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# Lipschitz connectivity and filling invariants in solvable groups and buildings

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Filling invariants of a group or space are quantitative versions of finiteness properties which measure the difficulty of filling a sphere in a space with a ball. Filling spheres is easy in nonpositively curved spaces, but it can be much harder in subsets of nonpositively curved spaces, such as certain solvable groups and lattices in semisimple groups. In this paper, we give some new methods for bounding filling invariants of such subspaces based on Lipschitz extension theorems. We apply our methods to find sharp bounds on higher-order Dehn functions of $\mathrm{Sol}_{2n+1}$, horospheres in euclidean buildings, Hilbert modular groups and certain $S$–arithmetic groups.

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## 1 Introduction

Many of the techniques used to find upper bounds on the Dehn function are difficult to generalize to higher-order Dehn functions. For example, one can prove that a nonpositively curved space has a Dehn function which is at most quadratic in a couple of lines: the fact that the distance function is convex implies that the disc formed by connecting every point on the curve to a basepoint on the curve has quadratically large area. On the other hand, proving that a nonpositively curved space has a $k^{\text{th}}$–order Dehn function bounded by $V^{(k+1)/k}$ takes several pages; see Gromov [13] and Wenger [22].

One reason for this is that the geometry of spheres is generally much more complicated than the geometry of curves. A closed curve is geometrically very simple. It has diameter bounded by its length, it has a natural parameterization by length, and a closed curve in a space with a geometric group action can be approximated by a word in the group. None of these hold for spheres. A $k$–sphere of volume $V$ may have arbitrarily large diameter, has no natural parameterization, and though it can often be approximated by a cellular or simplicial cycle, that cycle may have arbitrarily many cells of dimension less than $k$.

One way around this is to consider Lipschitz extension properties. A typical Lipschitz extension property is Lipschitz $k$–connectivity; we say that a space $X$ is *Lipschitz*

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*$k$–connected* (with constant $c$) if there is a $c$ such that for any $0 \le d \le k$ and any $l$–Lipschitz map $f\colon S^d \to X$, there is a $cl$–Lipschitz extension $\bar{f}\colon D^{d+1} \to X$. The advantage of dealing with Lipschitz spheres rather than spheres of bounded volume is that techniques for filling closed curves often generalize to Lipschitz spheres. For example, the same construction that shows that a nonpositively curved space has quadratic Dehn function shows that such a space is Lipschitz $k$–connected for any $k$. Any map $f\colon S^d \to X$ can be extended to a map $\bar{f}\colon D^{d+1} \to X$ by coning $f$ off to a point along geodesics, and if $f$ is Lipschitz, so is $\bar{f}$.

In this paper, we describe a way to use Lipschitz connectivity to prove bounds on higher-order filling functions of subsets of spaces with finite Assouad–Nagata dimension. These spaces include euclidean buildings and homogeneous Hadamard manifolds (see Lang and Schlichenmaier [17]) and we will show that a higher-dimensional analogue of the Lubotzky–Mozes–Raghunathan theorem holds for Lipschitz $n$–connected subsets of spaces with finite Assouad–Nagata dimension. Recall that Lubotzky, Mozes, and Raghunathan proved the following.

**Theorem 1.1** (Lubotzky, Mozes, and Raghunathan [19]) *If $\Gamma$ is an irreducible lattice in a semisimple group $G$ of rank greater than or equal to $2$, then the word metric on $\Gamma$ is quasi-isometric to the restriction of the metric on $G$ to $\Gamma$.*

One way to state this theorem is to say that the inclusion $\Gamma \hookrightarrow G$ does not induce any distortion of lengths. That is, there is a $c > 0$ such that if $x, y \in \Gamma$ are connected by a path of length $l$ in $G$, then they are connected by a path of length less than or equal to $cl$ in the Cayley graph of $\Gamma$. Bux and Wortman [6] conjectured that filling volumes should also be undistorted in lattices in higher-rank semisimple groups. We will state a version of this conjecture in terms of homological filling volumes; in a highly-connected space, these are equivalent to homotopical filling volumes in dimensions above 2; see [13], White [23] and Groft [12].

To state the conjecture, we introduce Lipschitz chains. A Lipschitz $d$–chain in $Y$ is a formal sum of Lipschitz maps $\Delta_d \to Y$. One can define the boundary of a Lipschitz chain as for singular chains, and this gives rise to a homology theory. If $\alpha$ is a Lipschitz $d$–cycle in $Y$, define

$$\mathrm{FV}_Y^{d+1}(\alpha) = \inf_{\partial\beta=\alpha} \operatorname{mass}\beta$$

to be the filling volume of $\alpha$ in $Y$. In particular, if $Y$ is a geodesic metric space and $\alpha$ is the $0$–cycle $x - y$, then $\mathrm{FV}_Y^1(\alpha) = d(x, y)$.

If $Z \subset X$, we say that $Z$ is *undistorted up to dimension n* if there is some $r \geq 0$ and $c$ such that if $\alpha$ is a Lipschitz $d$–cycle in $Z$ and $d < n$, then

$$\mathrm{FV}_Z^{d+1}(\alpha) \leq c\,\mathrm{FV}_X^{d+1}(\alpha) + c.$$

(Note that this differs from Bux and Wortman's definition in [6]; Bux and Wortman's definition deals with extending spheres in a neighborhood of $Z$ to balls in a larger neighborhood.)

**Conjecture 1.2** (cf [6, Question 1.6]) *If $\Gamma$ is an irreducible lattice in a semisimple group $G$ of rank $n$, then there is a nonempty $\Gamma$–invariant subset $Z \subset G$ such that $d_{\mathrm{Haus}}(Z, \Gamma) < \infty$ and $Z$ is* undistorted up to dimension $n-1$.

Here, $d_{\mathrm{Haus}}(Z, \Gamma)$ represents the Hausdorff distance between the two sets.

Theorem 1.1 is a special case of this conjecture, and the conjecture also generalizes conjectures of Gromov and Thurston on filling inequalities in lattices. Thurston famously conjectured that $\mathrm{SL}(n;\mathbb{Z})$ has quadratic Dehn function for $n \geq 4$ [11], and Gromov conjectured that the $(k-2)^{\text{th}}$ order Dehn function of a lattice in a symmetric space of rank $k$ should be bounded by a polynomial [14]. As Bestvina, Eskin and Wortman note in [5], Conjecture 1.2 would imply that the $k^{\text{th}}$–order Dehn function of $\Gamma$ is bounded by $V^{(k+1)/k}$. In recent years, a significant amount of progress has been made toward these conjectures. Druțu proved that a lattice of $\mathbb{Q}$–rank 1 in a symmetric space of $\mathbb{R}$–rank greater than or equal to 3 has a Dehn function bounded by $n^{2+\epsilon}$ for any $\epsilon > 0$ [9], Leuzinger and Pittet proved that, conversely, any irreducible lattice in a symmetric space of rank 2 which is not cocompact has an exponentially large Dehn function [18], and the author proved Thurston's conjecture in the case that $n \geq 5$ [24].

In this paper, we make a step toward proving Conjecture 1.2 by showing that, under some conditions on $G$ and $\Gamma$, undistortedness follows from a Lipschitz extension property.

We say that $Z$ is *Lipschitz n–connected* if there is a $c$ such that for any $0 \leq d \leq n$ and any $l$–Lipschitz map $f\colon S^d \to Z$, there is a $cl$–Lipschitz extension $\bar{f}\colon D^{d+1} \to Z$. If $X$ is a metric space, the *Assouad–Nagata dimension* of $X$ is the smallest integer $n$ such that there is a $c$ such that for all $s > 0$, there is a covering $\mathcal{B}_s$ of $X$ by sets of diameter at most $cs$ (a *$cs$–bounded covering*) such that any set with diameter less than or equal to $s$ intersects at most $n+1$ sets in the cover (ie $\mathcal{B}_s$ has *$s$–multiplicity* at most $n+1$).

**Theorem 1.3** *Suppose that $Z \subset X$ is a nonempty closed subset with metric given by the restriction of the metric of $X$. Suppose that $X$ is a geodesic metric space such that the Assouad–Nagata dimension $\dim_{\mathrm{AN}}(X)$ of $X$ is finite. Suppose that one of the following is true:*

- *$Z$ is Lipschitz $n$–connected.*
- *$X$ is Lipschitz $n$–connected, and if $X_p$, $p \in P$ are the connected components of $X \smallsetminus Z$, then the sets $H_p = \partial X_p$ are Lipschitz $n$–connected with uniformly bounded implicit constant.*

*Then $Z$ is undistorted up to dimension $n+1$.*

In the applications in this paper, $X$ will be a CAT(0) space (either a symmetric space or a building), and $Z$ will either be a horosphere of $X$ or the complement of a set of disjoint horoballs.

When $X$ is CAT(0), a theorem of Gromov [13; 22] implies that the $k^{\text{th}}$–order Dehn function of $X$ grows at most as fast as $V^{(k+1)/k}$ (ie if $\alpha$ is a Lipschitz $k$–cycle in $X$, there is a Lipschitz $(k+1)$–chain $\beta$ in $X$ such that $\partial\beta = \alpha$ and

$$\operatorname{mass}\beta \lesssim (\operatorname{mass}\alpha)^{(k+1)/k} + c.$$

Therefore:

**Corollary 1.4** *If $X$ is CAT(0) and the hypotheses above hold, the $k^{th}$–order Dehn function of $Z$ grows at most as fast as $V^{(k+1)/k}$ for $k \le n$.*

This bound is often sharp; for instance, if there is a rank-$(k+1)$ flat of $X$ contained in $Z$, then the $k^{\text{th}}$–order Dehn function of $Z$ grows at least as fast as $V^{(k+1)/k}$.

We will apply Theorem 1.3 to find fillings in a family of solvable groups and in the Hilbert modular groups:

**Theorem 1.5** *The group $\mathrm{Sol}_{2n-1} = \mathbb{R}^{n-1} \ltimes \mathbb{R}^n$ is Lipschitz $(n-2)$–connected, is undistorted in $(\mathbb{H}_2)^n$ up to dimension $(n-1)$, and its $k^{th}$–order Dehn function is asymptotic to $V^{(k+1)/k}$ for $k < n-1$.*

This is a higher-dimensional version of a theorem of Gromov [14, 5.A$_9$] which states that $\mathrm{Sol}_{2n-1}$ has quadratic Dehn function when $n > 2$. These bounds are sharp; there are $n$–spheres in $\mathrm{Sol}_{2n-1}$ with volume $V$ but filling volume exponential in $V$, so the $n^{\text{th}}$ order Dehn function of $\mathrm{Sol}_{2n-1}$ is exponential [14]. The same bounds apply to Hilbert modular groups:

**Theorem 1.6** *If $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^n$ is a Hilbert modular group, then the $k^{th}$–order Dehn function of $\Gamma$ is asymptotic to $V^{(k+1)/k}$ for $k < n-1$.*

Pittet showed that $\Gamma$ has an exponential $(n-1)^{\mathrm{th}}$ order Dehn function [20].

We will also apply the methods of Theorem 1.3 to horospheres in euclidean buildings and to the $S$–arithmetic groups considered by Bux and Wortman in [7].

Let $X$ be a thick euclidean building and $E \subset X$ be an apartment. Then the vertices of $E$ form a lattice, and if $r\colon [0,\infty) \to E$ is a geodesic ray, we say that $r$ has *rational slope* if it is parallel to a line segment connecting two vertices of $E$. This condition is independent of the choice of $E$, so if $r\colon [0,\infty) \to X$ is a geodesic ray, we say it has rational slope if it has rational slope considered as a ray in some apartment $E$. The boundary at infinity of $X$ consists of equivalence classes of geodesic rays, so if $\tau \in X_\infty$ is a point in the boundary at infinity of $X$, we say it has rational slope if one of the rays asymptotic to $\tau$ has rational slope. In particular, if the isometry group of $X$ acts cocompactly on a horosphere centered at $\tau$, then $\tau$ has rational slope.

**Theorem 1.7** *Let $X$ be a thick euclidean building and let $\tau \in X_\infty$ be a point in its boundary at infinity which has rational slope and is not contained in a factor of rank less than $n$ (in particular, $X$ has rank at least $n$). Let $Z$ be a horosphere in $X$ centered at $\tau$. Then $Z$ is Lipschitz $(n-2)$–connected, undistorted in $X$ up to dimension $n-1$, and its $k^{th}$–order Dehn function grows asymptotically like $V^{(k+1)/k}$ for $k \le n-2$.*

The horosphere $Z$ is not $(n-1)$–connected, so the bound on $k$ is sharp. Indeed, for every $r > 0$, there is a map $\alpha\colon S^{n-1} \to Z$ such that $\alpha$ is not nullhomotopic in the $r$–neighborhood of $Z$ (see Lemma 4.15).

Note that if $\tau$ does not have rational slope, then $Z$ may be $(n-2)$–connected and locally Lipschitz $(n-2)$–connected but not Lipschitz $(n-2)$–connected. Cells of $X$ may intersect $Z$ in arbitrarily small sets, and this can lead to arbitrarily small spheres which have small fillings in $X$ but not in $Z$.

Theorem 1.7 is similar to [7, Theorem 7.7], and gives a higher-order version of [9, Theorem 1.1] for buildings and products of buildings. (Though note that [9, Theorem 1.1] applies to $\mathbb{R}$–buildings as well as discrete buildings.)

The same methods lead to quantitative finiteness properties for $S$–arithmetic groups of $K$–rank 1.

**Theorem 1.8** (cf [7, Theorems 1.2, 8.1]) *Let $K$ be a global function field, $\boldsymbol{G}$ be a noncommutative, absolutely almost simple $K$–group of $K$–rank 1, let $S$ be a finite set of pairwise inequivalent valuations on $K$, and let $X$ be the associated euclidean building. Then the $k^{th}$–order Dehn function of the $S$–arithmetic group $G(\mathcal{O}_S)$ grows asymptotically like $V^{(k+1)/k}$ for $k \leq \dim X - 2$.*

Again, this is sharp; Bux and Wortman showed that $G(\mathcal{O}_S)$ is not of type $F_{\dim X}$, so its $(\dim X - 1)^{\text{th}}$ order Dehn function is undefined.

Some possible other applications of Theorem 1.3 include the study of higher-order fillings in, for instance, metabelian groups, as in de Cornulier and Tessera [8], lattices of $\mathbb{Q}$–rank 1 in semisimple groups, as in [9], and $S$–arithmetic lattices when $|S|$ is large, as in [5].

**Notational conventions** If $f$ and $g$ are expressions, we will write $f \lesssim g$ (or say that $f$ is of order at most $g$) if there is some constant $c$ such that $f \leq cg$. We write $f \sim g$ if there is some constant $c$ such that $c^{-1}g \leq f \leq cg$. When we wish to emphasize that $c$ depends on $x$ and $y$, we write $f \lesssim_{x,y} g$ or $f \sim_{x,y} g$. We give $S^k$ the round metric, scaled so that $\operatorname{diam} S^k = 1$, and we define the *standard $k$–simplex* to be the equilateral euclidean $k$–simplex, scaled to have diameter 1.

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## 2 Building fillings from simplices

The proof of Theorem 1.3 is based on the proof of a theorem of Lang and Schlichenmaier [17]. Lang and Schlichenmaier proved:

**Theorem 2.1** *Suppose $Z \subset X$ is a nonempty closed set and $\dim_{\mathrm{AN}} X \leq m < \infty$. If $Y$ is Lipschitz $(m-1)$–connected, then there is a $C$ such that any Lipschitz map $f\colon Z \to Y$ extends to a map $\bar{f}\colon X \to Y$ with $\operatorname{Lip}(\bar{f}) \leq C \operatorname{Lip}(f)$.*

Here, $\dim_{\mathrm{AN}}(X)$ is the Assouad–Nagata dimension of $X$.

One consequence of Theorem 2.1 is if $Z$ is Lipschitz $n$–connected for $n = \dim_{\mathrm{AN}}(X)$, then the identity map $Z \to Z$ can be extended to a Lipschitz map $\bar{f}\colon X \to Z$ and $Z$

is a Lipschitz retract of $X$. Consequently, if $\alpha$ is a $(k-1)$–cycle in $Z$ and $\beta$ is a chain in $X$ with boundary $\alpha$, then $\overline{f}_\sharp(\beta)$ is a chain in $Z$ with boundary $\alpha$, so

$$\mathrm{FV}^k_Z(\alpha) \le C^k \, \mathrm{FV}^k_X(\alpha),$$

and $Z$ is undistorted in $X$ up to dimension $n$. Theorem 1.3 claims that the same is true under the weaker condition that $X$ has finite Assouad–Nagata dimension.

Before we sketch the proof of Theorem 1.3, we need the notion of a quasiconformal complex. We define a *Riemannian simplicial complex* to be a simplicial complex with a metric which gives each simplex the structure of a Riemannian manifold with corners. We say that such a complex is quasiconformal (or that the complex is a *QC complex*) if there is a $c$ such that the Riemannian metric on each simplex is $c$–bilipschitz equivalent to a scaling of the standard simplex.

QC complexes are a compromise between the rigidity of simplicial complexes and the freedom of Riemannian simplicial complexes. A key feature of simplicial complexes is that curves and cycles can be approximated by simplicial curves and cycles. This is not true in Riemannian simplicial complexes, but it holds in QC complexes.

Specifically, a version of the Federer–Fleming deformation theorem holds in QC complexes. Recall that the Federer–Fleming theorem for simplicial complexes states that any Lipschitz cycle $a$ in a simplicial complex can be approximated by a simplicial cycle $P(a)$ whose mass is comparable to the mass of $a$. We will use the following variation of the Federer–Fleming theorem:

**Theorem 2.2** *Let $\Sigma$ be a finite-dimensional scaled simplicial complex, that is, a simplicial complex where each simplex is given the metric of the standard simplex of diameter $s$. There is a constant $c$ depending on* $\dim \Sigma$ *such that if $a \in C^{\mathrm{Lip}}_k(\Sigma)$ is a Lipschitz $k$–cycle, then there are $P(a) \in C^{\mathrm{cell}}_k(X)$ and $Q(a) \in C^{\mathrm{Lip}}_{k+1}(X)$ such that:*

(1) $\partial a = \partial P(a)$

(2) $\partial Q(a) = a - P(a)$

(3) $\mathrm{mass}\, P(a) \le c \cdot \mathrm{mass}(a)$

(4) $\mathrm{mass}\, Q(a) \le cs \cdot \mathrm{mass}(a)$

Epstein, Cannon, Holt, Levy, Paterson and Thurston proved this theorem for $s = 1$ in [10, Chapter 10.3]. A simple scaling argument proves the general case. Note that, while the bound on $\mathrm{mass}\, Q(a)$ depends on the size of the simplices, the bound on $\mathrm{mass}\, P(a)$ does not.

Because the bound on $\mathrm{mass}\, P(a)$ is independent of the size of the simplices in the complex, the following version of Theorem 2.2 holds for a QC complex:

**Theorem 2.3** *Let $\Sigma$ be a QC complex. There is a constant $c$ depending on $\dim \Sigma$ such that if $a \in C_k^{\mathrm{Lip}}(\Sigma)$ is a Lipschitz $k$–cycle, then there is a $P(a) \in C_k^{\mathrm{cell}}(X)$ such that $\partial a = \partial P(a)$ and $\operatorname{mass} P(a) \leq c \cdot \operatorname{mass}(a)$.*

Now we will sketch a proof of Theorem 1.3. Note that this sketch is incorrect due to some technical issues; we will fix these issues in the actual proof. In the proof of Theorem 1.5 of [17], Lang and Schlichenmaier show that, if $\dim_{\mathrm{AN}}(X) < \infty$, there are $a > 0$, $0 < b < 1$ and a cover $\mathcal{B} = (B_i)_{i \in I_0}$ of $X \setminus Z$ by subsets of $X \setminus Z$ such that

(1) $\operatorname{diam} B_i \leq a d(B_i, Z)$ for every $i \in I_0$,

(2) every set $D \subset X \setminus Z$ with $\operatorname{diam} D \leq b d(D, Z)$ meets at most $\dim_{\mathrm{AN}}(X) + 1$ members of $(B_i)_{i \in I_0}$.

They then define functions $\sigma_i \colon X \setminus Z \to \mathbb{R}$,

$$\sigma_i(x) = \max\{0, \delta d(B_i, Z) - d(x, B_i)\},$$

where $\delta = b/(2(b+1))$, and show that these have the property that for any $x$, there are no more than $\dim_{\mathrm{AN}}(X) + 1$ values of $i$ for which $\sigma_i(x) > 0$. Using these $\sigma_i$, they construct a Lipschitz map $g \colon X \setminus Z \to \Sigma_0$, where $\Sigma_0$ is the nerve of the supports of the $\sigma_i$. One can give $\Sigma_0$ the structure of a QC complex so that if $\Delta$ is a simplex of $\Sigma_0$ with a vertex corresponding to $\sigma_i$, then $\operatorname{diam} \Delta \sim \operatorname{diam} \operatorname{supp} \sigma_i$. Since the diameter of $\operatorname{supp} \sigma_i$ is proportional to $d(\sigma_i, Z)$, this means that the parts of $\Sigma_0$ which are close to $Z$ are given a fine triangulation and the parts of $\Sigma_0$ which are far from $Z$ are given a coarse triangulation.

Since $Z$ is Lipschitz $n$–connected, one can construct a Lipschitz map $h \colon \Sigma_0^{(n+1)} \to Z$, where $\Sigma_0^{(n+1)}$ is the $(n+1)$–skeleton of $\Sigma_0$. Then, if $\alpha$ is an $n$–cycle in $Z$, it has a filling $\beta$ in $X$. We can use the Federer–Fleming theorem to approximate $g_\sharp(\beta)$ by some simplicial $(n+1)$–chain $P(\beta)$ which lies in $\Sigma_0^{(n+1)}$. The pushforward of $P(\beta)$ under $h$ will then be a filling of $\alpha$.

The problem with this argument is twofold. First, since $g$ is only defined on $X \setminus Z$, we can't define $g_\sharp(\beta)$ without extending $g$ to $Z$. We could define an appropriate metric on the disjoint union $\Sigma_0 \amalg Z$ and a map $X \to \Sigma_0 \amalg Z$, but this is no longer a simplicial complex. Second, since the cells of $\Sigma$ get arbitrarily small close to $Z$, $P(\beta)$ may be an infinite sum of cells of $\Sigma$.

We know of two ways to fix this issue. First, one can make sense of infinite sums of cells of $\Sigma$ by introducing Lipschitz currents; see Ambrosio and Kirchheim [4]. The set of Lipschitz currents is a completion of the set of Lipschitz chains, and the $P(\beta)$ defined above is a current in $\Sigma_0 \amalg Z$. Its pushforward is then a filling of $\alpha$. Second, we can change the construction of $\Sigma_0$ to avoid the problem. We take this approach in the rest of this section.

All the constants and all the implicit constants in $\lesssim$ and $\sim$ in this section will depend on $X, Z$, and $n$.

First, we construct a QC complex $\Sigma$ which approximates $X$. This complex will have geometry similar to $\Sigma_0$ on $X \setminus Z$ and it will have $\epsilon$–small simplices on $Z$. For $t > 0$, let $N_t(Z) \subset X$ be the $t$–neighborhood of $Z$.

**Lemma 2.4** *There are $a, b, \gamma > 0$ such that if $\epsilon > 0$ and $\delta = b/(2(b+1))$, there is a covering $\mathcal{D}$ of $X$ by sets $D_k$, $k \in K$ and functions $r\colon K \to \mathbb{R}$, $\tau_k\colon X \to \mathbb{R}$*

$$r(k) = \max\{\delta d(D_k, Z), \epsilon\},$$
$$\tau_k(x) = \max\{0, r(k) - d(x, D_k)\},$$

*such that for any $k \in K$:*

(1) $\operatorname{diam} D_k \lesssim r(k)$
(2) $d(D_k, Z) \lesssim r(k)$
(3) *If $\rho = \epsilon\delta(1+a)$ and $d(D_k, Z) \geq \rho$, then $\operatorname{supp} \tau_k$ is contained in a connected component of $X \smallsetminus Z$.*
(4) *The cover of $X$ by the sets $\operatorname{supp} \tau_k$ has multiplicity at most $2\dim_{\mathrm{AN}}(X) + 2$.*
(5) *If $\operatorname{supp} \tau_k \cap \operatorname{supp} \tau_{k'} \neq \varnothing$, then*

$$\gamma^{-1} r(k') \leq r(k) \leq \gamma r(k')$$

**Proof** Let $a > 0$, $0 < b < 1$, and $\mathcal{B} = (B_i)_{i \in I_0}$ be as in the Lang–Schlichenmaier construction above. Let We may assume that each $B_i$ is contained in a connected component of $X \smallsetminus Z$. Let $\rho = \epsilon\delta(1+a)$, and let $I \subset I_0$ be the set

$$I := \{i \in I_0 \mid B_i \not\subset N_\rho(Z)\}.$$

Then $\bigcup_{i \in I} B_i \supset X \setminus N_\rho(Z)$. Since $\dim_{\mathrm{AN}}(X) \leq \infty$, we can let $\mathcal{C} = \{C_j\}_{j \in J}$ be a $2c_0\epsilon$–bounded covering of $N_\rho(Z)$ with $2\epsilon$–multiplicity at most $\dim_{\mathrm{AN}}(X) + 1$, where $c_0$ is the constant in the definition of $\dim_{\mathrm{AN}}(X)$. Let $\mathcal{D} = \mathcal{C} \cup \{B_i\}_{i \in I}$ and let $K = I \amalg J$.

Conditions (1) and (2) are easy to check. For (3), note that if $d(D_k, Z) \geq \rho$, then $k \in I$, so $D_i = B_i$ lies in a single connected component of $X \setminus Z$, and $\operatorname{supp} \tau_i$ lies in the same component. For (4), note that if $i \in I$, then $\tau_i = \sigma_i$, so the cover $\{\operatorname{supp} \tau_i\}_{i \in I}$ has multiplicity at most $\dim_{\mathrm{AN}}(X) + 1$. Likewise, if $x \in \operatorname{supp} \tau_j$ for some $j \in J$, then

$C_j \cap B(x,\epsilon) \neq \varnothing$, where $B(x,\epsilon)$ is the closed ball of radius $\epsilon$ around $x$. Since $\mathcal{C}$ has bounded $2\epsilon$–multiplicity, this can be true for only $\dim_{\mathrm{AN}}(X)+1$ values of $j$.

To check (5), suppose that $\operatorname{supp}\tau_k \cap \operatorname{supp}\tau_{k'} \neq \varnothing$. If $r(k') = \epsilon$, then $r(k') \le r(k)$. Otherwise, $r(k') = \delta d(D_{k'}, Z)$. But $d(D_k, D_{k'}) \lesssim r(k)$ and $\operatorname{diam} D_k \lesssim r(k)$, so $d(D_{k'}, Z) \lesssim r(k)$, and $r(k') \lesssim r(k)$. By symmetry, $r(k) \sim r(k')$. $\square$

Let $\Sigma$ be the nerve of the cover $\{\operatorname{supp}\tau_k\}_{k\in K}$, with vertex set $\{v_k\}_{k\in K}$ and let $s\colon \Sigma \to \mathbb{R}$ is the function such that $s(v_k) = r(k)$ and $s$ is linear on each simplex of $\Sigma$. Define a Riemannian metric $x_c$ on each simplex of $\Sigma$ by letting $dx_c^2 = s^2\, dx^2$. If $S = \langle v_{k_1}, \dots, v_{k_n}\rangle$ is a simplex of $\Sigma$, then $s$ varies between $\gamma^{-1} r(k_1)$ and $\gamma r(k_1)$ on $S$, so this metric makes $\Sigma$ a QC complex.

**Lemma 2.5** *There is a Lipschitz map $g\colon X \to \Sigma$ with Lipschitz constant $c_1$ independent of $\epsilon$. Furthermore, if $x \in \operatorname{supp}\tau_k$ for some $k \in K$, then $g(x)$ is in the star of $v_k$.*

**Proof** Consider the infinite simplex

$$\Delta_K := \{p\colon K \to [0,1] \mid \|p\|_1 = 1\}$$

with vertex set $K$, so that $\Sigma$ is a subcomplex of $\Delta_K$. Let

$$g(x)(k) = \frac{\tau_k(x)}{\overline{\tau}(x)},$$

where $\overline{\tau}(x) = \sum_{k\in K} \tau_k(x)$. The image of $g$ then lies in $\Sigma$, and we can consider $g$ as a function $X \to \Sigma$.

It remains to show $g$ is Lipschitz with respect to the QC metric on $\Sigma$. Since $X$ is geodesic, it suffices to show if $x, y \in X$ and $d(x,y) < \delta^2\epsilon < \epsilon$, then $d(g(x), g(y)) \lesssim d(x,y)$. Let $S$ and $T$ be the minimal simplices of $\Sigma$ which contain $g(x)$ and $g(y)$ respectively. First, we claim that $S$ and $T$ share some vertex $v_m$.

Let $\rho = \epsilon\delta(1+a)$ as above. If $d(x,Z) < \rho$, then there is some $j \in J$ such that $x \in C_j$ and $\tau_j(x) = \epsilon$. Since $\tau_j$ is 1–Lipschitz, $\tau_j(y) > 0$, so we can let $m = j$. Otherwise, if $d(x,Z) \ge \rho$, then there is some $i \in I$ such that $x \in B_i$. We have

$$d(x,Z) \le \operatorname{diam}(B_i) + d(B_i, Z) \le (a+1) d(B_i, Z),$$

so $\tau_i(x) = \delta d(B_i, Z) \ge \delta^2\epsilon$, and $\tau_i(y) > 0$ as desired. We let $m = i$.

Since $S$ and $T$ share $v_m$, the value of $s$ on $S \cup T$ is at most $\gamma r(m)$, and

$$\begin{aligned}
d(g(x), g(y)) &\le \gamma r(m) \sum_{k \in (S\cup T)^{(0)}} \left| \frac{\tau_k(x)}{\overline{\tau}(x)} - \frac{\tau_k(y)}{\overline{\tau}(y)} \right| \\
&\le \gamma r(m) \sum_{k \in (S\cup T)^{(0)}} \left| \frac{\tau_k(x)}{\overline{\tau}(x)} - \frac{\tau_k(y)}{\overline{\tau}(x)} \right| + \left| \frac{\tau_k(y)}{\overline{\tau}(x)} - \frac{\tau_k(y)}{\overline{\tau}(y)} \right| \\
&\le \gamma r(m) \sum_{k \in (S\cup T)^{(0)}} \frac{1}{\overline{\tau}(x)} \left( \left| \tau_k(x) - \tau_k(y) \right| + \frac{\tau_k(y)}{\overline{\tau}(y)} \left| \overline{\tau}(x) - \overline{\tau}(y) \right| \right) \\
&\le \gamma (2 \dim \Sigma + 1)(2 \dim \Sigma + 2) \frac{r(m)}{\overline{\tau}(x)} d(x, y).
\end{aligned}$$

Furthermore, if $x \in D_{m'}$, then

$$\overline{\tau}(x) \ge r(m') \ge \gamma^{-1} r(m),$$

so $g$ has Lipschitz constant at most

$$c_1 = \gamma^2 (2 \dim \Sigma + 1)(2 \dim \Sigma + 2). \qquad \square$$

Next, we construct a map $h\colon \Sigma^{(n+1)} \to Z$ on the $(n+1)$–skeleton of $\Sigma$. If $\Delta$ is a simplex of $\Sigma$, denote its vertex set by $\mathcal{V}(\Delta)$.

**Lemma 2.6** *For any $\epsilon' > 0$, there is a Lipschitz map $h^{(0)}\colon \Sigma^{(0)} \to Z$ with Lipschitz constant independent of $\epsilon$ which satisfies*

(1) $d(h^{(0)}(v_j), C_j) \lesssim \epsilon$ *for every $j \in J$,*

(2) *if $X_p$, $p \in P$ are the connected components of $X \smallsetminus Z$ and*

$$H_p(\epsilon') = \{x \in X \mid d(x, X_p) \le \epsilon'\} \cap Z,$$

*then for any simplex $\Delta \subset \Sigma$, we either have $\operatorname{diam} h^{(0)}(\mathcal{V}(\Delta)) \lesssim \epsilon$ (if $\Delta$ has a vertex of the form $v_j$ for some $j \in J$) or $h^{(0)}(\mathcal{V}(\Delta)) \subset H_p(\epsilon')$ for some $p \in P$ (otherwise).*

**Proof** For each vertex $v_k \in \Sigma$, choose a point $z_k \in Z$ such that $d(z_k, D_k) < d(Z, D_k) + \epsilon_H/2$, and let $h^{(0)}(v_k) = z_k$. If $k \in J$, then $d(Z, D_k) \lesssim \epsilon$, and so $d(z_k, D_k) \lesssim \epsilon$ and property (1) holds. We claim that this map is Lipschitz. Suppose that $v, w$ are vertices of $\Sigma$. Then there is a path $\gamma\colon [0,1] \to \Sigma$ between them of length $\ell(\gamma) \le 2d(v, w)$, and the Federer–Fleming theorem implies that this can be approximated by a path $\gamma'\colon [0,1] \to \Sigma^{(1)}$ in the 1–skeleton of $\Sigma$ with $\ell(\gamma') \lesssim \ell(\gamma)$.

So, to check that $h^{(0)}$ is Lipschitz, it suffices to show that if $v_k$ and $v_{k'}$ are connected by an edge $e$, then $d(z_k, z_{k'}) \lesssim \ell(e)$.

We may assume that $r(k) \geq r(k')$, so $\ell(e) \geq \gamma^{-1} r(k)$. Then we can bound $d(z_k, z_{k'})$ by

$$d(z_k, z_{k'}) \leq d(z_k, D_k) + \operatorname{diam}(D_k) + d(D_k, D_{k'}) + \operatorname{diam}(D_{k'}) + d(D_{k'}, z_{k'}).$$

Each term on the right is of order at most $r(k)$. For each term except $d(D_k, D_{k'})$, this follows from the remarks after the definition of $S$. To bound $d(D_k, D_{k'})$, note that since there is an edge from $v_k$ to $v_{k'}$, there is a $w \in \operatorname{supp} \tau_k \cap \operatorname{supp} \tau_{k'}$. Then $d(w, D_k) < r(k)$ and $d(w, D_{k'}) < r(k)$, so $d(D_k, D_{k'}) \leq 2r(k)$. Therefore, $h^{(0)}$ is Lipschitz.

It remains to check property (2). Let $\Delta = \langle v_{k_0}, \dots, v_{k_n} \rangle$ be a simplex of $\Sigma$ and suppose that $k_i \in J$ for some $i$. Then $r(k_i) \lesssim \epsilon$, so $\operatorname{diam} \Delta \lesssim \epsilon$, and therefore, $\operatorname{diam} h^{(0)}(\mathcal{V}(\Delta)) \lesssim \epsilon$.

Otherwise, $k_i \in I$ for all $i$. Then there is some $p \in P$ such that $\operatorname{supp} \tau_{k_i} \subset X_p$ for all $i$, and $h^{(0)}(\mathcal{V}(\Delta)) \subset H_p(\epsilon')$. □

If $\epsilon > 0$ and $n$ are such that whenever $k \leq n$ and $\tau\colon S^k \to Z$ is a map with $\operatorname{Lip} \tau \leq \epsilon$, there is an extension $\overline{\tau}\colon D^{k+1} \to Z$ with $\operatorname{Lip} \overline{\tau} \lesssim \operatorname{Lip} \tau$, we say that $Y$ is *$\epsilon$–locally Lipschitz $n$–connected.*

**Lemma 2.7** *If $X$ and $Z$ satisfy the conditions of Theorem 1.3 and $\epsilon$ is sufficiently small, then there is a Lipschitz extension $h\colon \Sigma^{(n+1)} \to Z$ with Lipschitz constant independent of $\epsilon$ such that $d(h(g(z)), z) \lesssim \epsilon$ for every $z \in Z$.*

**Proof** In this proof, it will be convenient to let $S^k$ be the boundary of the standard $(k+1)$–simplex and $D^k$ be the standard $k$–simplex. If $t > 0$, we let $tS^k$ and $tD^k$ be scalings of $S^k$ and $D^k$. If a space $Y$ is Lipschitz $n$–connected, there is a $c$ such that if $k \leq n$, any Lipschitz map $\tau\colon S^k \to Y$ can be extended to a Lipschitz map $\overline{\tau}\colon D^{k+1} \to Y$ with $\operatorname{Lip} \overline{\tau} \leq c \operatorname{Lip} \tau$. By scaling, any Lipschitz map $\tau\colon tS^k \to Y$ can be extended to a Lipschitz map $\overline{\tau}\colon tD^{k+1} \to Y$ with $\operatorname{Lip} \overline{\tau} \leq c \operatorname{Lip} \tau$

If $Z$ is Lipschitz $n$–connected, then we can use Lipschitz $n$–connectivity to extend $h^{(0)}$. That is, if we have already defined $h$ on $\Sigma^{(k)}$ and $\Delta \subset \Sigma$ is a $(k+1)$–simplex, then the Riemannian metric on $\Delta$ is bilipschitz equivalent to $s(x) D^{k+1}$ for any $x \in \Delta$. Since $h|_{\partial \Delta}$ is a Lipschitz map of a $k$–sphere, we can extend $h$ over $\Delta$, and the extension satisfies $\operatorname{Lip} h \lesssim \operatorname{Lip} h^{(0)}$.

If $Z$ is not Lipschitz $n$–connected, we need a more careful approach. By hypothesis, $X$ is Lipschitz $n$–connected; let $c > 0$ be the constant in the definition of Lipschitz $n$–connectivity.

Let $\epsilon' = \epsilon_H / c$ and let $k \leq n$. If $\tau \colon S^k \to Z$ is a map with $\operatorname{Lip} \tau \leq \epsilon'$, we claim that $\tau$ can be extended to a Lipschitz map on $D^{k+1}$. If $\tau(S^k) \subset H_p(\epsilon_H)$ for some $p \in P$, then we can extend $\tau$ to $D^{k+1}$ using the Lipschitz $n$–connectivity of $H_p(\epsilon_H)$. Otherwise, there is some $x \in S^k$ such that $d(\tau(x), X \setminus Z) > \epsilon_H$. Since $\operatorname{diam} \overline{\tau}_0(D^{k+1}) \leq \epsilon_H$, the image of $\overline{\tau}_0$ is contained in $Z$. Therefore, $Z$ is $\epsilon'$–locally Lipschitz $n$–connected.

If $\Delta \subset \Sigma$ is a simplex, we say that it is *coarse* if all its vertices are of the form $v_i$ for $i \in I$. We say that it is *fine* if it has a vertex of the form $v_j$ for some $j \in J$; all fine simplices have diameter of order at most $\epsilon$ and all coarse ones have diameter of order at least $\epsilon$. By the previous lemma, we can choose $h^{(0)}$ so that for every coarse simplex $\Delta$, there is some $p \in P$ such that $h^{(0)}(\mathcal{V}(\Delta)) \subset H_p(\epsilon_H)$. If $\Sigma_c \subset \Sigma$ is the subcomplex consisting of coarse simplices, we can extend $h^{(0)}$ to a map $h_c \colon \Sigma^{(0)} \cup \Sigma_c^{(n+1)} \to Z$ by induction; if $h_c|_{\partial\Delta}$ is defined, then $h_c(\partial\Delta) \subset H_p(\epsilon_H)$ for some $p \in P$. We extend $h_c$ over $\Delta$ using the Lipschitz $n$–connectivity of $H_p(\epsilon_H)$. The Lipschitz constant of $h_c$ is bounded independently of $\epsilon$.

Again by the previous lemma, we may choose $\epsilon$ sufficiently small that any fine simplex has diameter much less than $\epsilon' / \operatorname{Lip} h_c$. We can then extend $h_c$ over the fine simplices of $\Sigma$ using the local Lipschitz connectivity of $Z$ to get the desired map $h$.

In either case, if $z \in Z$, then $z \in \operatorname{supp} \tau_k$ only if $k \in J$. In particular, $g(z)$ is contained in a fine simplex of diameter of order at most $\epsilon$ and and $d(z, z_i) \lesssim \epsilon$, so

$$d(h(g(z)), z) \leq d(h(g(z)), h(v_i)) + d(z_i, z) \lesssim \epsilon$$

as desired. □

Therefore, $h \circ g$ has small displacement. To complete the proof of Theorem 1.3, we will need a lemma concerning such maps:

**Lemma 2.8** *Suppose that $m \leq n$, that $\alpha$ is a Lipschitz $m$–cycle in $Z$, that $Z$ is $\epsilon_0$–locally Lipschitz $n$–connected, and that $C > 0$. For any $\epsilon > 0$, there is a $\delta > 0$ such that if $f \colon Z \to Z$ is a $C$–Lipschitz map with displacement $\leq \delta$ (ie $d(f(z), z) \leq \delta$ for all $z \in Z$), then*

$$\mathrm{FV}_Z^{m+1}(f_\sharp(\alpha) - \alpha) \leq \epsilon.$$

**Proof** Since $Z$ is locally Lipschitz $n$–connected, if $M$ is a simplicial $(m+1)$–complex, $N$ is a subcomplex, and $f \colon N \to Z$ is a map with sufficiently small Lipschitz constant, then there is an extension $\bar{f}\colon M \to Z$ with $\mathrm{Lip}(\bar{f}) \sim \mathrm{Lip}(f)$. Write $\alpha$ as a sum $\alpha = \sum_{i=1}^{k} \alpha_i$ of Lipschitz maps $\alpha_i\colon \Delta^m \to Z$. Let $L$ be the maximum Lipschitz constant of the $\alpha_i$. In the following calculations, all our implicit constants will depend on $k$, $n$, $Z$, $C$, and $L$. We claim that

$$\mathrm{FV}_Z^{m+1}(f_\sharp(\alpha) - \alpha) \lesssim \delta.$$

First, we can subdivide $\Delta^m$ into roughly $\delta^{-m}$ simplices each with diameter less than or equal to $\delta/L$. We can use this subdivision to replace $\alpha$ with a sum $\alpha' = \sum_{i=1}^{k'} \alpha'_i$ where $k' \lesssim \delta^{-m}$ and each $\alpha'_i\colon \Delta^m \to Z$ has Lipschitz constant at most $\delta$.

There is a simplicial $m$–complex $A$ with at most $k'$ top-dimensional faces, a simplicial cycle $\omega$ on $A$, and a map $g\colon A \to Z$ with $\mathrm{Lip}(g) \le \delta$ such that the restriction of $g$ to each top-dimensional face of $A$ is one of the $\alpha'_i$ and $g_\sharp(\omega) = \alpha'$. Define $r_0\colon A \times \{0,1\} \to Z$ by letting $r_0|_{A\times 0} = g$ and $r_0|_{A \times 1} = f \circ g$. Then $\mathrm{Lip}(r_0) \lesssim \delta$, and if $\delta$ is sufficiently small, we can extend it to a Lipschitz map $r\colon A \times [0,1] \to Z$ with $\mathrm{Lip}\, r \sim \mathrm{Lip}\, r_0$. This is a homotopy from $g$ to $f \circ g$, so the pushforward of $\omega \times [0,1]$ is a filling of $f_\sharp(\alpha) - \alpha$ with mass

$$\mathrm{mass}\, r_\sharp(\omega \times [0,1]) \lesssim k' \delta^{m+1} \lesssim \delta$$

as desired. □

**Proof of Theorem 1.3** Suppose that $\alpha$ is a $(m-1)$–cycle in $Z$ and $\beta$ is a $m$–chain filling it. Let $\Sigma_J$ be the subcomplex of $\Sigma$ spanned by the vertices $v_j$, $j \in J$. Then $g(Z) \subset \Sigma_J$, and $g_\sharp(\alpha)$ is a cycle in $\Sigma_J$ with mass $\le \mathrm{Lip}(g)^{m-1} \,\mathrm{mass}\, \alpha$. Each simplex of $\Sigma_J$ has diameter approximately $\epsilon$, so by Theorem 2.2, there is a $c_3 > 0$ depending only on $X$, a simplicial cycle $P_\alpha := P_{\Sigma_J}(g_\sharp(\alpha))$ approximating $g_\sharp(\alpha)$, and a chain $Q_\alpha := Q_{\Sigma_J}(g_\sharp(\alpha))$ such that $\mathrm{mass}\, Q_\alpha \le c_3 \epsilon\, \mathrm{mass}(\alpha)$ and $\partial Q_\alpha = P_\alpha - g_\sharp(\alpha)$.

Then $g_\sharp(\beta) + Q_\alpha$ is a $m$–chain in $\Sigma$ with boundary $P_\alpha$ and mass

$$\mathrm{mass}(g_\sharp(\beta) + Q_\alpha) \le \mathrm{Lip}(g)^m \,\mathrm{mass}\, \beta + c_3 \epsilon\, \mathrm{mass}(\alpha).$$

Theorem 2.3 lets us approximate this by a chain

$$P_\beta := P_\Sigma(g_\sharp(\beta) + Q_\alpha)$$

with boundary $P_\alpha$.

By Lemma 2.8, if $\epsilon_0 > 0$, then for $\epsilon$ sufficiently small, there is a Lipschitz $(m+1)$–chain $R$ in $Z$ such that

$$\partial R = \alpha - (h \circ g)_\sharp(\alpha)$$

and mass $R \le \epsilon_0$. Let

$$B = R - h_\sharp(Q_\alpha) - h_\sharp(P_\beta).$$

Then $\partial B = \alpha$ and

$$\operatorname{mass} B \lesssim \operatorname{mass} \beta + \epsilon \operatorname{mass}(\alpha) + \epsilon_0,$$

so

$$\mathrm{FV}^k_Z(\alpha) \lesssim \operatorname{mass} \beta$$

as desired. □

The rest of this paper is dedicated to applying this theorem to horospheres and lattices in symmetric spaces and buildings.

# 3 Fillings in $\mathrm{Sol}_{2n-1}$

Theorem 1.3 is useful because it reduces a difficult to prove statement about the undistortedness of an inclusion to an easier to prove Lipschitz extension property. For example, in this section, we will prove:

**Theorem 3.1** *The solvable Lie group* $\mathrm{Sol}_{2n-1} = \mathbb{R}^{n-1} \ltimes \mathbb{R}^n$ *is Lipschitz* $(n-2)$*–connected.*

Theorem 1.5 follows as a direct application of Theorem 1.3.

We start by defining $\mathrm{Sol}_{2n-1}$, $n \ge 2$. This group is a solvable Lie group which can be written as a semidirect product of $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1}$ acts on $\mathbb{R}^n$ as the group of $n \times n$ diagonal matrices with positive coefficients and determinant 1. When $n = 2$, this is the three-dimensional solvable group corresponding to solvegeometry.

All the constants and implicit constants in this section will depend on $n$.

One feature of this group is that it can be realized as a horosphere in a product of hyperbolic planes. Let $\mathbb{H}_2$ be the hyperbolic plane and let $\beta\colon \mathbb{H}_2 \to \mathbb{R}$ be a Busemann function for $\mathbb{H}_2$. We can define Busemann functions $\beta_1, \dots, \beta_n$ in the product $\mathbb{H}_2^n$ by letting $\beta_i(x_1, \dots, x_n) = \beta(x_i)$. Then $b = n^{-1/2} \sum \beta_i$ is a Busemann function for $\mathbb{H}_2^n$, and $\mathrm{Sol}_{2n-1}$ acts freely, isometrically, and transitively on the resulting horosphere $b^{-1}(0)$. The metric induced on $\mathrm{Sol}_{2n-1}$ by inclusion in $\mathbb{H}_2^n$ is bilipschitz equivalent to a left-invariant Riemannian metric on $\mathrm{Sol}_{2n-1}$.

This group also appears as a subgroup of a Hilbert modular group. If $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^n$ is a Hilbert modular group and $X = (\mathbb{H}_2)^n$, then there is a collection $\mathcal{H}$ of disjoint open horoballs in $X$ such that the boundary of each horosphere is bilipschitz equivalent to $\mathrm{Sol}_{2n-1}$ and $\Gamma$ acts cocompactly on $X \smallsetminus \mathcal{H}$ [20]. Consequently, Theorem 1.6 is also a corollary of Theorem 3.1.

To prove Theorem 3.1, we will use the following condition, which is equivalent to Lipschitz connectivity (cf Gromov [15]):

**Lemma 3.2** *Let $Z$ be a metric space, let $\Delta_Z$ be the infinite-dimensional simplex with vertex set $Z$, and let $\Delta_Z^{(k)}$ be its $k$–skeleton. Let $\langle z_0, \dots, z_k\rangle$ denote the $k$–simplex with vertices $z_0, \dots, z_k$. Then $Z$ is Lipschitz $n$–connected if and only if there exists a map $\Omega\colon \Delta_Z^{(n+1)} \to Z$ such that:*

(1) *For all $z \in Z$, we have $\Omega(\langle z\rangle) = z$.*

(2) *There is a $c$ such that for any $d \le n+1$ and any simplex $\delta = \langle z_0, \dots, z_d\rangle$, we have*
$$\operatorname{Lip} \Omega|_\delta \le c \operatorname{diam}\{z_0, \dots, z_d\}.$$

**Proof** One direction is clear; if $Z$ is Lipschitz $n$–connected, then one can construct $\Omega$ by letting $\Omega(\langle z\rangle) = z$ for all $z \in Z$, then using the Lipschitz connectivity of $Z$ to extend $\Omega$ over each skeleton inductively.

The other direction is an application of Whitney decomposition. We view $D^{d+1}$ as a subset of $\mathbb{R}^{d+1}$; by the Whitney covering lemma, the interior of $D^{d+1}$ can be decomposed into a union of countably many dyadic cubes such that for each cube $C$, one has $\operatorname{diam} C \sim_d d(C, S^d)$. We can decompose each cube into boundedly many simplices to get a triangulation $\tau$ of the interior of $D^{d+1}$ where each simplex is bilipschitz equivalent to a scaling of the standard simplex.

We construct a map $h\colon D^{d+1} \to Z$ using this triangulation. For each vertex $v$ in $\tau$, let $h(v)$ be a point in $S^d$ such that $d(v, h(v))$ is minimized. One can check that $h$ is a Lipschitz map from $\tau^{(0)} \to S^d$, so $g_0 = \alpha \circ h\colon \tau^{(0)} \to Z$ is a Lipschitz map with $\operatorname{Lip}(g_0) \sim_{d,\Omega} \operatorname{Lip}(\alpha)$. We can extend $g_0$ to a map $g\colon \tau \to Z$ by sending the simplex $\langle v_0, \dots, v_k\rangle$ to the simplex $\Omega(\langle g_0(v_0), \dots, g_0(v_k)\rangle)$, and this is also Lipschitz with $\operatorname{Lip}(g) \sim_{d,\Omega} \operatorname{Lip}(\alpha)$.

Finally, we extend $g$ to a map $\beta\colon D^{d+1} \to Z$ by defining $g(v) = \alpha(v)$ when $v \in S^d$. Since the diameter of the simplices of $\tau$ goes to zero as one approaches the boundary, this extension is continuous and therefore Lipschitz, as desired. □

It therefore suffices to prove the following:

**Lemma 3.3** *Let $\Delta = \Delta_{\mathrm{Sol}_{2n-1}}$ be the infinite-dimensional simplex with vertex set $\mathrm{Sol}_{2n-1}$. There is a map $\Omega\colon \Delta^{(n-1)} \to \mathrm{Sol}_{2n-1}$ which satisfies the properties in Lemma 3.2. Therefore, $\mathrm{Sol}_{2n-1}$ is Lipschitz $(n-2)$–connected.*

Our construction is based on techniques from [5]; we will construct $\Omega$ using nonpositively curved subsets of $\mathrm{Sol}_{2n-1}$ called $k$–slices.

Recall that we defined $\mathrm{Sol}_{2n-1}$ as a horosphere in $(\mathbb{H}_2)^n$. Let $\beta\colon \mathbb{H}_2 \to \mathbb{R}$ be the Busemann function used to define $\mathrm{Sol}_{2n-1}$ and let $*$ be the corresponding point at infinity. If $\gamma$ is a geodesic in $\mathbb{H}_2$ which has one endpoint at $*$, we call $\gamma$ a *vertical geodesic*. For $i = 1, \dots, n$, let $s_i \subset \mathbb{H}_2$ be either a vertical geodesic or all of $\mathbb{H}_2$. If $k$ of the $s_i$ are equal to $\mathbb{H}_2$, we call the intersection $s_1 \times \dots \times s_n \cap \mathrm{Sol}_{2n-1}$ a *$k$–slice*.

Suppose that $k < n$ and that $S$ is a $k$–slice; without loss of generality, we may assume that

$$S = \mathbb{H}_2 \times \dots \times \mathbb{H}_2 \times \gamma_1 \times \dots \times \gamma_{n-k} \cap \mathrm{Sol}_{2n-1}.$$

Then the projection to the first $n-1$ factors (ie all but the last factor) is a homeomorphism from $S$ to $(\mathbb{H}_2)^k \times \mathbb{R}^{n-k-1}$. In fact, this map is bilipschitz, so $S$ is bilipschitz equivalent to a Hadamard manifold.

If $k < n$, then any $k$–slice is Lipschitz $d$–connected for any $d$:

**Lemma 3.4** *If $X$ is a Hadamard manifold, it is Lipschitz $n$–connected for any $n$.*

**Proof** Let $\alpha\colon S^n \to X$, and let $v \in S^n$. Let $(x, r) \in S^n \times [0, 1]$ be polar coordinates on $D^{n+1}$. We can construct a map $\bar{\alpha}\colon D^{n+1} \to X$ by letting $\bar{\alpha}(x, r)$ be the geodesic from $\alpha(v)$ to $\alpha(x)$. Because the distance function on $X$ is convex, this is a Lipschitz map, and $\mathrm{Lip}(\bar{\alpha}) \lesssim \mathrm{Lip}(\alpha)$. □

If $\tau$ is a polyhedral complex and $f\colon \tau \to \mathrm{Sol}_{2n-1}$, we say that $f$ is a *slice map* if the image of every cell $\delta$ of $\tau$ is contained in a $(\dim \delta)$–slice.

Our main tool in the proof of Lemma 3.3 is the following:

**Lemma 3.5** *Let $k < n$. Suppose that $\sigma$ is a polyhedral complex which is bilipschitz equivalent to $S^{k-1}$. Then there is a $c > 0$ and a polyhedral complex $\tau$ bilipschitz equivalent to $D^k$ which has boundary $\sigma$. Furthermore, if $f\colon \sigma \to \mathrm{Sol}_{2n-1}$ is a Lipschitz slice map, there is an extension $g\colon \tau \to \mathrm{Sol}_{2n-1}$ which is a slice map with $\mathrm{Lip}(g) \le c\,\mathrm{Lip}(f)$.*

The basic idea of the lemma is to first construct a family of projections along horospheres whose images lie in $(n-1)$–slices, then construct homotopies between $f$ and its projections. Gluing these homotopies together will give a map $\sigma \times [0,n] \to \mathrm{Sol}_{2n-1}$, and adding a final contraction will extend the map to all of $\tau$.

Let $\beta\colon \mathbb{H}_2 \to \mathbb{R}$ be the Busemann function used to define $\mathrm{Sol}_{2n-1}$. If $\gamma$ is a vertical geodesic in $\mathbb{H}_2$ and $x \in \mathbb{H}_2$, let $p(x)$ be the unique point on $\gamma$ such that $\beta(x) = \beta(p(x))$. This defines a map $p_\gamma\colon \mathbb{H}_2 \to \gamma$. It is straightforward to check that $p$ is distance-decreasing and that $d(x, p(x)) \le 2d(x,\gamma)$.

Suppose that $\boldsymbol{x} = (x_1,\dots,x_n) \in (\mathbb{H}_2)^n$. For $i = 1,\dots,n$, let $\gamma_i$ be a vertical geodesic containing $x_i$, and let $\beta\colon \mathbb{H}_2 \to \mathbb{R}$ be the Busemann function used to define $\mathrm{Sol}_{2n-1}$. For each $i$, let $p_i\colon \mathrm{Sol}^n_{2n-1} \to \mathrm{Sol}_{2n-1}$ be the map

$$p_i(y_1,\dots,y_n) = (y_1,\dots,y_{i-1}, p_{\gamma_i}(y_i), y_{i+1},\dots,y_n).$$

Let $S$ be the 0–slice

$$S = \gamma_1 \times \dots \times \gamma_n \cap \mathrm{Sol}_{2n-1}$$

and let $S_i$ be the $(n-1)$–slice

$$S_i = \mathbb{H}_2 \times \dots \times \gamma_i \times \dots \times \mathbb{H}_2 \cap \mathrm{Sol}_{2n-1},$$

where $\gamma_i$ occurs in the $i^{\text{th}}$ factor. It is easy to check the following properties:

- $p_i$ is distance-decreasing
- $d(y, p_i(y)) \le 2d(y,\boldsymbol{x})$ for all $y \in \mathrm{Sol}_{2n-1}$
- $p_i$ preserves $S$ pointwise
- If $S'$ is a $d$–slice, then $p_i(S')$ lies in a $d$–slice and $S'$ and $p_i(S')$ both lie in the same $(d+1)$–slice. In particular, $y$ and $p_i(y)$ lie in a 1–slice for every $y \in \mathrm{Sol}_{2n-1}$.

**Lemma 3.6** *For any $i$, if $\sigma$ is a polyhedral complex with $\dim\sigma < n$, $f\colon \sigma \to \mathrm{Sol}_{2n-1}$ is a Lipschitz slice map, and $s \in \sigma$ satisfies $f(s) = \boldsymbol{x}$, then there is a homotopy $g\colon \sigma \times [0,1] \to \mathrm{Sol}_{2n-1}$ from $f$ to $p_i \circ f$ which is a Lipschitz slice map with $\mathrm{Lip}(g) \lesssim \mathrm{Lip}(f)$.*

**Proof** We construct $g$ one skeleton at a time. For any cell $\delta$, the image $g(\delta \times [0,1])$ will be contained in the minimal slice that contains $f(\delta)$ and $p_i(f(\delta))$. Since $f(\delta)$ and $p_i(f(\delta))$ lie in a common $(\dim\delta + 1)$–slice, this ensures that $g$ is a slice map.

The map $g$ is already defined on the vertices of $\sigma \times [0,1]$ and we claim that it is Lipschitz on the 0–skeleton. If $l = \mathrm{Lip}(f)$, then $f$ and $p_i \circ f$ are $l$–Lipschitz, and if $v$ is a vertex of $\sigma$, then

$$d(f(v), p_i(f(v))) \le 2d(f(v),\boldsymbol{x}) \le 2l \operatorname{diam}\sigma,$$

so $g$ is Lipschitz on the vertex set and $\mathrm{Lip}(g) \lesssim l$.

Now suppose that we have defined $g$ on the $(d-1)$–cells of $\sigma \times [0,1]$ and that for any $(d-2)$–cell $\delta$, the image $g(\delta \times [0,1])$ is contained in the minimal slice that contains $f(\delta)$ and $p_i(f(\delta))$. Consider a cell of the form $\delta \times [0,1]$ for some $(d-1)$–cell $\delta$ in $\sigma$. Since $f$ is a slice map, $f(\delta)$ lies in some $(d-1)$–slice, so $f(\delta) \cup p_i(f(\delta))$ lies in some $d$–slice, and this $d$–slice also contains $g(\partial\delta \times [0,1])$ by the inductive hypothesis. Let $S'$ be the minimal slice that contains

$$g(\partial(\delta \times [0,1])) = f(\delta) \cup p_i(f(\delta)) \cup g(\partial\delta \times [0,1]).$$

By Lemma 3.4, we can extend $g$ over $\delta \times [0,1]$ so that it sends $\delta \times [0,1]$ to $S'$. The extension is Lipschitz and $\mathrm{Lip}\, g \lesssim \mathrm{Lip}\, f$. □

Now we can prove Lemmas 3.5 and 3.3

**Proof of Lemma 3.5** Let $\tau$ be the complex $\sigma \times [0,n] \cup C\sigma/\sim$, where $[0,n]$ is subdivided into $n$ unit-length edges, $C\sigma$ is the cone over $\sigma$ and $\sim$ is the relation gluing the base of $C\sigma$ to $\sigma \times \{n\}$. This is bilipschitz equivalent to $D^k$.

Choose a basepoint $v_* \in \sigma$ and suppose that $f(v_*) = \boldsymbol{x} = (x_1, \dots, x_n) \in (\mathbb{H}_2)^n$. For $i = 0, \dots, n$, let $f_i = p_i \times \cdots \times p_1 \circ f$. By Lemma 3.6, for $i = 1, \dots, n$, there is a homotopy $g_i \colon \sigma \times [i-1, i] \to \mathrm{Sol}_{2n-1}$ from $f_{i-1}$ to $f_i$ which is a Lipschitz slice map with $\mathrm{Lip}(g_i) \lesssim \mathrm{Lip}(f)$. Concatenating the $g_i$ gives a map $\sigma \times [0,n] \to \mathrm{Sol}_{2n-1}$ which is a Lipschitz homotopy from $f$ to $f_n$. To define $g$, it suffices to extend this map over $C\sigma$, but since the image of $f_n$ lies in $S$, we can use Lemma 3.4 to construct such an extension. Since this extension lies in a 0–slice, it is a slice map, so $g$ is a slice map and $\mathrm{Lip}(g) \lesssim \mathrm{Lip}(f)$. □

**Proof of Lemma 3.3** Let $\Delta_d$ be the standard $d$–simplex. We define a sequence $\tau_i, i = 0, \dots, n-1$ of polyhedral complexes homeomorphic to $\Delta_i$ and a sequence $\sigma_i, i = 0, \dots, n-1$ of polyhedral complexes homeomorphic to $\partial\Delta_{i+1}$ inductively. Let $\tau_0$ be a single point. For each $i \geq 0$, let $\sigma_i$ be the complex obtained by replacing each $i$–face of $\partial\Delta_{i+1}$ by a copy of $\tau_i$. Let $\tau_{i+1}$ be the complex obtained by applying Lemma 3.5 to $\sigma_i$. This is PL-homeomorphic to $\Delta_i$ and has boundary $\sigma_i$.

Let $\Delta'$ be the complex obtained by subdividing each $d$–simplex of $\Delta^{(n-1)}$ into a copy of $\tau_d$ and let $i \colon \Delta^{(n-1)} \to \Delta'$ be a bilipschitz equivalence taking each simplex to the corresponding copy of $\tau_d$. We can construct a slice map $\Omega' \colon \Delta' \to Z$ by defining $\Omega'(\langle x \rangle) = x$ for all $x \in \mathrm{Sol}_{2n-1}$ and using Lemma 3.5 inductively to extend $\Omega'$ over each of the $\tau_d$.

That is, if $\delta = \langle x_0, \dots, x_{d+1} \rangle \subset \Delta$ is a $(d+1)$–cell, $\Omega'$ is defined on $i(\partial\delta)$, and

$$\Omega|_{i(\partial\delta)}: \sigma_d \to \mathrm{Sol}_{2n-1}$$

is a slice map, and $\mathrm{Lip}(\Omega|_{i(\partial\delta)}) \lesssim \operatorname{diam}\{x_0, \dots, x_{d+1}\}$, we may extend it to a slice map on $i(\delta)$ using Lemma 3.5. The resulting map $\Omega|_\delta$ has

$$\mathrm{Lip}(\Omega|_\delta) \lesssim \operatorname{diam}\{x_0, \dots, x_{d+1}\}$$

as desired. □

Thus, by Lemma 3.2, $\mathrm{Sol}_{2n-1}$ is Lipschitz $(n-2)$–connected, and by Theorem 1.3, it is undistorted up to dimension $n$ inside $\mathbb{H}_2^n$. Consequently, if $k < n$ and if $\alpha$ is a Lipschitz $k$–cycle in $\mathrm{Sol}_{2n-1}$, then

$$\mathrm{FV}^{k+1}_{\mathrm{Sol}_{2n-1}}(\alpha) \lesssim \mathrm{FV}^{k+1}_{\mathbb{H}_2^n}(\alpha) \lesssim (\operatorname{mass}\alpha)^{(k+1)/k},$$

as desired.

## 4 Fillings in horospheres of euclidean buildings

In this section, we prove Theorem 1.7. A horosphere in a building of rank at least $n$ contains a flat of dimension $n-1$, so its $k^{\text{th}}$–order Dehn function grows at least as fast as $V^{(k+1)/k}$ when $k \le n-2$. It remains to show that it grows at most as fast as $V^{(k+1)/k}$.

If $Y$ is a spherical building modeled on a Coxeter complex $S$ with Weyl group $W$, let $\Delta_{\mathrm{mod}}(Y) = S/W$ be its *model chamber*. There is a canonical map $p: Y \to \Delta_{\mathrm{mod}}(Y)$ such that each chamber of $Y$ is sent to $M(Y)$ by an isometry, and we claim:

**Theorem 4.1** *Let $X$ be a thick euclidean building and let $X_\infty$ be the Bruhat–Tits building of $X$. If $X$ is reducible, then $X_\infty$ is a join of buildings; let $\tau$ be a point in $X_\infty$ which has rational slope and is not contained in a join factor of rank less than $n$. Let $Z$ be a horosphere in $X$ centered at $\tau$ and let $p: X_\infty \to \Delta_{\mathrm{mod}}(X_\infty)$ be the projection of $X_\infty$ to its model chamber. Then $Z$ is Lipschitz $(n-2)$–connected with implicit constant depending only on $X$ and $p(\tau)$.*

By Theorem 1.3 and Corollary 1.4, this implies Theorem 1.7.

Furthermore, if $K$ is a global function field, $\boldsymbol{G}$ is a noncommutative, absolutely almost simple $K$–group of $K$–rank 1, and $S$ is a finite set of pairwise inequivalent valuations on $K$, then $\Gamma = \boldsymbol{G}(\mathcal{O}_S)$ is an $S$–arithmetic group. If $X$ is the associated euclidean building and $n$ is its rank, then by [7, Theorem 3.7], there is a collection $\mathcal{H}$ of pairwise disjoint open horoballs in $X$ such that $X \smallsetminus \mathcal{H}$ is $\boldsymbol{G}(\mathcal{O}_S)$–invariant and cocompact. By Theorem 4.1, the boundary of each of these horoballs is Lipschitz $(n-2)$–connected with a uniform implicit constant, so Theorem 1.3 implies Theorem 1.8.

As in [9, Remark 4.2], it suffices to consider the case that $X$ is a thick euclidean building of rank $n$ and that $\tau$ is not parallel to any factor of $X$. If $X = X_1 \times X_2$, then $X_\infty = (X_1)_\infty * (X_2)_\infty$. If $\tau \in (X_1)_\infty$, then $Z = Z_1 \times X_2$, where $Z_1 \subset X_1$ is a horosphere of $X_1$ centered at $\tau$. If $\alpha\colon S^k \to Z$ is $c$–Lipschitz, we can replace it with its projection to $Z_1$ by using an homotopy with Lipschitz constant at most $(\operatorname{diam} S^k + 1)c$, so if $Z_1$ is Lipschitz $(n-2)$–connected, so is $Z$.

Therefore, in this section, we will let $X$ be a thick euclidean building of rank $n$ equipped with its complete apartment system, and let $X_\infty$ be its Bruhat–Tits building. We fix a direction at infinity $\tau \in X_\infty$ which is not contained in any factor of $X_\infty$, and let $h$ be a Busemann function centered at $\tau$, with corresponding horosphere $Z = h^{-1}(0)$. We orient $h$ so that $h(x)$ increases as $x$ approaches $\tau$; we use this orientation so that we can treat $h$ as a Morse function on $X$ more easily.

All the constants in this section and its subsections will depend on $X$ and $Z$.

The proof that $Z$ is Lipschitz $(n-2)$–connected is based on Lemma 3.2. Let $\Delta_Z$ be the infinite-dimensional simplex with vertex set $Z$, and let $\Delta_Z^{(k)}$ be its $k$–skeleton. We will show:

**Lemma 4.2** *There exists a map $\Omega\colon \Delta_Z^{(n-1)} \to Z$ such that:*

(1) *For all $z \in Z$, we have $\Omega(\langle z \rangle) = z$.*

(2) *For any $d \le n+1$ and any simplex $\delta = \langle z_0, \dots, z_d \rangle$, we have*

$$\operatorname{Lip} \Omega|_\delta \lesssim \operatorname{diam}\{z_0, \dots, z_d\} + 1.$$

The only difference between the map in Lemma 4.2 and the map in Lemma 3.2 is the bound on $\operatorname{Lip} \Omega|_\delta$. In Lemma 3.2, $\operatorname{Lip} \Omega|_\delta$ is bounded by a multiple of $\operatorname{diam}\{z_0, \dots, z_d\}$; in Lemma 4.2, it is bounded by a multiple of $\operatorname{diam}\{z_0, \dots, z_d\}$ and an additive constant.

As a corollary, we have:

**Lemma 4.3** *For any $t > 0$, there is a Lipschitz map $r_t\colon h^{-1}((\infty, t]) \cap X^{(n-1)} \to Z$ which restricts to the identity map on $Z$.*

**Proof** Define $r_t$ on $h^{-1}((\infty, 0])$ as the closest-point projection. Since horoballs are convex, this is a distance-decreasing map.

To define $r_t$ on $h^{-1}((0,t]) \cap X^{(n-1)}$, we view $X$ as a polyhedral complex, ie a complex whose faces consist of convex polyhedra in $\mathbb{R}^n$, glued along faces by isometries. Then $h$ is linear on each face of $X$, so if $P$ is a face of $X$, then the intersections $h^{-1}([0,t]) \cap P$, $Z \cap P$, and $h^{-1}(t) \cap P$ are convex polyhedra. Since $\tau$ has rational slope, the set $h(X^{(0)})$ of possible values of $h$ on the vertices of $X$ is discrete, so only finitely many isometry classes of polyhedra occur this way, and we can give $Z_t = h^{-1}([0,t]) \cap X^{(n-1)}$ the structure of a polyhedral complex with only finitely many isometry classes of cells. We subdivide each cell to make $Z_t$ into a simplicial complex. We define $r_t$ on the vertices of $Z_t$ so that $d(r_t(v), v)$ is minimized. If $\Delta$ is a simplex of $Z_t$ with vertices $v_0, \dots, v_k$, we define

$$r_t|_\Delta = \Omega|_{\langle r_t(v_0), \dots, r_t(v_k)\rangle}.$$

This gives a Lipschitz map with Lipschitz constant depending on the size of the smallest simplex in $Z_t$. □

The proof of this lemma is the only place that we use the assumption that $\tau$ has rational slope.

Given these lemmas, we prove Theorem 4.1 as follows.

**Proof of Theorem 4.1** Suppose that $\alpha\colon S^k \to Z$ is a Lipschitz map. If $\mathrm{Lip}(\alpha) \le 1$, we can extend $\alpha$ to a map $\beta\colon D^{k+1} \to X$ by coning $\alpha$ to a point along geodesics in $X$. Since $X$ is CAT(0), $\beta$ is Lipschitz and $\mathrm{Lip}(\beta) \sim \mathrm{Lip}(\alpha)$. Furthermore, the image of $\beta$ lies in $h^{-1}([-1,1])$, so $r_1 \circ \beta\colon D^{k+1} \to Z$ is an extension of $\alpha$ with $\mathrm{Lip}(r_1 \circ \beta) \sim \mathrm{Lip}(\alpha)$.

If $\mathrm{Lip}(\alpha) > 1$, let $L \in \mathbb{N}$ be the smallest integer such that $L \ge \mathrm{Lip}(\alpha)$, let $D^{k+1}(L)$ be the cube $[0,L]^{k+1} \subset \mathbb{R}^{k+1}$, and let $S^k(L) = \partial D^k(L)$. We view $\alpha$ as a map $S^k(L) \to Z$ such that $\mathrm{Lip}(\alpha) \sim 1$ and try to construct an extension to $D^{k+1}(L)$ with comparable Lipschitz constant.

As in the proof of Lemma 3.2, the Whitney covering lemma implies that $D^{k+1}(L)$ can be decomposed into a union of countably many dyadic cubes such that for each cube $C$, one has $\operatorname{diam} C \sim d(C, S^k(L))$. Since these cubes are dyadic, each cube of side length less than one is contained in a cube of side length 1. Let $\mathcal{C}$ be the cover of $D^{k+1}(L)$ obtained by combining cubes of side length less than 1 into cubes of side length 1. Then for each cube $C$ in $\mathcal{C}$, we have $\operatorname{diam} C \sim d(C, S^k(L)) + 1$, and each cube which touches $S^k(L)$ has side length 1. We call the cubes that touch $S^k(L)$ the *boundary cubes* and we call the rest *interior cubes*. We can decompose each cube into boundedly many simplices to get a triangulation $\tau$ of $D^{d+1}$ where each simplex is bilipschitz equivalent to a scaling of the standard simplex. Let $\tau_i$ be the subcomplex of $\tau$ contained in the interior cubes.

We construct a map $h\colon S^k(L) \cup \tau_i \to Z$ using this triangulation. If $x \in S^k(L)$, we define $h(x) = f(x)$. For each vertex $v$ in $\tau_i$, let $h(v)$ be a point in $S^d$ such that $d(v, h(v))$ is minimized, and if $\Delta = \langle v_0, \dots, v_k \rangle$ is a simplex of $\tau_i$, define

$$h|_\Delta = \Omega|_{\langle h(v_0),\dots,h(v_k)\rangle}.$$

Since $\operatorname{diam} \Delta \gtrsim 1$, this is Lipschitz with $\mathrm{Lip}(h) \sim 1$.

Since $X$ is CAT(0) and thus Lipschitz $n$–connected, we can extend $h$ over the boundary cubes inductively; if $C$ is a face of a boundary cube and $h$ is already defined on $\partial C$, we extend $h$ over $C$ by coning $h|_{\partial C}$ to a point along geodesics. This produces an extension $h\colon D^{k+1}(L) \to X$ with Lipschitz constant $\mathrm{Lip}(h) \sim 1$.

Finally, since the boundary cubes are all contained in a neighborhood of $S^k(L)$, their image is contained in a neighborhood of $Z$, so if $c$ is large enough, then

$$r_c \circ h\colon D^{k+1}(L) \to Z$$

is an extension of $\alpha$ and $\mathrm{Lip}(r_c \circ h) \sim 1$. □

In the rest of this section, we will prove Lemma 4.2. The proof is a quantitative Morse theory argument, like the "pushing" arguments by Abrams, Brady, Dani, Duchin and the author in [3]. Bux and Wortman [7] used a Morse theory argument to prove that $Z$ is $n$–connected; we sketch their proof in the case that $\tau$ is a generic direction. In general, $X$ is contractible, and $Z$ is the level set of $h$. If $\tau$ is generic, then $h$ is nonconstant on every edge of $X$, and we can treat it as a combinatorial Morse function.

That is, if $u$ is a vertex of $X$, then every vertex of its link $\mathrm{Lk}(u)$ corresponds to a vertex $v$ adjacent to $u$. We define the *downward link* $\mathrm{Lk}^{\downarrow} u \subset \mathrm{Lk}\, u$ to be the subcomplex spanned by vertices $v$ with $h(v) < h(u)$. By results of Schulz [21], $\mathrm{Lk}^{\downarrow} u$ is $(n-2)$–connected for all $u$, so combinatorial Morse theory implies that $Z$ is also $(n-2)$–connected. Bux and Wortman apply a similar argument in the general case, but with $h$ replaced by a more complicated function to deal with faces of dimension greater than $0$ that are orthogonal to $\tau$.

Arguments like this, however, give poor quantitative bounds. Given an $(n-2)$–sphere in $Z$, one can construct a filling in the horosphere $h^{-1}([0,\infty))$ and use Morse theory to homotope it to $Z$, but the filling may grow exponentially large in the process. The pushing methods in [3] avoid this sort of exponential growth by constructing maps from $\mathrm{Lk}^{\downarrow} u$ to $Z$, and we will apply similar methods here.

Let $\mathfrak{a}$ be a chamber of $X_\infty$ which contains $\tau$ in its closure and let

$$X^0_\infty(\mathfrak{a}) := \{\mathfrak{b} \mid \mathfrak{b} \text{ is a chamber of } X_\infty \text{ opposite to } \mathfrak{a}\}.$$

Abramenko [1] showed that if $Y$ is a sufficiently thick classical spherical building, then $Y^0(\mathfrak{a})$ is $(\operatorname{rank} Y - 2)$–connected for any chamber $\mathfrak{a}$ of $Y$. We will show that if $X$ is a thick euclidean building of rank $n$, then $X^0_\infty(\mathfrak{a})$ is $(n-2)$–connected.

Roughly, we show $(n-2)$–connectivity by showing that "most" pairs of chambers $\mathfrak{b}, \mathfrak{c} \subset X^0_\infty(\mathfrak{a})$ are opposite to one another and that if $E_{\mathfrak{b},\mathfrak{c}}$ is the apartment they span, then $\partial_\infty E_{\mathfrak{b},\mathfrak{c}} \subset X^0_\infty(\mathfrak{a})$. Then, for each sphere $\alpha\colon S^k \to X^0_\infty(\mathfrak{a})$ with $k < n-2$, we choose a $\mathfrak{c}$ such that for any $\mathfrak{b}$ in the support of $\alpha(S^{n-2})$, $\mathfrak{b}$ is opposite to $\mathfrak{c}$ and $\partial E_{\mathfrak{b},\mathfrak{c}} \subset X^0_\infty(\mathfrak{a})$. We can then contract $\alpha$ to a point in $\mathfrak{c}$ along geodesics. Since $X^0_\infty(\mathfrak{a})$ is $(n-2)$–connected, there is no obstruction to constructing a map

$$\Omega_\infty\colon \Delta_Z^{(n-1)} \to X^0_\infty(\mathfrak{a}).$$

Next, we construct a map to $Z$. Given a point $x \in X$ and a direction $\sigma \in X_\infty$, we let $r$ be the ray emanating from $x$ in the direction of $\sigma$. If $h(x) > 0$ and $\sigma \in X^0_\infty(\mathfrak{a})$, this ray will eventually intersect $Z$. This provides a map $X^0_\infty(\mathfrak{a}) \to Z$, but this map is not Lipschitz: a ray may travel a long distance before intersecting $Z$. To fix this, we define the downward link at infinity $\operatorname{Lk}^\downarrow_\infty(x)$ at $x$. This is a subset $\operatorname{Lk}^\downarrow_\infty(x) \subset X_\infty$ of directions that point "downward" from $x$ (ie away from $\mathfrak{a}$). Rays in these directions intersect $Z$ after traveling distance roughly $h(x)$, so we can define a map $i_x\colon \operatorname{Lk}^\downarrow_\infty(x) \to Z$ with $\operatorname{Lip}(i_x) \lesssim h(x)$ which sends each direction to the corresponding intersection point.

The sets $\operatorname{Lk}^\downarrow_\infty(x)$ get bigger as $x \to \mathfrak{a}$, and any finite subset of $X^0_\infty(\mathfrak{a})$ is contained in some $\operatorname{Lk}^\downarrow_\infty(x)$. This lets us convert $\Omega_\infty$ into a map to $Z$; for each simplex $\Delta$, we choose some $x_\Delta$ so that $\Omega_\infty(\Delta) \subset \operatorname{Lk}^\downarrow_\infty(x_\Delta)$ and define (after some patching around the edges)

$$\Omega|_\Delta = i_{x_\Delta} \circ \Omega_\infty.$$

Finally, we show that restrictions of $\Omega$ to simplices satisfy Lipschitz bounds. To do this, we need some control over the Lipschitz constants of the $i_{x_\Delta}$. We know that $\operatorname{Lip}(i_{x_\Delta}) \lesssim h(x_\Delta)$, so we try to bound the $h(x_\Delta)$ by controlling which chambers of $X^0_\infty(\mathfrak{a})$ we use in fillings of spheres. This proves the theorem.

The rest of this section is devoted to filling in the details of this sketch. First, in Sections 4.1 and 4.2, we describe our notation and define some maps and subsets that we will use in the rest of the proof. In Section 4.3, we construct $\operatorname{Lk}^\downarrow_\infty(x)$ and show that there are many apartments in $\operatorname{Lk}^\downarrow_\infty(x)$. In Section 4.4, we use this fact to show that $X^0_\infty(\mathfrak{a})$ is $(n-2)$–connected and to construct $\Omega_\infty$ and the $x_\Delta$. In Section 4.5, we use these to construct $\Omega$.

## 4.1 Preliminaries

In this section, we fix some notation for dealing with buildings, define some maps and subsets that will be important in the rest of the section, and prove some of their properties. Our primary reference is Abramenko and Brown [2].

As stated in the introduction to this section, we let $X$ be an irreducible thick euclidean building of rank $n$, equipped with its complete apartment system and let $X_\infty$ be its Bruhat–Tits building. If $E$ is an apartment of $X$, we can identify it with the Coxeter complex of a Euclidean reflection group $W$, and we can identify the corresponding apartment $\partial_\infty E \subset X_\infty$ with the Coxeter complex of $\overline{W}$, the reflection group corresponding to the linear parts of the elements of $W$.

Recall that $X_\infty$ can be defined as the set of classes of parallel unit-speed geodesic rays in $X$, where $r, r' \colon [0,\infty) \to X$ are parallel if $d(r(t), r'(t))$ is bounded as $t \to \infty$. For any $x \in X$ and any $\sigma \in X_\infty$, there is a unique ray based at $x$ and parallel to $\sigma$ [2, Lemma 11.72]. Given a subset $Y \subset X$, we define $\partial_\infty Y$ to be its boundary at infinity; for the subsets we will consider in this paper, $\partial_\infty Y$ consists of the set of parallelism classes of geodesic rays in $Y$. If $\mathfrak{d}$ is a chamber of $\partial_\infty E$, we say that $E$ is *asymptotic to* $\mathfrak{d}$.

If $x \in E$, there is a conical cell $x + \mathfrak{d}$ based at $x$ for every chamber $\mathfrak{d}$ of $\partial_\infty E$; we call these cells *sectors*. Note that $x + \mathfrak{d}$ doesn't depend on our choice of $E$; this construction gives the same result for any apartment $E'$ such that $\mathfrak{d} \subset \partial_\infty E'$ and $x \in E'$.

The codimension-1 cells of $E$ are called *panels*. Each panel is contained in a codimension-1 subspace of $E$ which we call a *wall*. Each wall divides $E$ into a pair of closed *half-apartments*. We say that $E'$ is a *ramification* of $E$ if either $E = E'$ or $E \cap E'$ is a half-apartment. Since $X$ is thick, each wall is the boundary of at least three half-apartments. We say that two chambers are *adjacent* if they have disjoint interiors and share a panel. A sequence of chambers $C_1, \dots, C_k$ such that $C_i$ and $C_{i+1}$ are adjacent is called a *gallery* of combinatorial length $k$. The minimal combinatorial length of a gallery connecting two chambers is called the *combinatorial distance* between them, and a gallery realizing this length is called a *minimal gallery*. We denote the combinatorial distance between $C$ and $C'$ by $d_{\mathrm{comb}}(C, C')$.

There is also a CAT(0) metric on $X$ which gives each apartment the metric of $\mathbb{R}^n$. We denote this metric by $d \colon X \times X \to \mathbb{R}$. Likewise, there is a CAT(1) metric (the angular metric) on $X_\infty$, which we also denote by $d$.

If $\mathfrak{c}, \mathfrak{d} \subset X_\infty$ are chambers and $d_{\mathrm{comb}}(\mathfrak{c}, \mathfrak{d}) = \mathrm{diam}_{\mathrm{comb}}(X_\infty)$, we say that $\mathfrak{c}$ and $\mathfrak{d}$ are opposite. Any pair of opposite chambers of $X_\infty$ determines a unique apartment of $X_\infty$ [2, Theorem 4.70]. Indeed, if $\mathfrak{c}, \mathfrak{d} \subset X_\infty$ are opposite chambers, then there is a unique apartment of $X$ which is asymptotic to $\mathfrak{c}$ and $\mathfrak{d}$ [2, Theorem 11.63].

## 4.2 Folded apartments

In order to prove Theorem 4.1, we will need to understand how apartments in $X$ are positioned relative to $\mathfrak{a}$. In this section, we describe some notions that will be useful to understand the arrangement of apartments in $X$.

Recall that if $E$ is an apartment of $X$ and $C \subset E$ is a chamber, there is a retraction $\rho_{E,C}\colon X \to E$ such that if $C = C_1, \dots, C_k$ is a minimal gallery in $X$, then $C = \rho(C_1), \dots, \rho(C_k)$ is a minimal gallery in $E$. We will use a related retraction which is based at a chamber of $X_\infty$ rather than a chamber of $X$.

Following Abramenko and Brown [2, 11.7], if $E$ is an apartment of $X$ and $\mathfrak{c}$ is a chamber of $\partial_\infty E$, we define $\rho_{E,\mathfrak{c}}\colon X \to E$ to be the map such that if $E'$ is an apartment of $X$ which is asymptotic to $\mathfrak{c}$, then $\rho_{E,\mathfrak{c}}|_{E'}$ is the isomorphism $\phi_{E'}\colon E' \to E$ which fixes $E \cap E'$ pointwise. (In the case that $X$ is a tree, this is the map obtained by "dangling" the tree from a point at infinity.)

Fix some apartment $F$ which is asymptotic to $\mathfrak{a}$ and let $\rho = \rho_{F,\mathfrak{a}}$. Note that changing the choice of $F$ changes $\rho$ by an isomorphism; if $F'$ is asymptotic to $\mathfrak{a}$ and $\phi_F\colon F \to F'$ is the isomorphism fixing $F \cap F'$ pointwise, then $\rho_{F',\mathfrak{a}} = \phi_F \circ \rho_{F,\mathfrak{a}}$. Furthermore, $\rho$ preserves Busemann functions centered at points in $\mathfrak{a}$. In particular, $h \circ \rho = h$.

If $E$ is an apartment of $X$, then $\rho$ maps $E$ to $F$ by a "folding" process. If $X$ is a tree, for instance, then either $\rho|_E$ is an isomorphism $E \to F$ or it folds $E$ once. In higher rank buildings, $\rho|_E$ can be more complicated. The following lemmas will help us describe these maps.

For any chamber $C$ of $X$ and any chamber $\mathfrak{c}$ of $X_\infty$, we define the direction $D_C(\mathfrak{c})$ of $\rho(\mathfrak{c})$ at $\rho(C)$ as follows. Let $\overrightarrow{xy}$ be a directed line segment in $C$ in the direction of an interior point of $\mathfrak{c}$. Then $\rho(\overrightarrow{xy})$ is a directed line segment in $F$ pointing toward the interior of some chamber of $\partial_\infty F$. We let $D_C(\mathfrak{c})$ be that chamber.

**Lemma 4.4** *Let $C$ be a chamber of an apartment $E$. Then $D_C|_{\partial_\infty E}\colon \partial_\infty E \to \partial_\infty F$ is a type-preserving isomorphism.*

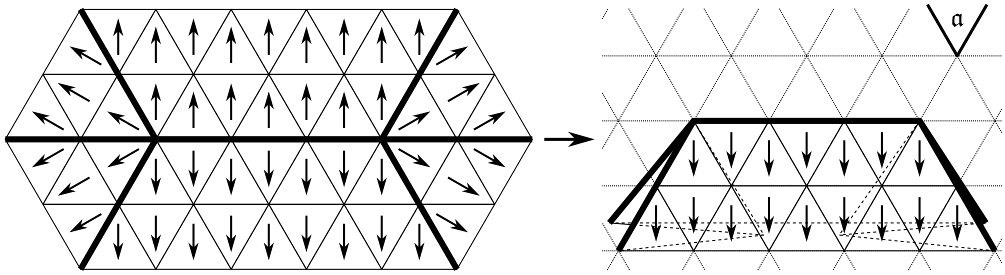


Figure 1: A subset of an apartment and its image under $\rho$. (The three-dimensional effect is for clarity: the map sends triangles to triangles.) Each triangle is $\mathfrak{a}$–characteristic for the chamber of $X_\infty$ in the direction of its arrow.

**Proof** If $E'$ is an apartment containing $C$ and asymptotic to $\mathfrak{a}$ and $\mathfrak{c}' \subset \partial_\infty E'$, we have $D_C(\mathfrak{c}') = \rho_\infty(\mathfrak{c}')$. If $\phi\colon E \to E'$ is the isomorphism fixing $E \cap E'$ pointwise, then $D_C(\mathfrak{c}) = D_C(\phi_\infty(\mathfrak{c}))$ for any $\mathfrak{c} \subset \partial_\infty E$, so

$$D_C|_{\partial_\infty E} = \rho_\infty|_{\partial_\infty E'} \circ \phi_\infty.$$

By [2, Proposition 11.87], $\phi_\infty$ is a type-preserving isomorphism. Likewise, since $\rho|_{E'}$ is the isomorphism fixing $E' \cap F$ pointwise, it induces a type-preserving isomorphism on $\partial_\infty E'$. □

If $C$ is a chamber of $X$, $x \in C$, and $\mathfrak{c} \subset X_\infty$, then there is some subsector $x' + \mathfrak{c}$ of $x + \mathfrak{c}$ such that some apartment of $X$ contains $x' + \mathfrak{c}$ and is asymptotic to $\mathfrak{a}$. The proof of Theorem 11.63(2) in [2] contains the following lemma, which gives us a criterion for when we can take $x' = x$.

**Lemma 4.5** *Suppose that $E$ is an apartment of $X$ and $\mathfrak{c}$ is a chamber in $\partial_\infty E$. If $C$ is a chamber of $E$ such that*

$$d_{\mathrm{comb}}(\mathfrak{a}, D_C(\mathfrak{c})) = \max_{B \subset E} d_{\mathrm{comb}}(\mathfrak{a}, D_B(\mathfrak{c}))$$

*and $x \in C$, then there is an apartment of $X$ containing $x + \mathfrak{c}$ and asymptotic to $\mathfrak{a}$.*

*In particular, if $\mathfrak{a}$ and $D_C(\mathfrak{c})$ are opposite, then $\mathfrak{a}$ and $\mathfrak{c}$ are opposite.*

If $C$ is a chamber of $X$ and $\mathfrak{c}$ is a chamber of $\partial_\infty X$ such that $\mathfrak{a}$ is opposite to $D_C(\mathfrak{c})$, we call $C$ an $\mathfrak{a}$*–characteristic chamber* for $\mathfrak{c}$.

**Lemma 4.6** *The following are equivalent:*

(1) $C$ *is an* $\mathfrak{a}$*–characteristic chamber for* $\mathfrak{c}$*.*

(2) $\mathfrak{a}$ *and* $\mathfrak{c}$ *are opposite and the unique apartment asymptotic to* $\mathfrak{a}$ *and* $\mathfrak{c}$ *contains* $C$*.*

(3) $\mathfrak{a}$ *and* $\mathfrak{c}$ point in opposite directions at $C$*, that is, whenever* $x$ *is in the interior of* $C$*, the rays from* $x$ *toward the barycenters of* $\mathfrak{a}$ *and* $\mathfrak{c}$ *point in opposite directions.*

**Proof** By Lemma 4.5, (1) implies (2). If (2) holds and $E$ is the unique apartment asymptotic to $\mathfrak{a}$ and $\mathfrak{c}$, then the rays toward the barycenters of $\mathfrak{a}$ and $\mathfrak{c}$ from any point in $E$ are rays in $E$ pointing in opposite directions, so (3) holds. Finally, if (3) holds, then $D_C(\mathfrak{a})$ and $D_C(\mathfrak{c})$ are opposite chambers of $\partial_\infty F$. Since $D_C(\mathfrak{a}) = \mathfrak{a}$, this implies (1). □

We can replace $\mathfrak{a}$ in the above constructions with any chamber $\mathfrak{d} \subset X_\infty$, so more generally, we may say that $C$ is an $\mathfrak{d}$–characteristic chamber for $\mathfrak{c}$ if $\mathfrak{d}$ and $\mathfrak{c}$ are opposite and the unique apartment asymptotic to $\mathfrak{d}$ and $\mathfrak{c}$ contains $C$. Then $C$ is an $\mathfrak{d}$–characteristic chamber for $\mathfrak{c}$ if and only if $\mathfrak{a}$ and $\mathfrak{c}$ point in opposite directions at $C$.

Similarly, we say that $\mathfrak{c}$ and $\mathfrak{c}'$ *point in the same direction at* $C$ if, whenever $x$ is in the interior of $C$, the rays from $x$ toward the barycenters of $\mathfrak{c}$ and $\mathfrak{c}'$ have the same tangent vector at $x$. It follows that we have:

**Lemma 4.7** *If* $\mathfrak{c}$ *and* $\mathfrak{c}'$ *point in the same direction at* $C$ *and* $C$ *is* $\mathfrak{d}$*–characteristic for* $\mathfrak{c}$*, then it is also* $\mathfrak{d}$*–characteristic for* $\mathfrak{c}'$*.*

We can apply this lemma to ramifications: if $C \subset E$ is $\mathfrak{a}$–characteristic for $\mathfrak{c} \subset \partial_\infty E$ and $E'$ is any apartment of $X$ that contains $C$, let $\phi\colon E \to E'$ be the isomorphism fixing $E \cap E'$ pointwise and let $\mathfrak{c}' = \phi_\infty(\mathfrak{c})$. Then $\mathfrak{c}$ and $\mathfrak{c}'$ point in the same direction at $C$, so $\mathfrak{c}'$ is opposite to $\mathfrak{a}$.

Figure 1 gives an example of the possible behavior of $\rho$ on an apartment; in the figure, $\rho$ "folds" $E$ along the thick lines. Each of the arrows is sent to an arrow pointing in the direction opposite $\mathfrak{a}$, so each chamber of $E$ is $\mathfrak{a}$–characteristic for the chamber of $E$ that its arrow points toward. Since there are arrows pointing toward every chamber of $\partial_\infty E$, we have $\partial_\infty E \subset X^0_\infty(\mathfrak{a})$. Any apartment $E'$ that contains the pictured portion of $E$ also satisfies $\partial_\infty E' \subset X^0_\infty(\mathfrak{a})$. In fact, if $E'$ is such an apartment, then $\rho$ "folds" $E'$ in the same way as $E$ (ie if $\phi\colon E \to E'$ is the isomorphism fixing $E \cap E'$ pointwise, then $\rho|_E = \rho|_{E'} \circ \phi$).

As the figure suggests, every apartment can be decomposed into $\mathfrak{a}$–characteristic chambers:

**Lemma 4.8** (cf [9, Lemma 3.1.1]) *If $E$ is an apartment of $X$ and $\mathfrak{c}_1, \dots, \mathfrak{c}_d \in \partial_\infty E$ are the chambers of $\partial_\infty E$ which are opposite to $\mathfrak{a}$, then $E$ is a union of subcomplexes $Y_1, \dots, Y_d$ such that the chambers of $Y_i$ are the chambers of $E$ that are $\mathfrak{a}$–characteristic for $\mathfrak{c}_i$. The $Y_i$ are convex in the sense that if $C, C' \subset Y_i$, then any minimal gallery from $C$ to $C'$ is contained in $Y_i$, and the restriction of $\rho$ to any of the $Y_i$ is an isomorphism.*

**Proof** For each $i$, let $E_i$ be the apartment asymptotic to $\mathfrak{a}$ and $\mathfrak{c}_i$. Then $Y_i = E \cap E_i$ is a convex subcomplex of $E$ consisting of the union of the chambers of $E$ that are $\mathfrak{a}$–characteristic for $\mathfrak{c}_i$. If $C$ is a chamber of $E$, let $\overrightarrow{xy}$ be a line segment in $\rho(C)$ in a direction opposite to $\mathfrak{a}$. We can pull it back under $\rho$ to a line segment in $C$ which points in the direction of a chamber $\mathfrak{c}_i \subset \partial_\infty E$. Then $C$ is an $\mathfrak{a}$–characteristic chamber for $\mathfrak{c}_i$ and $C \subset Y_i$. □

Even when $C$ is not $\mathfrak{a}$–characteristic for $\mathfrak{c}$, the direction $D_C(\mathfrak{c})$ still tells us about $\rho|_{x+\mathfrak{c}}$ for $x \in C$. The following lemma strengthens Lemma 4.5.

**Lemma 4.9** *Suppose that $\mathfrak{c}$ is a chamber in $\partial_\infty X$, that $C$ is a chamber of $X$, and $x_0 \in C$. Let $C'$ be a chamber which intersects the sector $x_0 + \mathfrak{c}$. Then either $D_C(\mathfrak{c}) = D_{C'}(\mathfrak{c})$ or $d_{\mathrm{comb}}(\mathfrak{a}, D_C(\mathfrak{c})) < d_{\mathrm{comb}}(\mathfrak{a}, D_{C'}(\mathfrak{c}))$.*

**Proof** We proceed similarly to [2, 11.63(2)].

Let $x \in C'$ be a point in $C' \cap x_0 + \mathfrak{c}$. We may choose $x$ so that the geodesic segment $\overrightarrow{x_0x}$ never crosses two walls simultaneously. Then $\overrightarrow{x_0x}$ passes through chambers $C = C_0, \dots, C_l = C'$ which all meet $x_0 + \mathfrak{c}$ and which form a minimal gallery in $X$. For each $i$, let $x_i$ be a point on $\overrightarrow{x_0x}$ which lies on the interior of $C_i$.

We proceed inductively. Suppose that the lemma is true for $C' = C_0, \dots, C_i$ and consider $C' = C_{i+1}$.

Let $E$ be an apartment containing $C_i$ and asymptotic to $\mathfrak{a}$. Let $A$ be the common panel between $C_i$ and $C_{i+1}$ and let $H$ be the wall of $E$ containing $A$. Let $E^+ \subset E$ be the half-apartment bounded by $H$ which is asymptotic to $\mathfrak{a}$ and let $E^- \subset E$ be the opposite half-apartment.

We consider two cases: $C_i \subset E^+$ and $C_i \subset E^-$.

If $C_i \subset E^+$, let $E'$ be a ramification of $E$ (possibly $E$ itself) which contains $E^+$ and $C_{i+1}$. This is an apartment asymptotic to $\mathfrak{a}$, so by the definition of $\rho$, the restriction $\rho|_{E'}$ is an isomorphism fixing $E' \cap F$ pointwise. This map sends the line segment $\overrightarrow{x_i x_{i+1}}$ to the line segment $\overrightarrow{\rho(x_i)\rho(x_{i+1})}$. Since $\overrightarrow{x_i x_{i+1}}$ is a line segment in the direction of an interior point of $\mathfrak{c}$, this implies that

$$D_{C_i}(\mathfrak{c}) = D_{C_{i+1}}(\mathfrak{c}). \tag{1}$$

If $C_i \subset E^-$, then we have two possibilities: either $C_{i+1} \subset E$ or $C_{i+1} \not\subset E$. If $C_{i+1} \subset E$, then the argument above, applied to $E$, shows that $D_{C_i}(\mathfrak{c}) = D_{C_{i+1}}(\mathfrak{c})$. Otherwise, let $E'$ be a ramification of $E$ which contains $E^-$ and $C_{i+1}$ and let $D = E' \smallsetminus E^-$. Then $D \cup E^+$ is an apartment asymptotic to $\mathfrak{a}$, so $\rho|_D$ is an isomorphism. Likewise, $\rho|_{E^-}$ is an isomorphism. In fact, the restriction of $\rho$ to $E' = E^- \cup D$ is a map $E' \to F$ which "folds" $E'$ along $H$, sending both $E^-$ and $D$ to $\rho(E^-)$.

If $s\colon F \to F$ is the reflection fixing $\rho(H)$, then $D_{C_{i+1}}(\mathfrak{c}) = s_\infty(D_{C_i}(\mathfrak{c}))$. But $\overrightarrow{x_0x}$ passes from $E^-$ to $E^+$, so $\mathfrak{c} \subset \partial_\infty E^+$ and $D_{C_i}(\mathfrak{c})$ is on the same side of $\partial_\infty \rho(H)$ as $\mathfrak{a}$. Therefore,

$$d_{\mathrm{comb}}(\mathfrak{a}, D_{C_i}(\mathfrak{c})) < d_{\mathrm{comb}}(\mathfrak{a}, D_{C_{i+1}}(\mathfrak{c})). \tag{2}$$

Either (1) or (2) holds for each $i$. The lemma follows by induction. □

We will also define some families of subsets of $X$ and $X_\infty$. Our argument is essentially a quantitative version of Morse theory, so for each point $x \in X$ with $h(x) \geq 0$, we will define a set $\mathrm{Lk}^{\downarrow}_\infty(x)$ of downward directions, the *downward link at infinity* and a map from that set to $Z$. By showing that the set of downward directions is highly connected, we will show that $Z$ is highly connected.

For any $x \in X$, let $S(x)$ be the union of the apartments $E$ such that $x \in E$ and $\mathfrak{a} \subset \partial_\infty E$. Let

$$\mathrm{Lk}^{\downarrow}_\infty(x) = \partial_\infty S(x) \cap X^0_\infty(\mathfrak{a}).$$

The following properties of $\mathrm{Lk}^{\downarrow}_\infty(x)$ will be helpful.

**Lemma 4.10** (1) *If $C$ is a chamber of $X$ and $x$ is in the interior of $C$, then $\mathfrak{c}$ is a chamber of $\mathrm{Lk}^{\downarrow}_\infty(x)$ if and only if $C$ is $\mathfrak{a}$–characteristic for $\mathfrak{c}$.*

(2) *If $C$ is a chamber of $X$, $x$ is in the interior of $C$, and $\mathfrak{c}, \mathfrak{c}' \subset \mathrm{Lk}^{\downarrow}_\infty(x)$, then $\mathfrak{c}$ and $\mathfrak{c}'$ point in the same direction at $C$.*

(3) *If $x' \in x + \mathfrak{a}$, then $\mathrm{Lk}^{\downarrow}_\infty(x) \subset \mathrm{Lk}^{\downarrow}_\infty(x')$.*

(4) *If $Q \subset X$ is a bounded subset, then there is an $x \in X$ such that $d(Q, x) \lesssim \operatorname{diam} Q$ and $x \in q + \mathfrak{a}$ for any $q \in Q$.*

(5) *If $r\colon [0, \infty) \to X$ is a unit-speed ray emanating from $x$ in the direction of a point $\sigma \in \mathrm{Lk}^{\downarrow}_\infty(x)$, then*

$$h(r(t)) = h(x) + t \cos d(\tau, \sigma).$$

*Furthermore, there is an $\epsilon > 0$ which depends on $X$ and $p(\tau)$ such that $-\cos d(\tau, \sigma) > \epsilon$.*

**Proof** The first property follows from the definition of $\mathrm{Lk}^{\downarrow}_{\infty}(x)$ and the fact that $C$ is an $\mathfrak{a}$–characteristic chamber for $\mathfrak{c}$ if and only if $\mathfrak{a}$ and $\mathfrak{c}$ are opposite and the unique apartment asymptotic to $\mathfrak{a}$ and $\mathfrak{c}$ contains $C$.

If $x$ is in the interior of $C$ and $\mathfrak{c}, \mathfrak{c}' \subset \mathrm{Lk}^{\downarrow}_{\infty}(x)$, then $C$ is $\mathfrak{a}$–characteristic for $\mathfrak{c}$ and $\mathfrak{c}'$. Consequently, $D_C(\mathfrak{c})$ and $D_C(\mathfrak{c}')$ are both the chamber of $\partial_\infty F$ opposite to $\mathfrak{a}$, so $\mathfrak{c}$ and $\mathfrak{c}'$ point in the same direction at $C$.

For the third property, we show that $S(x) \subset S(x')$. If $y \in S(x)$, then there is an apartment containing $x$ and $y$ and asymptotic to $\mathfrak{a}$. Since $x' \in x + \mathfrak{a}$, $x'$ lies in this apartment as well. It follows that $\mathrm{Lk}^{\downarrow}_{\infty}(x) \subset \mathrm{Lk}^{\downarrow}_{\infty}(x')$.

To prove the fourth property, for all $q \in Q$, let $r_q \colon [0, \infty) \to X$ be a ray emanating from $q$ in the direction of the barycenter of $\mathfrak{a}$. Let $E$ be an apartment asymptotic to $\mathfrak{a}$ that intersects $Q$ nontrivially. Then $d(q, E) \le \operatorname{diam} Q$ for any $q \in Q$, so by Kleiner and Leeb [16, Lemma 4.6.3], there is a $c$ such that if $t \ge c \operatorname{diam} Q$, then $r_q(t) \in E$. In particular, $V = \bigcap_q r_q(t) + \mathfrak{a}$ is a sector in $E$ that satisfies $V \subset q + \mathfrak{a}$ for all $q$ and $d(V, Q) \lesssim \operatorname{diam} Q$. Choose $x \in V$.

Finally, if $r$ is a ray in the direction of $\sigma$, let $E$ be an apartment which contains $x$ and is asymptotic to $\mathfrak{a}$ and to $\sigma$. Then $r$ is a geodesic ray in $E$, which makes an angle of $d(\tau, \sigma)$ with the ray emanating from $x$ in the direction of $\tau$. The formula for $h(r(t))$ follows by trigonometry.

To bound $d(\tau, \sigma)$, consider

$$m = \max_{\theta \in \mathfrak{a}} d(\tau, \theta).$$

If $\bar{\sigma}$ is the direction opposite to $\sigma$ in $\partial_\infty E$, then by the definition of $\mathrm{Lk}^{\downarrow}_{\infty}(x)$, we have $\bar{\sigma} \in \mathfrak{a}$, so $d(\tau, \sigma) = \pi - d(\tau, \bar{\sigma}) \ge \pi - m$. We claim that $m < \pi/2$.

By [7, Lemma 4.1], the diameter of $\mathfrak{a}$ is at most $\pi/2$, and if the diameter is equal to $\pi/2$, then $\mathfrak{a}$ is a nontrivial spherical join and $X$ is a nontrivial product of buildings. Furthermore, if $\theta \in \mathfrak{a}$ is such that $d(\tau, \theta) = \pi/2$, then we can write $X = X_1 \times X_2$ such that $\tau \in (X_1)_\infty, \theta \in (X_2)_\infty$. This contradicts the hypothesis that $\tau$ is not parallel to a factor of $X$, so $m < \pi/2$ and $-\cos d(\tau, \sigma) \ge -\cos m > 0$. □

## 4.3 Apartments in $X^0_\infty(\mathfrak{a})$

In this section, we use the tools of the previous section to construct apartments in $X^0_\infty(\mathfrak{a})$; in the next section, we will use these apartments to contract spheres in $X^0_\infty(\mathfrak{a})$. First, we show that every chamber in $X^0_\infty(\mathfrak{a})$ is part of some apartment in $X^0_\infty(\mathfrak{a})$:

**Lemma 4.11** *Suppose that $\mathfrak{c}$ is a chamber of $X_\infty$ opposite to $\mathfrak{a}$ and suppose that $C$ is an $\mathfrak{a}$–characteristic chamber for $\mathfrak{c}$. There is an apartment $E$ containing $C$ such that $E$ is asymptotic to $\mathfrak{c}$ and every chamber of $\partial_\infty E$ is opposite to $\mathfrak{a}$.*

*Furthermore, there is a $c > 0$ depending only on $X$ and an $\mathfrak{a}$–characteristic chamber $C_\mathfrak{b} \subset E$ for each chamber $\mathfrak{b} \subset \partial_\infty E$ such that $C_\mathfrak{c} = C$ and*

$$\operatorname{diam} \bigcup_{\mathfrak{b} \subset \partial_\infty E} C_\mathfrak{b} \le c.$$

We will prove this lemma by starting with an apartment $E \subset X$, then producing a series of ramifications of $E$ so that more and more chambers of $\partial_\infty E$ are opposite to $\mathfrak{a}$. Since $X$ is thick, if $\mathfrak{c}$ is a chamber of $\partial_\infty E$ which is not opposite to $\mathfrak{a}$, then there is some ramification $E'$ of $E$ that replaces $\mathfrak{c}$ with a chamber that is farther (in $X_\infty$) from $\mathfrak{a}$. This might replace a chamber of $\partial_\infty E$ which is already opposite to $\mathfrak{a}$ with a chamber which is not, but we avoid this by ensuring that $E'$ contains the same $\mathfrak{a}$–characteristic chambers as $E$.

The following lemma produces these ramifications:

**Lemma 4.12** *Let $E$ be an apartment of $X$ and let $\mathfrak{c} = \mathfrak{c}_1, \dots, \mathfrak{c}_k$ be chambers of $\partial_\infty E$ which are opposite to $\mathfrak{a}$. Let $C_i \subset E$ be a $\mathfrak{a}$–characteristic chamber for $\mathfrak{c}_i$ for each $i$. Let $\mathfrak{b}$ be a chamber of $\partial_\infty E$, distinct from the $\mathfrak{c}_i$, which is adjacent to $\mathfrak{c}$. There is a ramification $E_0$ of $E$ such that if $\phi$: $E \to E_0$ is the isomorphism fixing $E \cap E_0$ pointwise, then:*

- *$C_i \subset E \cap E_0$ for all $i$ (and thus $\phi_\infty(\mathfrak{c}_i)$ is opposite to $\mathfrak{a}$)*
- *$\phi_\infty(\mathfrak{b})$ is opposite to $\mathfrak{a}$*
- *There is an $\mathfrak{a}$–characteristic chamber $B_0 \subset E_0$ for $\phi_\infty(\mathfrak{b})$ so that $d(B_0, \bigcup C_i) \lesssim \operatorname{diam} \bigcup C_i$.*

**Proof** Let $C = C_1$ and let $x_0 \in C$. Let $H$ be a wall in $E$ such that $\partial_\infty H$ separates $\mathfrak{b}$ and $\mathfrak{c}$. Let $M, M' \subset E$ be the half-apartments of $E$ bounded by $H$. By translating $H$ and possibly switching $M$ and $M'$, we may arrange that:

- $\mathfrak{c} \in \partial_\infty M$ and $\mathfrak{b} \in \partial_\infty M'$
- $C_i \subset M$ for all $i$
- $d(H, C) \lesssim \operatorname{diam}(\bigcup C_i)$

We claim that there is a ramification $E_0$ of $E$ which contains $M$ and satisfies the conditions of the lemma.

By our choice of $H$, the intersection $x_0 + \mathfrak{b} \cap M'$ is a sector of $E$, and we can choose $B \subset x_0 + \mathfrak{b} \cap M'$ to be a chamber which borders $H$ and satisfies $d(x_0, B) \lesssim \operatorname{diam}(\bigcup C_i)$. Let $A$ be the panel of $H$ bordering $B$, let $D \subset M$ be the chamber of $E$ adjacent to $B$ along $A$, and let $B'$ be a chamber adjacent to $A$ and distinct from $B$ and $D$. Let $E'$ be a ramification of $E$ that contains $B'$ and let $\phi\colon E \to E'$ be the isomorphism fixing $E \cap E'$. We claim that either the lemma is satisfied for $E_0 = E$ and $B_0 = B$ or it is satisfied for $E_0 = E'$ and $B_0 = B'$.

Since $\mathfrak{a}$ is opposite to $D_C(\mathfrak{c})$ and $D_C(\mathfrak{b})$ is adjacent to $D_C(\mathfrak{c})$,

$$d_{\mathrm{comb}}(\mathfrak{a}, D_C(\mathfrak{b})) = d_{\mathrm{comb}}(\mathfrak{a}, D_C(\mathfrak{c})) - 1.$$

Lemma 4.9 implies that either $D_B(\mathfrak{b})$ is opposite to $\mathfrak{a}$ or $D_B(\mathfrak{b}) = D_C(\mathfrak{b})$. By Lemma 4.4, $D_B$ and $D_C$ are type-preserving isomorphisms from $\partial_\infty E$ to $\partial_\infty F$, so if $D_B(\mathfrak{b}) = D_C(\mathfrak{b})$, then $D_B = D_C$, and $B$ is $\mathfrak{a}$–characteristic for $\mathfrak{c}$. So $B$ is $\mathfrak{a}$–characteristic for either $\mathfrak{b}$ or $\mathfrak{c}$. In the first case, the lemma is satisfied for $E_0 = E$ and $B_0 = B$.

Likewise, if $\mathfrak{b}' = \phi_\infty(\mathfrak{b})$, then $D_C(\mathfrak{b}') = D_C(\mathfrak{b})$ is adjacent to $D_C(\mathfrak{c})$ and $B' \subset x_0 + \mathfrak{b}'$, so $B'$ is $\mathfrak{a}$–characteristic for either $\mathfrak{b}'$ or $\mathfrak{c}$. In the first case, the lemma is satisfied for $E_0 = E'$ and $B_0 = B'$.

Suppose by way of contradiction that $B$ and $B'$ are both $\mathfrak{a}$–characteristic for $\mathfrak{c}$. The union of the set of chambers of $X$ that are $\mathfrak{a}$–characteristic for $\mathfrak{c}$ is the unique apartment $E_{\mathfrak{a},\mathfrak{c}}$ asymptotic to $\mathfrak{a}$ and $\mathfrak{c}$, so in particular, it is convex. It contains $B$ and $C$, so it contains $D$ as well. But then $B$, $B'$, and $D$ are distinct chambers of $E_{\mathfrak{a},\mathfrak{c}}$ which are all adjacent to the same panel. This is impossible. □

**Proof of Lemma 4.11** Let $E_{\mathfrak{a},\mathfrak{c}} \subset X$ be the apartment spanned by $\mathfrak{a}$ and $\mathfrak{c}$, so that $C \subset E_{\mathfrak{a},\mathfrak{c}}$. By applying Lemma 4.12 to $E_{\mathfrak{a},\mathfrak{c}}$ repeatedly, we can construct an apartment $E$ such that for any chamber $\mathfrak{b} \in \partial_\infty E$, there is an $\mathfrak{a}$–characteristic chamber $C_\mathfrak{b}$ for $\mathfrak{b}$, and $\operatorname{diam} \bigcup_{\mathfrak{b} \subset \partial_\infty E'} C_\mathfrak{b}$ is bounded. □

In fact, we can find many apartments in $X^0_\infty(\mathfrak{a})$ simultaneously:

**Lemma 4.13** *Suppose that $E$ is an apartment of $X$ and suppose that for each chamber $\mathfrak{c} \subset \partial_\infty E$ there is a chamber $C_\mathfrak{c} \subset E$ which is $\mathfrak{a}$–characteristic for $\mathfrak{c}$ and a point $x_\mathfrak{c} \in C_\mathfrak{c}$. Let $\mathfrak{b}$ and $\overline{\mathfrak{b}}$ be two opposite chambers in $\partial_\infty E$. Suppose that $C$ is a chamber of $X$*

*and $x$ is a point in the interior of $C$ such that $x \in x_{\mathfrak{b}} + \mathfrak{b}$ and $C_{\mathfrak{c}} \subset x + \overline{\mathfrak{b}}$ for all $\mathfrak{c} \subset \partial_\infty E$. Then there is an $x' \in x + \mathfrak{a}$ such that*

$$d(x, x') \lesssim \operatorname{diam} \bigcup_{\mathfrak{c} \subset \partial_\infty E} C_{\mathfrak{c}}$$

*and for every chamber $\mathfrak{d} \subset \mathrm{Lk}^{\downarrow}_{\infty}(x)$:*

- *$\mathfrak{d}$ is opposite to $\overline{\mathfrak{b}}$.*
- *If $E_{\mathfrak{d},\overline{\mathfrak{b}}}$ is the apartment spanned by $\mathfrak{d}$ and $\overline{\mathfrak{b}}$, then $\partial_\infty E_{\mathfrak{d},\overline{\mathfrak{b}}} \subset \mathrm{Lk}^{\downarrow}_{\infty}(x')$.*

**Proof** Suppose that $\mathfrak{d} \subset \mathrm{Lk}^{\downarrow}_{\infty}(x)$. Then $C$ is $\mathfrak{a}$–characteristic for $\mathfrak{b}$ and $\mathfrak{d}$, so $\mathfrak{b}$ and $\mathfrak{d}$ point in the same direction at $C$. Since $\mathfrak{b}$ and $\overline{\mathfrak{b}}$ point in opposite directions at $C$, we conclude that $C$ is $\overline{\mathfrak{b}}$–characteristic for $\mathfrak{d}$. Thus, $\overline{\mathfrak{b}}$ and $\mathfrak{d}$ are opposite and $C \subset E_{\mathfrak{d},\overline{\mathfrak{b}}}$.

In particular, $x + \overline{\mathfrak{b}} \subset E_{\mathfrak{d},\overline{\mathfrak{b}}}$, so $C_{\mathfrak{c}} \subset E_{\mathfrak{d},\overline{\mathfrak{b}}}$ for all $\mathfrak{c} \subset \partial_\infty E$. Let $\phi\colon E_{\mathfrak{d},\overline{\mathfrak{b}}} \to E$ be the isomorphism fixing $E_{\mathfrak{d},\overline{\mathfrak{b}}} \cap E$ pointwise and suppose that $\mathfrak{c}' \subset \partial_\infty E_{\mathfrak{d},\overline{\mathfrak{b}}}$. If $\mathfrak{c} = \phi_\infty(\mathfrak{c}')$, then $C_{\mathfrak{c}}$ is an $\mathfrak{a}$–characteristic chamber for $\mathfrak{c}'$, so $\mathfrak{c}' \subset \mathrm{Lk}^{\downarrow}_{\infty}(x_{\mathfrak{c}})$.

By Lemma 4.10(3) and (4), there is an

$$x' \in \bigcap_{\mathfrak{c} \subset \partial_\infty E} x_{\mathfrak{c}} + \mathfrak{a}$$

such that $x' \in x + \mathfrak{a}$ and $\mathrm{Lk}^{\downarrow}_{\infty}(x_{\mathfrak{c}}) \subset \mathrm{Lk}^{\downarrow}_{\infty}(x')$ for every $\mathfrak{c} \subset \partial_\infty E$. □

Combining Lemmas 4.13 and 4.11 we get:

**Lemma 4.14** *For any $x \in X$, there is a chamber $\mathfrak{d} \subset X_\infty$ opposite to $\mathfrak{a}$ and an $x' \in x + \mathfrak{a}$ such that:*

- *If $\mathfrak{c} \subset \mathrm{Lk}^{\downarrow}_{\infty}(x)$ then $\mathfrak{d}$ is opposite to $\mathfrak{c}$.*
- *If $\mathfrak{c} \subset \mathrm{Lk}^{\downarrow}_{\infty}(x)$ and $E_{\mathfrak{c},\mathfrak{d}}$ is the apartment spanned by $\mathfrak{c}$ and $\mathfrak{d}$, then $\partial_\infty E_{\mathfrak{c},\mathfrak{d}} \subset \mathrm{Lk}^{\downarrow}_{\infty}(x')$.*
- *$d(x, x') \lesssim 1$*

**Proof** Let $\mathfrak{b} \subset \mathrm{Lk}^{\downarrow}_{\infty}(x)$ and let $E$ be the unique apartment asymptotic to $\mathfrak{a}$ and $\mathfrak{b}$. Since $\mathfrak{b} \subset \mathrm{Lk}^{\downarrow}_{\infty}(x)$, we have $x \in E$. We may perturb $x$ in the direction of $\mathfrak{a}$ to ensure that $x$ is in the interior of some chamber $C$ of $E$; this doesn't change $\mathrm{Lk}^{\downarrow}_{\infty}(x)$. Let $r$ be a unit-speed ray emanating from $x$ in the direction of the barycenter of $\mathfrak{a}$ and let

$0 < \theta < \pi/2$ be the minimum angle between the barycenter of $\mathfrak{a}$ and any point on its boundary. Let $c$ be the constant in Lemma 4.11 and let $t > c/\sin\theta$, so that

$$B_E(r(t), c) \subset x + \mathfrak{a},$$

where $B_E(r(t), c)$ is the ball in $E$ with center $r(t)$ and radius $c$. Let $x_0 = r(t)$.

Let $C_0 \subset E$ be a chamber such that $x_0 \in C_0$. Since $C_0 \subset E$, it is $\mathfrak{a}$–characteristic for $\mathfrak{b}$. By Lemma 4.11, there is an apartment $E'$ and a collection of $\mathfrak{a}$–characteristic chambers $C_{\mathfrak{c}} \subset E'$ for $\mathfrak{c} \subset \partial_\infty E'$ such that $x_0 + \mathfrak{b} \subset E'$ and

$$\bigcup_{\mathfrak{c} \subset \partial_\infty E'} C_{\mathfrak{c}} \subset B_{E'}(x_0, c).$$

Let $\overline{\mathfrak{b}}$ be the chamber of $\partial_\infty E'$ opposite to $\mathfrak{b}$. We claim that $x + \overline{\mathfrak{b}}$ contains all of the $C_{\mathfrak{c}}$.

Let $\phi\colon E \to E'$ be the isomorphism fixing $E \cap E'$ pointwise. Then $\phi$ fixes $C$ and $C_0$ and sends $\mathfrak{a}$ to $\overline{\mathfrak{b}}$, so $\phi(x + \mathfrak{a}) = x + \overline{\mathfrak{b}}$ and $\phi(B_E(x_0, c)) = B_{E'}(x_0, c)$. Therefore,

$$\bigcup_{\mathfrak{c} \subset \partial_\infty E'} C_{\mathfrak{c}} \subset B_{E'}(x_0, t) \subset x + \overline{\mathfrak{b}}.$$

By applying Lemma 4.13 to $E'$, we obtain an $x'$ that satisfies the required properties and has

$$d(x, x') \lesssim \operatorname{diam} \bigcup_{\mathfrak{c} \subset \partial_\infty E'} C_{\mathfrak{c}} \lesssim 1. \qquad \square$$

We can also use these techniques to construct $(n-1)$–spheres in $Z$ which are homotopically nontrivial in $Z$. This generalizes results of Bux and Wortman [6] on buildings acted on by $S$–arithmetic groups to arbitrary euclidean buildings.

**Lemma 4.15** *For any $r > 0$, there is a map $\alpha\colon S^{n-1} \to Z$ such that $\alpha$ is homotopically nontrivial in $N_r(Z)$, where $N_r(Z)$ is the $r$–neighborhood of $Z$.*

**Proof** Let $C$ be a chamber of $X$ such that $\min_{x \in C} h(x) > r$. Let $E$ be an apartment containing $C$ and asymptotic to $\mathfrak{a}$. If $\mathfrak{c} \subset \partial_\infty E$ is the chamber of $\partial_\infty E$ opposite to $\mathfrak{a}$, then $C$ is $\mathfrak{a}$–characteristic for $\mathfrak{c}$. Using Lemma 4.12, we can construct an apartment $E'$ such that $C \subset E'$ and $\partial_\infty E' \subset X^0_\infty(\mathfrak{a})$. In particular, the set of points $B = \{x \in E' \mid h(x) \geq 0\}$ is convex and compact and contains $C$, so $Z \cap E'$ is bilipschitz equivalent to the $(n-1)$–sphere. Let $\alpha\colon S^{n-1} \to Z \cap E'$ be a Lipschitz homeomorphism. We claim that $\alpha$ is homotopically nontrivial in $N_r(Z)$.

Let $\beta\colon D^n \to E$ be a homeomorphism from $D^n \to B$ which extends $\alpha$. This has degree 1 on any point in the interior of $C$. By way of contradiction, suppose that $\beta'\colon D^n \to N_r(Z)$ is another extension of $\alpha$. Then we can glue $\beta$ and $\beta'$ together to get a map $\gamma\colon S^n \to X$. Since $\beta'$ avoids $C$, this map has degree 1 on any point in the interior of $C$. Since $X$ is CAT(0), however, it is contractible, so $\gamma$ must be nullhomotopic, and $\gamma$ sends the fundamental class of $S^n$ to an $n$–boundary in $X$. This contradicts the fact that this map has degree 1 on any point in the interior of $C$, because $X$ is $n$–dimensional, and any $n$–boundary must be trivial. □

## 4.4 Proving $(n-2)$–connectivity for $X^0_\infty(\mathfrak{a})$ and constructing $\Omega_\infty$

The lemmas of the previous section will let us prove that $X^0_\infty(\mathfrak{a})$ is $(n-2)$–connected and construct a Lipschitz map

$$\Omega_\infty\colon \Delta_Z^{(n-1)} \to X^0_\infty(\mathfrak{a})$$

which we will use to construct $\Omega$.

Let $\Delta_Z$ be the infinite-dimensional simplex with vertex set $Z$. As before, we denote the simplex of $\Delta_Z$ with vertices $z_0,\dots,z_k$ by $\langle z_0,\dots,z_k\rangle$. If $\Delta$ is a simplex of $\Delta_Z$, we let $\mathcal{V}(\Delta) \subset Z$ be the vertex set of $\Delta$.

The main lemma of this section is the following:

**Lemma 4.16** *There is a cellular map*

$$\Omega_\infty\colon \Delta_Z^{(n-1)} \to X^0_\infty(\mathfrak{a}),$$

*a $c>0$ depending on $X$, and a family of points $x_\Delta \in X$, one for each simplex $\Delta \subset \Delta_Z^{(n-1)}$, such that:*

(1) $\operatorname{mass}(\Omega_\infty(\Delta)) \le c$

(2) $h(x_\Delta) \ge 0$

(3) $\Omega_\infty(\Delta) \subset \operatorname{Lk}^{\downarrow}_\infty(x_\Delta)$

(4) *If $\Delta' \subset \Delta$, then $x_\Delta \in x_{\Delta'} + \mathfrak{a}$*

(5) $d(x_\Delta, \mathcal{V}(\Delta)) \lesssim \operatorname{diam}\mathcal{V}(\Delta)+1$ *(consequently, $h(x_\Delta) \lesssim \operatorname{diam}\mathcal{V}(\Delta)+1$)*

*Furthermore, for any $z \in Z$, we have $x_{\langle z\rangle} = z$.*

The first condition is essentially a bound on the filling functions of $X^0_\infty(\mathfrak{a})$. The next three conditions ensure that the map $i_{x_\Delta}$ (as defined in the proof sketch at the beginning of the section) is defined on $\Omega_\infty(\Delta)$ and that $\operatorname{Lip}(i_{x_\Delta}) \lesssim \operatorname{diam} \mathcal{V}(\Delta)$. In order to construct $\Omega$ in the next section, we will glue maps of the form $i_{x_\Delta} \circ \Omega_\infty|_\Delta$, and we will use the last condition to perform this gluing.

First, we prove that $X^0_\infty(\mathfrak{a})$ is $(n-2)$–connected.

**Lemma 4.17** *If $k < n-1$, there is a $c > 0$ such that for every $x \in X$, there is a $x' \in X$ such that $x' \in x + \mathfrak{a}$, $d(x, x') \le c$, and if*

$$\alpha\colon S^k \to \operatorname{Lk}^\downarrow_\infty(x)^{(k)},$$

*then there is an extension*

$$\beta\colon B^{k+1} \to \operatorname{Lk}^\downarrow_\infty(x')^{(k+1)}$$

*such that $\operatorname{Lip} \beta \le c' \operatorname{Lip} \alpha + c'$.*

*Consequently, $X^0_\infty(\mathfrak{a})$ is $(n-2)$–connected.*

**Proof** Let $x' \in X$ and $\mathfrak{d} \subset \operatorname{Lk}^\downarrow_\infty(x')$ be opposite to every chamber of $\operatorname{Lk}^\downarrow_\infty(x)$ as in Lemma 4.14. Let $u$ be the barycenter of $\mathfrak{d}$. There is an $\epsilon > 0$ such that $d_{X_\infty}(u, v) < \pi - \epsilon$ for any $v \in \operatorname{Lk}^\downarrow_\infty(x)^{(k)}$. By our choice of $\mathfrak{d}$, the geodesic from $v$ to $u$ is contained in $\operatorname{Lk}^\downarrow_\infty(x')$.

Let

$$\gamma\colon \operatorname{Lk}^\downarrow_\infty(x)^{(k)} \times [0,1] \to \operatorname{Lk}^\downarrow_\infty(x')$$

be the map which sends $v \times [0,1]$ to the geodesic between $v$ and $u$. This is Lipschitz, with Lipschitz constant depending on $\epsilon$. Define $\beta_0\colon S^k \times [0,1] \to \operatorname{Lk}^\downarrow_\infty(x')$ by $\beta_0(v,t) = \gamma(\alpha(v), t)$. This is a nullhomotopy of $\alpha$, and

$$\operatorname{Lip} \beta_0 \le (\operatorname{Lip} \gamma)(1 + \operatorname{Lip} \alpha).$$

We obtain $\beta$ by approximating $\beta_0$ in $\operatorname{Lk}^\downarrow_\infty(x)^{(k+1)}$; this increases the Lipschitz constant by at most a multiplicative factor.

To conclude that $X^0_\infty(\mathfrak{a})$ is $(n-2)$–connected, consider a map $\alpha\colon S^k \to X^0_\infty(\mathfrak{a})$. This can be approximated by a simplicial map $\alpha'\colon S^k \to X^0_\infty(\mathfrak{a})^{(k)}$. The image of $\alpha'$ has finitely many simplices, and since every simplex of $X^0_\infty(\mathfrak{a})$ is contained in $\operatorname{Lk}^\downarrow_\infty(y)$ for some $y$, there is an $x \in X$ such that the image of $\alpha'$ is contained in $\operatorname{Lk}^\downarrow_\infty(x)^{(k)}$. Therefore, $\alpha'$ is nullhomotopic in $\operatorname{Lk}^\downarrow_\infty(x')^{(k)}$ for some $x'$, and $\operatorname{Lk}^\downarrow_\infty(x')^{(k)} \subset X^0_\infty(\mathfrak{a})$. $\square$

Next, we use this lemma to construct $\Omega_\infty$.

**Proof of Lemma 4.16** We construct $\Omega_\infty$ inductively. First, for each $z \in Z$, we let $\Omega_\infty(z)$ be an arbitrary vertex of $\mathrm{Lk}^{\downarrow}_\infty(z)$. Then choosing $x_{\langle z\rangle} = z$ satisfies the conditions of the lemma.

Now suppose that $\Delta$ is a simplex of $\Delta_Z$ with $1 \le \dim\Delta = k \le n-1$ and suppose that $\Omega_\infty$ is defined on $\partial\Delta$. Then, if

$$L = \bigcup_{\Delta' \subset \Delta} \mathrm{Lk}^{\downarrow}_\infty(x_{\Delta'}),$$

then $\Omega_\infty|_{\partial\Delta}$ is a map with image in $L$. By induction, we know that

$$d(x_{\Delta'}, \mathcal{V}(\Delta')) \lesssim_k \operatorname{diam}\mathcal{V}(\Delta') + 1,$$

with implicit constant depending on $k$, so

$$\operatorname{diam}\{x_{\Delta'}\}_{\Delta'\subset\Delta} \lesssim_k \operatorname{diam}\mathcal{V}(\Delta) + 1.$$

By Lemma 4.10(4), there is an $x_0 \in X$ such that $d(x_0, \mathcal{V}(\Delta)) \lesssim_k \operatorname{diam}\mathcal{V}(\Delta) + 1$ and $x_0 \in x_{\Delta'} + \mathfrak{a}$ for any face $\Delta'$ of $\Delta$. By Lemma 4.10(3), $L \subset \mathrm{Lk}^{\downarrow}_\infty(x_0)$.

By Lemma 4.17, there is an $x' \in X$ such that $x' \in x_0 + \mathfrak{a}$ and an extension

$$\beta\colon \Delta \to \mathrm{Lk}^{\downarrow}_\infty(x')^{(k+1)}$$

of $\alpha$ such that $\operatorname{Lip}\beta$ and $d(x_0, x')$ are bounded by a constant depending on $k$. If we define $\Omega_\infty|_\Delta = \beta$ and $x_\Delta = x'$, then

$$d(x_\Delta, \mathcal{V}(\Delta)) \lesssim_k \operatorname{diam}\mathcal{V}(\Delta) + 1.$$

Since $\Delta$ is finite-dimensional, we may drop the dependence on $k$, and the lemma holds. $\square$

## 4.5 Constructing $\Omega$

Finally, we construct a map $\Omega\colon \Delta_Z^{(n-1)} \to Z$ satisfying the hypotheses of Lemma 3.2. We will use a family of maps $i_x\colon \mathrm{Lk}^{\downarrow}_\infty(x) \to Z$ for $x \in X, h(x) \ge 0$.

For any $x \in X$ and $\sigma \in X_\infty$, there is a unit-speed ray $r_\sigma\colon [0,\infty) \to X$ emanating from $x$ and traveling in the direction of $\sigma$. Define

$$X^*_\infty = X_\infty \times [0,\infty)/X_\infty \times \{0\}$$

to be a space of "vectors" based at $x$. We can define an exponential map $e_x\colon X^*_\infty \to X$ by letting

$$e_x(\sigma, t) = r_\sigma(t).$$

For each chamber $\mathfrak{a}$ of $X_\infty$, this map sends the open cone $\mathfrak{a}\times[0,\infty)/\mathfrak{a}\times\{0\}$ to a sector corresponding to $\mathfrak{a}$; we give $X^*_\infty$ a metric so that this is an isometry. This makes $e_x$ a distance-decreasing map. Note also that, by the convexity of the distance function on $X$, we have

$$d(e_x(\sigma,t),e_{x'}(\sigma,t))\le d(x,x').$$

We can use $e_x$ to construct a map from $\mathrm{Lk}^{\downarrow}_\infty(x)$ to $Z$:

**Lemma 4.18** *Let $x\in X$ be such that $h(x)\ge 0$. Then there is a map $i_x\colon \mathrm{Lk}^{\downarrow}_\infty(x)\to Z$ given by*

$$i_x(\sigma)=e_x\left(\sigma,\frac{-h(x)}{\cos d(\tau,\sigma)}\right).$$

*This map has Lipschitz constant $\mathrm{Lip}(i_x)\lesssim h(x)$, with implicit constant depending on $X$ and $p(\tau)$.*

**Proof** By Lemma 4.10(5),

$$h(e_x(\sigma,t))=h(x)+t\cos d(\tau,\sigma)$$

for any $\sigma\in\mathrm{Lk}^{\downarrow}_\infty(x)$ and there is an $\epsilon$ such that $-\cos d(\tau,\sigma)\ge\epsilon>0$. The lemma follows. □

Furthermore, the map $(x,\sigma)\mapsto i_x(\sigma)$ is locally Lipschitz:

**Lemma 4.19** *Let $x,x'\in X$ be such that $h(x),h(x')\ge 0$. Let $\sigma,\sigma'\in\mathrm{Lk}^{\downarrow}_\infty(x)\cap\mathrm{Lk}^{\downarrow}_\infty(x')$. Then there is a $c>0$ depending on $X$ such that*

$$d(i_x(\sigma),i_{x'}(\sigma'))\le cd(x,x')+ch(x)d(\sigma,\sigma').$$

**Proof** By the previous lemma and the remark before it, there is a $c_0>0$ such that

$$\begin{aligned}d(i_x(\sigma),i_{x'}(\sigma'))&\le d(i_x(\sigma),i_x(\sigma'))+d(i_x(\sigma'),i_{x'}(\sigma'))\\&\le c_0h(x)d(\sigma,\sigma')+d\left(e_x\left(\sigma',\frac{-h(x)}{\cos d(\tau,\sigma')}\right),e_{x'}\left(\sigma',\frac{-h(x')}{\cos d(\tau,\sigma')}\right)\right)\\&\le c_0h(x)d(\sigma,\sigma')+d(x,x')+\frac{|h(x)-h(x')|}{-\cos d(\tau,\sigma')}.\end{aligned}$$

Since $d(x,x')\lesssim h(x')$, the lemma follows. □

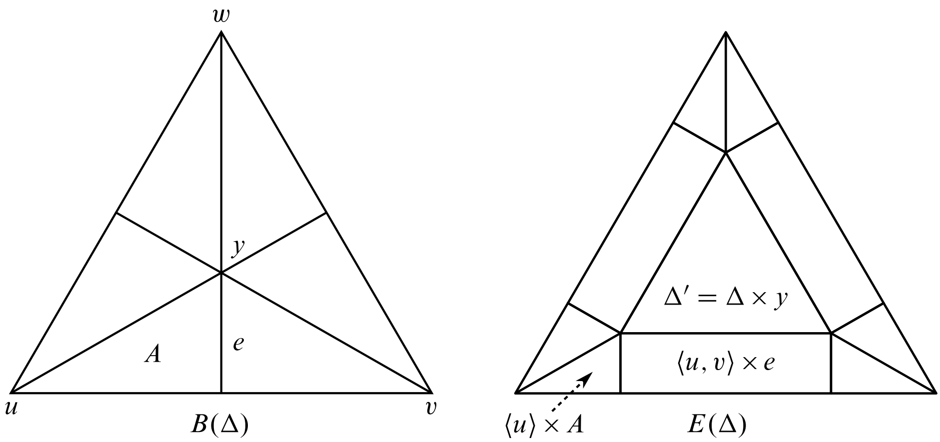


Figure 2: Cells of the "exploded simplex" $E(\Delta)$ are naturally products of cells of $\Delta$ and cells of $B(\Delta)$.

We construct $\Omega$ by piecing together maps of the form $i_{x_\Delta}(\Omega_\infty(\Delta))$, where $\Delta$ ranges over the simplices of $\Delta_Z$. The main problem is that if $\Delta'$ is a face of $\Delta$, the maps $i_{x_\Delta}(\Omega_\infty(\Delta))$ and $i_{x_{\Delta'}}(\Omega_\infty(\Delta'))$ need not agree, since $x_\Delta \neq x_{\Delta'}$, so we need to add some "padding" to make these maps agree.

Part of the construction is illustrated in Figure 2: for each simplex $\Delta$ of $\Delta_Z$, we "explode" the barycentric subdivision $B(\Delta)$ to get a complex $E(\Delta)$ by inserting a copy $\Delta'$ of $\Delta$ in the middle. Each cell in this subdivision is of the form $\Delta_1 \times \Delta_2$, where $\Delta_1$ is a face of $\Delta$ and $\Delta_2$ is a face of $B(\Delta)$. To be more specific, note that we can label each vertex of $B(\Delta)$ by a face $\delta$ of $\Delta$, and the vertex labels of a simplex $\langle \delta_0, \dots, \delta_k\rangle$ form a flag $\delta_0 \subset \dots \subset \delta_k$. Then each cell of $E(\Delta)$ is of the form

$$\delta \times \langle \delta_0, \dots, \delta_k\rangle,$$

for some flag $\delta_0 \subset \dots \subset \delta_k$ in $\Delta$ and some face $\delta$ of $\delta_0$. The map $\rho_1\colon E(\Delta) \to \Delta$ which projects each simplex to its first factor is a continuous map which sends $\Delta'$ homeomorphically to $\Delta$. Likewise, the map $\rho_2\colon E(\Delta) \to B(\Delta)$ which projects each cell to its second factor is a continuous map that collapses $\Delta'$ to the barycenter of $\Delta$.

We define a map $x\colon B(\Delta) \to X$ on the vertices of $B(\Delta)$ by sending the point $\langle\delta\rangle$ to the point $x_\delta$ for every face $\delta \subset \Delta$. We define $x$ on the rest of $B(\Delta)$ by linear interpolation. That is, if $\delta_0 \subset \dots \subset \delta_k$ is a flag of faces of $\Delta$, then we have $x_{\delta_i} \in x_{\delta_0} + a$ for all $i$. Therefore, all the $x_{\delta_i}$ lie in a common apartment, and we can define $x$ on $\langle \delta_0, \dots, \delta_k\rangle$ by linearly interpolating between the $x_{\delta_i}$. This map has Lipschitz constant $\mathrm{Lip}(x) \lesssim \operatorname{diam} \mathcal{V}(\Delta)$ on $\Delta$.

For any cell

$$\sigma = \delta \times \langle \delta_0, \dots, \delta_k \rangle$$

of $E(\Delta)$ and any $s \in \sigma$, let $x_s = x(\rho_2(s))$. We have $x_s \in x_\delta + a$ and therefore

$$\Omega_\infty(\delta) \subset \mathrm{Lk}_\infty^{\downarrow}(x_\delta) \subset \mathrm{Lk}_\infty^{\downarrow}(x_s).$$

This means that

$$i_{x_s}(\Omega_\infty(\rho_1(s)))$$

is defined for every $s \in \sigma$, so we define

$$\Omega(s) = i_{x_s}(\Omega_\infty(\rho_1(s))).$$

Finally, we check that this definition satisfies the conditions of Lemma 3.2. Since $x_{\langle z \rangle} = z$ for any $z \in Z$, we have $\Omega(\langle z \rangle) = z$, so the first condition is satisfied. Let $\sigma$ be a cell of $E(\Delta)$ as above and let $s, t \in \sigma$. Let $x_s = x(\rho_2(s))$, $x_t = x(\rho_2(t))$. By Lemma 4.19, we have

$$d(\Omega(s), \Omega(t)) \le c d(x_s, x_t) + c h(x_s) d(\Omega_\infty(\rho_1(s)), \Omega_\infty(\rho_1(t))).$$

Since $x_\Delta \in x_{\delta_i} + \mathfrak{a}$ for each $i = 1, \dots, k$, we have $x_\Delta \in x_s + \mathfrak{a}$ and thus $h(x_s) \lesssim \operatorname{diam} \mathcal{V}(\Delta)$. Since $\rho_1$, $\rho_2$, and $\Omega_\infty$ are Lipschitz with constants depending only on $X$ and $\mathrm{Lip}(x|_\Delta) \lesssim \operatorname{diam} \mathcal{V}(\Delta)$, each term in the inequality above is of order at most $\operatorname{diam} \mathcal{V}(\Delta) d(s,t)$. Therefore,

$$\mathrm{Lip}(\Omega_\infty|_\Delta) \lesssim \operatorname{diam} \mathcal{V}(\Delta)$$

for every simplex $\Delta \subset \Delta_Z^{(n-1)}$, as desired.

This proves Lemma 4.2. □

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