $n^i_{\mathcal{C}/B}$ is just the number of these curves. One exploits this observation by finding another way to express the Euler characteristics of the relative Hilbert schemes. In particular, the following two results have recently been established:

Theorem 1 (Shende [67]) *Fix a reduced plane curve C, a versal deformation Λ of its singularities, and the locus $\Lambda^\delta \subset \Lambda$ of cogenus δ curves. Then n^δ_C is the multiplicity of Λ^δ.*

Theorem 2 (Kool, Shende and Thomas [43]) *Let S be a surface, and L a δ–very-ample line bundle. Then the number of δ–nodal curves in a general $\mathbb{P}^\delta \subset |L|$ is $n^\delta_{\mathcal{C}/\mathbb{P}^\delta}$, which moreover is given by a certain explicit combination of integrals of Chern classes of $L^{[n]}$ and $TS^{[n]}$.*

[2] If B has many connected components, one should perform the change of variables component by component; on each g should be interpreted as the arithmetic genus of the fiber on the component in question. One can avoid this unpleasantness by indexing as in Pandharipande and Thomas [62] and elsewhere by the Euler characteristic of the ideal sheaf on the left-hand side and by the genus rather than cogenus on the right; however, for our purposes indexing by the cogenus is far more convenient.

The significance of the second result is that, due to a theorem of Ellingsrud, Lehn and Göttsche [20], such integrals depend in a universal way on the Chern classes of L and S. Such universality of the counts of nodal curves had been conjectured by Göttsche [27], and proven by Tzeng [70]. Note that in particular, this universality implies that the numbers $n^{\delta}_{\mathcal{C}/\mathbb{P}^{\delta}}$ will not vary under a deformation of the pair (S, L) which preserves the ampleness condition, since the integrals of Chern classes of S, L will not change. Likewise, Theorem 1 plus the fact that Euler numbers of the Hilbert schemes of singular curves may be recovered from the HOMFLY polynomials of their links (see Maulik [50]) implies that the numbers n^{δ}_{C} do not vary under an equisingular deformation.

The present article poses the following question: *Does replacing the topological Euler characteristic on the left-hand side by more sophisticated invariants have an enumerative counterpart on the right?*

We begin in Section 2 by studying the case of a single curve. At the outset, we work in the Grothendieck ring of varieties and consider $\sum_{n=0}^{\infty} q^{n+1-g}[C^{[n]}]$. However, this incorporates global information we would rather not consider, and in particular depends on the motive of the normalization $\widetilde{C}$. We can remove the global contributions by dividing out by the analogous series for $\widetilde{C}$. The quotient only depends on the singularities. We write $\widetilde{g} = g - \delta$ for the genus of $\widetilde{C}$, and show there exist classes $\widetilde{N}^{i}_{C}$ in the Grothendieck ring of varieties such that

$$\frac{\sum_{n=0}^{\infty} q^{n+1-g}[C^{[n]}]}{\sum_{n=0}^{\infty} q^{n+1-\widetilde{g}}[\widetilde{C}^{[n]}]} = \sum_{i=0}^{\delta} \widetilde{N}^{i}_{C} \cdot \left(\frac{q}{(1-q)(1-q[\mathbb{A}^1])}\right)^{i-\delta}.$$

The right-hand side moreover splits into a product over the singularities of C. See (1) and Corollary 20 below for proofs and further discussion.

According to Theorem 1, the Euler number $\chi(\widetilde{N}^{i}_{C})$ gives the multiplicity of a certain locus, and is in particular positive. In examples we see a stronger positivity:

Conjecture 3 *We have $\widetilde{N}^{i}_{C} \in \mathbb{Z}_{\geq 0}[\mathbb{A}^1]$.*

This conjecture is verified computationally for singularities of the form $x^p = y^q$, where $(p, q) = 1$ and $p < 12, q < 20$ using the formulas of Oblomkov, Rasmussen and Shende [58] for the classes of the Hilbert schemes.

The meanings of the $\widetilde{N}^{i}_{C}$ remain mysterious, but there is some evidence that they may be related to geometry over $\mathbb{R}$. We define real analogues $n^{i,\mathbb{R}}$ of the n^{i} by using the compactly supported Euler number of the real locus, and show these again vanish for $i > \delta(C)$. These have an interpretation analogous to Theorem 1. Recall that nodes of real curves come in three types: elliptic ($x^2+y^2=0$), hyperbolic ($x^2-y^2=0$), and pairs of complex conjugate nodes. Thus in the real deformation, the loci B_+^k of k–nodal curves split into components according to the types of the nodes.

Theorem 4 *Let C be a real reduced plane curve, and let $\mathcal{C}\to B$ be a locally versal deformation of its singularities. Let B_+^{δ,δ_-} be the locus of nodal curves with δ nodes of which δ_- are hyperbolic. Let D^j be a general real disc of dimension j passing near $[C]\in B$. Then*

$$n_C^{j,\mathbb{R}} = \sum_i (-1)^i D^j \cap B^{j,i}.$$

Note in particular that while the individual terms on the right-hand side of the formula may depend on the location of the disc, the theorem asserts that their sum does not.

For the simple singularities, we give a combinatorial formula for the $\widetilde{N}_C^i$ in terms of the Dynkin diagram. The formula refines the analogous prescription for the multiplicities given in [67]. Geometrically, this may be interpreted as the choice of a particular real form (in the unibranch case, there is no choice) and a particular disc D in the above statement so that the coefficient of $\mathbb{L}^i$ in N^j is $D^j\cap B^{j,i}$. For $j=\delta$, Duco van Straten has conjectured that such a disc may be found for any singularity.

In Section 3, we turn to the case of linear systems of curves on surfaces. From the point of view of the argument in [43], it is natural to refine the Euler characteristic to Hirzebruch's χ_{-y} genus, since the latter both factors through the Grothendieck ring of varieties and may be calculated in terms of Chern classes. Recall that the χ_{-y} genus is given by $\chi_{-y}(X)=\sum_{p,q}(-1)^{p+q}\mathrm{h}^{p,q}(X)$, where the $\mathrm{h}^{p,q}(X)$ are the Hodge numbers. We define invariants $N^i_{\mathcal{C}/\mathbb{P}^\delta}\in\mathbb{Z}[y]$ which refine the $n^i_{\mathcal{C}/\mathbb{P}^\delta}$ of Theorem 2 above. As before, we are motivated by the MacDonald formula, which in this case reads

$$\sum_{n=0}^{\infty} q^{n-g+1}\chi_{-y}(C^{[n]}) = \left(\frac{q}{(1-q)(1-qy)}\right)^{1-g}.$$

We *define* the $N^i_{\mathcal{C}/B}$ by equating terms in Laurent series expansions (see Definition 44)

$$\sum_{n=0}^{\infty} q^{n+1-g}\chi_{-y}(\mathcal{C}^{[n]}) = \sum_{i=0}^{\infty} N^i_{\mathcal{C}/B}\cdot\left(\frac{q}{(1-q)(1-qy)}\right)^{i+1-g}.$$

As before, if B were just isolated points and $\mathcal{C}$ were a collection of smooth curves, then $N^i_{\mathcal{C}/B}$ would be just the number of curves of cogenus i. In particular, for a single smooth curve, we have $N^0_C=1$ and $N^i_C=0$ for $i>0$. More generally, the N^i_C vanish for $i > \delta(C)$; this is particular to the χ_{-y} genus: the analogous statement does not hold for the virtual Poincaré polynomials.[3]

Let S be a surface, L a line bundle on it, $\mathbb{P}^\delta \subset |L|$ a linear system. Let $S^{[n]}$ be the Hilbert scheme of n points on S, and let $Z_n(S) \subset S \times S^{[n]}$ be the universal family, with the projections $q\colon Z_n(S) \to S$, $p\colon Z_n(S) \to S^{[n]}$. Let $L^{[n]} := p_* q^* L$. This is a vector bundle of rank n on $S^{[n]}$ with fiber $\mathrm{H}^0(Z, L|_Z)$ over $Z \in S^{[n]}$. Let $\mathcal{C}$ be the universal curve over $\mathbb{P}^\delta$ and $\mathcal{C}^{[n]}$ the relative Hilbert scheme of points. It is the scheme-theoretic zero locus of a tautological section of $L^{[n]} \boxtimes H$; when $\mathcal{C}^{[n]}$ is nonsingular this section is transverse. This allows us to compute $\chi_{-y}(\mathcal{C}^{[n]})$ as an intersection number on $S^{[n]}$ (see Proposition 52).

Experimental evidence suggests:

Conjecture 5 *Let L be a line bundle on a surface S, and $\mathbb{P}^\delta \subset |L|$ a linear subsystem with tautological curve $\mathcal{C} \to \mathbb{P}^\delta$. Assume that $\mathbb{P}^\delta$ contains no nonreduced curves, and that the total space of the relative Hilbert scheme $\mathcal{C}^{[n]}$ is smooth for all n. Then $N^i_{\mathcal{C}/\mathbb{P}^\delta}$ vanishes for $i > \delta$.*

Unlike in the Euler characteristic setting, one cannot prove this by "integrating over the base". Indeed, already in the smooth case ($\delta = 0$) Pandharipande and Fantechi [60] have found families of smooth curves over a smooth base curve with nonzero invariants N^i for some $i > 0$. The base curves are always of genus greater than 0, and indeed no such examples can exist over a simply connected base. We are not sure what further aspects of the geometry of families of curves on surfaces are implicated. A calculation of Migliorini has shown that the vanishing cannot be expected for the analogous expression involving Hodge polynomials, already over a one-dimensional base. Nevertheless, we have been able to show:

Theorem 6 *The conjecture holds when K_S is numerically trivial.*

We also give some evidence for the general statement. By [20], one can reduce the validity of the conjecture to the case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Here we may calculate in low degrees by equivariant localization.

In Section 4, we focus on the invariants $N^\delta_{\mathcal{C}/\mathbb{P}^\delta}$. Assuming Conjecture 5, this is the last nonvanishing N^i. Under this assumption we show that there are universal power

[3]This does not contradict the statement above that the $\widetilde{N}^i_C$ vanish even in the Grothendieck ring of varieties; the point being that the N^i_C and $\widetilde{N}^i_C$ have different virtual Poincaré polynomials but the same χ_{-y} genus. Ultimately this is because the χ_{-y} genus of an abelian variety vanishes.

series $A_1,\dots,A_4$ in $\mathbb{Q}[y][\![s]\!]$, such that whenever L is a k–very ample line bundle on a surface S we have

$$\sum_{\delta=0}^{\infty} N^{\delta}_{\mathcal{C}/\mathbb{P}^{\delta}} s^{\delta} \equiv A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)} + O(s^{k+1}).$$

We define polynomials $N^{\delta}_{\delta,[S,L]}$ by

$$\sum_{\delta=0}^{\infty} N^{\delta}_{\delta,[S,L]} s^{\delta} = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)}.$$

In particular $N^{\delta}_{\delta,[S,L]} = N^{\delta}_{\mathcal{C}/\mathbb{P}^{\delta}}$, if L is δ–very ample. The brackets in the notation serve to remind us that it depends only on the cobordism class of (S,L).

As in [27], it is easiest to express the A_i after a change of variable. Consider the following series in $\mathbb{Q}[y,y^{-1}][\![q]\!]$:

$$\widetilde{\Delta}(y,q) := q\prod_{n=1}^{\infty}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2,$$

$$\widetilde{DG}_2 := \sum_{m=1}^{\infty} mq^m \sum_{d|m} \frac{[d]_y^2}{d},$$

and let $D = q\frac{d}{dq}$. Above, $[n]_y := (y^{n/2}-y^{-n/2})/(y^{1/2}-y^{-1/2})$.

It is also convenient to introduce the notation $\overline{N}^{\delta}_{\delta,[S,L]} := y^{-\delta} N^{\delta}_{\delta,[S,L]}$; these invariants are symmetric under $y \to 1/y$.

Conjecture 7 *There exist power series $B_1(y,q)$, $B_2(y,q)$ in $\mathbb{Q}[y,y^{-1}][\![q]\!]$, such that*

$$\sum_{\delta\geq 0} \overline{N}^{\delta}_{\delta,[S,L]}(y)\widetilde{DG}_2^{\delta} = \frac{(\widetilde{DG}_2/q)^{\chi(L)} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q) D\widetilde{DG}_2/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

The first 11 terms of the series B_1, B_2 are given explicitly in Section 4.

Assuming Conjecture 5, the content of the above assertion is exhausted by the case where S is a $K3$ surface. Moreover as in [27] it may be reformulated without the expansion in powers of $\widetilde{DG}_2$:

Conjecture 8 *Let (S_g, L_g) be $K3$ surfaces of genus g with irreducible polarizations. Then for any k,*

$$\sum_{g=k}^{\infty} q^{g-1} \bar{N}^{g-k}_{g-k,[S_g,L_g]}(y) = \frac{\widetilde{DG}_2(y,q)^k}{\widetilde{\Delta}(y,q)}.$$

Similarly, if (A_g, L_g) are abelian surfaces of genus g with irreducible polarizations,

$$\sum_{g=k+2}^{\infty} \bar{N}^{g-k-2}_{g-k-2,[A_g,L_g]} q^{g-1} = \widetilde{DG}_2(y,q)^k D\widetilde{DG}_2(y,q).$$

Remark 9 Since the original version of this paper, we have proven Conjectures 8 and 10, and given more general formulas in [28]. In the current paper we show that Conjecture 8 follows from its validity for *both* $K3$ and abelian surfaces at $k=0$, and show the $k=0$ case for $K3$ surfaces. The proof uses the existence of $K3$ and abelian surfaces of all genera, and the multiplicative nature of the formulas. The proof of the conjecture in [28] relies on these results, and consists in establishing the validity of the conjecture at $k=0$ for abelian surfaces by studying moduli spaces of pairs on abelian surfaces and a crucial use of Theorem 6.

Conjecture 8 would also follow from its validity for all k for the $K3$ surface alone. Here we note a remarkable coincidence: the series on the right-hand side of the above formula appears in the work of Maulik, Pandharipande and Thomas [51] on computing descendant invariants in the (reduced) Gromov–Witten or stable pairs theory of a $K3$ surface. This leads to a further reformulation:

Conjecture 10 *Let (S, L) be a irreducibly polarized $K3$ surface of genus g, and let H be the hyperplane class on $|L|$. Then for all k,*

$$(y-2+y^{-1})^{k-1} \bar{N}^{g-k}_{g-k,[S,L]} = \sum_{n=0}^{\infty} y^{n+1-g} \int_{\mathcal{C}^{[n]}_{|L|}} c_{n+g-k}\big(T\mathcal{C}^{[n]}_{|L|}\big) \cdot \rho^*(H^k).$$

Thus far we have been discussing curves with a small number of nodes compared to the ampleness of the line bundle L; this is the regime to which the conjectures of [27] and the arguments of [43] apply. However, when the surface is $\mathbb{P}^2$, the recursion of Caporaso and Harris [12] determines the degrees of *all* such loci of nodal curves, without any such restriction on the ampleness. Indeed, it determines more: fix a line $H \subset \mathbb{P}^2$, and sequences α, β of integers specifying respectively fixed and moving tangency conditions to H. Then the Caporaso–Harris recursion determines the degrees $n^{d,\delta}(\alpha,\beta)$ of the loci of curves with δ nodes and satisfying the tangency conditions α,β.[4]

In Section 5 we study the following formal refinement of the Caporaso–Harris recursion. We take from [12] the notation $I\alpha = \sum i\alpha_i$ and $|\alpha| = \sum \alpha_i$; note that the curves counted by $n^{d,\delta}(\alpha,\beta)$ have degree $I\alpha + I\beta$.

Definition 11 The polynomials $\bar{N}^{d,\delta}(\alpha,\beta) \in \mathbb{Z}[y^{1/2}, y^{-1/2}]$ are defined by the recursion

$$\bar{N}^{d,\delta}(\alpha,\beta) = \sum_{k:\beta_k>0} [k]_y \cdot \bar{N}^{d,\delta}(\alpha+e_k,\beta-e_k) + \sum_{\alpha',\beta',\delta'} \Big(\prod_i [i]_y^{\beta'_i-\beta_i}\Big)\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}\bar{N}^{d-1,\delta'}(\alpha',\beta').$$

The limits on the sum and the initial conditions are the same as for the Caporaso–Harris recursion and are given explicitly in Section 5. The refined recursion immediately specializes to the Caporaso–Harris recursion upon setting $y=1$, so certainly $n^{d,\delta}(\alpha,\beta) = \bar{N}^{d,\delta}(\alpha,\beta)|_{y=1}$. On the other hand, we know that for $d \gg \delta$ the Severi degrees $n^{d,\delta}((0,0,\dots),(d,0,\dots))$ are given by the universal formulas, ie they are equal to the numbers $n^{\delta}_{\delta,[\mathbb{P}^2,\mathcal{O}(d)]}$. We conjecture a refined analogue:

Conjecture 12 *For $\delta \le 2d-2$, $\bar{N}^{d,\delta}((0,0,\dots),(d,0,\dots)) = y^{-\delta} N^{\delta}_{\delta,[\mathbb{P}^2,\mathcal{O}(d)]}$.*

The equality at $y=1$ was established by Kleiman and Shende [40]. At $y=0$, the recursion simplifies, allowing the right-hand side to be calculated explicitly; on the other hand, a result of Scala allows the left-hand side to be calculated as well (see Scala [65]); the answers match. We have verified the equality empirically for some small d,δ.

In Section 6, we note a connection to real enumerative geometry and to some ideas from tropical geometry. On a real toric surface S, there are real enumerative invariants counting real δ–nodal curves with suitable signs, the real analogues of the Severi degrees. If S is an unnodal del Pezzo surface, they coincide with the Welschinger invariants, real analogues of the Gromov–Witten invariants. Mikhalkin [55] has shown that the Severi degrees and the real enumerative invariants can be computed via tropical geometry: he introduces tropical Severi degrees and tropical Welschinger invariants by assigning Gromov–Witten and Welschinger multiplicities to tropical curves, and shows that they

[4] Vakil has generalized the Caporaso–Harris recursion to the case of rational ruled surfaces. In Section 5 we treat these as well; we have restricted in the introduction to $\mathbb{P}^2$ just for ease of notation.

coincide with the Severi degrees and the real enumerative invariants respectively. The Caporaso–Harris formula has been derived tropically by Gathmann and Markwig [23], and an analogue for the tropical Welschinger invariants by Itenberg, Kharlamov and Shustin [34]. These are specializations of the above recursion, specialized at 1 and -1 respectively. In particular the refined Severi degrees specialize to the Severi degrees and the tropical Welschinger invariants. In [9], Block and Göttsche define and study tropical refined Severi degrees by assigning polynomial multiplicities to the tropical curves which specialize to the Gromov–Witten and Welschinger multiplicities.

On the right-hand side of Conjecture 12, the specialization $y \mapsto -1$ has an entirely different meaning: we are taking the signatures of the relative Hilbert schemes and rearranging them in a certain way. On the other hand, a signed count of real nodal curves in a general $\mathbb{P}^\delta$ is obtained from the $n^{\delta,\mathbb{R}}_{\mathbb{P}^\delta}$ defined earlier. In order that $n^{\delta,\mathbb{R}}_{\mathbb{P}^\delta}$ match $\bar{N}^{\delta}_{\mathbb{P}^\delta}(-1)$, the following property would suffice:

Conjecture 13 *Let $\mathbb{P}^\delta \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ be a linear system determined by a subtropical collection of real points. If $\mathcal{C}^{[n]}/\mathbb{P}^\delta$ is smooth, then its signature is equal to the Euler characteristic of its real locus.*

Here roughly speaking a collection of points in $(\mathbb{R}^*)^2$ is called subtropical, if it can be degenerated to a tropical collection of points without crossing walls, for the precise definition see Itenberg and Mikhalkin [35, Lemma 2.7.(3)].

We remark briefly on related work. In the physics literature there is a notion of a refined topological string, which gives in some cases a one-parameter deformation of the various curve counting invariants; see Iqbal, Kozçaz and Vafa [33]. Notably it does not have a "worldsheet" definition, even in the sense of physics. Mathematically, the refined theory is supposed to correspond (see Dimofte and Gukov [18]) to the motivic DT theory (see Kontsevich and Soibelman [41]); the lack of a worldsheet definition corresponds to the fact that we do not know how to correspondingly refine the Gromov–Witten invariants. (For further discussions of motivic DT theory, see Kontsevich and Soibelman [42], Behrend, Bryan and Szendrői [6], Choi, Katz and Klemm [15], and Bussi, Joyce and Meinhardt [11].) There have also been intimations that a specialization of the refined theory is related to real invariants; see Krefl and Walcher [45, Section 5]. Our approach falls roughly into this paradigm insofar as $\bar{N}^i_{\mathcal{C}/\mathbb{P}^\delta}$ are assembled from χ_{-y} genera of relative Hilbert schemes, which under the relevant assumptions on $\mathbb{P}^\delta$ are just the same as stable pairs spaces. It might plausibly be hoped that the refined Severi degrees also admit an interpretation in the stable pairs theory (see Pandharipande and Thomas [61; 62]) or its surface variant; see Kool and Thomas [44].

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Notation We denote quantum numbers as

$$[n]_y := \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}.$$

By the Hirzebruch genus X_{-y} we mean the characteristic class which on a bundle E with Chern roots x_i takes the value

$$X_{-y}(E) = \prod \frac{x_i(1 - ye^{-x_i(1-y)})}{(1 - e^{-x_i(1-y)})} \in 1 + (x_i)\mathbb{Q}[y][\![x_i]\!].$$

Also let

$$\mathrm{ch}_{-y}(E) = \sum e^{x_i(1-y)}.$$

Setting by definition

$$\chi_{-y}(X, E) := \sum (-y)^p \chi(X, \Omega^p \otimes E),$$

we have according to Hirzebruch

$$\chi_{-y}(X, E) = \int_X \mathrm{ch}_{-y}(E) X_{-y}(TX).$$

When $E = \mathcal{O}_X$ we suppress it from the notation. Note that

$$\chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} y^q \mathrm{h}^{p,q}(X),$$

where $\mathrm{h}^{p,q}(X)$ are the Hodge numbers of X. Note the specializations to topological Euler characteristic $\chi_{-1}(X) = \chi(X)$, holomorphic Euler characteristic $\chi_0(X) = \chi(X, \mathcal{O}_X)$, and signature $\chi_1(X) = \sigma(X)$.

Glossary of notation

$X^{(n)}$	n^{th} symmetric power
$X^{[n]}$	Hilbert scheme of n points
$\mathcal{C}^{[n]}$	relative Hilbert scheme of points of a family of curves

$\mathbb{P}^\delta \subset \lvert L\rvert$	δ–dimensional sublinear system of a complete linear system
$\mathfrak{M}$	Grothendieck ring of varieties
$[X]$	class in $\mathfrak{M}$
$[S,L]$	cobordism class of a surface with a line bundle
$D^{S,L}(y,x,q)$	generating function for integrals on $S^{[n]}$ which determines the $N^i_{\delta,[S,L]}$ (6)
Q	$q/((1-q)(1-yq))$
$m(C)$	tropical multiplicity of a tropical curve
$r(C)$	$[m(C)]_{-1}$ Welschinger multiplicity of a tropical curve
$M(C)$	$[m(C)]_y$ refined multiplicity of a tropical curve

Curve counting invariants defined via generating functions of invariants of (relative) Hilbert schemes of points

n^i_C	invariants from Euler numbers for a single curve
$n^i_{\mathcal{C}}$	the same for a family of curves
$n^{i,\mathbb{R}}_C$	real invariant, from Euler number of real locus for a single curve
$n^{i,\mathbb{R}}_{\mathcal{C}}$	the same for a family of real curves
$\widetilde{N}^i_C$	invariants in the Grothendieck group of varieties for a single curve (see (1))
$\mathcal{N}^i_{\mathcal{C}/B}$	invariants for a family of curves in mixed Hodge modules (4)
$\boldsymbol{n}^i_{\mathcal{C}/B}$	invariants for a family of curves in Chern–Schwarz–Macpherson classes
$\boldsymbol{N}^i_{\mathcal{C}/B}$	Brasselet, Schürmann, Yokura refinement of the above
$N^i_{\mathcal{C}}$, $N^i_{\mathcal{C}/B}$	invariants for a family of curves from χ_{-y}–genus (see Definition 44)
$N_{\mathcal{C}/\mathbb{P}^\delta}$	the above for a linear system $\mathbb{P}^\delta \subset \lvert L\rvert$
$N^i_{\delta,[S,L]}$	polynomial in L^2, LK_S, K_S^2, $c_2(S)$ that equals $N_{\mathcal{C}/\mathbb{P}^\delta}$ for L sufficiently ample (see Definition 63)
$\boldsymbol{N}^i_{\delta,[S,L]}$	the same for $N_{\mathcal{C}/\mathbb{P}^\delta}$ (see Definition 63)
$\bar{N}^i_{\delta,[S,L]}$	$N^i_{\delta,[S,L]}/y^\delta$

Generalized Severi degrees

$n^{d,\delta}$	Severi degree on $\mathbb{P}^2$: number of δ–nodal curves in $\mathbb{P}^2$ of degree d
$n^{d,\delta}(\alpha,\beta)$	relative Severi degree: curves with contact conditions
$n^{L,\delta}(\alpha,\beta)$	relative Severi degree (14)
$N^{d,\delta}$, $N^{d,\delta}(\alpha,\beta)$	(relative) refined Severi degree on $\mathbb{P}^2$
$N^{L,\delta}$, $N^{L,\delta}(\alpha,\beta)$	(relative) refined Severi degree on toric surface 76
$\bar{N}^{d,\delta}$, $\bar{N}^{d,\delta}(\alpha,\beta)$	normalized refined Severi degrees (see Definition 11)
$\bar{N}^{L,\delta}$, $\bar{N}^{L,\delta}(\alpha,\beta)$	(see Definition 77)
$N_0^{L,\delta}$	refined Severi degree on $\mathbb{P}^2$ counting irreducible curves
$W^{L,\delta}$	real curve counting invariants of toric surface
$W_0^{L,\delta}$	real curve counting invariant for irreducible curves
$W_{\mathrm{trop}}^{d,\delta}$	tropical Welschinger invariants of degree d on $\mathbb{P}^2$
$W_{\mathrm{trop}}^{L,\delta}$	tropical Welschinger invariants on toric surface
$\bar{N}_{\mathrm{trop}}^{L,\delta}$	refined tropical Severi degrees on toric surface

2 Invariants of a single curve

2.1 Refined invariants

Let $\mathfrak{M}$ denote the Grothendieck ring of varieties; let $\mathbb{L}=[\mathbb{A}^1]$ denote the class of the affine line. Kapranov [36] introduced the motivic zeta function of a variety:

$$\zeta_X(q)=\sum_{n=0}^{\infty}\operatorname{Sym}^n X\cdot q^n\in\mathfrak{M}[\![q]\!].$$

When X is a smooth proper curve of genus g, he showed that $(1-q)(1-q\mathbb{L})\zeta_X(q)$ is a polynomial of degree $2g$, and that one has a functional equation

$$\zeta_X(1/q\mathbb{L})=\mathbb{L}^{1-g}q^{2-2g}\zeta_X(q).$$

Motivated by a circle of ideas relating curve counting, Hilbert schemes on singular curves, and knot invariants (see [25; 26; 43; 50; 58; 61; 62; 67], Katz, Klemm and Vafa [37], Oblomkov and Shende [59], Maulik and Yun [52], Migliorini and Shende [54]), we consider a "zeta function" defined using the Hilbert schemes rather than the symmetric products. The analogous rationality and functional equation continue to hold.[5]

Lemma 14 (Hartshorne [30]) *Let C be a Gorenstein curve, and let F be a torsion free sheaf on C. Write F^* for $\mathcal{H}om(F,\mathcal{O}_C)$. Then $\mathcal{E}xt^{\geq 1}(F,\mathcal{O}_C)=0$ and $F=(F^*)^*$. Serre duality holds in the form $\mathrm{H}^i(C,F)=\mathrm{H}^{1-i}(C,F^*\otimes\omega_C)^*$. For F of rank one and torsion free, define its degree $d(F):=\chi(F)-\chi(\mathcal{O}_C)$. This satisfies $d(F)=-d(F^*)$, and, for L any line bundle, $d(F\otimes L)=d(F)+d(L)$.*

Proposition 15 *Let C be a complete, reduced, irreducible Gorenstein, complex curve of arithmetic genus g. Then $f_C(q)=(1-q)(1-q\mathbb{L})\sum_d q^d[C^{[d]}]$ is a polynomial of degree $2g$, satisfying $q^{2g}\mathbb{L}^g f_C(1/q\mathbb{L})=f_C(q)$.*

Proof Fix a degree-one line bundle $\mathcal{O}(1)$ on C. Let $\bar{J}^0(C)$ denote the moduli space of rank-one, degree-zero, torsion free sheaves; see Altman and Kleiman [2]. We map $C^{[d]}\to\bar{J}^0(C)$ by sending the ideal $I\subset\mathcal{O}_C$ to the sheaf $I^*=\mathcal{H}om(I,\mathcal{O}_C)\otimes\mathcal{O}(-d)$; the fiber is $\mathbb{P}(\mathrm{H}^0(C,I^*))$. For $F\in\bar{J}^0(C)$, we write the Hilbert function as $h_F(d):=\dim\mathrm{H}^0(C,F\otimes\mathcal{O}(d))$.

Fix $h=h_F$ for some F. Evidently h is supported in $[0,\infty)$, and by Riemann–Roch and Serre duality is equal to $d+1-g$ in $(2g-2,\infty)$. Inside $[0,2g-2]$, it either increases by 0 or 1 at each step. Let $\phi_\pm(h)=\{d\mid 2h(d-1)-h(d-2)-h(d)=\pm1\}$; evidently $\phi_-\subset[0,2g]$ and $\phi_+\subset[1,2g-1]$. Consider

$$\begin{aligned}j_h(q)&:=(1-q)(1-q\mathbb{L})\sum_{d=0}^{\infty}q^d[\mathbb{P}^{h(d)-1}]\\&=\sum_{d=0}^{\infty}q^d([\mathbb{P}^{h(d)-1}]-(1+\mathbb{L})[\mathbb{P}^{h(d-1)-1}]+\mathbb{L}[\mathbb{P}^{h(d-2)-1}])\\&=\sum_{d\in\phi_-(h)}q^d\mathbb{L}^{h(d)-1}-\sum_{d\in\phi_+(h)}q^d\mathbb{L}^{h(d-1)}.\end{aligned}$$

This is a polynomial in q of degree at most $2g$.

[5] This was observed for the Euler characteristics in [62, Proposition 3.13]; the argument however is identical to that for the smooth case as in [36] which in turn is essentially the same argument as in Schmidt's original proof of the rationality and functional equation for the zeta function of a curve [66]. We nevertheless include (again the same) proof for completeness.

Now let $G=F^*\otimes\omega_C\otimes\mathcal{O}(2-2g)$, and $h^\vee=h_G$. By Serre duality and Riemann–Roch, $h^\vee(d)=h(2g-2-d)+d+1-g$, so in particular, $d\in\phi_\pm(h^\vee)$ if and only if $2g-d\in\phi_\pm(h)$. It follows that $q^{2g}\mathbb{L}^g j_{h^\vee}(1/q\mathbb{L}) = j_h(q)$.

Finally, stratify $\bar{J}^0(C)$ into strata over which h_F is constant. The restriction of $C^{[d]}$ to each stratum is the projectivization of a vector bundle of rank $h_F(d)$. Thus we have

$$\begin{aligned} f_C(q) &= (1-q)(1-q\mathbb{L})\sum_{d=0}^{\infty} q^d[C^{[d]}] \\ &= (1-q)(1-q\mathbb{L})\sum_h [\{F \mid h_F = h\}]\sum_{d=0}^{\infty} q^d[\mathbb{P}^{h(d)-1}] \\ &= \sum_h [\{F \mid h_F = h\}]\cdot j_h(q). \end{aligned}$$

Collecting together the terms for h and $h^\vee$ completes the proof. □

For us it is more convenient to have the symmetry without the shifting.

Definition 16 Let C be a complete, reduced, irreducible Gorenstein, complex curve of arithmetic genus g. Then we write

$$Z_C(q) := \sum_{n=0}^{\infty} C^{[n]} q^{n+1-g}.$$

Since $(\mathbb{P}^1)^{[n]} = \mathbb{P}^n$, we have

$$Z_{\mathbb{P}^1}(q) = \frac{q}{(1-q)(1-q\mathbb{L})}$$

and define classes $N_C^i\in\mathfrak{M}$ by the formula

$$\sum_{n=0}^{\infty} C^{[n]} q^{n+1-g} = \sum_{i=0}^{\infty} N_C^i \cdot Z_{\mathbb{P}^1}^{i+1-g}.$$

To be explicit, note that $Z := q/((1-q)(1-q\mathbb{L}))$ is a power series in $q(1+q\mathbb{Z}[\mathbb{L}][\![q]\!])$. It therefore has a compositional inverse $q(Z)\in\mathbb{Z}[\mathbb{L}][\![Z]\!]$ (given explicitly in Footnote 6). Thus the N_C^i are just defined by the change of variables

$$\sum_{i=0}^{\infty} N_C^i Z^{i+1-g} = \sum_{n=0}^{\infty} C^{[n]} q(Z)^{n+1-g}.$$

Corollary 17 *Let C be a complete, reduced, irreducible Gorenstein, complex curve of arithmetic genus g. Then $Z_C(q) = Z_C(1/q\mathbb{L})$, and $N_C^i = 0$ for $i > g$.*

Proof In terms of the f_C of Proposition 15, we have $Z_C(q) = q^{-g} Z_{\mathbb{P}^1}(q) f_C(q)$, so we may conclude the rationality and functional equation of Z_C from that for f_C.

Moreover $f_C(q)$ is a polynomial of degree $2g$. On the other hand we can expand it as $f_C(q) = q^g \sum_{i=0}^{\infty} N_C^i \cdot Z_{\mathbb{P}^1}^{i-g}$ and then further into

$$\left(\sum_{i=0}^{g} N_C^i \cdot q^i (1-q)^{g-i}(1-q\mathbb{L})^{g-i}\right) + \left(q^g \sum_{i=1}^{\infty} N_C^{g+i} \left(\frac{q}{(1-q)(1-q\mathbb{L})}\right)^i\right).$$

The first term is visibly a polynomial of degree at most 2g; for the second to be the same it must vanish. But the coefficient of N_C^{g+i} in the second term is $q^{g+i} + O(q^{g+i+1})$, so it follows that $N_C^{g+i} = 0$ for $i > 1$. □

Remark 18 We always have $N_C^0 = 1$ and $N_C^1 = [C] + (g-1)(1+\mathbb{L})$. Given the vanishing, only N_C^g contributes large powers of q to $Z_C(q)$; on the other hand when we have $n \gg 0$ the map $C^{[n]} \to \bar{J}^0(C)$ is a projective bundle. Comparison of these terms reveals $N_C^g = \bar{J}^0(C)$.

Example 19 Let $\mathbb{P}^1$, A_1, A_2 be rational curves that are smooth, have one node and have one cusp respectively. Then:

- $N_{\mathbb{P}^1}^0 = 1$.
- $N_{A_1}^0 = 1$ and $N_{A_1}^1 = \mathbb{L}$.
- $N_{A_2}^0 = 1$ and $N_{A_2}^1 = 1 + \mathbb{L}$.

In each case Corollary 17 ensures the higher N^i vanish.

As the Euler characteristic factors through $\mathfrak{M}$, it makes sense to write $\chi(N_C^i)$. When C is smooth of genus g, it follows from Macdonald's calculation of the cohomology of symmetric products that $\chi(N_C^i) = 0$ for $i > 0$. However, $N_C^1 = [C] + (g-1)(1+\mathbb{L})$ is never zero. To avoid this incursion of the global geometry of C, we remove the contribution of the normalization $\widetilde{C}$ of C. We define $\widetilde{Z}_C := Z_C / Z_{\widetilde{C}}$, and $\widetilde{N}_C^i$ by the formula

$$\widetilde{Z}_C = \sum_{i=0}^{\infty} \widetilde{N}_C^i Z_{\mathbb{P}^1}^{i-\delta}.$$

For $p \in C$, let $C_p^{[n]}$ be the subvariety of $C^{[n]}$ parameterizing subschemes supported at p. Let $\delta(p)$ denote the δ invariant of the singularity at p, and $b(p)$ the number of analytic local branches at p. Note $C_p^{[n]}$ only depends on the analytic local structure of C at p. A stratification argument gives the product expansion

$$(1) \qquad \widetilde{Z}_C = \frac{\sum_{n=0}^{\infty} q^{n+1-g}[C^{[n]}]}{\sum_{n=0}^{\infty} q^{n+1-\widetilde{g}}[\widetilde{C}^{[n]}]} = \prod_{p\in C} \frac{\sum_{n=0}^{\infty} q^{n-\delta(p)}[C_p^{[n]}]}{(1-q)^{-b(p)}}.$$

The product is written over all points in C, but may as well be written over only singular points as the smooth points contribute 1.

We define $Z_{C,p} := (1-q)^{b(p)} q^{-\delta(p)} \sum_{n=0}^{\infty}[C_p^{[n]}]$ so that the above can be written $\widetilde{Z}_C = \prod Z_{C,p}$. Similarly, define $N^i_{C,p}$ by the expansion $Z_{C,p} := \sum_{i=0}^{\infty} N^i_{C,p} Z_{\mathbb{P}^1}^{i-\delta(p)}$.

Corollary 20 *Let C be a curve, let $p \in C$. Then $N^i_{C,p} = 0$ for $i > \delta(p)$, and $\widetilde{N}^i_C = 0$ for $i > \delta(C)$.*

Proof Let C', p' be a rational curve with a unique singularity at p' analytically isomorphic to the singularity of C at p; such a curve exists by Laumon [46, Proposition 2.1.1]. Then

$$N^i_{C,p} = N^i_{C',p'} = \widetilde{N}^i_{C'} = N^i_{C'}.$$

Since $\delta(p) = \delta(p') = g(C')$, the vanishing $N^i_{C,p}$ for $i > \delta(p)$ follows from Corollary 17 applied to C'.

Then we have

$$\sum_{i=0}^{\infty} \widetilde{N}^i_C Z_{\mathbb{P}^1}^{i-\delta(C)} = \widetilde{Z}_C = \prod_{p\in C} \sum_{i=0}^{\delta(p)} N^i_{C,p} Z_{\mathbb{P}^1}^{i-\delta(p)},$$

giving the desired vanishing of $\widetilde{N}^i_C$ for $i > \delta(C)$. □

Conjecture 21 *For all h, we have $\widetilde{N}^h_C \in \mathbb{Z}_{\geq 0}[\mathbb{L}]$.*

Remark Theorem 1 realizes $\chi(\widetilde{N}^i_C)$ as the multiplicity of the stratum of curves of cogenus i inside the versal deformation of C, whence it follows that $\chi(\widetilde{N}^i_C) > 0$ for $i \leq \delta(C)$. It may be hoped that this conjecture indicates a refinement of this geometric structure, ie that the coefficients of the $\widetilde{N}^i_C$ count something.

For rational C, we have $\widetilde{N}^\delta_C = [\bar{J}(C)]$; by [46] these Jacobians are known to receive a bijective morphism from an affine Springer fiber for $\mathfrak{gl}$. It is known that such affine Springer fibers in other types *are not necessarily* in $\mathbb{Z}[\mathbb{L}]$ (see Kazhdan and Lusztig [39, Appendix]), however according to Lusztig the status of the $\mathfrak{gl}$ affine

Springer fibers is unknown. From the work of Piontkowski [63] it follows that for unibranch singularities with a single Puiseux pair, and for unibranch singularities whose links are two-cablings of links of simple unibranch singularities, one has at least $\widetilde{N}_C^\delta \in \mathbb{Z}[\mathbb{L}]$. The stated positivity has been checked for unibranch singularities with a single Puiseux pair (eg $x^m = y^n$) for $m < 14$ and $n < 20$ using the explicit formula for $Z(C)$ given in [58, Theorem 5].

From the fact that $\chi(Z_{\widetilde{C}}) = \chi(Z_{\mathbb{P}^1})^{1-g}$, we see that $n_C^i := \chi(N_C^i) = \chi(\widetilde{N}_C^i)$, and in particular that the n_C^i vanish for $i > \delta(C)$. This fact was used in [62] as evidence that the n_C^i were in fact the Gopakumar–Vafa invariants, and was exploited in [43] to count curves on surfaces. Here we note the Hirzebruch χ_{-y} genus has the same property, which suggests that it is a more sensible refinement than working in the ring of varieties.

Lemma 22 *For a smooth curve C of genus g, $\chi_{-y}(Z_C) = \chi_{-y}(Z_{\mathbb{P}^1})^{1-g}$.*

Proof The Hodge structures of symmetric products are known explicitly [48]. □

Corollary 23 *We have $\chi_{-y}(N_C^i) = \chi_{-y}(\widetilde{N}_C^i)$. In particular, $\chi_{-y}(N_C^i) = 0$ for $i > \delta(C)$.*

Remark 24 According to Conjecture 21, the $\widetilde{N}_C^i$ may be recovered from $\chi_{-y}(N_C^i)$ by $y \mapsto \mathbb{L}$. Conversely, one may separate Conjecture 21 into two pieces, one asking that $\widetilde{N}_C^i \in \mathbb{Z}[\mathbb{L}]$ and the second asking that $\chi_{-y}(N_C^i)$ have positive coefficients.

2.2 Curves with simple singularities

Here we express the refined BPS invariants of curves with simple singularities in terms the associated Dynkin diagram. In Figure 1, we recall the ADE classification of simple singularities.

Lemma 25 *We define A_∞ to be the germ at the origin of the curve cut out by $y^2 = 0$. Similarly we define D_∞ for $xy^2 = 0$ and E_∞ for $y^3 = 0$. Then we have an equality $X_\mu^{[i]} = X_\infty^{[i]}$ as subsets of $(\mathbb{C}^2)^{[i]}$ for $X = A, D, E$ and for any i up to the delta invariant of X_μ.*

Proof Any subscheme of length at most δ supported at the origin is annihilated by $(x, y)^\delta$. In each case, the right-hand side of the equation of X_μ already belongs to this ideal. □

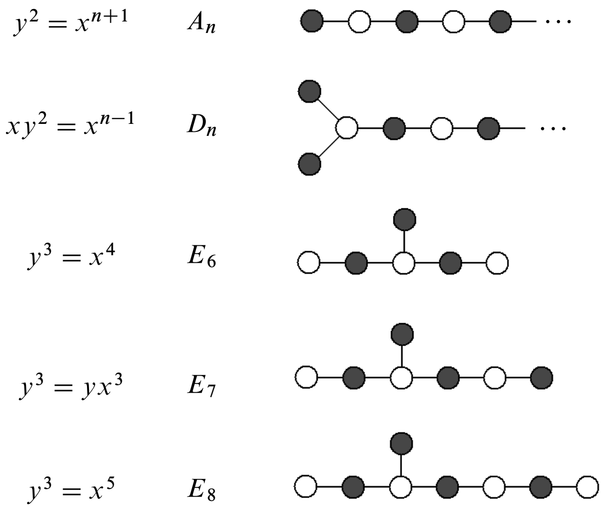

Figure 1: The ADE singularities and associated colored diagrams: the subscript gives the total number of vertices of the diagram and the Milnor number of the singularity. The coloring is characterized by requiring the colors to alternate, and requiring that the total number of black vertices is the delta invariant. Except for A_1, the number of filled vertices of valence one is the number of analytic local branches.

Proposition 26 *We have*

$$\sum [A_\infty^{[n]}] q^n = \frac{1}{(1-q)(1-q^2\mathbb{L})},$$

$$\sum [D_\infty^{[n]}] q^n = \frac{1-q+q^3\mathbb{L}^2}{(1-q)^2(1-q^2\mathbb{L})},$$

$$\sum [E_\infty^{[n]}] q^n = \frac{1}{(1-q)(1-q^2\mathbb{L})(1-q^3\mathbb{L}^2)}.$$

Proof We fix the monomial order $1 < x < x^2 < \cdots < xy < x^2y < x^3y < \cdots$. Recall that in the setting of power series rings, the theory of Gröbner bases (or 'standard bases') is developed with respect to the *lowest* rather than highest order terms of series. Thus let $lt(f)$ of $f \in \mathbb{C}[\![x,y]\!]$ be the lowest degree monomial appearing. A generating set $I = (i_1, i_2, \dots, i_k)$ is a Gröbner basis when $lt(I) = (lt(i_1), \dots, lt(i_k))$. It is reduced when the $lt(i_t)$ are a minimal generating set for $lt(I)$, ie when $k = \dim_{\mathbb{C}} lt(I)/(x,y)lt(I)$, and moreover

$$i_t = lt(i_t) + \text{monomials not in } lt(I).$$

As is well known, every ideal admits a unique reduced Gröbner basis.

Let us first consider the case A_∞. Ideals of $\mathbb{C}[\![x,y]\!]/y^2$ are the same as ideals of $\mathbb{C}[\![x,y]\!]$ containing y^2. The *lt* ideal of a finite colength such ideal must be of the form (y^2, x^a) or (y^2, yx^b, x^a) for some $b < a$. In the first case, a reduced Gröbner basis of the original ideal takes the form

$$\Big(y^2, x^a + y\sum_{i=0}^{a-1} c_i x^i\Big),$$

whereas in the second it takes the form

$$\Big(y^2, yx^b, x^a + y\sum_{i=0}^{b-1} c_i x^i\Big).$$

These sum as

$$\sum [A_\infty^{[n]}]q^n = \sum_{a=0}^{\infty}\sum_{b=0}^{a} q^{a+b}\mathbb{L}^b = \sum_{a=0}^{\infty} q^a \frac{1-(q\mathbb{L})^{a+1}}{1-q\mathbb{L}} = \frac{1}{1-q\mathbb{L}}\Big(\frac{1}{1-q} - \frac{q\mathbb{L}}{1-q^2\mathbb{L}}\Big),$$

which simplifies to the stated expression. The E_∞ case can be treated similarly, and in any event the statements for A_∞, E_∞ are special cases of [58, Proposition 6].

We turn to D_∞. Let $I \subset \mathbb{C}[\![x,y]\!]$ be an ideal which contains xy^2 and is of finite colength. We have already counted all the ideals containing y^2; these contribute $1/(1-q)(1-q^2\mathbb{L})$. We must account for the remaining ideals, and show they contribute $q^3\mathbb{L}^2/(1-q)^2(1-q^2\mathbb{L})$. So let $I \subset \mathbb{C}[\![x,y]\!]$ be an ideal containing xy^2 but not y^2. Then $lt(I)$ contains xy^2; it may be written in the form $(y^a, xy^2, x^b y, x^c)$ for some $b \le c$. A corresponding Gröbner basis is

$$\Big(y^a, xy^2, x^b y + \sum_{i=2}^{a-1} c_i y^i, x^c + y\sum_{j=1}^{b-1} d_j x^j + \sum_{k=1}^{a-1} e_k y^k\Big).$$

We start the sum over j from 1 since the monomial y is already accounted for in the sum over k. Note that $a \ge 3$ and $b, c \ge 1$ because we assumed $y^2 \notin I$. There are further constraints on which such things may be Gröbner bases. For instance, multiplying the third generator by y and subtracting a multiple of xy^2 gives $\sum c_i y^{i+1}$; by our assumption on $lt(I)$ the lowest term of this must be $c_{a-1}y^a$, ie, the other c_i must vanish. So the basis takes the form

$$\Big(y^a, xy^2, x^b y + c' y^{a-1}, x^c + y\sum_{j=1}^{b-1} d_j x^j + \sum_{k=1}^{a-1} e_k y^k\Big).$$

Multiplying the fourth generator by y^2 and arguing similarly, we see in fact e_k vanishes for $k < a-2$, so the basis takes the form

$$\Big(y^a, xy^2, x^b y + c' y^{a-1}, x^c + y \sum_{j=1}^{b-1} d_j x^j + e y^{a-2} + e' y^{a-1}\Big).$$

Multiplying the fourth element by y and subtracting from the third element times x^{c-b} gives $ey^{a-1} + e'y^a - c'x^{c-b}y^{a-1} = y^{a-1}(e + e'y - c'x^{c-b})$. If $e \neq 0$, then the term in parenthesis is invertible and hence y^{a-1} is in the ideal, a contradiction. Finally we are reduced to the form

$$\Big(y^a, xy^2, x^b y + c' y^{a-1}, x^c + y \sum_{j=1}^{b-1} d_j x^j + e' y^{a-1}\Big).$$

We leave it to the reader to check that there are no other constraints, and moreover that every ideal admits a unique basis of this form (even though it is not literally a reduced Gröbner basis, eg when $b = c$). We count them:

$$\sum_{a=3}^{\infty}\sum_{c=1}^{\infty}\sum_{b=1}^{c} \mathbb{L}^{b-1+2} q^{a+b+c-2} = q^3\mathbb{L}^2 \sum_{a'=0}^{\infty}\sum_{c'=0}^{\infty}\sum_{b'=0}^{c'} \mathbb{L}^{b'} q^{a'+b'+c'} = \frac{q^3\mathbb{L}^2}{(1-q)^2(1-q^2\mathbb{L})}.$$

This completes the proof. □

Let X be a simple singularity type. Let C be a curve and p a point at which C has a!singularity analytically of type X. We will then write $X^{[n]} := C_p^{[n]}$, and similarly $\delta(X)$ and $b(X)$ for what we have before written as $\delta(p)$ and $b(p)$; and similarly $Z_X := Z_{C,p}$ and $N_X^i := N_{C,p}^i$. Recall these are related by

$$\sum_{i=0}^{\delta(X)} N_X^i \cdot Z_{\mathbb{P}^1}^{i-\delta} = Z_X = (1-q)^{b(X)} q^{-\delta(X)} \sum_{n=0}^{\infty} q^n X^{[n]}.$$

Theorem 27 *Let X be a simple singularity. Color the dots of the associated Dynkin diagram as in Figure 1. Let $n_X^{w,b}$ be the number of ways to choose w white dots and b black ones such that no two dots of the same color are adjacent. Then*

$$N_X^h = \sum_{w+b=h} n_X^{w,b} \mathbb{L}^b.$$

Proof We temporarily write

$$M_X^h := \sum_{w+b=h} n_X^{w,b} \mathbb{L}^b$$

and $Y_X := \sum_{i=0}^{\delta(X)} M_X^i \cdot Z_{\mathbb{P}^1}^{i-\delta}$. We will show that $Y_X = Z_X$. For small cases, say $X = A_1, A_2, A_3 = D_3, D_4, E_6, E_7, E_8$, we have explicit formulas on both sides so this may be verified by hand or by computer.

It remains to treat in general A_n, D_n. The argument will reveal that in fact, Lemma 25 in a sense determined the values of the series in Proposition 26; in particular, we will not use that Proposition again. (We used it above to check the base cases; of course these could have been done without appealing to its full strength.)

We begin with the case of A_n. By considering what happens according as the right end dot of the Dynkin diagram — the one not seen in the pictures above, which is white if $n = 2k$ and black if $n = 2k-1$ — is chosen or not, we have

$$n_{A_{2k}}^{w,b} = n_{A_{2k-1}}^{w,b} + n_{A_{2k-2}}^{w-1,b}, \quad n_{A_{2k-1}}^{w,b} = n_{A_{2k-2}}^{w,b} + n_{A_{2k-3}}^{w,b-1}.$$

Summing these, we have

$$M_{A_{2k}}^h = M_{A_{2k-1}}^h + M_{A_{2k-2}}^{h-1}, \quad M_{A_{2k-1}}^h = M_{A_{2k-2}}^h + \mathbb{L} M_{A_{2k-3}}^{h-1}.$$

Noting that $\delta(A_{2k}) = \delta(A_{2k-1}) = k$ and summing,

$$Y_{A_{2k}} = Y_{A_{2k-1}} + Y_{A_{2k-2}}, \quad Y_{A_{2k-1}} = Y_{\mathbb{P}^1}^{-1} Y_{A_{2k-2}} + \mathbb{L} Y_{A_{2k-3}}.$$

It remains to show the same for the Z. Since both Y_X and Z_X are symmetric Laurent polynomials in q, it suffices to check the recurrence holds modulo q. Writing $\xi_A = \sum q^n A_\infty^{[n]}$ we had from Lemma 25 that $Z_{A_n} = (1-q)^{b(A_n)} q^{-\delta(A_n)} \xi_A \pmod{q}$. We are left to show that

$$q^{-k}(1-q)\xi_A = q^{-k}(1-q)^2\xi_A + q^{-k+1}(1-q)\xi_A \pmod{q},$$
$$q^{-k}(1-q)^2\xi_A = q^{-k}(1-q)^2(1-q\mathbb{L})\xi_A + \mathbb{L}q^{-k+1}(1-q)^2\xi_A \pmod{q}.$$

But these are both formal identities, independent of the value of ξ_A.

An identical argument suffices to treat the D_n. We remark in passing that the recursion above corresponds [50; 58; 59] to the skein relation on the HOMFLY invariants of the links of the singularities. □

Example 28 For A_9, the nonvanishing invariants are:

$$\begin{aligned}
\widetilde{N}^4 &= 1+\mathbb{L}+\mathbb{L}^2+\mathbb{L}^3+\mathbb{L}^4\\
\widetilde{N}^3 &= 4+6\mathbb{L}+6\mathbb{L}^2+4\mathbb{L}^3\\
\widetilde{N}^2 &= 6+9\mathbb{L}+6\mathbb{L}^2\\
\widetilde{N}^1 &= 4+4\mathbb{L}\\
\widetilde{N}^0 &= 1
\end{aligned}$$

The $\widetilde{N}^i$ are not generally symmetric.

Example 29 For E_6, the nonvanishing invariants are:

$$\begin{aligned}
\widetilde{N}^3 &= 1+\mathbb{L}+2\mathbb{L}^2+\mathbb{L}^3\\
\widetilde{N}^2 &= 3+4\mathbb{L}+3\mathbb{L}^2\\
\widetilde{N}^1 &= 3+3\mathbb{L}\\
\widetilde{N}^0 &= 1
\end{aligned}$$

Example 30 For E_8, the nonvanishing invariants are:

$$\begin{aligned}
\widetilde{N}^4 &= 1+\mathbb{L}+2\mathbb{L}^2+2\mathbb{L}^3+\mathbb{L}^4\\
\widetilde{N}^3 &= 4+6\mathbb{L}+7\mathbb{L}^2+4\mathbb{L}^3\\
\widetilde{N}^2 &= 6+9\mathbb{L}+6\mathbb{L}^2\\
\widetilde{N}^1 &= 4+4\mathbb{L}\\
\widetilde{N}^0 &= 1
\end{aligned}$$

For the simple singularities, we have the following remarkable statement, which may be proven by comparing Theorem 27 to the description of the versal deformation of a simple singularity as the quotient of the hyperplane arrangement of the same name by the Weyl group; see Arnold, Gusein-Zade and Varchenko [4].

Theorem 31 *Let X be a simple singularity. Then there exists some curve C containing as its unique singularity a real form of X, and a real disc D^j in the real locus of the versal deformation of X such that*

$$\chi_{-y}(N_C^j) = \sum_i y^i \cdot \#(D^j \cap B_+^{j,i}),$$

where $B_+^{j,i}$ is the locus of real nodal curves with j total nodes of which i are hyperbolic.

This result was known to van Straten when $j=\delta(C)$; he had conjectured in this case that it holds for all singularities (see [68, Conjecture 4.7]) on the evidence of its validity for the simple singularities and its validity at $y=1$ for all singularities [22]. Theorem 1 asserts the validity of the above statement at $y=1$ for all singularities, so one might analogously conjecture that the statement of Theorem 31 holds always. Note that for unibranch singularities, there is a unique topological type of real form.

2.3 Real invariants

We recall that the Betti numbers may depend on the choice of coefficients for cohomology: for instance, we have

$$\sum q^n \dim \mathrm{H}^k(\mathbb{P}^2(\mathbb{R}),\mathbb{Q}) = 1 \neq 1+q+q^2 = \sum q^n \dim \mathrm{H}^k(\mathbb{P}^2(\mathbb{R}),\mathbb{Z}/2\mathbb{Z}).$$

Note that nonetheless we have $\chi(\mathbb{P}^2(\mathbb{R}),\mathbb{Q}) = 1 = \chi(\mathbb{P}^2(\mathbb{R}),\mathbb{Z}/2\mathbb{Z})$. In fact, as is well known to follow from the universal coefficient theorem, the Euler characteristic does not depend on the choice of coefficients. The same holds for compactly supported cohomology, for instance because this can be described as the relative cohomology of an appropriate pair. We therefore suppress the coefficients from the notation for Euler characteristics.

In particular, since any (not necessarily oriented or closed) manifold M enjoys Poincaré duality between $\mathrm{H}^*(M,\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{H}^*_c(M,\mathbb{Z}/2\mathbb{Z})$, we have $\chi_c(M) = (-1)^{\dim M}\chi(M)$.

As pointed out by Macdonald [48], for any reasonable space X, we have

$$\mathrm{H}^*(X^n/\mathfrak{S}_n,\mathbb{Q}) = \mathrm{H}^*(X^n,\mathbb{Q})^{\mathfrak{S}_n} = (\mathrm{H}^*(X,\mathbb{Q})^{\otimes n})^{\mathfrak{S}_n}$$

and it follows formally that

$$\sum_{n=0}^{\infty} q^n \chi(X^n/\mathfrak{S}_n) = \left(\frac{1}{1-q}\right)^{\chi(X)}.$$

We note this is false in general for the compactly supported Euler characteristic, for instance, one can show $\sum_{n=0}^{\infty} q^n \chi_c(\mathbb{R}^n/\mathfrak{S}_n) = 1$. Nonetheless, for real surfaces we have:

Lemma 32 *Let Σ be a manifold without boundary of (real) dimension 2; it may be nonorientable and noncompact. Then*

$$\sum_{n=0}^{\infty} q^n \chi_c(\Sigma^n/\mathfrak{S}_n) = \left(\frac{1}{1-q}\right)^{\chi_c(\Sigma)}.$$

Proof Both $\Sigma^n/\mathfrak{S}_n$ and Σ are real even-dimensional manifolds, and so we have $\chi = \chi_c$. □

Now let C be an algebraic curve defined over $\mathbb{R}$. We consider

$$Z_C^{\mathbb{R}} := \sum_{n=0}^{\infty} q^{n+1-g} \chi_c(C^{[n]}(\mathbb{R})).$$

Note there are *two* different projective lines over $\mathbb{R}$: one which has a real point (and hence whose real locus is a circle), say $X^2 - Y^2 + Z^2 = 0$, and one which has no real points $X^2 + Y^2 + Z^2 = 0$. We call them $\mathbb{P}^1_-$ and $\mathbb{P}^1_+$. Their symmetric powers likewise carry different complex conjugations:

$$(\mathbb{P}^1_-)^{[n]}(\mathbb{R}) = \mathbb{R}\mathbb{P}^n, \quad (\mathbb{P}^1_+)^{[2k+1]}(\mathbb{R}) = \varnothing, \quad (\mathbb{P}^1_+)^{[2k]}(\mathbb{R}) = \mathbb{R}\mathbb{P}^{2k}.$$

Nonetheless, since $\chi_c(\mathbb{R}\mathbb{P}^{2k+1}) = 0$, we have

$$Z^{\mathbb{R}}_{\mathbb{P}^1_+} = q/(1-q^2) = Z^{\mathbb{R}}_{\mathbb{P}^1_-}.$$

In fact, for a smooth curve C, the series $Z_C^{\mathbb{R}}$ does not depend on the real structure of C. We can determine it explicitly:

Lemma 33 *Let C be a curve defined over $\mathbb{R}$ such that $C \otimes_{\mathbb{R}} \mathbb{C}$ is a smooth curve of genus g. Then $Z_C^{\mathbb{R}} = (Z^{\mathbb{R}}_{\mathbb{P}^1})^{1-g}$.*

Proof As C is smooth, the Hilbert schemes agree with the symmetric products. Let σ denote the complex conjugation. We may stratify the locus $(\mathrm{Sym}^n(C))(\mathbb{R})$ according to the number of pairs of complex conjugate points. We parameterize the real points by the symmetric powers of the real locus, and n pairs of complex conjugate points in C by $\mathrm{Sym}^n((C(\mathbb{C}) \setminus C(\mathbb{R}))/\sigma)$. Thus compatibility of compactly supported Euler characteristic with cut-and-paste gives

$$\sum_{n=0}^{\infty} q^n \chi_c(\mathrm{Sym}^n(C)(\mathbb{R})) = \Big(\sum_{n=0}^{\infty} q^n \chi_c(\mathrm{Sym}^n(C(\mathbb{R})))\Big)\Big(\sum_{n=0}^{\infty} q^{2n} \chi_c\Big(\mathrm{Sym}^n\Big(\frac{C(\mathbb{C}) \setminus C(\mathbb{R})}{\sigma}\Big)\Big)\Big).$$

The loci $\mathrm{Sym}^n(C(\mathbb{R}))$ are products of symmetric products of the connected components of $C(\mathbb{R})$, ie of circles; it is easy to see (for instance by using a circle action) that these have Euler characteristic and compactly supported Euler characteristic zero. On the other hand, the second term was computed above in Lemma 32. We conclude

$$\sum_{n=0}^{\infty} q^n \chi_c(\mathrm{Sym}^n(C)(\mathbb{R})) = \sum_{n=0}^{\infty} q^{2n}\chi_c\left(\mathrm{Sym}^n\left(\frac{C(\mathbb{C})\setminus C(\mathbb{R})}{\sigma}\right)\right) = \left(\frac{1}{1-q^2}\right)^{\chi_c((C(\mathbb{C})\setminus C(\mathbb{R}))/\sigma)}.$$

Finally, since a circle has vanishing Euler characteristic, and since the Euler characteristic of a $n:1$ étale cover is just n times the Euler characteristic of the base, we have $\chi_c((C(\mathbb{C})\setminus C(\mathbb{R}))/\sigma) = \chi(C(\mathbb{C}))/2 = 1-g$. □

This motivates the definition of integers $n_C^{i,\mathbb{R}}$ by the formula

$$\sum_{n=0}^{\infty} \chi_{\mathbb{R}}(C^{[n]}) q^{n-g+1} = \sum_{i=0}^{\infty} n_C^{i,\mathbb{R}} \left(\frac{q}{1-q^2}\right)^{i-g+1}. \tag{2}$$

In this notation, Lemma 33 asserts that, for C a smooth curve defined over $\mathbb{R}$, we have $n_C^{0,\mathbb{R}} = 1$ and $n_C^{i,\mathbb{R}} = 0$ for $i > 0$.

For a point p on a curve C, write (C,p) for the germ at p, and $(C,p)^{[n]}$ for the locus in the Hilbert scheme of points on C of subschemes set-theoretically supported at p. For $p \in C(\mathbb{R})$, we write $rb(p)$ and $cb(p)$ for respectively the number of real and complex points above p in the normalization of C.

We have by stratification

$$\frac{\sum q^{n+1-g}\chi_c(C^{[n]}(\mathbb{R}))}{\sum q^{n+1-\tilde{g}}\chi_c(\tilde{C}^{[n]}(\mathbb{R}))} = \prod_{p\in C^{\mathrm{sing}}(\mathbb{R})} q^{-\delta(p)} \frac{\sum q^n \chi_c((C,p)^{[n]}(\mathbb{R}))}{(1-q^2)^{-cb(p)/2}(1-q)^{-rb(p)}} \times \prod_{p\in C^{\mathrm{sing}}(\mathbb{C})/\sigma} q^{-2\delta(p)} \frac{\sum q^{2n}\chi_c((C,p)^{[n]})}{(1-q^2)^{cb(p)}}.$$

We now compute $n_C^{\delta,\mathbb{R}}$ for a curve C with δ nodes and no other singularities. By the product formula above, it suffices to do this in three special cases.

Lemma 34 *The only nonvanishing invariants of a curve $c_\pm$ of arithmetic genus 1 with a single node analytically of the form $\mathbb{R}[\![x,y]\!]/(x^2\pm y^2)$ are $n^{0,\mathbb{R}} = 1$ and $n^{1,\mathbb{R}} = \pm 1$.*

(Caution: c_- is the one with a node that looks like $+$.)

Lemma 35 *The only nonvanishing invariants of a curve of arithmetic genus two with a pair of complex conjugate nodes are $n^{0,\mathbb{R}} = 1$ and $n^{1,\mathbb{R}} = 0$ and $n^{2,\mathbb{R}} = 1$.*

We conclude:

Proposition 36 *For a nodal curve C with $\delta = \delta_+ + \delta_- + 2\delta_0$ nodes, where $\delta_\pm$ are of the form $\mathbb{R}[\![x,y]\!]/(x^2 \pm y^2)$ and the $2\delta_0$ are complex conjugates,*

$$n_C^{\delta,\mathbb{R}} = (-1)^{\delta_-} = (-1)^{\delta_+ - \delta}.$$

Moreover, $n_C^{i,\mathbb{R}} = 0$ for $i > \delta$.

Theorem 37 *Let C be a real reduced plane curve, and let $\mathcal{C} \to B$ be a versal deformation of its singularities. Let $B_+^{\delta,\delta_-} \subset B(\mathbb{R})$ be the locus of nodal curves with δ nodes of which δ_- are of the form $\mathbb{R}[\![x,y]\!]/(x^2 - y^2)$. Let D^j be a general disc of dimension j, preserved by complex conjugation, passing near $[C] \in B$. Then*

$$n_C^{j,\mathbb{R}} = \sum_k (-1)^k D(\mathbb{R})^j \cap B_+^{j,k}.$$

In particular, $n_C^{j,\mathbb{R}} = 0$ for $j > \delta(C)$.

Proof In the following we write $\delta := \delta(C)$. View $n^{j,\mathbb{R}}$ as a constructible function on $B(\mathbb{R})$ taking $b \mapsto n_{\mathcal{C}_b}^{j,\mathbb{R}}$.

Let $D_0^i \subset B$ be a general (complex but preserved by conjugation) disc of dimension i containing $[C]$. Then by [22; 67], $\mathcal{C}^{[\leq i]}{}_{D_0^i}$ are all smooth, and if $i \geq \delta$ then all $\mathcal{C}^{[n]}{}_{D_0^i}$ are smooth. The real locus of a smooth variety is smooth, so the same holds upon passing to real points. Taking a disc $D^{\delta+1}$ containing D_0^δ and D_1^δ a sufficiently nearby slice, the spaces $\mathcal{C}^{[n]}{}_{D_0^\delta}(\mathbb{R})$ and $\mathcal{C}^{[n]}{}_{D_1^\delta}(\mathbb{R})$ are diffeomorphic (by smoothness) and therefore have the same compactly supported Euler characteristics. Note that the first j Hilbert schemes suffice to determine $n^{j,\mathbb{R}}$. By additivity it follows that

$$\text{(3)} \qquad \int_{D_0^i(\mathbb{R})} n^{j,\mathbb{R}} d\chi = \int_{D_1^i(\mathbb{R})} n^{j,\mathbb{R}} d\chi \quad \text{for any } j \leq i, \text{ and for all } j \text{ if } i \geq \delta.$$

We first show the vanishing of the $n^{j,\mathbb{R}}$ for $j > \delta(C)$. Note we have already established it for smooth and nodal curves. We induct on $\delta(C)$. Take $i = \delta$ in (3). By Diaz and Harris [17] and Teissier [69], the locus of curves of cogenus at least δ is of codimension δ and is the closure of the locus of δ–nodal curves. Thus by genericity its only intersection with D_0^δ is at the central point $[C]$, and its only intersection with D_1^δ is in finitely many nodal curves. Thus by induction and our explicit verification in the case of nodal curves, the integral on the right vanishes for $j > \delta$, hence so does the integral on the left, which is again equal to $n_C^{j,\mathbb{R}}$ by induction.

Now we consider the remaining n^j. Take $i = j$ in the above equality. Then by the same reasoning the only contribution to the integral on the left is $n_C^{i,\mathbb{R}}$, and the only contribution to the integral on the right is the j–nodal curves. By our previous calculation, these contribute as required. □

Remark 38 If C has a real line bundle of degree 1 then the proof of Proposition 15 and Corollary 17 carries through in $\mathrm{K}_0(\mathrm{var}/\mathbb{R})$; we may use this instead to conclude that $n_C^{i,\mathbb{R}} = 0$ for $i > g(C)$. However we have not found an analogous argument if C has no such bundle.

Remark 39 In [67], the proof of Theorem 1 used as an input the vanishing of the n_C^i for $i > \delta$; the argument given above shows this was unnecessary.

Remark 40 Thus a necessary condition for the statement of Theorem 31 to hold for a singularity X is the existence a curve C containing a real form of X with $\chi_1(Z_C) = Z_C^{\mathbb{R}}$.

Corollary 41 *Let $\mathcal{C} \to B$ be a family of reduced real plane curves in which all curves have cogenus less than or equal to δ and there are finitely many curves of cogenus δ, all nodal. Then defining $n_{\mathcal{C}/B}^{i,\mathbb{R}}$ by*

$$\sum_{n=0}^{\infty} \chi_c(\mathcal{C}_B^{[n]}(\mathbb{R}))q^{n-g+1} = \sum_{i=0}^{\infty} n_{\mathcal{C}/B}^{i,\mathbb{R}} \left(\frac{q}{1-q^2}\right)^{i-g+1},$$

the number of δ–nodal curves counted with signs $(-1)^{\delta_-} = (-1)^{\delta_+ - \delta}$ is $n_{\mathcal{C}/B}^{\delta,\mathbb{R}}$.

3 Refined invariants of linear systems

Let $\pi\colon \mathcal{C} \to B$ be a family of plane curves, and $\pi^{[n]} : \mathcal{C}^{[n]} \to B$ the relative Hilbert schemes. Denote by $\mathrm{MHM}(B)$ the category of mixed Hodge modules (see Saito [64]) over B. We define

$$\mathcal{Z}_{\mathcal{C}/B} = \sum_{n=0}^{\infty} \pi_!^{[n]} \mathbb{Q}_{\mathcal{C}^{[n]}} q^{n+1-g} \in K_0(\mathrm{MHM}(B))[\![q]\!].$$

We define invariants $\mathcal{N}_{\mathcal{C}/B}^i \in K_0(\mathrm{MHM}(B))$ by the formula

$$\sum_{n=0}^{\infty} \pi_!^{[n]} \mathbb{Q}_{\mathcal{C}^{[n]}} q^{n+1-g} = \sum_{i=0}^{\infty} \mathcal{N}_{\mathcal{C}/B}^i \times \mathcal{Z}_{\mathbb{P}^1/\mathrm{pt}}^{i+1-g}. \tag{4}$$

Proposition 42 *If $\pi\colon \mathcal{C} \to B$ is a family of integral plane curves and moreover $\mathcal{C}^{[n]}$ is smooth for all n, then $\mathcal{N}^i_{\mathcal{C}/B} = 0$ for $i > g$.*

Proof In fact, according to [52; 54] we know much more. Writing $\widetilde{\pi}$ for the restriction of the map to the locus on the base where it is smooth, and (1) for the Tate twist (it decreases weights by 2, so $[2](1)$ preserves weights) we have

$$(1-q)(1-q\mathbb{Q}_B[2](1)) \sum_{n=0}^{\infty} q^n \pi_!^{[n]} \mathbb{Q}_{\mathcal{C}^{[n]}}[\dim B + n] = \sum_{n=0}^{2g} q^n \mathrm{IC}(B, \Lambda^i R^1 \widetilde{\pi}_* \mathbb{Q}_{\mathcal{C}}[\dim B + g])[-i].$$

The right-hand side of the above equation is a polynomial of degree $2g$ which enjoys the symmetry $\Lambda^i R^1 \cong \Lambda^{2g-i} R^1$ by hard Lefschetz; we deduce the vanishing of the $\mathcal{N}^i_{\mathcal{C}/B}$ as in Corollary 17. □

Remark 43 One may write the same definition in $\mathrm{K}_0(\mathrm{var}/B)$, but then we do not know whether $\mathcal{N}^i$ vanishes for $i > g$. The problem arises already for families of smooth curves; the question here is whether a family of Jacobians is equivalent to its torsors in the Grothendieck group of varieties.

We may take pointwise Euler characteristic in order to define an integer valued constructible function $n^i := \chi(\mathcal{N}^i)$, or global Euler characteristic to define $n^i_{\mathcal{C}/B} \in \mathbb{Z}$. (By proper base-change, these are the same n^i as in the introduction.) Since we have that n^i is supported on the locus of curves of cogenus i, certainly $n^k_{\mathcal{C}/B}$ vanishes if k is greater than the maximum cogenus of any curve in the family. These are the same as the constructible functions of the introduction, that is, they satisfy (and could take as their definition) the formula

$$\sum_{n=0}^{\infty} \chi(\mathcal{C}^{[n]}) q^{n+1-g} = \sum_{i=0}^{\infty} n^i_{\mathcal{C}/B} \left(\frac{q}{(1-q)^2} \right)^{i+1-g}.$$

Definition 44 We define $N^i_{\mathcal{C}/B}(y) := \chi_{-y}((B \to \mathrm{pt})_! \mathcal{N}^i)$. Equivalently we can apply $\chi_{-y} \circ (B \to \mathrm{pt})_!$ to both sides of (4) in order to directly define $N^i_{\mathcal{C}/B}(y)$ by the formula

$$\sum_{n=0}^{\infty} \chi_{-y}(\mathcal{C}^{[n]}) q^{n+1-g} = \sum_{i=0}^{\infty} N^i_{\mathcal{C}/B}(y) \left(\frac{q}{(1-q)(1-qy)} \right)^{i+1-g}.$$

From Proposition 42, we see that $N^i_{\mathcal{C}/B}(y)$ vanishes for i greater than the maximum genus of any curve in a family whose relative Hilbert schemes are nonsingular. But for both the χ_{-y} invariants of a single curve, and the χ invariants of families, we had vanishing beyond the maximum *cogenus*. Thus we may at least plausibly ask whether this holds for $N^i_{\mathcal{C}/B}$. In fact it need not: Fantechi and Pandharipande observed that this vanishing can fail already for B a curve of positive genus and $\mathcal{C} \to B$ a family of smooth curves. However, empirically, the situation appears to be better for linear systems of curves in surfaces. We have the following conjecture:

Conjecture 45 *Let L be a line bundle on a surface S, $\mathcal{C} \to |L|$ the tautological family of curves, and $\mathbb{P}^\delta \subset |L|$ be a linear subsystem. Assume the relative Hilbert schemes $\mathcal{C}^{[n]}_{\mathbb{P}^\delta}$ are nonsingular for all $n \geq 0$. Then $N^i_{\mathcal{C}/\mathbb{P}^\delta}(y) = 0$ for $i > \delta$.*

From the smoothness criterion in [22] one may deduce that the maximum cogenus of any curve in any such $\mathbb{P}^\delta$ is δ. The assumption holds in the following situations:

Theorem 46 *Let L be a line bundle on a surface S, $\mathcal{C} \to |L|$ the tautological family of curves, and $\mathbb{P}^\delta \subset |L|$ a general linear subsystem. Then all relative Hilbert schemes $\mathcal{C}^{[n]}_{\mathbb{P}^\delta}$ are nonsingular in the following situations:*

- *S is arbitrary and L is δ–very ample [43; 22; 67].*[6]
- *S is a $K3$ or abelian surface and L is irreducible (see Mukai [56]).*
- *S is a rational surface and $\mathbb{P}^\delta$ contains no nonreduced curves, and no curves with components which intersect K_S nonnegatively [40]; in particular, for a general $\mathbb{P}^{\leq 2d-2} \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$.*

Rather than simply integrate to get $n^i_{\mathcal{C}/B} = \int_B n^i \, d\chi$, we can extract more refined information by taking Chern–Schwarz–Macpherson classes c_*^{SM}. We recall that c_*^{SM} is the unique map from constructible functions to homology which commutes with pushforward and satisfies the normalization $c_*^{SM}(1_X) = c(TX) \cap [X]$ when X is smooth projective; its existence was conjectured by Deligne and Grothendieck and established by Macpherson [49]. Now taking $\boldsymbol{n}^i := c_*^{SM}(n^i) \in H_*(B)$, we see by Macpherson's theorem that

$$\sum_{n=0}^{\infty} \pi_*^{[n]}(c(T\mathcal{C}^{[n]}) \cap [\mathcal{C}^{[n]}])q^{n+1-g} = \sum_{i=0}^{\infty} \boldsymbol{n}^i_{\mathcal{C}/B} \left(\frac{q}{(1-q)^2} \right)^{i+1-g}.$$

Of course we are also free to take this as the definition of the $\boldsymbol{n}^i_{\mathcal{C}/B}$ and conclude from Macpherson's theorem that $\boldsymbol{n}^i_{\mathcal{C}/B}$ vanishes if i is greater than the maximum cogenus

[6] What is actually observed in [43] is the (obvious) fact that the first δ relative Hilbert schemes are nonsingular under this hypothesis, and the less obvious fact that the assumption implies that no nonreduced curves or curves of cogenus greater than δ occur in the $\mathbb{P}^\delta$. Consequently, we have that the smoothness criteria in [22; 67] may be used to establish smoothness of the remaining relative Hilbert schemes.

of any curve in the family. In good cases, the constructible function n^i and the class $\boldsymbol{n}^i$ carry singularity-theoretic meaning:

Theorem 47 *Let $\mathcal{C}/B$ be a family of reduced plane curves with all $\mathcal{C}_B^{[n]}$ nonsingular. Let B^i be the locus of curves of cogenus i, and let B^i_+ be the sublocus of curves smooth away from i nodes. Assume $B^j \subset \overline{B}^i_+$ for all $j \geq i$. Then*

$$\mathrm{mult}_b(\overline{B}^i_+) = n^i(b) = \mathrm{Eu}_b(\overline{B}^i_+).$$

Here Eu is the local Euler obstruction. Moreover, $\boldsymbol{n}^i_{\mathcal{C}/B}$ is the Chern–Mather class of $\overline{B}^i_+$.

Proof The first equality was asserted in [67] in the case of a locally versal family, but in fact the same argument applies in the above generality.

For the second, it is shown by Migliorini and Shende in [53] that for *any* proper map $f\colon X \to Y$ of algebraic varieties, the varieties V^α which appear in the expansion $f_*1 = \sum c_\alpha \cdot \mathrm{Eu}(V^\alpha)$ are all components of the higher discriminants (introduced in [53]) of the morphism f. It was shown in [67] that the higher discriminants of the map $\mathcal{C}_B^{[n]} \to B$ are precisely the loci $\overline{B}^i$, which by assumption agree with $\overline{B}^i_+$. It remains to determine the coefficients c_α; this may be done at the general point of each $\overline{B}^i_+$, ie in B^i_+. Now we can either make a calculation for nodal curves, or alternatively observe that since the loci B^i_+ are all immersed in B their Euler obstructions agree with their multiplicities.

The identification of $\boldsymbol{n}^i$ with the Chern–Mather class of $\overline{B}^i_+$ now follows from Macpherson's construction of the functorial Chern class. □

Remark 48 The assumption on genericity of nodal curves in Theorem 47 holds for a locally versal family by [17; 69], for the general $\mathbb{P}^\delta \subset |L|$ when L is δ–very-ample by [43], for the general $\mathbb{P}^\delta \subset |L|$ when L is irreducible on a general $K3$ surface (the genericity of nodal curves in maximal cogenus is shown in Chen [14]), and it was known classically for the general $\mathbb{P} \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ containing no nonreduced curves.

Remark 49 Aluffi [3] has shown that the multiplicity and Euler obstruction of the discriminant of cubic curves on $\mathbb{P}^2$ differ at a triple line. The argument above fails here because the total space of the restriction of the universal family to a one-dimensional disc passing through this point is necessarily singular. Aluffi has also managed to extract enumerative information about curves with singularities more complicated than nodes from the Chern–Mather and Chern–Schwarz–Macpherson classes of discriminants; perhaps the same can be done with the higher Severi strata.

The map c_*^{SM} admits a refinement due to Brasselet, Schürmann and Yokura. We denote it by $X_{-y}^{\mathrm{BSY}}\colon \mathrm{MHM}(\cdot) \to \mathrm{H}_*(\cdot)[y]$. It commutes with pushforward and obeys the normalization $X_{-y}^{\mathrm{BSY}}(\mathbb{Q}_M) = X_{-y}(TM) \cap [M]$ for M proper smooth. Thus we may apply their functor to the $\mathcal{N}^i_{\mathcal{C}/B}$ and conclude that there are $\boldsymbol{N}^i_{\mathcal{C}/B}(y) = X_{-y}^{\mathrm{BSY}}(\mathcal{N}^i) \in \mathrm{H}_*(B)[y]$ such that $\boldsymbol{N}^i_{\mathcal{C}/B}(y) = 0$ for i greater than the arithmetic genus of the curves (in a family of integral curves with nonsingular relative Hilbert schemes), and

$$\sum_{n=0}^{\infty} \pi_*^{[n]}(X_{-y}(T\mathcal{C}^{[n]}) \cap [\mathcal{C}^{[n]}])q^{n+1-g} = \sum_{i=0}^{\infty} \boldsymbol{N}^i_{\mathcal{C}/B}(y)\left(\frac{q}{(1-q)(1-qy)}\right)^{i+1-g}.$$

In the case of interest when $B = \mathbb{P}^\delta$, we denote by abuse $\boldsymbol{N}^i_{\mathcal{C}/B}(y, H) \in \mathrm{H}^*(\mathbb{P}^\delta)$ the Poincaré dual class of $\boldsymbol{N}^i_{\mathcal{C}/B}(y)$. Since $\boldsymbol{N}^i_{\mathcal{C}/B}(1)$ is, in good cases, the Chern–Mather class of the (codimension i) Severi variety of cogenus i curves, we might expect:

Conjecture 50 *Let L be a line bundle on a surface S, and let $\mathbb{P}^\delta \subset |L|$ be a linear subsystem of reduced curves over which the relative Hilbert schemes $\mathcal{C}^{[n]} \to \mathbb{P}^\delta$ are nonsingular for all $n \geq 0$. Then $\boldsymbol{N}^i_{\mathcal{C}/\mathbb{P}^\delta}(y, H)$ is a polynomial of minimal degree i in H, and in particular vanishes for $i > \delta$.*

We will see shortly that this conjecture is in fact equivalent to Conjecture 45. First we recall how the Hirzebruch genera of the relative Hilbert schemes may be computed.

3.1 Genera of relative Hilbert schemes

Following Hirzebruch [31], we take a (normalized) genus to mean a natural transformation of contravariant functors $\Phi\colon K^0(\cdot) \to \mathrm{H}^*(\cdot, \Lambda)$ (where Λ is a commutative ring) such that:

- For the trivial bundle $\mathbb{C}$, we have $\Phi(\mathbb{C}) = 1$.
- Sums go to products: $\Phi(E \oplus F) = \Phi(E)\Phi(F)$.
- There is a power series $f_\Phi \in 1 + z\Lambda[\![z]\!]$ such that for a line bundle L, we have $\Phi(L) = f_\Phi(c_1(L))$.

In the remainder of the paper we will be concerned only with the Hirzebruch genus $\Phi = X_{-y}$, for which $\Lambda = \mathbb{Q}[\![y]\!]$ and $f(z) = (z(1-ye^{-z(1-y)}))/(1-e^{-z(1-y)})$. In any case fix some Λ, Φ. We write $\phi(X) := \Phi(TX)$.

Let S be a surface, L a line bundle on it, $\mathbb{P}^\delta \subset |L|$ some linear system, $H = \mathcal{O}_{\mathbb{P}^\delta}(1)$. Let $S^{[n]}$ be the Hilbert scheme of n points on S, and let $Z_n(S) \subset S \times S^{[n]}$ be the universal family, with the projections $q\colon Z_n(S) \to S$, $p : Z_n(S) \to S^{[n]}$. Let

$L^{[n]} := p_* q^* L$. This is a vector bundle of rank n on $S^{[n]}$ with fiber $\mathrm{H}^0(Z, L|_Z)$ over $Z \in S^{[n]}$. Let $\pi\colon \mathcal{C}_{\mathbb{P}^\delta} \to \mathbb{P}^\delta$ be the universal curve over $\mathbb{P}^\delta$ and denote by $\pi^{[n]}\colon \mathcal{C}^{[n]}_{\mathbb{P}^\delta} \to \mathbb{P}^\delta$ the relative Hilbert scheme of points. The relative Hilbert scheme always has the expected dimension $\delta+n$ (see Altman, Iarrobino and Kleiman [1]), and is the scheme-theoretic zero locus of a tautological section of $L^{[n]} \boxtimes H$; this section is transverse when $\mathcal{C}^{[n]}_{\mathbb{P}^\delta}$ is nonsingular.

As a bookkeeping device, let e^x denote a trivial line bundle with nontrivial $\mathbb{C}^*$ action giving equivariant first Chern class x, ie $\Phi(e^x) = f_\Phi(x)$.

Definition 51 Let

$$D_n^{S,L,\Phi}(x) := \int_{S^{[n]}} \Phi(TS^{[n]}) \frac{c_n(L^{[n]} \otimes e^x)}{\Phi(L^{[n]} \otimes e^x)} \in \Lambda[\![x]\!].$$

Proposition 52 *Assume $\mathcal{C}^{[n]}_{\mathbb{P}^\delta}$ is nonsingular. Then*

$$\pi^{[n]}_*\big(\Phi\big(T\mathcal{C}^{[n]}_{\mathbb{P}^\delta}\big) \cap [\mathcal{C}^{[n]}]\big) = f_\Phi(H)^{\delta+1} D_n^{S,L,\Phi}(H) \cap [\mathbb{P}^\delta].$$

In particular,

$$\phi\big(\mathcal{C}^{[n]}_{\mathbb{P}^\delta}\big) = \mathrm{Res}_{x=0}\left(\frac{f_\Phi(x)}{x}\right)^{\delta+1} D_n^{S,L,\Phi}(x).$$

Proof Denote $q\colon S^{[n]} \to \mathrm{pt}$. Then

$$\begin{aligned}\pi^{[n]}_*&\big(\Phi\big(T\mathcal{C}^{[n]}_{\mathbb{P}^\delta}\big) \cap [\mathcal{C}^{[n]}]\big)\\ &= (q \times 1_{\mathbb{P}^\delta})_*\left(c_n(L^{[n]} \boxtimes H) \frac{\Phi(TS^{[n]})\Phi(T\mathbb{P}^{[\delta]})}{\Phi(L^{[n]} \boxtimes H)} \cap [S^{[n]}]\right) \cap [\mathbb{P}^\delta]\\ &= \Phi(H)^{\delta+1} q_*\left[\Phi(TS^{[n]}) \frac{c_n(L^{[n]} \otimes e^x)}{\Phi(L^{[n]} \otimes e^x)}\right]\Bigg|_{x=H} \cap [\mathbb{P}^\delta].\end{aligned}$$

This completes the proof. □

Remark 53 If $h^0(L) > \delta$, the formula makes sense without requiring smoothness, if we view it as a virtual contribution. The description of $\mathcal{C}^{[n]}_\delta$ as zero locus of a section of $L^{[n]} \boxtimes H$ gives it a virtual fundamental class and a virtual tangent bundle (see eg Fantechi and Göttsche [21]). Thus independent of the singularities of $\mathcal{C}^{[n]}_\delta$, what is computed here is $\phi(\mathcal{C}^{[n]})$ with this virtual structure. Without any assumption on L, we can view the second equality in Proposition 52 as a definition of $\phi(\mathcal{C}^{[n]})$.

The $D_n^{S,L,\Phi}$ enjoy a certain multiplicativity. Introduce the series

(5) $$D^{S,L,\Phi} = \sum D_n^{S,L,\Phi} q^n.$$

We denote by $[S,L]$ the algebraic cobordism class of the pair of the surface S and the line bundle L. For us[7] this is the equivalence class of pairs (S,L), where two such (S_1,L_1), (S_2,L_2) are equivalent if the numbers L_i^2, $L_iK_{S_i}$, ${K_{S_i}}^2$, $c_2(S_i)$ coincide for $i=1,2$. These form a group where in particular $[S_1,L_1]+[S_2,L_2]=[S_1\sqcup S_2,L]$ with L the line bundle which is L_1 on S_1 and L_2 on S_2.

Proposition 54 *The map $[S,L]\mapsto D^{S,L,\Phi}$ is a homomorphism from the cobordism group of surfaces with bundles to the multiplicative group of invertible power series in q with coefficients in Λ. In particular there exist universal power series $D_1,D_2,D_3,D_4\in\Lambda[\![q]\!]$ such that*

$$D^{S,L,\Phi}=D_1^{L^2}D_2^{LK_S}D_3^{K_S^2}D_4^{c_2(S)}.$$

Proof The $D_n^{S,L,\Phi}(x)$ are defined by a genus applied to $L^{[n]}$ and $T_{S^{[n]}}$, so the first statement follows from the arguments in [20].

For the second statement, let (S_i,L_i), $i=1,\dots,4$ be chosen such that the four vectors $(L_i^2,L_iK_{S_i},K_{S_i}^2,c_2(S_i))$ are linearly independent. Then for any (S,L) we can write $[S,L]=a_1[S_1,L_1]+a_2[S_2,L_2]+a_3[S_3,L_3]+a_4[S_4,L_4]$, with $a_i\in\mathbb{Q}$ and thus

$$D^{S,L,\Phi}=\prod_{i=1}^{4}(D^{S,L_i,\Phi})^{a_i}.$$

For simplicity we write $[S_i,L_i]$ also for the corresponding vector. Let $e_1=(1,0,0,0)$, $e_2=(0,1,0,0)$, $e_3=(0,0,1,0)$, $e_4=(0,0,0,1)$. Choosing $(a_{i,j})_{i,j=1}^4$ in $\mathbb{Q}$ with $\sum_j a_{i,j}[S_j,L_j]=e_i$ for all i, we put $D_i=\prod_{j=1}^4(D^{S,L_i,\Phi})^{a_{i,j}}$. As $[S,L]=L^2e_1+LK_Se_2+K_S^2e_3+e(S)e_4$, this gives

$$D^{S,L,\Phi}=D_1^{L^2}D_2^{LK_S}D_3^{K_S^2}D_4^{c_2(S)}.\qquad\square$$

The multiplicativity of Proposition 54 allows $D_n^{S,L,\Phi}(x)$ (and therefore also $\phi(\mathcal{C}^{[n]})$ and the BPS invariants) to be computed by localization in the following standard way.

By the above argument to compute $D^{S,L,\Phi}$ for any (S,L) it is enough to compute them for pairs (L_i,S_i) whose corresponding vectors are linearly independent. We can choose the (S_i,L_i) as $(\mathbb{P}^2,\mathcal{O})$, $(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O})$, $(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1,0))$, $(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1,-1))$. In this case $S=S_i$ is a toric surface, ie it has an action by a torus $(\mathbb{C}^*)^2$ with finitely many fixed points, and $L=L_i$ has an natural equivariant lifting. The action of $(\mathbb{C}^*)^2$ on S induces in a natural way an action on $S^{[n]}$, and the equivariant lifting of L

[7] For a geometric account of algebraic cobordism of varieties with bundles see Lee and Pandharipande [47]; we however do not require any results of this theory and only use "cobordism class" as a convenient shorthand.

induces an equivariant lifting of $L^{[n]}$. Thus we can apply equivariant localization to compute $D_n^{S,L,\Phi}(x)$, in terms of the weights of the action on the fibers of $T_{S^{[n]}}$ and $L^{[n]}$ at the fixed points. The fixed points are parametrized by tuples of Young diagrams and the weights of the action can be expressed explicitly in terms of this data. For more details in a slightly different situation see eg Ellingsrud and Göttsche [19], Nakajima and Yoshioka [57] and Carlsson and Okounkov [13].

From now on we specialize to $\Phi = X_{-y}$ and abbreviate $D_n^{S,L}(y,x) := D_n^{S,L,X_{-y}}(x)$, $D^{S,L}(y,x,q) := \sum_{n\geq 0} D_n^{S,L}(y,x)q^n$. In this case a computer calculation yields the $D_n^{S,L}(y,x)$ for $n \leq 10$ and modulo x^{14}. The $\chi_{-y}(\mathcal{C}^{[n]}_{\mathbb{P}^\delta})$ are computed from this by Proposition 52.

3.2 A reformulation of the conjectures, and evidence.

Let t_i be the Chern roots of the tangent bundle of the Hilbert scheme, and let l_i be the Chern roots of the bundle $L^{[i]}$. In the previous subsection we introduced the series

(6)
$$D^{S,L}(y,x,q) := \sum q^n \int_{S^{[n]}} \prod_{i=1}^{2n} \frac{t_i(1-ye^{-t_i(1-y)})}{(1-e^{-t_i(1-y)})} \cdot \prod_{j=1}^{n} \frac{(1-e^{-(l_j+x)(1-y)})}{(1-ye^{-(l_j+x)(1-y)})} \in \mathbb{Q}[y][\![x]\!][\![q]\!].$$

For convenience we write $Q = q/((1-q)(1-qy))$.[8]

By Proposition 52 we have

(7)
$$\sum_{i=0}^{\infty} N^i_{\mathcal{C}/\mathbb{P}^\delta}(y,H)Q^i = \left(\frac{q}{Q}\right)^{1-g} \left(\frac{H(1-ye^{-H(1-y)})}{1-e^{-H(1-y)}}\right)^{\delta+1} D^{S,L}(y,H,q).$$

[8] We record here that the compositional inverse is given by what are called the Narayana numbers,

$$q(Q) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} Q^n y^{k-1}((-1)^{n-1}/n)\binom{n}{k}\binom{n}{k-1},$$

which specializes to the following formulas involving Catalan numbers,

$$q(Q)|_{y=1} = \sum_{n=1}^{\infty} Q^n((-1)^{n-1}/(n+1))\binom{2n}{n},$$
$$q(Q)|_{y=0} = Q/(1+Q),$$
$$q(Q)|_{y=-1} = \sum_{n=0}^{\infty} Q^{2n+1}(-1)^n/(2n+1)\binom{4n}{2n}.$$

Note also that $Q(q) \in q\mathbb{Z}[\![q,qy]\!]$ and $q(Q) \in Q\mathbb{Z}[\![Q,Qy]\!]$.

Conjecture 55 *For any surface S and line bundle L, we have*

$$(q/Q)^{1-g(L)} D^{S,L}(y,x,q) \in \mathbb{Q}[y][\![x, xQ]\!].$$

Proposition 56 *Conjectures 45, 50 and 55 are equivalent.*

Proof Note $f(x) := X_{-y}(e^x) \in 1 + x\mathbb{Q}[y][\![x]\!]$ is invertible. From (7), we see that Conjecture 55 implies Conjectures 45 and 50.

Assume Conjecture 45. Now consider some fixed linear system $\mathbb{P}^\delta \subset |L|$ on some surface S such that all the relative Hilbert schemes $\mathcal{C}^{[n]} \to \mathbb{P}^\delta$ are nonsingular. Conjecture 50 amounts to the statement

$$(Q/q)^{g-1} f(x)^{\delta+1} D^{S,L}(y,x,q) \in \mathbb{Q}[\![x, xQ]\!] + O(x^{\delta+1}).$$

This obviously holds at $\delta = 0$; let us prove it holds at δ by induction. If we know this statement holds for some $\delta = r-1$ and wish to check it for $\delta = r$, since $f(x) \in \mathbb{Q}[\![x]\!]$ we already know the statement modulo x^r. So we need only check

$$\deg_Q \operatorname{Coeff}_{x^r} (Q/q)^{g-1} f_\Phi(x)^{r+1} D^{S,L}(y,x,q) \leq r.$$

But this is precisely the assertion of Conjecture 45 for $\mathbb{P}^r$. And since all the relative Hilbert schemes will also be smooth over a general $\mathbb{P}^r \subset \mathbb{P}^\delta$ for any $r \leq \delta$, the hypothesis of Conjecture 45 is satisfied and we may deduce Conjecture 50.

Finally, Conjecture 50 asserts that Conjecture 55 holds modulo $x^{\delta+1}$. We have an expression

$$D^{S,L} = D_1^{L^2} D_2^{L.K_S} D_3^{K_S^2} D_4^{c_2(S)},$$

where the D_i are power series starting with 1. Thus we compare $D^{S,L}$ for various surfaces and line bundles to conclude the statement of Conjecture 55 for the series D_i, modulo some x^k. Taking $k \to \infty$ by choosing increasingly ample line bundles recovers the statement for $D^{S,L}$. □

Remark 57 The above argument implies in particular that Conjecture 55 holds in the Euler characteristic limit $y = 1$. From this it follows formally that for *any* $\mathbb{P}^\delta \subset |L|$, with no assumptions on the reducedness or irreducibility of the curves that appear or on the smoothness of the relative Hilbert schemes, there are integers $n^i_{\mathcal{C}/\mathbb{P}^\delta}$ such that

$$\sum_{n=0}^{\infty} q^{n+1-g} \int_{[\mathcal{C}^{[n]}]^{\mathrm{vir}}} c_{\mathrm{top}}(T^{\mathrm{vir}}\mathcal{C}^{[n]}) = \sum_{i=0}^{\delta} n^i_{\mathcal{C}/\mathbb{P}^\delta} \left(\frac{q}{(1-q)^2}\right)^{i+1-g}.$$

Recall that $D^{S,L}$ can be expressed in four universal power series,

$$D^{S,L} = D_1^{L^2} D_2^{K_S.L} D_3^{K_S^2} D_4^{c_2(S)}.$$

To avoid writing $(Q/q)^{g-1}$ we adjust these series slightly.

Definition 58 We write $\widetilde{D}^{S,L} := (Q/q)^{g-1} D^{S,L}$. We also take $\widetilde{D}_1 := (Q/q)^{1/2} D_1$, $\widetilde{D}_2 := (Q/q)^{1/2} D_2$ and $\widetilde{D}_3 = D_3$, $\widetilde{D}_4 = D_4$, so that

$$\widetilde{D}^{S,L} = \widetilde{D}_1^{L^2} \widetilde{D}_2^{K_S.L} \widetilde{D}_3^{K_S^2} \widetilde{D}_4^{c_2(S)}.$$

We have $\widetilde{D}^{S,L} \in 1 + (y, x, Q)\mathbb{Q}[y][\![x, Q]\!]$ for all S, L, hence the same is true for the $\widetilde{D}_i$. Similarly, Conjecture 55 is equivalent to the assertion that $\widetilde{D}_i \in \mathbb{Q}[y][\![x, xQ]\!]$ for all i.

Theorem 59 *We have* $\widetilde{D}_1, \widetilde{D}_4 \in \mathbb{Q}[y][\![x, xQ]\!]$.

Proof Let (A, L) be a primitively polarized abelian surface of Picard rank 1. If $L^2 = 2k + 2$ then $\dim |L| = k$ and the curves in $|L|$ have arithmetic genus $k + 2$. Note such (A, L) exist for all k. By [56] the relative Hilbert schemes are smooth, and so from Proposition 42 we find that the N^i vanish beyond the arithmetic genus. By this vanishing and the formula (7) extracting the N^i from $D^{S,L}$,

$$\deg_Q \operatorname{Coeff}_{x^k} f_\Phi(x)^{k+1} \widetilde{D}_1^{2k+2} \le k + 2.$$

We write $\xi(x, Q) = f_\Phi(x)\widetilde{D}_1^2 \in 1 + (x, Q)\mathbb{Q}[y][\![x, Q]\!]$. We want to show that $\widetilde{D}_1 \in \mathbb{Q}[y][\![x, xQ]\!]$; since this evidently holds for $f_\Phi(x)$ and we may take roots of power series starting with 1, it suffices to show this for ξ. So we have that $\deg_Q \operatorname{Coeff}_{x^k} \xi(x, Q)^{k+1} \le k + 2$.

The following argument is completely formal and does not involve the geometric meaning of ξ. We write $d_Q(k) := \deg_Q \operatorname{Coeff}_{x^k} \xi(x, Q)^{k+1}$. Let $k_1 = \min\{k \mid d_Q(k) > k\}$, assuming this set is nonempty. Then $d_Q(k_1) = \deg_Q \operatorname{Coeff}_{x^{k_1}} \xi(x, Q)^{k_1+1}$, since no lower (in x) degree term can contribute such a high power of Q. There are two cases, $d_Q(k_1) = k_1 + 2$ or $d_Q(k_1) = k_1 + 1$. In the first case, consider $\operatorname{Coeff}_{x^{2k_1}} \xi(x, Q)^{2k_1+1}$. There will be a contribution from products of two terms of the form $Q^{k_1+2} x^{k_1}$, which gives the highest possible power of Q and thus cannot be canceled. But then $\deg_Q \operatorname{Coeff}_{x^{2k_1}} \xi(x, Q)^{2k_1+1} = 2k_1 + 4$, which is a contradiction. In the second case, consider $\operatorname{Coeff}_{x^{3k_1}} \xi(x, Q)^{3k_1+1}$. In order that the degree $3k_1 + 3$ contribution from products of three terms $Q^{k_1+1} x^{k_1}$ be canceled, there must be some $h + h' = 3k_1$ with $d_Q(h) = h + 2$ and $d_Q(h') > h'$. By minimality of k_1, we have $h' > k_1$ hence $h < 2k_1$. Let $k_2 = \min\{k \mid d_Q(k) > k + 1\} \le h < 2k_1$. Finally consider $\mathrm{Coeff}_{x^{k_1+k_2}}\,\xi(x,Q)^{k_1+k_2+1}$. There is a contribution from products of terms the form $x^{k_1}Q^{k_1+1}$ and $x^{k_2}Q^{k_2+2}$; since $k_2 < 2k_1$ this contribution cannot be canceled. This is a contradiction. So finally we must have $d_Q(k) \le k$ for all k, hence $\xi(x,Q) \in \mathbb{C}[y][\![x, xQ]\!]$, hence the same holds for $\widetilde{D}_1$.

Now let (K, L) be a primitively polarized $K3$ surface of Picard rank 1. If $L^2 = 2g-2$ then $\dim |L| = g$ and the curves in $|L|$ have genus g. Such (K, L) exist for all g. By vanishing of the N^i beyond the arithmetic genus we have

$$\deg_Q \mathrm{Coeff}_{x^g}\, f_\Phi(x)^{g+1}\,\widetilde{D}_1^{2g-2} D_4^{24} \le g.$$

Since we know $f_\Phi(x), \widetilde{D}_1 \in \mathbb{Q}[y][\![x, xQ]\!]$, we may conclude the same for D_4. □

Corollary 60 *Conjectures 45, 50 and 55 hold for surfaces with numerically trivial canonical class.*

Remark 61 Note the slightly curious nature of the proof of the theorem and corollary: for geometric reasons, namely smoothness of the relative Hilbert schemes and the near equality of the genus and dimension of the linear system for certain line bundles on $K3$ and abelian surfaces, we know the conjecture for *complete* linear systems on $K3$ surfaces and something close for abelian surfaces. Then by leveraging the universality of the expressions, and the existence of $K3$ and abelian surfaces of all genera, we can conclude the result also for *not necessarily complete* linear systems.

This sort of approach was suggested to the authors by Pandharipande [60], who further suggested that the other power series may be similarly constrained by finding enough other surfaces with nontrivial canonical class but for which nonetheless the genus and dimension of linear systems are close. However to use these (or any) surfaces for the present purposes, one must establish smoothness of the relative Hilbert schemes, which we do not know how to do.

Using the localization calculation described in Section 3.1, we can give evidence for Conjecture 55 for arbitrary surfaces.

Proposition 62 *Conjecture 55 holds modulo q^{11} and x^{14}. Therefore, if $0 \le \delta \le 13$ and $\mathbb{P}^\delta \subset |L|$ is a linear system over which the relative Hilbert scheme is smooth, there exist polynomials $N^i_{\mathcal{C}/\mathbb{P}^\delta}(y) \in \mathbb{Z}[y]$, where $i = 0, \dots, \delta$, such that*

$$\sum_{n\ge 0} \chi_{-y}(\mathcal{C}^{[n]})q^{n+1-g} \equiv \sum_{l=0}^{\delta} N^i_{\mathcal{C}/\mathbb{P}^\delta}(y)\,Q^{l+1-g} \mod O(q^{11+1-g}),$$

and furthermore these polynomials are explicitly computed. If moreover $g < 11$ and all curves are irreducible, then the equality is established to all orders.

For example, Conjecture 45 holds for a general $\mathbb{P}^4$ in $|\mathcal{O}_{\mathbb{P}^2}(6)|$.

The relation between the various $n, \boldsymbol{n}, N, \boldsymbol{N}$ and the series $D^{S,L}$ has in this section always been contingent on the smoothness of the relative Hilbert schemes over the appropriate linear subsystem $\mathbb{P}^\delta \subset |L|$. To avoid continually making this hypothesis, we introduce the following:

Definition 63 For a surface S and a line bundle L, we define $\boldsymbol{N}^i_{\delta,[S,L]}$, $N^i_{\delta,[S,L]}$, by the formulas

$$\sum_{i=0}^{\infty} \boldsymbol{N}^i_{\delta,[S,L]} Q^i = \left(\frac{H(1-ye^{-H(1-y)})}{1-e^{-H(1-y)}}\right)^{\delta+1} \widetilde{D}^{S,L}(y,H,q),$$

$$\sum_{i=0}^{\infty} N^i_{\delta,[S,L]} Q^i = \mathrm{Res}_{x=0}\left[\left(\frac{1-ye^{-x(1-y)}}{1-e^{-x(1-y)}}\right)^{\delta+1} \widetilde{D}^{S,L}(y,x,q)\right],$$

and similarly for the specializations $n, \boldsymbol{n}$.

By comparison with (7), we see that for a linear system $\mathbb{P}^\delta \subset |L|$ containing only reduced curves, and whose relative Hilbert schemes are smooth, $\boldsymbol{N}^i_{\mathbb{P}^\delta \subset |L|} = \boldsymbol{N}^i_{\delta,[S,L]}$.

4 The term of the deepest stratum

For a linear system $\mathbb{P}^\delta \subset |L|$, the numbers $n^\delta_{\mathcal{C}/\mathbb{P}^\delta}$ have the clearest enumerative significance, counting the number of δ–nodal curves in the linear system. Thus we might also hope that the $N^\delta_{\mathcal{C}/\mathbb{P}^\delta}(y)$ have an enumerative meaning refining this. In any case, assuming Conjecture 55, the $N^\delta_{\delta,[S,L]}(y)$ are the easiest to compute and their generating function is multiplicative.

We have been in the meantime in able to prove the main conjecture of this section (Conjecture 67) in case K_S is numerically trivial. The proof appears in [28], and depends on several results from this paper, Theorem 59 in particular.

Proposition 64 *Assume Conjecture 55, or $K_S = 0$. View $\widetilde{D}^{S,L}$ as an element of $\mathbb{Q}[y][\![x,s]\!]$ with $s = xQ$. Then*

$$\sum_{\delta\geq 0} N^\delta_{\delta,[S,L]} s^\delta = \widetilde{D}^{S,L}(y, x=0, s).$$

Corollary 65 *Assume Conjecture 55, or $K_S = 0$. Then there exist series $A_i \in \mathbb{Q}[y][\![s]\!]$ such that*

$$\sum_{\delta\geq 0} N^\delta_{\delta,[S,L]}(y) s^\delta = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)}.$$

Proof Viewing $\widetilde{D}_i \in \mathbb{Q}[y][\![x,s]\!]$, where $s = xQ$, take $A_i := \widetilde{D}_i|_{x=0}$. □

In the unrefined ($y=1$) setting, more explicit formulas were expressed after substituting for s a certain quasimodular form. Specifically, in [27, Conjecture 2.4], the following expansion was proposed:

$$\sum_{\delta\geq 0} n^{\delta}_{\delta,[S,L]}\cdot (DG_2)^{\delta} = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\Delta\cdot DDG_2/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{8}$$

Here G_2 is the Eisenstein series, Δ is the discriminant:

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^k)^{24}, \quad G_2(q) = \frac{1}{24} + \sum_{m=1}^{\infty} q^m \sum_{d|m} d.$$

The series $B_1, B_2 \in 1 + q\mathbb{Q}[\![q]\!]$ are not known explicitly, although their first several coefficients may be computed and are given in [27]. We also write $D = q\frac{d}{dq}$. The above formula is by now a theorem, since the existence of universal formulas has been established [70; 43], and the case of the $K3$ surface (where $K_S = 0$ hence the B_i do not appear) was solved explicitly; see Bryan and Leung [10].

Notation 66 We write $\bar{N}^{\delta}_{\delta,[S,L]} := N^{\delta}_{\delta,[S,L]}/y^{\delta}$.

Now we give a conjectural refinement of Equation (8). The series Δ, DG_2 are refined as follows:[9]

$$\widetilde{\Delta}(y,q) := q\prod_{n=1}^{\infty}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2,$$

$$\widetilde{DG}_2 := \sum_{m=1}^{\infty} mq^m \sum_{d|m}\frac{[d]_y^2}{d}.$$

[9] These functions are related to certain Jacobi forms. Let $q = e^{2\pi i\tau}$, $y = e^{2\pi i z}$. (1) Then $\widetilde{\Delta}(y,q) = \phi_{10,1}(\tau,z)/(y^{1/2}-y^{-1/2})^2$. Here $\phi_{10,1}(\tau,z) = \eta(\tau)^{18}\theta(\tau,z)^2$ is up to normalization the unique Jacobi cusp form on $\mathrm{Sl}_2(\mathbb{Z})$ of weight 10 and index 1. (2) We can write

$$\begin{aligned}(y-2+y^{-1})\widetilde{DG}_2 &= \sum_{m=1}^{\infty} q^m \sum_{d|m}\frac{m}{d}(y^d-2+y^{-d})\\ &= -2(G_2(\tau)+1/24) + \sum_{d,e>0} e(y^d-y^{-d})q^{de}\\ &= -\tfrac{1}{2}D\log\left(\frac{\phi_{10,1}(\tau,z)}{\Delta(\tau)}\right) = -\tfrac{1}{2}D\log(\phi_{-2,1}(\tau,z)).\end{aligned}$$

Here $\Delta(\tau)$ is the discriminant function and $\phi_{-2,1} = \phi_{10,1}/\Delta$ is the up to normalization unique weak Jacobi cusp form of weight -2 and index 1 on $\mathrm{Sl}_2(\mathbb{Z})$.

Conjecture 67 *There exist universal power series $B_1(y,q), B_2(y,q) \in \mathbb{Q}[y,y^{-1}][\![q]\!]$ such that*

$$\sum_{\delta\geq 0} \overline{N}^{\delta}_{\delta,[S,L]}(y)(\widetilde{DG}_2)^{\delta} = \frac{(\widetilde{DG}_2/q)^{\chi(L)} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q) D\widetilde{DG}_2/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{9}$$

Here, to make the change of variables, view all functions as elements of $\mathbb{Q}[y,y^{-1}][\![q]\!]$.

This conjecture is again checked modulo q^{11}, and we get that

$$\begin{aligned}
B_1(y,q) = {} & 1-q-((y^2+3y+1)/y)q^2+((y^4+10y^3+17y^2+10y+1)/y^2)q^3 \\
& -((18y^4+87y^3+135y^2+87y+18)/y^2)q^4+((12y^6+210y^5 \\
& +728y^4+1061y^3+728y^2+210y+12)/y^3)q^5-((2y^8+259y^7 \\
& +2102y^6+5952y^5+8236y^4+5952y^3+2102y^2+259y+2)/y^4)q^6 \\
& +((162y^8+3606y^7+19668y^6+48317y^5+64253y^4+48317y^3 \\
& +19668y^2+3606y+162)/y^4)q^7-((47y^{10}+3789y^9+41999y^8 \\
& +177800y^7+392361y^6+505678y^5+392361y^4+177800y^3 \\
& +41999y^2+3789y+47)/y^5)q^8+((5y^{12}+2416y^{11}+60202y^{10} \\
& +445989y^9+1576410y^8+3197831y^7+4018919y^6+3197831y^5 \\
& +1576410y^4+445989y^3+60202y^2+2416y+5)/y^6)q^9 \\
& -((896y^{12}+58504y^{11}+793194y^{10}+4483755y^9+13818256y^8 \\
& +26192369y^7+32243357y^6+26192369y^5+13818256y^4 \\
& \qquad +4483755y^3+793194y^2+58504y+896)/y^6)q^{10}+O(q^{11}),
\end{aligned}$$

$$\begin{aligned}
B_2(y,q) = {} & \frac{1}{(1-yq)(1-q/y)}(1+3q-((3y^2+y+3)/y)q^2+((y^4+8y^3+18y^2 \\
& +8y+1)/y^2)q^3-((13y^4+53y^3+76y^2+53y+13)/y^2)q^4 \\
& +((7y^6+100y^5+316y^4+455y^3+316y^2+100y+7)/y^3)q^5 \\
& -((y^8+112y^7+779y^6+2076y^5+2819y^4+2076y^3+779y^2 \\
& +112y+1)/y^4)q^6+((67y^8+1243y^7+6129y^6+14386y^5 \\
& +18870y^4+14386y^3+6129y^2+1243y+67)/y^4)q^7-((19y^{10} \\
& +1281y^9+12417y^8+48879y^7+104034y^6+132579y^5+104034y^4 \\
& +48879y^3+12417y^2+1281y+19)/y^5)q^8+((2y^{12}+822y^{11}
\end{aligned}$$

$$+17542y^{10}+117829y^9+393703y^8+775411y^7+965540y^6$$
$$+775411y^5+393703y^4+117829y^3+17542y^2+822y+2)/y^6)q^9$$
$$-((310y^{12}+17206y^{11}+207074y^{10}+1085712y^9+3197506y^8$$
$$+5913778y^7+7223539y^6+5913778y^5+3197506y^4+1085712y^3$$
$$+2070742y^2+17206y+310)/y^6)q^{10}+O(q^{11})).$$

At $y=1$, we recover modulo q^{11} the functions $B_1(q)$, $B_2(q)$ of [27].

As in [27, Remark 2.6], the expansion in $\widetilde{DG}_2$ may be exchanged for an expansion in q while simultaneously trading a sum over varying numbers of point conditions while fixing the line bundle for a sum over line bundles while fixing the point conditions. Note the latter form is more natural from the point of view of the GW/DT/pairs theories, and indeed this is the form in which the $K3$ is solved in [10].

In detail this procedure is as follows. For any power series $f \in R[\![q]\!]$ and $g \in q+q^2R[\![q]\!]$, we may expand f in terms of g by the residue formula:

$$f(q)=\sum_{l=0}^{\infty} g(q)^l \operatorname{Coeff}_{q^0}\left[\frac{f(q)Dg(q)}{g(q)^{l+1}}\right].$$

Conjecture 67 asserts that $\overline{N}^{\delta}_{\delta,[S,L]}$ is the coefficient of $\widetilde{DG}_2^{\delta}$ of a certain expression; taking this coefficient by the above residue formula gives the equivalent formulation

$$\begin{aligned}\overline{N}^{\delta}_{\delta,[S,L]}(y)&\\ &=\operatorname{Coeff}_{q^0}\left[\widetilde{DG}_2^{-\delta-1}D\widetilde{DG}_2\frac{(\widetilde{DG}_2/q)^{\chi(L)}B_1(y,q)^{K_S^2}B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q)D\widetilde{DG}_2/q^2)^{\chi(\mathcal{O}_S)/2}}\right]\\ &=\operatorname{Coeff}_{q^{(L^2-LK_S)/2}}\left(\frac{(\widetilde{DG}_2)^{\chi(L)-1-\delta}D\widetilde{DG}_2B_1(y,q)^{K_S^2}B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q)D\widetilde{DG}_2)^{\chi(\mathcal{O}_S)/2}}\right).\end{aligned}$$

We would now like to collect coefficients of q to write the entire series in the $(\cdot)$ in terms of the $\overline{N}$. So we must choose some values of $\delta,[S,L]$ such that $K_S^2, LK_S, \chi(\mathcal{O}_S)$ and $k:=\chi(L)-1-\delta$ remain constant, but $(L^2-LK_S)/2$ assumes every integer value starting from $k+1-\chi(\mathcal{O}_S)$. In other words, the cobordism class of the surface, the number of point conditions $k=\chi(L)-1-\delta$, and LK_S are fixed, and L^2 varies. Note that it is not necessarily possible to find a fixed surface S and honest line bundles L_i which realize all the desired values. This causes no difficulties as the $\overline{N}^{\delta}_{\delta,[S,L]}$ may be viewed as just functions of the four values $L^2, LK_S, K_S^2, c_2(S)$. Making this dependence explicit we write

$$\overline{M}_{k,[S]}((L^2-LK_S)/2, LK_S):=\overline{N}^{\chi(L)-1-k}_{\chi(L)-1-k,[S,L]},$$

where the right-hand side is viewed just as a function of two integers and is determined by evaluating the left-hand side on the cobordism class $[S,L]$ with the specified invariants. In terms of the $\overline{M}$, we have

$$\sum_{l=k+1-\chi(\mathcal{O}_S)}^{\infty} \overline{M}_{k,[S]}(l, LK_S)q^l = \widetilde{DG}_2(y,q)^k \frac{B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} D\widetilde{DG}_2(y,q)}{(\widetilde{\Delta}(y,q) D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_S)/2}}. \tag{10}$$

We can also express this in a slightly different way, fixing LK_S, k and varying δ: write $\bar{N}_{k,[S]}(\delta, LK_S) := \bar{N}^{\delta}_{\delta,[S,L]}$ with $k = \chi(L) - 1 - \delta$. Then

$$\sum_{\delta=0}^{\infty} \bar{N}_{k,[S]}(\delta, LK_S)q^{\delta} = (\widetilde{DG}_2(y,q)/q)^k \frac{B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} (D\widetilde{DG}_2(y,q)/q)}{(\widetilde{\Delta}(y,q) \cdot D\widetilde{DG}_2(y,q)/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{11}$$

When S is a $K3$ surface, the right-hand side simplifies dramatically. Moreover, on the left-hand side, for each term we may choose a representative $K3$ surface S_g of genus g, with an irreducible line bundle L_g giving the polarization. That is,

$$\overline{M}_{k,[K3]}(g-1, 0) = \bar{N}^{g-k}_{g-k,[S_g,L_g]}.$$

The relative Hilbert schemes over the general $\mathbb{P}^{\delta} \subset |L_g|$ are all smooth [56], so the $\bar{N}^{\delta}_{\delta,[S_g,L_g]}$ are equal to the geometric $\bar{N}^{\delta}_{\mathcal{C}/\mathbb{P}^{\delta}}$. Summarizing the preceding discussion:

Conjecture 68 *For any k,*

$$\sum_{g=k}^{\infty} q^{g-1} \bar{N}^{g-k}_{g-k,[S_g,L_g]}(y) = \frac{\widetilde{DG}_2(y,q)^k}{\widetilde{\Delta}(y,q)}.$$

Proposition 69 *Assume Conjecture 55. Then Conjectures 67 and 68 are equivalent.*

Proof According to Corollary 65, from Conjecture 55 we can deduce that the $\bar{N}$ have a multiplicative generating series. The two of its components which are made explicit in Conjecture 67 are determined by the case when S is a $K3$ surface, and we have seen that the posited explicit formula is equivalent to that given in Conjecture 68. □

Proposition 70 *Conjecture 68 is true at $k = 0$.*

Proof The relative Hilbert schemes over the general $\mathbb{P}^{\delta} \subset |L_g|$ are smooth, so the quantity $\bar{N}^{g}_{g,[S_g,L_g]}$ is the χ_{-y} genus of the relative compactified Jacobian of the tautological family of curves $\mathcal{C}/|L_g|$. In fact, Kawai and Yoshioka compute the Hodge polynomial of this space [38]. Alternatively, as in [5] one may note that the relative compactified Jacobian over $|L_g|$ is birational to the Hilbert scheme of points $S_g^{[g]}$; since both are hyper-Kähler they are deformation equivalent by Huybrechts [32], and the Hodge polynomial of the Hilbert scheme of points was computed by Göttsche and Soergel in [29]. In either case the result is given by the right-hand side. □

We do not know how to compute the χ_{-y} genera of relative Hilbert schemes over linear subsystems for $K3$ surfaces. In fact, while the Euler numbers of these spaces are known, the only known calculation of them (due to Maulik, Pandharipande and Thomas) is extremely indirect, and involves at least two uses of the Gromov–Witten/Pairs correspondence [51]. Rather remarkably, the same series which we have conjectured describes the $\bar{N}_\delta^\delta$ is found in [51] to encode *all* the Euler numbers of relative Hilbert schemes over linear subsystems.

Theorem 71 [51] *For each g, let (S_g, L_g) be a $K3$ surface of genus g, and assume L_g is irreducible. Let H be the hyperplane class on $|L_g|$. Then for any k,*

$$(y-2+y^{-1})^{k-1}\frac{\widetilde{DG}_2^k}{\widetilde{\Delta}} = \sum_{g=k}^{\infty} q^{g-1}\sum_{n=0}^{\infty} y^{n+1-g}\int_{\mathcal{C}^{[n]}_{|L_g|}} c_{n+g-k}(T\mathcal{C}^{[n]}_{|L_g|})\cdot\rho^*(H^k).$$

Comparing powers of q, we see that given [51], Conjecture 68 is equivalent to the following statement:

Conjecture 72 *Let (S, L) be a $K3$ surface of genus g with L irreducible. For all k,*

$$(y-2+y^{-1})^{k-1}\bar{N}^{g-k}_{g-k,[S,L]} = \sum_{n=0}^{\infty} y^{n+1-g}\int_{\mathcal{C}^{[n]}_{|L|}} c_{n+g-k}(T\mathcal{C}^{[n]}_{|L|})\cdot\rho^*(H^k).$$

Remark 73 The statement of Theorem 71 in [51] differs slightly; there are some signs owing to the use of Ω rather than T, and it is formulated in terms of the space of "stable pairs" rather than the relative Hilbert scheme. But since L_g is irreducible, these are the same space as per [62, Appendix B]. In the language of [51], Conjecture 72 may be viewed as asserting that the $\bar{N}$ encode certain descendent integrals in the stable pairs theory, or equivalently in the Gromov–Witten theory.

We may instead specialize to abelian surfaces: let (A, L_g) denote a primitively polarized abelian surface with $L_g^2 = 2g-2$, hence $\chi(L) = g-1$ and $g(L) = g$; assume L is irreducible. Note such surfaces exist for all $g \geq 1$. Equation (10) specializes to

$$\sum_{g=k+2}^{\infty} \bar{N}^{g-k-2}_{g-k-2,[A,L_g]} q^{g-1} = \sum_{l=k+1}^{\infty} \bar{M}_{k,[A]}(l,0)q^l = \widetilde{DG}_2(y,q)^k D\widetilde{DG}_2(y,q).$$

The above formula is not equivalent to Conjecture 67, since $\chi(\mathcal{O}_A)=0$. Moreover we do not know how to establish it, even at $k=0$. However:

Proposition 74 *Assume Conjecture 55 or $K_S=0$. Conjecture 67 is equivalent to the formulas*

(12) $$\sum_{g=0}^{\infty} \bar{N}^g_{g,[K3,L_g]} q^{g-1} = \widetilde{\Delta}(y,q)^{-1},$$

(13) $$\sum_{g=2}^{\infty} \bar{N}^{g-2}_{g-2,[A,L_g]} q^{g-1} = D\widetilde{DG}_2(y,q).$$

Proof It is enough to show that the given invariants of the complete linear system suffice to determine the series A_1, A_4 and then apply the residue trick explained above. But $\bar{N}^{g-2}_{g-2,[A,L_g]}(y) = \mathrm{Coeff}_{s^{g-2}}\, A_1(y,s)^{2g-2}$. Beginning at $g=2$ this allows us to iteratively determine the coefficients of A_1. Then $\bar{N}^g_{g,[K3,L_g]}(y) = \mathrm{Coeff}_{s^g}\, A_1(y,s)^{2g-2} A_4(y,s)^{24}$; since we now know A_1 this determines A_4. □

As we have established (12), it remains only to compute the invariants for complete linear systems on abelian surfaces. In the meantime in [28] we have proved a generalized version of (13) using the methods of [38]. By the above this gives a proof of Conjecture 67 in the case K_S numerically trivial. This also provides an entirely sheaf theoretic proof of the formula [51, Theorem 6] for linear subsystems for irreducible line bundles on $K3$ surfaces.

5 Refined Severi degrees

The *Severi degrees* $n^{d,\delta}$ are the numbers of δ–nodal reduced degree d curves in $\mathbb{P}^2$ though $\binom{d+2}{2}-1-\delta$ general points. The famous Caporaso–Harris formula [12] gives a recursive method of computing the Severi degrees. The recursion involves the *relative Severi degrees* $n^{d,\delta}(\alpha,\beta)$ (we sketch their definition below) which count δ–nodal curves with tangency conditions along a fixed line in $\mathbb{P}^2$. More generally, for a line bundle L on a surface S, one can define the *Severi degree* $n^{L,\delta}$ as the number of δ–nodal reduced curves in $|L|$ though $\dim|L|-\delta$ general points, provided this number is finite.

We begin by a review of the Caporaso–Harris recursion formula; we will use the more general formulation of Vakil [71], which also applies to rational ruled surfaces. By a *sequence* we mean a collection $\alpha=(\alpha_1,\alpha_2,\dots)$ of nonnegative integers, almost all of which are zero. We write d for the sequence $(d,0,0,\dots)$ and e_k for the sequence whose k^{th} element is 1 and all other ones 0. For two sequences α, β we define $|\alpha| = \sum_i \alpha_i$, $I\alpha = \sum_i i\alpha_i$, $\alpha+\beta = (\alpha_1+\beta_1, \alpha_2+\beta_2, \dots)$, $\binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}$. We write $\alpha \le \beta$ to mean $\alpha_i \le \beta_i$ for all i.

Throughout this section we take S to be $\mathbb{P}^2$ or a rational ruled surface. In case $S = \mathbb{P}^2$, let E be a line in $\mathbb{P}^2$, in case S is a rational ruled surface $\Sigma_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, let E be the class of the section with $E^2 = -e$. We denote by H the hyperplane class on $\mathbb{P}^2$, F the class of a fiber on Σ_e.

Let L be a line bundle on S and let α, β be sequences with $I\alpha + I\beta = EL$, and let $\delta \ge 0$ be an integer. Let $\gamma(L,\beta,\delta) := \dim|L| - EL + |\beta| - \delta$. The *relative Severi degree* $n^{L,\delta}(\alpha,\beta)$ is the number of δ–nodal curves C in $|L|$, which do not contain E as a component and for each k with α_k points of contact of order k to E at given points of E and β_k points of contact of order k to E at variable points of E, and passing through $\gamma(L,\beta,\delta)$ general points of S (see [71] for a more formal definition).

Recursion 75 [12; 71] The relative Severi degrees $n^{L,\delta}(\alpha,\beta)$ are recursively given as follows: $n^{L,\delta}(\alpha,\beta) = 0$ if $\gamma(L,\beta,\delta) < 0$. If $\gamma(L,\beta,\delta) > 0$, then

$$(14)\quad n^{L,\delta}(\alpha,\beta) = \sum_{k:\beta_k>0} k \cdot n^{L,\delta}(\alpha+e_k, \beta-e_k) + \sum_{\alpha',\beta',\delta'} \prod_i i^{\beta'_i - \beta_i} \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} n^{L-E,\delta'}(\alpha',\beta').$$

Here the second sum runs through all α', β', δ' satisfying the conditions

$$(15)\quad \begin{aligned} &\alpha' \le \alpha, \beta' \ge \beta, I\alpha' + I\beta' = E(L-E), \\ &\delta' = \delta + g(L-E) - g(L) + |\beta'-\beta| + 1 = \delta - E(L-E) + |\beta'-\beta|. \end{aligned}$$

Initial conditions If $\gamma(L,\beta,\delta) = 0$ we have $n^{L,\delta}(\alpha,\beta) = 0$ unless we are in one of the following cases.

(1) In case $S = \mathbb{P}^2$ we put $n^{H,0}(1,0) = 1$.

(2) In case $S = \Sigma_e$, let F be the class of a fiber of the ruling; we put $n^{kF,0}(k,0) = 1$.

We put $n^{L,\delta} := n^{L,\delta}(0, LE)$. In case $S = \mathbb{P}^2$, we write $n^{d,\delta}(\alpha,\beta) := n^{dH,\delta}(\alpha,\beta)$ and $n^{d,\delta} := n^{dH,\delta}(0,d)$.

5.1 Refined Severi degrees

We formally introduce a refinement of this recursion.

Recursion 76 With the same notation, assumptions, limits of summation and initial values as for the relative Severi degrees in Recursion 75, we define the refined relative Severi degrees $N^{L,\delta}(\alpha,\beta)(y)$ for $\gamma(L,\beta,\delta)>0$ by

$$(16)\quad N^{L,\delta}(\alpha,\beta)(y) = \sum_{k\mid\beta_k>0} \frac{1-y^k}{1-y}\cdot N^{L,\delta}(\alpha+e_k,\beta-e_k)(y) + \sum_{\alpha',\beta',\delta'} y^{I\alpha'+I\beta} \cdot\prod_i\Big(\frac{1-y^i}{1-y}\Big)^{\beta'_i-\beta_i}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}N^{L-E,\delta'}(\alpha',\beta')(y).$$

We abbreviate $N^{L,\delta}:=N^{L,\delta}(0,LE)$, and, in case $S=\mathbb{P}^2$, then $N^{d,\delta}(\alpha,\beta):=N^{dH,\delta}(\alpha,\beta)$, $N^{d,\delta}:=N^{dH,\delta}(0,d)$. As with the refined invariants, we define normalized refined relative Severi degrees which are Laurent polynomials in $y^{1/2}$, symmetric under $y\mapsto 1/y$.

Definition 77 The *normalized (relative) refined Severi degrees* $\bar{N}^{L,\delta}(y)$ are defined by

$$\bar{N}^{L,\delta}(\alpha,\beta)(y) = N^{L,\delta}(\alpha,\beta)(y)/y^{\delta+(I\beta-|\beta|)/2},\quad \bar{N}^{L,\delta}(y)=N^{L,\delta}/y^{\delta}.$$

Proposition 78 *The $\bar{N}^{L,\delta}(\alpha,\beta)(y)$ are determined by the same initial conditions as the $N^{L,\delta}(\alpha,\beta)(y)$ and the recursion*

$$(17)\quad \bar{N}^{L,\delta}(\alpha,\beta) = \sum_{k:\beta_k>0}[k]_y\cdot\bar{N}^{L,\delta}(\alpha+e_k,\beta-e_k) + \sum_{\alpha',\beta',\delta'}\Big(\prod_i [i]_y^{\beta'_i-\beta_i}\Big)\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}\bar{N}^{L-E,\delta'}(\alpha',\beta')$$

with the same conditions on α',β',δ' as above. In particular $\bar{N}^{L,\delta}(\alpha,\beta)(y)$ is symmetric under $y\mapsto 1/y$.

Proof It is enough to prove that every summand on the right-hand side of (16) is obtained from the corresponding summand of (17) by multiplying by $y^{\delta+(I\beta-|\beta|)/2}$. Each summand in the first sum is multiplied by y^m with

$$m=(k-1+I(\beta-e_k)-(|\beta-e_k|))/2+\delta=\delta+(I\beta-|\beta|)/2.$$

Each summand in the second sum is multiplied by y^m with

$$m=I\alpha'+I\beta+(I(\beta'-\beta)-|\beta'-\beta|)/2+\delta'+(I\beta'-|\beta'|)/2=\delta+(I\beta-|\beta|)/2,$$

where we use $I\alpha'+I\beta'=E(L-E)=\delta-\delta'+|\beta'-\beta|$. □

It is clear that the recursions for the refined Severi degrees specialize at $y=1$ to the recursion for the usual Severi degrees. Thus:

Proposition 79 *We have* $N^{L,\delta}(\alpha,\beta)(1)=\bar{N}^{L,\delta}(\alpha,\beta)(1)=n^{L,\delta}(\alpha,\beta)$.

According to [40], if the general $\mathbb{P}^\delta\subset|L|$ contains no nonreduced curves and no curves containing components with negative self intersection, the Severi degrees are computed by the universal formulas: $n^{L,\delta}=n^{\delta}_{\delta,[S,L]}$. We expect the same for refined Severi degrees.

Conjecture 80 *Let* S *be* $\mathbb{P}^2$ *or a rational ruled surface, let* L *be a line bundle, and assume* $\mathbb{P}^\delta\subset|L|$ *contains no nonreduced curves and no curves containing components with negative self intersection. Then the refined Severi degrees are computed by the universal formulas:* $N^{L,\delta}=N^{\delta}_{\delta,[S,L]}$. *Explicitly:*

(1) *On* $\mathbb{P}^2$, $N^{d,\delta}=N^{\delta}_{\delta,[\mathbb{P}^2,dH]}$ *for* $\delta\le 2d-2$.

(2) *On* $\mathbb{P}^1\times\mathbb{P}^1$, $N^{nF+mG,\delta}=N^{\delta}_{\delta,[\mathbb{P}^1\times\mathbb{P}^1,nF+mG]}$ *for* $\delta\le\min(2n,2m)$.

(3) *On* Σ_e *with* $e>0$, $N^{nF+mE,\delta}=N^{\delta}_{\delta,[\Sigma_e,nF+mE]}$ *for* $\delta\le\min(2m,n-em)$.

Directly from the defining Recursion 76 we have computed all the $N^{d,\delta}(y)$ for $d\le 15$ and $\delta\le 30$. Assuming the vanishing Conjecture 55 and part (1) of Conjecture 80, the refined Severi degrees suffice to determine all the power series in Corollary 65 or equivalently in Conjecture 67. Note that Recursion 76 is much more computationally tractable than equivariant localization. Thus under the above assumption, we have verified Conjecture 67 modulo q^{29} and determined $B_1(y,q)$ and $B_2(y,q)$ modulo q^{29}.

5.2 Irreducible refined Severi degrees

Denote by $n_0^{L,\delta}$ the irreducible Severi degrees, ie informally the number of irreducible δ–nodal curves in $|L|\neq|E|$ passing though $\dim|L|-\delta$ general points. In [24] Getzler observes in the case $S=\mathbb{P}^2$ that the $n_0^{d,\delta}$ can be expressed in terms of the Severi degrees $n^{d,\delta}$ by the relation

$$\sum_{d,\delta}\frac{z^{\binom{d+2}{2}-\delta-1}}{(\binom{d+2}{2}-\delta-1)!}q^d n_0^{d,\delta}=\log\left(1+\sum_{d,\delta}\frac{z^{\binom{d+2}{2}-\delta-1}}{(\binom{d+2}{2}-\delta-1)!}q^d n^{d,\delta}\right).$$

The generalization of this to $n_0^{L,\delta}$ is in [71]. The same formula can be used to define the irreducible normalized refined Severi degrees $\bar{N}_0^{L,\delta}(y)$ by

$$\sum_{L,\delta}\frac{z^{\dim|L|-\delta}}{(\dim|L|-\delta)!}v^L\bar{N}_0^{L,\delta}(y)=\log\left(1+\sum_{L,\delta}\frac{z^{\dim|L|-\delta}}{(\dim|L|-\delta)!}v^L\bar{N}^{L,\delta}(y)\right),$$

and the irreducible refined Severi degrees by $N_0^{L,\delta}(y) := y^\delta \bar{N}_0^{L,\delta}(y)$. Here we have that

$$\{v^L\}_{L \text{ effective}, L \neq E}$$

are elements of the Novikov ring, ie $v^{L_1} v^{L_2} = v^{L_1+L_2}$. Evidently $\bar{N}_0^{L,\delta}(y)$ is a Laurent polynomial in y invariant under $y \mapsto 1/y$, and

$$N_0^{L,\delta}(1) = \bar{N}_0^{L,\delta}(1) = n_0^{L,\delta}.$$

Theorem 81 [9] *The polynomial $N_0^{L,\delta}$ has nonnegative integer coefficients.*

From this positivity, one can conclude vanishing results for $N_0^{L,\delta}$ from the analogous (known) results for $n_0^{L,\delta}$. For instance

$$N_0^{L,\delta}(y) = n_0^{L,\delta} = 0 \quad \text{for } \delta > g(L)$$

since there are no irreducible curves of cogenus greater than $g(L)$.

We list the first few of the $N_0^{d,\delta}(y)$. Write

$$N_0^d(y,t) := \sum_{\delta \geq 0} N_0^{d,\delta}(y) t^\delta.$$

We have computed the $N_0^d(y,t)$ for $d \leq 14$. We have

$$
\begin{aligned}
N_0^1(y,t) &= 1,\\
N_0^2(y,t) &= 1,\\
N_0^3(y,t) &= 1 + (y^2 + 10y + 1)t,\\
N_0^4(y,t) &= 1 + (3y^2 + 21y + 3)t + (3y^4 + 33y^3 + 153y^2 + 33y + 3)t^2\\
&\qquad + (y^6 + 13y^5 + 94y^4 + 404y^3 + 94y^2 + 13y + 1)t^3,\\
N_0^5(y,t) &= 1 + (6y^2 + 36y + 6)t + (15y^4 + 156y^3 + 540y^2 + 156y + 15)t^2\\
&\quad + (20y^6 + 268y^5 + 1555y^4 + 4229y^3 + 1555y^2 + 268y + 20)t^3\\
&\quad + (15y^8 + 228y^7 + 1674y^6 + 7407y^5 + 18207y^4 + 7407y^3\\
&\quad + 1674y^2 + 228y + 15)t^4 + (6y^{10} + 96y^9 + 792y^8 + 4398y^7\\
&\quad + 17190y^6 + 42228y^5 + 17190y^4 + 4398y^3 + 792y^2 + 96y + 6)t^5\\
&\quad + (y^{12} + 16y^{11} + 139y^{10} + 867y^9 + 4203y^8 + 16377y^7\\
&\quad + 44098y^6 + 16377y^5 + 4203y^4 + 867y^3 + 139y^2 + 16y + 1)t^6.
\end{aligned}
$$

On $\mathbb{P}^1 \times \mathbb{P}^1$ we have computed the $N_0^{nF+mG,\delta}$ for $1 \le, n, m \le 8$. We list the first few; write $N_0^{n,m} := \sum_{\delta \ge 0} N_0^{n,m,\delta}(y)t^\delta$. We have

$$
\begin{aligned}
N_0^{1,k}(y,t) &= 1 \text{ for all } k,\\
N_0^{2,2}(y,t) &= 1 + (y^2 + 10y + 1)t,\\
N_0^{2,3}(y,t) &= 1 + (2y^2 + 16y + 2)t + (y^4 + 12y^3 + 79y^2 + 12y + 1)t^2,\\
N_0^{2,4}(y,t) &= 1 + (3y^2 + 22y + 3)t + (3y^4 + 36y^3 + 174y^2 + 36y + 3)t^2\\
&\qquad + (y^6 + 14y^5 + 117y^4 + 596y^3 + 117y^2 + 14y + 1)t^3,\\
N_0^{3,3}(y,t) &= 1 + (4y^2 + 26y + 4)t + (6y^4 + 64y^3 + 256y^2 + 64y + 6)t^2\\
&\qquad + (4y^6 + 52y^5 + 332y^4 + 1168y^3 + 332y^2 + 52y + 4)t^3\\
&\qquad + (y^8 + 14y^7 + 109y^6 + 636y^5 + 2430y^4 + 636y^3\\
&\qquad + 109y^2 + 14y + 1)t^4.
\end{aligned}
$$

5.3 The refined invariants at $y = 0$

Now we compute the specialization of the refined Severi degrees at $y = 0$.

Notation 82 For a sequence β we write $\binom{|\beta|}{\beta} := |\beta|!/\prod_i \beta_i!$.

Proposition 83 *We have that $N^{L,\delta}(\alpha,\beta)(0) = \binom{|\beta|}{\beta}\binom{g(L)}{\delta}$. In particular, we have that $N^{L,\delta}(0) = \binom{g(L)}{\delta}$.*

Proof It is easy to see that the statement holds for the initial values. Setting $y = 0$ in the recursion formula (16) gives the two recursion formulas

$$
\begin{aligned}
N^{L,\delta}(\alpha,\beta)(0) &= \sum_{k \mid \beta_k > 0} N^{L,\delta}(\alpha + e_k, \beta - e_k)(0) \quad \text{if } \beta \neq 0,\\
N^{L,\delta}(\alpha,0)(0) &= \sum_{I\beta = E(L-E)} N^{L-E,\delta - E(L-E) + |\beta|}(0,\beta)(0).
\end{aligned} \tag{18}
$$

The first formula gives

$$
N^{L,\delta}(\alpha,\beta)(0) = \binom{|\beta|}{\beta} N^{L,\delta}(\alpha + \beta, 0)(0). \tag{19}
$$

The second formula shows in particular that $N^{L,\delta}(\alpha,0)(0)$ is independent of α, thus $N^{L,\delta}(\alpha,0)(0) = N^{L,\delta}(LE,0)(0) = N^{L,\delta}(0)$; the last equality is by (19). Thus $N^{L,\delta}(\alpha,\beta)(0) = \binom{|\beta|}{\beta} N^{L,\delta}(0)$.

Finally, writing $L = L_0 + aE$ with $L_0 = H$ in case $S = \mathbb{P}^2$ and $L_0 = kF$ in case S is a rational ruled surface, we prove $N^{L,\delta} = \binom{g(L)}{\delta}$ by induction over a. It is easy to see that the claim is true for L_0. By induction the second equation of (18) becomes

$$N^{L,\delta}(0) = \sum_{I\beta=E(L-E)} \binom{g(L-E)}{\delta - E(L-E) + |\beta|}\binom{|\beta|}{\beta}.$$

Thus we need to show the identity

$$(1+t)^{g(L)} = \sum_{\delta\geq 0}\sum_{I\beta=E(L-E)} \binom{g(L-E)}{\delta - I\beta + |\beta|}\binom{|\beta|}{\beta} t^{\delta}. \tag{20}$$

Note that by the multinomial formula we have

$$\frac{x}{1-x(1+t)} = \frac{1}{1-\frac{x}{1-tx}} - 1 = \sum_{n>0}(x + tx^2 + t^2x^3 + \cdots)^n = \sum_{\beta\neq 0}\binom{|\beta|}{\beta} t^{I\beta-|\beta|}x^{I\beta}.$$

Thus the right-hand side of (20) becomes

$$\mathrm{Coeff}_{x^{E(L-E)}}\left[(1+t)^{g(L-E)}\frac{x}{1-x(1+t)}\right] = (1+t)^{g(L)}. \qquad \square$$

Using the tropical interpretation of the $N^{L,\delta}(\alpha,\beta)$ of [9] using refined multiplicities (see also Section 6), this result has been generalized in [35] to arbitrary toric surfaces.

For M a line bundle on S, let $M_n := f^* g_*(\bigotimes_{i=1}^n \mathrm{pr}_i^* M)^{\mathfrak{S}_n} \in \mathrm{Pic}\, S^{[n]}$, where $f\colon S^{[n]} \to S^{(n)}$ and $g\colon S^n \to S^{(n)}$ are the natural morphisms, and $\mathrm{pr}_i\colon S^n \to S$ is the i^{th} projection. It is well known that $K_{S^{[n]}} = (K_S)_n$, and it is also standard that $\det M^{[n]} = M_n \otimes \det \mathcal{O}_S^{[n]}$ (see eg [20]).

Lemma 84 *We have*

$$\sum_{n,k\geq 0} \chi(S^{[n]}, \Lambda^k (L^{[n]})^\vee) x^k q^n = \frac{(1+xq)^{\chi(L^\vee)}}{(1-q)^{\chi(\mathcal{O}_S)}}.$$

Proof This is a corollary to [65, Theorem 5.2.1], which implies for line bundles L, M on S that

$$\chi(S^{[n]}, \Lambda^k L^{[n]} \otimes M_n) = \binom{\chi(L\otimes M)}{k}\binom{\chi(M)+n-k-1}{n-k}.$$

We apply this with $M=K_S$. Thus, by applying Serre duality on $S^{[n]}$ and on S, we have

$$\begin{aligned}\chi(S^{[n]},\Lambda^k(L^{[n]})^\vee)&=\chi(S^{[n]},\Lambda^k L^{[n]}\otimes(K_S)_n)\\&=\binom{\chi(L\otimes K_S)}{k}\binom{\chi(K_S)+n-k-1}{n-k}\\&=\binom{\chi(L^\vee)}{k}\binom{\chi(\mathcal{O}_S)+n-k-1}{n-k},\end{aligned}$$

which is equivalent to the statement of the lemma. □

Proposition 85 *We have $N^l_{\delta,[S,L]}(0)=0$ for $l>\delta$, and for all $0\le l\le\delta$ we have*

$$N^l_{\delta,[S,L]}(0)=\binom{\chi(L^\vee)}{l}=\binom{g(L)+\chi(\mathcal{O}_S)-1}{l}.$$

In particular if S is a rational surface, then $N^\delta_{\delta,[S,L]}(0)=\binom{g(L)}{\delta}$.

Proof By Proposition 52 and Definition 63, we have

$$\operatorname{Coeff}_{q^n}\Biggl[\sum_{l\ge 0}N^l_{\delta,[S,L]}(0)\frac{q^l}{(1-q)^{l+1-g(L)}}\Biggr]\\=\operatorname{res}_{x=0}\int_{S^{[n]}}\Bigl(\frac{1}{1-e^{-x}}\Bigr)^{\delta+1}\frac{c_n(L^{[n]}\cdot e^x)\operatorname{td}(S^{[n]})}{\operatorname{td}(L^{[n]}\cdot e^x)}\,dx.$$

Note that by definition $c_n(L^{[n]}\cdot e^x)/\operatorname{td}(L^{[n]}\cdot e^x)=\sum_{k=0}^n(-e^{-x})^k\operatorname{ch}(\Lambda^k(L^{[n]})^\vee)$. Thus by Riemann–Roch the right-hand side is

$$\operatorname{res}_{x=0}\Biggl[\Bigl(\frac{1}{1-e^{-x}}\Bigr)^{\delta+1}\sum_{k=0}^n(-e^{-x})^k\chi(S^{[n]},\Lambda^k(L^{[n]})^\vee)\,dx\Biggr].$$

We put $T=e^{-x}$ and apply Lemma 84 to obtain

$$\begin{aligned}\sum_{l\ge 0}N^l_{\delta,[S,L]}(0)\frac{q^l}{(1-q)^{l+1-g(L)}}&=-\operatorname{res}_{T=1}\Biggl[\Bigl(\frac{1}{1-T}\Bigr)^{\delta+1}\sum_{n\ge 0}\sum_{k=0}^n(-T)^kq^n\chi(S^{[n]},\Lambda^k(L^{[n]})^\vee)\frac{dT}{T}\Biggr]\\&=-\operatorname{res}_{T=1}\Biggl[\Bigl(\frac{1}{1-T}\Bigr)^{\delta+1}\frac{(1-Tq)^{\chi(L^\vee)}}{(1-q)^{\chi(\mathcal{O}_S)}}\frac{dT}{T}\Biggr].\end{aligned}$$

Substituting $T = 1-\alpha$ the right-hand side becomes

$$\operatorname{res}_{\alpha=0}\left[\frac{1}{\alpha^{\delta+1}}\frac{(1-q+\alpha q)^{\chi(L^\vee)}}{(1-q)^{\chi(\mathcal{O}_S)}}\sum_{l\geq 0}\alpha^l d\alpha\right] = \sum_{l=0}^{\delta}\binom{\chi(L^\vee)}{l}q^l(1-q)^{\chi(L^\vee)-l-\chi(\mathcal{O}_S)}$$
$$= \sum_{l=0}^{\delta}\binom{\chi(L^\vee)}{l}q^l(1-q)^{g(L)-l-1}.$$

This completes the proof. □

5.4 Conjectural generalization to higher powers of y

Propositions 85 and 83 can be subsummed in the following statements:

(1) For any line bundle L on a surface S we have

$$\sum_{\delta\geq 0} N^{\delta}_{\delta,[S,L]}(0)q^\delta = (1+q)^{\chi(L^\vee)}.$$

(2) If L is an effective divisor on $\mathbb{P}^2$ or a rational ruled surface S, then

$$\sum_{\delta\geq 0} N^{\delta}_{\delta,[S,L]}(0)N^{L,\delta}(0)q^\delta = (1+q)^{g(L)} = \sum_{\delta\geq 0} N^{\delta}_{\delta,[S,L]}(0)q^\delta.$$

We want to give a conjectural extension of these two statements to higher powers of y. We start with the analogue of (1):

Conjecture 86 *Let L be a line bundle on a surface S. Then we have for all $i\geq 0$*

$$\operatorname{Coeff}_{y^i}\left[\sum_{\delta\geq 0} N^{\delta}_{\delta,[S,L]}(y)q^\delta\right] = (1+q)^{\chi(L^\vee)-3i}P^i_L(q).$$

Here $P^i_L(q)$ is a polynomial in q of degree at most $3i$. In particular if $\chi(L^\vee)\geq 3i$ then $\operatorname{Coeff}_{y^i} N^{\delta}_{\delta,[S,L]}(y) = 0$ for $\delta > \chi(L^\vee)$.

Assuming Conjecture 55 we get by Corollary 65

$$\sum_{\delta\geq 0} N^{\delta}_{\delta,[S,L]}(y)q^\delta = F_1^{\chi(L^\vee)}F_2^{LK_S/2}F_3^{K_S^2}F_4^{\chi(\mathcal{O}_S)},$$

with $F_i\in\mathbb{Q}[y][\![q]\!]$. We put $C_1 = (F_1/(1+q))|_{y\mapsto y(1+q)^3}$, $C_2 = F_2|_{y\mapsto y(1+q)^3}$, $C_3 = F_3|_{y\mapsto y(1+q)^3}$, $C_4 = F_4|_{y\mapsto y(1+q)^3}$.

Conjecture 87 *For $i = 1,\dots,4$ we have $C_i\in\mathbb{Q}[y^{1/3}][\![y^{1/3}q]\!]\cap\mathbb{Q}[q][\![y]\!]$.*

Proposition 88 *Conjecture 87 implies Conjecture 86. Furthermore*

$$\sum_{i\geq 0} P_L^i(q)y^i = C_1^{\chi(L^\vee)} C_2^{LK_S/2} C_3^{K_S^2} C_4^{\chi(\mathcal{O}_S)}.$$

Proof By definition

$$(1+q)^{\chi(L^\vee)} C_1^{\chi(L^\vee)} C_2^{LK_S} C_3^{K_S^2} C_4^{\chi(\mathcal{O}_S)} = \sum_{\delta\geq 0} N^\delta_{\delta,[S,L]}(y(1+q)^3)q^\delta.$$

Therefore

$$\mathrm{Coeff}_{y^i}\Big[\sum_{\delta\geq 0} N^\delta_{\delta,[S,L]}(y)q^\delta\Big] = (1+q)^{\chi(L^\vee)-3i}\,\mathrm{Coeff}_{y^i}\Big[C_1^{\chi(L^\vee)} C_2^{LK_S/2} C_3^{K_S^2} C_4^{\chi(\mathcal{O}_S)}\Big].$$

As by Conjecture 87 all C_i are in $\mathbb{Q}[y^{1/3}][\![y^{1/3}q]\!]$, we see that the coefficient of y^i is a polynomial of degree at most $3i$ in q. □

Conjecture 87 has been verified modulo q^{11}. Assuming Conjecture 80 it has been verified modulo q^{29}. We list the power series C_1, C_2, C_3, C_4 modulo y^4:

$$\begin{aligned}
C_1 &= 1+(4q+2q^2)y+(q-7q^2+12q^3+15q^4+3q^5)y^2\\
&\qquad +(-6q^2+56q^3-104q^4-112q^5+26q^6+32q^7+4q^8)y^3+O(y^4),\\
C_2 &= 1+(-2q-6q^2-2q^3)y+(5q^2+48q^3+35q^4+6q^5+q^6)y^2\\
&\qquad +(14q^3-390q^4-286q^5+60q^6+52q^7)y^3+O(y^4),\\
C_3 &= 1+(-q-3q^2-q^3)y+(q^2+16q^3+2q^4-6q^5-q^6)y^2\\
&\qquad +(15q^3-130q^4+66q^5+199q^6+65q^7)y^3+O(y^4),\\
C_4 &= 1+(6q+18q^2+10q^3)y+(18q^2+64q^3+219q^4+222q^5+67q^6)y^2\\
&\qquad +(-44q^3+336q^4+72q^5+952q^6+2328q^7+1608q^8+352q^9)y^3+O(y^4).
\end{aligned}$$

Now we formulate the conjectural analogue of (2). For simplicity we only deal with the case of $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$.

Conjecture 89 (1) *Let $S=\mathbb{P}^2$ and assume $d\geq i+2$, then*

$$\mathrm{Coeff}_{y^i}\Big[\sum_{\delta\geq 0} N^{d,\delta}(y)q^\delta\Big] = (1+q)^{g(dH)-3i} P^i_{dH}(q).$$

(2) *Let $S=\mathbb{P}^1\times\mathbb{P}^1$ and assume $n,m\geq i+1$, then*

$$\mathrm{Coeff}_{y^i}\Big[\sum_{\delta\geq 0} N^{nF+mG,\delta}(y)q^\delta\Big] = (1+q)^{g(nF+mG)-3i} P^i_{nF+mG}(q).$$

For $d \le 14$, and for $n, m \le 8$ this conjecture has been checked modulo q^{11} and, assuming Conjecture 80, modulo q^{29}.

6 Refined, real and tropical

Mikhalkin [55] has shown that the Severi degrees of projective toric surfaces can also be computed using tropical geometry: the Severi degrees $n^{L,\delta}$ count — with multiplicities — simple tropical curves through $\dim|L|-\delta$ points in $\mathbb{R}^2$ in tropical general position. Roughly speaking a simple tropical curve C is a trivalent graph Γ immersed in $\mathbb{R}^2$ together with some extra data. From this data, one assigns to each vertex v of Γ a multiplicity $m(v) \in \mathbb{Z}_{\ge 0}$ and defines the *complex multiplicity* $m(C)$ as the product of the $m(v)$ over the vertices of Γ. In [23] a proof of the Caporaso–Harris recursion formula is given via tropical geometry.

The analogues of the Gromov–Witten invariants in real algebraic geometry are the Welschinger invariants [72]. These were originally defined to count real pseudoholomorphic curves in real symplectic manifolds. We restrict attention to the case that S is a smooth projective toric surface. As toric varieties are defined over $\mathbb{Z}$, they certainly carry a real structure, and we write σ for the associated antiholomorphic involution. A real curve in S is an algebraic curve $C \subset S$ with $C = \sigma(C)$, and the real locus of C is C^σ. Fix a generic set Σ of $\dim|L|-\delta$ general real[10] points on S. The real enumerative invariant is $W^{L,\delta}(\Sigma) := \sum_C (-1)^{s(C)}$, where C runs through the possibly reducible real curves $C \in |L|$ of geometric genus $g(L)-\delta$, passing through all the points of Σ, and $s(C)$ is the number of isolated real nodes of C, ie the points where C analytically locally has the equation x^2+y^2. We denoted by $W_0^{L,\delta}(\Sigma)$ the corresponding sum for irreducible curves. If S is an unnodal (ie it contains no rational curve with self intersection $-n$, with $n \ge 2$) del Pezzo surface then the real enumerative invariants coincide with the Welschinger invariants. In [72] it was proven that $W_0^{L,g(L)}(\Sigma)$, ie the count of curves of geometric genus 0, is independent of the generic Σ. We will denote it just by $W_0^{L,g(L)}$. In general $W^{L,\delta}(\Sigma)$ and $W_0^{L,\delta}(\Sigma)$ will depend on Σ via a system of walls and chambers.

In a sense, we have already seen these invariants. For a family of real curves $\mathcal{C}/B$, let $n^{i,\mathbb{R}}_{\mathcal{C}/B}$ be defined by the same formula as the $n^{i,\mathbb{R}}_C$ introduced for individual curves in Section 2. Then we have:

[10] We are here only considering the so-called *totally real* Welschinger invariants. More generally one could consider for any $0 \le l \le (\dim|L|-\delta)/2$ the numbers $W^{L,\delta,l}(\Sigma)$ which count real curves passing through $\dim|L|-\delta-2l$ real points and l pairs of complex conjugate points.

Proposition 90 *Let L be a real line bundle on S, and let $\mathbb{P}^\delta \subset |L|$ be a linear subsystem determined by the real point conditions Σ. Assume that all curves in $\mathbb{P}^\delta$ are reduced, that no curves have cogenus greater than δ, and that all curves of cogenus δ are nodal. Then*

$$(-1)^\delta n^{\delta,\mathbb{R}}_{\mathcal{C}/\mathbb{P}^\delta} = W^{L,\delta}(\Sigma).$$

The real enumerative invariants of toric surfaces can also be computed via tropical geometry [55, Theorem 6]. For any real line bundle L and any $\delta \geq 0$, the tropical Welschinger invariant $W^{L,\delta}_{\mathrm{trop}}$ counts simple tropical curves in C in $|L|$ passing through $\dim|L|-\delta$ points in $\mathbb{R}^2$ in tropically general position. Here the tropical curves C are counted with the *Welschinger multiplicity* $r(C)$:

$$r(C) = \prod_{\text{vertices } v} r(v), \quad r(v) = \begin{cases} (-1)^{(m(v)-1)/2} & m(v) \text{ odd}, \\ 0 & m(v) \text{ even}. \end{cases}$$

The irreducible Welschinger invariants $W^{L,\delta}_{0,\mathrm{trop}}$ are defined by summing over only irreducible curves. It is proven in [34] that this is independent of the points as long as they are in tropical general position. Finally [55] shows that there exists a set Σ of $\dim|L|-\delta$ real points of S, so that $W^{L,\delta}(\Sigma) = W^{L,\delta}_{\mathrm{trop}}$, and $W^{L,\delta}_0(\Sigma) = W^{L,\delta}_{0,\mathrm{trop}}$.

If S is $\mathbb{P}^2$ or a rational ruled surface, there is a recursion for the tropical Welschinger invariants [34]. We write it in a modified form which makes the close relation to the recursion for the Severi degrees more evident.[11]

Definition 91 A sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ is called *odd* if $\alpha_i = 0$ for all even i.

Recursion 92 Let L be a line bundle on S and let α, β be odd sequences with $I\alpha + I\beta = EL$, and a let $\delta \geq 0$ be an integer. With the same notation and assumptions and initial values as for the relative Severi degrees in Recursion 75 the relative tropical Welschinger invariants $W^{L,\delta}_{\mathrm{trop}}(\alpha,\beta)(y)$ are given by the following recursion formula: if $\gamma(L,\beta,\delta) > 0$,

$$\text{(21)} \quad W^{L,\delta}_{\mathrm{trop}}(\alpha,\beta) = \sum_{k \text{ odd}:\beta_k>0} (-1)^{(k-1)/2} \cdot W^{L,\delta}_{\mathrm{trop}}(\alpha+e_k, \beta-e_k)(y) + \sum_{\alpha',\beta',\delta'} \prod_{i \text{ odd}} ((-1)^{(i-1)/2})^{\beta'_i-\beta_i} \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} W^{L-E,\delta'}_{\mathrm{trop}}(\alpha',\beta').$$

Here the second sum runs through all odd sequences α', β' and all δ' satisfying (15).

[11] Also the multiplicity assigned in [34] differs from those given above which we have taken from [55], but it can be shown they are equivalent.

We put $W^{L,\delta}_{\mathrm{trop}} := W^{L,\delta}_{\mathrm{trop}}(0, LE)$, and in the case $S = \mathbb{P}^2$,

$$W^{d,\delta}_{\mathrm{trop}}(\alpha,\beta) = W^{dH,\delta}_{\mathrm{trop}}(\alpha,\beta), \quad W^{d,\delta}_{\mathrm{trop}} = W^{dH,\delta}_{\mathrm{trop}}(0,d).$$

Note the following specialization:

$$[k]_{-1} = \frac{y^{k/2} - y^{-k/2}}{y^{1/2} - y^{-1/2}}\bigg|_{y=-1} = \begin{cases} (-1)^{(k-1)/2} & k \text{ odd}, \\ 0 & k \text{ even}. \end{cases} \tag{22}$$

In particular, the recursion for the refined Severi degrees interpolates between the Caporaso–Harris recursion for Severi degrees and the Itenberg–Kharlamov–Shustin recursion for tropical Welschinger invariants. Thus:

Proposition 93 *We have that*

$$\bar{N}^{L,\delta}(\alpha,\beta)(1) = n^{L,\delta}(\alpha,\beta) \quad \textit{and} \quad \bar{N}^{L,\delta}(\alpha,\beta)(-1) = W^{L,\delta}_{\mathrm{trop}}(\alpha,\beta).$$

In [9], Block and Göttsche relate the refined Severi degrees to tropical geometry and study them by the methods of tropical geometry. They introduce the refined multiplicity $M(v) := [m(v)]_y$, which specializes to $m(v)$ at $y = 1$ and to $r(v)$ at $y = -1$. Then the refined tropical Severi degrees $\bar{N}^{L,\delta}_{\mathrm{trop}}(\Sigma)$ are defined by counting curves with multiplicity $M(C) = \prod M(v)$. Note this definition applies to any smooth toric surface. It is shown that, for $S = \mathbb{P}^2$ or a rational ruled surface, and Σ a "vertically stretched" configuration of points, the $\bar{N}^{L,\delta}_{\mathrm{trop}}(\Sigma)$ satisfy the recursion (17). Thus $\bar{N}^{L,\delta} = \bar{N}^{L,\delta}_{\mathrm{trop}}(\Sigma)$.

This is the analogue of the tropical proof of the Caporaso–Harris recursion formula in [23], and like the original proof of Caporaso and Harris it can be viewed as a proof by degeneration. For a vertically stretched configuration of points the tropical curve degenerates, so that it can be described in terms of tropical curves of lower degree, and this gives the recursion both for the Severi degrees and the refined Severi degrees.

Itenberg and Mikhalkin have in the meantime shown in [35] that $\bar{N}^{L,\delta}_{\mathrm{trop}}(\Sigma)$ is independent of Σ, and so we drop it from the notation. For $S = \mathbb{P}^2$ or a rational ruled surface, Conjecture 80 then implies that the $\bar{N}^{L,\delta}_{\mathrm{trop}}$ agree with the $\bar{N}^{\delta}_{\delta,[S,L]}$ when L is δ very ample. More generally one expects:

Conjecture 94 [9] *Let S be a smooth projective toric surface and L a real line bundle on S. If L is δ–very ample, then $\bar{N}^{\delta}_{\delta,[S,L]} = \bar{N}^{\delta,L}_{\mathrm{trop}}$.*

Using the refined multiplicity, in [9] the $\bar{N}^{L,\delta}_{\mathrm{trop}}(\Sigma)$ are studied using methods similar to those employed by Block in [8] for the nonrefined Severi degrees. In particular it is shown that, for L sufficiently ample with respect to δ, they are given by refined node polynomials, and the Conjecture says that these agree with the $\bar{N}^{\delta}_{\delta,[S,L]}$.

According to Conjecture 94 and Proposition 93, we expect:

Conjecture 95 *Let L be a δ very ample real line bundle on a toric surface S. Then $\bar{N}^{\delta}_{\delta,[S,L]}(-1) = W^{\delta,L}_{\mathrm{trop}}$.*

For convenience we record the corresponding specialization of Conjecture 67 at $y=-1$. Consider $\eta(\tau) := q^{1/24}\prod_{n>0}(1-q^n)$ the Dirichlet eta function and write

$$\bar{G}_2(\tau) := G_2(\tau) - G_2(2\tau) = \sum_{n>0}\Big(\sum_{d|n, d \text{ odd}} \frac{n}{d}\Big)q^n.$$

Conjecture 96 *We have*

$$\sum_{\delta\geq 0} \bar{N}^{\delta}_{\delta,[S,L]}(-1)\bar{G}_2(\tau)^{\delta} = \frac{(\bar{G}_2(\tau)/q)^{\chi(L)} B_1(-1,q)^{K_S^2} B_2(-1,q)^{LK_S}}{(\eta(\tau)^{16}\eta(2\tau)^4 D\bar{G}_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{23}$$

This conjecture has been checked modulo q^{15} and the coefficients of $B_1(-1,q)$ and $B_2(-1,q)$ have been determined modulo q^{15}. (These computations are numerically easier than those involving the indeterminate y, thus we get to a higher order in q). The values of the series B_i are computed to be

$$\begin{aligned} B_1(-1,q) &= 1 - q - q^2 - q^3 + 3q^4 + q^5 - 22q^6 + 67q^7 - 42q^8 - 319q^9 \\ &\quad + 1207q^{10} - 1409q^{11} - 3916q^{12} + 20871q^{13} - 34984q^{14} + O(q^{15}), \\ B_2(-1,q) &= 1 + q + 2q^2 - q^3 + 4q^4 + 2q^5 - 11q^6 + 24q^7 + 4q^8 - 122q^9 \\ &\quad + 313q^{10} - 162q^{11} - 1314q^{12} + 4532q^{13} - 4746q^{14} + O(q^{15}). \end{aligned}$$

When $S = \mathbb{P}^2$ or a rational ruled surface, the Severi degrees $n^{\delta,L}$ agree with the universal numbers $n^{\delta}_{\delta,[S,L]}$ somewhat beyond the regime where L is δ very ample. Specifically, it is conjectured in [27] and proven in [40] that it suffices for the general $\mathbb{P}^{\delta} \subset |L|$ to contain no nonreduced curves, and no curves containing components with negative self intersection. We expect the same to hold for the comparison between refined Severi degrees $N^{\delta,L}$ and the universal numbers $N^{\delta}_{\delta,[S,L]}$, and a fortiori for the specialization at -1. However for this specialization more seems to be true:

Conjecture 97 *Assume $S = \mathbb{P}^2$ or $S = \Sigma_e$, and the following subloci of $|L|$ have codimension more than δ:*

(1) *The nonreduced curves with a component of multiplicity at least 3.*

(2) *Curves containing a component with negative self intersection.*

Then $W^{L,\delta}_{\mathrm{trop}} = \bar N^{\delta}_{\delta,[S,L]}(-1)$. *Explicitly the condition amounts to:*

(1) *On* $\mathbb{P}^2$, $W^{d,\delta}_{\mathrm{trop}} = \bar N^{\delta}_{dH}(-1)$ *if* $\delta \le 3d-3$.

(2) *On* $\mathbb{P}^1 \times \mathbb{P}^1$, $W^{nF+MG,\delta}_{\mathrm{trop}} = \bar N^{\delta}_{nF+mG}(-1)$ *if* $\delta \le 3\min(n,m)$.

(3) *On* Σ_e *with* $e>0$, $W^{nF+mE,\delta}_{\mathrm{trop}} = \bar N^{\delta}_{nF+mE}(-1)$, *if* $\delta \le \min(3m, n-em)$.

Using the recursion formula (21) this conjecture has been checked for $d,\delta \le 14$. Assuming (1) of Conjecture 97, and using the recursion formula (21), Conjecture 96 has been checked modulo q^{67} and $B_1(-1,q)$ and $B_2(-1,q)$ have been determined modulo q^{67}. Note that the recursion for $W^{d,\delta}_{\mathrm{trop}}(\alpha,\beta)$ is much more efficient than those of the $N^{d,\delta}(\alpha,\beta)(y)$ or $n^{d,\delta}(\alpha,\beta)$ because only odd sequences α and β occur.

We have seen $W^{d,\delta}_{\mathrm{trop}} = \bar N^{d,\delta}(-1)$. In the sufficiently ample setting, taking a linear system $\mathbb{P}^\delta$ determined by subtropical point conditions, and assuming all conjectures, this implies $N^{\delta}_{\mathcal{C}/\mathbb{P}^\delta}(-1) = (-1)^\delta \bar N^{\delta}_{\mathcal{C}/\mathbb{P}^\delta}(-1) = n^{\delta,\mathbb{R}}_{\mathcal{C}/\mathbb{P}^\delta}$. More generally we conjecture:

Conjecture 98 *Let* L *be a sufficiently ample real line bundle on a real toric surface, and let* $\mathbb{P}^\delta \subset |L|$ *be determined by a subtropical collection of point conditions. Then the signatures of the relative Hilbert schemes agree with the compactly supported Euler characteristics of their real loci. That is,*

$$\chi_1\big(\mathcal{C}^{[n]}_{\mathbb{P}^\delta}\big) = \chi_c\big(\mathcal{C}^{[n]}_{\mathbb{P}^\delta}(\mathbb{R})\big).$$

Making the BPS change of variables, it follows that $N^{\delta}_{\mathcal{C}/\mathbb{P}^\delta}(-1) = n^{\delta,\mathbb{R}}_{\mathcal{C}/\mathbb{P}^\delta}$.

More generally one may consider the question:

Question 99 Let X be a smooth real variety. When is $\chi_1(X) = \chi_c(X(\mathbb{R}))$?

This has been the subject of some classical study, one general result being that for an "M–variety", ie one for which the total dimension of the $\mathbb{Z}/2\mathbb{Z}$ cohomology is equal for the real and complex locus, the equality holds modulo 16; see Degtyarev and Kharlamov [16].

Evidently the equality holds for any variety whose class in the Grothendieck ring of varieties over $\mathbb{R}$ lies inside $\mathbb{Z}[\mathbb{A}^1_{\mathbb{R}}]$. In particular, $\mathbb{RP}^n$, toric surfaces, and Hilbert schemes of points on toric surfaces qualify. The relative Hilbert schemes are cut out of a product of these by a section of a vector bundle, and the signature behaves predictably under taking such sections. Thus, to study Conjecture 98 it would also suffice to give criteria for the Euler characteristic of the real locus to exhibit the same behavior. In [7], Bertrand and Bihan show this holds in a tropical sense for complete intersections in toric varieties.

Remark 100 The conjectured relation between the refined invariants at $y = -1$ and the Welschinger invariants is in a sense a global analogue of a conjecture of van Straten [68, Conjectures 4.6, 4.7] (see also Theorems 31 and 37 above and the nearby discussion).

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