Geometry & Topology 18 (2014) 2747–2821

Gromov–Witten/pairs descendent correspondence for toric 3–folds

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We construct a fully equivariant correspondence between Gromov–Witten and stable pairs descendent theories for toric 3–folds X. Our method uses geometric constraints on descendents, $\mathcal{A}_n$ surfaces and the topological vertex. The rationality of the stable pairs descendent theory plays a crucial role in the definition of the correspondence. We prove our correspondence has a non-equivariant limit.

As a result of the construction, we prove an explicit non-equivariant stationary descendent correspondence for X (conjectured by MNOP). Using descendent methods, we establish the relative GW/Pairs correspondence for X/D in several basic new log Calabi–Yau geometries. Among the consequences is a rationality constraint for non-equivariant descendent Gromov–Witten series for $\mathbf{P}^3$.

14N35; 14H60

0 Introduction

0.1 Descendents in Gromov–Witten theory

Let X be a nonsingular projective 3–fold. Gromov–Witten theory is defined via integration over the moduli space of stable maps. Let $\overline{M}_{g,r}(X,\beta)$ denote the moduli space of r–pointed stable maps from connected genus-g curves to X representing the class $\beta \in H_2(X,\mathbb{Z})$. Let

$$\mathrm{ev}_i : \overline{M}_{g,r}(X,\beta) \to X,$$
$$\mathbb{L}_i \to \overline{M}_{g,r}(X,\beta)$$

denote the evaluation maps and the cotangent line bundles associated to the marked points. Let $\gamma_1,\ldots,\gamma_r \in H^*(X,\mathbb{Q})$ and let

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}_{g,n}(X,\beta),\mathbb{Q}).$$

Published: 1 December 2014 DOI: 10.2140/gt.2014.18.2747

The *descendent* fields, denoted by $\tau_k(\gamma)$, correspond to the classes $\psi_i^k \mathrm{ev}_i^*(\gamma)$ on the moduli space of maps. Let

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\rangle_{g,\beta} = \int_{[\overline{M}_{g,r}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i)$$

denote the descendent Gromov–Witten invariants. Foundational aspects of the theory are treated, for example, in [1; 2; 13].

Let C be a possibly disconnected curve with at worst nodal singularities. The genus of C is defined by $1-\chi(\mathcal{O}_C)$. Let $\overline{M}'_{g,r}(X,\beta)$ denote the moduli space of maps with possibly disconnected domain curves C of genus g with *no* collapsed connected components. The latter condition requires each connected component of C to represent a nonzero class in $H_2(X,\mathbb{Z})$. In particular, C must represent a nonzero class β.

We define the descendent invariants in the disconnected case by

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\rangle'_{g,\beta} = \int_{[\overline{M}'_{g,r}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i).$$

The associated partition function is defined by[1]

$$\mathsf{Z}'_{\mathsf{GW}}\biggl(X;u \;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\biggr)_\beta = \sum_{g\in\mathbb{Z}} \Bigl\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Bigr\rangle'_{g,\beta} u^{2g-2}. \tag{1}$$

Since the domain components must map nontrivially, an elementary argument shows the genus g in the sum (1) is bounded from below. The descendent insertions in (1) should match the (genus-independent) virtual dimension,

$$\dim\bigl[\overline{M}'_{g,r}(X,\beta)\bigr]^{\mathrm{vir}} = \int_\beta c_1(T_X) + r. \tag{2}$$

If X is a nonsingular toric 3–fold, then the descendent invariants can be lifted to equivariant cohomology. Let

$$\mathbf{T} = (\mathbb{C}^*)^3$$

be the 3–dimensional algebraic torus acting on X. Let s_1, s_2, s_3 be the equivariant first Chern classes of the standard representations of the three factors of $\mathbf{T}$. The equivariant cohomology of the point is well known to be

$$H^*_{\mathbf{T}}(\bullet) = \mathbb{Q}[s_1, s_2, s_3].$$

[1] Our notation follows [15] and emphasizes the role of the moduli space $\overline{M}'_{g,r}(X,\beta)$. The degree-0 collapsed contributions will not appear anywhere in our paper.

For equivariant classes $\gamma_i \in H^*_{\mathbf{T}}(X, \mathbb{Q})$, the descendent invariants

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle'_{g,\beta} = \int_{[\overline{M}'_{g,r}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \in H^*_{\mathbf{T}}(\bullet)$$

are well-defined. In the equivariant setting, the descendent insertions may exceed the virtual dimension (2). The equivariant partition function

$$\mathsf{Z}'_{\mathsf{GW}}\left(X; u \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\right)^{\mathbf{T}}_{\beta} \in \mathbb{Q}[s_1, s_2, s_3]((u))$$

is a Laurent series in u with coefficients in $H^*_{\mathbf{T}}(\bullet)$.

If X is a nonsingular quasi-projective toric 3–fold, the equivariant Gromov–Witten invariants of X are still well-defined by localization residues [3]. In the quasi-projective case,

$$\mathsf{Z}'_{\mathsf{GW}}\left(X; u \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\right)^{\mathbf{T}}_{\beta} \in \mathbb{Q}(s_1, s_2, s_3)((u)).$$

For the study of the Gromov–Witten theory of toric 3–folds, the open geometries play an important role.

0.2 Descendents in the theory of stable pairs

Let X be a nonsingular projective 3–fold, and let $\beta \in H_2(X, \mathbb{Z})$ be a nonzero class. We consider next the moduli space of stable pairs

$$\left[\mathcal{O}_X \xrightarrow{s} F\right] \in P_n(X, \beta),$$

where F is a pure sheaf supported on a Cohen–Macaulay subcurve of X, s is a morphism with 0–dimensional cokernel, and

$$\chi(F) = n, \quad [F] = \beta.$$

The space $P_n(X, \beta)$ carries a virtual fundamental class obtained from the deformation theory of complexes in the derived category [30].

Since $P_n(X, \beta)$ is a fine moduli space, there exists a universal sheaf

$$\mathbb{F} \to X \times P_n(X, \beta);$$

see [30, Section 2.3]. For a stable pair $[\mathcal{O}_X \to F] \in P_n(X, \beta)$, the restriction of $\mathbb{F}$ to the fiber

$$X \times [\mathcal{O}_X \to F] \subset X \times P_n(X, \beta)$$

is canonically isomorphic to F. Let

$$\pi_X\colon X \times P_n(X,\beta) \to X,$$
$$\pi_P\colon X \times P_n(X,\beta) \to P_n(X,\beta)$$

be the projections onto the first and second factors. Since X is nonsingular and $\mathbb{F}$ is π_P–flat, $\mathbb{F}$ has a finite resolution by locally free sheaves. Hence the Chern character of the universal sheaf $\mathbb{F}$ on $X \times P_n(X,\beta)$ is well-defined. By definition, the operation

$$\pi_{P*}\big(\pi_X^*(\gamma)\cdot \mathrm{ch}_{2+i}(\mathbb{F}) \cap \pi_P^*(\,\cdot\,)\big)\colon H_*(P_n(X,\beta)) \to H_*(P_n(X,\beta))$$

is the action of the descendent $\tau_i(\gamma)$, where $\gamma \in H^*(X,\mathbb{Z})$.

For nonzero $\beta \in H_2(X,\mathbb{Z})$ and arbitrary $\gamma_i \in H^*(X,\mathbb{Q})$, define the stable pairs invariant with descendent insertions by

$$\begin{aligned}\big\langle \tau_{k_1}(\gamma_1)\dots\tau_{k_r}(\gamma_r)\big\rangle_{n,\beta} &= \int_{[P_n(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r \tau_{k_i}(\gamma_i)\\ &= \int_{P_n(X,\beta)} \prod_{i=1}^r \tau_{k_i}(\gamma_i)\big([P_n(X,\beta)]^{\mathrm{vir}}\big).\end{aligned}$$

The partition function is

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q \;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta = \sum_n \Big\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big\rangle_{n,\beta} q^n.$$

Since $P_n(X,\beta)$ is empty for sufficiently negative n, the partition function is a Laurent series in q. The following conjecture was made in [29].

Conjecture 1 *The partition function $\mathsf{Z}_{\mathsf{P}}(X;q \mid \prod_{i=1}^r \tau_{k_i}(\gamma_i))_\beta$ is the Laurent expansion of a rational function in q.*

Let X be a nonsingular quasi-projective toric 3–fold. The stable pairs descendent invariants can be lifted to equivariant cohomology (and defined by residues in the open case). For equivariant classes $\gamma_i \in H^*_{\mathbf{T}}(X,\mathbb{Q})$, we see that

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q \;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)^{\mathbf{T}}_\beta \in \mathbb{Q}(s_1,s_2,s_3)((q))$$

is a Laurent series in q with coefficients in $H^*_{\mathbf{T}}(\bullet)$. A central result of [27; 28] is the following rationality property.

Rationality *Let X be a nonsingular quasi-projective toric 3–fold. The partition function*

$$\mathsf{Z}_P\Big(X;q \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_\beta^{\mathbf{T}}$$

is the Laurent expansion in q of a rational function in the field $\mathbb{Q}(q,s_1,s_2,s_3)$.

The above rationality result implies Conjecture 1 when X is a nonsingular projective toric 3–fold. The corresponding statement for the equivariant Gromov–Witten descendent partition function is expected (from calculational evidence) to be false.

0.3 Correspondence

Let X be a nonsingular quasi-projective toric 3–fold, and let $p_1,\dots,p_m \in X$ be the distinct $\mathbf{T}$–fixed points. Let $\mathsf{p}_j \in H^*_{\mathbf{T}}(X,\mathbb{Q})$ be the class of the $\mathbf{T}$–fixed point p_j. Let α be a partition,

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\ell > 0),$$

of size[2] $|\alpha|$ and length ℓ. Define the descendent insertion

$$\tau_\alpha(\mathsf{p}_j) = \tau_{\alpha_1-1}(\mathsf{p}_j)\tau_{\alpha_2-1}(\mathsf{p}_j)\cdots\tau_{\alpha_\ell-1}(\mathsf{p}_j). \tag{3}$$

Since the classes of the $\mathbf{T}$–fixed points span a basis of localized equivariant cohomology

$$H^*_{\mathbf{T}}(X,\mathbb{Q}) \otimes \mathbb{Q}(s_1,s_2,s_3),$$

we can consider equivariant descendents of X in the form

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q \;\Big|\; \prod_{j=1}^{m} \tau_{\alpha^{(j)}}(\mathsf{p}_j)\Big)_\beta^{\mathbf{T}}, \quad \mathsf{Z}'_{\mathsf{GW}}\Big(X;u \;\Big|\; \prod_{j=1}^{m} \tau_{\alpha^{(j)}}(\mathsf{p}_j)\Big)_\beta^{\mathbf{T}}$$

for partitions $\alpha^{(1)},\dots,\alpha^{(m)}$ associated to the $\mathbf{T}$–fixed points.

A central result of the paper is the construction of a universal correspondence matrix K indexed by partitions α and $\widehat{\alpha}$ of positive size with[3]

$$\mathsf{K}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_3]((u))$$

and $\mathsf{K}_{\alpha,\widehat{\alpha}} = 0$ unless $|\alpha| \geq |\widehat{\alpha}|$. The coefficients $\mathsf{K}_{\alpha,\widehat{\alpha}}$ are *symmetric* in the variables w_i. The matrix K is used to define a correspondence rule

$$\tau_\alpha(\mathsf{p}_j) \mapsto \widehat{\tau}_\alpha(\mathsf{p}_j) = \sum_{|\alpha|\geq|\widehat{\alpha}|} \mathsf{K}_{\alpha,\widehat{\alpha}}(w_1^j,w_2^j,w_3^j)\tau_{\widehat{\alpha}}(\mathsf{p}_j),$$

[2]The unique partition of size 0 is the empty partition of length $\ell = 0$. In the empty case, $\tau_\varnothing(\mathsf{p}_j) = 1$.

[3]As usual, $i^2 = -1$.

where w_1^j, w_2^j, w_3^j are the tangent weights of X at p_j. The symmetry of K in the variables w_i is required for the correspondence rule to be well-defined. If $\alpha = \varnothing$, we formally set

$$\widehat{\tau}_{\varnothing}(\mathsf{p}_j) = \widehat{1} = 1.$$

To state the correspondence property of K, the basic degree

$$d_\beta = \int_\beta c_1(X) \in \mathbb{Z}$$

associated to the class $\beta \in H_2(X, \mathbb{Z})$ will be required.

Theorem 1 *There exists a universal correspondence matrix* K *(symmetric in the variables* w_i*) satisfying*

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\Big(X; q \;\Big|\; \prod_{j=1}^m \tau_{\alpha^{(j)}}(\mathsf{p}_j)\Big)^{\mathbf{T}}_\beta = (-iu)^{d_\beta} \mathsf{Z}'_{\mathsf{GW}}\Big(X; u \;\Big|\; \prod_{j=1}^m \widehat{\tau}_{\alpha^{(j)}}(\mathsf{p}_j)\Big)^{\mathbf{T}}_\beta$$

under the variable change $-q = e^{iu}$ *for all nonsingular quasi-projective toric 3–folds* X.

The variable change in the descendent correspondence is well-defined by the rationality result for the stable pairs partition function. However, much of the u dependence of K remains mysterious.[4] A central point of the paper is to show the consequences which can be derived from various accessible properties of the u dependence.

We will construct the matrix K from the study of 1–leg equivariant descendent invariants. A geometric argument using capped descendent vertices following [27] is used to prove the 2–leg and then the complete 3–leg result of Theorem 1. The argument uses the full force of the equivariant Gromov–Witten/Pairs correspondence for primary fields in [18; 20].

Along with the construction of K, we prove several basic properties. A uniqueness statement for K in the context of capped vertices appears in Theorem 8 of Section 1. The leading terms of K are determined by the following result.

Theorem 2 *For partitions* α *of positive size,* $\mathsf{K}_{\alpha,\alpha} = (iu)^{\ell(\alpha)-|\alpha|}$ *and*

$$\mathsf{K}_{\alpha,\widehat{\alpha}\neq\alpha} = 0 \quad \textit{if} \quad |\alpha| \le |\widehat{\alpha}| + |\ell(\alpha) - \ell(\widehat{\alpha})|.$$

[4]Conjectural formulas for a partial descendent correspondence between Gromov–Witten theory and the Donaldson–Thomas theory of ideal sheaves are proposed in a forthcoming article by Oblomkov, Okounkov and the first author [21]. The investigation of the relationship between descendents for stable pairs and ideal sheaves is an interesting direction for further study. Though not fully equivariant, the formulas of [21] should partially constrain K.

In other words, we can write the correspondence as

$$\widehat{\tau}_\alpha(\mathsf{p}) = (iu)^{\ell(\alpha)-|\alpha|}\tau_\alpha(\mathsf{p}) + \cdots,$$

where the dots stand for terms $\tau_{\widehat{\alpha}}$ with partitions $\widehat{\alpha}$ of positive size satisfying

$$|\alpha| > |\widehat{\alpha}| + |\ell(\alpha) - \ell(\widehat{\alpha})|.$$

Theorem 2, proven in Section 2, plays an important role in the applications. We prove the u coefficients of $\mathsf{K}_{\alpha,\widehat{\alpha}}$ are symmetric polynomials in the variables w_i in Section 4.

Theorem 3 *The u coefficients of $\mathsf{K}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_3]((u))$ are symmetric and homogeneous in the variables w_i of degree $|\alpha| + \ell(\alpha) - |\widehat{\alpha}| - \ell(\widehat{\alpha})$.*

0.4 Consequences

We derive several implications of our descendent correspondence which require only basic properties of K.

A first consequence is the following result for the non-equivariant partition functions with primary fields $\tau_0(\gamma)$ and stationary descendents $\tau_k(\mathsf{p})$.

Theorem 4 *Let X be a nonsingular projective toric 3–fold. After the variable change $-q = e^{iu}$, we have*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X;q \;\Big|\; \prod_{i=1}^{r}\tau_0(\gamma_i)\prod_{j=1}^{s}\tau_{k_j}(\mathsf{p})\Big)_\beta = (-iu)^{d_\beta}(iu)^{-\sum k_j}\mathsf{Z}'_{\mathsf{GW}}\Big(X;u \;\Big|\; \prod_{i=1}^{r}\tau_0(\gamma_i)\prod_{j=1}^{s}\tau_{k_j}(\mathsf{p})\Big)_\beta,$$

where $\gamma_i \in H^*(X,\mathbb{Q})$ are classes of positive degree.

Theorem 4 was conjectured for arbitrary nonsingular projective 3–folds in [15] for the Donaldson–Thomas theory of ideal sheaves. Our proof, via Theorem 1, uses only the leading terms of the u dependence of correspondence matrix K. The non-equivariant limit plays an important role in the simple form of the descendent correspondence in Theorem 4. If fully $\mathbf{T}$–equivariant partition functions are considered, then complete knowledge of the matrix K is required.

By Theorem 4 and the rationality result for stable pairs descendents, we conclude that

$$e^{-iud_\beta/2}\cdot(-iu)^{d_\beta}(iu)^{-\sum k_j}\mathsf{Z}'_{\mathsf{GW}}\Big(X;u \;\Big|\; \prod_{i=1}^{r}\tau_0(\gamma_i)\prod_{j=1}^{s}\tau_{k_j}(\mathsf{p})\Big)_\beta$$

is a rational function of e^{-iu}. We know no other approaches to such rationality results for descendents in Gromov–Witten theory.

In a different direction, we can also prove the Gromov–Witten/Pairs correspondences for primary fields in several new relative cases. The first is a non-equivariant log Calabi–Yau geometry with the relative divisor given by a $K3$ surface.

Theorem 5 *Let X be a nonsingular projective Fano toric 3–fold, and let $K3 \subset X$ be a nonsingular anti-canonical $K3$ surface. After the variable change $-q = e^{iu}$, we have*

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\big(X/K3; q \mid \tau_0(\gamma_1)\cdots\tau_0(\gamma_r) \mid \mu\big)_\beta = (-iu)^{d_\beta+\ell(\mu)-|\mu|} \mathsf{Z}'_{\mathsf{GW}}\big(X/K3; u \mid \tau_0(\gamma_1)\cdots\tau_0(\gamma_r) \mid \mu\big)_\beta,$$

where $\gamma_i \in H^*(X,\mathbb{Q})$ are arbitrary classes.

Relative Gromov–Witten and stable pairs theory are reviewed in Section 1.1. Our standard conventions for the boundary conditions μ along the relative divisors are explained there. The rationality of the stable pairs series of Theorem 5 has been established earlier in Section 4 of [27]. Theorem 5 can be used to prove a rationality constraint for the Gromov–Witten descendent series of $\mathbf{P}^3$.

Let $\mathbb{Q}(-q,i)[u,\frac{1}{u}]$ be the ring of Laurent polynomials in u with coefficients given by rational functions in $-q$ over $\mathbb{Q}[i]$. For example

$$\frac{q-1/q}{2i}u^{-2} + \frac{q+1/q}{2}u^4 \in \mathbb{Q}(-q,i)[u,1/u].$$

Corollary 3 *For the non-equivariant descendent series, we have*

$$\mathsf{Z}'_{\mathsf{GW}}\left(\mathbf{P}^3; u \;\Bigg|\; \prod_{j=1}^{s} \tau_{k_j}(\gamma_j)\right)_\beta \in \mathbb{Q}(-q=e^{iu},i)[u,1/u],$$

where $\gamma_j \in H^*(\mathbf{P}^3,\mathbb{Q})$ are classes of positive degree.

Let $\mathcal{A}_n$ be the minimal toric resolution of the standard A_n–singularity obtained by a quotient of a cyclic $\mathbb{Z}_{n+1}$–action on $\mathbb{C}^2$, see Section 4.2 for a review. Consider the $(\mathbb{C}^*)^2$–equivariant geometry relative geometry

$$\mathcal{A}_n \times \mathbf{P}^1/D = \mathcal{A}_n \times \mathbf{P}^1/(\mathcal{A}_n)_{x_1} \cup (\mathcal{A}_n)_{x_2} \cup (\mathcal{A}_n)_{x_3}$$

relative to the fibers over the distinct point $x_1,x_2,x_3 \in \mathbf{P}^1$. Here $(\mathbb{C}^*)^2$ acts only on the toric surface $\mathcal{A}_n$. We prove the following result.

Theorem 6 *After the variable change $-q = e^{iu}$, we have*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\big(\mathcal{A}_n\times\mathbf{P}^1/D;q\mid\mu^{(1)},\mu^{(2)},\mu^{(3)}\big)_\beta^{(\mathbb{C}^*)^2}$$
$$= (-iu)^{d_\beta+\sum_{i=1}^k \ell(\mu^{(i)})-|\mu^{(i)}|}\mathsf{Z}'_{\mathsf{GW}}\big(\mathcal{A}_n\times\mathbf{P}^1/D;u\mid\mu^{(1)},\mu^{(2)},\mu^{(3)}\big)_\beta^{(\mathbb{C}^*)^2},$$

where the $\mu^{(i)}$ are arbitrary $(\mathbb{C}^*)^2$–equivariant relative conditions along the fibers $(\mathcal{A}_n)_{x_i}$.

Theorem 6 resolves questions left open in [16; 17]. In fact, Theorem 6 would follow from the results of [16; 17] if certain conjectured invertibilities were established; see [16, Section 8.3]. Our proof of the 3–point Gromov–Witten/Pairs correspondence for $\mathcal{A}_n$–local curves completely bypasses such invertibility issues.

The results stated above are the first applications of the equivariant descendent correspondence. The main application will be to establish the Gromov–Witten/Pairs correspondence for several basic families of compact Calabi–Yau 3–folds. The strategy is to follow the methods of [19] which determine the Gromov–Witten theory of the quintic 3–fold

$$X_5 \subset \mathbf{P}^4$$

and to take parallel geometric steps for the stable pairs theory. A non-equivariant Gromov–Witten descendent correspondence is necessary for the argument.

A basic result of the present paper is a non-equivariant formulation of the Gromov–Witten/pairs descendent correspondence. The application to compact Calabi–Yau 3–folds will be taken up in [26].

0.5 Non-equivariant limit

Let X be a nonsingular quasi-projective toric 3–fold with $\mathbf{T}$–fixed points $p_1,\dots,p_m$ and inclusions

$$\iota_j : p_j \hookrightarrow X, \quad \iota : X^{\mathbf{T}} \hookrightarrow X.$$

The pull-back of the top Chern class of the tangent bundle,

$$\iota_j^*(c_3(T_X)) = e(\mathrm{Tan}_j) \in H^*_{\mathbf{T}}(p_j,\mathbb{Q}),$$

is the Euler class of the tangent representation at p_j.

Theorem 1 establishes a descendent correspondence for $\mathbf{T}$–equivariant Gromov–Witten and stable pairs theories. Does Theorem 1 define a correspondence for non-equivariant theories? Certainly every non-equivariant descendent $\tau_k(\gamma)$ can be lifted to a combination of $\mathbf{T}$–equivariant descendents of the form $\tau_k(\mathsf{p}_j)$ by localization

$$\widetilde{\gamma} = \iota_* \sum_{j=1}^{m} \frac{\iota_j^*(\widetilde{\gamma})}{\iota_j^*(c_3(T_X))} \mathsf{p}_j, \tag{4}$$

where $\widetilde{\gamma}$ is any $\mathbf{T}$–equivariant lift of γ. Theorem 1 can then be applied. However, the coefficients

$$\frac{\iota_j^*(\widetilde{\gamma})}{e(c_3(T_X))} \in \mathbb{Q}(s_1, s_2, s_3)$$

which appear on the right of (4) are rational functions of the s_i (almost always with poles). After the application of Theorem 1, poles in s_i will occur on the Gromov–Witten side of the correspondence. Whether the resulting combination of Gromov–Witten invariants can be rewritten in non-equivariant terms is not immediately clear. The outcome depends upon properties of the correspondence matrix K.

We prove a Gromov–Witten/Pairs descendent correspondence for X which admits a non-equivariant limit. In order to state the answer, we will require the following notation. Let $\widehat{\alpha}$ be a partition of length $\widehat{\ell}$ as in Section 0.3. Let Δ be the cohomology class of the small diagonal in the product $X^{\widehat{\ell}}$. For a cohomology class γ of X, let

$$\gamma \cdot \Delta = \sum_{j_1,\dots,j_{\widehat{\ell}}} \theta_{j_1}^{\gamma} \otimes \cdots \otimes \theta_{j_{\widehat{\ell}}}^{\gamma}$$

be the Künneth decomposition of $\gamma \cdot \Delta$ in the cohomology of $X^{\widehat{\ell}}$. We define the descendent insertion $\tau_{\widehat{\alpha}}(\gamma)$ by

$$\tau_{\widehat{\alpha}}(\gamma) = \sum_{j_1,\dots,j_{\widehat{\ell}}} \tau_{\widehat{\alpha}_1 - 1}(\theta_{j_1}^{\gamma}) \cdots \tau_{\widehat{\alpha}_{\widehat{\ell}} - 1}(\theta_{j_{\widehat{\ell}}}^{\gamma}). \tag{5}$$

For example, if γ is the class of a point, then

$$\tau_{\widehat{\alpha}}(\mathsf{p}) = \tau_{\widehat{\alpha}_1 - 1}(\mathsf{p}) \cdots \tau_{\widehat{\alpha}_{\widehat{\ell}} - 1}(\mathsf{p})$$

in accordance with convention (3). Definition (5) is valid for both the standard and the $\mathbf{T}$–equivariant cohomology of X.

We construct a second correspondence matrix $\widetilde{\mathsf{K}}$ indexed by partitions α and $\widehat{\alpha}$ of positive size with

$$\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u))$$

and $\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} = 0$ unless $|\alpha| \geq |\widehat{\alpha}|$. Via the substitution

$$c_i = c_i(T_X),$$

the elements of $\widetilde{\mathsf{K}}$ act on the cohomology (both standard and $\mathbf{T}$–equivariant) of X with $\mathbb{Q}[i]$–coefficients. Of course, we take the canonical lift of $\mathbf{T}$ to the tangent bundle T_X in the equivariant case.

The matrix $\widetilde{\mathsf{K}}$ is used to define a new correspondence rule

$$\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \mapsto \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}. \tag{6}$$

The formula for the right side of (6) requires a sum over all set partitions P of $\{1,\dots,\ell\}$. For such a set partition P, each element $S\in P$ is a subset of $\{1,\dots,\ell\}$. Let α_S be the associated subpartition of α, and let

$$\gamma_S = \prod_{i\in S}\gamma_i.$$

We define the right side of (6) by

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{\substack{P \text{ set partition of} \\ \{1,\dots,\ell\}}} \prod_{S\in P}\sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}\big(\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}}\cdot\gamma_S\big). \tag{7}$$

Theorem 7 *There exists a universal correspondence matrix $\widetilde{\mathsf{K}}$ satisfying*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\big(X;q \mid \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\big)^{\mathbf{T}}_\beta = (-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\big(X;u \mid \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}\big)^{\mathbf{T}}_\beta$$

under the variable change $-q=e^{iu}$ for all nonsingular quasi-projective toric 3–folds X.

We prove Theorem 7 by constructing $\widetilde{\mathsf{K}}$ canonically from K. Divisibility properties of the coefficients of K, required for the construction of $\widetilde{\mathsf{K}}$, are proven geometrically. From our construction of $\widetilde{\mathsf{K}}$, we will see Theorem 7 specializes to Theorem 1. The main advantage of Theorem 7 over Theorem 1 is the obvious existence of a non-equivariant limit. In fact, since (7) makes sense for the standard cohomology of *any* nonsingular projective 3–fold, we conjecture the following.

Conjecture 2 *For any nonsingular projective 3–fold X, we have*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\big(X;q \mid \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\big)_\beta = (-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\big(X;u \mid \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}\big)_\beta$$

under the variable change $-q=e^{iu}$.

By Conjecture 1, the stable pairs descendent series on the left is expected to be a rational function in q, so the change of variables is well-defined. Conjecture 2 is a consequence of Theorem 7 in case X is toric by taking the non-equivariant limit. In the non-toric case, Conjecture 2 predicts the correspondence is the same.

Formula (7) assumes all the cohomology classes γ_j are even. In the presence of odd cohomology, a natural sign must be included in (7). We may write set partitions P of $\{1,\dots,\ell\}$ indexing the sum on the right side of (7) as

$$S_1 \cup \ldots \cup S_{|P|} = \{1,\dots,\ell\}.$$

The parts S_i of P are unordered, but we choose an ordering for each P. We then obtain a permutation of $\{1,\dots,\ell\}$ by placing the elements in the ordered parts S_i (and respecting the original order in each part). The permutation determines a sign $\sigma(P)$ by the anti-commutation of the associated odd classes. We then write

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{\substack{P \text{ set partition of} \\ \{1,\dots,\ell\}}} (-1)^{\sigma(P)} \prod_{S_i \in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}\big(\widetilde{\mathsf{K}}_{\alpha_{S_i},\widehat{\alpha}} \cdot \gamma_{S_i}\big).$$

0.6 Plan of the paper

We start in Section 1 by reviewing relative Gromov–Witten and stable pairs theories. The capped descendent vertex of [27], defined in Section 1.3, will play a central role in the construction of the correspondence matrix K. Theorem 1 is implied by a relative descendent correspondence stated in Theorem 8 of Section 1.3.

To construct K, we proceed leg by leg for capped descendent vertices. The study of the 1–leg case in Section 2 uniquely determines K by the invertibility of the associated descendent/relative matrices. We define K via the 1–leg geometry. After the proof of the 1–leg descendent correspondence in Section 2.3, the initial terms of K are calculated in Section 2.4 to prove Theorem 2. The symmetry between the variables s_i is broken in the 1–leg geometry, so the symmetry of K is not immediate.

We review the technique of capped localization in Section 3. The capped descendent correspondence in the 2–leg case is established in Section 4 using the geometry of $\mathcal{A}_1$ surfaces (a strategy already employed in [18; 27]). Crucial here is a new invertibility proven in Section 4.5. A consequence of the 2–leg correspondence is the symmetry of K in the variables w_i proven in Section 4.7. The 3–leg case is obtained by a parallel argument in Section 5, completing the proof of Theorem 8 and thus of Theorem 1.

The first applications of the descendent correspondence are taken up in Section 6. The easiest is Theorem 6 proven in Section 6.2. After a further study of the initial terms of K, we prove Theorem 4 in Section 6.6.

Section 7 concerns the non-equivariant formulation of the descendent correspondence. After delicate divisibility properties for the coefficients of K are established, the formula for $\widetilde{\mathsf{K}}$ in terms of K is given in Section 7.3. Theorem 7 is then a consequence of Theorem 1. The final applications of the paper, in Section 8 to log Calabi–Yau geometries, require the non-equivariant correspondence. Theorem 5 is proven in Section 8.3.

0.7 Acknowledgements

Discussions with J Bryan, D Maulik, A Oblomkov, A Okounkov and R Thomas about Gromov–Witten theory, stable pairs, and descendent invariants played an important role. The study of descendents for 3–fold sheaf theories in [15; 29] motivated several aspects of the paper.

R P was partially supported by NSF grants DMS-0500187 and DMS-1001154, SNF grant 200021-143274 and ERC grant 2012-AdG-320368-MCSK. A P was supported by a NDSEG graduate fellowship. The paper was started in the summer of 2011 while visiting the Instituto Superior Técnico in Lisbon where R P was supported by a Marie Curie fellowship and a grant from the Gulbenkian foundation. Much of the work was done during visits of A P to the Forschungsinstitut für Mathematik at ETH Zürich during the 2011–2012 year.

1 Capped descendent vertex

1.1 Relative theories

Let $D \subset X$ be a nonsingular divisor. Relative Gromov–Witten and relative stable pairs theories enumerate curves with specified tangency to the divisor D. See [15; 28] for a technical discussion of relative theories.

In Gromov–Witten theory, relative conditions are represented by a partition μ of the integer $\int_\beta [D]$, each part μ_i of which is marked by a cohomology class $\gamma_i \in H^*(D,\mathbb{Z})$. The numbers μ_i record the multiplicities of intersection with D while the cohomology labels γ_i record where the tangency occurs. More precisely, let $\overline{M}'_{g,r}(X/D,\beta)_\mu$ be the moduli space of stable relative maps with tangency conditions μ along D. To impose the full boundary condition, we pull-back the product $\prod_i \gamma_i$ via the evaluation maps

$$\overline{M}'_{g,r}(X/D,\beta)_\mu \to D$$

at the points of tangency. By convention, an absent cohomology label stands for $1 \in H^*(D,\mathbb{Z})$. Also, the tangency points are considered to be unordered.

In the stable pairs theory, the relative moduli space admits a natural morphism to the Hilbert scheme of d points in D,

$$P_n(X/D,\beta) \to \mathrm{Hilb}\Big(D, \int_\beta [D]\Big).$$

Cohomology classes on $\mathrm{Hilb}(D, \int_\beta [D])$ may thus be pulled-back to the relative moduli space. We will work in the *Nakajima basis* of $H^*(\mathrm{Hilb}(D, \int_\beta [D]), \mathbb{Q})$ indexed by a partition μ of $\int_\beta [D]$ labeled by cohomology classes of D. For example, the class

$$|\mu\rangle \in H^*\Big(\mathrm{Hilb}\Big(D, \int_\beta [D]\Big), \mathbb{Q}\Big),$$

with all cohomology labels equal to the identity, is $\prod \mu_i^{-1}$ times the Poincaré dual of the closure of the subvariety formed by unions of schemes of length

$$\mu_1, \dots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points of D.

The conjectural relative GW/Pairs correspondence for primary fields [15] equates the partition functions of the theories.

Conjecture 3 *We have*

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\big(X/D; q \mid \tau_0(\gamma_1)\cdots\tau_0(\gamma_r) \mid \mu\big)_\beta \\ = (-iu)^{d_\beta+\ell(\mu)-|\mu|} \mathsf{Z}'_{\mathsf{GW}}\big(X/D; u \mid \tau_0(\gamma_1)\cdots\tau_0(\gamma_r) \mid \mu\big)_\beta,$$

after the change of variables $e^{iu} = -q$.

As before, $\mathsf{Z}_{\mathsf{P}}(X/D; q \mid \tau_0(\gamma_1)\cdots\tau_0(\gamma_r) \mid \mu)_\beta$ is conjectured to be a rational function of q.

1.2 Degeneration formulas

Relative theories satisfy degeneration formulas. Let

$$\mathfrak{X} \to B$$

be a nonsingular 4–fold fibered over an irreducible and nonsingular base curve B. Let X be a nonsingular fiber, and let

$$X_1 \cup_D X_2$$

be a reducible special fiber consisting of two nonsingular 3–folds intersecting transversally along a nonsingular surface D.

If all insertions $\gamma_1, \dots, \gamma_r$ lie in the image of

$$H^*(X_1 \cup_D X_2, \mathbb{Z}) \to H^*(X, \mathbb{Z}),$$

the degeneration formula in Gromov–Witten theory takes the form [8; 10; 12]

$$\begin{aligned}(8)\quad & \mathsf{Z}'_{\mathsf{GW}}\big(X \mid \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\big)_\beta \\ &= \sum \mathsf{Z}'_{\mathsf{GW}}\big(X_1 \mid \cdots \mid \mu\big)_{\beta_1} \mathfrak{z}(\mu) u^{2\ell(\mu)} \mathsf{Z}'_{\mathsf{GW}}\big(X_2 \mid \cdots \mid \mu^\vee\big)_{\beta_2},\end{aligned}$$

where the summation is over all curve splittings $\beta = \beta_1 + \beta_2$, all splittings of the insertions $\tau_{k_i}(\gamma_i)$, and all relative conditions μ.

In (8), the cohomological labels of $\mu^\vee$ are Poincaré duals of the labels of μ. The gluing factor $\mathfrak{z}(\mu)$ is the order of the centralizer in the symmetric group $S(|\mu|)$ of an element with cycle type μ.

The degeneration formula in the stable pairs theory takes a very similar form,

$$\begin{aligned}&\mathsf{Z}_{\mathsf{P}}\big(X \mid \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\big)_\beta \\ &\quad= \sum \mathsf{Z}_{\mathsf{P}}\big(X_1 \mid \cdots \mid \mu\big)_{\beta_1} (-1)^{|\mu|-\ell(\mu)} \mathfrak{z}(\mu) q^{-|\mu|} \mathsf{Z}_{\mathsf{P}}\big(X_2 \mid \cdots \mid \mu^\vee\big)_{\beta_2};\end{aligned}$$

see [15; 28]. The sum over the relative conditions μ is interpreted as the coproduct of 1,

$$\Delta 1 = \sum_\mu (-1)^{|\mu|-\ell(\mu)} \mathfrak{z}(\mu)\, |\mu\rangle \otimes |\mu^\vee\rangle,$$

in the tensor square of $H^*(\mathrm{Hilb}(D, \int_\beta [D]), \mathbb{Z})$. Conjecture 3 is easily seen to be compatible with degeneration.

1.3 Definition of the capped vertex

Bare capped vertices were first considered in [18] in the context of Donaldson–Thomas theory. The capped descendent vertex was introduced in [27] to prove the rationality of the stable pairs theory of toric 3–folds. Capped vertices will play a basic role in the construction and proof of the descendent correspondence. We review the definitions here.

Let $\mathbf{T}$ be a 3–dimensional algebraic torus. As before, let $s_1, s_2, s_3 \in H^*_{\mathbf{T}}(\bullet)$ be the first Chern classes of the standard representations of the three factors of $\mathbf{T}$. Let $\mathbf{T}$ act diagonally on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$,

$$(\xi_1, \xi_2, \xi_3) \cdot ([x_1, y_1], [x_2, y_2], [x_3, y_3]) = ([x_1, \xi_1 y_1], [x_2, \xi_2 y_2], [x_3, \xi_3 y_3]).$$

Let $0, \infty \in \mathbf{P}^1$ be the points $[1,0]$ and $[0,1]$ respectively. The tangent weights[5] of $\mathbf{T}$ at the point

$$\mathsf{p} = (0,0,0) \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

are s_1, s_2 and s_3.

Let $U \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ be the $\mathbf{T}$–invariant 3–fold obtained by removing the three $\mathbf{T}$–invariant lines

$$L_1, L_2, L_3 \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

passing through the point (∞, ∞, ∞),

$$U = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \setminus \bigcup_{i=1}^{3} L_i. \tag{9}$$

Let $D_i \subset U$ be the divisor with i^{th} coordinate ∞. For $i \neq j$, the divisors D_i and D_j are disjoint in U.

The capped descendent vertex is a partition function of U with integrand

$$\tau_\alpha(\mathsf{p}) = \tau_{\alpha_1 - 1}(\mathsf{p}) \cdots \tau_{\alpha_\ell - 1}(\mathsf{p})$$

and free relative conditions imposed at the divisors D_i. While the relative geometry $U/\bigcup_i D_i$ is not compact, the moduli spaces $P_n(U/\bigcup_i D_i, \beta)$ have compact $\mathbf{T}$–fixed loci. The stable pairs invariants of $U/\bigcup_i D_i$ are well-defined by $\mathbf{T}$–equivariant residues. In the localization formula for the residue theories of $U/\bigcup_i D_i$, nonzero degrees can occur *only* on the edges meeting the origin $\mathsf{p} \in U$.

We denote the capped descendent vertex by

$$\mathsf{C}\big(\tau_\alpha(\mathsf{p}) \mid \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big) = \mathsf{Z}\Big(U/\bigcup_i D_i, \prod_{i=1}^{\ell} \tau_{\alpha_i - 1}(\mathsf{p}) \,\Big|\, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\Big)_\beta^{\mathbf{T}}, \tag{10}$$

where the partition α specifies the descendent integrand and the partitions $\lambda^{(1)}, \lambda^{(2)}$, $\lambda^{(3)}$ denote relative conditions imposed at D_1, D_2, D_3. While α must be non-empty as before, the partitions $\lambda^{(i)}$ are permitted to be empty. However, we require the condition

$$|\lambda^{(1)}| + |\lambda^{(2)}| + |\lambda^{(3)}| > 0 \tag{11}$$

to hold. The number of legs of the vertex is the number of non-empty partitions among $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$.

The curve class β in (10) is determined by the relative conditions: β is the sum of the three axes passing through $\mathsf{p} \in U$ with coefficients $|\lambda^{(1)}|$, $|\lambda^{(2)}|$ and $|\lambda^{(3)}|$

[5] Our sign conventions here follow [25] and disagree with [28].

respectively. The superscript $\mathbf{T}$ after the bracket denotes $\mathbf{T}$–equivariant integration on $P_n(U/\bigcup_i D_i, \beta)$. The condition (11) implies β is nonzero.

The above definition is valid for both Gromov–Witten and stable pairs theories. The relative conditions are interpreted as tangency in Gromov–Witten theory and as element of the Nakajima basis in the theory of stable pairs. We denote the vertices in the two theories by

$$\mathsf{C}_{\mathsf{P}}\big(\tau_\alpha(\mathsf{p}) \mid \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big), \quad \mathsf{C}_{\mathsf{GW}}\big(\tau_\alpha(\mathsf{p}) \mid \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big).$$

We will prove Theorem 1 by a refined correspondence result for the capped descendent vertex.

Theorem 8 *There exists a unique correspondence matrix* K *satisfying*

$$(-q)^{-\sum_i |\lambda^{(i)}|}\mathsf{C}_{\mathsf{P}}\big(\tau_\alpha(\mathsf{p}) \mid \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big) \\ = (-iu)^{\sum_i |\lambda^{(i)}|+\ell(\lambda^{(i)})}\mathsf{C}_{\mathsf{GW}}\big(\widehat{\tau}_\alpha(\mathsf{p}) \mid \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big)$$

under the variable change $-q = e^{iu}$.

In case no descendents are present, the basic equality of the equivariant capped vertices

$$(-q)^{-\sum_i |\lambda^{(i)}|}\mathsf{C}_{\mathsf{P}}\big(1 \mid \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big) = (-iu)^{\sum_i |\lambda^{(i)}|+\ell(\lambda^{(i)})}\mathsf{C}_{\mathsf{GW}}\big(1 \mid \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big)$$

is the main result of [18]. The number of legs of the descendent vertex refers to the number of non-empty partitions among $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$.

By capped localization discussed in Section 3, we will easily derive Theorem 1 from Theorem 8. The proof of Theorem 8 will be given leg by leg starting with the 1–leg case.

2 Descendent correspondence: 1–leg

2.1 Construction of K

We construct the matrix K in several steps. The first is very simple. Let $d > 0$ be an integer, and let $\mathcal{P}_d$ be the set of partitions of positive size at most d. Let

$$\mathsf{C}^d_{\mathsf{P}}(\alpha, \lambda) = \mathsf{C}_{\mathsf{P}}(\tau_\alpha(\mathsf{p}) \mid \varnothing, \varnothing, \lambda), \quad \alpha, \lambda \in \mathcal{P}_d$$

be a matrix with both rows and columns indexed by $\mathcal{P}_d$. The coefficients of $\mathsf{C}^d_{\mathsf{P}}$ are rational functions in q.

Lemma 1 *For all $d > 0$, the matrix $\mathsf{C}^d_{\mathsf{P}}$ is invertible.*

Similarly, we define the corresponding matrix using the 1–leg Gromov–Witten descendent vertices,

$$\mathsf{C}^d_{\mathsf{GW}}(\alpha,\lambda) = \mathsf{C}_{\mathsf{GW}}(\tau_\alpha(\mathsf{p}) \mid \varnothing,\varnothing,\lambda), \quad \alpha,\lambda \in \mathcal{P}.$$

The coefficients of $\mathsf{C}^d_{\mathsf{P}}$ are Laurent series in u.

Lemma 2 *For all $d > 0$, the matrix $\mathsf{C}^d_{\mathsf{GW}}$ is invertible.*

By the invertibility of Lemmas 1 and 2, there exists a unique correspondence matrix K^d indexed by $\mathcal{P}_d$ with coefficients in $\mathbb{Q}(i,s_1,s_2,s_3)((u))$ which satisfies the condition

$$(-q)^{-|\lambda|}\mathsf{C}_{\mathsf{P}}(\tau_\alpha(\mathsf{p}) \mid \varnothing,\varnothing,\lambda) = (-iu)^{|\lambda|+\ell(\lambda)}\mathsf{C}_{\mathsf{GW}}\bigg(\sum_{\widehat{\alpha}\in\mathcal{P}_d} \mathsf{K}^d_{\alpha,\widehat{\alpha}}\tau_{\widehat{\alpha}}(\mathsf{p}) \,\bigg|\, \varnothing,\varnothing,\lambda\bigg)$$

under the variable change $-q = e^{iu}$ for all $\alpha,\lambda \in \mathcal{P}_d$.

Definition *The correspondence matrix K is defined by the rule*

$$\mathsf{K}_{\alpha,\widehat{\alpha}} = \mathsf{K}^{|\alpha|}_{\alpha,\widehat{\alpha}}$$

if $|\alpha| \geq |\widehat{\alpha}|$ and $\mathsf{K}_{\alpha,\widehat{\alpha}} = 0$ otherwise.

After proving Lemmas 1 and 2 in Section 2.2 below, we will use a geometric argument in Section 2.3 to prove the following compatibility statement.

Proposition 3 *For all $d \geq |\alpha|$, $\mathsf{K}^d_{\alpha,\widehat{\alpha}} = \mathsf{K}^{|\alpha|}_{\alpha,\widehat{\alpha}}$ for all $\widehat{\alpha}$.*

Lemmas 1–2 and Proposition 3 together yield a unique correspondence matrix

$$\mathsf{K}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}(i,s_1,s_2,s_3)((u))$$

satisfying Theorem 8 in the 1–leg case. The proof of Theorem 8 in the 2– and 3–leg cases will be presented in Sections 4 and 5. Theorem 1 will be derived as a consequence.

By our construction, the u coefficients of $\mathsf{K}_{\alpha,\widehat{\alpha}}$ are easily seen to be homogeneous rational functions in the variables s_i of degree $|\alpha| + \ell(\alpha) - |\widehat{\alpha}| - \ell(\widehat{\alpha})$. The claim follows from the homogeneity of the coefficients of the matrices

$$\mathsf{C}^d_{\mathsf{P}}(\alpha,\lambda), \quad \mathsf{C}^d_{\mathsf{GW}}(\alpha,\lambda)$$

obtained from geometric dimensional analysis.

Theorem 3 asserts further the symmetry and polynomiality of the coefficients $\mathsf{K}_{\alpha,\widehat{\alpha}}$. Unfortunately, the construction of K from the 1–leg geometry breaks the symmetry between the variables s_i. The symmetry of K will be established in Section 4.7 as a step in the proof of Theorem 8. The restriction of the coefficients of K to the subring

$$\mathbb{Q}[i, s_1, s_2, s_3]((u)) \subset \mathbb{Q}(i, s_1, s_2, s_3)((u))$$

will also be proven in Section 4.7, completing the proof of Theorem 3.

2.2 Proof of Lemmas 1 and 2

On the set $\mathcal{P}_d$, we define two partial orderings

$$\begin{aligned} \alpha \trianglerighteq \widetilde{\alpha} \quad &\Longleftrightarrow \quad |\alpha| - \ell(\alpha) \geq |\widetilde{\alpha}| - \ell(\widetilde{\alpha}), \\ \alpha \succcurlyeq \widetilde{\alpha} \quad &\Longleftrightarrow \quad \ell_+(\alpha) \geq \ell_+(\widetilde{\alpha}), \end{aligned}$$

where $\ell_+(\alpha)$ is the number of parts of α which are strictly greater than 1. The conditions $\alpha \rhd \widetilde{\alpha}$ and $\alpha \succ \widetilde{\alpha}$ are defined via the corresponding strict inequalities.

Lemma 4 *If $\alpha \lhd \lambda$, then $\mathsf{C}^d_{\mathsf{P}}(\alpha, \lambda) = 0$ and $\mathsf{C}^d_{\mathsf{GW}}(\alpha, \lambda) = 0$.*

Proof The result is a consequence of a dimension count. The 1–leg vertex can be studied via the *cap* geometry,

$$N = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \to \mathbf{P}^1,$$

relative to the fiber

$$N_\infty \subset N$$

over $\infty \in \mathbf{P}^1$. The total space N naturally carries an action of a 3–dimensional torus $\mathbf{T}$ where the first two factors scale the components of the rank 2 trivial bundle and last factor acts on $\mathbf{P}^1$ with fixed points $0, \infty \in \mathbf{P}^1$. Let the scaling weights be s_1 and s_2, and let the tangent weight along $\mathbf{P}^1$ at the fixed point $p_0 \in N$ over $0 \in \mathbf{P}^1$ be s_3. The $\mathbf{T}$–action preserves the relative divisor N_∞. The equivariant relative geometry N/N_∞ is equivalent to (9) in the 1–leg case.

Let $N_0 \subset N$ be the fiber over $0 \in \mathbf{P}^1$ of the cap, and let N_0 be the associated class in equivariant cohomology. Similarly, let p_0 be the equivariant cohomology class of p_0. We have the relation

$$\mathsf{N}_0 = \frac{\mathsf{p}_0}{s_1 s_2}$$

in the equivariant cohomology of the cap. For the relative conditions, we can weight each part of λ by the fixed point p_∞ over $\infty \in \mathbf{P}^1$. Let $\lambda[\mathsf{p}_\infty]$ denote the resulting weighted partition. We have the relation

$$1 = \frac{\mathsf{p}_\infty}{s_1 s_2}$$

in the equivariant cohomology of the divisor over $\infty \in \mathbf{P}^1$. Hence, we can write

$$\mathsf{C}^d_{\mathsf{P}}(\alpha,\lambda) = (s_1s_2)^{\ell(\alpha)-\ell(\lambda)} \mathsf{Z}_{\mathsf{P}}\bigg(\mathsf{Cap}; q \;\bigg|\; \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i-1}(\mathsf{N}_0) \;\bigg|\; \lambda[\mathsf{p}_\infty]\bigg)^{\mathbf{T}}_{|\lambda|},$$
$$\mathsf{C}^d_{\mathsf{GW}}(\alpha,\lambda) = (s_1s_2)^{\ell(\alpha)-\ell(\lambda)} \mathsf{Z}'_{\mathsf{GW}}\bigg(\mathsf{Cap}; u \;\bigg|\; \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i-1}(\mathsf{N}_0) \;\bigg|\; \lambda[\mathsf{p}_\infty]\bigg)^{\mathbf{T}}_{|\lambda|}.$$

The moduli spaces of relative stable pairs and relative stable maps to the cap with boundary condition $\lambda[\mathsf{p}_\infty]$ are both compact of virtual dimension $|\lambda| - \ell(\lambda)$. The integrand

$$\prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i-1}(\mathsf{N}_0)$$

imposes $|\alpha| - \ell(\alpha)$ condition in both theories. If the dimension of the integrand is strictly less than the virtual dimension, the equivariant integral vanishes for compact moduli spaces. □

Lemma 5 *If $|\alpha| - \ell(\alpha) = |\lambda| - \ell(\lambda)$ and $\alpha \prec \lambda$, then*

$$\mathsf{C}^d_{\mathsf{P}}(\alpha,\lambda) = 0, \quad \mathsf{C}^d_{\mathsf{GW}}(\alpha,\lambda) = 0.$$

Proof If $\lambda = (\lambda_1)$ has a single part (which must exceed 1 by the second hypothesis), then α must have all parts equal to 1. The first hypothesis

$$|\alpha| - \ell(\alpha) = |\lambda| - \ell(\lambda) \tag{12}$$

then can not hold. Hence, the lemma is true if the length of λ is 1. We will assume the length of λ is at least 2 and proceed by induction.

Consider the equivariant geometry of $\mathbf{P}^2 \times \mathbf{P}^1$ relative to the fiber

$$\mathbf{P}^2_\infty = \mathbf{P}^2 \times \{\infty\} \subset \mathbf{P}^2 \times \mathbf{P}^1.$$

Let $\mathsf{L} \in H_2(\mathbf{P}^2 \times \mathbf{P}^1, \mathbb{Z})$ be the class of the section $\mathbf{P}^1$ contracted over $\mathbf{P}^2$. The first two factors of $\mathbf{T}$ acts on $\mathbf{P}^2$ with fixed points $\xi_0, \xi_1, \xi_2 \in \mathbf{P}^2$. The tangent weights can be chosen as

$$s_1, s_2 \quad \text{for} \quad \xi_0, \qquad -s_1, s_2 - s_1 \quad \text{for} \quad \xi_1, \qquad s_1 - s_2, -s_2 \quad \text{for} \quad \xi_2.$$

The last factor of $\mathbf{T}$ acts on $\mathbf{P}^1$ as before with weight s_3 at $0 \in \mathbf{P}^1$. Let $\widetilde{\lambda}$ be the cohomology weighted partition obtained from λ with all parts weighted by $[\xi_0] \in H^*_{\mathbf{T}}(\mathbf{P}^2, \mathbb{Q})$. The integral

$$\int_{[P_n(\mathbf{P}^2 \times \mathbf{P}^1 / \mathbf{P}^2_\infty, |\lambda|\mathsf{L})_{\widetilde{\lambda}}]^{\mathrm{vir}}} \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i - 1}(\mathbf{P}^2_0) \in \mathbb{Q}$$

has integrand dimension equal to the virtual dimension and hence is independent of equivariant lift.

Let $\lambda_1 > 1$ be the largest part of λ. Let

$$\lambda' = \lambda \setminus \{\lambda_1\},$$

which is not empty since $\ell(\lambda)$ at least 2. Let $\widehat{\lambda}$ be the cohomology weighted partition with the parts λ_i weighted by $[\xi_0] \in H^*_{\mathbf{T}}(\mathbf{P}^2, \mathbb{Q})$ except λ_1 which is weighted by $[\xi_1] \in H^*_{\mathbf{T}}(\mathbf{P}^2, \mathbb{Q})$. By the independence of the choice of equivariant lift,

$$\int_{[P_n(\mathbf{P}^2 \times \mathbf{P}^1 / \mathbf{P}^2_\infty, |\lambda|\mathsf{L})_{\widetilde{\lambda}}]^{\mathrm{vir}}} \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i - 1}(\mathbf{P}^2_0) = \int_{[P_n(\mathbf{P}^2 \times \mathbf{P}^1 / \mathbf{P}^2_\infty, |\lambda|\mathsf{L})_{\widehat{\lambda}}]^{\mathrm{vir}}} \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i - 1}(\mathbf{P}^2_0). \tag{13}$$

The left side of (13) is exactly the q^n coefficient of

$$\mathsf{Z}_{\mathsf{P}}\Big(\mathsf{Cap}; q \;\Big|\; \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i - 1}(\mathsf{N}_0) \;\Big|\; \lambda[\mathsf{p}_\infty]\Big)^{\mathbf{T}}_{|\lambda|}.$$

Similarly the right side of (13) is the q^n coefficient of

$$\sum_{\alpha' \cup \alpha'' = \alpha} \frac{|\mathrm{Aut}(\lambda')|}{|\mathrm{Aut}(\lambda)|} \mathsf{Z}_{\mathsf{P}}\Big(\mathsf{Cap}; q \;\Big|\; \prod_{i=1}^{\ell(\alpha')} \tau_{\alpha'_i - 1}(\mathsf{N}_0) \;\Big|\; \lambda'[\mathsf{p}_\infty]\Big)^{\mathbf{T}}_{|\lambda'|}$$
$$\times \Big[\mathsf{Z}_{\mathsf{P}}\Big(\mathsf{Cap}; q \;\Big|\; \prod_{i=1}^{\ell(\alpha'')} \tau_{\alpha''_i - 1}(\mathsf{N}_0) \;\Big|\; \lambda_1[\mathsf{p}_\infty]\Big)^{\mathbf{T}}_{\lambda_1}\Big]_{\substack{s_1 = -s_1 \\ s_2 = s_2 - s_1}},$$

where the sum is over disjoint splittings of α. For the right side, the support of the stable pairs is disconnected. Hence, the universal sheaf $\mathbb{F}$ (used in the definition of the descendents in Section 0.2) splits as a direct sum. Since the Chern character of a sum is the sum of Chern characters, we obtain the above descendent distribution.

The hypothesis (12) implies

$$|\alpha'| - \ell(\alpha') + |\alpha''| - \ell(\alpha'') = |\lambda'| - \ell(\lambda') + |\lambda_1| - 1.$$

For nonvanishing terms of the sum, by Lemma 4, we must have

$$|\alpha'| - \ell(\alpha') = |\lambda'| - \ell(\lambda'), \quad |\alpha''| - \ell(\alpha'') = |\lambda_1| - 1.$$

At least one of the conditions $\alpha' \prec \lambda'$ or $\alpha'' \prec (\lambda_1)$ must hold. Since $\alpha'' \prec (\lambda_1)$ is impossible, the condition $\alpha' \prec \lambda'$ must hold. The induction statement is established.

The argument for Gromov–Witten theory is literally identical. The formal properties used above hold also for Gromov–Witten theory. □

We define an equivalence relation $\sim$ on $\mathcal{P}_d$ by the following rule: $\alpha \sim \widetilde{\alpha}$ if α and $\widetilde{\alpha}$ differ only by parts of size 1. For example,

$$(4,4,3,1) \sim (4,4,3,1,1,1).$$

The proof of Lemma 5 in fact yields a refined result.

Lemma 6 *If $|\alpha| - \ell(\alpha) = |\lambda| - \ell(\lambda)$, $\ell_+(\alpha) = \ell_+(\lambda)$, and $\alpha \nsim \lambda$ then*

$$\mathsf{C}^d_{\mathsf{P}}(\alpha,\lambda) = 0, \quad \mathsf{C}^d_{\mathsf{GW}}(\alpha,\lambda) = 0.$$

By Lemma 4, the matrices $\mathsf{C}^d_{\mathsf{P}}$ and $\mathsf{C}^d_{\mathsf{GW}}$ are block lower-triangular with respect to the partial ordering $\trianglerighteq$. In order to establish invertibility, we need only study the blocks where

$$|\alpha| - \ell(\alpha) = |\lambda| - \ell(\lambda) \tag{14}$$

is fixed. By Lemma 5, the above blocks themselves are block lower-triangular with respect to the partial ordering $\succcurlyeq$. So, we need only study blocks where both (14) and

$$\ell_+(\alpha) = \ell_+(\lambda)$$

are fixed. By Lemma 6, we finally restrict ourselves to the square blocks where the equivalence class under $\sim$ is fixed.

Let $\gamma \in \mathcal{P}_d$ be a partition with no parts equal to 1. The evaluation

$$\mathsf{Z}_{\mathsf{P}}\left(\mathsf{Cap}; q \;\middle|\; \prod_{i=1}^{\ell(\gamma)} \tau_{\gamma_i - 1}(\mathsf{N}_0) \;\middle|\; \gamma[\mathsf{p}_\infty]\right)^{\mathbf{T}}_{|\gamma|} = \frac{q^{|\gamma|}}{|\mathrm{Aut}(\gamma)|} \prod_{i=1}^{\ell(\gamma)} \frac{1}{\gamma_i!}, \tag{15}$$

has been computed in [25] and does not vanish. The cardinality of the equivalence class under $\sim$ determined by γ is $d - |\gamma|$. By the divisor equation, the block of $\mathsf{C}^d_{\mathsf{P}}$

corresponding to the equivalence class of γ is, up to harmless $(s_1s_2)^{\ell(\alpha)-\ell(\lambda)}$ factors, the matrix

$$\begin{pmatrix} \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(d-|\gamma|)!} \\ \frac{|\gamma|}{0!} & \frac{|\gamma|+1}{1!} & \frac{|\gamma|+2}{2!} & \cdots & \frac{d}{(d-|\gamma|)!} \\ \frac{|\gamma|^2}{0!} & \frac{(|\gamma|+1)^2}{1!} & \frac{(|\gamma|+2)^2}{2!} & \cdots & \frac{d^2}{(d-|\gamma|)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{|\gamma|^{d-|\gamma|}}{0!} & \frac{(|\gamma|+1)^{d-|\gamma|}}{1!} & \frac{(|\gamma|+2)^{d-|\gamma|}}{2!} & \cdots & \frac{d^{d-|\gamma|}}{(d-|\gamma|)!} \end{pmatrix}$$

with every element multiplied by (15). Invertiblity is immediate from the Vandermonde determinant.

The argument for the Gromov–Witten matrix $\mathsf{C}^d_{\mathsf{GW}}$ is identical. The replacement for (15) is the evaluation

$$\mathsf{Z}'_{\mathsf{GW}}\Big(\mathsf{Cap}; u \,\Big|\, \prod_{i=1}^{\ell(\gamma)} \tau_{\gamma_i-1}(\mathsf{N}_0) \,\Big|\, \gamma[\mathsf{p}_\infty]\Big)^{\mathbf{T}}_{|\gamma|} = \frac{u^{-2\ell(\gamma)}}{|\mathrm{Aut}(\gamma)|} \prod_{i=1}^{\ell(\gamma)} \frac{1}{\gamma_i!} \tag{16}$$

obtained[6] from [22, Lemma 7]. The proofs of Lemmas 1 and 2 are complete. □

We define $\mathsf{K}^{|\alpha|}_{\alpha,\widehat{\alpha}}$ using the invertibility of $\mathsf{C}^{|\alpha|}_{\mathsf{P}}$ and $\mathsf{C}^{|\alpha|}_{\mathsf{P}}$. A direct consequence of Lemma 4 is the following vanishing.

Lemma 7 *If $\alpha \lhd \widehat{\alpha}$, then $\mathsf{K}^{|\alpha|}_{\alpha,\widehat{\alpha}} = 0$.*

We will later require an invertibility result which is clear from our matrix analysis here.

Lemma 8 *The submatrix of $\mathsf{C}^d_{\mathsf{P}}(\alpha,\lambda)$ determined by the conditions*

$$d = |\alpha| = |\lambda|$$

is invertible (even after the restriction $s_3 = 0$).

Proof By Lemmas 4–6, the submatrix is lower-triangular with respect to the partial ordering. Moreover the diagonal elements are nonzero with no s_3 dependence. The evaluation $s_3 = 0$ is well-defined since the dependence of the descendents of the cap have no poles along s_3; see [28, Lemma 1]. □

[6] Beware of the typographical error of a factor of d in Lemma 7 of [22].

2.3 Proof of Proposition 3

The proposition will follow once we establish the identity

$$(17)\quad (-q)^{-|\lambda|}\mathsf{C}_{\mathsf{P}}(\tau_\alpha(\mathsf{p}) \mid \varnothing,\varnothing,\lambda) = (-iu)^{|\lambda|+\ell(\lambda)}\mathsf{C}_{\mathsf{GW}}\Bigg(\sum_{\widehat{\alpha}\in\mathcal{P}_{|\alpha|}} \mathsf{K}^{|\alpha|}_{\alpha,\widehat{\alpha}}\tau_{\widehat{\alpha}}(\mathsf{p}) \,\Bigg|\, \varnothing,\varnothing,\lambda\Bigg)$$

for all nonempty partitions λ.

Let $d = |\lambda|$. If $d \le |\alpha|$, the identity holds by the definition of $\mathsf{K}^{|\alpha|}$. To prove the identity for $d > |\alpha|$, we employ a geometric relation using the relative space

$$\mathbf{P}^2 \times \mathbf{P}^1/\mathbf{P}^2_\infty$$

introduced in the proof of Lemma 5. Once identity (17) is proven, the proposition follows from the invertibility of K^d.

Let $\widehat{\lambda}$ be the cohomology weighted partition with the largest part λ_1 weighted by $[\xi_0] \in H^*_{\mathbf{T}}(\mathbf{P}^2,\mathbb{Q})$ and all the remaining parts $\lambda_2,\dots,\lambda_{\ell(\lambda)}$ weighted by $1 \in H^*_{\mathbf{T}}(\mathbf{P}^2,\mathbb{Q})$. We will consider the partition functions

$$\mathsf{Z}_{\mathsf{P}}\big(\alpha,\widehat{\lambda}\big) = \mathsf{Z}_{\mathsf{P}}\Bigg(\mathbf{P}^2\times\mathbf{P}^1/\mathbf{P}^2_\infty; q \,\Bigg|\, \Delta\cdot\prod_{i=1}^{\ell(\alpha)}\tau_{\alpha_i-1}(\mathsf{N}_0) \,\Bigg|\, \widehat{\lambda}\Bigg)^{\mathbf{T}}_{d\mathsf{L}},$$

$$\mathsf{Z}_{\mathsf{GW}}\big(\alpha,\widehat{\lambda}\big) = \mathsf{Z}_{\mathsf{GW}}\Bigg(\mathbf{P}^2\times\mathbf{P}^1/\mathbf{P}^2_\infty; u \,\Bigg|\, \Delta\cdot\prod_{i=1}^{\ell(\alpha)}\tau_{\alpha_i-1}(\mathsf{N}_0) \,\Bigg|\, \widehat{\lambda}\Bigg)^{\mathbf{T}}_{d\mathsf{L}},$$

where Δ is the small diagonal condition obtained from $\mathbf{P}^2$.

To explain the small diagonal class Δ in the case of stable pairs, a recasting of the descendents is required. Let X/D be a 3–fold relative geometry, and let $\beta \in H_2(X,\mathbb{Z})$. Let

$$\mathcal{X} \to P_n(X/D,\beta)$$

be the universal space. We consider the r^{th} fiber product

$$\pi_P : \mathcal{X}^r \to P_n(X/D,\beta)$$

of $\mathcal{X}$ over $P_n(X/D,\beta)$ with projections

$$\pi_i : \mathcal{X}^r \to \mathcal{X}$$

for $1 \le i \le r$ onto the i^{th} factor. After composing with the canonical contraction $\mathcal{X} \to X$, we obtain

$$\pi_{i,X} : \mathcal{X}^r \to X.$$

The Chern character of the universal sheaf $\mathbb{F} \to \mathcal{X}$ on the universal space $\mathcal{X}$ is well-defined. The operation

$$\pi_{P*}\left(\prod_{i=1}^{r} \pi_{i,X}^*(\gamma_i) \cdot \pi_i^*(\mathrm{ch}_{2+k_i}(\mathbb{F})) \cap \pi_P^*(\,\cdot\,)\right) \tag{18}$$

on $H_*(P_n(X/D,\beta)$ is defined to be the action of the descendent $\prod_{i=1}^r \tau_{k_i}(\gamma_i)$, where $\gamma_i \in H^*(X,\mathbb{Z})$. By the push-pull formula, definition (18) agrees with the descendents constructed in Section 0.2.

The advantage here of definition (18) is the existence of a morphism

$$\pi_{X^r} : \mathcal{X}^r \to X^r, \quad \pi_{X^r} = (\pi_{1,X}, \dots, \pi_{r,X}).$$

Any class in $\delta \in H^*(X^r,\mathbb{Q})$ can be included in the descendent as

$$\pi_{P*}\left(\pi_{X^r}^*(\delta) \prod_{i=1}^{r} \pi_{i,X}^*(\gamma_i) \cdot \pi_i^*(\mathrm{ch}_{2+k_i}(\mathbb{F})) \cap \pi_P^*(\,\cdot\,)\right).$$

Of course, $\pi_{X^r}^*(\delta)$ can be incorporated in the γ_i by the Künneth decomposition. However, in the equivariant case, the Künneth decomposition requires inversion of the equivariant parameters and interferes with dimension arguments.

In the case relevant to the proof of Proposition 3,

$$X/D = \mathbf{P}^2 \times \mathbf{P}^1/\mathbf{P}^2_\infty,$$

and Δ is the class on the $\ell(\alpha)$–fiber product of the universal space $\mathcal{X}$ obtained by pulling back the class of the small diagonal of $(\mathbf{P}^2)^{\ell(\alpha)}$,

$$\mathcal{X}^{\ell(\alpha)} \to (\mathbf{P}^2 \times \mathbf{P}^1)^{\ell(\alpha)} \to (\mathbf{P}^2)^{\ell(\alpha)}.$$

Since the moduli space of maps has marked points, the parallel construction for Gromov–Witten theory is immediate.

Lemma 9 *If $d > |\alpha|$, then*

$$\mathsf{Z}_{\mathsf{P}}(\alpha, \widehat{\lambda}) = \mathsf{Z}_{\mathsf{GW}}(\alpha, \widehat{\lambda}) = 0.$$

Proof The virtual dimension of the stable pairs and stable maps moduli spaces with the relative condition $\widehat{\lambda}$ imposed is

$$d + \ell(\lambda) - 2.$$

The dimension of the integrand in both cases is

$$|\alpha| - \ell(\alpha) + 2(\ell(\alpha) - 1),$$

with the last term accounting for the small diagonal Δ. If

$$|\alpha| + \ell(\alpha) - 2 < d + \ell(\lambda) - 2 \tag{19}$$

then the lemma is obtained from dimension constraints for the compact geometry.

We may also express the theories of $\mathbf{P}^2 \times \mathbf{P}^1 / \mathbf{P}^2_\infty$ by localization in terms of the 1–leg descendent vertex. A simple analysis using Lemma 4 shows the vanishing of the lemma holds if

$$|\alpha| - \ell(\alpha) < d - \ell(\lambda). \tag{20}$$

Finally, we observe if neither condition (19) nor condition (20) are satisfied, then

$$2|\alpha| - 2 \geq 2d - 2$$

which violates the hypothesis $d > |\alpha|$. □

The constraint on 1–leg descendent vertices obtained by localization of the vanishing of Lemma 9 expresses

$$\mathsf{C}_{\mathsf{P}}(\tau_\alpha(\mathsf{p}) \mid \lambda, \varnothing, \varnothing) \quad \text{and} \quad \mathsf{C}_{\mathsf{GW}}(\tau_\alpha(\mathsf{p}) \mid \lambda, \varnothing, \varnothing)$$

in terms of

$$\mathsf{C}_{\mathsf{P}}(\tau_\alpha(\mathsf{p}) \mid \lambda', \varnothing, \varnothing) \quad \text{and} \quad \mathsf{C}_{\mathsf{GW}}(\tau_\alpha(\mathsf{p}) \mid \lambda', \varnothing, \varnothing),$$

where $\lambda' \subset \lambda$ is a strict subset. Moreover, the reduction equation respects identity (17) since all the $\tau_{\widehat{\alpha}}$ which appear in

$$\widehat{\tau}_\alpha = \sum_{\widehat{\alpha} \in \mathcal{P}_{|\alpha|}} \mathsf{K}^{|\alpha|}_{\alpha,\widehat{\alpha}} \tau_{\widehat{\alpha}}$$

also satisfy $d > |\widehat{\alpha}|$. By induction, we have established identity (17) and completed the proof of Proposition 3. □

The result of Proposition 3 is simultaneously the construction of the matrix K and the proof of Theorem 8 in the 1–leg case.

2.4 Basic properties of K

By construction, we have the vanishing

$$|\alpha| < |\widehat{\alpha}| \quad \Longrightarrow \quad \mathsf{K}_{\alpha,\widehat{\alpha}} = 0.$$

We have already established the vanishing

$$\alpha \lhd \widehat{\alpha} \quad \Longrightarrow \quad \mathsf{K}_{\alpha,\widehat{\alpha}} = 0 \tag{21}$$

in Lemma 7. A more subtle result is the following.

Proposition 10 *We have*

$$\widehat{\tau}_\alpha(\mathsf{p}) = (iu)^{\ell(\alpha)-|\alpha|}\tau_\alpha(\mathsf{p}) + \cdots ,$$

where the dots stand for terms $\tau_{\widehat{\alpha}}$ *with* $\alpha \rhd \widehat{\alpha}$.

Proof By the vanishing (21), we need only consider $\widehat{\alpha}$ for which

$$|\alpha| - \ell(\alpha) = |\widehat{\alpha}| - \ell(\widehat{\alpha}).$$

By Proposition 3, we have

$$(-q)^{-|\widehat{\alpha}|}\mathsf{Z}_{\mathsf{P}}(\mathsf{Cap}; q|\tau_\alpha(\mathsf{p}_0)|\widehat{\alpha})^{\mathbf{T}}_{|\widehat{\alpha}|} = (-iu)^{|\widehat{\alpha}|+\ell(\widehat{\alpha})}\mathsf{Z}'_{\mathsf{GW}}\Big(\mathsf{Cap}; u \;\Big|\; \sum_\mu \mathsf{K}_{\alpha,\mu}\tau_\mu(\mathsf{p}_0) \;\Big|\; \widehat{\alpha}\Big)^{\mathbf{T}}_{|\widehat{\alpha}|}.$$

Using the proven invertibilities, we need only match

$$(-q)^{-|\widehat{\alpha}|}\mathsf{Z}_{\mathsf{P}}\big(\mathsf{Cap}; q \mid \tau_\alpha(\mathsf{p}_0) \mid \widehat{\alpha}\big)^{\mathbf{T}}_{|\widehat{\alpha}|} \tag{22}$$

with the series

$$(-iu)^{|\widehat{\alpha}|+\ell(\widehat{\alpha})}\mathsf{Z}'_{\mathsf{GW}}\big(\mathsf{Cap}; u \mid (iu)^{\ell(\alpha)-|\alpha|}\tau_\alpha(\mathsf{p}_0) \mid \widehat{\alpha}\big)^{\mathbf{T}}_{|\widehat{\alpha}|}. \tag{23}$$

The required matching is established in the next lemma. □

After trading p_0 insertions on the cap for N_0 via

$$\mathsf{p}_0 = s_1 s_2 \mathsf{N}_0$$

and using standard dimension arguments, we reduce the necessary matching to the following result.

Lemma 11 *Let* α *be a partition of positive size, and let* $d-1 = |\alpha| - \ell(\alpha)$. *Then,*

$$\mathsf{Z}_{\mathsf{P}}\Big(\mathsf{Cap}; q \;\Big|\; \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i-1}(\mathsf{N}_0) \;\Big|\; d[\mathsf{p}_\infty]\Big)^{\mathbf{T}}_d = q^d \frac{d^{\ell(\alpha)-2}}{\prod_{i=1}^{\ell(\alpha)}(\alpha_i-1)!},$$

$$\mathsf{Z}'_{\mathsf{GW}}\Big(\mathsf{Cap}; u \;\Big|\; \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i-1}(\mathsf{N}_0) \;\Big|\; d[\mathsf{p}_\infty]\Big)^{\mathbf{T}}_d = u^{-2} \frac{d^{\ell(\alpha)-2}}{\prod_{i=1}^{\ell(\alpha)}(\alpha_i-1)!}.$$

Proof The Gromov–Witten calculation is well-known. The result follows directly from [22, Lemma 7] after translation of notation (and accounting of a factor d typographical error in the formula in [22]).

The stable pairs evaluation can be computed by localization using the same methods as in [25, Lemma 4]. By dimension counting, the result has no dependence on s_1 and s_2. Therefore, we can work mod s_1+s_2. We obtain the sum

$$\frac{(-1)^{d+\ell(\alpha)-1)}q^d}{d\cdot d!\cdot\prod_{i=1}^{\ell(\alpha)}(\alpha_i+1)!}\sum_{a+b=d-1}(-1)^a\binom{d-1}{a}P_\alpha(a),$$

where

$$P_\alpha(a)=\prod_{i=1}^{\ell(\alpha)}\bigl(-(-b-1)^{\alpha_i+1}+(-b)^{\alpha_i+1}+a^{\alpha_i+1}-(a+1)^{\alpha_i+1}\bigr)$$

is a polynomial in a with leading term[7]

$$\prod_{i=1}^{\ell(\alpha)}(-d(\alpha_i+1)\alpha_i)a^{d-1}.$$

The stable pairs evaluation is then

$$\frac{(-1)^{d+\ell(\alpha)-1}q^d}{d\cdot d!\cdot\prod_{i=1}^{\ell(\alpha)}(\alpha_i+1)!}(-1)^{d-1}(d-1)!\prod_{i=1}^{\ell(\alpha)}(-d(\alpha_i+1)\alpha_i)=\frac{d^{\ell(\alpha)-2}q^d}{\prod_{i=1}^{\ell(\alpha)}(\alpha_i-1)!}.\qquad\square$$

We define yet another partial ordering $\trianglerighteq^*$ on partitions by

$$\alpha\trianglerighteq^*\widetilde{\alpha}\quad\Longleftrightarrow\quad|\alpha|+\ell(\alpha)\geq|\widetilde{\alpha}|+\ell(\widetilde{\alpha}).$$

The conditions $\alpha\triangleright^*\widetilde{\alpha}$ is defined via the corresponding strict inequalities. In Section 6.3, we will prove the following result parallel to Proposition 10.

Proposition 12 *We have*

$$\widehat{\tau}_\alpha(\mathsf{p})=(iu)^{\ell(\alpha)-|\alpha|}\tau_\alpha(\mathsf{p})+\cdots,$$

where the dots stand for terms $\tau_{\widehat{\alpha}}$ with $\alpha\triangleright^*\widehat{\alpha}$.

Propositions 10 and 12 together immediately imply Theorem 2 constraining the initial terms of K.

[7]The lower terms do not contribute since $\sum_{a+b=d-1}(-1)^a\binom{d-1}{a}a^k=0$ for $k<d-1$.

As a Corollary of Proposition 10, we see the simple form

$$\widehat{\tau}_{1^\ell}(\mathsf{p}) = \tau_{1^\ell}(\mathsf{p})$$

holds for $\alpha = (1^\ell)$ since no partition satisfies $(1^\ell) \rhd \widehat{\alpha}$.

Let $(1) + \alpha$ be the partition obtained by adding a part equal to 1 to α. The part 1 corresponds to a $\tau_0(\mathsf{p})$ factor in $\tau_{(1)+\alpha}(\mathsf{p})$. We can write

$$\tau_0(\mathsf{p}_0) = s_1 s_2 \tau_0(\mathsf{N}_0)$$

in the cap geometry. Using the **T**–equivariant divisor equations[8] for the cap,

$$\Big\langle \tau_0(\mathsf{N}_0) \prod_{i=1}^r \tau_{k_i}(\mathsf{p}_0) \,\Big|\, \mu \Big\rangle^{\mathsf{P}}_{n,d} = d \Big\langle \tau_0(\mathsf{N}_0) \prod_{i=1}^r \tau_{k_i}(\mathsf{p}_0) \,\Big|\, \mu \Big\rangle^{\mathsf{P}}_{n,d},$$

$$\begin{aligned}\Big\langle \tau_0(\mathsf{N}_0) \prod_{i=1}^r \tau_{k_i}(\mathsf{p}_0) \,\Big|\, \mu \Big\rangle^{\mathsf{P}}_{g,d} &= d \Big\langle \tau_0(\mathsf{N}_0) \prod_{i=1}^r \tau_{k_i}(\mathsf{p}_0) \,\Big|\, \mu \Big\rangle^{\mathsf{GW}}_{g,d} \\ &\quad + \sum_{j=1}^r s_3 \Big\langle \tau_{k_j-1}(\mathsf{p}_0) \prod_{i \neq j} \tau_{k_i}(\mathsf{p}_0) \,\Big|\, \mu \Big\rangle^{\mathsf{GW}}_{g,d},\end{aligned}$$

we can easily understand how the matrix K treats the part 1. To state the answer, let

$$\Phi\big(\tau_{k_1}(\mathsf{p}) \cdots \tau_{k_r}(\mathsf{p})\big) = \sum_{j=1}^r \tau_{k_j-1}(\mathsf{p}) \prod_{i \neq j} \tau_{k_i}(\mathsf{p})$$

and extend Φ linearly to linear combinations of monomials in $\tau_k(\mathsf{p})$.

Proposition 13 *For partitions α,*

$$\widehat{\tau}_{(1)+\alpha}(\mathsf{p}) = \tau_0(\mathsf{p}) \cdot \widehat{\tau}_\alpha - s_1 s_2 s_3 \cdot \Phi(\widehat{\tau}_\alpha(\mathsf{p})).$$

Proof The divisor equations show the proposed formula for $\widehat{\tau}_{1+\alpha}(\mathsf{p})$ respects the 1–leg correspondence of Theorem 8 which uniquely defines K. □

2.5 Example

We calculate the first coefficients of K. The terms

$$\mathsf{K}_{(1),(1)} = 1, \quad \mathsf{K}_{(1),\widehat{\alpha} \neq (1)} = 0.$$

[8] In the Gromov–Witten case, if $k_j - 1 < 0$, the summand is omitted in the divisor equation.

have already been established. More interesting are the coefficients $\mathsf{K}_{(2),\widehat{\alpha}}$. By Proposition 10, we have

$$\mathsf{K}_{(2),(2)} = \frac{1}{iu}$$

and the only other non-vanishing coefficients are possibly $\mathsf{K}_{(2),(1^2)}$ and $\mathsf{K}_{(2),(1)}$. However, $\mathsf{K}_{(2),(1^2)}$ vanishes by Proposition 12. A degree-1 calculation below yields

$$\mathsf{K}_{(2),(1)} = \frac{s_1+s_2+s_3}{iu},$$

so we see that

$$\widehat{\tau}_{(2)}(\mathsf{p}) = \frac{1}{iu}\tau_{(2)}(\mathsf{p}) + \frac{s_1+s_2+s_3}{iu}\tau_{(1)}(\mathsf{p}). \tag{24}$$

Note, $\mathsf{K}_{(2),(1)}$ is symmetric in the s_i.

To prove (24), we need only check a single correspondence (as only the single coefficient $\mathsf{K}_{(2),(1)}$ is unknown). In [25], we have already calculated that[9]

$$\mathsf{Z}_{\mathsf{P}}\big(\mathsf{Cap}; q \mid \tau_{(2)}(\mathsf{p}_0) \mid (1)[\mathsf{p}_\infty]\big)_1^{\mathbf{T}} = -q\Big(\frac{s_1+s_2}{2}\Big)\frac{1-q}{1+q}.$$

There is not much difficulty in calculating the corresponding Gromov–Witten series

$$\mathsf{Z}'_{\mathsf{GW}}\big(\mathsf{Cap}; q \mid \tau_{(2)}(\mathsf{p}_0) \mid (1)[\mathsf{p}_\infty]\big)_1^{\mathbf{T}} \\ = -s_3u^{-2} + \frac{1}{u}\frac{d}{du}\bigg(s_3\Big(\frac{u/2}{\sin(u/2)}\Big)^{-\frac{(s_1+s_2)}{s_3}}\bigg)\cdot\Big(\frac{u/2}{\sin(u/2)}\Big)^{\frac{(s_1+s_2)}{s_3}}$$

by the Hodge integral methods of [4; 6]. The descendent is inserted via the dilation equation which appears as differentiation of the vertex term. The factor furthest to the right is the rubber contribution. The series

$$\mathsf{Z}'_{\mathsf{GW}}\big(\mathsf{Cap}; q \mid \tau_{(1)}(\mathsf{p}_0) \mid (1)[\mathsf{p}_\infty]\big)_1^{\mathbf{T}} = u^{-2}$$

is simple. After including the $(-q)^{-1}$ and $(-iu)^2$ scalings of the 1–leg descendent correspondence (17), we check that

$$\Big(\frac{s_1+s_2}{2}\Big)\frac{1-q}{1+q} \\ = -\frac{1}{iu}\Bigg((s_1+s_2) + u\frac{d}{du}\bigg(s_3\Big(\frac{u/2}{\sin(u/2)}\Big)^{-\frac{(s_1+s_2)}{s_3}}\bigg)\cdot\Big(\frac{u/2}{\sin(u/2)}\Big)^{\frac{(s_1+s_2)}{s_3}}\Bigg)$$

after $-q = e^{iu}$.

[9]The tangent weight conventions of [25] differ from the conventions here by a sign.

In the above example, the stable pairs descendent had been exactly calculated and the dilation equation at the vertex could be used to handle the Gromov–Witten side. While the stable pairs descendents series are difficult to calculate, at least methods exist [28; 25]. At the moment, there is no reasonable way to calculate the Gromov–Witten descendent series except, of course, order by order in u.

3 Capped localization

3.1 Toric geometry

Let X be a nonsingular toric 3–fold. Virtual localization with respect to the action of the full 3–dimensional torus $\mathbf{T}$ reduces all stable pairs and Gromov–Witten invariants of X to local contributions of the vertices and edges of the associated toric polytope. We will use the regrouped localization procedure introduced in [18] with capped vertex and edge contributions. The capped vertex and edge terms are equivalent building blocks for global toric calculations, but are much better behaved.

Let Δ denote the polytope associated to X. The vertices of Δ are in bijection with $\mathbf{T}$–fixed points $X^{\mathbf{T}}$. The edges e correspond to $\mathbf{T}$–invariant curves

$$C_e \subset X.$$

The three edges incident to any vertex carry canonical $\mathbf{T}$–weights; the tangent weights of the torus action.

We will consider both compact and noncompact toric varieties X. In the latter case, edges may be compact or noncompact. Every compact edge is incident to two vertices.

3.2 Capping

Capped localization expresses the $\mathbf{T}$–equivariant stable pairs descendents of X as a sum of capped descendent vertex and capped edge data.

A half-edge $h = (e, v)$ is a compact edge e together with the choice of an incident vertex v. A partition assignment

$$h \mapsto \lambda(h)$$

to half-edges is *balanced* if the equality

$$|\lambda(e, v)| = |\lambda(e, v')|$$

always holds for the two halves of e. For a balanced assignment, let

$$|e| = |\lambda(e, v)| = |\lambda(e, v')|$$

denote the *edge degree*.

The outermost sum in the capped localization formula runs over all balanced assignments of partitions $\lambda(h)$ to the half-edges h of Δ satisfying

$$\beta = \sum_e |e| \cdot [C_e] \in H_2(X, \mathbb{Z}). \tag{25}$$

Such a partition assignment will be called a *capped marking* of Δ. The weight of each capped marking in the localization sum for the stable pairs descendent partition function equals the product of three factors:

(i) Capped descendent vertex contributions.

(ii) Capped edge contributions.

(iii) Gluing terms.

Each vertex determines up to three half-edges specifying the partitions for the capped vertex. Each compact edge determines two half-edges specifying the partitions of the capped edge. The capped edge contributions (ii) and gluing terms (iii) here are *exactly* the same as for the capped localization formula in [18]. Precise formulas are written in Section 3.3.

The capped localization formula is easily derived from the standard localization formula (with roots in [6; 14]). Indeed, the capped objects are obtained from the uncapped objects by rubber integral[10] factors. The rubber integrals cancel in pairs in capped localization to yield standard localization.

3.3 Formulas

The $\mathbf{T}$–equivariant cohomology of X is generated (after localization) by the classes of the $\mathbf{T}$–fixed points $X^{\mathbf{T}} \subset X$. Let α be a partition with parts $\alpha_1, \ldots, \alpha_\ell$ and let

$$\sigma : \{1, \ldots, \ell\} \to X^{\mathbf{T}}.$$

Let $\mathsf{p}_{\sigma(i)} \in H^*_{\mathbf{T}}(X, \mathbb{Q})$ denote the class of the $\mathbf{T}$–fixed point $\sigma(i)$. We consider the capped localization formula for the $\mathbf{T}$–equivariant stable pairs and Gromov–Witten descendent partition functions

$$\mathsf{Z}_{\mathsf{P}}\Big(X; q \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\mathsf{p}_{\sigma(i)})\Big)^{\mathbf{T}}_{\beta}, \quad \mathsf{Z}'_{\mathsf{GW}}\Big(X; u \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\mathsf{p}_{\sigma(i)})\Big)^{\mathbf{T}}_{\beta}. \tag{26}$$

[10] Rubber integrals $\langle \lambda \mid 1/(1-\psi_\infty) \mid \mu \rangle^{\sim}$ arise in the localization formulas for relative geometries. See [28] for a discussion.

We will indicate the slight differences between the formula for stable pairs and stable maps below.

Let $\mathcal{V}$ be the set of vertices of Δ which we identify with $X^{\mathbf{T}}$. For each vertex $v \in \mathcal{V}$, let h_1^v, h_2^v, h_3^v be the associated half-edges[11] with tangent weights s_1^v, s_2^v, s_3^v respectively. Let Γ_β be the set of capped markings satisfying the degree condition (25). Each $\Gamma \in \Gamma_\beta$ associates a partition $\lambda(h)$ to every half-edge h. Let

$$|h| = |\lambda(h)|$$

denote the half-edge degree.

For each $v \in \mathcal{V}$, the assignments σ and Γ determine an evaluation of the capped vertex,

$$\mathsf{C}(v,\sigma,\Gamma) = \mathsf{C}\bigg(\prod_{i\in\sigma^{-1}(v)} \tau_{k_i}(\mathsf{p}_v) \,\Big|\, \lambda(h_1^v), \lambda(h_2^v), \lambda(h_3^v)\bigg)\bigg|_{s_1=s_1^v, s_2=s_2^v, s_3=s_3^v}.$$

Let h_1^e and h_2^e be the half-edges associated to the edge e. The assignment Γ also determines an evaluation of the capped edge,

$$\mathsf{E}(e,\Gamma) = \mathsf{E}(\lambda(h_1^e), \lambda(h_2^e)).$$

The capped edge geometry is discussed in [18]. A gluing factor is specified by Γ at each half-edge $h_i^v \in \mathcal{H}$. For stable pairs

$$\mathsf{G}_{\mathsf{P}}(h_i^v,\Gamma) = (-1)^{|h_i^v| - \ell(\lambda(h_i^v))} \mathfrak{z}(\lambda(h_i^v)) \bigg(\frac{\prod_{j=1}^3 s_j^v}{s_i^v}\bigg)^{\ell(\lambda(h_i^v))} \cdot q^{-|h_i^v|},$$

where $\mathfrak{z}(\lambda)$ is the order of the centralizer in the symmetric group of an element with cycle type λ. For Gromov–Witten theory,

$$\mathsf{G}_{GW}(h_i^v,\Gamma) = \mathfrak{z}(\lambda(h_i^v)) \bigg(\frac{\prod_{j=1}^3 s_j^v}{s_i^v}\bigg)^{\ell(\lambda(h_i^v))} \cdot u^{2\ell(\lambda(h_i^v))}.$$

The capped localization formula for stable pairs can be written exactly in the form presented in Section 3.2,

$$\mathsf{Z}_{\mathsf{P}}\bigg(X, \prod_{i=1}^r \tau_{k_i}(\mathsf{p}_{\sigma(i)})\bigg)_\beta^{\mathbf{T}} = \sum_{\Gamma\in\Gamma_\beta} \prod_{v\in\mathcal{V}} \prod_{e\in\mathcal{E}} \prod_{h\in\mathcal{H}} \mathsf{C}_{\mathsf{P}}(v,\sigma,\Gamma)\mathsf{E}_{\mathsf{P}}(e,\Gamma)\mathsf{G}_{\mathsf{P}}(h,\Gamma),$$

[11] For simplicity, we assume X is projective so each vertex is incident to 3 compact edges.

where the product is over the sets of vertices $\mathcal{V}$, edges $\mathcal{E}$ and half-edges $\mathcal{H}$ of the polytope Δ. Similarly,

$$\mathsf{Z}'_{\mathsf{GW}}\left(X, \prod_{i=1}^{r} \tau_{k_i}\left(\mathsf{p}_{\sigma(i)}\right)\right)_{\beta}^{\mathbf{T}} = \sum_{\Gamma\in\Gamma_\beta} \prod_{v\in\mathcal{V}} \prod_{e\in\mathcal{E}} \prod_{h\in\mathcal{H}} \mathsf{C}_{\mathsf{GW}}(v,\sigma,\Gamma)\mathsf{E}_{\mathsf{GW}}(e,\Gamma)\mathsf{G}_{\mathsf{GW}}(h,\Gamma).$$

An immediate consequence of the above capped localization formulas for stable pairs and Gromov–Witten theories is the implication of Theorem 1 by Theorem 8 and the symmetry of K in the variables s_i. Theorem 8 is applied to the capped descendent vertices on the right side of the formula. The capped edge correspondences have already been proven in [18] (with the stable pairs case discussed in [20, Section 5]). Tracing the factors of q and u here is an easy exercise.

4 Descendent correspondence: 2–leg

4.1 Overview

Our goal here is to prove Theorem 8 in the 2–leg case. Consider the capped 2–leg descendent vertices

$$\mathsf{C}_{\mathsf{P}}(\tau_\alpha(\mathsf{p}) \mid \varnothing, \mu, \lambda), \quad \mathsf{C}_{\mathsf{GW}}(\tau_\alpha(\mathsf{p}) \mid \varnothing, \mu, \lambda).$$

Our proof of Theorem 8 will be by induction on the complexity of the legs. The descendent insertion $\tau_\alpha(\mathsf{p})$ will be fixed for the argument. If $\mu = \varnothing$, we are in the 1–leg case where Theorem 8 has already been established in Section 2. The 1–leg case will be the base of the induction.

Define a partial ordering on pairs of partitions (μ,λ) satisfying the condition $(\mu,\lambda) \neq (\varnothing,\varnothing)$ by the following rules. We say

$$(\mu,\lambda) \triangleright (\mu',\lambda')$$

if we have $|\mu| > |\mu'|$. The proof of Theorem 8 in the 2–leg case is by induction with respect to the partial ordering $\triangleright$.

Along the way, we will also establish basic properties of the correspondence matrix K constructed in Section 2.1. The following result, completing the proof of Theorem 3, will be proven in Section 4.7:

The coefficients of K *lie in the subring*

$$\Lambda_3((u)) \subset \mathbb{Q}(i, s_1, s_2, s_3)((u)), \tag{27}$$

where Λ_3 *is the ring of symmetric* polynomials *in* s_1, s_2, s_3 *over the field* $\mathbb{Q}[i]$.

4.2 $\mathcal{A}_1$ geometry

Let ζ be a primitive $(n+1)^{\text{th}}$ root of unity, for $n \geq 0$. Let the generator of the cyclic group $\mathbb{Z}_{n+1}$ act on $\mathbb{C}^2$ by

$$(z_1, z_2) \mapsto (\zeta z_1, \zeta^{-1} z_2).$$

Let $\mathcal{A}_n$ be the minimal resolution of the quotient

$$\mathcal{A}_n \to \mathbb{C}^2/\mathbb{Z}_{n+1}.$$

The diagonal $(\mathbb{C}^*)^2$–action on $\mathbb{C}^2$ commutes with the action of $\mathbb{Z}_n$. As a result, the surfaces $\mathcal{A}_n$ are toric.

The surface $\mathcal{A}_1$ is isomorphic to the total space of

$$\mathcal{O}(-2) \to \mathbf{P}^1$$

and admits a toric compactification

$$\mathcal{A}_1 \subset \mathbf{P}(\mathcal{O} + \mathcal{O}(-2)) = \mathcal{F}_2$$

by the Hirzebruch surface.

Let $C \subset \mathcal{A}_1$ be the 0–section of $\mathcal{O}(-2)$, and let $\star, \bullet \in C$ be the $(\mathbb{C}^*)^2$–fixed points. Let

$$\overline{\star}, \overline{\bullet} \in \mathcal{F}_2 \setminus \mathcal{A}_1$$

be the $(\mathbb{C}^*)^2$–fixed points lying above $\star, \bullet$ respectively. We fix our $(\mathbb{C}^*)^2$–action by specifying tangent weights at the four $(\mathbb{C}^*)^2$–points

$$\begin{aligned}
T_\star(\mathcal{F}_2)&: \quad s_1 - s_2, \quad 2s_2,\\
T_\bullet(\mathcal{F}_2)&: \quad s_2 - s_1, \quad 2s_1,\\
T_{\overline{\star}}(\mathcal{F}_2)&: \quad s_1 - s_2, \ -2s_2,\\
T_{\overline{\bullet}}(\mathcal{F}_2)&: \quad s_2 - s_1, \ -2s_1.
\end{aligned} \tag{28}$$

No tangent weight here is divisible by $s_1 + s_2$.

Consider the nonsingular projective toric variety $\mathcal{F}_2 \times \mathbf{P}^1$. The 3–torus

$$\mathbf{T} = (\mathbb{C}^*)^3$$

acts on $\mathcal{F}_2$ as above via the first two factors and acts on $\mathbf{P}^1$ via the third factor with tangent weights s_3 and $-s_3$ at the points $0, \infty \in \mathbf{P}^1$ respectively. The two $\mathbf{T}$–invariant divisors of $\mathcal{F}_2 \times \mathbf{P}^1$

$$\mathcal{D}_0 = \mathcal{F}_2 \times \{0\}, \quad \mathcal{D}_\infty = \mathcal{F}_2 \times \{\infty\}$$

will play a basic role. The 3–fold $\mathcal{F}_2 \times \mathbf{P}^1$ has eight $\mathbf{T}$–fixed points which we denote by

$$\star_0,\ \overline{\star}_0,\ \bullet_0,\ \overline{\bullet}_0,\ \star_\infty,\ \overline{\star}_\infty,\ \bullet_\infty,\ \overline{\bullet}_\infty \in \mathcal{F}_2 \times \mathbf{P}^1,$$

where the subscript indicates the coordinate in $\mathbf{P}^1$.

Let $L_0 \subset \mathcal{F}_2 \times \mathbf{P}^1$ be the $\mathbf{T}$–invariant line connecting $\star_0$ and $\overline{\star}_0$. Similarly, let $L_\infty \subset \mathcal{F}_2 \times \mathbf{P}^1$ be the $\mathbf{T}$–invariant line connecting $\star_\infty$ and $\overline{\star}_\infty$. The lines L_0 and L_∞ are $\mathbf{P}^1$–fibers of the Hirzebruch surfaces $\mathcal{D}_0$ and $\mathcal{D}_\infty$. We have

$$H_2(\mathcal{F}_2 \times \mathbf{P}^1, \mathbb{Z}) = \mathbb{Z}[C] \oplus \mathbb{Z}[L_0] \oplus \mathbb{Z}[P],$$

where P is the fiber of the projection to $\mathcal{F}_2$.

4.3 Integration

We will find relations which express $\mathsf{C}(\tau_\alpha(\mathsf{p}) \mid \varnothing, \mu, \lambda)$ in terms of inductively treated vertices for stable pairs and Gromov–Witten theory. The inductive equations will respect the correspondence claimed in Theorem 8.

Let μ' be a partition. The relations will be obtained from vanishing invariants of the relative geometry $\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty$ in curve class

$$\beta = |\mu| \cdot [C] + (|\lambda| + |\mu'|) \cdot [P].$$

The virtual dimensions of the associated moduli spaces are

$$\dim^{\mathrm{vir}} P_n(\mathcal{F}_2 \times \mathbf{P}^1, \beta) = 2|\lambda| + 2|\mu'|,$$
$$\dim^{\mathrm{vir}} \overline{M}'_g(\mathcal{F}_2 \times \mathbf{P}^1, \beta) = 2|\lambda| + 2|\mu'|.$$

Relative conditions in $\mathrm{Hilb}(\mathcal{D}_\infty, |\lambda| + |\mu'|)$ are best expressed in terms of the Nakajima basis given by a $\mathbf{T}$–equivariant cohomology weighted partition of $|\lambda| + |\mu'|$. We impose the relative condition determined by the partition

$$\lambda \cup \mu' = \lambda_1 + \cdots + \lambda_{\ell(\lambda)} + \mu'_1 + \cdots + \mu'_{\ell(\mu')}$$

weighted by $[\star_\infty] \in H^*_{\mathbf{T}}(\mathcal{D}_\infty, \mathbb{Q})$ for the parts of λ and $[\bullet_\infty] \in H^*_{\mathbf{T}}(\mathcal{D}_\infty, \mathbb{Q})$ for the parts of μ'. We denote the relative condition by $\mathsf{r}(\lambda, \mu')$. The same data expresses relative conditions in Gromov–Witten theory. After imposing $\mathsf{r}(\lambda, \mu')$, the virtual dimension drops to

$$\dim^{\mathrm{vir}} P_n(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \mathsf{r}(\lambda, \mu'))_\beta = |\lambda| - \ell(\lambda) + |\mu'| - \ell(\mu'),$$
$$\dim^{\mathrm{vir}} \overline{M}'_g(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \mathsf{r}(\lambda, \mu'))_\beta = |\lambda| - \ell(\lambda) + |\mu'| - \ell(\mu').$$

To define an equivariant integral, we specify the descendent insertion by

$$\tau_\alpha([\star_0]) = \tau_{\alpha_1 - 1}([\star_0]) \cdots \tau_{\alpha_{\ell(\alpha)}-1}([\star_0]).$$

The descendent insertion imposes $|\alpha| + \ell(\alpha)$ conditions. Therefore, the integrals

$$\int_{[P_n(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \mathsf{r}(\lambda,\mu'))_\beta]^{\mathrm{vir}}} \tau_\alpha([\star_0]), \quad \int_{[\overline{M}'_{g,\ell(\alpha)}(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \mathsf{r}(\lambda,\mu'))_\beta]^{\mathrm{vir}}} \tau_\alpha([\star_0]), \tag{29}$$

viewed as $\mathbf{T}$–equivariant push-forwards to a point, both have dimension

$$|\lambda| - \ell(\lambda) + |\mu'| - \ell(\mu') - |\alpha| - \ell(\alpha).$$

We conclude the following result.

Proposition 14 *If $|\mu'| - \ell(\mu') > |\alpha| + \ell(\alpha)$, then the $\mathbf{T}$–equivariant integrals* (29) *vanish for all Euler characteristics n and genera g.*

4.4 Relations

We consider first the stable pairs case. Define the $\mathbf{T}$–equivariant series

$$\mathsf{Z}_{\mathsf{P}}(\alpha, \lambda, \mu')_\beta = \sum_n q^n \int_{[P_n(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \mathsf{r}(\lambda,\mu'))_\beta]^{\mathrm{vir}}} \tau_\alpha([\star_0])$$

obtained from the stable pairs integrals (29). By Proposition 14, the series $\mathsf{Z}_{\mathsf{P}}(\alpha, \lambda, \mu')_\beta$ vanishes identically if $|\mu'| - \ell(\mu') > |\alpha| + \ell(\alpha)$. We will calculate the left side of

$$\mathsf{Z}_{\mathsf{P}}(\alpha, \lambda, \mu')_\beta = 0 \tag{30}$$

by capped localization to obtain a relation constraining the stable pairs capped descendent vertices.

The stable pairs theory of the relative geometry $\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty$ admits a capped localization formula. Over $0 \in \mathbf{P}^1$, capped descendent vertices occur as in the capped localization formula of Section 3.3. Over $\infty \in \mathbf{P}^1$, capped rubber terms for $\mathbf{T}$–equivariant localization in the relative geometry arise. Capped rubber is discussed in [18, Section 3.4]. Since all our descendent insertions lie over $0 \in \mathbf{P}^1$, our capped rubber has the same definition as the capped rubber of [18].

By the curve choice β and the relative constraints $\mathsf{r}(\lambda, \mu')$, the only capped rubber contributions of $\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty$ which arise over $\infty \in \mathbf{P}^1$ in the $\mathbf{T}$–equivariant localization formula for $\mathsf{Z}_{\mathsf{P}}(\alpha, \lambda, \mu')_\beta$ lie in

$$\mathcal{A}_1 \times \mathbf{P}^1 \subset \mathcal{F}_2 \times \mathbf{P}^1.$$

The capped rubber contributions of $\mathcal{A}_1 \times \mathbf{P}^1/\mathcal{D}_\infty$ are proven to satisfy the GW/DT correspondence in [18, Lemma 6] relying on the results of [16; 17]. See [20, Section 5] for GW/Pairs correspondence for the $\mathcal{A}_1$ capped rubber.

We now analyze the capped localization of $\mathsf{Z}_{\mathsf{P}}(\alpha, \lambda, \mu')_\beta$ over $0 \in \mathbf{P}^1$. A term in the capped localization formula is said to be *principal* if not all the capped descendent vertices which arise are lower than (μ, λ) in the partial ordering $\rhd$. Our first task now is to identify the principal terms.

First consider the descendent insertions. The descendents

$$\tau_{\alpha_1 - 1}([\star_0]) \cdots \tau_{\alpha_{\ell(\alpha)}-1}([\star_0])$$

all lie on $\star_0$. Hence, the only capped vertex with non-trivial descendents is $\star_0$. The tangent weights at $\star_0$ are

$$s_1 - s_2, \quad 2s_2, \quad s_3, \tag{31}$$

where the first two lie along $\mathcal{D}_0$. For the capped vertices occurring at $\star_0$, the weights (31) are substituted

$$\mathsf{C}_{\mathsf{P}}\big(\tau_\alpha([\star_0]) \mid \varnothing, \lambda^{(2)}, \lambda^{(3)}\big) = \mathsf{C}_{\mathsf{P}}\big(\tau_\alpha(\mathsf{p}) \mid \varnothing, \lambda^{(2)}, \lambda^{(3)}\big)|_{s_1 = s_1 - s_2, s_2 = 2s_2, s_3 = s_3}$$

into the standard capped vertex defined in Section 1.3. An equivariant vertex with no descendents occurs at $\bullet_0$. By the choice of β and $\mathsf{r}(\lambda, \mu')$, no vertices can only occur at $\overline{\star}_0$ and $\overline{\bullet}_0$.

Next consider the edge degree d of C over $0 \in \mathbf{P}^1$ in the capped localization formula. If $d < |\mu|$, then the capped descendent vertex at $\star_0$ is lower than (μ, λ) in the partial ordering $\rhd$. We restrict ourselves to the principal terms where $d = |\mu|$.

Since all of $|\mu| \cdot [C]$ occurs over $0 \in \mathbf{P}^1$, the rubber over $\infty \in \mathbf{P}^1$ is all 1–leg. The relative conditions are determined by λ with weights $[\star_\infty]$ and μ' with weights $[\bullet_\infty]$. In the principal terms of the capped localization of (30), precisely the following set of capped 2–leg descendent vertices occur at $\star_0$:

$$\{\mathsf{C}_{\mathsf{P}}(\tau_\alpha([\star_0]) \mid \varnothing, \widehat{\mu}, \lambda) \mid |\widehat{\mu}| = |\mu|\}. \tag{32}$$

The principal terms arise as displayed in Figure 1. In addition to the vertex $\mathsf{C}(\alpha|\lambda, \widehat{\mu}, \varnothing)$ at $\star_0$, there is a capped edge with partitions

$$|\widehat{\mu}| = |\widehat{\mu}'|$$

along the curve C over $0 \in \mathbf{P}^1$. Finally, there is a capped 2–leg vertex with no descendents at $\bullet_0$ with outgoing partitions $\widehat{\mu}'$ and μ'.

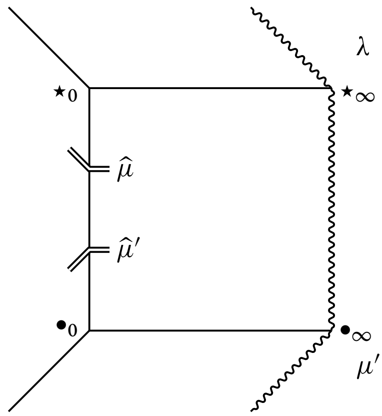

Figure 1: Principal terms

The system of equations (30) as the partition μ' varies has unknowns (32) parameterized by partitions of $|\mu|$. However, the number of equations is infinite. The induction step is established if the set of equations as μ' varies subject to the condition

$$|\mu'| - \ell(\mu') > |\alpha| + \ell(\alpha)$$

has maximal rank (equal to the number of partitions of size $|\mu|$) with respect to the unknowns (32).

4.5 Maximal rank

The capped edge matrix along C has maximal rank [18]. The main difficulty is to prove the matrix of capped 2–leg vertices

$$\mathsf{C}_{\mathsf{P}}(1 \mid \varnothing, \widehat{\mu}', \mu')$$

has maximal rank when $\widehat{\mu}'$ varies among partitions of size $|\mu|$ and μ' varies among the infinite set of partitions satisfying

$$|\mu'| - \ell(\mu') > |\alpha| + \ell(\alpha).$$

Proposition 15 *For any positive integers d and N, the matrix*

$$\mathsf{C}_{\mathsf{P}}(1 \mid \varnothing, \delta, \nu)$$

determined as δ varies among partition of d and ν varies among partitions satisfying

$$|\nu| - \ell(\nu) \geq N - 1$$

is of maximal rank.

Proof We may prove the maximal rank condition after the *topological vertex* specialization

$$s_1 + s_2 + s_3 = 0.$$

The capped vertex is related to the standard uncapped vertex by invertible capped rubbers. The uncapped vertex may be evaluated directly; see [14; 24]. Up to further invertible factors, the matrix to consider becomes

$$\sum_\eta s_{\delta^t/\eta}(q^\rho)\, s_{\nu/\eta}(q^\rho),$$

where $s_{\delta^t/\eta}$ and $s_{\nu/\eta}$ are skew Schur functions evaluated at

$$q^\rho = (q^{-1/2}, q^{-3/2}, \dots).$$

As δ and η vary over all partitions[12] of size at d and at most d respectively, the matrix of skew Schur functions $s_{\delta^t/\eta}$ is of maximal rank. We are thus reduced to proving that the matrix

$$\big[s_{\nu/\eta}(q^\rho)\big], \quad |\nu| - \ell(\nu) \geq N-1, |\eta| \leq d, \tag{33}$$

has maximal rank.

We select a square minor from (33) by the following construction. For every partition

$$\eta = (e_1 \geq e_2 \geq \cdots \geq e_{\ell(\eta)})$$

of size at most d, define

$$\eta^+ = (e_1 + N \geq e_2 \geq \cdots \geq e_{\ell(\eta)}).$$

In case $\eta = \varnothing$, then $\eta^+ = (N)$ has length 1. As η varies among partitions of size at most d, η^+ varies among partitions which satisfy

$$|\eta^+| - \ell(\eta^+) \geq N-1$$

with equality achieved for $\eta = \varnothing$. We will prove the matrix

$$\big[s_{\nu^+/\eta}(q^\rho)\big], \quad |\nu|, |\eta| \leq d, \tag{34}$$

is invertible.

We define a partial ordering $\geq$ on partitions η of size at most d by the following rule. Let η_- be the partition obtained by removing the largest part of η,

$$\eta_- = (e_2 \geq \cdots \geq e_{\ell(\eta)}).$$

[12] η is permitted here to be empty.

In case $\eta = \varnothing$, then $\eta_- = \varnothing$. Define $\eta \geq \widetilde{\eta}$ if

$$\eta_- \supset \widetilde{\eta}_-,$$

or equivalently,

$$e_2 \geq \widetilde{e}_2, e_3 \geq \widetilde{e}_3, e_4 \geq \widetilde{e}_4, \dots.$$

Unwinding the definitions, we immediately find that

$$s_{\nu^+/\eta}(q^\rho) = 0 \quad \text{unless} \quad \nu \geq \eta$$

since the skew Schur function vanishes unless $\eta \subset \nu^+$. We conclude the matrix (34) is block triangular with respect to the ordering $\geq$.

For any matrix with a block triangular structure with respect to a partial ordering, invertibility is equivalent to the invertibility of the blocks. In the case at hand, a block is specified by a partition

$$\theta = (t_1 \geq t_2 \geq \cdots \geq t_{\ell(\theta)})$$

satisfying $d - |\theta| \geq t_1$. The block corresponding to θ is indexed by the partitions

$$P_\theta = \{\eta \mid \eta_- = \theta\}.$$

The cardinality of P_θ is $M+1$, where

$$M = d - |\theta| - t_1.$$

In fact, the elements of P_θ are simply the partitions

$$(t_1, t_1, t_2, \dots, t_{\ell(\theta)}), \quad (t_1+1, t_1, t_2, \dots, t_{\ell(\theta)}), \quad \dots, \quad (t_1+M, t_1, t_2, \dots, t_{\ell(\theta)}).$$

The associated block is

$$B_\theta = \big[s_{\nu^+/\eta}(q^\rho)\big], \quad \nu, \eta \in P_\theta.$$

To proceed further, we recall the definition of the skew Schur functions,

$$s_{\lambda/\mu} = \det\big(h_{\lambda_i - \mu_j + j - i}\big), \tag{35}$$

where h_k is the complete symmetric function of degree k. If $\mu = \varnothing$,

$$s_{\lambda/\varnothing} = \det\big(h_{\lambda_i + j - i}\big)$$

is the standard Schur function associated to λ. The elements of P_θ are

$$\{t_1 + i\} \cup \theta$$

for $0 \le i \le M$. Expanding definition (35), we see

$$s_{\{t_1+i+N\}\cup\theta/\{t_1+j\}\cup\theta} = h_{N+i-j}.$$

Therefore, we can write the determinant of the block as

$$\det B_\theta = \det\bigl(h_{N+i-j}(q^\rho)\bigr), \tag{36}$$

where $0 \le i, j \le M$ on the right. Fortunately, we recognize the determinant (36) as the Schur function $s_{(N,\dots,N)}(q^\rho)$ associated to the partition $(N,\dots,N)$ with $M+1$ parts equal to N. The evaluation q^ρ does not vanish on the Schur functions. □

4.6 Proof of Theorem 8 in the 2–leg case

We have already seen the integration relations determine $\mathsf{C}_{\mathsf{P}}(\tau_\alpha(\mathsf{p}) \mid \varnothing, \mu, \lambda)$ by induction on the complexity of the legs. We will now study the parallel integration relations in Gromov–Witten theory. Our goal is to determine $\mathsf{C}_{\mathsf{GW}}(\widehat{\tau}_\alpha(\mathsf{p}) \mid \varnothing, \mu, \lambda)$ by the *same* induction and compatible with the correspondence of Theorem 8 for capped 2–leg descendent vertices,

We will consider integration relations from Gromov–Witten theory for

$$\mathsf{Z}_{\mathsf{GW}}(\alpha, \lambda, \mu')_\beta = \sum_g u^{2g-2} \sum_{\widehat{\alpha}\in\mathcal{P}_{|\alpha|}} \mathsf{K}_{\alpha,\widehat{\alpha}}(s_1-s_2, 2s_2, s_3) \times \int_{[\overline{M}'_{g,\ell(\widehat{\alpha})}(\mathcal{F}_2\times\mathbf{P}^1/\mathcal{D}_\infty, \mathsf{r}(\lambda,\mu'))_\beta]^{\mathrm{vir}}} \tau_{\widehat{\alpha}}([\star_0]). \tag{37}$$

The first issue to confront is the broken symmetry in the definition of K in Section 2.1. While K is symmetric in s_1 and s_2, the variable s_3 is treated differently. We orient K in (37) by setting the s_3 direction lie along $\mathbf{P}^1$ (which, conveniently, is also s_3 in the conventions of Section 4.2).

The formal analysis of the Gromov–Witten relations is identical to the above stable pairs analysis. For fixed α, there are only finitely many $\widehat{\alpha}$ which occur on the right side of

$$\widehat{\tau}_\alpha = \sum_{\widehat{\alpha}\in\mathcal{P}_{|\alpha|}} \mathsf{K}_{\alpha,\widehat{\alpha}}\tau_{\widehat{\alpha}}.$$

In order to make the integration relations compatible with Theorem 8, we must consider the matrix of capped 2–leg vertices

$$\mathsf{C}_{\mathsf{GW}}(1 \mid \varnothing, \widehat{\mu}', \mu') \tag{38}$$

when $\widehat{\mu}'$ varies among partitions of size $|\mu|$ and μ' varies among the infinite set of partitions satisfying

$$|\mu'| - \ell(\mu') > \mathrm{Max}\{|\widehat{\alpha}| + \ell(\widehat{\alpha}) \mid \mathsf{K}_{\alpha,\widehat{\alpha}} \neq 0\}.$$

As Proposition 15 still applies,[13] the matrix (38) is of maximal rank.

The inductive determination of the 2–leg descendent vertex via the integration relations therefore respects the correspondence of Theorem 8. □

4.7 Proof of Theorem 3

Since the homogeneity assertion was established in Section 2.1, the inclusion (27) is the only part of Theorem 3 which remains to be proven.

The base of the induction in the proof of Theorem 8 is the $\mu = \varnothing$ case of 1–leg established in Section 2. The orientation of K in (37) is perfect for the base case (as the 1–leg direction is then along P). Applying the induction, we prove Theorem 8 for the case where μ is arbitrary and $\lambda = \varnothing$. In other words, the 1–leg correspondence holds for the *same* matrix K when s_2 and s_3 are interchanged! By the uniqueness statement for the 1–leg descendent correspondence in Section 2.1, we conclude K is symmetric.

The u^k coefficients of K are polynomials in s_3 (having only poles at along s_1 and s_2). The proof is obtained from two observations. First, we have basic $\mathbf{T}$–equivariant proper maps from the stable pairs and stable maps spaces to the symmetric product of $\mathbb{C}^2$,

$$P_n(\mathsf{Cap} \mid \lambda)_d \to \mathrm{Sym}^d(\mathbb{C}^2),$$
$$\overline{M}_{g,\ell}(\mathsf{Cap} \mid \lambda)_d \to \mathrm{Sym}^d(\mathbb{C}^2);$$

see [28, Lemma 1]. The $\mathbf{T}$–action of the third torus factor corresponding to s_3 on $\mathrm{Sym}^d(\mathbb{C}^2)$ is trivial. Pushing-forward the integrals in both cases shows the u^k coefficients of the capped descendent invariants are polynomial in s_3. The matrix K is obtained in Section 2.1 from the matrices of 1–leg capped descendents after inversion and product. The second observation is that the determinants of the 1–leg capped descendent matrices (see the proofs of Lemmas 1 and 2) have no s_3 dependence. Hence, the u^k coefficients of K are polynomials in s_3.

By the symmetry, the u^k coefficients of K are polynomial in all the variables s_1, s_2 and s_3. □

[13] The correspondence for capped stable pair and Gromov–Witten vertices *without* descendents is used here.

5 Descendent correspondence: 3–leg

5.1 Overview

We now prove Theorem 8 in the 3–leg case. Consider the capped 3–leg descendent vertices

$$\mathsf{C}_{\mathsf{P}}(\tau_\alpha(\mathsf{p}) \mid \nu, \mu, \lambda), \quad \mathsf{C}_{\mathsf{GW}}(\tau_\alpha(\mathsf{p}) \mid \nu, \mu, \lambda).$$

Our proof of Theorem 8 will again be by induction on the complexity of the legs. The descendent insertion $\tau_\alpha(\mathsf{p})$ will be fixed for the argument. If $\nu = \varnothing$ or $\mu = \varnothing$, we are in the 2–leg case[14] where Theorem 8 has been established in Section 4. The 2–leg case will be the base of the induction.

Define a partial ordering on pairs of partitions (ν, μ, λ) satisfying the condition $(\nu, \mu, \lambda) \neq (\varnothing, \varnothing)$ by the following rules. We say

$$(\nu, \mu, \lambda) \rhd (\nu', \mu', \lambda')$$

if we have $|\nu| + |\mu| > |\nu'| + |\mu'|$. The proof of Theorem 8 in the 3–leg case is by induction with respect to the partial ordering $\rhd$.

The argument for descendent correspondence for the capped 3–leg vertex closely follows the 2–leg case. The main difference is to replace $\mathcal{A}_1$ geometry by $\mathcal{A}_2$ geometry.

5.2 $\mathcal{A}_2$ geometry

Let $\mathcal{A}_2 \subset \mathcal{F}$ be any nonsingular projective toric compactification. We will only be interested in the two (-2)–curves of $\mathcal{A}_2$,

$$C, \widehat{C} \subset \mathcal{A}_2.$$

No other curves of $\mathcal{F}$ will play a role in the construction.

Let $\widehat{\bullet}, \star, \bullet \in \mathcal{A}_2$ be the $(\mathbb{C}^*)^2$–fixed points. The curve $\widehat{C}$ connects $\widehat{\bullet}$ to $\star$ and C connects $\star$ to $\bullet$. The other $(\mathbb{C}^*)^2$–fixed points in $\mathcal{F} \setminus \mathcal{A}_2$ will not play an important role.

Consider the nonsingular projective toric variety $\mathcal{F} \times \mathbf{P}^1$. The 3–torus

$$\mathbf{T} = (\mathbb{C}^*)^3$$

acts on $\mathcal{F}$ via the first two factors and acts on $\mathbf{P}^1$ via the third factor with tangent weights s_3 and $-s_3$ at the points $0, \infty \in \mathbf{P}^1$ respectively. Let

$$\mathcal{D}_0 = \mathcal{F} \times \{0\}, \quad \mathcal{D}_\infty = \mathcal{F} \times \{\infty\}$$

[14] Since the symmetry of K has been proven, all the 2–leg cases are equivalent.

be $\mathbf{T}$–invariant divisors of $\mathcal{F}_2 \times \mathbf{P}^1$. The 3–fold $\mathcal{F} \times \mathbf{P}^1$ has six important $\mathbf{T}$–fixed points which we denote by

$$\widehat{\bullet}_0,\ \star_0,\ \bullet_0,\ \widehat{\bullet}_\infty,\ \star_\infty,\ \bullet_\infty \in \mathcal{F} \times \mathbf{P}^1,$$

where the subscript indicates the coordinate in $\mathbf{P}^1$.

Let $L_\infty \subset \mathcal{F} \times \mathbf{P}^1$ be the $\mathbf{T}$–invariant line connecting $\star_\infty$ to $(\mathcal{F} \setminus \mathcal{A}_2)_\infty$. We have

$$H_2(\mathcal{F} \times \mathbf{P}^1, \mathbb{Z}) \supset \mathbb{Z}[C] \oplus \mathbb{Z}[\widehat{C}] \oplus \mathbb{Z}[P],$$

where P is the fiber of the projection to $\mathcal{F}$.

5.3 Integration

We will find relations which express $\mathsf{C}(\tau_\alpha(\mathsf{p}) \mid \nu, \mu, \lambda)$ in terms of inductively treated vertices for stable pairs and Gromov–Witten theory. The inductive equations will respect the correspondence claimed in Theorem 8.

Let ν' and μ' be partitions. The relations will be obtained from vanishing invariants of the relative geometry $\mathcal{F} \times \mathbf{P}^1/\mathcal{D}_\infty$ in curve class

$$\beta = |\nu| \cdot [C] + |\mu| \cdot \widehat{C} + (|\lambda| + |\mu'| + |\nu'|) \cdot [P].$$

The virtual dimensions of the associated moduli spaces are

$$\dim^{\mathrm{vir}} P_n(\mathcal{F} \times \mathbf{P}^1, \beta) = 2|\lambda| + 2|\mu'| + 2|\nu'|,$$
$$\dim^{\mathrm{vir}} \overline{M}'_g(\mathcal{F} \times \mathbf{P}^1, \beta) = 2|\lambda| + 2|\mu'| + 2|\nu'|.$$

Relative conditions in $\mathrm{Hilb}(\mathcal{D}_\infty, |\lambda| + |\mu'| + |\nu'|)$ in the Nakajima basis are given by a $\mathbf{T}$–equivariant cohomology weighted partition of $|\lambda| + |\mu'| + |\nu'|$. We impose the relative condition determined by the partition

$$\lambda \cup \mu' \cup \nu' = \lambda_1 + \cdots + \lambda_{\ell(\lambda)} + \mu'_1 + \cdots + \mu'_{\ell(\mu')} + \nu'_1 + \cdots + \nu'_{\ell(\nu')}$$

weighted by $[\star_\infty] \in H^*_{\mathbf{T}}(\mathcal{D}_\infty, \mathbb{Q})$ for the parts of λ, $[\widehat{\bullet}_\infty] \in H^*_{\mathbf{T}}(\mathcal{D}_\infty, \mathbb{Q})$ for the parts of μ' and $[\bullet_\infty] \in H^*_{\mathbf{T}}(\mathcal{D}_\infty, \mathbb{Q})$ for the parts of ν'. We denote the relative condition by $\mathsf{r}(\lambda, \mu', \nu')$. After imposing $\mathsf{r}(\lambda, \mu', \nu')$, the virtual dimension drops to

$$\dim^{\mathrm{vir}} P_n(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \mathsf{r})_\beta = |\lambda| - \ell(\lambda) + |\mu'| - \ell(\mu') + |\nu'| - \ell(\nu'),$$
$$\dim^{\mathrm{vir}} \overline{M}'_g(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \mathsf{r})_\beta = |\lambda| - \ell(\lambda) + |\mu'| - \ell(\mu') + |\nu'| - \ell(\nu').$$

To define an equivariant integral, we specify the descendent insertion by

$$\tau_\alpha([\star_0]) = \tau_{\alpha_1 - 1}([\star_0]) \cdots \tau_{\alpha_{\ell(\alpha)} - 1}([\star_0]).$$

The descendent insertion imposes $|\alpha| + \ell(\alpha)$ conditions. Therefore, the integrals

$$\int_{[P_n(\mathcal{F}\times\mathbf{P}^1/\mathcal{D}_\infty,\mathsf{r})_\beta]^{\mathrm{vir}}} \tau_\alpha([\star_0]), \qquad \int_{[\overline{M}'_{g,\ell(\alpha)}(\mathcal{F}\times\mathbf{P}^1/\mathcal{D}_\infty,\mathsf{r})_\beta]^{\mathrm{vir}}} \tau_\alpha([\star_0]) \tag{39}$$

viewed as $\mathbf{T}$–equivariant push-forwards to a point, both have dimension

$$|\lambda| - \ell(\lambda) + |\mu'| - \ell(\mu') + |\nu'| - \ell(\nu') - |\alpha| - \ell(\alpha).$$

We conclude the following result.

Proposition 16 *If either* $|\mu'| - \ell(\mu')$ *or* $|\nu'| - \ell(\nu')$ *exceeds* $|\alpha| + \ell(\alpha)$, *then the* $\mathbf{T}$*–equivariant integrals* (39) *vanish for all Euler characteristics* n *and genera* g.

5.4 Proof of Theorem 8

Define the $\mathbf{T}$–equivariant series

$$\mathsf{Z}_{\mathsf{P}}(\alpha,\lambda,\mu',\nu')_\beta = \sum_n q^n \int_{[P_n(\mathcal{F}\times\mathbf{P}^1/\mathcal{D}_\infty,\mathsf{r}(\lambda,\mu',\nu'))_\beta]^{\mathrm{vir}}} \tau_\alpha([\star_0])$$

obtained from the stable pairs integrals (39). On the Gromov–Witten side, we consider the integrals

$$\mathsf{Z}_{\mathsf{GW}}(\alpha,\lambda,\mu',\nu')_\beta = \sum_g u^{2g-2} \sum_{\widehat{\alpha}\in\mathcal{P}_{|\alpha|}} \mathsf{K}_{\alpha,\widehat{\alpha}} \int_{[\overline{M}'_{g,\ell(\widehat{\alpha})}(\mathcal{F}\times\mathbf{P}^1/\mathcal{D}_\infty,\mathsf{r}(\lambda,\mu',\nu'))_\beta]^{\mathrm{vir}}} \tau_{\widehat{\alpha}}([\star_0]).$$

Since we have already established the symmetry of K in the variable s_i, we no longer need to worry about the orientation of K. When both $|\mu'| - \ell(\mu')$ and $|\nu'| - \ell(\nu)$ exceed

$$\mathrm{Max}\{|\widehat{\alpha}| + \ell(\widehat{\alpha}) | \mathsf{K}_{\alpha,\widehat{\alpha}} \neq 0\},$$

Proposition 16 implies

$$\mathsf{Z}_{\mathsf{P}}(\alpha,\lambda,\mu',\nu')_\beta = 0, \quad \mathsf{Z}_{\mathsf{GW}}(\alpha,\lambda,\mu',\nu')_\beta = 0. \tag{40}$$

The inductive analysis of the capped localization of stable pairs and Gromov–Witten relations (40) exactly follows the treatment given in Section 4.4 for the 2–leg case. The outcome is an inductive determination of the capped descendent 3–leg vertices in terms of the capped descendent 2–leg vertices which respects the correspondence of Theorem 8. The maximal rank result of Proposition 15 is used twice.

6 First consequences

6.1 Descendents over $\mathbf{0} \in \mathbf{P}^1$

Since Theorems 3 and 8 together imply Theorem 1, we have proven the matrix K determines a Gromov–Witten/Pairs descendent correspondence for all nonsingular quasi-projective toric 3–folds X.

Recall the surfaces $\mathcal{A}_n$ defined in Section 4.2. As before, let

$$\mathcal{D}_\infty \subset \mathcal{A}_n \times \mathbf{P}^1$$

be the fiber over $\infty \in \mathbf{P}^1$. The relative geometry $\mathcal{A}_n \times \mathbf{P}^1/\mathcal{D}_\infty$ is equivariant with respect to the action of 3–dimensional torus

$$\mathbf{T} = T \times \mathbb{C}^*,$$

where the 2–dimensional torus T acts on $\mathcal{A}_n$ and $\mathbb{C}^*$ acts on $\mathbf{P}^1$ with fixed points $0, \infty \in \mathbf{P}^1$. The weights associated to T are s_1, s_2, and the weight associated to $\mathbb{C}^*$ is s_3. Let

$$p_0, \dots, p_n \in \mathcal{D}_0$$

be the $\mathbf{T}$–fixed points lying over $0 \in \mathbf{P}^1$.

Using the correspondence of Theorem 8 for capped descendent vertices over $0 \in \mathbf{P}^1$ and the correspondence for capped $\mathcal{A}_n$–rubber ([18, Lemma 6] together with [20, Section 5]), we immediately conclude the following result.

Proposition 17 *For the $\mathbf{T}$–equivariant relative geometry $\mathcal{A}_n \times \mathbf{P}^1/\mathcal{D}_\infty$, after the variable change $-q = e^{iu}$, we have*

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\Big(\mathcal{A}_n \times \mathbf{P}^1/\mathcal{D}_\infty; q \;\Big|\; \prod_{j=0}^{n} \tau_{\alpha^{(j)}}(\mathsf{p}_j) \;\Big|\; \mu\Big)^{\mathbf{T}}_\beta$$
$$= (-iu)^{d_\beta + \ell(\mu) - |\mu|} \mathsf{Z}'_{\mathsf{GW}}\Big(\mathcal{A}_n \times \mathbf{P}^1/\mathcal{D}_\infty; u \;\Big|\; \prod_{j=0}^{n} \widehat{\tau}_{\alpha^{(j)}}(\mathsf{p}_j) \;\Big|\; \mu\Big)^{\mathbf{T}}_\beta,$$

where $\alpha^{(0)}, \dots, \alpha^{(n)}$ are partitions, μ is a $\mathbf{T}$–equivariant relative condition along $\mathcal{D}_\infty$, and $\beta \in H_2(\mathcal{A}_n \times \mathbf{P}^1, \mathbb{Z})$ is any curve class.

Since the coefficients of the matrix K have no poles in the s_i by Theorem 3, we can restrict the correspondence of Proposition 17 to $s_3 = 0$ (so long as the relative conditions μ have no denominators in s_3). We will typically take μ to have no s_3 dependence at all. As an application of Proposition 17, we will prove Theorem 6.

6.2 Proof of Theorem 6

Let $\mathsf{L} \in H_2(\mathcal{A}_n \times \mathbf{P}^1, \mathbb{Z})$ be the class of the factor $\mathbf{P}^1$. For the 3–fold $\mathcal{A}_n \times \mathbf{P}^1$, we can uniquely write a curve class β as

$$\beta = d\mathsf{L} + \mathsf{F},$$

where $\mathsf{F} \in H_2(\mathcal{A}_n, \mathbb{Z})$ is a fiber class and

$$d_\beta = \int_\beta c_1(\mathcal{A}_n \times \mathbf{P}^1) = 2d.$$

For fixed d, we define a matrix square matrix

$$M_{\mathsf{P},d}(\alpha^{(0)}, \dots, \alpha^{(n)} \mid \mu)$$

with columns indexed by $(n+1)$–tuples of partitions

$$\alpha^{(0)}, \dots, \alpha^{(n)}, \quad \sum_{j=0}^{n} |\alpha|^{(j)} = d,$$

rows indexed by relative conditions μ in the Nakajima basis[15] of the T–equivariant cohomology of $\mathrm{Hilb}(\mathcal{A}_n, d)$ with respect to the classes $\mathsf{p}_0, \dots, \mathsf{p}_n$, and with matrix coefficients

$$\sum_{F \in H_2(\mathcal{A}_n, \mathbb{Z})} Q^F (-q)^{-d} \mathsf{Z}_{\mathsf{P}}\left(\mathcal{A}_n \times \mathbf{P}^1/\mathcal{D}_\infty; q \;\Big|\; \prod_{j=0}^{n} \tau_{\alpha^{(j)}}(\mathsf{p}_j) \;\Big|\; \mu\right)^{\mathbf{T}}_{d\mathsf{L}+\mathsf{F}}.$$

Lemma 18 *For all $d > 0$, the matrix $M_{\mathsf{P},d}$ is invertible and remains invertible after restriction $M_{\mathsf{P},d}|_{s_3=0}$.*

Proof We may prove invertibility after restriction to $Q = 0$. The issue then separates to the invertibility of matrices of caps determined at each p_i. The capped geometry $\mathbf{P}^1/\infty$ associated to p_i is the line

$$p_i \times \mathbf{P}^1 \subset \mathcal{A}_n \times \mathbf{P}^1.$$

The required invertibility is then obtained from Lemma 8. □

By the correspondence of Proposition 17, the Gromov–Witten matrix $M_{\mathsf{GW},d}$ with the same indices and coefficients

$$\sum_{F \in H_2(\mathcal{A}_n, \mathbb{Z})} Q^F (-iu)^{2d+\ell(\mu)-|\mu|} \mathsf{Z}_{\mathsf{GW}}\left(\mathcal{A}_n \times \mathbf{P}^1/\mathcal{D}_\infty; u \;\Big|\; \prod_{j=0}^{n} \widehat{\tau}_{\alpha^{(j)}}(\mathsf{p}_j) \;\Big|\; \mu\right)^{\mathbf{T}}_{d\mathsf{L}+\mathsf{F}}$$

[15] The Nakajima basis here is given by assigning a partition to each p_i.

is also invertible.

To prove Theorem 6, we restrict to the fiberwise T–action. Let p_j^0, p_j^1, p_j^∞ be the T–fixed points of $\mathcal{A}_n$ corresponding to p_j in the fibers over the points $0, 1, \infty \in \mathbf{P}^1$ respectively. We consider the stable pairs series for $\mathcal{A}_n \times \mathbf{P}^1$

$$\sum_{F \in H_2(\mathcal{A}_n, \mathbb{Z})} Q^F (-q)^{-d} \mathsf{Z}_{\mathsf{P}} \Big(\prod_{j=0}^n \tau_{\alpha^{(j)}}(\mathsf{p}_j^0) \prod_{j=0}^n \tau_{\beta^{(j)}}(\mathsf{p}_j^1) \prod_{j=0}^n \tau_{\gamma^{(j)}}(\mathsf{p}_j^\infty) \Big)_{d\mathsf{L}+\mathsf{F}}^T,$$

where we have

$$\sum_{j=0}^n |\alpha^{(j)}| = \sum_{j=0}^n |\beta^{(j)}| = \sum_{j=0}^n |\gamma^{(j)}| = d.$$

By the correspondence of Theorem 1, the above series equals

$$\sum_{F \in H_2(\mathcal{A}_n, \mathbb{Z})} Q^F (-iu)^{2d} \mathsf{Z}_{\mathsf{GW}} \Big(\prod_{j=0}^n \widehat{\tau}_{\alpha^{(j)}}(\mathsf{p}_j^0) \prod_{j=0}^n \widehat{\tau}_{\beta^{(j)}}(\mathsf{p}_j^1) \prod_{j=0}^n \widehat{\tau}_{\gamma^{(j)}}(\mathsf{p}_j^\infty) \Big)_{d\mathsf{L}+\mathsf{F}}^T.$$

We degenerate the descendents over $0, 1, \infty$ to three $\mathcal{A}_n$–caps. Using the compatibility of the above correspondences with the degeneration formula and the invertibility of $M_{\mathsf{P},d}|_{s_3=0}$, we obtain the correspondence of Theorem 6. □

6.3 Descendents on $S \times \mathbf{P}^1/S_\infty$

Let S be a nonsingular quasi-projective toric surface. Let $S_\infty \subset S \times \mathbf{P}^1$ be the fiber over $\infty \in \mathbf{P}^1$. The basic relative geometry $S \times \mathbf{P}^1/S_\infty$ is equivariant with respect to the full 3–dimensional torus

$$\mathbf{T} = T \times \mathbb{C}^*,$$

where the 2–dimensional torus T acts on S and $\mathbb{C}^*$ acts on $\mathbf{P}^1$ with fixed points $0, \infty \in \mathbf{P}^1$. The weights associated to T are s_1, s_2 and the weight associated to $\mathbb{C}^*$ is s_3.

Let $p_1, \dots, p_m$ be the $\mathbf{T}$–fixed points of $S \times \mathbf{P}^1$ lying over $0 \in \mathbf{P}^1$. As before, let $\mathsf{L} \in H_2(S \times \mathbf{P}^1, \mathbb{Z})$ be the curve class of the factor $\mathbf{P}^1$. For the class $d\mathsf{L}$, we have

$$d_{d\mathsf{L}} = \int_{d\mathsf{L}} c_1(S \times \mathbf{P}^1) = 2d.$$

Proposition 19 *For the* $\mathbf{T}$*–equivariant relative geometry* $S\times\mathbf{P}^1/S_\infty$*, after the variable change* $-q=e^{iu}$*, we have*

$$(-q)^{-d}\mathsf{Z}_{\mathsf{P}}\left(S\times\mathbf{P}^1/S_\infty;q\;\Big|\;\prod_{j=1}^{m}\tau_{\alpha^{(j)}}(\mathsf{p}_j)\;\Big|\;\mu\right)^{\mathbf{T}}_{d\mathsf{L}} = (-iu)^{d+\ell(\mu)}\mathsf{Z}'_{\mathsf{GW}}\left(S\times\mathbf{P}^1/S_\infty;u\;\Big|\;\prod_{j=1}^{m}\widehat{\tau}_{\alpha^{(j)}}(\mathsf{p}_j)\;\Big|\;\mu\right)^{\mathbf{T}}_{d\mathsf{L}},$$

where $\alpha^{(0)},\dots,\alpha^{(n)}$ *are partitions,* μ *is a* $\mathbf{T}$*–equivariant relative condition along* S_∞*, and* $d>0$*.*

Proof The result is immediate from the 1–leg correspondence of Theorem 8. Each fixed point p_i of S is contained in a torus invariant open $U\cong\mathbb{C}^2$. The open set

$$U\times\mathbf{P}^1/U_\infty\subset S\times\mathbf{P}^1/S_\infty$$

is simply the cap. Since the curve class is $d\mathsf{L}$, we can directly reduce the proposition to the correspondence for 1–leg capped descendent vertices. □

We will apply Proposition 19 in case $S=\mathbf{P}^1\times\mathbf{P}^1$ to study the non-equivariant limit of the descendent correspondence in Section 7. We will require there also the following technical divisibility result valid for all projective S.

Let S be a nonsingular projective toric surface, and let $p\in S\times\mathbf{P}^1$ be a toric fixed point lying over $0\in\mathbf{P}^1$. Let

$$F(\tau(\mathsf{p}))=\sum_{\alpha}\mathsf{C}_\alpha\tau_\alpha(\mathsf{p})$$

be a finite sum over α of positive size and $\mathsf{C}_\alpha\in\mathbb{Q}[i,s_1,s_2,s_3]((u))$.

Proposition 20 *If the divisibility*

$$s_1^k\mid\mathsf{Z}'_{\mathsf{GW}}\big(S\times\mathbf{P}^1/S_\infty;u\mid F(\tau(\mathsf{p}))\mid\mu\big)^{\mathbf{T}}_{d\mathsf{L}}$$

holds for all $d>0$ *and all relative conditions* μ *with cohomology weights in* $H^*_{\mathbf{T}}(S,\mathbb{Q})$*, then* s_1^k *divides* F*.*

Since S is projective, the series $\mathsf{Z}'_{\mathsf{GW}}(S\times\mathbf{P}^1/S_\infty;u\mid F(\tau(\mathsf{p}))\mid\mu)^{\mathbf{T}}_{d\mathsf{L}}$ has no poles in the s_i, so the divisibility hypothesis in Proposition 20 is sensible.

Proof We can write F in the form[16]

$$F=\sum_{m>0} F_m, \quad F_m=\sum_{|\alpha|+\ell(\alpha)=m} \mathsf{C}_\alpha \tau_\alpha(\mathsf{p}).$$

We argue by contradiction. Let F_M be the largest M for which F_M is not divisible by s_1^k. Since the higher $F_{m>M}$ are divisible by s_1^k, the hypothesis implies

$$s_1^k \mid \mathsf{Z}'_{\mathsf{GW}}\Bigl(S\times\mathbf{P}^1/S_\infty; u \Bigm| \sum_{m=2}^{M} F_m \Bigm| \mu\Bigr)^{\mathbf{T}}_{d\mathsf{L}}$$

for all $d>0$ and all relative conditions μ.

The next step is a simple dimension analysis. The codimension of the class $\tau_\alpha(\mathsf{p})$ is $|\alpha|+\ell(\alpha)$. To each partition γ, we associate the relative condition γ along S_∞ with all cohomology weights equal to $1\in H^*_{\mathbf{T}}(S,\mathbb{Q})$. The virtual dimension of the relative moduli space of maps to $S\times\mathbf{P}^1/S_\infty$ of class $|\gamma|\mathsf{L}$ satisfying the relative condition γ is $|\gamma|+\ell(\gamma)$. By compactness and dimension constraints, we obtain the vanishing

$$\mathsf{Z}'_{\mathsf{GW}}\bigl(S\times\mathbf{P}^1/S_\infty; u \mid \tau_\alpha(\mathsf{p}) \mid \gamma\bigr)^{\mathbf{T}}_{|\gamma|\mathsf{L}}=0$$

when $|\alpha|+\ell(\alpha)<|\gamma|+\ell(\gamma)$. Hence, we see that

$$s_1^k \mid \mathsf{Z}'_{\mathsf{GW}}\bigl(S\times\mathbf{P}^1/S_\infty; u \mid F_M \mid \gamma\bigr)^{\mathbf{T}}_{|\gamma|\mathsf{L}} \tag{41}$$

for all partitions γ satisfying $|\gamma|+\ell(\gamma)=M$.

Next, we consider the matrix indexed by partitions α and γ satisfying

$$|\alpha|+\ell(\alpha)=M, \quad |\gamma|+\ell(\gamma)=M \tag{42}$$

with coefficients

$$\mathsf{Z}'_{\mathsf{GW}}\bigl(S\times\mathbf{P}^1/S_\infty; u \mid \tau_\alpha(\mathsf{p}) \mid \gamma\bigr)^{\mathbf{T}}_{|\gamma|\mathsf{L}}. \tag{43}$$

By the dimension constraints, the coefficient (43) is independent of the equivariant parameters; we can treat the coefficient as a non-equivariant integral. Hence, we can calculate (43) by separating the points in the descendent insertion. We replace $\tau_\alpha(\mathsf{p})$ by

$$\tau_{a_1-1}(\mathsf{p}'_1)\cdots\tau_{a_{\ell(\alpha)}-1}(\mathsf{p}'_{\ell(\alpha)}) \tag{44}$$

for distinct points $p'_1,\dots,p'_{\ell(\alpha)}\in S$. If $\ell(\alpha)>\ell(\gamma)$, then the corresponding coefficient (43) certainly vanishes as the relative condition does not have enough parts[17] to satisfy the incidences with $p'_1,\dots,p'_{\ell(\alpha)}$.

[16] Note the minimum value of $|\alpha|+\ell(\alpha)$ for α of positive size is 2.

[17] Remember the class of the curve is degree 0 when projected to S.

The matrix (43) is block triangular with blocks given by the equal length condition

$$\ell(\alpha) = \ell(\gamma).$$

The equal length condition implies $|\alpha| = |\gamma|$ by (42). Using the separated insertion (44) and further dimension counting, we conclude the coefficient (43) vanishes in the block unless $\alpha = \gamma$.

We have proven the matrix (43) is triangular. The diagonal elements $\triangle_\alpha$ are determined by (16) and are non-zero (with no s_1 dependence). The divisibility of the coefficients of F_M by s_1^k then immediately follows from (41). □

The proof of Proposition 20 provides the first step of the proof[18] of Proposition 12. Recall the partial ordering $\trianglerighteq^*$ on partitions:

$$\alpha \trianglerighteq^* \widetilde{\alpha} \quad \Longleftrightarrow \quad |\alpha| + \ell(\alpha) \geq |\widetilde{\alpha}| + \ell(\widetilde{\alpha})$$

Proposition 12 *We have*

$$\widehat{\tau}_\alpha(\mathsf{p}) = (iu)^{\ell(\alpha)-|\alpha|}\tau_\alpha(\mathsf{p}) + \cdots,$$

where the dots stand for terms $\tau_{\widehat{\alpha}}$ with $\alpha \triangleright^* \widehat{\alpha}$.

Proof Let S be any nonsingular projective toric surface, and let $p \in S$ be a toric fixed point. By Proposition 19,

$$(-q)^{-d}\mathsf{Z}_{\mathsf{P}}\big(S \times \mathbf{P}^1/S_\infty; q \mid \tau_\alpha(\mathsf{p}) \mid \mu\big)^{\mathbf{T}}_{d\mathsf{L}} = (-iu)^{d+\ell(\mu)}\mathsf{Z}'_{\mathsf{GW}}\bigg(S \times \mathbf{P}^1/S_\infty; u \,\bigg|\, \sum_{\widehat{\alpha}} \mathsf{K}_{\alpha,\widehat{\alpha}}\tau_{\widehat{\alpha}}(\mathsf{p}) \,\bigg|\, \mu\bigg)^{\mathbf{T}}_{d\mathsf{L}}$$

for all d.

Consider the set of $\mathcal{P}$ of partitions $\widehat{\alpha}$ with $|\alpha| \geq |\widehat{\alpha}|$ which maximize $|\widehat{\alpha}| + \ell(\widehat{\alpha})$ subject to the condition $\mathsf{K}_{\alpha,\widehat{\alpha}} \neq 0$. Let $\gamma \in \mathcal{P}$ minimize $\ell(\gamma)$. We view γ as a relative condition along S_∞ with all cohomology weights equal to $1 \in H^*_{\mathbf{T}}(S,\mathbb{Q})$. If $\gamma \triangleright^* \alpha$, then

$$(-q)^{-d}\mathsf{Z}_{\mathsf{P}}\big(S \times \mathbf{P}^1/S_\infty; q \mid \tau_\alpha(\mathsf{p}) \mid \gamma\big)^{\mathbf{T}}_{|\gamma|\mathsf{L}} = 0,$$

$$(-iu)^{d+\ell(\mu)}\mathsf{Z}'_{\mathsf{GW}}\bigg(S \times \mathbf{P}^1/S_\infty; u \,\bigg|\, \sum_{\widehat{\alpha}} \mathsf{K}_{\alpha,\widehat{\alpha}}\tau_{\widehat{\alpha}}(\mathsf{p}) \,\bigg|\, \gamma\bigg)^{\mathbf{T}}_{|\gamma|\mathsf{L}} = \mathsf{K}_{\alpha,\gamma} \cdot \triangle_\gamma,$$

[18] Proposition 12 was stated in Section 2.3, but not used except as part of the proof of Theorem 2. The proof of Proposition 12 here also completes the proof of Theorem 2.

where $\triangle_\gamma$ is non-zero. Here, we have used the geometric vanishing arguments of the proof of Proposition 20 for both stable pairs and stable maps. Hence, $\mathsf{K}_{\alpha,\gamma} = 0$ which is a contradiction. We conclude that

$$\widehat{\alpha} \rhd^* \alpha \quad \Longrightarrow \quad \mathsf{K}_{\alpha,\widehat{\alpha}} = 0.$$

To prove the proposition, we need now only consider $\widehat{\alpha}$ for which

$$|\alpha| + \ell(\alpha) = |\widehat{\alpha}| + \ell(\widehat{\alpha}).$$

Specifically, we need to exactly match

$$(-q)^{-|\widehat{\alpha}|} \mathsf{Z}_{\mathsf{P}}\big(S \times \mathbf{P}^1 / S_\infty; q \mid \tau_\alpha(\mathsf{p}) \mid \widehat{\alpha}\big)^{\mathbf{T}}_{|\widehat{\alpha}|\mathsf{L}} \tag{45}$$

with the series

$$\begin{aligned}(-iu)^{|\widehat{\alpha}|+\ell(\widehat{\alpha})} \mathsf{Z}'_{\mathsf{GW}}\big(S \times \mathbf{P}^1 / S_\infty; u \mid (iu)^{\ell(\alpha)-|\alpha|} \tau_\alpha(\mathsf{p}) \mid \widehat{\alpha}\big)^{\mathbf{T}}_{|\widehat{\alpha}|\mathsf{L}} \\ = (-1)^{|\alpha|} u^{2\ell(\alpha)} \mathsf{Z}'_{\mathsf{GW}}\big(S \times \mathbf{P}^1 / S_\infty; u \mid \tau_\alpha(\mathsf{p}) \mid \widehat{\alpha}\big)^{\mathbf{T}}_{|\widehat{\alpha}|\mathsf{L}}.\end{aligned} \tag{46}$$

The exact matching is proven in Section 6.4. □

6.4 Matching

Let Y be a nonsingular surface, and let $E \subset Y$ be a nonsingular curve. Let $\mathsf{L} \in H_2(Y \times \mathbf{P}^1, \mathbb{Z})$ be the curve class of the factor $\mathbf{P}^1$. The divisor

$$E \times \mathbf{P}^1 \subset Y \times \mathbf{P}^1$$

intersects the divisor $Y_\infty \subset Y \times \mathbf{P}^1$ lying over $\infty \in \mathbf{P}^1$. We will consider here the relative geometry

$$Y \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup Y_\infty \tag{47}$$

for the curve classes $d\mathsf{L}$ in both stable pairs and Gromov–Witten theory.

For normal crossings boundary, the most promising approach to the relative theories is via log geometry [7]. However, our case is very simple since we are only considering the curve classes $d\mathsf{L}$. Since

$$\mathsf{L} \cdot [E \times \mathbf{P}^1] = 0,$$

our curves never meet $E \times \mathbf{P}^1$ and the delicate choices required for curves passing through the singularities of $E \times \mathbf{P}^1 \cup Y_\infty$ can be completely avoided. The moduli spaces

$$P_n(Y \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup Y_\infty, d\mathsf{L})_\mu,$$
$$\overline{M}'_{g,r}(Y \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup Y_\infty, d\mathsf{L})_\mu$$

are easily defined. In both cases, the projections of the curves to Y are never allowed to meet E; bubbling occurs along E to keep the projections away. The relative boundary conditions over $\infty \in \mathbf{P}^1$ are then imposed as usual. The points of the resulting moduli spaces corresponds to stable pairs or stable maps which meet (the degeneration of) Y_∞ away from the intersection with $E \times \mathbf{P}^1$. Hence, the deformation theories, virtual classes and degeneration formulas are all standard. Further details concerning the moduli problem here are given in Section 6.5.

Let α and $\widehat{\alpha}$ be two partitions of positive size satisfying

$$|\alpha| \geq |\widehat{\alpha}|, \quad |\alpha| + \ell(\alpha) = |\widehat{\alpha}| + \ell(\widehat{\alpha})$$

as in the proof of Proposition 12. The required matching of (45) and (46) concerns the relative geometry $S \times \mathbf{P}^1 / S_\infty$. By a dimension analysis, the series (45) and (46) have no dependence on the equivariant parameters s_i. Therefore, we can replace $\tau_\alpha(\mathsf{p})$ by

$$\tau_{a_1-1}(\mathsf{p}'_1) \cdots \tau_{a_{\ell(\alpha)}-1}(\mathsf{p}'_{\ell(\alpha)}),$$

where the points $p'_i \in S$ are distinct. Furthermore, we can degenerate to the normal cone of $p'_i \subset S$ for each p'_i. The limit of p'_i then lies on surface $\mathbf{P}^2/E$ where $E \subset \mathbf{P}^2$ is a line. We immediately conclude

$$\mathsf{Z}_{\mathsf{P}}\big(S \times \mathbf{P}^1 / S_\infty; q \mid \tau_\alpha(\mathsf{p}) \mid \widehat{\alpha}\big)^{\mathbf{T}}_{|\widehat{\alpha}|\mathsf{L}} = \sum_{\widehat{\alpha} = \bigcup_i \gamma^{(i)}} \prod_{i=1}^{\ell(\alpha)} \mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}^2 \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup \mathbf{P}^2_\infty; q \mid \tau_{\alpha_i - 1}(\mathsf{p}'_i) \mid \gamma^{(i)}\big)^{\mathbf{T}}_{|\gamma^{(i)}|\mathsf{L}},$$

where the sum on the right is over all ways of writing $\widehat{\alpha}$ as a union of $\ell(\alpha)$ disjoint subpartitions $\gamma^{(i)}$ satisfying

$$\alpha_i + 1 = |\gamma^{(i)}| + \ell(\gamma^{(i)}).$$

Another degeneration argument implies

$$\mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}^2 \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup \mathbf{P}^2_\infty; q \mid \tau_{\alpha_i - 1}(\mathsf{p}'_i) \mid \gamma^{(i)}\big)^{\mathbf{T}}_{|\gamma^{(i)}|\mathsf{L}} = \mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}^2 \times \mathbf{P}^1 / \mathbf{P}^2_\infty; q \mid \tau_{\alpha_i - 1}(\mathsf{p}'_i) \mid \gamma^{(i)}\big)^{\mathbf{T}}_{|\gamma^{(i)}|\mathsf{L}}$$

If $\gamma^{(i)}$ equals $\mu^{(i)} \cup (1^{m_i})$, where $\mu^{(i)}$ has no parts equal to 1, then localization with respect to the torus action on $\mathbf{P}^2$ yields

$$\mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}^2 \times \mathbf{P}^1/\mathbf{P}^2_\infty; q \mid \tau_{a_i-1}(\mathsf{p}_i') \mid \gamma^{(i)}\big)^{\mathbf{T}}_{|\gamma^{(i)}|\mathsf{L}} = \sum_{\substack{e_0+e_1+\cdots+e_j=m_i \\ e_0\ge 0,\, e_k>0 \text{ for } k>0}} \Big((-1)^j \mathsf{Z}_{\mathsf{P}}\big(\mathsf{Cap}; q \mid \tau_{\alpha_i-1}(\mathsf{p}) \mid \mu^{(i)} \cup (1^{e_0})\big)^{\mathbf{T}}_{(|\mu^{(i)}|+e_0)\mathsf{L}} \times \prod_{k=1}^{j} \mathsf{Z}_{\mathsf{P}}\big(\mathsf{Cap}; q \mid 1 \mid (1^{e_k})\big)^{\mathbf{T}}_{e_k\mathsf{L}}\Big).$$

The parallel formulas hold in Gromov–Witten theory.

The matching of (45) and (46) is then a consequence of the following calculation.

Lemma 21 *Let $\gamma = \mu \cup (1^m)$ be a partition of positive size where μ has no parts equal to 1. Let $a+1 = |\gamma| + \ell(\gamma)$. Then*

$$\sum_{\substack{e_0+e_1+\cdots+e_j=m \\ e_0\ge 0,\, e_k>0 \text{ for } k>0}} \Big((-1)^j \mathsf{Z}_{\mathsf{P}}\big(\mathsf{Cap}; q \mid \tau_{a-1}(\mathsf{p}) \mid \mu \cup (1^{e_0})\big)^{\mathbf{T}}_{(|\mu|+e_0)\mathsf{L}} \times \prod_{k=1}^{j} \mathsf{Z}_{\mathsf{P}}\big(\mathsf{Cap}; q \mid 1 \mid (1^{e_k})\big)^{\mathbf{T}}_{e_k\mathsf{L}}\Big) = q^{|\gamma|} \frac{(-1)^{\ell(\gamma)-1}}{|\gamma|!|\mathrm{Aut}(\gamma)|},$$

$$\sum_{\substack{e_0+e_1+\cdots+e_j=m \\ e_0\ge 0,\, e_k>0 \text{ for } k>0}} \Big((-1)^j \mathsf{Z}'_{\mathsf{GW}}\big(\mathsf{Cap}; q \mid \tau_{a-1}(\mathsf{p}) \mid \mu \cup (1^{e_0})\big)^{\mathbf{T}}_{(|\mu|+e_0)\mathsf{L}} \times \prod_{k=1}^{j} \mathsf{Z}'_{\mathsf{GW}}\big(\mathsf{Cap}; q \mid 1 \mid (1^{e_k})\big)^{\mathbf{T}}_{e_k\mathsf{L}}\Big) = u^{-2} \frac{1}{|\gamma|!|\mathrm{Aut}(\gamma)|}.$$

Proof The Gromov–Witten calculation is well-known. In fact, the result is just a genus-0 connected Gromov–Witten invariant determined as a special case of [23, Theorem 2].

We require the stable pairs evaluation. We know that the expression has no dependence on s_1 and s_2, actually state why somewhere so we can work mod s_1+s_2 as in [25] and Lemma 11. Localization yields the formula (true mod s_1+s_2)

$$\mathsf{Z}_{\mathsf{P}}\big(\mathsf{Cap}; q \mid \tau_{a-1}(\mathsf{p}) \mid \mu \cup (1^{e_0})\big)^{\mathbf{T}}_{(|\mu|+e_0)\mathsf{L}} \equiv \frac{(-1)^{\ell(\gamma)-1}(s_1s_2)^{e_1+\cdots+e_j} q^{|\mu|+e_0}}{(a+1)!(|\mu|+e_0)!e_0!\mathfrak{z}(\mu)} \sum_{\sigma\vdash|\mu|+e_o} \chi_\sigma\big((1)^{|\mu|+e_0}\big)\chi_\sigma\big(\mu\cup(1^{e_0})\big) \times \sum_{\substack{\square\in\sigma \\ c=c(\square)}} \big((c-1)^{a+1} - 2c^{a+1} + (c+1)^{a+1}\big),$$

where χ_σ is the character of the irreducible representation of the symmetric group corresponding to σ and $c(\Box)$ is the content of a square in a partition.

The crucial step in the evaluation is the application of the character sum identity

$$\begin{aligned}(48)\qquad &\sum_{\sigma\vdash|\mu|+e} \chi_\sigma\big((1)^{|\mu|+e}\big)\chi_\sigma\big(\mu\cup(1^e)\big) \sum_{\substack{\Box\in\sigma\\ c=c(\Box)}} X^c \\ &= \bigg(\sum_{i=0}^{e} i!\binom{e}{i}\binom{|\mu|+e}{i}\big(X^{\frac12}-X^{-\frac12}\big)^{|\mu|+2e-2i-2}\bigg)\prod_{k=1}^{\ell(\mu)}\big(X^{\frac{\mu_k}{2}}-X^{-\frac{\mu_k}{2}}\big),\end{aligned}$$

valid for μ with no part of size 1. Before proving this identity, we will complete the proof of Lemma 21. Using (48) along with the basic evaluation

$$\mathsf{Z}_{\mathsf{P}}\big(\mathsf{Cap}; q \mid 1 \mid (1^e)\big)^{\mathbf{T}}_{e\mathsf{L}} = \frac{1}{e!(s_1s_2)^e},$$

we see the stable pair expression we need to compute is given by replacing X^c with $(c-1)^{a+1}-2c^{a+1}+(c+1)^{a+1}$ in the Laurent polynomial

$$\begin{aligned}&(-1)^{\ell(\gamma)-1}q^{|\gamma|} \sum_{\substack{e_0+e_1+\cdots+e_j=m\\ e_0\geq 0,\, e_k>0 \text{ for } k>0}} \frac{(-1)^j}{\prod_{k=1}^j e_k!}\frac{1}{(a+1)!(|\mu|+e_0)!e_0!\mathfrak{z}(\mu)} \\ &\times\bigg(\sum_{i=0}^{e_0} i!\binom{e_0}{i}\binom{|\mu|+e_0}{i}\big(X^{\frac12}-X^{-\frac12}\big)^{|\mu|+2e_0-2i-2}\bigg)\prod_{k=1}^{\ell(\mu)}\big(X^{\frac{\mu_k}{2}}-X^{-\frac{\mu_k}{2}}\big).\end{aligned}$$

Fortunately, most of the terms in this double sum cancel. If $0\leq r\leq m$, then the coefficient of

$$q^{|\gamma|}\big(X^{\frac12}-X^{-\frac12}\big)^{|\mu|+2m-2r-2}\prod_{k=1}^{\ell(\mu)}\big(X^{\frac{\mu_k}{2}}-X^{-\frac{\mu_k}{2}}\big)$$

appearing above is

$$\begin{aligned}&\frac{(-1)^{\ell(\gamma)-1}}{(a+1)!(m-r)!(|\mu|+m-r)!\mathfrak{z}(\mu)} \sum_{\substack{e_0+e_1+\cdots+e_j=r\\ e_0\geq 0,\, e_k>0 \text{ for } k>0}} \frac{(-1)^j}{\prod_{k=0}^j e_k!}. \\ &\qquad = \begin{cases} \dfrac{(-1)^{\ell(\gamma)-1}}{(a+1)!m!(|\mu|+m)!\mathfrak{z}(\mu)} & \text{if } r=0,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

because terms in the sum with $e_0=0$ can be paired off with those with $e_0>0$.

Thus, we just need to compute the result of replacing X^c with

$$(c-1)^{a+1} - 2c^{a+1} + (c+1)^{a+1}$$

in the Laurent polynomial

$$\frac{(-1)^{\ell(\gamma)-1}}{(a+1)!m!(|\mu|+m)!\mathfrak{z}(\mu)} q^{|\gamma|} \big(X^{\frac{1}{2}} - X^{-\frac{1}{2}}\big)^{|\mu|+2m-2} \prod_{k=1}^{\ell(\mu)} i\big(X^{\frac{\mu_k}{2}} - X^{-\frac{\mu_k}{2}}\big).$$

Since the above polynomial is divisible by $(X-1)^{a-1}$, this is simply a matter of differentiating $a-1$ times with respect to X, setting $X=1$, and then multiplying by $(a+1)!$. We find

$$\frac{(-1)^{\ell(\gamma)-1}}{m!(|\mu|+m)!\mathfrak{z}(\mu)} q^{|\gamma|} \prod_{k=1}^{\ell(\mu)} \mu_k = q^{|\gamma|} \frac{(-1)^{\ell(\gamma)-1}}{|\gamma|!\,|\mathrm{Aut}(\gamma)|}$$

as desired.

We now return to (48). The proof of this identity requires only a slight modification of the arguments used in the proof of Theorem A.1 in [5], so we will only give a sketch here.

Let $n = |\mu| + e$ and define the Jucys–Murphy elements L_i $(1 \le i \le n)$ as sums of transpositions

$$L_i = (1,i) + (2,i) + \cdots + (i-1,i)$$

in the group algebra $\mathbb{C}[S_n]$ of the symmetric group S_n. The elements L_i have the following property: For any polynomial f, the element $f(L_1) + \cdots + f(L_n)$ acts as the scalar $\sum_{\square\in\lambda} f(c(\square))$ on the irreducible representation V^λ of S_n corresponding to a partition λ of n.

If we let $\tau \in S_n$ be any permutation with cycle type $\mu \cup (1^e)$, then the trace of the element $\tau^{-1}(f(L_1) + \cdots + f(L_n))$ acting on $\mathbb{C}[S_n]$ will be equal to

$$\sum_{\sigma \vdash |\mu|+e} \chi_\sigma\big((1)^{|\mu|+e}\big)\chi_\sigma\big(\mu \cup (1^e)\big) \sum_{\substack{\square\in\sigma \\ c=c(\square)}} f(c).$$

But the trace is also equal to $n!$ times the coefficient of τ in the element $f(L_1) + \cdots + f(L_n)$, which can be computed easily using the Lascoux–Thibon formula to expand the power sum $L_1^r + \cdots + L_n^r$; see [5, Theorem A.4]. The substitution $q = e^t$ appearing there corresponds precisely to replacing the X^c in (48) with c^r. □

6.5 Moduli spaces for $Y \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup Y_\infty$

Following Section 6.4, Let Y be a nonsingular surface, and let $E \subset Y$ be a nonsingular curve. Let $\mathsf{L} \in H_2(Y \times \mathbf{P}^1, \mathbb{Z})$ be the curve class of the factor $\mathbf{P}^1$. We consider the relative geometry

$$Y \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup Y_\infty \tag{49}$$

for the curve classes $d\mathsf{L}$ in both stable pairs and Gromov–Witten theory.

Consider first the simple relative geometry $Y \times \mathbf{P}^1 / E \times \mathbf{P}^1$. Jun Li has constructed the associated Artin stack of degenerations $\mathcal{A}$ in [11]. The morphism defined by the universal target

$$\pi : \mathcal{Y} \to \mathcal{A}$$

is *not* smooth. The morphism has singularities over the boundary of $\mathcal{A}$ along the internal divisors (all isomorphic to $E \times \mathbf{P}^1$) of the degenerations. Let

$$\mathcal{Y}^o \subset \mathcal{Y}$$

denote the smooth locus of π,

$$\pi^o : \mathcal{Y}^o \to \mathcal{A}.$$

There is a divisor $\mathcal{Y}^o_\infty \subset \mathcal{Y}^o$ corresponding to $Y_\infty \subset Y \times \mathbf{P}^1$. Since π^o is a smooth morphism, the moduli spaces of stable pairs and stable maps to the relative geometry $\mathcal{Y}^o / \mathcal{Y}^o_\infty$ relative to the morphism π^o are well-defined. The moduli spaces

$$P_n(Y \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup Y_\infty, d\mathsf{L}),$$
$$\overline{M}'_{g,r}(Y \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup Y_\infty, d\mathsf{L})_\mu$$

are defined to the open subsets with finite stabilizers. Properness is proven as usual by semistable reduction. The virtual class is constructed relative over $\mathcal{A}$, and the degeneration formula used in Section 6.4 are proven exactly following [12].

The points of the moduli space $P_n(Y \times \mathbf{P}^1 / E \times \mathbf{P}^1 \cup Y_\infty, d\mathsf{L})$ are easily described. We start with a k–step degeneration of $Y \times \mathbf{P}^1$ along $E \times \mathbf{P}^1$. Such degenerations have finitely many additional components which are all isomorphic to the $\mathbf{P}^1$–bundle $\mathbf{P}^1(N_{Y/E} \oplus \mathcal{O}_E) \times \mathbf{P}^1$ over $E \times \mathbf{P}^1$. The result is the space

$$Y \times \mathbf{P}^1 \cup \mathbf{P}^1(N_{Y/E} \oplus \mathcal{O}_E) \times \mathbf{P}^1 \cup \cdots \cup \mathbf{P}^1(N_{Y/E} \oplus \mathcal{O}_E) \times \mathbf{P}^1 \tag{50}$$

with $k+1$ components. The components (50) are attached along the internal divisors $E \times \mathbf{P}^1$. The original space $Y \times \mathbf{P}^1$ is the 0^{th} step of the degeneration. Next, we take an l–step degeneration along the fiber over $\infty \in \mathbf{P}^1$. Let $\widetilde{Y \times \mathbf{P}^1}$ denote the l–step degeneration

$$Y \times \mathbf{P}^1 \cup Y_\infty \times \mathbf{P}^1 \cup \cdots \cup Y_\infty \times \mathbf{P}^1.$$

Similarly, let $\widetilde{\mathbf{P}^1_E \times \mathbf{P}^1}$ denote the l–step degeneration

$$\mathbf{P}^1(N_{Y/E} \oplus \mathcal{O}_E) \times \mathbf{P}^1 \cup \mathbf{P}^1(N_{Y_\infty/E_\infty} \oplus \mathcal{O}_{E_\infty}) \times \mathbf{P}^1 \cup \cdots \cup \mathbf{P}^1(N_{Y_\infty/E_\infty} \oplus \mathcal{O}_{E_\infty}) \times \mathbf{P}^1$$

with attachment along the internal divisors Y_∞. The stable pairs we consider lie on the target

$$\widetilde{Y \times \mathbf{P}^1} \cup \widetilde{\mathbf{P}^1_E \times \mathbf{P}^1} \cup \cdots \cup \widetilde{\mathbf{P}^1_E \times \mathbf{P}^1} \tag{51}$$

with attachment along the degenerations of the internal divisors $E \times \mathbf{P}^1$. The curve class $d\mathsf{L}$ is distributed to each term in (51). By stability (finiteness of automorphisms), a nonzero degree must be distributed to each term other than

$$\widetilde{Y \times \mathbf{P}^1}.$$

On each term of (51), we specify a stable pair relative to the fiber over infinity and disjoint from the internal divisors obtained from degenerations of $E \times \mathbf{P}^1$. By stability, we require a nontrivial stable pair in *at least one* of the k–steps over *each* of the positive steps of the degeneration over $\infty \in \mathbf{P}^1$. The description in the stable map case is identical.

6.6 Proof of Theorem 4

Let X be a nonsingular projective toric 3–fold with $\mathbf{T}$–fixed points

$$p_1, \ldots, p_m \in X.$$

Let $\gamma_1, \ldots, \gamma_r \in H^*(X, \mathbb{Q})$ be classes of positive degree. Since $H^*(X, \mathbb{Q})$ is generated by divisors, we may regard γ_i as a polynomial of positive degree in the divisor classes.

Lemma 22 *Every divisor class in $H^2(X, \mathbb{Q})$ can be lifted to $H^2_{\mathbf{T}}(X, \mathbb{Q})$ with trivial restriction to p_m.*

Proof Since the divisor classes of X are spanned by toric divisors, all divisor classes can be lifted to $\mathbf{T}$–equivariant cohomology. After further tensoring with a 1–dimension representation of $\mathbf{T}$, the restriction to $H^2_{\mathbf{T}}(p_m, \mathbb{Q})$ can be set to 0. □

After lifting each γ_i to a polynomial $\widetilde{\gamma}_i$ in $\mathbf{T}$–equivariant divisor classes vanishing at p_m, we can lift the 0–descendent insertions as a finite sum

$$\prod_{i=1}^{r} \tau_0(\gamma_i) = \sum_{k=1}^{m-1} \sum_{l \geq 0} f_{k,l}(s_1, s_2, s_3)\tau_0(\mathsf{p}_k)^l,$$

where $f_{k,l} \in \mathbb{Q}(s_1, s_2, s_3)$. Hence

$$\begin{aligned}(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}&\Big(X; q \;\Big|\; \prod_{i=1}^{r} \tau_0(\gamma_i) \prod_{j=1}^{s} \tau_{k_j}(\mathsf{p})\Big)_\beta \\ &= (-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X; q \;\Big|\; \Big(\sum_{k=1}^{m-1} \sum_{l \geq 0} f_{k,l}(s_1, s_2, s_3)\tau_0(\mathsf{p}_k)^l\Big) \prod_{j=1}^{s} \tau_{k_j}(\mathsf{p}_m)\Big)_\beta^{\mathbf{T}}.\end{aligned}$$

We now apply Theorem 1 to the right hand side using the relation

$$\widehat{\tau}_{1^\ell}(\mathsf{p}_i) = \tau_{1^\ell}(\mathsf{p}_i)$$

proven in Section 2.4. After the variable change $-q = e^{iu}$, we obtain

$$\begin{aligned}(-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}&\Big(X; u \;\Big|\; \Big(\sum_{k=1}^{m-1} \sum_{l \geq 0} f_{k,l}(s_1, s_2, s_3)\tau_0(\mathsf{p}_k)^l\Big)\widehat{\tau}_\kappa(\mathsf{p}_m)\Big)_\beta^{\mathbf{T}} \\ &= (-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\Big(X; u \;\Big|\; \prod_{i=1}^{r} \tau_0(\widetilde{\gamma}_i) \cdot \widehat{\tau}_\kappa(\mathsf{p}_m)\Big)_\beta^{\mathbf{T}},\end{aligned}$$

where $\kappa = (k_1 + 1, \dots k_s + 1)$. By Proposition 12, we have

$$\widehat{\tau}_\kappa(\mathsf{p}) = (iu)^{\ell(\kappa)-|\kappa|}\tau_\kappa(\mathsf{p}) + \cdots,$$

where the dots stand for terms $\tau_{\widehat{\alpha}}$ with $\alpha \rhd^* \widehat{\alpha}$. Finally, using the dimension constraint for non-equivariant integrals, we obtain

$$(-iu)^{d_\beta}(iu)^{-\sum_j k_j}\mathsf{Z}'_{\mathsf{GW}}\Big(X; u \;\Big|\; \prod_{i=1}^{r} \tau_0(\gamma_i) \prod_{j=1}^{s} \tau_{k_j}(\mathsf{p})\Big)_\beta,$$

which is the claimed correspondence. $\square$

7 Non-equivariant limit

7.1 Overview

Our goal here is to prove the correspondence of Theorem 1 can be written completely in non-equivariant terms. The outcome is a descendent correspondence which makes sense for any (not necessarily toric) nonsingular projective 3–fold. In the toric case, the non-equivariant correspondence is a consequence of Theorem 1. For general 3–folds, the correspondence is conjectural.

7.2 Notation

We denote the set of descendent symbols by

$$\tau = \{\tau_0, \tau_1, \tau_2, \ldots\}.$$

Let σ be a partition with positive parts $\sigma_1, \ldots, \sigma_\ell$. We associate a polynomial τ_σ in the symbols τ to σ by

$$\tau_\sigma = \tau_{\sigma_1 - 1}\tau_{\sigma_2 - 1}\cdots\tau_{\sigma_l - 1}$$

following the conventions of Section 0.3. Using the correspondence matrix, we define

$$\widehat{\tau}_\sigma = \sum_{\widehat{\sigma}} \mathsf{K}_{\sigma,\widehat{\sigma}}\tau_{\widehat{\sigma}}$$

for non-empty partitions σ.

For subsets $S \subset \{1, \ldots, \ell\}$, we let σ_S be the subpartition consisting of the parts σ_i for $i \in S$. The definition

$$\widetilde{\tau}_\sigma = \sum_{\substack{P \text{ set partition} \\ \text{of } \{1,\ldots,\ell\}}} (-1)^{|P|-1}(|P|-1)! \prod_{S\in P} \widehat{\tau}_{\sigma_S} \tag{52}$$

is crucial to our study. Here $\widetilde{\tau}_\sigma$ lies in the polynomial ring in the symbols τ with coefficients in the ring $Q[i, s_1, s_2, s_3]((u))$. The polynomials $\widehat{\tau}_{\sigma_S}$ on the right carry the complexity of the correspondence matrix K.

7.3 Divisibility

In order to obtain a non-equivariant formulation of Theorem 1, our first step is to prove a divisibility result constraining the coefficients

$$\mathsf{K}_{\sigma,\widehat{\sigma}} \in \mathbb{Q}[i, s_1, s_2, s_3]((u))$$

of the correspondence matrix.

Proposition 23 *Let σ be a partition of positive length ℓ. Then*

$$\widetilde{\tau}_\sigma \equiv 0 \mod (s_1s_2s_3)^{\ell-1} \tag{53}$$

as a polynomial in the descendent symbols τ with coefficients in $\mathbb{Q}[i, s_1, s_2, s_3]((u))$.

Proof We will use the relative correspondence established in Proposition 19 for the geometry
$$S \times \mathbf{P}^1/S_\infty = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1/(\mathbf{P}^1 \times \mathbf{P}^1)_\infty.$$
The surface $S = \mathbf{P}^1 \times \mathbf{P}^1$ is viewed as the first two factors of the 3–fold $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. We will consider $\mathbf{T}$–equivariant stationary descendents at
$$p_\bullet = (0,0,0), \quad p_\star = (\infty, 0, 0).$$
Let the tangent weights at $p_\bullet$ be s_1, s_2, s_3 respectively along the three $\mathbf{P}^1$ factors. Then
$$\mathsf{p}_\bullet - \mathsf{p}_\star = s_1 \mathsf{P}_{\bullet\star},$$
where $\mathsf{P}_{\bullet\star}$ is the class of the line $P_{\bullet\star}$ connecting the two points.

We will prove the proposition together with the divisibility claim
$$\mathsf{Z}'_{\mathsf{GW}}\big(S \times \mathbf{P}^1/S_\infty; u \mid \widetilde{\tau}_\sigma(\mathsf{p}_\bullet) \mid \mu\big)^{\mathbf{T}}_{d\mathsf{L}} \equiv 0 \mod s_1^{\ell-1} \tag{54}$$
for every curve class $d\mathsf{L}$ and relative condition μ. We prove the proposition and the divisibility (54) simultaneously by induction on the length ℓ of σ. For $\ell = 1$, both statements are trivial. We assume $\ell > 1$.

If divisibility (54) holds for partitions σ of length ℓ, then $\widetilde{\tau}_\sigma$ must be divisible by $s_1^{\ell-1}$ by Proposition 20. By the symmetry of the coefficients of K in the variables s_i, we conclude $\widetilde{\tau}_\sigma$ is divisible by $(s_1 s_2 s_3)^{\ell-1}$. We have proven claim (54) for length ℓ implies claim (53) for length ℓ.

Finally, we show if (53) holds for partitions of length $1, 2, \dots, \ell-1$, then divisibility (54) holds for length ℓ. For any set partition
$$Q = \{Q_1, \dots, Q_k\} \quad \text{of} \quad \{1, \dots, \ell\}$$
with $1 \in Q_1$ and $k > 1$, consider the Gromov–Witten series for $S \times \mathbf{P}^1/S_\infty$
$$\mathsf{Z}'_{\mathsf{GW}}\bigg(\widetilde{\tau}_{\sigma_{Q_1}}(\mathsf{p}_\bullet) \prod_{j=2}^{k} \big(\widetilde{\tau}_{\sigma_{Q_j}}(\mathsf{p}_\bullet) + (-1)^{|Q_j|}\widetilde{\tau}_{\sigma_{Q_j}}(\mathsf{p}_\star)\big) \,\bigg|\, \mu\bigg)^{\mathbf{T}}_{d\mathsf{L}} \tag{55}$$
lying in $\mathbb{Q}[i, s_1, s_2, s_3]((u))$. By the inductive hypothesis, we pick up a factor of $s_1^{|Q_j|-1}$ for each part in the set partition. Also, each part in the set partition after the first contributes an additional factor of s_1 because

(i) the correspondence matrices K at the points $p_\bullet$ and $p_\star$ are equal after changing the sign of s_1,

(ii) s_1 divides $\mathsf{p}_\bullet - \mathsf{p}_\star$.

Thus, we see (55) is divisible by $s_1^{\ell-1}$.

Using again the divisibility of $\mathsf{p}_\bullet - \mathsf{p}_\star$ by s_1 in $\mathbf{T}$–equivariant cohomology, we see

$$\mathsf{Z}_{\mathsf{P}}\Big(\tau_{\sigma_1}(\mathsf{p}_\bullet)\prod_{j=2}^{\ell}\tau_{\sigma_j}(\mathsf{p}_\bullet - \mathsf{p}_\star)\ \Big|\ \mu\Big)^{\mathbf{T}}_{d\mathsf{L}} \equiv 0 \mod s_1^{\ell-1}.$$

From the descendent correspondence of Proposition 19, we immediately conclude

$$\mathsf{Z}'_{\mathsf{GW}}\Big(\widehat{\tau_{\sigma_1}(\mathsf{p}_\bullet)\prod_{j=2}^{\ell}\tau_{\sigma_j}(\mathsf{p}_\bullet - \mathsf{p}_\star)}\ \Big|\ \mu\Big)^{\mathbf{T}}_{d\mathsf{L}} \equiv 0 \mod s_1^{\ell-1}, \tag{56}$$

where $\widehat{\tau_\delta(\mathsf{p}_\bullet)\tau_\eta(\mathsf{p}_\star)} = \widehat{\tau}_\delta(\mathsf{p}_\bullet)\widehat{\tau}_\eta(\mathsf{p}_\star)$.

The desired divisibility (54) now follows from the basic identity

$$\sum_{\substack{Q \text{ set partition}\\ \text{of } \{1,\dots,\ell\}\\ 1\in Q_1}} \widetilde{\tau}_{\sigma_{Q_1}}(\mathsf{p}_\bullet)\prod_{j=2}^{|Q|}\big(\widetilde{\tau}_{\sigma_{Q_j}}(\mathsf{p}_\bullet) + (-1)^{|Q_j|}\widetilde{\tau}_{\sigma_{Q_j}}(\mathsf{p}_\star)\big) = \widehat{\tau_{\sigma_1}(\mathsf{p}_\bullet)\prod_{j=2}^{\ell}\tau_{\sigma_j}(\mathsf{p}_\bullet - \mathsf{p}_\star)}\,. \tag{57}$$

Since $s_1^{\ell-1}$ divides both (55) and (56), we conclude the claim (54) holds for length ℓ from the identity (57). The term $\tau_\sigma(\mathsf{p}_\bullet)$ occurs on the left side of (57) when $Q_1 = \{1,\dots,\ell\}$.

We now check identity (57) by computing the coefficient of

$$\prod_{S\in A}\widehat{\tau_{\sigma_S}(\mathsf{p}_\bullet)}\prod_{T\in B}\widehat{\tau_{\sigma_T}(\mathsf{p}_\star)} \tag{58}$$

appearing when the $\widetilde{\tau}$ on the left side are expanded in terms of $\widehat{\tau}$. Here, $A\cup B = C$ is a partition of a set partition C of $\{1,\dots,\ell\}$ into two nonempty subsets, with 1 belonging to one of the parts in A. Let

$$a = \Big|\bigcup_{S\in A} S\Big| \quad \text{and} \quad b = \Big|\bigcup_{T\in B} T\Big|,$$

so the coefficient of this term on the right side of (57) is equal to $(-1)^b$ if $|A| = 1$ and $|B| \le 1$ and 0 otherwise.

The term on the left side of (57) given by a set partition Q of $\{1,\dots,\ell\}$ will contribute to the coefficient of (58) if and only if $Q = Q_A \cup Q_B$ with A a refinement of Q_A and

B a refinement of Q_B. Such Q are parametrized by choosing one set partition of A and one of B, so we compute the coefficient to be

$$\sum_{\substack{P_A \text{ set partition of } A \\ P_B \text{ set partition of } B}} \left(\prod_{S \in P_A} (-1)^{|S|-1}(|S|-1)!\right) \cdot (-1)^b \cdot \left(\prod_{T \in P_B} (-1)^{|T|-1}(|T|-1)!\right).$$

Finally, we need only prove the fundamental identity

$$\sum_{\substack{P \text{ set partition} \\ \text{of } \{1,\dots,k\}}} \prod_{S \in P} (-1)^{|S|-1}(|S|-1)! = \begin{cases} 1 & \text{if } k = 0, 1, \\ 0 & \text{if } k > 1, \end{cases}$$

which follows immediately from the observation that each term is counting the permutations of $\{1, \dots, k\}$ that yield a given orbit partition P, with sign equal to the sign of the permutations of this type. $\square$

We define the correspondence matrix $\widetilde{\mathsf{K}}$ which we will use for the non-equivariant limit by

$$\widetilde{\mathsf{K}}_{\sigma,\widehat{\sigma}} = \frac{1}{(s_1 s_2 s_3)^{\ell(\sigma)-1}} \mathrm{Coeff}_{\tau_{\widehat{\sigma}}}(\widetilde{\tau}_\sigma). \tag{59}$$

By the vanishing $\mathsf{K}_{\sigma,\widehat{\sigma}} = 0$ unless $|\sigma| \geq |\widehat{\sigma}|$, we deduce the vanishing

$$\widetilde{\mathsf{K}}_{\sigma,\widehat{\sigma}} = 0 \quad \text{unless} \quad |\sigma| \geq |\widehat{\sigma}|.$$

By the divisibility of Proposition 23,

$$\widetilde{\mathsf{K}}_{\sigma,\widehat{\sigma}} \in \mathbb{Q}[i, s_1, s_2, s_3]((u)).$$

In fact, since K is symmetric in the s_i, we may view

$$\widetilde{\mathsf{K}}_{\sigma,\widehat{\sigma}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u)),$$

where the c_i are elementary symmetric functions in the s_i.

Proposition 24 *The u coefficients of $\widetilde{\mathsf{K}}_{\sigma,\widehat{\sigma}} \in \mathbb{Q}[i, s_1, s_2, s_3]((u))$ are symmetric and homogeneous in the variables s_i of degree*

$$|\sigma| + \ell(\sigma) - |\widehat{\sigma}| - \ell(\widehat{\sigma}) - 3(\ell(\sigma) - 1).$$

Proof The result follows from Theorem 3 and definitions (52) and (59). $\square$

7.4 Proof of Theorem 7

Let X be a nonsingular quasi-projective toric 3–fold. Let α be a partition of length ℓ and positive parts. Let

$$\gamma_1, \dots, \gamma_\ell \in H^*_{\mathbf{T}}(X, \mathbb{Q})$$

be $\mathbf{T}$–equivariant classes. We can express

$$\mathsf{Z}_{\mathsf{P}}\big(X; q \mid \tau_{a_1-1}(\gamma_1) \cdots \tau_{a_\ell-1}(\gamma_\ell)\big)_\beta \tag{60}$$

in terms of Gromov–Witten theory by writing each class γ_l as a combination of the $\mathbf{T}$–fixed points via (4) and then applying the descendent correspondence of Theorem 1.

Let $\mathcal{P}^{\mathrm{set}}_\ell$ be the set of set partitions of $\{1, \dots, \ell\}$. For a partition $P \in \mathcal{P}^{\mathrm{set}}_\ell$, each $S \in P$ is a subset of $\{1, \dots, \ell\}$. Let

$$\gamma_S = \prod_{i \in S} \gamma_i \quad \text{and} \quad \tau_{\alpha_S} = \prod_{i \in S} \tau_{a_i-1}.$$

A first formula for the Gromov–Witten descendent corresponding to the stable pairs integral (60) is given by

$$\sum_{P \in \mathcal{P}^{\mathrm{set}}_\ell} \sum_{\substack{\text{Injective } \phi, \\ \phi: P \to \{1,\dots,m\}}} \prod_{S \in P} \frac{\iota^*_{\phi(S)}(\gamma_S)}{\iota^*_{\phi(S)}(c_3(X)^{|S|})} \widehat{\tau}_{\alpha_S}(\mathsf{p}_{\phi(S)}). \tag{61}$$

Here, the $\mathbf{T}$–fixed points of X are $p_1, \dots, p_m$, and we follow the notation of the localization identity (4).

We may extend the second sum in (61) to run over all functions

$$\phi : P \to \{1, \dots, m\}$$

(rather than just the injective ones) by rewriting the formula as

$$\sum_{P \in \mathcal{P}^{\mathrm{set}}_\ell} \sum_{\phi: P \to \{1,\dots,m\}} \prod_{S \in P} \frac{\iota^*_{\phi(S)}(\gamma_S)}{\iota^*_{\phi(S)}(c_3(X)^{|S|})} \widetilde{\tau}_{\alpha_S}(\mathsf{p}_{\phi(S)}) \tag{62}$$

using definition (52). Finally, the cohomological identity

$$\sum_{j=1}^m \frac{\iota^*_j(\gamma)}{\iota^*_j(c_3(X))} \mathsf{p}_j \otimes \cdots \otimes \mathsf{p}_j = \gamma \cdot \Delta \in H^*_{\mathbf{T}}(X \times \cdots \times X, \mathbb{Q})$$

allows us to rewrite (62) efficiently as

$$\sum_{P \in \mathcal{P}^{\mathrm{set}}_\ell} \prod_{S \in P} \frac{1}{c_3(X)^{|S|-1}} \widetilde{\tau}_{\alpha_S}(\gamma_S) \tag{63}$$

following convention (5). Theorem 7 then follows from Theorem 1, formula (7) and our definition of $\widetilde{\mathsf{K}}$. $\square$

For a nonsingular quasi-projective toric 3–fold X with $\mathbf{T}$–fixed points $p_1, \dots, p_m$, we have two descendent correspondences for the stable pairs series

$$\mathsf{Z}_{\mathsf{P}}\Big(X; q \;\Big|\; \prod_{j=1}^{m} \tau_{\alpha^{(j)}}(\mathsf{p}_j)\Big)_{\beta}^{\mathbf{T}} \tag{64}$$

of Section 0.3. We can apply Theorem 1 or Theorem 7. In fact, the result is the same.

Lemma 25 *Theorem 7 applied to* (64) *specializes exactly to Theorem 1.*

Proof The claim reduces to the inversion formula

$$\sum_{\substack{Q \text{ set partition} \\ \text{of } \{1,\dots,\ell(\alpha^{(j)})\}}} \prod_{S \in Q} \widetilde{\tau}_{\alpha_S^{(j)}}(\mathsf{p}_j) = \widehat{\tau}_{\alpha^{(j)}}(\mathsf{p}_j)$$

obtained from the specialization $\mathsf{p}_\star = 0$ of (57). $\square$

Proposition 24 implies the descendent correspondence of Theorem 7 respects the dimensions of the insertions.

7.5 Relative descendent correspondence

Let X be a nonsingular projective 3–fold, and let $D \subset X$ be a nonsingular divisor. Let $\Omega_X[D]$ denote the locally free sheaf of differentials with logarithmic poles along D. Let

$$T_X[-D] = \Omega_X[D]^\vee$$

denote the dual sheaf of tangent fields with logarithmic zeros.

For the relative geometry X/D, we let the coefficients of $\widetilde{\mathsf{K}}$ act on the cohomology of X via the substitution

$$c_i = c_i(T_X[-D])$$

instead of the substitution $c_i = T_X$ used in the absolute case. Then, we would like to define

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{\substack{P \text{ set partition} \\ \text{of } \{1,\dots,l\}}} \prod_{S \in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}(\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} \cdot \gamma_S).$$

as before. The correct definition is subtle for arbitrary classes γ_i. A full discussion of the descendent correspondence for relative geometries will be given in [26]. However, a restricted case in which the above definition is appropriate will be relevant for Section 8.

Conjecture 4 *Let $\gamma_1, \ldots, \gamma_l \in H^*(X, \mathbb{Q})$ be classes which restrict to 0 on D, then we have*

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\big(X/D; q \mid \tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell) \mid \mu\big)_\beta \\ = (-iu)^{d_\beta+\ell(\mu)-|\mu|} \mathsf{Z}'_{\mathsf{GW}}\big(X/D; u \mid \overline{\tau_{a_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)} \mid \mu\big)_\beta$$

under the variable change $-q = e^{iu}$.

In addition, the stable pairs descendent series on the left is conjectured to be a rational function in q, so the change of variables is well-defined. Conjecture 4 is open.

8 Log Calabi–Yau 3–folds

8.1 Overview

Let X be a nonsingular projective toric Fano 3–fold with a nonsingular irreducible anti-canonical divisor S (necessarily isomorphic to a $K3$ surface). The relative geometry X/S is *log Calabi–Yau* since the sheaf of differentials of X with logarithmic poles along S has trivial determinant. In order to prove the Gromov–Witten/Pair correspondence of Theorem 5 for X/S, we will require results about projective bundles over S.

Let L_0 and L_∞ be two line bundles on S. The projective bundle

$$\mathbf{P}_S = \mathbf{P}(L_0 \oplus L_\infty) \to S$$

admits sections

$$S_i = \mathbf{P}(L_i) \subset \mathbf{P}_S.$$

Before proving Theorem 5, we will establish the relative descendent correspondence of Conjecture 4 for $\mathbf{P}_S/S_\infty$ for descendent insertions supported on S_0. While the result goes beyond toric varieties, the vanishings which hold for $K3$ geometries make $\mathbf{P}_S/S_\infty$ accessible. The descendent correspondences for projective bundles over surfaces will be studied in more detail in [26].

Theorem 5 and Corollary 3 will follow easily from Theorem 7, degeneration, and the descendent correspondence for $\mathbf{P}_S/S_\infty$.

8.2 Descendent correspondence for $\mathbf{P}_S/S_\infty$

Let $\phi_1,\dots,\phi_\ell$ be cohomology classes on $\mathbf{P}_S$ supported[19] on the section S_0. Let μ be a relative condition along S_∞. Our first step is to prove the non-equivariant descendent correspondence of Conjecture 4 for the classes ϕ_i.

Proposition 26 *We have*

$$(65)\quad (-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}_S/S_\infty;q \mid \tau_{\alpha_1-1}(\phi_1)\cdots\tau_{\alpha_\ell-1}(\phi_\ell)\mid \mu\big)_\beta \\ = (-iu)^{d_\beta+\ell(\mu)-|\mu|}\mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}_S/S_\infty;u \mid \overline{\tau_{\alpha_1-1}(\phi_1)\cdots\tau_{\alpha_\ell-1}(\phi_\ell)}\mid \mu\big)_\beta$$

after the variable change $-q=e^{iu}$.

The log tangent bundle of $\mathbf{P}_S/S_\infty$ restricts to the standard tangent bundle of $\mathbf{P}_S$ on the section S_0. Since the classes ϕ_i are supported on S_0, the descendent correspondence matrix $\widetilde{\mathsf{K}}$ for $\mathbf{P}_S/S_\infty$ is the same as the matrix for $\mathbf{P}_S$.

Proof By the vanishings in stable pair and Gromov–Witten theory obtained from the holomorphic symplectic structure of $K3$ surfaces, only invariants of $\mathbf{P}_S$ in multiples of the fiber class

$$\mathsf{L}\in H_2(\mathbf{P}_S,\mathbb{Z})$$

contracted over S are non-zero. Moreover, for the stable pairs series, only the initial q–coefficient is non-zero.

Let X be any nonsingular projective surface equipped with line bundles L_0 and L_∞. Let $\phi_1,\dots,\phi_\ell$ be cohomology classes on

$$\mathbf{P}_S=\mathbf{P}(L_0\oplus L_\infty)\to S$$

supported on the section X_0. Consider the Gromov–Witten series

$$(66)\qquad (-iu)^{|\mu|+\ell(\mu)}\mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}_X/X_\infty;u\mid \overline{\tau_{\alpha_1-1}(\phi_1)\cdots\tau_{\alpha_\ell-1}(\phi_\ell)}\mid\mu\big)_{|\mu|\mathsf{L}}.$$

Each u–coefficient of (66) can be expressed by an explicit study of the moduli space of stable maps to the fiber classes of $\mathbf{P}_X\to X$. By a standard analysis (see [19, Section 1.2]), each u–coefficient is a universal polynomial over $\mathbb{Q}$ in the all classical pairings

$$(67)\qquad \int_X \Theta\big(c(T_X),c(L_0),c(L_\infty)\big)\cup\prod_{i\in I}\phi_i,$$

[19] Each ϕ_i is push-forward of a class on S. Since $K3$ surfaces have only even cohomology, the ϕ_i have even degrees.

where Θ is a monomial in the Chern classes of the bundles

$$T_X, L_0, L_\infty \to X$$

of bounded degree (determined by the descendent partition $\alpha = (\alpha_1, \dots, \alpha_\ell)$ and the degrees of ϕ_i). In the product on the right side of (67), $I \subset \{1, \dots, \ell\}$ is a subset.

Let X be a nonsingular projective toric surface with toric line bundles L_0 and L_∞. For fixed α and ϕ_i, there are only finitely many classical pairings (67). Moreover, as we vary the toric surface X and the toric line bundle L_i, we easily see a Zariski dense set of possible classical pairings is achieved.[20] Hence, the u–coefficient polynomials of the Gromov–Witten series are fully determined by the toric examples.

If X is a nonsingular projective toric surface with toric line bundles L_i, then the relation

$$\begin{aligned}(-q)^{-|\mu|}\mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}_X/X_\infty; q \mid \tau_{\alpha_1-1}(\phi_1)\cdots\tau_{\alpha_\ell-1}(\phi_\ell) \mid \mu\big)_{|\mu|\mathsf{L}} \\ = (-iu)^{|\mu|+\ell(\mu)}\mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}_X/X_\infty; u \mid \overline{\tau_{\alpha_1-1}(\phi_1)\cdots\tau_{\alpha_\ell-1}(\phi_\ell)} \mid \mu\big)_{|\mu|\mathsf{L}}\end{aligned}$$

is a direct consequence, by localization, of the descendent correspondence for the cap. When we localize with respect to the 2–dimension torus T acting on X and the L_i, the result is cap for each T–fixed point of X. The stable pairs series

$$(-q)^{-|\mu|}\mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}_X/X_\infty; q \mid \tau_{\alpha_1-1}(\phi_1)\cdots\tau_{\alpha_\ell-1}(\phi_\ell) \mid \mu\big)_{|\mu|\mathsf{L}} \tag{68}$$

is thus also determined by the classical pairings (67) in the toric case. In fact, using the denominator results proven in Theorem 5 of [28], the q–coefficients of (68) are polynomials in the pairings (67).

Next, let the nonsingular projective surface X with line bundles L_0 and L_∞ be arbitrary. The q^0–coefficient of the stable pairs series (68) is special. The associated moduli space of stable pairs is simply the Hilbert scheme of $|\mu|$ point of X. The stable pairs invariant then can be calculated by Hilbert scheme techniques [9]. The result is also a polynomial in classical pairings (67). Hence, we have two polynomials in the classical pairing (67):

(i) The q^0–coefficient of the Gromov–Witten series (66) for X after the variable change $-q = e^{iu}$.

(ii) The polynomial obtained from the Hilbert scheme of points calculation of the q^0–coefficient of the stable pairs series (68) for X.

[20] Since the classes ϕ_i we consider are of even degrees, the degrees can be matched in toric geometry.

The two polynomials are equal when evaluated in the toric geometry and thus must be identical (by Zariski denseness).

The polynomials (i) and (ii) are therefore equal for the $K3$ geometry $\mathbf{P}_S/S_\infty$. To complete the proof of the correspondence (65), we must only prove the higher q–coefficients, obtained after the variable change $-q = e^{iu}$ for Gromov–Witten series (66) for $\mathbf{P}_S/S_\infty$, all vanish.

Let X be a nonsingular quasi-projective toric surface with toric line bundles L_0 and L_∞. Consider the T–equivariant Gromov–Witten series

$$(-iu)^{|\mu|+\ell(\mu)} \mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}_X/X_\infty; u \mid \overline{\tau_{\alpha_1-1}(\phi_1)\cdots\tau_{\alpha_\ell-1}(\phi_\ell)} \mid \mu\big)^T_{|\mu|\mathsf{L}}, \tag{69}$$

where T is the 2–dimensional torus acting on X and the L_i. As before, each u–coefficient is a universal polynomial over $\mathbb{Q}$ in the all classical T–equivariant pairings

$$\int_X \Theta\big(c(T_X), c(L_0), c(L_\infty)\big) \cup \prod_{i\in I} \phi_i, \tag{70}$$

where Θ is a monomial in the Chern classes of the bundles

$$T_X, L_0, L_\infty \to X$$

of bounded degree (determined by the descendent partition α and the degrees of ϕ_i). In T–equivariant geometry, more pairings may be non-zero. Otherwise, the situation is exactly the same as in the non-equivariant case. The universal polynomials in the T–equivariant geometry restrict to the universal polynomials in the non-equivariant geometry.

We finally specialize X to the quasi-projective surfaces $\mathcal{A}_n$. If we restrict to the sub-torus $\mathbb{C}^* \subset T$ which preserves the holomorphic form, then

$$c_1(T_X) = 0.$$

The $\mathcal{A}_n$ geometries, as n varies, provide a rich supply of $\mathbb{C}^*$–equivariant pairings (70) subject to the vanishing of $c_1(T_X)$, The T–equivariant correspondence

$$\begin{aligned}(-q)^{-|\mu|} \mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}_X/X_\infty; q \mid \tau_{a_1-1}(\phi_1)\cdots\tau_{a_\ell-1}(\phi_\ell) \mid \mu\big)^T_{|\mu|\mathsf{L}} \\ = (-iu)^{|\mu|+\ell(\mu)} \mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}_X/X_\infty; u \mid \overline{\tau_{a_1-1}(\phi_1)\cdots\tau_{a_\ell-1}(\phi_\ell)} \mid \mu\big)^T_{|\mu|\mathsf{L}}\end{aligned}$$

has already been proven for $X = \mathcal{A}_n$. The higher q–coefficients of the stable pairs side above vanish (since $\mathcal{A}_n$ has a holomorphic symplectic form invariant under $\mathbb{C}^*$). Hence, the higher q–coefficients obtained after the change of variables $-q = e^{iu}$ for the Gromov–Witten series (69) for $\mathcal{A}_n$ all vanish. By the universality of the polynomials and the sufficient Zariski density of the $\mathcal{A}_n$ geometries (subject to the vanishing of the first Chern class of the tangent bundle), we conclude the necessary vanishing of the higher q–coefficients for Gromov–Witten series (66) for $\mathbf{P}_S/S_\infty$. $\square$

8.3 Proof of Theorem 5

Let X be a nonsingular projective Fano toric 3–fold, and let $S \subset X$ be a nonsingular anti-canonical $K3$ surface. Let N be the normal bundle of S in X. Let

$$S_0, S_\infty \subset \mathbf{P}(\mathcal{O}_S \oplus N)$$

be the sections determined by the summand $\mathcal{O}_S$ and N respectively. Let

$$\iota_0 : S \hookrightarrow \mathbf{P}(\mathcal{O}_S \oplus N)$$

be the inclusion of S_0.

Let $\mathcal{B}$ be a fixed self-dual basis of the cohomology of S. Recall a Nakajima basis element in the Hilbert scheme $\mathrm{Hilb}(S,n)$ is a cohomology weighted partition μ of n,

$$((\mu_1,\phi_1),\dots,(\mu_\ell,\phi_\ell)), \quad n=\sum_{i=1}^{\ell}\mu_i, \quad \phi_i \in \mathcal{B}.$$

Such a weighted partition determines a descendent insertion

$$\tau[\phi] = \prod_{i=1}^{\ell} \tau_{\mu_i-1}(\iota_{\infty *}(\phi_i)).$$

By standard $K3$ vanishing arguments [20], the stable pairs invariants of the relative 3–fold geometry $\mathbf{P}(\mathcal{O}_S \oplus N)/S_\infty$ are nontrivial only for curves classes in the fibers of

$$\mathbf{P}(\mathcal{O}_S \oplus N)/S_\infty \to S.$$

Define the partition function for the relative geometry by

$$\mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}(\mathcal{O}_S \oplus N)/S_\infty; q \mid \tau[\phi] \mid \mu\big)_{d\mathsf{L}}, \tag{71}$$

where ϕ and μ are both partitions of d weighted by $\mathcal{B}$. By further vanishing, only the leading q^d terms of (71) are possibly nonzero. The following result is proven in [27, Section 4.1].

Proposition 27 *Let $d>0$ be an integer. The square matrix indexed by $\mathcal{B}$–weighted partitions of d with coefficients*

$$\mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}(\mathcal{O}_S \oplus N)/S_\infty; q \mid \tau[\phi] \mid \mu\big)_{d\mathsf{L}} \tag{72}$$

has maximal rank.

We can also consider the Gromov–Witten analogue of Proposition 27. By Proposition 26, we have a descendent correspondence

$$(-q)^{-d}\mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}(\mathcal{O}_S\oplus N)/S_\infty;q\mid\tau[\phi]\mid\mu\big)_{d\mathsf{L}} = (-iu)^{|\nu|+\ell(\nu)}\mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}(\mathcal{O}_S\oplus N)/S_\infty;u\mid\overline{\tau[\phi]}\mid\mu\big)_{d\mathsf{L}} \tag{73}$$

after the variable change $-q=e^{iu}$. In particular, the Gromov–Witten matrix corresponding to (72) is also invertible.

Let $\beta\in H_2(X,\mathbb{Z})$ be a curve class, and let

$$d_\beta=\int_\beta c_1(X)=\int_\beta [S].$$

Consider the descendent correspondence of Theorem 7,

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\big(X;q\mid\tau[\phi]\big)_\beta=(-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\big(X;u\mid\overline{\tau[\phi]}\big)_\beta, \tag{74}$$

where ϕ is a partition of d_β weighted by $\mathcal{B}$. Since all the cohomology classes of the descendent $\tau[\phi]$ lie on S, we can degenerate to the normal cone. The resulting degeneration formula[21] in stable pairs theory for $\mathsf{Z}_{\mathsf{P}}(X\mid\tau[\phi])_\beta$ is

$$\sum \mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}(\mathcal{O}_S\oplus N)/S_\infty\mid\tau[\phi]\mid\mu\big)_{d_\beta\mathsf{L}}(-1)^{|\mu|-\ell(\mu)}\mathfrak{z}(\mu)q^{-|\mu|}\mathsf{Z}_{\mathsf{P}}\big(X/S\mid\ \mid\mu^\vee\big)_\beta,$$

where the sum is over all elements μ of the Nakajima basis of cohomology of $\mathrm{Hilb}(S,d_\beta)$. The parallel degeneration formula for Gromov–Witten theory together with Propositions 26 and 27 imply Theorem 5 in case there are *no descendent insertions*.

Consider now the correspondence of Theorem 5 with the full descendent insertion

$$\tau_0(\gamma_1)\cdots\tau_0(\gamma_r). \tag{75}$$

Since X is a toric variety, the cohomological degree of each γ_i must be even. Degrees 0 and 2 can be removed from both stable pairs and Gromov–Witten theory by the fundamental class and divisor equations. We need only consider insertions γ_i of degree 4 or 6. The divisor

$$\iota_S:S\subset X$$

is ample since X is Fano. Hence, classes $\gamma_i\in H^*(X,\mathbb{Q})$ of degrees 4 and 6 can be written as

$$\iota_{S*}(\phi_i)=\gamma_i \tag{76}$$

[21] We follow here the notation of Section 1.2.

for $\phi_i \in H^*(S,\mathbb{Q})$ by hard Lefschetz. We can write the insertion (75) as

$$\tau_0\big(\iota_{S*}(\phi_1)\big)\cdots\tau_0\big(\iota_{S*}(\phi_r)\big).$$

We now reduce correspondence of Theorem 5 with the full insertion (75) to Theorem 5 with no insertions. Via degeneration to the normal cone of S, we can write

$$\mathsf{Z}_{\mathsf{P}}\big(X \mid \tau_0(\iota_{S*}(\phi_1))\cdots\tau_0(\iota_{S*}(\phi_r))\big)_\beta$$

in terms of the relative geometries as

$$\sum \mathsf{Z}_{\mathsf{P}}\big(\mathbf{P}(\mathcal{O}_S\oplus N)/S_\infty \mid \tau_0(\iota_{S_0*}(\phi_1))\cdots\tau_0(\iota_{S_0*}(\phi_r)) \mid \mu\big)_{d_\beta \mathsf{L}} \times(-1)^{|\mu|-\ell(\mu)}\mathfrak{z}(\mu)q^{-|\mu|}\mathsf{Z}_{\mathsf{P}}\big(X/S \mid \ \mid \mu^\vee\big)_\beta,$$

where the sum is as before. The parallel degeneration formula for Gromov–Witten theory together with Proposition 26 achieves the desired reduction. □

8.4 Proof of Corollary 3

Let $S\subset\mathbf{P}^3$ be a nonsingular quartic surface (anti-canonical and $K3$). Let $\beta\in H_2(\mathbf{P}^3,\mathbb{Z})$. Since $c_1(T_{\mathbf{P}^3})$ is even, d_β is even. Then, by Theorem 5, we have

$$\mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}^3/S \mid \ \mid \mu^\vee\big)_\beta \in \mathbb{Q}(-q=e^{iu},i)[u,1/u] \tag{77}$$

by the rationality in q of the corresponding stable pairs series [27].

Since the classes $\gamma_j\in H^*(\mathbf{P}^3,\mathbb{Q})$ are assumed to be of positive degree, we can write

$$\iota_{S*}(\phi_j)=\gamma_j$$

for classes $\phi_j\in H^*(S,\mathbb{Z})$. After replacing the descendent insertion with

$$\tau_{k_1}\big(\iota_{S*}(\phi_1)\big)\cdots\tau_{k_s}\big(\iota_{S*}(\phi_s)\big),$$

we can degenerate to the normal cone of S. We find that

$$\mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}^3 \mid \tau_{k_1}(\iota_{S*}(\phi_1))\cdots\tau_{k_s}(\iota_{S*}(\phi_s))\big)_\beta$$

is equal to

$$\sum \mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}(\mathcal{O}_S\oplus N)/S_\infty \mid \tau_{k_1}(\iota_{S_0*}(\phi_1))\cdots\tau_{k_s}(\iota_{S_0*}(\phi_s)) \mid \mu\big)_{d_\beta\mathsf{L}} \times\mathfrak{z}(\mu)u^{2\ell(\mu)}\mathsf{Z}'_{\mathsf{GW}}\big(\mathbf{P}^3/S \mid \ \mid \mu^\vee\big)_\beta. \tag{78}$$

The terms of (78) which are invariants of $\mathbf{P}(\mathcal{O}_S\oplus N)/S_\infty$ are Laurent polynomials in u and $1/u$ by $K3$ vanishings (the only connected contributions are of genus 0 and 1). The terms with are invariants of $\mathbf{P}^3/S$ are constrained by (77). The claim of the corollary then follows immediately. □

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Proposed: Jim Bryan Received: 4 December 2012
Seconded: Lothar Goettsche, Richard Thomas Revised: 25 October 2013