

A knot characterization and 1–connected nonnegatively curved 4–manifolds with circle symmetry

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We classify nonnegatively curved simply connected 4–manifolds with circle symmetry up to equivariant diffeomorphisms. The main problem is to rule out knotted curves in the singular set of the orbit space. As an extension of this work we classify all knots in \mathbb{S}^3 that can be realized as an extremal set with respect to an inner metric on \mathbb{S}^3 that has nonnegative curvature in the Alexandrov sense.

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It is known that the only (closed simply connected) positively curved 4–manifolds with infinite isometry group (equivalently, having circle symmetry) are \mathbb{S}^4 and $\mathbb{C}\mathbb{P}^2$. In the case of nonnegative curvature additionally only $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}\mathbb{P}^2 \# \pm \mathbb{C}\mathbb{P}^2$ will occur. Topologically this is due, via Freedmann’s work [4], to Hsiang and Kleiner [9] in positive curvature and to Kleiner [13] and Searle and Yang [19] in nonnegative curvature; and differentiably it follows via the Poincaré conjecture by their work and work of Fintushel [2; 3] and Pao [16].

Our main purpose here is to provide a classification of (closed) simply connected nonnegatively curved 4–manifolds with an isometric circle action up to equivariant diffeomorphism. In particular, we have:

Theorem A *A closed positively curved 4–manifold M with an isometric S^1 action is equivariantly diffeomorphic to a linear action on \mathbb{S}^4 , $\mathbb{R}\mathbb{P}^4$ or $\mathbb{C}\mathbb{P}^2$.*

This actually is a consequence of the following general result.

Theorem B *A closed nonnegatively curved simply connected 4–manifold M with an isometric S^1 action is diffeomorphic to \mathbb{S}^4 , $\mathbb{C}\mathbb{P}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or one of $\mathbb{C}\mathbb{P}^2 \# \pm \mathbb{C}\mathbb{P}^2$, and the action extends to a smooth T^2 action.*

All such T^2 actions have been classified by Orlik and Raymond in [15], and it turns out that each such action admits an invariant metric of nonnegative curvature. In fact,

in the cases of Euler characteristic 2 or 3 such an action is linear, and if the Euler characteristic is 4 any such T^2 action can be obtained as the induced quotient action on a T^2 quotient of the standard T^4 action of $S^3 \times S^3$ [6].

Our proof uses Alexandrov geometry and the solution of the Poincaré conjecture in essential ways (see Remark 5-6(b) though). The point of departure for the latter is the simple fact that the orbit space $M^* = M/S^1$ is a simply connected topological 3-manifold, and is hence S^3 , \mathbb{D}^3 or $S^2 \times \mathbb{D}^1$. It is intriguing that the presumably more complicated case of Euler characteristic 4 presents more geometric rigidity and as a consequence, the Poincaré conjecture is not needed there (see Section 4). In other words, we offer a purely geometric proof of the equivariant classification when the Euler characteristic is four.

By work of Pao [16] it is known that there are S^1 -actions on S^4 such that the singular set of the orbit space S^4/S^1 is given by a knotted closed curve, and in fact any locally flat knot can be realized in this way. Hence our equivariant classification relies on showing that there can be no knotted circles c in the singular set of M^* . For this we consider the canonical two-fold branched cover of M^* branched along a circle in the singular set c , denoted by $M_2^*(c)$. The following characterization is pivotal.

Theorem C *Let c be a locally flat embedded S^1 in S^3 and let $S_2^3(c)$ denote the corresponding canonical 2-fold branched cover. Then $S_2^3(c)$ is the 3-sphere if c is unknotted, and otherwise $\pi_1(S_2^3(c))$ is infinite or has order at least 3.*

The Alexandrov geometry of the orbit space $M^* = M/S^1$ is already instrumental for the topological classification alluded to above. Our use of the above knot characterization is based on the key observation that $M_2^*(c)$ is also an Alexandrov space with the same lower curvature bound as that of M^* . When M^* has no boundary, ie M^* is homeomorphic to the 3-sphere, we prove that any closed curve $c \subset M^*$ consisting of nonprincipal orbits is unknotted when M^* has nonnegative curvature in distance-comparison sense. A curve c consisting of nonprincipal orbits provides an example of a so-called *extremal set* in the Alexandrov space X .

This naturally leads to the following question: Which knots c in S^3 can arise as extremal subsets when S^3 is equipped with the structure of an Alexandrov space with nonnegative curvature? We characterize those knots, which leads to the following result via the equivariant Poincaré conjecture.

Theorem D *A knot c in S^3 can be an extremal set relative to a nonnegatively curved Alexandrov metric on the sphere if and only if it is a so-called spherical Montesinos knot of cyclic, tetrahedral or icosahedral type. In particular all these knots arise as singular sets with respect to a constant curvature 1 orbifold metric on S^3 .*

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1 A knot characterization

In this section we will see that **Theorem C** in the introduction is a simple consequence of the solution of the (equivariant) Poincaré conjecture.

If c is a closed embedded smooth circle in \mathbb{S}^3 , it is well known that the complement of c admits a canonical 2-fold cover corresponding to the 2-fold cover of the normal circle to c . By gluing back c we get the 2-fold branched cover $\mathbb{S}_2^3(c)$ alluded to in the introduction.

If c is the unknot it is clear that $\mathbb{S}_2^3(c)$ is the 3-sphere.

Now suppose c is knotted (the subsequent construction works without this assumption). Choose an orbifold metric on \mathbb{S}^3 with normal cone angle π along c . This can be done by choosing a metric invariant under the antipodal map of the normal bundle in a small tubular neighborhood of c , taking the quotient by this isometric involution and gluing it back via a partition of unity. The induced metric g on $\mathbb{S}_2^3(c)$ is Riemannian and c is a geodesic in $\mathbb{S}_2^3(c)$ fixed by a global isometric involution ι , the covering transformation on $\mathbb{S}_2^3(c) - c$.

The following lemma is crucial.

Lemma 1-1 *If $\pi_1(\mathbb{S}_2^3(c))$ is finite, then $\mathbb{S}_2^3(c)$ admits a (positive) constant curvature metric in which ι remains an isometry.*

This is a consequence of the “equivariant” Poincaré conjecture due to Dinkelbach and Leeb [1]. In particular, if $\pi_1(\mathbb{S}_2^3(c))$ has order two, $\mathbb{S}_2^3(c)$ must be \mathbb{RP}^3 and the involution ι is linear. This is already a contradiction, since such an ι will have two circles as fixed point set. This proves **Theorem C** modulo the following lemma.

Lemma 1-2 (Smoothing) *Any locally flat embedded circle c in \mathbb{S}^3 can be smoothed. That is, there is a homeomorphism of \mathbb{S}^3 whose image of c is a smooth submanifold.*

This, however, is an immediate consequence of the fact that any topological 3–manifold has a unique smooth structure up to diffeomorphism, combined with the theorem proved in [12] that c has a normal bundle neighborhood which, since \mathbb{S}^3 is orientable, is homeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2$. In contrast to the case of locally flat embeddings of \mathbb{S}^1 in \mathbb{S}^3 , a nonsmoothable locally flat embedding of \mathbb{S}^3 in \mathbb{S}^5 was constructed in [14].

2 Orbit spaces and branched covers

We begin this section with an analysis of the orbit space $M^* = M/S^1$ with its induced orbital metric. We denote by M^{S^1} the fixed-point set of the action. The projection map $M \rightarrow M^*$ is a submetry and M^* is a 3–dimensional Alexandrov space with nonnegative curvature (positive if M has positive curvature). Unless otherwise explicitly stated, we assume throughout that M is a closed simply connected 4–manifold of nonnegative sectional curvature. In [13] and [19] it was shown that the Euler characteristic of M , $\chi(M) = \chi(M^{S^1})$, is at most 4 (3 if M has positive curvature; see [9]). Note that M^{S^1} is also naturally a subset of M^* .

The orbit space has nonempty boundary ∂M^* if and only if M^{S^1} is 2–dimensional. In this case, a 2–dimensional component of M^{S^1} is also a component of ∂M^* and M is said to be *fixed-point homogeneous*. By the soul theorem for orbit spaces, ∂M^* has at most two components, and in the case of two components, M^* is isometric to the product of an interval and a boundary component. A complete classification of the possible actions in this fixed-point homogeneous case was done for positive curvature in [7], and for nonnegative curvature in [5] and [6].

Since the claims in our main theorem have been proved already in the fixed-point homogeneous case, we assume from now on that M^* has no boundary, ie M^{S^1} consists of 2, 3 or 4 isolated points.

The isotropy representation at an isolated fixed point $p \in M^{S^1}$ has the form $e^{i\theta}(z_1, z_2) = (e^{ik\theta}z_1, e^{i\ell\theta}z_2)$, where $T_p M$ has been identified with \mathbb{C}^2 , and $k \geq \ell \geq 1$ are relatively prime. In particular, the action on the unit sphere \mathbb{S}^3 of $T_p M$ is either free, has one isotropy group \mathbb{Z}_k , or two isotropy group \mathbb{Z}_k and \mathbb{Z}_ℓ . The corresponding orbit space $\mathbb{S}^3/S^1 =: \mathbb{S}_{k,\ell}$ is isometric to the space of directions at $p \in M^*$, is a (singular) surface of revolution and is topologically the 2–sphere. At orbits with isotropy group \mathbb{Z}_k the space of directions is the suspension of $\mathbb{S}^1/\mathbb{Z}_k$ (cf discussion below); in particular M^* is a topological manifold, and since it is clearly simply connected, the Poincaré conjecture tells us that:

Lemma 2-1 M^* is the 3–sphere.

Metrically it is important to note that

$$\mathbb{S}_{1,1} = \mathbb{S}^2(\frac{1}{2})$$

is the standard 2-sphere with radius $\frac{1}{2}$, and

$$(2-2) \quad \mathbb{S}_{1,1} \geq \mathbb{S}_{k,1} \geq \mathbb{S}_{k,\ell}$$

in the sense that there are natural (\mathbb{S}^1 equivariant) distance-decreasing maps to the smaller space. We will refer to the geodesics orthogonal to the \mathbb{S}^1 orbits in each of these spaces as *great circles*. The bound 4 (3 in positive curvature) on the number of fixed points is simply achieved by observing that there can be at most 4 (resp. 3) points in M^* as singular as $\mathbb{S}^2(\frac{1}{2})$ when the curvature is nonnegative (resp. positive); see Lemma 2-6 and [21].

A stratum in the orbit space M^* corresponding to points with isotropy group \mathbb{Z}_n forms a geodesic arc γ whose closure joins two different isolated fixed points. The space of directions at a point of γ is isometric to the spherical suspension of a circle of length $2\pi/n$, ie to

$$\mathbb{S}^2(1)/\mathbb{Z}_n =: \mathbb{S}_n.$$

Note that the spaces of directions with two singular points, ie $\mathbb{S}_{k,\ell}$ and \mathbb{S}_n , with $\ell, n \geq 2$, have natural two-fold branched covers along the two singular points, denoted by $\mathbb{S}_{k/2,\ell/2}$ and $\mathbb{S}_{n/2}$ respectively.

Also note that at most two geodesic strata corresponding to finite isotropy groups can end at a given fixed point, there making a right angle. The union of closures of two, three, or four such geodesic strata may form a closed curve c in M^* . Our main objective is to show that when this happens, such a closed curve in the 3-sphere M^* is not knotted. The following is a key observation towards this.

Lemma 2-3 (Branched cover) *Suppose c is a closed curve in M^* formed by the closure of geodesic strata corresponding to finite isotropy groups. Then the two-fold branched cover $M_2^*(c)$ is an Alexandrov space with the same lower curvature bound as M^* . Moreover, the space of directions at any point $p \in c$ of $M_2^*(c)$ is the canonical two-fold branched cover of the corresponding space of directions of M^* . In particular, for any point $p \in c \cap M^{\mathbb{S}^1}$ the space of directions satisfies $\Sigma_p M_2^*(c) \leq \mathbb{S}^2(\frac{1}{2})$.*

Proof We let $c_2 \subset c$ denote the closed subset of those orbits whose isotropy group is not \mathbb{Z}_2 . Notice that $M_2^*(c)$ is a smooth Riemannian manifold in a neighborhood of $c - c_2$. It is well known that the isotropy group at interior points of a minimal geodesic between two orbits is the same as the isotropy group of the whole geodesic. In particular, the set of regular points of the orbit space corresponding to principal

orbits is a convex set. For the same reason, $M^* - c$ is convex in our case. Thus it suffices to see that $M_2^*(c) - c_2$ is also convex in $M_2^*(c)$, since any geodesic triangle in $M_2^*(c)$ is the limit of geodesic triangles in $M_2^*(c) - c_2$.

To prove the convexity claim above it suffices to see what the geodesics in $M_2^*(c)$ are. Here only those emanating at points in c are an issue. By construction, however, it is clear that for each geodesic emanating at points of c in M^* there are exactly two emanating from the same point in $M_2^*(c)$. This shows that the spaces of directions at points along c are as claimed in the lemma, and also shows that minimal geodesics between points of $M_2^*(c) - c$ can cross c only at points with isotropy \mathbb{Z}_2 when viewed in M^* .

Finally, if $k, \ell \geq 2$ are relatively prime, then it is easy to see that

$$\mathbb{S}_{k/2, \ell/2} \leq \mathbb{S}_{1,1} = \mathbb{S}^2(\frac{1}{2}). \quad \square$$

To use [Theorem C](#) we also need the following lemma.

Lemma 2-4 (Local flatness) *A closed curve c consisting of singular points of M^* is a locally flat 1-dimensional submanifold of M^* .*

Proof We will denote the orbit of a point p by $p^* \in M^*$. It is immediate from the slice theorem that the exponential map at any $p \in M$ with $p^* \in c$ induces a homeomorphism from a sufficiently small ball in the tangent cone at p^* onto the corresponding ball centered at p^* . Since both $\mathbb{S}_{k,\ell}$ and \mathbb{S}_n with $\ell, n \geq 2$ admit homeomorphisms to $\mathbb{S}^2(1)$ taking the pair singular points to a pair of antipodal points the claim follows. \square

We are now in position to prove our key result.

Theorem 2-5 (Unknot) *Let M be a simply connected nonnegatively curved 4-manifold with an isometric S^1 action with isolated fixed points only. If c is a circle in M^* consisting of nonregular points, then c is unknotted, there is at most one such curve, and all fixed points are on the curve forming a biangle, triangle, or quadrangle corresponding to 2, 3 or 4 fixed points.*

We recall the following (see also [\[21\]](#)).

Lemma 2-6 *A three-dimensional nonnegatively curved Alexandrov space A has at most four points for which the space of directions is not larger than $\mathbb{S}^2(\frac{1}{2})$.*

Proof Suppose, to the contrary, that we can find 5 distinct points p_1, \dots, p_5 with $\Sigma_{p_i} A \leq \mathbb{S}^2(\frac{1}{2})$. For any pair of points $\{p_i, p_j\}$ in $\{p_1, \dots, p_5\}$, choose a minimal

geodesic c_{ij} and consider the corresponding 10 geodesic triangles. The sum of the 30 angles adds up to at least 10π .

We may assume after reordering that the sum of the 6 angles based at p_1 is at least 2π . Next consider the initial directions of the chosen minimal geodesics based at p_1 . They are four distinct points v_1, \dots, v_4 in the space of directions $\Sigma_{p_1}A$. We will refer to the distances in the space of directions as *angles*, and denote them by \angle .

Using that $\Sigma_{p_1}A \leq \mathbb{S}^2(\frac{1}{2})$ it is clear that the angles between any three points in $\{v_1, \dots, v_4\}$ add up to at most π . This of course shows that the sum of the 6 angles is indeed equal to 2π . From the equality discussion it is now easy to deduce that after possibly renumbering v_1, \dots, v_4 we can assume that $\angle(v_1, v_2) = \angle(v_3, v_4) = \pi/2$. Moreover, it is now clear that the point p_1 was arbitrary. Thus there are exactly 10 right angles, and in each of the 10 triangles in A exactly one angle equals $\pi/2$.

We may assume now that $d(p_1, p_2) = \min_{i \neq j} \{d(p_i, p_j)\}$. We choose a point $p_i \in \{p_3, p_4, p_5\}$ such that neither the angle between c_{12} and c_{1i} nor angle between c_{12} and c_{i2} is equal to $\pi/2$. This implies that the angle between c_{1i} and c_{2i} based at p_i equals $\pi/2$. Because of the equality in Toponogov's theorem, $d(p_1, p_2)^2 = d(p_1, p_i)^2 + d(p_2, p_i)^2$, which contradicts our choice above. \square

Proof of the unknot theorem As we know M has Euler characteristic 2, 3 or 4 corresponding to 2, 3 or 4 isolated fixed points of S^1 . Since the points of c that correspond to fixed points of S^1 remain more singular than $\mathbb{S}^2(\frac{1}{2})$ according to our description above, and those possibly outside c keep their size but double in numbers when we pass to the two-fold branched cover $M_2^*(c)$, we know that there are at most 4 (resp. 3 in positive curvature) such points also in $M_2^*(c)$ as well as in the universal cover of $M_2^*(c)$. If c were knotted we could use **Theorem C** to see that the number of singular points would at least triple when we passed to the universal cover.

Thus c is unknotted. Moreover, applying **Lemma 2-6** to $M_2^*(c)$ we deduce that there is at most one such curve c , and all the fixed points are on this curve c , except possibly when there are three fixed points, the curvature is only nonnegative and the singularities of M^* are given by one isolated fixed point and two fixed points contained in a singular (unknotted) biangle c . It is a simple consequence of [2, Lemma 5.1] that no circle action on $\mathbb{C}\mathbb{P}^2$ has an orbit space of this type. \square

By Fintushel [2, Theorem 7.1], **Theorem B**, and hence also **Theorem A** when M is simply connected, follow.

In the next section we will give a direct geometric proof adapted to our assumptions, which also yields the extension to a T^2 action (independent of [2]) when the Euler characteristic is four.

3 Double disc bundle decomposition

In this section we will use the Poincaré conjecture to analyze M with its circle action further. Specifically we will use the fact that M^* is the 3–sphere, and the unknot theorem of the previous section to decompose M^* in a specific way into two 3–discs respecting the strata, which in turn will yield a decomposition of M into two invariant disc bundles over points and or 2–spheres. We assume that M and hence $M^* = M/S^1$ is simply connected and nonnegatively curved, but see [Remark 3-1](#) below.

Consider the orbit space M^* . By the unknot theorem, the singular set in M^* either forms a closed unknotted curve c , or we can extend the singular set to a closed embedded unknotted curve also denoted by c , such that any two arcs of c make the maximal angle $\pi/2$ at each fixed point and are geodesics near each fixed point. (We chose such an extension only to make all arguments uniform). Note that the inverse image of each arc σ of c joining two fixed points form a smooth invariant 2–sphere Σ in M , actually the fixed-point set of a finite isotropy group (or a component thereof) if the arc is a geodesic strata. To make it clear, c is a right angled biangle, triangle or quadrilangle/rectangle corresponding to the action having 2, 3 or 4 fixed points respectively.

The decomposition is now achieved as follows. Let A and B be invariant smooth “dual” submanifolds of M corresponding either to the two fixed points when c is a biangle, one fixed point and the inverse image of the opposite edge when c is a triangle, and the inverse image of two opposite edges when c is a rectangle. In a small ϵ neighborhood U of the inverse image C of c we construct a smooth S^1 –invariant horizontal vector field V which is “normally radial” near A and B and tangential to the inverse image of the remaining two edges of c . This descends to a “smooth” vector field \bar{V} in the ϵ neighborhood U^* of c in M^* which is “normal” near the image of the boundaries of the tubular neighborhoods of A and B and for which the remaining two edges of c are integral curves. Using the fact that the ϵ neighborhoods of the images of A and B are 3–balls as are their complements, and the fact that c is unknotted, it follows that \bar{V} can be extended to a smooth nonvanishing vector field on the complement of U^* respecting this ball decomposition of M^* . The extension of \bar{V} obviously uniquely lifts to an invariant extension of V providing the desired decomposition of M into tubular neighborhoods of A and of B .

When A is a point this immediately yields a proof of [Theorem A](#).

In the remaining cases, $\chi(M) = 4$, c is a “rectangle” and A and B are 2–spheres. It is easy to see that the vector field \bar{V} on M^* can be chosen so that the flow lines emanating from each point of one edge will meet at a point of the other edge to form

a 2-sphere unless the points are vertices of the rectangle, in which case there is only one flow line. There is an S^1 action on M^* preserving these spheres with orbit space a 2-dimensional rectangle. This action clearly lifts to an action on M whose orbits near A and B are the normal circles in a tubular neighborhood. It follows that this lift commutes with the given isometric S^1 action on M .

Remark 3-1 Note that if M in [Theorem A](#) is not simply connected it has fundamental group \mathbb{Z}_2 by the Sygne theorem. Consider the lifted S^1 action on the universal cover \tilde{M} . It follows that M has Euler characteristic 1, and we conclude that this action on \tilde{M} is either the linear action on $S^2 * S^1$ with fixed-point set S^2 (fixed-point homogeneous), or the suspension of a linear almost free action on S^3 (as above). In the first case our claim follows directly as in [\[7\]](#). In the second case, the covering group \mathbb{Z}_2 interchanges the two fixed points and preserves a 3-sphere in $\tilde{M} = S^4$ invariant under the S^1 action as well. From the equivariant Poincaré conjecture it follows that indeed \mathbb{Z}_2 acts as the antipodal map and we are done.

4 Rigidity for Euler characteristic four

The main aim of this section is to prove the following equality discussion in Kleiner’s estimate of the Euler characteristic of a nonnegatively curved 4-manifold with circle symmetry.

Theorem 4-1 *Let (M, g) be a nonnegatively curved 4-manifold of Euler characteristic four endowed with an isometric S^1 action which has only isolated fixed points. Then one of the following holds:*

- (a) *There is a totally geodesic immersed compact flat surface F which is horizontal with respect to the S^1 -action and projects to an embedded 2-sphere $S^2 \subset M^*$ endowed with a flat orbifold metric.*
- (b) *There are two closed intervals $I, J \subset \mathbb{R}$ and a submetry $\sigma: M^4 \rightarrow I \times J$. The fibers of σ are given by the orbits of a (not necessarily isometric) T^2 action that extends the given S^1 -action.*

Moreover (b) holds if one of the following is true:

- (i) *In M^* there are two fixed points which can be connected by more than one minimal geodesic.*
- (ii) *There is a fixed point whose space of directions in M^* has two orbifold singularities.*

For the proof we need a lemma.

Lemma 4-2 *Let p_1, p_2, p_3, p_4 be the four isolated fixed points and let c_{ij} be a minimal geodesic between p_i and p_j .*

- (a) *If the minimal geodesic in M^* between p_1, p_3 is not unique then the minimal geodesics $c_{12}, c_{14}, c_{23}, c_{34}$ are unique and $\angle(c_{12}, c_{14}) = \angle(c_{34}, c_{23}) = \pi/2$.*
- (b) *If $\angle(c_{12}, c_{14}) = \pi/2$ then c_{24} is not unique. In fact the initial directions of all minimal geodesics from p_2 to p_4 form a circle.*

It will be clear from the proof of (b) that the number $\angle(c_{23}, c_{24})$ is independent of the choice of c_{24} . The fact that the initial directions form a circle implies that this is indeed the only constraint for initial directions of minimal geodesics from p_2 to p_3 .

Proof of Lemma 4-2 (a) The sum of the angles between any three of the points is at least π . Thus the 12 angles add up to at least 4π . For each fixed point p_i , the three angles based at p_i form a triangle in $\Sigma_{p_i} M^* \leq \mathbb{S}^2(\frac{1}{2})$. Thus the three angles can add up to at most π . Clearly equality must hold everywhere. Let v_2, v_3, v_4 be the initial vectors of c_{12}, c_{13}, c_{14} and let \tilde{v}_3 denote the initial direction of a different minimal from p_1 to p_3 . The sum of the angles is in each case π .

The distances between three points in $\mathbb{S}^2(\frac{1}{2})$ only add up to π if the three points are on a great circle but not on an open semicircle. Moreover $\angle(v_2, v_3) = \angle(v_2, \tilde{v}_3)$ because both numbers are equal to the angle in the comparison triangle of p_1, p_2, p_3 . Using that $\Sigma_{p_i} M^* \leq \mathbb{S}^2(\frac{1}{2})$ this readily implies $\angle(v_2, v_4) = \pi/2$. If v_2 were not unique one could prove similarly that $\angle(v_3, v_4) = \pi/2$, which is clearly impossible.

(b) Let c_{12}^h and c_{14}^h be arbitrary lifts of c_{12} and c_{14} . Using $\angle(c_{12}, c_{14}) = \pi/2$ we deduce that the initial directions of c_{12}^h and c_{14}^h can be chosen arbitrarily in two two-dimensional orthogonal subspaces. In particular, for any choice, $\angle(c_{12}^h, c_{14}^h) = \pi/2$. By the equality discussion in Toponogov’s theorem there is a flat triangle in M which has c_{12}^h and c_{14}^h in its boundary. The remaining side is a minimal geodesic c_{24} from p_2 to p_4 in M . Moreover the triangle must be horizontal. It is clear that this gives a two-dimensional (\mathbb{T}^2) family of minimal geodesics from p_2 to p_4 and thus c_{24} is not unique in M^* . □

Proof of Theorem 4-1 The lemma implies in particular that if there is one right angle then there are exactly four and the configurations form a unique rectangle with many choices for the diagonals. From the proof it is then easy to see that any geodesic leaving perpendicular to one side of the rectangle meets the opposite side of the rectangle after the exact same time. Therefore, if E_1 and E_2 are two sides of the triangle meeting in p_1 , then $M^* \rightarrow \mathbb{R}^2, p \mapsto (d(E_1, p), d(E_2, p))$ is a submetry onto its image.

If there is one fixed point p for which the space of direction $\Sigma_p(M^*)$ is given by $S_{k,\ell}$ with $k, \ell \geq 2$, then we consider the three initial directions v_1, v_2, v_3 of minimal geodesics to the other fixed points. Using that $\sum_{i < j} \angle(v_i, v_j) = \pi$ we then deduce that the two orbifold singularities in $\Sigma_p X = S_{k,\ell}$ must be contained in $\{v_1, v_2, v_3\}$ and thus a right angle occurs.

This proves the theorem as long as one of the geodesics is not unique. In the remaining case all geodesics are unique and there are no right angles. Then each of the four triangles can be filled uniquely with a flat convex set which forms a totally geodesic S^2 in M^* .

It remains to show that a horizontal lift of S^2 “closes up” to a compact immersed flat surface. We look at a fixed point $p_1 \in M^*$. Let v_1 be the initial direction to another fixed point and v_2 and v_3 the other two. In case that $\Sigma_{p_1} M^*$ has one singular point it must be an initial direction and we assume it to be v_1 . From the equality discussion it is clear that the initial directions of geodesics in S^2 can be parametrized by a geodesic loop $c: [0, \pi] \rightarrow \Sigma_{p_1} M^*$ of length π with $c(0) = v_1$ (a great circle). We can assume that $c(\angle(v_1, v_2)) = v_2$ and $c(\pi - \angle(v_1, v_3)) = v_3$. We now look at the unit sphere $T_{p_1}^1 M$ and the natural (orbifold) Riemannian submersion $T_{p_1}^1 M \rightarrow \Sigma_{p_1} M^*$. We let γ denote a horizontal lift of c . Clearly γ is a semicircle and $\gamma(0), \gamma(\angle(v_1, v_2)), \gamma(\pi - \angle(v_1, v_3))$ as well as $\gamma(\pi) = -\gamma(0)$ are initial directions of minimal geodesics to the other fixed points. We must have equality in Toponogov’s theorem and thus the two-dimensional half plane σ^+ containing the semicircle γ is tangent to a flat surface with boundary. Moreover the element $-1 \in S^1$ maps the half plane σ^+ to $\sigma^- = \{-w \mid w \in \sigma^+\}$. Therefore it is clear that $\sigma = \sigma^+ \cup \sigma^-$ is tangent to a totally geodesic flat surface. The action of -1 on σ is either given by $-\text{id}|_\sigma$ or by a reflection at the line $\mathbb{R}\gamma(0)$, depending on whether $v_1 = \alpha(\gamma(0))$ corresponds in $\Sigma_{p_1} M^*$ to an odd or even orbifold singularity.

Since we can do this at each fixed point it is clear that we get a totally geodesic immersion of \mathbb{R}^2 into M . If we keep track of the (unoriented) tangent space at each point, the immersion induces an immersion of \mathbb{R}^2 into the Grassmannian of planes $\text{Gr}_2 TM$ of M whose image is given by an injectively immersed flat manifold $F \subset \text{Gr}_2 TM$. We want to show that F is compact. The group S^1 acts via the differentials of isometries on $\text{Gr}_2 TM$ and from the above discussion it is clear that $-1 \in S^1$ leaves F invariant. Furthermore, if σ is a plane in F based at one of the fixed points p_i then σ is fixed by -1 . If $\Sigma_{p_i} M^*$ has no or only an odd orbifold singularity then the fixed point σ is isolated in F . Otherwise the fixed-point component is 1-dimensional.

It is clear that the number l of points in p_1, \dots, p_4 whose space of direction in M^* has an even orbifold singularity is even. If $l = 0$, then $F/\pm 1$ is a two-dimensional

flat orbifold with at least four isolated singularities, since F intersects each of the four submanifolds $\text{Gr}_2 T_{p_i} M$ at least once. It is elementary (and also follows from the above discussion) that there can only be four such singularities and equality can only occur in the compact case. Thus F is compact and the projection map $F \rightarrow M$ is injective.

If $l = 2$, then $F/\pm 1$ is a two-dimensional flat orbifold with nontrivial totally geodesic boundary and at least two orbifold singularities in the interior. It is again clear (by doubling) that no additional singularities can occur and $F/\pm 1$ hence F has to be compact.

If $l = 4$ then $F/\pm 1$ corresponds to smooth flat surface with two totally geodesic boundary components, such a manifold is isometric to $I \times S^1$ or $I \times \mathbb{R}$, where I is a compact interval. We have to show that the latter cannot occur. Going back to the above notation this case corresponds to the case that $\gamma(0)$ is fixed by -1 . Thus $\gamma(0)$ is tangent to the fixed-point set -1 which is a totally geodesic 2-sphere endowed with a cohomogeneity-one metric (coming from the S^1 action). This implies that the geodesic $h(s) = \exp(s\gamma(0))$ closes up, $h'(l) = \gamma(0)$ for some l .

The parallel transport along $h_{[0,l]}$ maps the plane σ to another plane $\tilde{\sigma}$, that like σ has to be spanned by two minimal geodesics to other fixed points which are tangent to a horizontal flat hinge. If we have $\gamma(0) = h'(l)$ as one initial vector, the choice of such a hinge amounts to choosing the different possible lifts of the geodesic loop $c \in \Sigma_{p_1} M^*$ with initial point $\gamma(0)$. Since an orbifold singularity is present, there is more than one choice but there are clearly only finitely many. This shows that for some positive integer k the parallel transport of $h_{[0,kl]}$ maps $\gamma'(0)$ to itself. Therefore the lift of α to the Grassmannian $\text{Gr}_2 M$ also closes. Hence the fixed-point set of -1 in F contains a circle and we can not have $F/\pm 1 \cong I \times \mathbb{R}$. \square

Remark 4-3 (a) In the situation of [Theorem 4-1\(a\)](#) one can use the soul theorem to see (without using the Poincaré conjecture) that either side of the \mathbb{S}^2 is given by a 3-disc, see also [Remark 5-6\(b\)](#).

(b) There are nonnegatively curved metrics (Müter metrics) on $\mathbb{S}^2 \times \mathbb{S}^2$ with an S^1 -symmetry for which $\mathbb{S}^2 \times \mathbb{S}^2/S^1$ has positive curvature outside a set of codimension 1.

(c) [Theorem 4-1](#) remains valid if M is replaced by a nonnegatively curved 4-dimensional orbifold with Euler characteristic four. Recall that by a result of Martin Kerin [[11](#), [Theorem 2.4](#)] there are a lot of 4-dimensional orbifolds of Euler characteristic four which have positive sectional curvature on an open dense set. These orbifolds have an S^1 symmetry with four isolated fixed points.

5 Knots arising as extremal subsets in 3-dimensional Alexandrov spaces

Throughout this section we let X denote the 3-sphere endowed with an arbitrary Alexandrov metric of nonnegative curvature, and $c \subset X$ an extremal subset homeomorphic to \mathbb{S}^1 . Recall that c being extremal means that for any point $p \in X \setminus c$, any point $q \in c$ with $d(p, q) = d(p, c)$ is a critical point of the distance function $d(p, \cdot)$. Such sets play an important role in Alexandrov geometry; see [17] for further information.

The question we answer in this section is: Which knots can arise in this way? We will first show that the question is equivalent to asking for which knots c in X is the two-fold branched cover $X_2(c)$ a spherical space form.

To see this we need generalizations of Lemmas 2-3 and 2-4.

Lemma 5-1 *An extremal closed curve $c \subset X$ is a locally flat 1-dimensional submanifold.*

Proof As in the proof of Lemma 2-4 it suffices to see that for each point of c , a sufficiently small ball is homeomorphic to the corresponding ball in its tangent cone by a homeomorphism taking the intersection of c with the ball to the corresponding extremal curve in the tangent cone. The latter is of course the intersection of the ball with the cone of the space of directions of c , which is a pair of extremal points in the space of directions of X at the point. The claim is a direct consequence of the relative stability theorem due to Kapovitch [10, Theorem 9.2] via scaling. \square

Lemma 5-2 *$X_2(c)$ is an Alexandrov space with nonnegative curvature.*

Proof For $a \in c$, the tangent cone $T_a X_2(c)$ exists and is a two-fold branched cover of $T_a X$. It is easy to see that the space of directions $\Sigma_a X_2(c)$ of $X_2(c)$ at a is an Alexandrov space of curvature ≥ 1 whose projection to the space of directions of X at a is a two-fold branched cover. The branching locus of the cover in $\Sigma_a X_2(c)$ is given by two points corresponding to the two directions in $T_a c$.

Clearly geodesics in the two-fold branched cover $\sigma: X_2(c) \rightarrow X$ are nonbranching and even more two minimal geodesics $c_1, c_2: [0, 1] \rightarrow X_2(c)$ with the same initial direction coincide.

Step 1 Toponogov's theorem holds for any hinge in $X_2(c)$ based at a point $a \in c$.

We consider the gradient exponential map $\text{gexp}_a: T_a X \rightarrow X$ due to Petrunin [17]. It is 1-Lipschitz and equal to the usual exponential map at all points where the latter is

defined. Since the gradient of the distance function $d(q, \cdot)$ is tangential to c along $c \setminus q$, it follows that if $v \in \Sigma_a X$ satisfies $\text{gexp}(t_0 v) \in c$ for some $t_0 > 0$ then $\text{gexp}(tv) \in c$ for all $t > t_0$. This in turn shows that for any lift $\hat{v} \in \Sigma_a X_2(c)$ of v we can find a unique continuous lift of $\text{gexp}(tv)$ ($t \geq 0$) to a continuous curve $\widehat{\text{gexp}}(tv)$. It is then easy to see that $\widehat{\text{gexp}}_a: T_a X_2(c) \rightarrow X_2(c)$ is 1-Lipschitz. In fact one can distinguish between two cases. If $w \in T_a X_2(c)$ with $\widehat{\text{gexp}}(w) \notin c$ then $\widehat{\text{gexp}}$ is 1-Lipschitz in a neighborhood of w since it is a lift of gexp . If $w \in T_a X_2(c)$ with $\widehat{\text{gexp}}(w) \in c$ then

$$d(\widehat{\text{gexp}}(w), \widehat{\text{gexp}}(v)) = d(\text{gexp}(\sigma_* w), \text{gexp}(\sigma_* v)) \leq d(\sigma_* v, \sigma_* w) \leq d(v, w).$$

Clearly $\widehat{\text{gexp}}_a$ coincides with the usual exponential map for all points where the latter is defined. If $v, w \in T_a X_2(c)$ are the initial vectors of a hinge in $X_2(c)$, then the Euclidean comparison hinge is isometric to the hinge spanned by v and w in $T_a X_2(c)$ (based at the cone point). Since $\widehat{\text{gexp}}_a$ is 1-Lipschitz it maps the opposite side of the hinge to a shorter curve and the comparison follows.

Step 2 Let $q \in X_2(c) \setminus c$ and S the set of all points $s \in X_2(c)$ for which a minimal geodesic from q to s passes through c . Then S has measure zero.

We let c^h denote the subset of all points $s \in c$ such that the minimal geodesic γ_{qs} from q to s continues to be minimal if we extend it by $h > 0$.

There is a map $f_h: c^h \rightarrow S$ that sends $s = \gamma_{qs}(d(q, s))$ to the point $\gamma_{qs}(d(q, s) + h)$.

We finish by showing this map is locally Lipschitz. Put $\varepsilon_0 = \frac{1}{3} \min\{1, h, d(q, c)\}$. More precisely we plan to show that for two points $s, s' \in c^h$ with $d(s, s') \leq \varepsilon_0$ we have $d(f_h(s), f_h(s')) \leq \frac{4h}{\varepsilon_0} d(s, s')$.

Let $v \in T_s X_2(c)$ be the initial vector of the minimal geodesic to $f_h(s)$, $u \in T_s X_2(c)$ the initial vector of the minimal geodesic to q and $x \in T_s X_2(c)$ the initial vector of a minimal geodesic from s to s' . Of course the minimal geodesic in $T_s X_2(c)$ passes through the cone point and therefore the triangle spanned by u, v and x is Euclidean. By assumption we know that $|x| = d(s, s') \leq \varepsilon_0 \leq \frac{1}{3} \min\{|u|, |v|\}$. This in turn implies that we can estimate the defect in the triangle inequality

$$d(u, x) + d(v, x) - (|u| + |v|) \leq \frac{|x|^2}{\varepsilon_0}.$$

Since the g exponential map $\widehat{\text{gexp}}_s$ is 1-Lipschitz, we obtain

$$d(s', f_h(s)) + d(s', q) - d(q, f_h(s)) \leq \frac{d(s, s')^2}{\varepsilon_0}.$$

By Step 1, we can apply the hinge version of Toponogov’s theorem to the triangle $(f_s(h), q, s')$ to see that the angle between the minimal geodesic from s' to $f_s(h)$ and

the minimal geodesic from s' to q is given by $\pi - \varphi$ with

$$\varphi \leq \frac{2d(s, s')}{\varepsilon_0}.$$

Next notice that φ is the angle between the minimal geodesic from s' to $f_h(s)$ and the minimal geodesic from s' to $f_h(s')$. Using Step 1 once more and $d(s', f_h(s)) \leq h + d(s, s')$ we see that

$$d(f_h(s), f_h(s')) \leq \frac{2d(s, s')h}{\varepsilon_0} + d(s, s') \leq 4 \frac{d(s, s')h}{\varepsilon_0}.$$

Step 3 Toponogov's theorem holds in $X_2(c)$.

Notice that any geodesic in $X_2(c)$ is the limit of geodesics which do not meet c . It is well known that Toponogov's theorem is equivalent to saying that for all q in $X_2(c)$ the function $f(p) = d(p, q)^2$ satisfies $\text{Hess}(f) \leq 2$ in the sense of support functions, ie $f \circ \gamma(t) - t^2$ is concave for any unit speed geodesic $\gamma : [a, b] \rightarrow X_2(c)$.

Clearly we may assume that γ and q are generic. Thus we may assume that $\gamma(t)$ does not meet the branching locus and (by Step 2) that the set of parameters t for which a minimal geodesic from q to $\gamma(t)$ intersects c forms a set of measure zero.

Since $X_2(c)$ is in a neighborhood of the image of c an Alexandrov space it follows that $\text{Hess}(f)(\gamma(t)) \leq C$ for all $t \in [a, b]$ for some large C . Thus it suffices to prove that $\text{Hess}(f)(\gamma(t)) \leq 2$ for all t for which a minimal geodesic α from q to $\gamma(t)$ does not meet the branching locus $c \subset X_2(c)$. This in turn one can establish analogous to the usual globalization theorem of Toponogov's comparison statement. \square

We are now ready to analyze our question. As mentioned above our point of departure is the following lemma.

Lemma 5-3 *The two-fold branched cover $X_2(c)$ of X has finite fundamental group and $H_1(X_2(c), \mathbb{Z}_2) = 0$.*

Proof We view $H_1(X_2(c), \mathbb{Z}_2)$ as vector space over \mathbb{Z}_2 . Suppose for the moment it is not 0-dimensional.

Let ι denote the involutive isometry of $X_2(c)$ given by the nontrivial deck transformation of the branched cover. The automorphism $H_1(\iota)$ of $H_1(X_2(c), \mathbb{Z}_2)$ induced by ι then leaves some subspace $V \subset H_1(X_2(c), \mathbb{Z}_2)$ of codimension 1 invariant. Since $H_1(X_2(c), \mathbb{Z}_2) \cong \Pi / \langle \{g^2 \mid g \in \Pi\} \rangle$ is a quotient of $\Pi := \pi_1(X_2(c))$, this in turn shows that Π contains a subgroup Π' of index 2 which is invariant under the natural action of $\pi_1(\iota)$.

Consider next the action of Π on the universal cover $\widehat{X_2(c)}$. We can also lift ι to an isometry $\tilde{\iota}$ of the universal cover with a nontrivial fixed-point set.

Let Γ be the group generated by $\tilde{\iota}$ and Π . By construction $\Gamma/\Pi \cong \mathbb{Z}_2$ acts as ι on $X_2(c)$. Let $g \in \Gamma$ be an element with a nontrivial fixed-point component L . The image of L in $X_2(c)$ must coincide with the image of the fixed-point set of ι . This shows that there must be a deck transformation $h \in \Pi$ which maps L to the fixed-point set of $\tilde{\iota}$. In other words $h^{-1}\tilde{\iota}h$ fixes L . But then the differential of g and $h^{-1}\tilde{\iota}h$ must coincide along L , namely they are given by the identity on T_pL and by $-\text{id}$ on the normal bundle of L . Thus $g = h^{-1}\tilde{\iota}h$ is in the same conjugacy class as $\tilde{\iota}$.

Let $\hat{\Gamma}$ be the normal subgroup generated by the conjugacy class of $\tilde{\iota}$ in Γ . Finally let Γ' be the group generated by $\tilde{\iota}$ and the subgroup Π' which is normalized by $\tilde{\iota}$. Since Γ' has index two in Γ it is normal and thus $\hat{\Gamma} \subset \Gamma' \neq \Gamma$.

As shown above any element in Γ with a nontrivial fixed-point set is in the conjugacy class of $\tilde{\iota}$ and hence in $\hat{\Gamma}$. Thus the group $\Gamma/\hat{\Gamma}$ acts freely on $\widehat{X_2(c)}/\hat{\Gamma}$ and $X = \widehat{X_2(c)}/\Gamma$ is not simply connected; a contradiction.

Hence $H_1(X_2(c), \mathbb{Z}_2) = 0$. Using that $X_2(c)$ is a topological manifold with a non-negatively curved Alexandrov metric we see that either $\pi_1(X_2(c))$ is finite, $X_2(c)$ is two-fold covered by $\mathbb{S}^1 \times \mathbb{S}^2$ or $X_2(c)$ is a flat Riemannian manifold. Since $H_1(X_2(c), \mathbb{Z}_2)$ is nontrivial for the latter two cases (see [22, Theorem 3.5.5]) we conclude that $\pi_1(X_2(c))$ is finite. □

Using the equivariant elliptization conjecture we know that $X_2(c)$ is a spherical space form endowed with a linear involution whose fixed-point set is a circle. In addition $X_2(c)$ is a \mathbb{Z}_2 -homology 3-sphere.

Lemma 5-4 *Let \mathbb{S}^3/Π be a spherical space form which is also a \mathbb{Z}_2 homology sphere. Then there is a linear involution ι (unique up to conjugation) whose fixed-point set is a circle. Moreover the underlying topological space of the quotient $\mathbb{S}^3/\langle \Pi, \iota \rangle$ is the 3-sphere.*

Proof By assumption the abelianization $\Pi/[\Pi, \Pi]$ has no two torsion. Recall that the finite groups $\Pi \subset \text{SO}(4)$ that act freely on \mathbb{S}^3 are conjugate to subgroups of $U(2)$. If we divide out the center of $U(2)$ we get a homomorphism $\Pi \rightarrow \text{SO}(3)$ whose kernel is central in Π . By the classification of finite subgroups of $\text{SO}(3)$ the image is either cyclic, dihedral or given as the orientation-preserving symmetries of one of the platonic solids.

If the image is cyclic then Π is abelian (as a central extension of a cyclic group), and since Π acts freely this in turn implies that Π itself is cyclic. Since $\Pi/[\Pi, \Pi]$ has no

two-torsion, the image of Π in $SO(3)$ can not be dihedral and it can neither be given as the orientation-preserving symmetries of a octagon (or the dual cube).

This leaves us with three cases, Π is either cyclic (Case 1), of tetrahedral type (Case 2) or of icosahedral type (Case 3). In the last two cases we will also use that the greatest common divisor of the orders of the subgroups $\Pi \cap SU(2)$ and $\Pi \cap \text{Center}(U(2))$ is at most 2, as otherwise the action would not be free.

In all cases we establish first the existence of ι and it will be clear from the construction that ι fixes at least one circle. In all cases this implies that the fixed-point set in S^3/Π is equal to a circle because by a theorem of Floyd the total \mathbb{Z}_2 -Betti number of the fixed-point set of ι is bounded by the \mathbb{Z}_2 -Betti number of S^3/Π , which is 2 by assumption.

Case 1 (Cyclic case) Here Π is a cyclic group of odd order m . Let $\zeta \in S^1$ be a primitive m -root of unity We may assume that for some integer p (prime to m) the action of Π is generated by $\zeta \star (z_1, z_2) = (\zeta z_1, \zeta^p z_2)$, where $z_1, z_2 \in \mathbb{C}$ with $|z_1|^2 + |z_2|^2 = 1$.

We let $\tilde{\tau}$ denote complex conjugation. Clearly $\tilde{\tau}$ normalizes the action of Π . The group generated by Π and $\tilde{\tau}$ is a dihedral group which is generated by elements which are conjugate to $\tilde{\tau}$.

Since Π is generated by elements with a nontrivial fixed-point set the underlying topological space of $S^3/\langle \tilde{\tau}, \Pi \rangle$ is simply connected and hence a 3-sphere. The involution $\iota: S^3/\Pi \rightarrow S^3/\Pi$ induced by $\tilde{\tau}$ fixes at least one circle. As explained above this implies that the fixed-point set is given by a circle. The uniqueness of ι follows from the fact that ι has to anticommute with the action of Π .

Case 2 (Tetrahedral case) We can assume $\Pi \subset U(2) \subset SO(4)$. We let $\hat{\Pi} \subset S^3 \times S^1 \subset S^3 \times S^3 = \text{Spin}(4)$ denote the inverse image. We assume that $S^1 \subset S^3$ is the image of the one-parameter group $e^{i\varphi}$.

Then we can find a number m coprime to 6 and an integer $k \geq 0$ with $k \neq 1$ such that $\hat{\Pi}$ is the smallest group satisfying the following:

- (i) $\hat{\Pi}$ contains the cyclic subgroup of order $2m$ in $1 \times S^1$.
- (ii) For one primitive $(3^k)^{\text{th}}$ root of unity $\zeta \in S^1$ (if $k = 0$ this means $\zeta = 1$), Π contains the element $(\frac{1}{2}(1 + i + j + k), \zeta) \in S^3 \times S^1$.
- (iii) Π contains the group $\{\pm 1, \pm i, \pm j, \pm k\} \times \{1\} \subset S^3 \times S^1$.

We put $\hat{\iota} := ((i + j)/\sqrt{2}, j) \in S^3 \times S^3$. It is straightforward to check that $\hat{\iota}$ normalizes $\hat{\Pi}$. Clearly the image $\tilde{\iota}$ of $\hat{\iota}$ in $SO(4)$ and corresponds again to a complex conjugation and thus fixes a circle in S^3 . It is easy to see that the group $\Gamma := \langle \Pi, \tilde{\iota} \rangle$ is generated by the conjugacy class of $\tilde{\iota}$.

Thus $S^3/\langle \Pi, \tilde{\iota} \rangle$ is again a sphere.

As before the map $\iota: S^3/\Pi \rightarrow S^3/\Pi$ has the desired properties.

The uniqueness follows from the fact that $\tilde{\iota}$ normalizes Π and that the conjugacy class of $\tilde{\iota}$ generates $\langle \Pi, \tilde{\iota} \rangle$.

Case 3 (Icosahedral case) We can assume $\Pi \subset U(2) \subset SO(4)$. We let $\hat{\Pi} \subset S^3 \times S^1 \subset S^3 \times S^3 = Spin(4)$ denote the inverse image. We assume that $S^1 \subset S^3$ is the image of the one-parameter group $e^{i\varphi}$. In this case $\hat{\Pi}$ is a product subgroup $\hat{\Pi} = \Pi_1 \times \Pi_2 \subset S^3 \times S^1$, where Π_1 is the binary icosahedral group and Π_2 is a cyclic subgroup of order $2m$ with m being an integer coprime to 30. The binary icosahedral group Π_1 is the group generated by

$$\pm i, \pm j, \pm k, \pm 1, i + j\sqrt{2}, \frac{1}{2}(1+i+j+k), i + \frac{1}{2}(1+\sqrt{5})j + \frac{2}{1+\sqrt{5}}k.$$

It has order 120 and its only normal subgroup is given by ± 1 .

We put $\hat{\iota} := (j, j)$. Clearly $\hat{\iota}$ normalizes $\hat{\Pi}$. We consider in $\langle \hat{\Pi}, \hat{\iota} \rangle$ the subgroup $\hat{\Gamma}$ generated by the conjugacy class of $\hat{\iota}$. The projection of $\hat{\Gamma}$ to $S^3 \times \{1\}$ is given by a normal subgroup of Π_1 generated by the conjugacy class of j in Π_1 . Since Π_1 has no nontrivial normal subgroups of order larger than 2 we deduce that $\hat{\Gamma}$ projects surjectively to Π_1 . Using that the commutator group of Π_1 is equal to Π_1 and that the second commutator group of $\hat{\Gamma}$ is contained in $S^3 \times 1$ we see that $\hat{\Gamma}$ contains $\Pi_1 \times \{1\}$. Moreover it is clear the projection of $\hat{\Gamma}$ to $\{1\} \times S^3$ contains Π_2 . In summary this proves $\hat{\Gamma} = \langle \hat{\Pi}, \hat{\iota} \rangle$. The image $\tilde{\iota}$ of $\hat{\iota}$ in $SO(4)$ is again a complex conjugation and one can finish the argument as before. □

Corollary 5-5 *If c is a closed embedded curve in $A = S^3$ which is an extremal set with respect to a nonnegatively curved Alexandrov metric then we can find an orbifold metric S^3 of constant curvature 1 such that the only orbifold singularity is of \mathbb{Z}_2 type and the singular locus is given by c . Moreover these orbifolds are classified in the proof of Lemma 5-4.*

Remark 5-6 (a) In the special case that Γ is a cyclic group of odd order, the corresponding knots arising are, by work of Seifert, the so called two bridge knots. By Milnor, two bridge knots in \mathbb{R}^3 are exactly those knots which for any $\varepsilon > 0$ are isotopic to knots with total curvature $\leq 4\pi + \varepsilon$. In all cases, the Hopf S^1 action gives rise to a

Seifert fibered structure on the space form $\mathbb{S}^3/\Pi = X_2(c)$ preserved by the involution ι induced from complex conjugation whose fixed-point set is c . The knot c in the 3-sphere $X_2(c)/\iota$ is explicitly described by Montesinos via the Seifert invariants of the fibration (see [18]). In particular we mention that c is the $(3, 5)$ -torus knot in the special case where $X_2(c) = \mathbb{S}^3/\Pi$ is the Poincaré homology sphere.

(b) In his PhD thesis, Wolfgang Spindeler [20] showed that a quotient X of a simply connected nonnegatively curved 4-manifold by an isometric S^1 action is the Gromov–Hausdorff limit of positively curved three manifolds. If X contains a closed curve c in its singular set then one can also find a sequence of smooth positively curved orbifold structures on X (Gromov–Hausdorff converging to the quotient metric) such that the only orbifold singularity is a \mathbb{Z}_2 singularity along c . In particular each of these metrics induces a smooth Riemannian metric on $X_2(c)$. This in turn allows that the equivariant classification of the corresponding 4-manifolds can also be proved without using Perelman’s solution of the Poincaré conjecture and the \mathbb{Z}_2 -equivariant version. Instead one can then just refer to Hamilton’s classification [8] of positively curved 3-manifolds.

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