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Totally twisted Khovanov homology

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We define a variation of Khovanov homology formally similar to totally twisted Heegaard–Floer homology. Over a certain field, this version of Khovanov homology has a completely explicit description in terms of the spanning trees of a link projection. We prove that this new theory is a link invariant and describe some of its properties. Finally, we provide the results of some computer computations of the invariant.

57M27; 57M25

1 Introduction

1.1 Background

In [8], M Khovanov introduced his well-known construction of a homology theory for a link $\mathcal{L}$ in S^3 whose Euler characteristic encodes a version of the Jones polynomial for L (see also Bar-Natan [3] and Viro [12]). This construction used the exponentially many ways a link diagram can be resolved, in a manner analogous to Kauffman's state summation approach to the Jones polynomial. However, Thisthlethwaite and others had established a more efficient means for computing the Jones polynomial by using the spanning trees of the Tait graphs for the link diagram. Trying to repeat this process for Khovanov homology led to papers by A Champanerkar and I Kofman [5] and S Wehrli [13]. Both papers show that, in principle, the Khovanov homology can be computed from a complex whose generators are the spanning trees for one of the Tait graphs of L by demonstrating that the Khovanov homology deformation retracts to a subcomplex whose generators are in one-to-one correspondence with the spanning trees. However, while these constructions identified the generators of the complex with spanning trees, they only demonstrated the existence of the differential for a spanning tree complex, remaining mute about how to fully and explicitly compute it. In [5] A Champanerkar and I Kofman describe parts of the differential, but not the entire structure.

In this paper, we describe a new structure in (reduced, characteristic 2) Khovanov homology which leads to a homology theory the author calls totally twisted Khovanov

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homology (due to formal analogies with the "totally twisted" Heegaard–Floer homology). It arises by deforming the Khovanov differential, and is a singly graded theory. It shares many of the properties of Khovanov homology. In particular, it provides a new invariant homology theory for links. However, there is one notable difference: taken with the correct coefficients, the totally twisted homology deformation retracts to a spanning tree complex with a completely explicit differential.

An almost complete account of the construction of totally twisted Khovanov homology, the main results of the paper and some of the author's computerized comparison to Khovanov homology are presented in the remainder of this introduction. This account also forms the main narrative of the paper, and should be read before the remaining sections. The proofs of the theorems stated in this introduction occur in the remaining sections. The paper concludes by proving the basic results common to most knot homology theories using the spanning tree perspective. In the preprint version of this paper originally posted to the arXiv, substantial computer calculations were cited to suggest that for *knots* the spanning tree complex had homology identical with the characteristic 2, δ–graded, reduced Khovanov homology. Shortly after the appearance of that preprint, T Jaeger proved this result in [6].

1.2 The construction

Throughout, $\mathcal{L}$ will be an *oriented* link in S^3, equipped with a marked point $p \in \mathcal{L}$. We will study $\mathcal{L}$ through a link diagram: a generic projection L of $\mathcal{L}$ into S^2, taking p to a noncrossing point. We will always use script letters to denote links and nonscript letters to denote link diagrams.

Definition 1.1 For an oriented link diagram,

(1) $\mathrm{CR}(L)$ denotes the set of crossings in L,

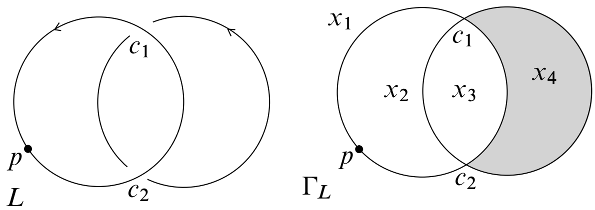
(2) $n_\pm(L)$ is the number of right-handed (positive)/left-handed (negative) crossings.

It will be convenient to let Γ_L be the image of the projection of L in S^2; Γ_L is the four-valent graph found by stripping L of its crossing data.

Definition 1.2 The components of $S^2 \backslash \Gamma_L$ are the *faces* of L. The set of faces will be denoted $\mathfrak{F}_L$. Let $\mathbb{P}_L = \mathbb{Z}/2\mathbb{Z}[x_f \mid f \in \mathfrak{F}_L]$ be the polynomial ring which assigns a formal variable to each $f \in \mathfrak{F}_L$. The field of fractions of $\mathbb{P}_L$ will be denoted $\mathbb{F}_L$.

We will make essential use of the requirement that every coefficient ring have characteristic 2. We will usually enumerate the faces of L and then identify the faces of L with the corresponding formal variable x_i.

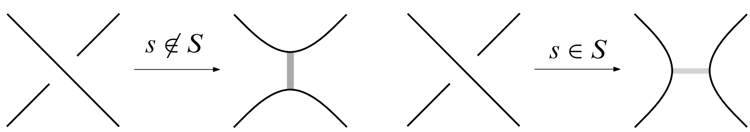
Example As an illustration, to which we return repeatedly, consider the following diagram for the two-component unlink:



On the left is the oriented diagram L with a marked point p, and two labeled crossings. We will identify $\mathrm{CR}(L)$ with $\{c_1, c_2\}$. On the right is the four-valent graph Γ_L. The vertices are identified with the crossings, and the marked point is not at a crossing. The shaded region is one face of L labeled with the formal variable x_4. There are three other faces in $\mathfrak{F}_L$, labeled by their formal variables x_1, x_2, x_3. Thus, $\mathbb{P}_L = \mathbb{Z}/2\mathbb{Z}[x_1, x_2, x_3, x_4]$ and $\mathbb{F}_L$ is the field of binary rational functions in four variables.

1.3 Resolutions

To each subset $S \subset \mathrm{CR}(L)$, we define the resolution L_S of L to be the *diagram* obtained by locally resolving each crossing in $\mathrm{CR}(L)$ according to the rule:[1]



The diagram L_S includes only the solid lines, and thus is a collections of disjoint circles embedded in S^2. One of the circles contains the image of the marked point p and will be called the *marked circle* for L_S. If we include the dark gray lines for those crossings $s \notin S$ and the light gray lines for $s \in S$, we obtain a 1–complex embedded in S which will be denoted Γ_S.

We can partially order the resolutions by asserting $L_S \leq L_{S'}$ when $S \subset S'$.

Definition 1.3 (1) $R(L)$ is the set of resolutions L_S, $S \subset \mathrm{CR}(L)$. It is in one-to-one correspondence with the power set of $\mathrm{CR}(L)$.

(2) Given a resolution L_S, $\delta(L_S) = |S|$.

(3) For $i \in \mathbb{Z}$, $R_i(L)$ is the subset of $R(L)$ consisting of those L_S with $\delta(L_S) = i$.

$R_i(L)$ consists of those resolution diagrams adorned with exactly i green arcs.

[1] In the more common language of 0– and 1–resolutions used for Khovanov homology, the 0–, 1–values correspond to the indicator function for S as a subset of $\mathrm{CR}(L)$.

Example (continued) The four resolutions for our example unlink are depicted with their colored arcs in Figure 1.

1.4 Discs and formal areas

To each subset $T \subset \mathfrak{F}_L$ we can associate a special element of $\mathbb{P}_L$:

$$[T] = \sum_{f \in T} x_f.$$

We will use the following shorthand: if $\{i_1, \dots, i_k\} \subset \mathfrak{F}_L$ then $[i_1 \dots i_k]$ will also denote $[\{f_{i_1}, \dots, f_{i_k}\}]$. In particular, both equal $x_{i_1} + \cdots + x_{i_k}$. Thus, $[2] = x_2$ and will be preferred to $[\{2\}]$.

For a resolution S of L, we use the additional arcs in Γ_S to divide the components of $S^2 \backslash L_S$ into a collection of smaller regions that are in one-to-one correspondence with $\mathfrak{F}_L$. We call these components the faces of L_S and label them with the same formal variable as the corresponding face of L. If R is a component of $S^2 \backslash L_S$ then it is a union of faces $f_{i_1}, \dots, f_{i_k}$ of L_S, and we assign R the following element of $\mathbb{P}_L$:

$$[R] = [\{f_{i_1}, f_{i_2}, \dots, f_{i_k}\}] = [i_1 \dots i_k] = x_{i_1} + \dots + x_{i_k}.$$

Using this data and the marked point p, we can assign a formal area to each of the circles in a resolution L_S. Every unmarked circle C in L_S bounds two discs in S^2 which are unions of faces in L_S. These two discs can be distinguished by which one contains the marked point p.

Definition 1.4 Given $S \subset \text{CR}(L)$, let $\text{CIR}(L_S)$ be the set of circles in the resolution L_S. Given an unmarked circle $C \in \text{CIR}(L_S)$ let

(1) $A_p(C)$ be the component of $S^2 \backslash C$ which does not contain p; $A_p(C)$ will be called the *interior* of C;

(2) the *exterior* of C in L_S be the component of $S^2 \backslash C$ which contains p;

(3) $[C]$ be $[A_p(C)]$, the formal area of the interior of C, ie the sum of the formal variables assigned to the faces contained in C.

1.5 Complexes associated to resolutions

As with reduced Khovanov homology we will associate an algebraic object to each resolution L_S of a diagram L. In this paper, we assign a Koszul complex to L_S. In our setting we need Koszul complexes over $\mathbb{P}_L$ which take the following form.

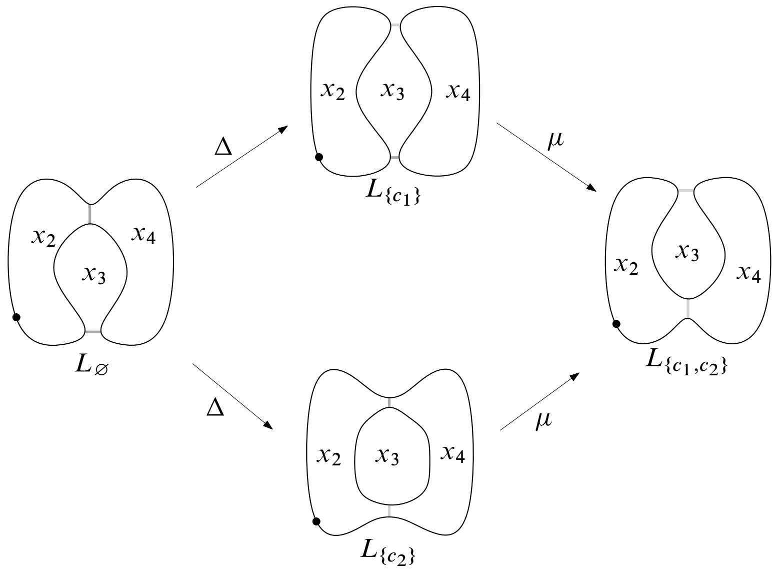


Figure 1: The four resolutions for the unlink considered in the text: from left to right we have the resolutions with δ equal to $0, 1$ and 2 respectively. The marked point is depicted with a darkened disc. The labels for each of the faces in each resolution are given as a formal variable in $\mathbb{P}_L$. In $L_{\{c_2\}}$ the unmarked circle has area x_3 while the bounded region inside the marked circle has area $x_2 + x_3 + x_4$. The μ and Δ maps indicate whether we use a multiplication or comultiplication when constructing the Khovanov complex.

Definition 1.5 Let R be a ring of characteristic 2. For $r \in R$ let $\mathcal{K}_R(r)$ be the complex

$$0 \longrightarrow Rv_+ \xrightarrow{\cdot r} Rv_- \longrightarrow 0,$$

where $v_\pm$ occur in gradings ± 1. Given elements $r_1, \dots, r_k \in \mathbb{P}_L$ the Koszul complex $\mathcal{K}_R(r_1, \dots, r_k)$ is the tensor product complex

$$\mathcal{K}(r_1) \otimes_R \mathcal{K}(r_2) \otimes_R \cdots \otimes_R \mathcal{K}(r_k).$$

Our definition differs from the normal one in three respects: we only define the complex in characteristic 2 and thus do not specify a sign convention for the differential in the tensor product complex; the gradings are different than usual but will be more convenient for our purposes; we choose to explicitly label a basis for the rank-1 free modules in $\mathcal{K}(r)$.

We will use the ring associated to the faces of L_S as well as the marked point to associate a Koszul type complex to the resolution L_S:

Definition 1.6 Given a link diagram L and a subset $S \subset \textsc{cr}(L)$, let $\textsc{Cir}(L_S) = \{C_0, C_1, \dots, C_k\}$ with C_0 being the marked circle. Define

$$\mathcal{V}(L_S) = \mathbb{P}_L v_0 \otimes_{\mathbb{P}_L} \mathcal{K}_{\mathbb{P}_L}([C_1], \dots, [C_k]),$$

where v_0 is in grading 0. The grading on any $\mathcal{V}(L_S)$ will be called a q–grading.

To summarize, we use the formal areas of the unmarked circles in L_S as the sequence of elements of $\mathbb{P}_L$ in forming a Koszul complex.[2] However, we do not use the rank-2 complex for C_0. Instead, let $\mathcal{K}_{C_0}$ be the trivial complex $0 \to \mathbb{P}_L v_0 \to 0$, supported in degree 0. This is the first factor in the tensor product defining $\mathcal{V}(L_S)$.

Definition 1.7 Let C be an unmarked circle. The differential in $\mathcal{K}([C])$ will be denoted ∂_C. Thus the differential in $\mathcal{V}(L_S)$ will be $\partial_{\mathcal{V}(L_S)} = \sum_{i>0} \partial_{C_i}$. It changes the q–grading by -2.

As usual, we represent the basis for $\mathcal{V}(L_S)$ through decorations on the diagram L_S: each basis element can be represented as L_S with each circle C_i, $i > 0$, adorned with either a $+$ or a $-$, depending on whether the element v_+ or v_- is in the i^{th}–factor of the basis element [3].

Example In Figure 1, $L_{\{c_2\}}$ consists of two circles: C_0 and C_1. Then $\mathcal{K}(C_1)$ is the complex $\mathcal{K}_{\mathbb{P}_L}([3])$ and $\mathcal{V}(L_S)$ is the complex $\mathbb{P}_L v_0 \otimes_{\mathbb{P}_L} \mathcal{K}_{\mathbb{P}_L}([3])$. Each of the complexes $\mathcal{V}(L_S)$ associated to the four resolutions in Figure 1 is depicted in Figure 2 (with the factor from the marked circle suppressed).

Before proceeding we note that if we mod out by the ideal $(x_f \mid f \in \mathfrak{F}_L)$, then the complex $\mathcal{V}(L_S)$ is precisely the module associated to L_S by Khovanov in defining the characteristic 2, reduced Khovanov homology [8; 3]. This motivates the use of $v_\pm$ and the definition of the q–grading, as these will then be identical to Khovanov's.

1.6 Totally twisted Khovanov homology

We can now describe the *totally twisted Khovanov complex*. The reader familiar with reduced Khovanov homology in characteristic 2 will immediately recognize the procedure as that of [8], but with significant additions. Before we start, we will need notation for shifting the gradings in the Koszul complexes above. We use the following notation throughout the paper:

[2] One needs a convention for which circle to choose as C_1, C_2, etc. Such a convention can be found in [3] and is the one used for computations described later in this paper.

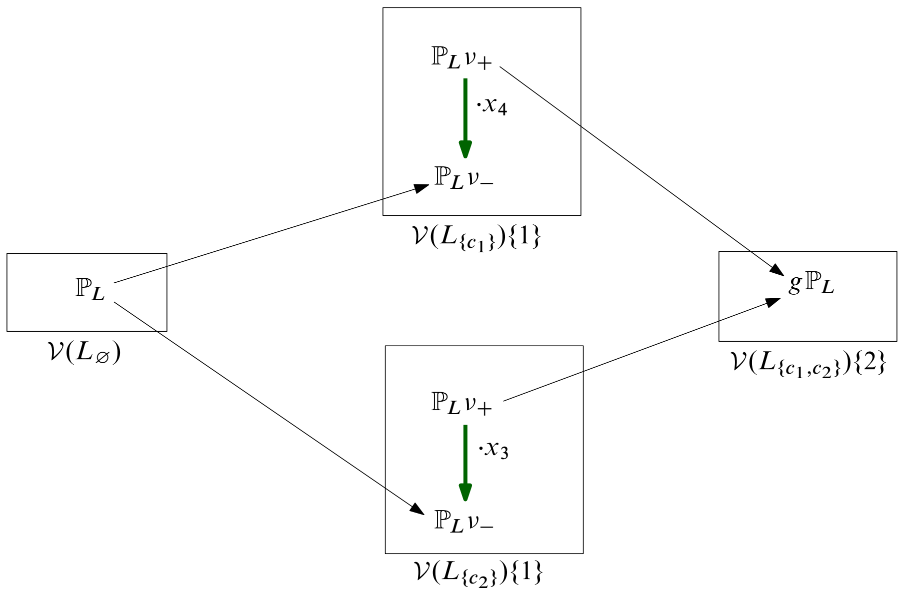


Figure 2: In each box is an example complex of the form $\mathcal{V}(L_S)$ for the resolutions in Figure 1. The relative positions of the complexes are identical to the relative positions of the resolutions in that figure. The thick, green vertical arrows correspond to the differentials in the Koszul complex for that resolution, using the formal areas of the unmarked circles in that resolution. Furthermore, we have amassed the complexes for each resolution into an example of a totally twisted Khovanov complex.

Definition 1.8 Let $M = \bigoplus_{\vec{v}\in\mathbb{Z}^k} M_{\vec{v}}$ be a $\mathbb{Z}^k$–graded R–module. Then $M[\vec{w}]$ denotes the $\mathbb{Z}$–graded module with $(M[\vec{w}])_{\vec{v}} \cong M_{\vec{v}-\vec{w}}$.

Consequently, $\operatorname{grad}_{M[\vec{w}]}(m) = \operatorname{grad}_M(m) + \vec{w}$ on homogeneous elements $m \in M$.

If we follow Khovanov's construction, we should combine the complexes $\mathcal{V}(L_S)$ into a bigraded module $\mathrm{KH}_{\mathrm{red}}(L)$ by taking $\mathrm{KH}_{\mathrm{red}}(L) = \bigoplus K^i(L)$, where for each $i \in \mathbb{Z}$,

$$K^i(L) = \bigoplus_{S\subset \mathrm{CR}(L),\ |S|=i} \mathcal{V}(L_S)[(i,i)]$$

is the q–grading from $\mathcal{V}(L_S)$ corresponds to the second element of the bigrading. Thus each K^i is a q graded chain complex. Since $\mathcal{V}(L_S)$ has a differential, $\partial_{\mathcal{V}(L_S)}$, $\mathrm{KH}(L)$ inherits a differential, $\partial_{\mathcal{V}}$, which reduces the q–grading by 2. The other grading on $\mathrm{KH}_{\mathrm{red}}(L)$ is $K^i(L) \to i$, and will be called the h–grading. The homogeneous elements of $\mathrm{KH}_{\mathrm{red}}(L)$ in h–grading i and q–grading j will be denoted $\mathrm{KH}^{i,j}_{\mathrm{red}}(L)$.

As a bigraded module $\mathrm{KH}_{\mathrm{red}}(L)$ is that used for the (unshifted, reduced) Khovanov chain complex (tensored with $\mathbb{P}_L$). Khovanov discovered a $(+1,0)$ differential $\widetilde{\partial}_{\mathrm{KH}}$ on this bigraded module, which defines the reduced, unshifted Khovanov complex $\widetilde{\mathrm{CKH}}_u^{*,*}(L) = (\mathrm{KH}_{\mathrm{red}}(L), \widetilde{\partial}_{\mathrm{KH}})$. Note that $\widetilde{\mathrm{CKH}}_u^{*,*}(L)$ is a direct sum of chain complexes, one in each q–grading.

The differential $\widetilde{\partial}_{\mathrm{KH}}$ is constructed from the multiplication $\mu\colon V \otimes V \to V$ and the comultiplication $\Delta\colon V \to V \otimes V$ for a certain commutative Frobenius algebra, $V = \mathbb{Z}/2\mathbb{Z}v_+ \oplus \mathbb{Z}/2\mathbb{Z}v_-$, graded as above. On the graded basis, these maps are given by

$$\mu\colon \begin{cases} v_+ \otimes v_+ \to v_+, \\ v_+ \otimes v_- \to v_-, \\ v_- \otimes v_- \to 0, \end{cases} \qquad \Delta\colon \begin{cases} v_+ \to v_+ \otimes v_- + v_- \otimes v_+, \\ v_- \to v_- \otimes v_-, \end{cases}$$

where each map shifts the q–grading by -1. To obtain a differential preserving the q–grading the image, thus, needs to be shifted by 1.

If we tensor V with $\mathbb{P}_L$ we obtain the module underlying the complex $\mathcal{K}([C])$ for C an unmarked circle in some resolution L_S. μ and Δ can be extended to maps on the modules $\mathcal{K}([C])$, but when we try to extend to the chain complexes we need to account for the different areas incorporated in each circle. Nevertheless, in Section 2 we prove:

Proposition 1.9 *If C is a circle in $L_{S\cup\{i\}}$ formed by merging the circles C_1 and C_2 in L_S, then*

$$\mu\colon \mathcal{K}([C_1]) \otimes \mathcal{K}([C_2]) \to \mathcal{K}([C])[1]$$

is a chain map on the Koszul complexes over $\mathbb{P}_L$. Likewise, if C_1 and C_2 arise in $L_{S\cup\{i\}}$ from dividing a circle C in L_S, then

$$\Delta\colon \mathcal{K}([C]) \to \big(\mathcal{K}([C_1]) \otimes \mathcal{K}([C_2])\big)[1]$$

is a chain map over $\mathbb{P}_L$.

On a summand $\mathcal{V}(L_S)$, $\widetilde{\partial}_{\mathrm{KH}}$ is a sum of chain maps $\mathcal{V}(L_S) \to \mathcal{V}(L_{S\cup\{i\}})[1]$ for each $i \notin S$. The addition of i to the set S corresponds to changing L_S at one crossing, and thus either merges two circles in L_S — in which case the map uses μ on the corresponding factors and the identity on the others — or divides a circle, in which case we use Δ on the factor corresponding to the splitting circle. The remaining factors are mapped by the identity. Thus $\widetilde{\partial}_{\mathrm{KH}}$ is a chain map when considering unmarked circles.

For the marked circle, we treat v_0 as a shifted v_-, so

$$\mu\colon \begin{cases} v_+ \otimes v_0 \to v_0, \\ v_- \otimes v_0 \to 0, \end{cases} \qquad \Delta\colon v_0 \to v_0 \otimes v_-,$$

since dividing the marked circle results in a marked circle and a new unmarked circle. These also extend to chain maps on the Koszul complexes above.

Together these results imply:

Theorem 1.10 *Let* $\widetilde{\partial}_{\mathrm{KH}}\colon \mathrm{KH}^{i,j}_{\mathrm{red}}(L) \to \mathrm{KH}^{i+1,j}_{\mathrm{red}}(L)$ *be the Khovanov differential and let* $\partial_{\mathcal{V}}\colon \mathrm{KH}^{i,j}_{\mathrm{red}}(L) \to \mathrm{KH}^{i,j-2}_{\mathrm{red}}(L)$ *be the Koszul differential. Then* $\underline{\partial} = \partial_{\mathrm{KH}} + \partial_{\mathcal{V}}$ *is a differential on* $\mathrm{KH}_{\mathrm{red}}(L)$*.*

Thus, $\underline{\mathrm{CKH}}^*_u(L) = (\mathrm{KH}_{\mathrm{red}}(L), \underline{\partial})$ is a chain complex, the *unshifted totally twisted Khovanov complex* for L.

Example (continued) Figure 2 depicts the unshifted totally twisted Khovanov complex for the two-component unlink we introduced earlier. Each of the horizontal arrows is an isomorphism in this complex.

Gradings The (unshifted) reduced Khovanov complex over $\mathbb{P}_L$ is $\widetilde{\mathrm{CKH}}^{*,*}_u(L) = (\mathrm{KH}_{\mathrm{red}}(L), \widetilde{\partial}_{\mathrm{KH}})$. Since $\widetilde{\partial}_{\mathrm{KH}}$ is a $(+1,0)$–differential on the bigraded module, the homology of this complex is also bigraded, with the h–grading being the homology grading. With the addition of $\partial_{\mathcal{V}}$ the homology is no longer naturally bigraded. Instead, we equip $\mathrm{KH}_{\mathrm{red}}(L)$ with a single grading:

Definition 1.11 The δ–grading on $\mathrm{KH}_{\mathrm{red}}(L)$ is $\delta\colon \mathrm{KH}^{i,j}_{\mathrm{red}}(L) \to 2i - j$.

The differential $\widetilde{\partial}_{\mathrm{KH}}$ is a $(+1,0)$ map, and thus changes δ by $+2$. The differential $\partial_{\mathcal{V}}$ is a $(0,-2)$ map, so it also shifts the δ–grading by $+2$. Thus, δ provides a grading to the complex $\underline{\mathrm{CKH}}^*_u(L)$. The δ–grading will be written as a subscript, to distinguish it from the q and h–gradings.[3]

Shifting In addition, the homology of the unshifted complex is not quite an invariant of $\mathcal{L}$. This is also true for the reduced Khovanov homology $\widetilde{\mathrm{CKH}}^{*,*}_u$. To define a complex whose homology is an invariant of $\mathcal{L}$, Khovanov shifts the bigraded complex $\widetilde{\mathrm{CKH}}^{*,*}_u$ by $[(-n_-(L), n_+(L) - 2n_-(L))]$. We will make the same shift, which changes δ by $2(-n_-(L)) - (n_+ - 2n_-)(L) = -n_+(L)$. Once we make this shift, the resulting complex will be (almost) an invariant of the link $\mathcal{L}$.

[3] There are several definitions of the δ grading in the literature of Khovanov homology. Our definition is -2 times the definition of the δ–grading used by J Baldwin in [1] and -1 times J Rasmussen's definition in [11].

Definition 1.12 The *unshifted totally twisted Khovanov complex* for a link diagram L is the complex $\underline{\mathrm{CKH}}^*_u(L) = (\mathrm{KH}_{\mathrm{red}}, \underline{\partial})$, where

$$\mathrm{KH}_{\mathrm{red}}(L) = \bigoplus_{S \subset \mathrm{CR}(L)} \mathcal{V}(L_S)[(|S|, |S|)]$$

is equipped with the δ–grading, and $\underline{\partial} = \widetilde{\partial}_{\mathrm{KH}} + \partial_{\mathcal{V}}$.

The *totally twisted Khovanov complex*, $\underline{\mathrm{CKH}}^*(L)$ is $\underline{\mathrm{CKH}}^*_u(L)[-n_+(L)]$, the complex resulting from shifting the δ–grading by $-n_+(L)$. We will denote the homology of this complex by $\underline{\mathrm{KH}}^*(L)$.

Comment The name "totally twisted" comes from assigning a formal variable to every face in L. The construction works equally well if we only assign formal variables to some of the faces of L, which is equivalent to modding out by the ideal generated by the remaining faces. Therefore, between the reduced Khovanov complex over $\mathbb{P}_L$ and $\underline{\mathrm{KT}}_*(L)$ there are many twisted complexes, one for each subset of faces, with $\underline{\mathrm{KT}}_*(L)$ being the most twisted. We can make this more algebraic through the following definition:

Definition 1.13 Let L be a link diagram and let M be a module over $\mathbb{P}_L$. Then $\underline{\mathrm{KH}}^*(L; M)$ is the homology of the chain complex $\underline{\mathrm{CKH}}^*(L) \otimes_{\mathbb{P}_L} M$ equipped with the differential $\underline{\partial} \otimes \mathbb{I}_M$.

We can obtain the intermediate twisted homologies by choosing different modules for M. For example:

Proposition 1.14 *Let $M = \mathbb{Z}/2\mathbb{Z}$ be the module over $\mathbb{P}_L$ with trivial action, ie $x_i \cdot 1 = 0$ for each variable x_i. Then*

$$\underline{\mathrm{KH}}^*(L; M) \cong \widetilde{\mathrm{KH}}^*(L),$$

the delta graded, reduced Khovanov homology.

1.7 Invariance

In Section 4 we prove the fundamental result for the invariance of the totally twisted Khovanov homology:

Theorem 1.15 *Let L be a diagram for $\mathcal{L}$ with marked point p, and let L' be another diagram obtained from L by Reidemeister I, II, and III moves, conducted in open discs which do not include the marked point. Then the chain complexes $\underline{\mathrm{KH}}^*(L)$ and $\underline{\mathrm{KH}}^*(L')$ are stably chain homotopy equivalent.*

In that section we also explain the notion of stable isomorphism needed in this paper. This notion relates the rings $\mathbb{P}_L$ and $\mathbb{P}_{L'}$ even though $\mathfrak{F}_L$ and $\mathfrak{F}_L$ may not be in one-to-one correspondence.

It should be noted that the author does not know if a similar result, or just isomorphism of the corresponding homology modules, occurs when changing the marked point, in particular if the marked point changes between components of a link. However, we will change coefficients in a moment, and then the invariance under change of marked point can be established.

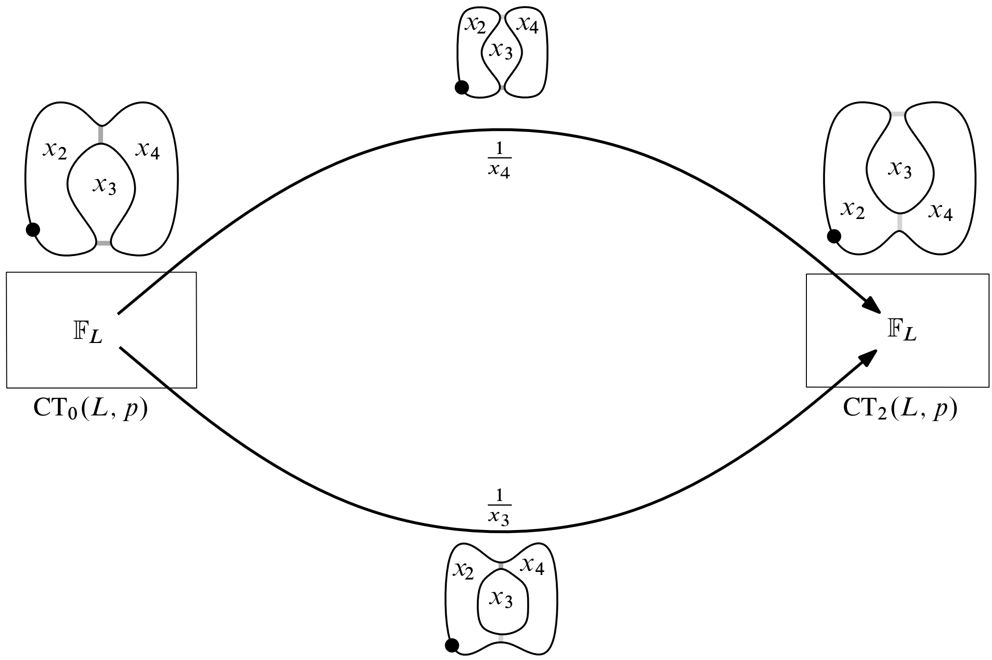


Figure 3: The unshifted spanning tree complex for the unlink example in Figure 2. Over $\mathbb{F}_L$ the vertical arrows in Figure 2 are isomorphisms. Simplifying both of these leaves the left and right diagrams, as these consist only of a single marked circle. The expense is a change in coefficient, written next to the top and bottom arrows. The differential in the new complex is the sum of the maps corresponding to the two arrows. The top arrow comes from changing the resolution at c_1 first, followed by c_2. This alteration cleaves from the marked circle the region marked with x_4 and then rejoins it. Call this $B_{T,T'}$, so $[B_{T,T'}] = x_4$. This means $W_{T,T'}$ is the region cleaved off by changing the resolution at c_2 first and then at c_1. This is the region contained in the inner circle, so $[W_{T,T'}] = x_3$.

1.8 The spanning tree deformation

As stated above, our interest in the totally twisted Khovanov homology comes from the existence of an homotopy equivalent spanning tree model for the chain complex $\underline{\mathrm{CKH}}^*(L)$. However, to realize this model it is necessary to use elements of the field $\mathbb{F}_L$ as coefficients. Doing so makes each of the complexes $\mathcal{K}([C])$ acyclic as the nontrivial map is an isomorphism. For example, in Figure 2 the thickened vertical arrows are isomorphisms over $\mathbb{F}_L$. We can use these isomorphisms to simplify the chain complex using the following analog of Gaussian elimination:

For a chain complex C over a field, if $\partial v = \lambda w + z$ with $\lambda \neq 0$ (and w and z linearly independent), there is a chain homotopy equivalent complex defined on $C/\operatorname{Span}\{v, w\}$ with differential ∂' defined by this rule: If $\partial u = \nu w + \eta v + r$ with r linearly independent of v and w, then $\partial' u = r - \nu\lambda^{-1} z$, where, for grading reasons, either nu or η (or both) will be zero.

The result of applying this simplification to both vertical arrows in Figure 2 appears in Figure 3. For example, the bottom arrow in Figure 3 arises from the formula with $v = v_+$ in $\mathcal{V}(L_{\{c_2\}})$, $w = v_-$ and $\lambda = x_3$ For more general diagrams we use a standard result about Koszul complexes: that $\mathcal{K}([C_1], \dots, [C_k])$ is acyclic over $\mathbb{F}_L$; see Matsumura [10]. Consequently, when we simplify along nonzero components of $\partial_{\mathcal{V}}$ only those resolutions with $\mathcal{V}(L_S) = \mathbb{F}_L v_0$ will remain to contribute. These are precisely the resolutions which consist of a single circle.

Resolutions consisting of a single circle are in one-to-one correspondence with pairs of complementary spanning trees for the Tait graphs of L. These graphs are found by first bicoloring the faces of L in a checkerboard fashion. Taking all the black faces as vertices, each crossing in $\mathrm{CR}(L)$ provides an edge since it abuts one or two black faces. The Tait graphs for L are the two planar graphs obtained by repeating this construction for both the black and white faces. Furthermore, the marked point on L abuts one black and one white region. These regions identify a root vertex in each graph. Resolutions L_S which consist of a single circle divide S^2 into two discs, one of each color, which are composed of the black faces and the white faces. If we take only those crossings which are resolved in L_S as a merging of two black faces, we obtain a spanning tree for the black graph. Likewise, if we take the complementary set of edges, we obtain a spanning tree for the white graph.

Thus, in the homotopy equivalent complex, only those resolutions contribute generators which correspond to the rooted spanning trees. We now explicitly describe the chain complex in terms of the rooted spanning trees.

Within $R(L)$ we distinguish those resolutions which result in a single circle in S^2:

$$O(L) = \{S \subset \mathrm{CR}(L) \mid L_S \text{ is connected}\}.$$

Furthermore, let $O_i(L) = O(L) \cap R_i(L)$. Elements of $O(L)$ will be typically be denoted by T, or a decorated variant. Given $T \in O(L)$ we let

$$O(T, L) = \{T' \in O(L) \mid T \subset T'\}$$

and $O_i(T, L) = O(T, L) \cap O_i(L)$. If $\delta(T) = i$ then $O_{i+k}(T, L)$ are those resolutions such that $L_{T'}$ is a single (marked) circle and $T' \backslash T$ is a k crossing subset of $\mathrm{CR}(L) \backslash T$.

We now define the chain complex: Take

$$\mathrm{CT}^i(L) = \mathrm{Span}_{\mathbb{F}_L}\{T \in O_i(L)\}$$

for each $i \in \mathbb{Z}$. Note that if L is a split diagram, then $\mathrm{CT}^*(L) \cong 0$ as there are no such resolutions. There is a boundary map

$$\partial_{i,L}\colon \mathrm{CT}^i(L) \to \mathrm{CT}^{i+2}(L).$$

For each $T' \in O_{i+2}(S, L)$, $T' \backslash T = \{c_1, c_2\}$ for two crossings $c_1, c_2 \in \mathrm{CR}(L) \backslash S$. In L_T these are depicted with resolution arcs a_1 and a_2. For $L_{T'}$ to be a single circle, a_1 and a_2 must have interlocking feet along the circle L_T. Since all the arcs are disjoint, one of the arcs must lie in each region of $S^2 \backslash L_T$. Between T and T' there are two elements $b, w \in R(L)$, with $b = T \cup \{c_1\}$ and $w = T \cup \{c_2\}$. L_b consists of two circles, found by surgering the arc a_1. One of these circles contains the marked point while the other bounds a region $B_{T,T'} \subset S^2$ *disjoint* from the marked point p, which is cleaved from L_T by the change in resolution at c_1. Likewise, L_w consists of two circles, one marked and the other containing a subset $W_{T,T'}$ disjoint from p. For the case of our extended example, see Figure 3.

To $B_{T,T'}$ we assign the formal area $[B_{T,T'}]$ in $\mathbb{F}_L$ which is the sum of the formal variables for the faces in $B_{T,T'}$. Similarly we can define an area $[W_{T,T'}]$ for $W_{T,T'}$. These define the boundary map $\partial_{i,L}$

$$\partial_{i,L} T = \sum_{T' \in O_{i+2}(T,L)} \langle T, T' \rangle T', \tag{1}$$

where

$$\langle T, T' \rangle = \frac{1}{[B_{T,T'}]} + \frac{1}{[W_{T,T'}]}. \tag{2}$$

This differential emerges from the reduction process previously described, but given its form we can also verify that it is a boundary map directly. This argument is combinatorial and is presented in Section 6. Moreover, we can verify that this complex *is invariant under changes of the marked point*. Together, these statements form:

Theorem 1.16 *Let L be the diagram for an oriented link with a marked point p. Let $\mathrm{CT}^*(L, p) = \bigoplus_{i\in\mathbb{Z}} \mathrm{CT}^i(L, p)$, and ∂_L be the map $\bigoplus \partial_{i,L}$. Then $(\mathrm{CT}^*(L), \partial_L)$ is a chain complex, whose homology will be denoted $\mathrm{HT}^*_u(L)$. The isomorphism type of the chain complex is invariant under changes of the marked point.*

The arguments for invariance of the twisted Khovanov homology are readily extended to $\mathbb{F}_L$, and thus apply to the homotopy equivalent spanning tree complex. Consequently, we obtain the main result of this paper:

Theorem 1.17 *Let L be the diagram for an oriented link, $\mathcal{L}$. Then the (stable) isomorphism class of $\mathrm{HT}^*_u(L)[-n_+]$ is an oriented link invariant, denoted $\mathrm{HT}^*(\mathcal{L})$.*

1.9 Properties

The reduced Khovanov homology has Euler characteristic equal to the Jones polynomial (for a suitable convention on the coefficients of the polynomial). The spanning tree complex also has a classical knot invariant as its Euler characteristic:

Theorem 1.18 *For a link $\mathcal{L}$ in S^3, let*

$$P(t) = \sum_{j\in\mathbb{Z}} \mathrm{rk}_{\mathbb{F}_L}(\mathrm{HT}^j(\mathcal{L}))t^j.$$

Then $\det(\mathcal{L}) = |P(i)|$, where $i = \sqrt{-1}$.

The strange form of the Euler characteristic comes from ∂_L being a $+2$–differential. However, using a $+2$–differential ensures that our gradings will occur in $\mathbb{Z}$ and not $\frac{1}{2}\mathbb{Z}$. This theorem can be derived either from the relationship with Khovanov homology, where one interprets the addition of $\partial_{\mathcal{V}}$ as corresponding to evaluating the Jones polynomial at -1, or from the known relationship between the spanning trees of the Tait graph and the determinant of a link. We will opt for the latter in Section 8.

Second, we highlight the long exact sequence arising from the resolutions of a crossing in L.

Proposition 1.19 *Let L be oriented link diagram with crossing c. Let L_0 be the resolution of L at c according to the rule $c \notin S$, and let L_1 be the resolution of L at c according to the rule $c \in S$. If c is a positive crossing, and if $e = n_+(L) - n_+(L_1)$ (for any orientation on L_1), then*

$$\cdots \to \mathrm{HT}^{i+e-1}(\mathcal{L}_1) \otimes_{\mathbb{F}_{L_1}} \mathbb{F}_L \to \mathrm{HT}^i(\mathcal{L}) \to \mathrm{HT}^{i+1}(\mathcal{L}_0) \otimes_{\mathbb{F}_{L_0}} \mathbb{F}_L \to \mathrm{HT}^{i+e+1}(\mathcal{L}_1) \otimes_{\mathbb{F}_{L_1}} \mathbb{F}_L \to \cdots . \tag{3}$$

However, if c is negative, and $f = n_+(L) - n_+(L_0)$, then

$$\cdots \to \mathrm{HT}^{i-1}(\mathcal{L}_1) \otimes_{\mathbb{F}_{L_1}} \mathbb{F}_L \to \mathrm{HT}^i(\mathcal{L}) \to \mathrm{HT}^{i+f}(\mathcal{L}_0) \otimes_{\mathbb{F}_{L_0}} \mathbb{F}_L \to \mathrm{HT}^{i+1}(\mathcal{L}_1) \otimes_{\mathbb{F}_{L_1}} \mathbb{F}_L \to \cdots . \tag{4}$$

The groups in these long exact sequences are tensored with $\mathbb{F}_L$ to have all groups over the same field. The proof of the proposition as well as details of the actions of $\mathbb{F}_{L_0}$ and $\mathbb{F}_{L_1}$ are in Section 9.1.

The long exact sequence can be used to replicate an argument of Manolescu and Ozsváth [9]. In particular:

Theorem 1.20 *If $\mathcal{L}$ represents a (quasi)alternating link with a connected diagram L, then $\mathrm{HT}^i(\mathcal{L}) \cong 0$ when $i \neq \sigma(L)$ and has rank $\det(\mathcal{L})$ when $i = \sigma(\mathcal{L})$, the signature of $\mathcal{L}$.*

We will use the convention that the signature of the right-handed trefoil is -2. This result can also be proved from the more detailed connection with Khovanov homology given below. We also note the spanning tree homology has two properties similar to those for other knot homologies.

Theorem 1.21 *Let $\mathcal{L}$ be an oriented link. Then $\mathrm{HT}^i(\mathcal{L}) \cong \mathrm{HT}^{-i}(\overline{\mathcal{L}})$.*

Theorem 1.22 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two nonsplit oriented links, and let $\mathcal{L} = \mathcal{L}_1 \# \mathcal{L}_2$, in some manner. Then*

$$\mathrm{HT}^k(\mathcal{L}) \cong \bigoplus_{i+j=k} \mathrm{HT}^i(\mathcal{L}_1) \otimes \mathrm{HT}^j(\mathcal{L}_2),$$

where $\cong$ denotes stable equivalence.

Again, similar results should be provable directly from the totally twisted Khovanov homology. However, each of these three properties is proved in Section 9 using the spanning tree formalism.

1.10 Relationship with Khovanov homology

There is a more precise relationship between the (characteristic 2, reduced) Khovanov homology and the totally twisted Khovanov homology, which extends also to the spanning tree homology. The q–grading, after we add $\partial_{\mathcal{V}}$, defines a filtration on the totally twisted Khovanov homology. We can examine the induced Leray spectral sequence to derive a relationship with Khovanov homology.

Theorem 1.23 *The spectral sequence induced by the filtration from the q–grading has E^0 page isomorphic to the δ–graded reduced Khovanov complex $\widetilde{\mathrm{CKH}}^*(L)$ and converges, in finitely many steps, to $\mathrm{HT}^*(\mathcal{L})$.*

Proof On the unshifted, twisted Khovanov complex, the map $(i,j)\to j$ is a filtration and $\delta(i,j)=2i-j$ is a grading. To compute the E^0–page we ignore the portion of the differential which changes the j–value. For us, this is the $(1,0)$ portion of the differential, $\widetilde{\partial}_{\mathrm{KH}}$. Consequently, the E^0 page is just the reduced Khovanov complex over the field $\mathbb{F}_L$. The complex is bounded, so the corresponding spectral sequence converges to the total homology of the complex. The total homology when using the δ–grading is isomorphic to $\mathrm{HT}^*_u(L)$, since we are working over a field. Finally, when we shift the Khovanov complex by $(-n_-, n_+ - 2n_-)$ to obtain the invariant homology, the δ–grading shifts by $-n_+ + 2n_- + 2(-n_-) = -n_+$, which is how we calculated the shift to apply to $\mathrm{HT}^*_u(L)$ to obtain $\mathrm{HT}^*(\mathcal{L})$. Thus in the spectral sequence above, we may use the Khovanov shifts on each bigraded page, and this appropriately shifts the total grading so that the direct sum of the pieces with $\delta = k$ converges to $\mathrm{HT}^k(\mathcal{L})$. □

Corollary 1.24 *If $\widetilde{\mathrm{KH}}^s(L)$ is the portion of the (reduced) Khovanov homology over $\mathbb{F}_L$ in δ–grading s, then*

$$\widetilde{\mathrm{KH}}^s(L) \geq \mathrm{rk}\,\mathrm{HT}^s(\mathcal{L}).$$

This corollary implies bounds on the Khovanov width of links in S^3, a result we examine in more depth in a later paper. For now, it is a natural question whether more can be said. To this end we relate some computations and the results of computer calculations of $\mathrm{HT}^*(\mathcal{L})$ and compare them with the results for characteristic 2, reduced Khovanov homology.

1.11 Computations

1.11.1 Unlinks To finish our extended example let $\mathcal{L}$ be the two-component unlink above. We compute the spanning tree homology $\mathrm{HT}^*(\mathcal{L})$ for the unlink $\mathcal{L}$ in the example. The spanning tree homology is trivial. Both of the arrows in Figure 3 are multiplication by a nonzero element of $\mathbb{F}_L$; the boundary operator is multiplication by their sum: $(x_3 + x_4)/x_3x_4$. Since this element is also nonzero, the corresponding boundary map is an isomorphism, and the homology is trivial. This confirms the intuition that the homology should be trivial since the simplest diagram for the two-component unlink is disconnected and thus unable to support any spanning trees for both its Tait graphs. On the other hand, for the twisted homology $\underline{\mathrm{KH}}^*(\mathcal{L})$, the situation is more complicated. For the complex in Figure 2, we can reduce two of the horizontal isomorphisms — one out of $L_\varnothing$ and the other into $L_{\{c_1,c_2\}}$ — to be left with

$$0 \longrightarrow \mathbb{P}_L \xrightarrow{\cdot(x_3+x_4)} \mathbb{P}_L \longrightarrow 0$$

with homology $\mathbb{Z}/2\mathbb{Z}[x_1', x_2', x_4']$, as a $\mathbb{P}_L$–module where x_3 acts by multiplication by x_4'. We can compare this with the usual geometrically split diagram for the two-component unlink. Then we would have two circles, enclosing regions y_1 and y_2, where we choose the labels so that the circle enclosing y_1 is the marked circle. The homology module would then be the homology of the complex

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z}[y_1, y_2] \xrightarrow{\cdot y_2} \mathbb{Z}/2\mathbb{Z}[y_1, y_2] \longrightarrow 0.$$

The homology of this complex is $\mathbb{Z}/2\mathbb{Z}[y_1]$ as a $\mathbb{Z}/2\mathbb{Z}[y_1, y_2]$–module with y_2 acting by multiplication by 0. This complex and its homologies will be related through the notion of stable isomorphism over polynomial rings to the complex for the diagram in our example, under which they are equivalent.

Using the invariance results above, the completely split diagram for the n–component unlink will give a complex isomorphic to $\mathcal{K}(y_2, \ldots, y_n)$ over $P = \mathbb{Z}/2\mathbb{Z}[y_1, \ldots, y_n]$, where y_1 corresponds to the bounded region enclosed by the marked circle. Since $y_2, \ldots, y_n$ is a regular sequence over P, standard results about Koszul complexes (see for example [10, Theorem 16.5]) imply

$$\underline{\mathrm{KH}}^k(\mathcal{L}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}[y_1] & k = 0, \\ 0 & k \neq 0, \end{cases}$$

which determines the stable isomorphism class of the totally twisted Khovanov homology. In fact, we can take this homology to be $\mathbb{Z}/2\mathbb{Z}$ in grading 0, as a $\mathbb{Z}/2\mathbb{Z}$–module, considered up to stable equivalence.

1.11.2 Example calculations for knots A computer can quickly calculate the ranks of the spanning tree homology for knots up to 15 crossings. There is a mild difficulty in that computations over $\mathbb{F}_L$ are not very efficient due to the large number of variables that may need to be tracked. The details of the workaround and results of these computer surveys will be highlighted in the sequel to this paper, as will various generalizations. Here we relate some of the data with an eye to understanding $\mathrm{HT}^*(\mathcal{K})$ more fully, where K is a *knot*.

When this paper originally appeared, these computations were more extensively described and provided evidence to conjecture that for a knot K, $\mathrm{HT}^*(\mathcal{K})$ is (stably) isomorphic to $\widetilde{\mathrm{KH}}^*_{\mathrm{red}}(K)$. This result has been proved by T Jaeger in [6]. Since the conjecture has been proven we give a shorter list.

Caution As mentioned above, in the Khovanov homology literature, there are several different definitions of the δ–grading. Below we implicitly convert the results in other papers to the δ–grading employed in this paper.

We will describe the homology $\mathrm{HT}^*(\mathcal{K})$ by its Poincaré polynomial,

$$\sum_{j\in\mathbb{Z}} \mathrm{rk}_{\mathbb{F}_L}(\mathrm{HT}^j(\mathcal{L}))\delta^j,$$

which indicates the ranks and the gradings. Since we are currently interested in stable equivalence over fields, the "graded ranks" are all that remain.

(1) Theorem 1.20 shows that for (quasi)alternating knots and links, $\mathrm{HT}^*(\mathcal{L})$ has the same rank in each δ–grading as $\widetilde{\mathrm{KH}}^*_{\mathrm{red}}(L)$. The analogous theorem for $\widetilde{\mathrm{KH}}^*_{\mathrm{red}}(L)$ was proven for alternating links by E S Lee, and for quasialternating links by Manolescu and Ozsváth [9].

(2) The torus knots $T_{5,3}$, $T_{7,3}$, and $T_{5,4}$ also have the rank of $\mathrm{HT}^*(\mathcal{K})$ in each δ–grading the same as that for $\widetilde{\mathrm{KH}}^*_{\mathrm{red}}(K)$. For $T_{5,3}$ the common Poincaré polynomial is

$$\mathrm{HT}_*(T_{5,3}):\quad 4\delta^{-8}+3\delta^{-6},$$

found from a diagram yielding 27 spanning trees as generators. Likewise, for $T_{7,3}$ both theories have Poincaré polynomial

$$\mathrm{HT}_*(T_{7,3}):\quad 4\delta^{-12}+4\delta^{-10}+\delta^{-8},$$

found from a diagram yielding 841 spanning trees as generators. For $T_{5,4}$ the homology is

$$\mathrm{HT}_*(T_{5,4}):\quad 4\delta^{-12}+4\delta^{-10}+5\delta^{-8},$$

found from a diagram yielding 1805 spanning trees as generators. It is worth describing the chain complex for $T_{5,4}$. The table below lists the nonzero number of generators in each δ–grading for the diagram used in the computation:

δ	-12	-10	-8	-6	-4
# gens	125	500	700	400	80

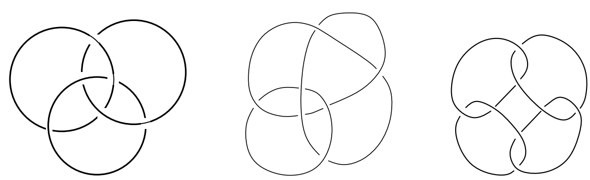
(3) In [11], there are several examples of knots where the knot Floer homology differs from $\widetilde{\mathrm{KH}}^*_{\mathrm{red}}(K)$, we consider the $(2,5)$–cable of the positive trefoil, $C_{2,5}T$, and the $(2,7)$–cable of the positive trefoil, $C_{2,7}T$. These are the knots 13_n4639 and 13_n4587, respectively. For these we find

$$\mathrm{HT}_*(C_{2,5}T): \quad 4\delta^{-8} + 7\delta^{-6} + 8\delta^{-4},$$
$$\mathrm{HT}_*(C_{2,7}T): \quad 4\delta^{-10} + 5\delta^{-8} + 8\delta^{-6}.$$

In addition, the torus knot $T_{4,5}$ also has different δ–graded knot Floer and Khovanov homology. Nevertheless, all of these still have identical Poincaré polynomials for the spanning tree and reduced, characteristic 2, δ–graded Khovanov homologies.

1.11.3 Results for links One can make the same comparison for links as for knots. It is straightforward to see that the homologies for the two-component unlinks are different (see Section 1.11.1). However, even among nonsplit links there are discrepancies between the homologies as soon as nonalternating links appear in the link tables. Of the 1424 links with 11 or fewer crossings found on the KnotAtlas website, 200 have different Poincaré polynomials for the spanning tree and reduced, characteristic 2 Khovanov homology.

For instance, $L6_n1$, $L7_n1$, and $L8_n8$ are depicted from left to right below:



$\mathrm{HT}_*(L6_n1)$ has Poincaré polynomial $4\delta^0$, whereas for the δ–graded, reduced, characteristic 2 Khovanov homology has polynomial $\delta^{-2} + 5\delta^0$. Thus there is a higher differential in the spectral sequence in Theorem 1.23. For $L7_n1$ a similar reduction occurs: the spanning tree homology has polynomial $4\delta^5$ whereas the Khovanov homology has $\delta^3 + 5\delta^5$. For $L8_n8$ we have $4\delta^{-1} + 4\delta^1$ for the spanning tree homology, whereas the Khovanov homology is $6\delta^{-1} + 6\delta$. For $L8_n6$ we have spanning tree homology $4\delta^2 + 4\delta^4$ whereas the Khovanov homology is $5\delta^2 + 5\delta^4$. No obvious property of a link explains these reductions. $L6_n1$ has three components whereas $L7_n1$ has only two; nevertheless, they have the same rank difference when compared to their Khovanov homologies. $L8_n8$ and $L8_n6$ both have nullity 1, but their rank comparisons are different. On the other hand, $L6_n1$ has nullity 0 but the same rank reduction as $L8_n8$. The author has no ready explanation for the occurrence of the higher differentials for links, nor an explanation for why they seem to occur frequently.

Acknowledgments The author did not discover the idea of "twisting" a link homology chain complex to deform it to a spanning tree complex. As he understands it, the idea first emerged in unpublished work of P Ozsváth and Z Szabó in the context of Heegaard–Floer homology. The author learned the Heegaard–Floer idea from John Baldwin while at the Mathematical Sciences Research Institute for the program on Homology theories of knots and links in the spring of 2010. While at MSRI, he stumbled on to the constructions in this paper while trying to understand what he was being told, completing the proof of invariance in fall of 2010. John Baldwin and Adam Levine have used this idea, in conjunction with a construction of C Manolescu, to describe Ozsváth and Szabó's knot Floer homology using spanning trees of a link diagram; see Baldwin and Levine [2]. The author would like to thank John Baldwin for those conversations, as well as P Ozsváth and Z Szabó for the great idea. He would also like to thank Liam Watson, Matt Hedden and Tom Mark for listening as he worked out some of the details while at MSRI. The author would also like to thank MSRI for the great semester. Finally, the author would like to thank the referee for many thoughtful suggestions, which contributed meaningfully to the improvement of this paper.

2 Totally twisted Khovanov homology

Theorem 2.1 *Let $\widetilde{\partial}_{\mathrm{KH}}\colon \mathrm{KH}^{*,*}(L) \to \mathrm{KH}^{*+1,*}(L)$ be the Khovanov differential, and let $\partial_{\mathcal{V}}\colon \mathrm{KH}^{*,*}(L) \to \mathrm{KH}^{*,*-2}(L)$ be the map $\bigoplus \partial_{\mathcal{V}(L_S)}$. Then $\underline{\partial} = \widetilde{\partial}_{\mathrm{KH}} + \partial_{\mathcal{V}}$ is a boundary map on $\mathrm{KH}(L)$.*

Proof To show $(\widetilde{\partial}_{\mathrm{KH}} + \partial_{\mathcal{V}})^2 \equiv 0$ we need to show $\widetilde{\partial}_{\mathrm{KH}} \circ \partial_{\mathcal{V}} = \partial_{\mathcal{V}} \circ \widetilde{\partial}_{\mathrm{KH}}$. However,

$$\widetilde{\partial}_{\mathrm{KH}}\colon \mathcal{V}(L_S) \to \bigoplus_{i \in \mathrm{CR}(L)\backslash S} \mathcal{V}(L_{\{i\}\cup S})[+1],$$

so it suffices to verify that $\widetilde{\partial}_{\mathrm{KH},S} = \mathrm{pr}_{\mathcal{V}(L_{\{i\}\cup S})[+1]} \circ \widetilde{\partial}_{\mathrm{KH}}$ is a chain map $\mathcal{V}(L_S) \to \mathcal{V}(L_{\{i\}\cup S})[\cdot] + 1$ for each $i \in \mathrm{CR}(L)\backslash S$. Furthermore, since $\mathcal{V}(L_S)$ is itself a tensor product, we can verify that $\widetilde{\partial}_{\mathrm{KH}}$ is a chain map through three lemmas addressing its effect on each of the factors. These verify that $\widetilde{\partial}_{\mathrm{KH}}$ is a chain map for factors corresponding to circles which are not merging or dividing, and for factors corresponding to circles that merge or that divide. Furthermore, implicitly each argument will allow one of the circles to be the marked circle.

Lemma 2.2 *If C is a circle which is unaffected by changing the resolution from L_S to $L_{\{i\}\cup S}$, then $\widetilde{\partial}_{\mathrm{KH}}$ induces a chain map on $\mathcal{K}([C])$.*

Proof The map $\widetilde{\partial}_{\mathrm{KH}}$ induces the identity map on the module underlying $\mathcal{K}([C])$. Furthermore, $[C]$ is the same for both L_S and $L_{S\cup\{i\}}$ since the circle is unchanged. Thus, the map $\widetilde{\partial}_{\mathrm{KH}}$ is the identity on the chain complex $\mathcal{K}([C])$ as well. □

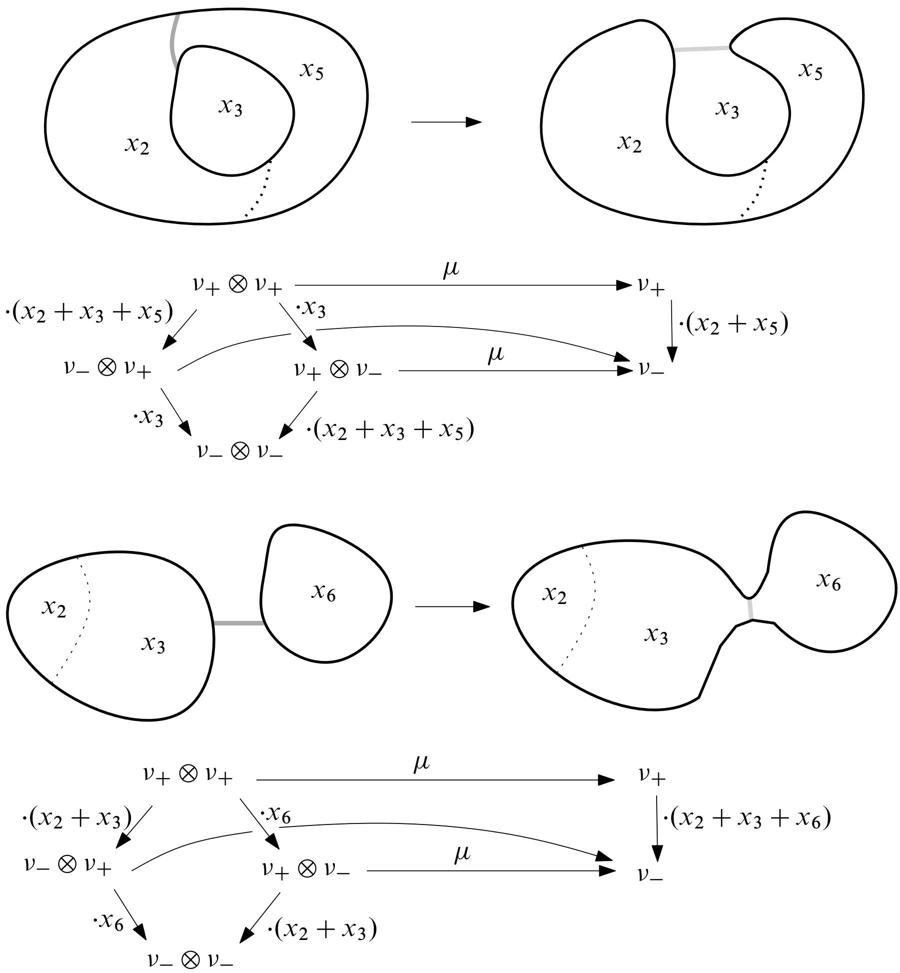


Figure 4: Examples of the two cases considered in the proof of Lemma 2.3: the first figure depicts case (1), $A_p(C_1) \subset A_p(C_2)$ along with a simplified version of the chain complexes and maps involved, while the bottom figure depicts case (2), $A_p(C_1) \cap A_p(C_2) = \varnothing$. The labels are not consecutive to indicate each may be only a portion of a larger diagram.

Lemma 2.3 *Let C_1 and C_2 be circles in L_S which merge into a single circle C in $L_{\{i\}\cup S}$. The map*

$$\mu\colon \mathcal{K}([C_1]) \otimes \mathcal{K}([C_2]) \to \mathcal{K}([C])[+1]$$

is a chain map.

Proof When C_1 and C_2 are both unmarked circles, there are two cases to consider: $A_p(C_1) \subset A_p(C_2)$ (or $A_p(C_2) \subset A_p(C_1)$), and $A_p(C_1) \cap A_p(C_2) = \varnothing$. Examples of the argument in the two cases are given in Figure 4. In each case we start by computing $[C]$. If $A_p(C_1) \cap A_p(C_2) = \varnothing$ then merging C_1 and C_2 produces a circle C with $A_p(C) = A_p(C_1) \natural A_p(C_2)$, the boundary connect sum of $A_p(C_1)$ and $A_p(C_2)$. But then any face $f \in \mathfrak{F}_L$ with $f \cap A_p(C) \neq \varnothing$ has either $f \cap A_p(C_1) \neq \varnothing$ or $f \cap A_p(C_2) \neq \varnothing$, but *not both.* Thus $[C] = [C_1] + [C_2]$. In the other case, if $A_p(C_1) \subset A_p(C_2)$, merging C_1 and C_2 results in $A_p(C) = A_p(C_2) \backslash A_p(C_1)$, but $[A_p(C_2) \backslash A_p(C_1)] + [C_1] = [C_2]$. Since we are working in characteristic 2,

$$[C] = [A_p(C_2) \backslash A_p(C_1)] = [C_1] + [C_2].$$

Thus in both cases, $[C] = [C_1] + [C_2]$.

With this result, we will now verify that $\partial_{\mathcal{K}([C])} \circ \mu = \mu \circ (\partial_{\mathcal{K}([C_1])} \otimes \mathbb{I} + \mathbb{I} \otimes \partial_{\mathcal{K}([C_2])})$. The map $\partial_{\mathcal{K}([C])} \circ \mu$ computed on generators of $\mathcal{K}([C_1]) \otimes \mathcal{K}([C_2])$ equals

$$v_+ \otimes v_+ \xrightarrow{\mu} v_+ \xrightarrow{\partial_{\mathcal{K}([C])}} [C]v_-,$$

$$v_+ \otimes v_-, v_- \otimes v_+ \xrightarrow{\mu} v_- \xrightarrow{\partial_{\mathcal{K}([C])}} 0,$$

$$v_- \otimes v_- \xrightarrow{\mu} 0 \xrightarrow{\partial_{\mathcal{K}([C])}} 0.$$

On the other hand, if we apply $\partial = \partial_{\mathcal{K}([C_1])} \otimes \mathbb{I} + \mathbb{I} \otimes \partial_{\mathcal{K}([C_2])}$ first, and then μ, we obtain

$$v_+ \otimes v_+ \xrightarrow{\partial} ([C_1]v_- \otimes v_+ + [C_2]v_+ \otimes v_-) \xrightarrow{\mu} ([C_1] + [C_2])v_-,$$

$$v_+ \otimes v_- \xrightarrow{\partial} [C_1]v_- \otimes v_- \xrightarrow{\mu} 0,$$

$$v_- \otimes v_+ \xrightarrow{\partial} [C_2]v_- \otimes v_- \xrightarrow{\mu} 0,$$

$$v_- \otimes v_- \xrightarrow{\partial} 0 \xrightarrow{\mu} 0.$$

Since $[C] = [C_1] + [C_2]$ these two maps are equal. Now let C_1 be the marked circle. Then $\mathcal{K}([C_1])$ is spanned by v_0. v_0 behaves identically to v_- in μ, but both of the above maps have image equal to 0 if one of the generators equals v_-. Thus μ is also a chain map when merging the marked circle. □

Lemma 2.4 *Let C be an unmarked circle in L_S which divides into two circles C_1 and C_2 in $L_{\{i\}\cup S}$. The map*

$$\Delta\colon \mathcal{K}([C]) \to \big(\mathcal{K}([C_1]) \otimes \mathcal{K}([C_2])\big)[+1]$$

is a chain map.

Proof The module V underlying the complex $\mathcal{K}([C])$ is a Frobenius algebra with multiplication μ and counit $\epsilon\colon V \to \mathbb{F}$ given by $\epsilon(v_-) = 1$ and $\epsilon(v_+) = 0$. That V is a Frobenius algebra implies that $\epsilon \circ \mu$ is a nondegenerate bilinear form which induces an isomorphism $\lambda\colon V \to V^*$. The comultiplication Δ for a Frobenius algebra is the map obtained from $\mu\colon V \otimes V \to V$ by dualizing, $V^* \otimes V^* \leftarrow V^*$, and then identifying V^* with V using λ^{-1} to obtain $V \to V \otimes V$. Let $f_\pm$ be the basis dual to $v_\pm$ in V^*. We consider the differential on V under this duality. The map $\partial_{\mathcal{V}}$ induces a map ∂^* which can be computed as $\partial^*(f_-) = [C] f_+$, and otherwise 0. $\lambda(v_\pm) = f_\mp$, so λ is a chain map $\mathcal{K}([C]) \to \mathcal{K}([C])^*$. Since μ is a chain map, it follows easily that Δ is likewise a chain map. □

Of course, this last lemma can also be verified directly, using the same method as in Lemma 2.3 and the result on $[C]$.

These three lemmas imply that the building blocks of $\widetilde{\partial}_{\mathrm{KH}}$ are chain maps on the factors in $\mathcal{V}(L_S)$. Consequently, $\widetilde{\partial}_{\mathrm{KH}} + \partial_{\mathcal{V}}$ is a differential on $\mathrm{KH}(L)$. □

3 Before proving invariance: stable equivalence

Our goal is to show that $\underline{\mathrm{KH}}^*(L)$ is almost a link invariant. We will not quite get a link invariant due to the presence of the additional marked point. Instead, in the next section, we will prove:

Theorem 3.1 *Let L be the diagram for a link $\mathcal{L}$ in S^2, equipped with a marked point p. The (stable) chain homotopy type of $\underline{\mathrm{KH}}^*(L)$ is an invariant of L under Reidemeister moves and planar isotopies in $S^2 \backslash \{p\}$.*

In other words, as long as the isotopies do not cross the marked point, the homology is an invariant. The author does not know if the twisted homology is invariant under changes of marked point. However, for a different set of coefficients, we will be able to prove this.

In this section, we collect some algebraic constructions that will help us prove the theorem in the next section. We need these constructions because different projections on $\mathcal{L}$ can have different numbers of faces, and thus the corresponding twisted complexes occur with nonisomorphic coefficient rings, $\mathbb{P}_L$. Thus, in this section we describe an appropriate algebraic equivalence for relating the complexes for different projections. These constructions will allow us to prove the invariance theorem by adapting the usual proofs of invariance for Khovanov homology; see [8] and Bar-Natan [4].

Let W be a vector space over $\mathbb{F}$, and let $P_W = \mathrm{Sym}(W)$ be its symmetric algebra. The algebra P_W is an integral domain, so we may find its field of fractions F_W. A basis for W identifies P_W with a commutative polynomial ring generated by the basis elements, and F_W with the corresponding field of rational functions. Thus, we will sometimes refer to F_W as $\mathrm{Rat}(W)$ when we wish to emphasize this connection. Any linear map $A\colon W \to W'$ induces a map $\mathrm{Sym}(A)\colon P_W \to P_{W'}$. If A is also an injection, then A also induces a map $\mathrm{Rat}(A)\colon F_W \to F_{W'}$ since $\mathrm{Sym}(A)$ has trivial kernel. When A is an isomorphism $\mathrm{Sym}(A)$ and $\mathrm{Rat}(A)$ are also isomorphisms in the appropriate category.

If M is a module over P_W, and $A\colon W \to W'$, then $M \otimes_{P_W} P_{W'}$ is a module over $P_{W'}$ where $(p \cdot m) \otimes p' = m \otimes (\mathrm{Sym}(A)(p) \cdot p')$ and the action of $P_{W'}$ occurs on the second factor. Likewise if V is a vector space over F_W, then $V \otimes_{F_W} F_{W'}$ is a vector space over F'.

Definition 3.2 Let W and W' be two $\mathbb{F}$–vector spaces. A module M over P_W is stably isomorphic to a module M' over $P_{W'}$ if there is an $\mathbb{F}$–vector space W'', and injections $i, i'\colon W, W' \hookrightarrow W''$ which induce an isomorphism

$$M \otimes_{P_W} P_{W''} \cong M' \otimes_{P_{W'}} P_{W''}$$

as $P_{W''}$ modules. When we wish to identify W'' and the injections we will say that W is stably isomorphic to W' through (W'', i, i').

We will consider this as a relation on pairs (M, W), although we will often omit reference to W when it is clear in the context.

Lemma 3.3 *Stable isomorphism of pairs (M, W) is an equivalence relation on modules M over the rings P_W, when W is an $\mathbb{F}$–vector space.*

Proof The identity and symmetry of the relation are clear in the definition. We need only verify transitivity. Suppose that (M, W) is stably isomorphic to (M', W') through

$(\widetilde{W}_1, i, i_1')$ and (M', W') is stably isomorphic to (M'', W'') through $(\widetilde{W}_2, i_2', i'')$. Let $\widetilde{W}$ be the quotient of $\widetilde{W}_1 \oplus \widetilde{W}_2$ by the subspace

$$\{i_1'(w) \oplus \vec{0} - \vec{0} \oplus i_2'(w) \mid w \in W'\}.$$

Then projection onto $\widetilde{W}$ composed with $i \oplus \vec{0}$ is an injection $W \hookrightarrow \widetilde{W}$, since it maps entirely into the first factor, while no nontrivial element in the subspace is entirely in the first factor. Likewise, $\vec{0} \oplus i''$ induces an injection $W'' \hookrightarrow \widetilde{W}$. Furthermore, $i_1' \oplus 0$ and $0 \oplus i_2'$ induce injections of W' into $\widetilde{W}$ with the same image. We now consider $T = (M \otimes_{P_W} P_{\widetilde{W}_1}) \otimes_{P_{\widetilde{W}_1}} P_{\widetilde{W}}$. On the one hand, T is isomorphic to $M \otimes_{P_W} (P_{\widetilde{W}_1} \otimes_{P_{\widetilde{W}_1}} P_{\widetilde{W}})$. Here the action of P_W on $P_{\widetilde{W}}$ is given by symmetric power of the composition of the inclusion maps. On the other hand stable equivalence implies that T is isomorphic to $T' = (M' \otimes_{P_{W'}} P_{\widetilde{W}_1}) \otimes_{P_{\widetilde{W}_1}} P_{\widetilde{W}}$ where the action of $P_{\widetilde{W}_1}$ on $P_{\widetilde{W}}$ is by the inclusion $\mathrm{Id} \oplus \vec{0}$ followed by projection, and the action of $P_{W'}$ on $P_{\widetilde{W}_1}$ is by i_1'. Reorganizing the tensor product using associativity, as before we obtain an isomorphism with $M' \otimes_{P_{W'}} P_{\widetilde{W}}$ where the action of $P_{W'}$ on $P_{\widetilde{W}}$ is given by the symmetric power of $\mathrm{pr} \circ (i_1' \oplus \vec{0})$.

We can perform the same argument starting with $(M'' \otimes_{P_{W''}} P_{\widetilde{W}_2}) \otimes_{P_{\widetilde{W}_2}} P_{\widetilde{W}}$. This is isomorphic to $M'' \otimes_{P_{W''}} P_{\widetilde{W}}$ with action given by the symmetric power of the composition $W'' \hookrightarrow \widetilde{W}_2 \hookrightarrow \widetilde{W}$. It is, as above, also isomorphic to $M' \otimes_{P_{W'}} P_{\widetilde{W}}$ with action of $P_{W'}$ on $P_{\widetilde{W}}$ given by the symmetric power of $\mathrm{pr} \circ (\vec{0} \oplus i_2')$. However, we have $\mathrm{pr} \circ (\vec{0} \oplus i_2') = \mathrm{pr} \circ (i_1' \oplus \vec{0})$, so this is also isomorphic to $M' \otimes_{P_{W'}} P_{\widetilde{W}}$ where the action of $P_{W'}$ on $P_{\widetilde{W}}$ is given by the symmetric power of $\mathrm{pr} \circ (i_1' \oplus \vec{0})$. From the preceding paragraph, we can conclude that $M \otimes_{P_W} P_{\widetilde{W}}$ using the composition $I\colon W \hookrightarrow \widetilde{W}_1 \hookrightarrow \widetilde{W}$ is isomorphic, as a $P_{\widetilde{W}}$–module, to $M'' \otimes_{P_{W''}} P_{\widetilde{W}}$ using the inclusion $I''\colon W'' \hookrightarrow \widetilde{W}_2 \hookrightarrow \widetilde{W}$. In particular, (M, W) and (M'', W'') are stably isomorphic through $(\widetilde{W}, I, I'')$. □

Lemma 3.4 *If M is a free module over P_W, M' is a free module over $P_{W'}$ and (M, W) is stable isomorphic to (M', W'), then $\dim_{P_W} M = \dim_{P_{W'}} M'$.*

Proof Let $\{e_i \in M \mid i \in \Lambda\}$ be a basis for M over P_W. Then since $(\sum a_i e_i) \otimes w$ equals $\sum e_i \otimes \mathrm{Sym}(I_1)(a_i) \cdot w = \sum (e_i \otimes 1) \cdot (\mathrm{Sym}(I_1)(a_i) \cdot w)$, we see that $\{e_i \otimes 1 \mid i \in \Lambda\}$ is a basis for $M \otimes_{P_W} P_{\widetilde{W}}$ over $P_{\widetilde{W}}$. Consequently, it has the same rank. Performing the same calculation for M' we see that the ranks must be equal. □

A similar equivalence relation holds for vector spaces over F_W. When two vector spaces are stably isomorphic their dimensions over their respective fields are equal. If V is a graded vector space over F_W, stable isomorphism induces stable isomorphism in each grading, and the rank equality holds in each grading.

We will use stable isomorphisms to relate chain complexes of modules defined over P_W for differing vector spaces W. To do this we note a commutative algebra result:

Lemma 3.5 *Let $I: W \hookrightarrow W'$ be an injection of finite-dimensional vector spaces. Then $P_{W'}$ is free, hence flat, as a P_W–module.*

Proof We select a basis for W, $\{w_1, \dots, w_k\}$ and consider $S = \{I(w_1), \dots, I(w_k)\}$ in W'. S is linearly independent over $\mathbb{F}$, and thus can be extended to a basis for W' by appending some vectors $\{y_{k+1}, \dots, y_l\}$. With these choices, there are ring isomorphisms $P_W \cong \mathbb{F}[w_1, \dots, w_k]$ and $P_{W'} \cong \mathbb{F}[I(w_1), \dots, I(w_k), y_{k+1}, \dots, y_l]$. As a module over P_W, $P_{W'}$ is thus isomorphic to $P_W[y_{k+1}, \dots, y_l]$ with basis given by the monomials in $y_{k+1}, \dots, y_l$. Thus $P_{W'}$ is free over P_W. □

Consequently, we can define stable isomorphism for chain complexes where every chain group is free.

Definition 3.6 Let $\mathcal{C}$ be a chain complex with chain groups free over P_W and let $\mathcal{C}'$ be similarly defined for $P_{W'}$. We will say that $\mathcal{C}$ is stably isomorphic to $\mathcal{C}'$ if there are injections $I, I': W, W' \hookrightarrow W''$ such that $\mathcal{C} \otimes_I P_{W''}$ is isomorphic to $\mathcal{C}' \otimes_{I'} P_{W''}$ as chain complexes over $P_{W''}$. Likewise, we will say that $\mathcal{C}$ is stably chain homotopic to $\mathcal{C}'$ if there is a W'' where $\mathcal{C} \otimes_I P_{W''}$ is chain homotopic to $\mathcal{C}' \otimes_{I'} P_{W''}$.

Due to the flatness, if $\mathcal{C}$ is stably isomorphic to $\mathcal{C}'$ then $H_i(\mathcal{C})$ is stably isomorphic to $H_i(\mathcal{C}')$ through (W'', I, I'). Consequently, there is a well-defined notion of stable rank, and stably isomorphic complexes will have identical Euler characteristics.

Definition 3.7 Let $v \in W$ with W an $\mathbb{F}$–vector space. $\mathcal{K}_W(v)$ is the complex

$$0 \longrightarrow P_W \xrightarrow{\cdot v} P_W \longrightarrow 0$$

supported in gradings $+1$ and -1. Let $v_1, \dots, v_k$ be vectors in W. Then

$$\mathcal{K}_W(v_1, \dots, v_k) = \mathcal{K}_W(v_1) \otimes_{P_W} \mathcal{K}_W(v_2) \otimes_{P_W} \cdots \otimes_{P_W} \mathcal{K}_W(v_k)$$

denotes the Koszul complex for $v_1, \dots, v_K$.

Example Let L be a link diagram, and let W be the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the faces of L. Then $P_W = \mathbb{P}_L$, and the Koszul complexes in the definition correspond to those in the definition of the totally twisted Khovanov homology in Section 1.2.

The following is a straightforward exercise in definitions:

Proposition 3.8 *Let $I: W \hookrightarrow W'$ be an injection of vector spaces, and $v_1, \dots, v_k \in W$. Then $\mathcal{K}(v_1, \dots, v_k)$ is stably chain isomorphic to $\mathcal{K}(I(v_1), \dots, I(v_k))$.*

In addition, some of our chain complexes will be defined over fields with large isomorphism groups. We can use these isomorphisms to adjust our chain complexes. Let C_* be a chain complex over a field $\mathbb{F}$, and let $\sigma\colon \mathbb{F} \to \mathbb{F}'$ be a field homomorphism, then we have new chain complex over F': $C'_* = C_* \otimes_{\mathbb{F}} \mathbb{F}'$ where the action of $\mathbb{F}$ on $\mathbb{F}'$ is given by $(\lambda, f) \to \sigma(\lambda) \cdot f$. In effect this construction just applied σ to all the coefficients. That is, if we have a basis $\{x_i\}$ given in C_* and $\partial x' = \sum f_i x_i$, then C'_* will be spanned by the same basis elements but with differential map $\partial' x = \sum \sigma(f_i) x_i$. ∂' is easily verified to be a differential from σ being a field map. In particular, when $\mathbb{F} = \mathrm{Rat}(W)$ and $\mathbb{F}' = \mathrm{Rat}(W')$ with $I\colon W \hookrightarrow W'$, C_* and C'_* will be stably isomorphic. We will also denote the tensor product as $C_* \otimes_\sigma \mathbb{F}'$ when we wish to emphasize the homomorphism.

4 Invariance for the totally twisted Khovanov homology

Given a set, A, let W_A be the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the elements of A. We will write x_s for the element in P_{W_A} which corresponds to the basis vector for $s \in A$. Recall that we will use a special notation for certain elements in P_{W_A}. For each $T \subset A$ let

$$[T] = \sum_{s \in T} x_s.$$

We will also denote this element by x_T when it is convenient. For $i_1, \dots, i_k \in S$ we will shorten $[\{i_1, i_2, \dots, i_k\}]$ to $[i_1 i_2 \dots i_k]$, so that both will denote $x_{i_1} + x_{i_2} + \dots + x_{i_k}$.

4.1 Two auxiliary constructions

4.1.1 Dissection In constructing $\mathbb{P}_L$ we associated a formal variable to each face in $\mathfrak{F}_L$. If we consider modules over these rings up to stable equivalence, we can make this more flexible. In particular, we can partition the faces in L and assign formal variables to each of the new components. Usually we will do this by embedding arcs in S^2 with endpoints on L and interiors mapped to $S^2 \backslash \Gamma_L$, as in Figure 5. If we let S be this collection of arcs, we can use the set of components $A = S^2 \backslash (\Gamma_L \cup S)$ to form the vector space W_A, as above. We will denote this vector space by $W_{L,S}$ and the corresponding symmetric algebra $P_{W_{L,S}}$ by $\mathbb{P}_{L,S}$. We can make $\mathbb{P}_{L,S}$ into a module over $\mathbb{P}_L$. For each face $R_i \in \mathfrak{F}_L$, the arcs of S partition R_i into a set of subcomponents $S_{1,i}, \dots, S_{k,i}$. If $S_{j,i}$ is associated to the formal variable $y_{j,i}$ in $\mathbb{P}_{L,S}$, we can define a map $I\colon \mathbb{P}_L \hookrightarrow \mathbb{P}_{L,S}$ by the decomposition relations

$$x_i \xrightarrow{I} y_{1,i} + y_{2,i} + \dots + y_{k,i}$$

corresponding to $R_i = S_{1,i} \cup \cdots \cup S_{k,i}$. Define $\underline{\mathrm{CKH}}^*_u(L;S)$ to be $\underline{\mathrm{CKH}}^*_u(L;S) \otimes_I \mathbb{P}_{L,S}$ and $\underline{\mathrm{KH}}^*_u(L;S)$ to be the homology $\underline{\mathrm{KH}}^*_u(l;\mathbb{P}_{L,S})$. We can use this complex to compute the twisted Khovanov homologies:

Lemma 4.1 *$\underline{\mathrm{KH}}^*_u(L)$ is stably isomorphic to $\underline{\mathrm{KH}}^*_u(L;S)$.*

Proof The module $\mathbb{P}_{L,S}$ is flat over $\mathbb{P}_L$, and thus $\underline{\mathrm{KH}}^*_u(L;S) \cong \underline{\mathrm{KH}}^*_u(L) \otimes_I \mathbb{P}_{L,S}$. The stable isomorphism follows directly. □

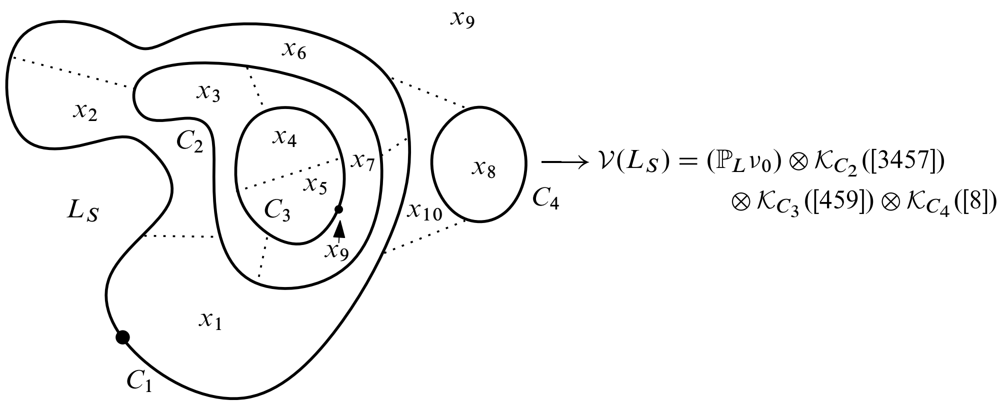


Figure 5: The complex assigned to a resolution in which some regions have been dissected and some edges have received weights

4.1.2 Edge weighting In addition to dissecting the regions, we will also find it useful to attach a weight to an edge, which can then be incorporated into the formal area of a circle in resolutions L_S. We depict this by adding a dot to the edge and labeling the dot with an element of $\mathbb{P}_L$ or with a new formal variable (and then enlarging $\mathbb{P}_L$). Diagrammatically we will draw an arrow from the weight to the point, to distinguish it from the weights assigned to the faces. For an example, see Figure 5.

A weight w assigned to a point on an edge should be interpreted as adding w to the weights of each of the regions adjacent to the edge. Thus, if C is a circle in a diagram for L and we add a weight w to a point in an edge then the area of C in the new diagram is

(1) $[C]$ if the closure of $A_p(C)$ contains neither or both of the faces adjacent to the edge,

(2) $[C] + w$ if $A_p(C)$ contains one of the adjacent faces but not the other.

Each edge occurs in only one circle C in a resolution L_S. Given L_S, write the states as $v \otimes v'$ where $v' = v_\pm$ is the decoration on the circle containing the weighted edge. If $\partial_{\mathcal{V}(L_S)}$ is the Koszul differential without the edge weight, then the complex with the edge weight has differential $\partial_{\mathcal{V}(L_S)} + wD_C$ where $D_C(v \otimes v_+) = v \otimes v_-$ and $D_C(v \otimes v_-) = 0$. We will call this *distributing the weights*, and note that by iterating the process we can distribute the weights on multiple edges.

If there are multiple points on the same edge, we can coalesce the points into one point and add their weights to get the weight of the new point. Once we have the adjusted area we can form the twisted Khovanov complex as before. For an example, see Figure 5. In particular, the area for C_3 is $x_4 + x_5 + x_9$ due to the edge weight, while that for C_2 is $x_3 + x_4 + x_5 + x_7$ with no edge contribution, since C_3 contains both regions which receive the extra x_9 term.

Let E be the data consisting of the weights on the edges. We will denote an edge weighted diagram for L by (L, E), and the corresponding chain groups, homology groups, etc by $\underline{\mathrm{KH}}^*_u(L; E)$.

Lemma 4.2 *Let (L, E) be an edge weighted diagram for a link $\mathcal{L}$. Suppose edges $e_1, \dots, e_n$ are weighted with $w_1, \dots, w_n$ and let E' be edge weight data identical to E away from the e_i, but assigning weight 0 to each e_i. Suppose that after distributing the weights on $e_1, \dots, e_n$, the new areas of each face are still linearly independent of all the other weights in (L, E'), then $\underline{\mathrm{KH}}^*_u(L; E)$ is stably isomorphic to $\underline{\mathrm{KH}}^*_u(L, E')$.*

Proof Let x'_A be the variable in (L, E') assigned to face A, and let $x'_A \to x_A + \sum w_j$ where the sum is over all the edges e_j in the boundary of A. This defines a change of variables from the ring for (L, E') to that for (L, E) which identifies the complexes for (L, E') and (L, E). The assumption on the linear independence guarantees that $x_A + \sum w_j$ is not equal to the adjusted area assigned to any other face in (L, E) since the adjusted areas remain linearly independent vectors over $\mathbb{F}$. Consequently, the map is an injection $\mathbb{P}_{L,E'} \to \mathbb{P}_{L,E}$ which can be used to establish the stable isomorphism. □

4.2 Deforming the chain complex

In proving that stable chain homotopy class of $CT_*(L)$ is invariant under the Reidemeister moves we will make use of the following formulation of a well-known lemma in homological algebra, which follows from a graded version of Gaussian elimination.

Lemma 4.3 *Let (M, d) be a differential module over a ring R. Suppose $M \cong M_1 \oplus M_2 \oplus M_3$ as an R–module and that $d = [L_{ij}]_{i,j=1,2,3}$ with respect to this decomposition. If L_{32} is an R–isomorphism, then there is a submodule $D \subset M$ with $d|_D \colon D \to D \subset M$ such that*

(1) $(D, d|_D)$ *is a deformation retract of* (M, d) *and*

(2) $(D, d|_D)$ *is isomorphic to* (M_1, d'), *where* $d' = L_{11} - L_{12} \circ L_{32}^{-1} \circ L_{31}$.

We will refer to the process of cutting down from (M, d) to $(D, d|_D)$ as *reduction*, and we will say that we are *canceling* L_{32}. It will often be the case that $L_{11}^2 = 0$, and so L_{11} is a differential on M_1. For this reason, we will call $-L_{12} \circ L_{32}^{-1} \circ L_{31}$ a *perturbation* term and d' the perturbed differential. As the lemma is well known, we will omit some of the computations underlying the proof in favor of indicating which computations should be performed.

Proof Let $Q = M_2 + d(M_2)$. Then $(Q, d|_Q)$ is an acyclic subcomplex of M. Let $x \in M_1$. Under $\pi\colon M \to M/Q$, $d(x)$ is mapped to $L_{11}(x) - L_{12}(y) = (L_{11} - L_{12}L_{32}^{-1}L_{31})(x)$. Thus the quotient M/Q is the complex in item ii) of the lemma since L_{32} is an isomorphism. In particular, this shows that d' is a differential, which can also be verified directly using the nine relations between the L_{ij} found from $d^2 \equiv 0$. On the other hand, there is a chain map $(M_1, d') \to (M, d)$ defined as $\iota(x) = x \oplus -L_{32}^{-1}L_{31}(x) \oplus 0$. That this is a chain map is an exercise in using the entries of $d^2 \equiv 0$. Furthermore, ι is injective, due to the form of the first summand. We let D be $\operatorname{im}\iota$, so $(D, d|_D)$ is chain isomorphic to (M_1, d'). It remains to verify that $(D, d|_D)$ is a deformation retract of (M, d). However,

$$\pi \circ \iota = \mathrm{Id}_{M_1}$$

and if we let

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_{32}^{-1} \\ 0 & 0 & 0 \end{bmatrix}$$

it is then easy to verify (once again using $d^2 = 0$) that

$$\iota \circ \pi - \mathrm{Id}_M = dH + Hd$$

so that $(D, d|_D)$ is a deformation retract of (M, d). In particular, D is chain homotopy equivalent to M. □

4.3 Invariance under the first Reidemeister move

Convention All gradings in the following sections are δ gradings.

Proposition 4.4 *Let* c *be a crossing in an oriented link diagram* L *which can be removed by a local Reidemeister I move. If* L' *is the diagram after the move, then* $\underline{\mathrm{KH}}^*(L)$ *is (stably) chain homotopy equivalent to* $\underline{\mathrm{KH}}^*(L')$.

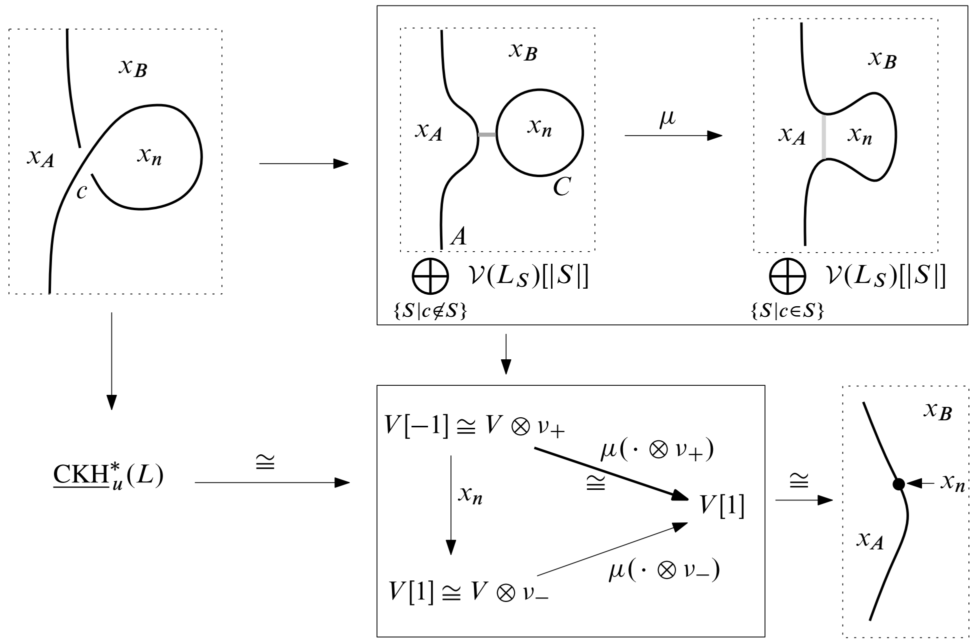


Figure 6: A schematic representation of the proof of invariance under the first Reidemeister move for a positive crossing: we prove that the twisted complex for the diagram L in the upper left is homotopy equivalent to the edge weighted diagram in the lower right. This, in turn, is stably isomorphic to the twisted homology for L', the diagram resulting from the Reidemeister move. The complex is reduced along the thickened arrow, which is an isomorphism of the respective submodules of the chain modules.

Proof There are two cases to consider, based on the handedness of the crossing. Let L' be the diagram resulting after the Reidemeister move. The argument for Case I below is directly reflected in the diagrams in Figure 6.

Case I: c is right-handed Our aim is to show that $\underline{\mathrm{CKH}}^*_u(L)$, for L with a local swatch as in the upper left corner of the figure, is chain homotopy equivalent to complex for the edge weighted diagram in the lower right. The edge weighted diagram is stably isomorphic to $\underline{\mathrm{CKH}}^*_u(L')$ by Lemma 4.2. We can divide $\underline{\mathrm{CKH}}^*_u(L)$ as a direct sum of $\bigoplus_{c\notin S} \mathcal{V}(L_S)[|S|]$ and $\bigoplus_{c\in S} \mathcal{V}(L_S)[|S|]$. This is depicted in the top right of Figure 6. When $c \notin S$, there is a complete circle, C, in the local diagram used in the Reidemeister move. Thus $\mathcal{V}(L_S) \cong \mathcal{V}(L'_S) \otimes \mathcal{K}(C)$. Let x_n be the formal variable associated with the region, $A_p(C)$, so that ∂_C is multiplication by x_n. The complex $\underline{\mathrm{CKH}}^*_u(L)$ can be decomposed further. Let $V = \underline{\mathrm{CKH}}^*_u(L')$, then $\bigoplus_{c\in S} \mathcal{V}(L_S) \cong V[1]$, where the grading shift comes from the additional resolution at c when compared with L', and

$\bigoplus_{c \notin S} \mathcal{V}(L_S) \cong V_+ \oplus V_-$ where $V_\pm = V \otimes v_\pm$ and the last factor is that from C. Then $\underline{\mathrm{CKH}}^*_u(L) \cong V_- \oplus V_+ \oplus V[1] \cong V[1] \oplus V[-1] \oplus V[1]$ since V_+ has Khovanov bigrading shifted by $(0,1)$ compared with V and that corresponds to a δ–shift of -1. The twisted differential, when written to respect this decomposition, is the sum of two maps:

$$\begin{aligned}\underline{\partial}_L = \partial_{\mathrm{KH},L} + \partial_{\mathcal{V},L} &= \begin{bmatrix} \partial_{\mathrm{KH},L'} & 0 & 0 \\ 0 & \partial_{\mathrm{KH},L'} & 0 \\ \mu(\cdot \otimes v_-) & \mu(\cdot \otimes v_+) & \partial_{\mathrm{KH},L'} \end{bmatrix} + \begin{bmatrix} \partial_{\mathcal{V},L'} & x_n D_C & 0 \\ 0 & \partial_{\mathcal{V},L'} & 0 \\ 0 & 0 & \partial_{\mathcal{V},L'} \end{bmatrix} \\ &= \begin{bmatrix} \underline{\partial}_{L'} & x_n D_C & 0 \\ 0 & \underline{\partial}_{L'} & 0 \\ \mu(\cdot \otimes v_-) & \mu(\cdot \otimes v_+) & \underline{\partial}_{L'} \end{bmatrix}.\end{aligned}$$

In particular, $\mu(\cdot \otimes v_+)\colon V_+ \cong V[-1] \xrightarrow{\mu} V[1]$ is an isomorphism, since v_+ acts as the identity element in the Frobenius algebra. The inverse map H is simply $\xi \to \xi \otimes v_+$. This decomposition is depicted as the middle bottom diagram of Figure 6.

If we consider the decomposition of $\underline{\mathrm{CKH}}^*_u(L)$ above as that into $M_1 \oplus M_2 \oplus M_3$ as in Lemma 4.3, then we can cancel the isomorphism $L_{32} = \mu(\cdot \otimes v_+)$. Consequently, we obtain the perturbed differential on V_-:

$$\begin{aligned}(L_{11} + L_{12} L_{32}^{-1} L_{31})(\xi \otimes v_-) &= (\underline{\partial}_{L'} \otimes I)(\xi \otimes v_-) + x_n D_C \circ H \circ \mu(\xi \otimes v_-) \\ &= (\underline{\partial}_{L'}(\xi) + x_n \mu(\xi \otimes v_-)) \otimes v_- .\end{aligned}$$

Under the identification $V_- \cong V[1]$, we can drop the last v_- factor. Given a full resolution, L'_S, let C' be the circle containing the local arc, then the perturbed differential on V_- is then $\partial_V + x_n D_{C'}$. This is the edge weighted differential occurring in the last diagram in Figure 6. By Lemma 4.2 the last complex is stably chain isomorphic to $V_- \cong \underline{\mathrm{CKH}}^*_u(L')[1]$.

Finally, we address the grading shifts. We know that $n_+(L) = n_+(L') + 1$. Therefore we have that $\underline{\mathrm{CKH}}^*(L) \cong \underline{\mathrm{CKH}}^*_u(L)[-n_+(L)]$ is stably chain homotopic to $\underline{\mathrm{CKH}}^*_u(L')[1][-n_+(L') - 1] \cong \underline{\mathrm{CKH}}^*_u(L')[-n_+(L')]$. The latter is $\underline{\mathrm{CKH}}^*(L')$ by definition, and we have verified (stable) chain homotopy invariance in this case.

Note There is a special case implicitly handled in the above argument: if the arc in the local diagram belongs to the marked circle. This is addressed by noting that if the circle C' is assigned v_-, then by case (I) deforming the differential has no effect on the image of that state.

Case II: c is a left-handed crossing We decompose $\underline{\mathrm{CKH}}^*_u(L)$ as in the right-handed case, although the gradings and differential are different. Indeed, $\bigoplus_{c \in S} \mathcal{V}(L_S) \cong$

$V_+[1] \oplus V_-[1] \cong V \oplus V[2]$ and the map $\Delta\colon V \to V_+[1] \oplus V_-[1]$ followed by projection onto $V_-[1]$ is surjective. There is still a map

$$V_+[1] \xrightarrow{\cdot x_n} V_-[1]$$

as before, and elements in V with nontrivial image under the composition of Δ with projection to $V_+[1]$. These occur when the circle C' containing the local arc in V is adorned with a v_+. Thus, we may deform the complex to one supported on $V_+[1]$ with a perturbed differential. Namely, for states marked $v_+ \otimes v_+$ (arc $\otimes$ C) in $V_+[1]$ the image under ∂_C is $x_n(v_+ \otimes v_-)$. This is canceled using Δ since $\Delta(x_n v_+) = x_n(v_- \otimes v_+) + x_n(v_+ \otimes v_-)$. But this map also has the component $x_n(v_- \otimes v_+)$ in $V_+[1]$. Consequently, the effect of the cancellation is to perturb the differential in $V_+[1]$ by adding terms $v_+ \otimes v_+ \to x_n(v_- \otimes v_+)$. Once again, this has the effect of adding x_n to the formal area of the region bounded by C'. The result of this deformation is a chain complex in the same δ–gradings as $(V \otimes v_+)[1]$. This complex is isomorphic, under the change of variables imposed by Lemma 4.1, to V, with the same grading since the tensor product with v_+ contributes a -1 to the δ–grading. On the other hand, the number of positive crossings is not altered by the Reidemeister move, hence $\underline{\mathrm{CKH}}^*(L)$ is (stably) chain homotopy equivalent to $\underline{\mathrm{CKH}}^*(L')$. □

4.4 Invariance under the second Reidemeister move

Proposition 4.5 *If L is a diagram for $\mathcal{L}$ and L' is another diagram differing from L only by a local Reidemeister II move, then $\underline{\mathrm{KH}}^*(L)$ is (stably) chain homotopy equivalent to $\underline{\mathrm{KH}}^*(L')$.*

Proof The argument refers to Figure 7. We show that the $\underline{\mathrm{CKH}}^*_u(L)$ for L as in the upper left diagram of Figure 7 is chain homotopy equivalent to complex for the edge weighted and dissected diagram in the lower right, by canceling the thickened arrows in the lower diagram. This diagram, in turn, represents a complex that is stably isomorphic to the complex for L' by Lemmas 4.1 and 4.2. In particular, we let $V_{c_1} = \bigoplus_{\{S \mid c_1 \notin S, c_2 \in S\}} \mathcal{V}(L_S)$ and note that V_{c_1} as a graded module is identical with $\mathrm{KH}(L')[1]$, while the restricted twisted differential differs from that on for L' only in the area contributions. The proof consists of canceling portions of $\underline{\mathrm{CKH}}^*_u(L)$ using Lemma 4.3, to obtain a perturbed differential on V_{c_1} complex for the edge weighted and dissected diagram in the lower right.

Let

$$V_\varnothing = \bigoplus_{\{S:\, c_1, c_2 \notin S\}} \mathcal{V}(L_S), \quad V_{c_2} = \bigoplus_{\{S:\, c_1 \in S, c_2 \notin S\}} \mathcal{V}(L_S), \quad V_{c_1,c_2} = \bigoplus_{\{S:\, c_1, c_2 \in S\}} \mathcal{V}(L_S).$$

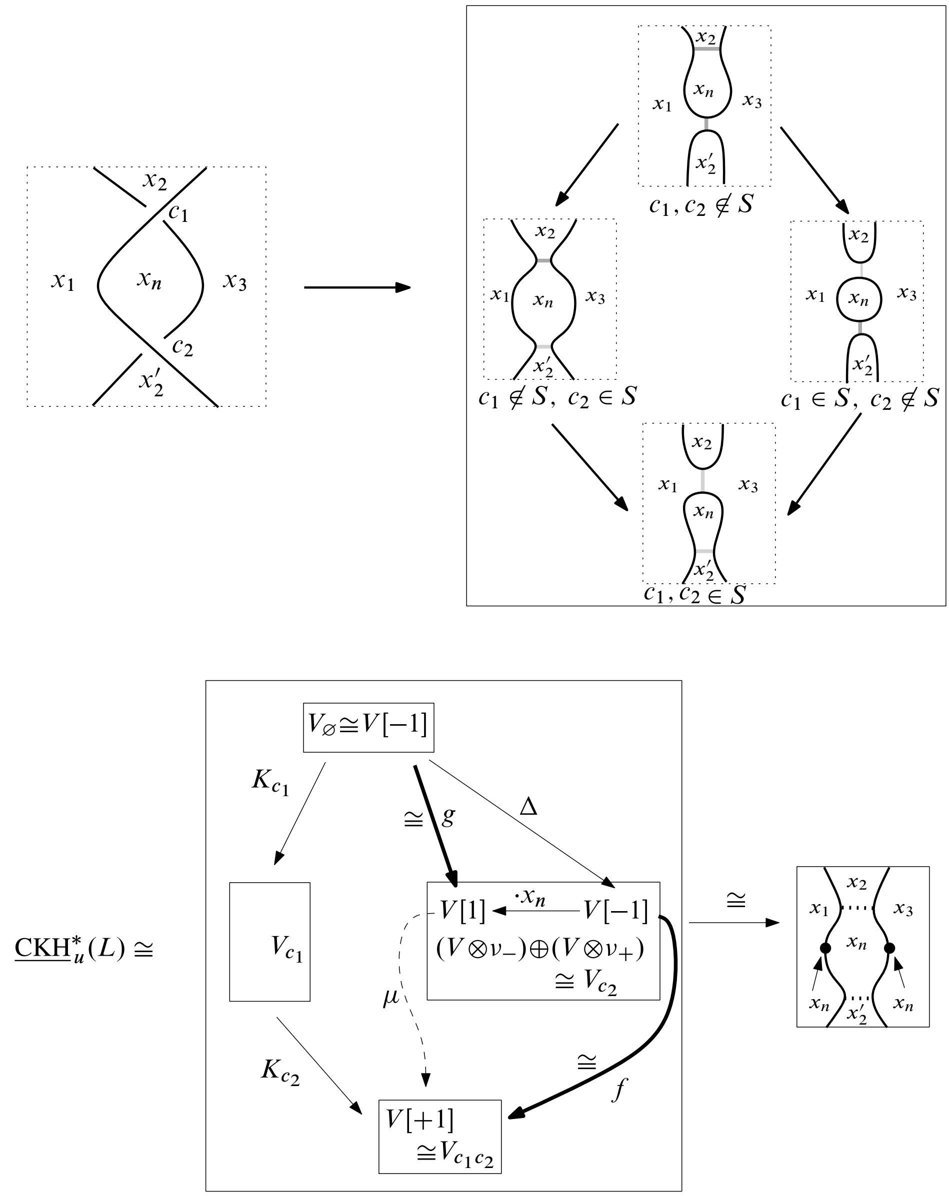


Figure 7: A representation of the proof of invariance under the second Reidemeister move: the complex is reduced along the thickened arrows, which are both isomorphisms. This is essentially the proof in [8], but the map which multiplies by x_n introduces a nontrivial perturbation to the differential on the complex after the reduction. The alteration is depicted through the edge weighting in the last diagram. We then verify that the twisted chain complex for the edge weighted diagram is stably isomorphic to that for L'.

Then $\mathrm{KH}(L) \cong V_\varnothing \oplus V_{c_1} \oplus V_{c_2} \oplus V_{c_1c_2}$, as depicted in Figure 7.

Furthermore, V_{c_2} can be decomposed as $(V \otimes v_+) \oplus (V \otimes v_-)$, where the second factor is the decoration on the circle, C, in the local diagram for V_{c_2} (in Figure 7), and V is the complex $V_{c_1c_2}[-1]$. The map $V_{c_2} \to V_{c_1c_2}$ in $\underline{\mathrm{CKH}}^*_u(L)$ is identical to the map in the Khovanov complex: it is μ applied to the last factor and a factor corresponding to the arc C merges into. In particular, on $V \otimes v_+$, μ restricts to an isomorphism $f\colon V \otimes v_+ \to V_{c_1c_2}$. Consequently, given any state s in $V_{c_1c_2}$, there is a canceling state, $f^{-1}(s)$ in V_{c_2}. Likewise, the Khovanov division map $\Delta\colon V[-1] \cong V_\varnothing \to V_{c_2}$ followed by projection to $V \otimes v_-$ is an isomorphism, since $s \to L(s) \oplus (s \otimes v_-)$ where $L(s) \in V \otimes v_+$ may or may not be zero, depending on the state s. Call this isomorphism g.

The twisted differential provides a map $\partial_C\colon V \otimes v_+ \xrightarrow{\cdot x_n} V \otimes v_-$ which takes $s \otimes v_+$ to $x_n(s \otimes v_-)$.

We can cancel the isomorphisms f and g simultaneously, since their domains and images are disjoint. More precisely, $\underline{\mathrm{CKH}}^*_u(L) \cong M_1 \oplus M_2 \oplus M_3$, where $M_1 = V_{c_1}$, $M_2 = (V \otimes v_+) \oplus V_\varnothing$ and $M_3 = (V \otimes v_-) \oplus V_{c_1c_2}$, then:

$$\underline{\partial}_L = \left[\begin{array}{c|cc|cc} \partial_{V_{c_1}} & 0 & K_{c_1} & 0 & 0 \\ \hline 0 & \partial_{V \otimes v_+} & \Delta & 0 & 0 \\ 0 & 0 & \partial_{V_\varnothing} & 0 & 0 \\ \hline 0 & \cdot x_n & g & 0 & 0 \\ K_{c_2} & f & 0 & \mu & \partial_{V_{c_1c_2}} \end{array}\right]$$

Then $L_{32} = \left(\begin{smallmatrix} \cdot x_n & g \\ f & 0 \end{smallmatrix}\right)$, which is an isomorphism since f and g are. Canceling along L_{32} produces a deformation equivalent chain complex on V_{c_1} with a perturbed differential. By Lemma 4.3, this perturbed differential is given by

$$\begin{aligned} L_{11} + L_{12} L_{32}^{-1} L_{31} &= \partial_{V_{c_1}} + \begin{bmatrix} 0 & K_{c_1} \end{bmatrix} \begin{bmatrix} 0 & f^{-1} \\ g^{-1} & g^{-1}(\cdot x_n) f^{-1} \end{bmatrix} \begin{bmatrix} c0 \\ K_{c_2} \end{bmatrix} \\ &= \partial_{V_{c_1}} + K_{c_1} g^{-1} (\cdot x_n) f^{-1} K_{c_2}. \end{aligned}$$

If ξ is a generator of $V_{c_1c_2}$ then $f^{-1}(\xi)$ is the generator for V_{c_2} with the same decorations on the common circles, and a $+$ on the circle C, which we will denote $\xi \otimes v_+$. Thus, $(\cdot x_n) f^{-1}(\xi) = x_n(\xi \otimes v_-)$. Then $g^{-1}(\xi \otimes x_-) = \xi$, where ξ is the corresponding state in $V_\varnothing$. Thus, under this identification, $g^{-1}(\cdot x_n) f^{-1} K_{c_2} = x_n K_{c_2}$ and the perturbed differential is $\partial_{V_{c_1}} + x_n K_{c_1} K_{c_2}$. We will now examine the maps K_{c_1} and K_{c_2} more closely.

For any resolution L_S with $c_1 \notin S$, $c_2 \in S$, we will let C_L and C_R be the circles containing the left and right arcs in the local picture for V_{c_1} in Figure 7. Note that C_L and C_R *may be the same circle in the larger diagram* L_S. The map $K_{c_1} K_{c_2}$ is just a Khovanov homology map, and can be computed by considering two cases.

Case 1: $C_L = C_R$ Then K_{c_2} is a copy of Δ and K_{c_1} is a copy of μ applied to the circles resulting from the division at c_2 that gives Δ. But in characteristic 2, $\mu \circ \Delta\colon \mathcal{V} \to \mathcal{V}$ is the zero map. Consequently there is no perturbation to the differential applied to any state from this case.

Case 2: $C_L \neq C_R$ Then K_{c_2} is a merge map, and we use μ. K_{c_1} comes from dividing this same circle, and uses Δ. Thus we need to compute $\Delta \circ \mu\colon \mathcal{V}_{C_L} \otimes \mathcal{V}_{C_R} \to \mathcal{V}_{C_L} \otimes \mathcal{V}_{C_R}$. It is straightforward to verify that

$$\begin{aligned} v_+ \otimes v_+ &\to v_- \otimes v_+ + v_- \otimes v_+, \\ v_+ \otimes v_-, v_+ \otimes v_- &\to v_- \otimes v_-, \\ v_- \otimes v_- &\to 0. \end{aligned}$$

The perturbation is x_n times this map, and thus equals $x_n(D_{C_L} \otimes \mathbb{I} + \mathbb{I} \otimes D_{C_R})$, where D_C is the isomorphism $V \otimes v_+ \to V \otimes v_-$ where $\mathcal{V}_C = \mathbb{F} v_+ \oplus \mathbb{F} v_-$. We can interpret this formula as endowing each of the two arcs in the local diagram with the additional weight x_n, and including that weight in the formal area used in the vertical differentials for C_L and C_R. Indeed, when C_L and C_R coincide, the weight is added twice, once for each arc, and thus cancels so the vertical differential doesn't change. When C_L and C_R are distinct, x_n is added to each of $[C_L]$ and $[C_R]$. Note also that if one or both arcs is contained in the marked circle, it will act as if adorned by v_- and the additional contribution will not appear for that circle. By Lemmas 4.1 and 4.2 the complex for the diagram in the lower right is stably isomorphic to the complex for L'. □

4.5 Invariance under the third Reidemeister move

Proposition 4.6 *If L is a diagram for $\mathcal{L}$ and L' is another diagram differing from L only by a local Reidemeister III move, then $\underline{\mathrm{KH}}^*(L)$ is (stably) chain homotopy equivalent to $\underline{\mathrm{KH}}^*(L')$.*

Proof We consider the case of a third Reidemeister move from the upper left diagram in Figure 8 to the upper left diagram in Figure 9. The resolutions, S, with $d, e \notin S$ are in the upper layer of the cubes (shown to the left in the diagrams) while those with $d, e \in S$ are in the bottom layer (shown to the right). As with the proof of

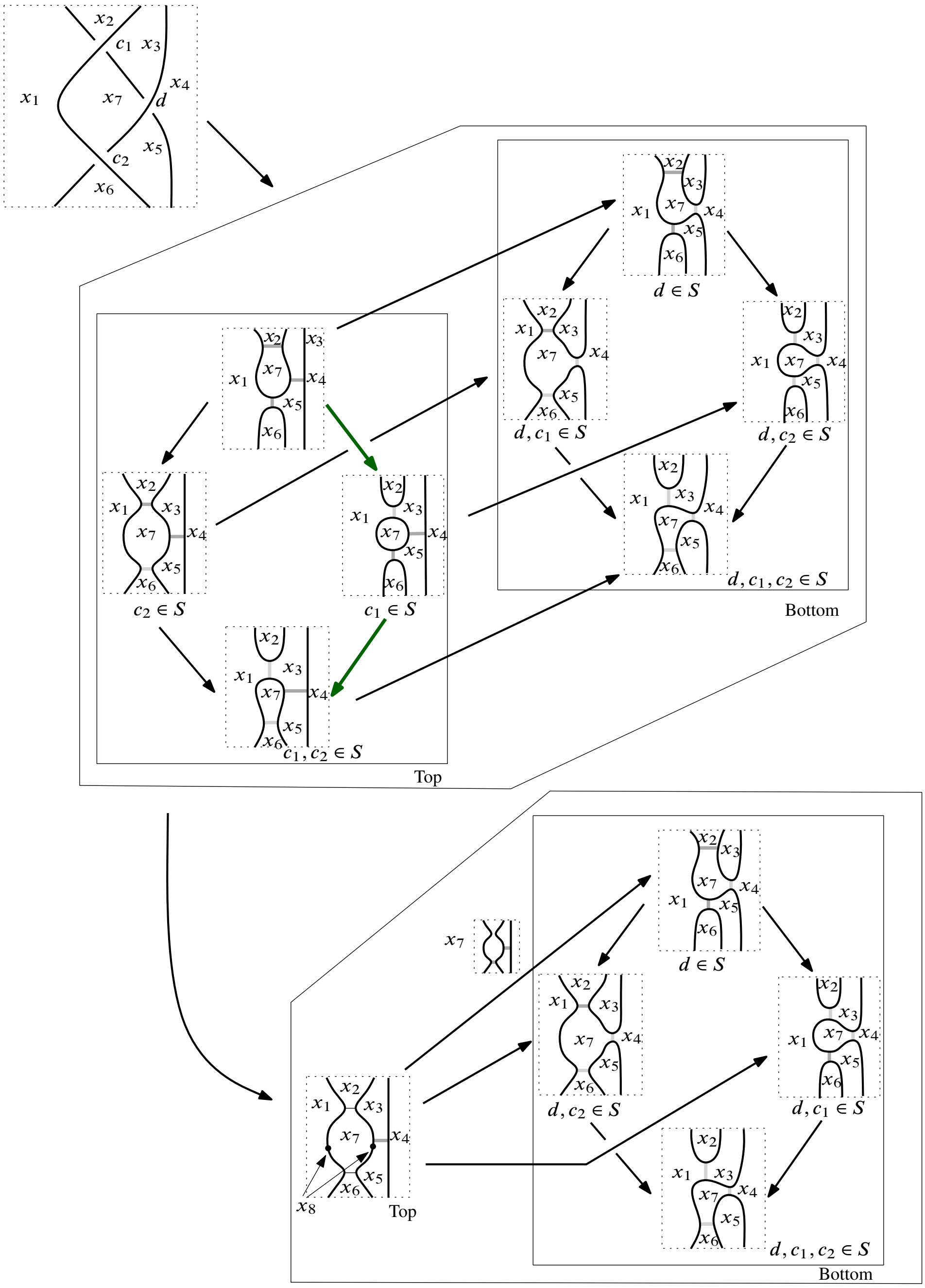


Figure 8: Diagrams for the link L in the proof of RIII invariance: notice the small diagram over an arrow in the bottom picture. This depicts the surface used in the proof of RIII invariance.

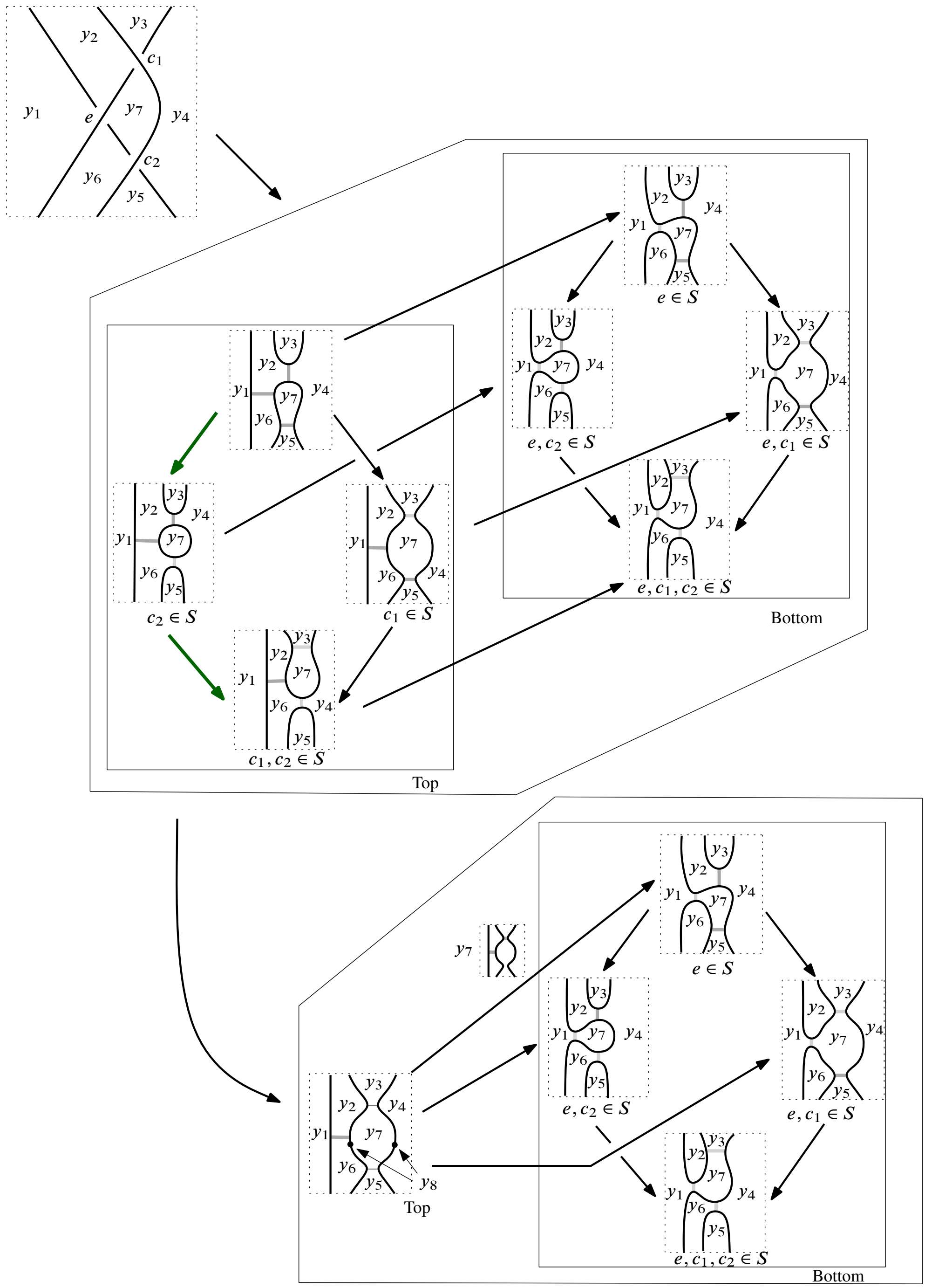


Figure 9: Diagrams for the link L' in the proof of RIII invariance

RII–invariance, we group according to the local resolution pattern; thus we can think of $\underline{\mathrm{CKH}}^*_u(L)$ as a standard mapping cone from homological algebra,

$$\underline{\mathrm{CKH}}^*_u(L) \cong \mathrm{MC}\Bigg(\bigoplus_{\{S \mid d \notin S\}} \mathcal{V}(L_S) \to \bigoplus_{\{S \mid d \in S\}} \mathcal{V}(L_S) \Bigg),$$

where the homomorphism comes from changing the resolution at d. Thus the homomorphism consists entirely of terms from the Khovanov differential, since the vertical differentials do not alter the resolutions. A similar result holds for L' and e.

As in Figures 8 and 9, we will further decompose the terms in the mapping cone, based on the choices of resolutions at c_1 and c_2. We will refer to the eight different summands in the decomposition of $\underline{\mathrm{CKH}}^*_u(L)$ shown in the middle of these figures by subscripts indicating which crossings are 1–resolved to obtain that summand: for example, $\mathcal{V}_{d,c_1} = \bigoplus_{\{S \mid d,c_1 \in S, c_2 \notin S\}} \mathcal{V}(L_S)$, while $\mathcal{V} \cong \bigoplus_{\{S \mid d,c_1,c_2 \notin S\}} \mathcal{V}(L_S)$.

The bottom layers in Figures 8 and 9, when $e \in S$, $d \in S$, are identical except for the presence of weight x_7 in one and y_7 in the other. These weights are treated differently since they correspond to the faces in the diagrams which are eliminated and created by the RIII move. Otherwise, for $i = 1, \dots, 7$, the weights x_i and y_i correspond to faces that can be identified based on how they intersect the bounding boxes. In the resolutions for L with $d \in S$, however, the region labeled by x_7 is always included in the same component as that for x_4, while the region labeled by y_7 in the resolutions for L' with $e \in S$ is always in the same component as y_1. Thus, defining Φ by $x_i \to y_i$, $i \neq 1, 4$, $x_1 \to y_1 + y_7$, and $x_4 \to y_4 + y_7$ will be a chain isomorphism from the subcomplex with $e \in S$ (the "bottom" in Figure 9) to that with $d \in S$ (the "bottom" in Figure 8).

We would like to extend Φ to an isomorphism of the whole complex. However, without alteration Φ is not a chain isomorphism on the summands where $d \notin S$ (the "top" diagrams in the middle of Figure 8), nor will it correctly map the connecting homomorphism in the mapping cone for L to that for L'. However, $\bigoplus_{\{S \mid d \notin S\}} \mathcal{V}(L_S)$ is isomorphic to the complex for a diagram with only d resolved, so that a local RII–move can be performed. A similar observation holds for e and L'. Consequently, repeating the cancellations performed in the proof of RII–invariance will simplify the top layer (as is done in [4]) to a deformed complex $\mathcal{D}_L$.

However, after the simplifications of the upper layer, we obtain a complex $\mathcal{D}_L$ in the upper layer which is the perturbed complex from the proof of RII invariance, and a new map to $\mathcal{V}_d$ (see Figure 8). This map comes from states, s, in $\mathcal{V}_{c_2}$ mapped to $\mathcal{V}_{c_1 c_2}$ whose image is then canceled. In particular, let $K_{c_2;c_1}\colon \mathcal{V}_{c_2} \to \mathcal{V}_{c_1 c_2}$ be the Khovanov map. If we write

$$K_{c_2;c_1}(s) = \sum a_j t_j \in \mathcal{V}_{c_1 c_2}$$

we see that $K_{c_2;c_1}(s)$ is the image of $\sum a_j(t_j \otimes v_+)$ under the canceling isomorphism from $\mathcal{V}_{c_1}$, where v_+ is the generator assigned to the local circle in the diagram for $\mathcal{V}_{c_1}$ and we use the same decorations on the local arcs for each term of $K(s)$. After this cancellation, the boundary of s is perturbed by the image of $K_{c_2;c_1}(s)$ under the vertical differential $\partial_{\mathcal{V}}$ added to the image under the Khovanov map $K_{c_1;d}$, ie by

$$x_7\Big(\sum a_j(t_j \otimes v_-)\Big) + K_{c_1;d}\Big(\sum a_j(t_j \otimes v_+)\Big).$$

Since merging a $+$ circle does not change any of the decorations on the arcs, the second term is just $\sum a_j t_j \in \mathcal{V}_{c_1,d}$, under the identification of generators, and is thus the image under the Khovanov surface map arising from adding a one-handle to the two leftmost arcs in the diagram for $\mathcal{V}_{c_2}$. Since s already maps to $K_{c_2;d}(s) \in \mathcal{V}_{c_2,d}$, the $\underline{\partial} s$ consists of a sum

$$\partial_{\mathcal{V}_{c_2}}(s) + G(s) + x_7\Big(\sum a_j(t_j \otimes v_-)\Big),$$

where G is the sum of Khovanov maps $K_{c_2;d} + K_{c_1;d}$. This sum is exactly the sum of Khovanov maps obtained from the analogous argument for RIII–invariance for Khovanov homology found in [4], and are known to be equal to the corresponding maps found from simplifying the diagrams for L'. In particular, there is no dependence on the weights and thus these maps will remain equal under the application of Φ.

In canceling the map $\mathcal{V} \to \mathcal{V}_{c_1}$, the term $x_7(\sum a_j(t_j \otimes v_-))$ will be the image of $x_7(\sum a_j t_j) \in \mathcal{V}$ where we again use the same decorations on local arcs to identify the generators. This perturbs the boundary map again to give

$$\partial_{\mathcal{V}_{c_2}}(s) + G(s) + x_7(K_{\varnothing;d} + K_{\varnothing;c_2}) \circ J \circ K_{c_2;c_1}(s),$$

where J is the identification of $\mathcal{V}_{c_1,c_2}$ with $\mathcal{V}$ from the diagrams being isotopic. Regrouping gives

$$\big(\partial_{\mathcal{V}_{c_2}}(s) + x_7(K_{\varnothing;c_2} \circ J \circ K_{c_2;c_1})(s)\big) + G(s) + x_7(K_{\varnothing;d} \circ J \circ K_{c_2;c_1})(s).$$

The first sum in parentheses is the perturbed differential on $\mathcal{V}_{c_2}$ arising in the proof of RII–invariance. The last term is x_7 times the Khovanov map for the surface which attaches two 1–handles to the diagram with three vertical lines. This surface is depicted in Figure 8 above the corresponding arrow. The gray lines indicate where the cores of the 1–handles will project to. A similar argument for the diagram in Figure 9 produces an analogous result. The surface map is likewise depicted in the figure. Due to the gray lines being isotopic in the two diagrams, the corresponding surfaces are isotopic, while preserving boundaries. Thus, the Khovanov maps for these surfaces are the same [4].

Consequently, the three maps composing the connecting homomorphism of the mapping cone will be identified: the two maps $\mathcal{D}_L \to \mathcal{V}_{d,c_1}, \mathcal{V}_{d,c_2}$ will be identified with $\mathcal{D}_{L'} \to \mathcal{V}_{e,c_1}, \mathcal{V}_{e,c_2}$ because they are Khovanov maps which do not depend on the formal variables, and the last map $\mathcal{D}_L \to \mathcal{V}_d$ will be identified with $\mathcal{D}_{L'} \to \mathcal{V}_e$ because $\Phi(x_7) = y_7$ and the planar isotopy identified the Khovanov maps. It remains to see that Φ also identifies the perturbed complexes $\mathcal{D}_L$ and $\mathcal{D}_{L'}$. These are depicted as edge weighted diagrams in Figure 10.

If we distribute the edge weightings in each diagram of Figure 10, we obtain the following local contributions to the vertical strips:

	Left diagram	Right diagram
I	$x_1 + x_7$	y_1
II	$x_2 + x_7 + x_6$	$y_2 + y_7 + y_6$
III	$x_3 + x_7 + x_5$	$y_3 + y_7 + y_5$
IV	x_4	$y_4 + y_7$

It is straightforward to check that Φ taking $x_i \to y_i$, $i \neq 1, 4$, $x_1 \to y_1 + y_7$ and $x_4 \to y_4 + y_7$ will map the terms in the left column to those in the right column. Thus, Φ maps the vertical differentials in $\mathcal{D}_L$ to the corresponding differential in $\mathcal{D}_{L'}$. Consequently, after simplifying the top layers, Φ is a isomorphism of the whole complex, built out of an automorphism of the coefficient rings. Since the simplifications themselves are chain homotopy equivalences, and the automorphism is a chain isomorphism, the two-complexes are (stably) chain homotopy equivalent. □

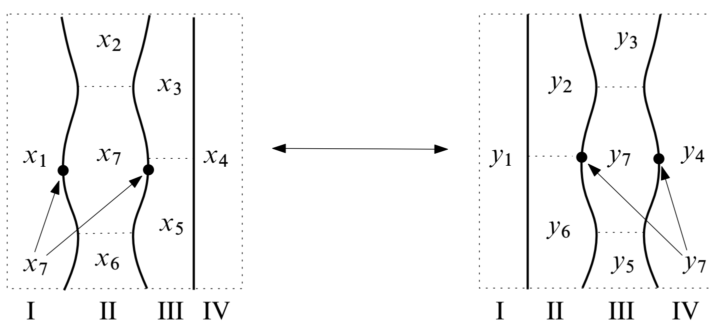


Figure 10

5 The spanning tree deformation of $\underline{\mathrm{CKH}}^*(L, \mathbb{F}_L)$

In the remaining sections we will work over the *field* $\mathbb{F}_L$, the field of fractions of $\mathbb{P}_L$. The complex $\underline{\mathrm{CKH}}^*_u(L)$ is defined over $\mathbb{F}_L$, and the arguments for invariance transfer directly. Thus the homotopy equivalence class of $\underline{\mathrm{CKH}}^*_u(L)$ is invariant up to changes of marked point. Over $\mathbb{F}_L$, however, we will be able to show the invariance under changes of marked point as well. In fact, the complex as a whole can be deformed using Lemma 4.3 into $\mathrm{CT}^*(L)$ as defined in Section 1.8 . We start by deriving the form in the introduction, and then prove the irrelevance of changes in the marked point.

Theorem 5.1 *The complex* $\mathrm{CT}^*(L)$ *is homotopy equivalent to* $\underline{\mathrm{CKH}}^*_u(L,\mathbb{F}_L)$.

Proof For $S \subset \mathrm{CR}(L)$ with $|S| = i$, the complex, $\mathcal{V}(L_S)$ is of the form $\mathbb{F}_L \otimes \mathcal{V}_{C_2} \otimes \cdots \otimes \mathcal{V}_{C_{k(S)}}$, where $k \geq 1$ and C_1 is the marked circle. Since these complexes are over a field, the homology of the tensor product is the tensor product of the homologies. Since $[C_i]$ is invertible for all i we have ∂_{C_i} is an isomorphism from $\mathbb{F}_L v_+$ to $\mathbb{F}_L v_-$. Recall that $O(L)$ is the set of $S \subset \mathrm{CR}(L)$ such that the associated resolution L_S consists of a single circle. We can decompose $\underline{\mathrm{CKH}}^*_u(L,\mathbb{F}_L)$ into a direct sum of three pieces

$$K_0 = \bigoplus_{S \in O(L)} \mathcal{V}(L_S),$$

$$K_+ \cong \bigoplus_{S \notin O(L)} \widetilde{v}_0 \otimes v_+^2 \otimes \mathcal{V}_{C_3} \otimes \cdots \otimes \mathcal{V}_{C_{k(S)}},$$

$$K_- \cong \bigoplus_{S \notin O(L)} \widetilde{v}_0 \otimes v_-^3 \otimes \mathcal{V}_{C_2} \otimes \cdots \otimes \mathcal{V}_{C_{k(S)}}.$$

Then the differential induces an isomorphism $K_+ \to K_-$ since it contains a term

$$v_0 \otimes v_+^2 \otimes W \xrightarrow{\cdot [C_2]} v_0 \otimes v_-^2 \otimes W$$

which is an isomorphism of vector spaces, and which is the only map preserving the value of $|S|$ with this image. Consequently, we can cancel K_+ and K_- through this map, leaving K_0.

We now compute the differential map which results from the cancellation of K_+ and K_-. Since K_0 consists of single circle resolutions, the only generator for the chain group $\mathcal{V}(L_S)\{(|S|,|S|)\}$ occurs in Khovanov bigrading $(|S|,|S|)$ (for the unshifted complex). If we start with a single circle resolution S, then for there to be a nontrivial term in the deformed complex supported on another single circle resolution S' we will have $\Delta i = \Delta j = |S'| - |S|$, where $(\Delta i, \Delta j)$ is the change in the Khovanov bigrading. Note that any nontrivial contribution to the perturbed boundary map must increase $\delta = 2i - j$ only by 2 since δ is a grading and we canceled terms in the differential. Consequently, $2\Delta i - \Delta j = 2$ as well, and thus $|S'| - |S| = 2$.

To compute the differential, note that to get a single circle resolution S' with $|S'|-|S|=2$ and $S'>S$ it must be that $S'\backslash S=\{c_1,c_2\}$. Both resolutions $S\cup\{c_1\}$ and $S\cup\{c_2\}$ must have two circles in their diagrams. Then $L_{S\cup\{c_i\}}=C_*\cup C_2^i$, where C_* is the marked circle, and changing the resolution on the other crossing merged C_2^i into C_*. Now consider the image $\partial_{\mathrm{KH}} v_0^S$ in

$$\mathcal{V}(L_{S\cup\{c_1\}})\cong v_0^{S,i}\otimes(\mathbb{F}_L v_+\oplus\mathbb{F}_L v_-).$$

In each resolution $S\cup\{c_i\}$, we have $\partial_{\mathrm{KH}}(v_0^S)=v_0^{S,i}\otimes v_-$ which is canceled by $(1/[C_2^i])(v_0^{S,i}\otimes v_+)$. Doing this for both c_1 and c_2 results in

$$\partial^{\mathrm{pert}} v_0^S=\Big(\frac{1}{[C_2^1]}+\frac{1}{[C_2^2]}\Big)v_0^{S'}$$

which is easily seen to agree with the differential for $\mathrm{CT}^*(L)$ described in the introduction. □

We can interpret this complex in terms of a checkerboard coloring of the regions in the diagram L. The Tait graphs for L are two planar graphs, one for each color. The black Tait graph has the black colored regions as vertices. Each crossing c of L provides an edge joining the vertices containing the two diagonally opposite black quadrants at c. We will usually draw the black Tait graph embedded in the union of the black regions and the projection of L (see Figure 11). The white Tait graph is defined similarly, using the white regions as vertices and all the crossings to provide edges. The colors will be used only to identify the graph, and will not otherwise be prescribed. Let $S\subset\mathrm{CR}(L)$ such that L_S is a single circle. At each crossing c the resolution bridges either the two black quadrants or the two white quadrants. If we let T be the subset of $\mathrm{CR}(L)$ which bridge opposite black quadrants, we can consider the corresponding edges in the black Tait graph. The subgraph formed by these edges is a deformation retract of the union of the black regions in L_S, which is a disc. Consequently, T determines a subtree of the black Tait graph, which is necessarily spanning since there is only one disc. Identically, the subset of $\mathrm{CR}(L)$ where the resolution L_S bridges the white quadrants can be identified with a (dual) spanning tree for the white Tait graph. Furthermore, we will take the white and black regions on either side of the marked point, $p\in L$, as roots for the Tait graphs, and thus for all the spanning trees.

On the other hand, given a partition of $\mathrm{CR}(L)$ into two sets T and T' with T determining a spanning tree for the black Tait graph and T' a dual spanning tree for the white Tait graph, we can resolve L so that the edges in T correspond to those crossings where the resolution bridges black quadrants, and the edges of T' correspond to those crossings where the resolution bridges the white quadrants. The resulting diagram L_S will consist of a single circle (note that S must then be determined by which crossings are resolved according to the rule in the introduction). Consequently, there is a one-to-one correspondence between spanning trees for the black Tait graph of a link projection and the generators of $\mathrm{CT}^*(L)$. For an illustration, see Figure 11. We will often use these trees as generators spanning the chain groups.

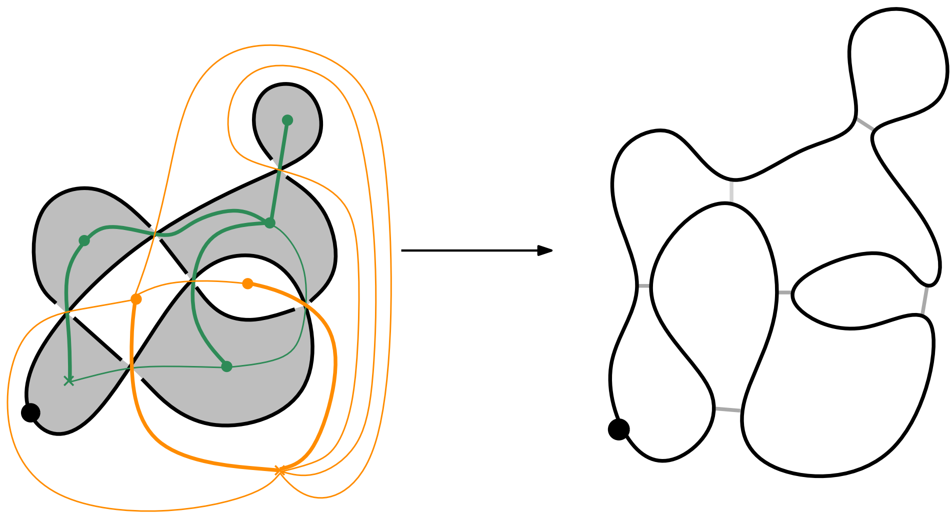


Figure 11: The dual spanning trees for the white and black Tait graphs on the left correspond to the single circle resolution diagram on the right.

Note also that when we change the resolution at a crossing c_1 of a single circle resolution, we cut off either a black disc region from the black disc or a white disc region from the white disc. To get back to a single circle resolution changing the second crossing c_2 must rejoin these. Switching the order of the crossing changes — and thus changing the resolution at c_2 first followed by changing it at c_1 — will cut off a disc of the opposite color and rejoin it.

With this deformation equivalent complex in hand, we can resolve the difficulties surrounding invariance with respect to the choice of basepoint.

Lemma 5.2 *Let (L, p) and (L, p') be two marked projections of $\mathcal{L}$ which differ only in the marked point. Then there is a field isomorphism $I\colon \mathbb{F}_L \to \mathbb{F}_L$ such that $\mathrm{CT}^p(L) \otimes_I \mathbb{F}_L$ is isomorphic to $\mathrm{CT}^{p'}(L)$. In short, the stable isomorphism class of $\mathrm{CT}^*(L)$ does not depend upon the choice of marked point on L.*

Proof of Lemma 5.2 Suppose the black face abutting the old marked point, p, is s_B and the black face abutting the new marked point, p', is s'_B (these could be the same).

Suppose similarly that s_W is the white face abutting the old marked point and s'_W is the white face abutting the new one. For the black faces there are two cases to consider for any two generators $T \in O_i(L)$ and T' with $T' \in O_{i+2}(T, L)$. Let $T < b < T'$ where b alters the resolution on the arc crossing a black face. Recall from the introduction that there is the disk $B^p_{T,T'}$ in the diagram for b whose interior and boundary are disjoint from the marked point p.

(1) If both s_B and s'_B are in the same region of L_b, $B^p_{T,T'}$ is the same as $B^{p'}_{T,T'}$, and neither contains s_B or s'_B.

(2) If each of the two circles in L_b contains one of s_B and s'_B, then

$$B^{p'}_{T,T'} = B \backslash B^p_{T,T'},$$

where B is the union of all the black faces. Furthermore, we have that $[B^{p'}_{T,T'}]$ contains a single x_{s_B} summand and $[B^p_{T,T'}]$ contains a single $x_{s'_B}$ summand.

A similar pattern holds for the white faces if we let W be the union of all the white faces. We will define a field isomorphism of $\mathbb{F}_L$ by taking

$$\begin{aligned} x_{s'_B} &\to x_{s'_B} + [B], \\ x_{s'_W} &\to x_{s'_W} + [W], \\ x_j &\to x_j \quad \text{when } j \neq s'_B, s'_W. \end{aligned}$$

In the first case above, $[B^p_{T,T'}]$ is fixed by this automorphism, and equals $[B^{p'}_{T,T'}]$. In the second case, $[B^p_{T,T'}] = x_{s'_B} + [B^p_{T,T'} \backslash \{s'_B\}]$ is mapped to

$$x_{s'_B} + [B] + [B^p_{T,T'} \backslash \{s'_B\}] = [B] + [B^p_{T,T'}] = [B \backslash B^p_{T,T'}] = [B^{p'}_{T,T'}].$$

Thus the coefficient of T' in $\partial^p_{i,L}$ coming from the black regions is mapped, under the automorphism, to the coefficient of T' in $\partial^{p'}_{i,L}$ defined from the black regions. Mutatis mutandis, the result also holds for the white regions. □

Thus, at this point we have established that $\mathrm{CT}^*(L)$, up to stable homotopy equivalence, is a link invariant, and thus its homology is also a link invariant.

6 Verifying $\partial^2 = 0$ for the spanning tree differential without reference to Khovanov homology

One can prove that $\mathrm{CT}^*(L)$ is a chain complex directly from the explicit representation of its differential, and without the circuitous route through twisted Khovanov homology. This is done in the next proposition, which serves as a good introduction to the combinatorial complexities we avoided (somewhat) by using twisted Khovanov homology. It is easy to also prove invariance under the Reidemeister I moves directly from the definition of $\mathrm{CT}^*(L)$. It is also possible to directly prove invariance under the RII move, although this is a substantially more involved combinatorial proof. However, the author has not been able to prove RIII invariance without using twisted Khovanov homology. For now we content ourselves with proving, directly from the definition, that the boundary map for $\mathrm{CT}^*(L)$ really is a differential.

Proposition 6.1 *For the map ∂_L we have $\partial_L^2 \equiv 0$.*

Proof of Proposition 6.1 Let $T \in O_i(L)$. Take p, the marked point, and move it to infinity. Then we may think of the single circle in L_T as the y–axis in the plane and the arcs from the resolutions as semicircles whose ends lie on this axis, and which are wholly contained either in $x \geq 0$ or $x \leq 0$. We may choose the black region inside L_T to correspond to the set $x \geq 0$. The endpoints of each arc c cut the y–axis into three segments: the segment unbounded towards $\pm\infty$, and $\gamma(c)$, the bounded segment.

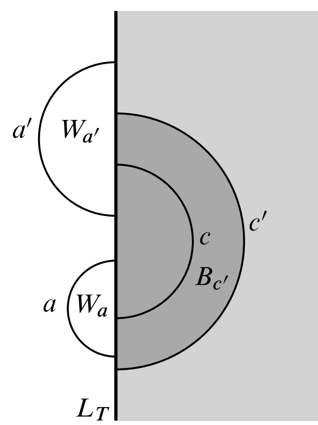


Figure 12: The arcs a and a' are peers for the diagram L_T. The arcs c and c' are parallel. The arc a' interleaves with both c and c', while a interleaves only with c. Thus this configuration is in Case IV of the proof of Proposition 6.1. The regions W_a and $W_{a'}$, as used in the proof, are drawn on the left. The region $B_{c'}$ is the darker gray region on the right. It includes B_c.

Definition 6.2 Two disjoint arcs c_1 and c_2 in $x \geq 0$ will be called *parallel* if $\gamma(c_2) \subset \gamma(c_1)$, or vice-versa. If $\gamma(c_1) \cap \gamma(c_2) = \varnothing$ then we will call the arcs *peers*. An arc c in $x \geq 0$ and an arc a in $x \leq 0$ will be said to *interleave* if $\gamma(a) \cap \gamma(c) \neq \varnothing$ but $\gamma(a) \not\subset \gamma(c)$ and $\gamma(c) \not\subset \gamma(a)$.

The configuration of the resolution diagram along the y–axis and the preceding definition are illustrated in Figure 12.

To show that $\partial^2 = 0$ we compute

$$\partial^2 T = \sum_{T' \in O_{i+4}(T,L)} \Big(\sum_{T' > T_r > T} \langle T', T_r \rangle \langle T_r, T \rangle \Big) T',$$

where the inner sum is over $T_r \in O_{i+2}(T, L)$. We will show, for each $T' \in O_{i+4}(L)$, that the inner summation equals zero in $\mathbb{F}_L$. Each term in this sum corresponds to four arcs, each coded with a 0, on the diagram for T considered in $\mathbb{R}^2$, two arcs in $x \leq 0$ and two in $x \geq 0$. Let these arcs be $\{a, a'\}$ and $\{c, c'\}$ respectively. Then each T_r in the summation corresponds to two interleaved arcs, $\{a_r, c_r\}$ with $a_r \in \{a, a'\}$ and $c_r \in \{c, c'\}$. We now forget the remainder of the arcs and concentrate only on these configurations.

We will analyze configurations in the following cases: there is a labeling of the arcs in the white region as a and a' and the arcs in the black region as c and c' such that

(I) a does not interleave with either c or c';

(II) each of $\{a, c\}$,$\{a', c'\}$, $\{a', c\}$ and $\{a, c'\}$ interleave;

(III) $\{a, c\}$ and $\{a', c'\}$ interleave, but $\{a, c'\}$ and $\{a', c\}$ do not;

(IV) $\{a, c\}$ and $\{a', c'\}$ and $\{a', c\}$ interleave, but $\{a, c'\}$ does not.

The remaining cases can be obtained by either by switching the roles of a' and a, c' and c, in the interleaving of the last case, or by arguing by symmetry between the white and black regions. Note, however, that there are actually several different types of configurations which can occur in Case IV, although the argument we give applies to all of them. We now analyze each of the cases.

Case I When a does not interleave with either c or c', resolving a results in a new circle component which cannot be rejoined to the other components by resolving along either c or c'. Consequently, the result of resolving all four arcs is not a single circle, and there is no contribution to ∂_L^2.

Case II When a and a' each interleave with both c and c': if either a and a' are parallel or c and c' are parallel, then resolving along all four arcs does not result in a single circle resolution, T', and thus this case does not contribute to $\partial_L^2 T$. To see this, suppose c and c' are parallel and $\gamma(c') \subset \gamma(c)$. Their mutual resolution results in a new circle component between the two arcs, which intersects the y–axis in segments $s = \gamma(c) \backslash \gamma(c')$. The endpoints of a cannot be on the segments in s: if there were an endpoint in one of the segments, then for a and c' to interleave, the other endpoint would need to be in $\gamma(c')$, but then a and c would not interleave. The same argument applies to a'. Consequently, resolving along a and a' does not affect the new circle component and T' is not a single circle. By symmetry, if a and a' are parallel then this case does not contribute to ∂^2. So assume that c and c' are peers. Since a and a' interleave both, a must have an endpoint in $\gamma(c)$ and in $\gamma(c')$ as these segments are disjoint. So must a', and since a and a' are disjoint they will have to be parallel. Thus, one or both pairs $\{a, a'\}$ or $\{c, c'\}$ are parallel and this case does not contribute to ∂_L^2.

Case III In this case, the arc pairs $\{a, c\}$ and $\{a', c'\}$ are independent. Let T_r be the result of resolving along $\{a, c\}$ and T_r' be the result of resolving along $\{a', c'\}$. If c and c' are peers, then B_{T,T_r} and $B_{T,T_r'}$ are disjoint. If we have resolved c and then resolve c', the region cut off is the same as if we resolve T along c', ie $B_{T_r,T'} = B_{T,T_r'}$. Likewise $B_{T_r',T'} = B_{T,T_r}$. By symmetry, the same argument holds for a and a' when they are peers. Now suppose c and c' are parallel with $\gamma(c') \subset \gamma(c)$. Then $B_{T,T_r'} \subset B_{T,T_r}$. If we resolve first along $\{a, c\}$, then we rejoin B_{T,T_r} to the unbounded black region, without affecting the region cut out by the arc c' since the endpoints of a do not intersect $\gamma(c')$. Consequently, $[B_{T_r,T'}] = [B_{T,T_r'}]$ since the formal variables are unchanged. Furthermore, if we first resolve $\{a', c'\}$, since a' does not interleave c, one endpoint of a' is in $\gamma(c')$ and the other is in $\gamma(c)\backslash\gamma(c')$. Thus the region $B_{T,T_r'}$ is rejoined to $B_{T,T_r}\backslash B_{T,T_r'}$ by a', so that $[B_{T_r',T'}] = [B_{T,T_r}]$. A similar argument applies to the white regions. Considerations of this type for each of the possible peer/parallel configurations yields that $[B_{T,T_{r_2}}] = [B_{T_{r_1},T'}]$ and $[B_{T_{r_2},T'}] = [B_{T,T_{r_1}}]$, and likewise $[W_{T,T_{r_2}}] = [W_{T_{r_1},T'}]$ and $[W_{T_{r_2},T'}] = [W_{T,T_{r_1}}]$, and it is straightforward to see that the contribution to $\langle \partial_L^2 T, T' \rangle$ coming from these arc pairs cancels in the summation.

Case IV This case is illustrated in Figure 13. Let W_a will be the white, bounded region of $\mathbb{R}^2\backslash(L_T \cup a)$, and similarly for $W_{a'}$. Likewise B_c and $B_{c'}$ will be the bounded, black region cut out by the arc. Let T_r result when resolving $\{a, c\}$, T_r' result when resolving $\{a', c'\}$ and T_s result when resolving $\{a', c\}$. For each of these circles, resolving the remaining two arcs results in the same circle T'. There are several possible cases to consider.

(1) For $T \to T_r \to T'$: first we cut off region B_c and W_a when changing from T to T_r. Since a interleaves with c, but not with c', B_c is rejoined to the same component of $\{x \geq 0\}\backslash c'$ as it was cut from. Consequently, resolving c' on T_r cuts off a region with formal representative $[B_{c'}]$. Meanwhile, resolving c joins the region W_a to:

(a) $W_{a'}$ in the case that a and a' are peers; when resolving a' we will cut off a region with formal area $[W_a]+[W_{a'}]$;

(b) the unbounded white region if a and a' are parallel with $\gamma(a) \subset \gamma(a')$; thus resolving a' will cut off the region between a' and a which has formal area $[W_{a'}]-[W_a]$. Since we are working in characteristic two, this equals $[W_a]+[W_{a'}]$;

(c) $W_{a'}$ in the case that a and a' are parallel with $\gamma(a') \subset \gamma(a)$; resolving a' will cut off the region between a and a' with area $[W_a]-[W_{a'}]$ which is the same as $[W_a]+[W_{a'}]$.

For all three cases the contribution to ∂_L^2 is

$$\Big(\frac{1}{[B_c]}+\frac{1}{[W_a]}\Big)\Big(\frac{1}{[B_{c'}]}+\frac{1}{[W_a]+[W_{a'}]}\Big).$$

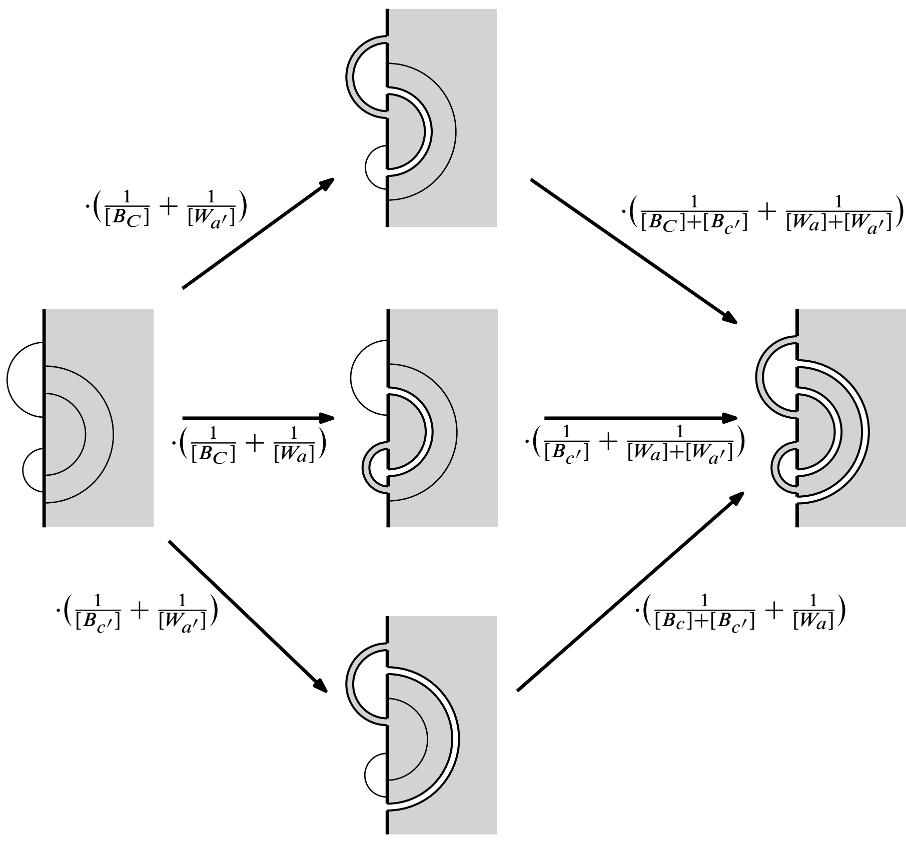


Figure 13: Illustration of why $\partial^2=0$ for the arcs in Figure 12, as in Case IV of Proposition 6.1

(2) For $T \to T_r' \to T'$: First we cut off $B_{c'}$ and $W_{a'}$. By a similar argument as above, the exact configuration of parallel and peer arcs for a and a' and c and c' will not matter. So we can do the calculation when each pair of arcs is peer. This results in the contribution

$$\Big(\frac{1}{[B_{c'}]}+\frac{1}{[W_{a'}]}\Big)\Big(\frac{1}{[B_c]+[B_{c'}]}+\frac{1}{[W_a]}\Big).$$

(3) For $T \to T_s \to T'$: first we cut off B_c and $W_{a'}$. Again we can do the calculation only for the peer case and get

$$\Big(\frac{1}{[B_c]}+\frac{1}{[W_{a'}]}\Big)\Big(\frac{1}{[B_c]+[B_{c'}]}+\frac{1}{[W_a]+[W_a']}\Big).$$

We will now combine fractions, multiply, and simplify. To aid us, note that

$$\frac{1}{XY}+\frac{1}{X(X+Y)}+\frac{1}{Y(X+Y)}=\frac{(X+Y)+Y+X}{XY(X+Y)}=0$$

in $\mathbb{F}_L$ for any nonzero elements X and Y we choose. Observe that in the products above, if we take only those terms using for the black regions we obtain such a sum with $X=B_c$ and $Y=B_{c'}$. Likewise if we take only those terms involving the white regions, we get such a sum with $X=W_a$ and $Y=W_{a'}$. So we only need to consider the sum of the cross-terms

$$\frac{1}{[W_a][B_{c'}]}+\frac{1}{[B_c]([W_a]+[W_{a'}])}+\frac{1}{[W_{a'}]([B_c]+[B_{c'}])} \\ +\frac{1}{[B_{c'}][W_a]}+\frac{1}{[W_{a'}]([B_c]+[B_{c'}])}+\frac{1}{[B_c]([W_a]+[W_a'])}.$$

However, these come in canceling pairs, so the sum of the three products is zero.

Consequently, for each of the four cases we have $\sum_{T'>T_r>T}\langle T',T_r\rangle\langle T_r,T\rangle=0$, and thus $\partial_L^2=0$. □

7 Results for mirrors and connected sums

We record two results which facilitate the calculation of the homology for knots and links built out of mirrors and connected sums. We then do a calculation which identifies the homology exactly for the class of quasialternating links. The results precisely reflect those for other knot homology theories.

Theorem 7.1 *Let $\mathcal{L}$ be an oriented link and $\overline{L}$ be the mirror of $\mathcal{L}$. Then* $\mathrm{HT}^i(\mathcal{L})\cong \mathrm{HT}^{-i}(\overline{\mathcal{L}})$.

Proof Let L be a diagram for $\mathcal{L}$ and $\bar{L}$ be the mirror diagram. We can use the same set of regions in forming $\mathbb{F}_L$ and $\mathbb{F}_{\bar{L}}$, and can thus identify the coefficient fields. Furthermore, $n_+(\bar{L}) = n_-(L)$ and $n_-(\bar{L}) = n_+(L)$. Every resolution $T \in O_i(L)$ corresponds to a resolution $\bar{T} \in O_{|\mathrm{CR}(L)|-i}(\bar{L})$ where $c \in T$ for the resolution of L corresponds to $c \notin \bar{T}$ for the resolution of $\bar{L}$. A tree $T' \in O_{i+2}(T, L)$ corresponds to a tree $\bar{T}' \in O_{|\mathrm{CR}(L)|-i-2}(\bar{L})$ and $\bar{T} \in O_{|\mathrm{CR}(L)|-i}(\bar{T}', \bar{L})$. Thus $\bar{T}$ can appear in $\partial_{\bar{L}}\bar{T}'$. $T' = T \cup \{c_1, c_2\}$ with $c_1, c_2 \notin T$ corresponds to $\bar{T}' = \bar{T} \setminus \{c_1, c_2\}$, or $\bar{T} = \bar{T}' \cup \{c_1, c_2\}$ with $c_1, c_2 \notin \bar{T}'$. When these alterations are performed the same two regions will be cut off, $B_{\bar{T}', \bar{T}} = B_{T,T'}$ and $W_{\bar{T}', \bar{T}} = W_{T,T'}$, and thus

$$\langle \partial_L T, T' \rangle = \langle \bar{T}, \partial_{\bar{L}} T' \rangle.$$

Consequently, the differential $\partial_{\bar{L}}$ corresponds to the cohomology differential ∂_L^*. Thus the unshifted cohomology for L in degree i is isomorphic to $\mathrm{HT}_u^{|\mathrm{CR}(L)|-i}(\bar{L})$. Since we are working with coefficients in a field, we have

$$\begin{aligned} \mathrm{HT}^i(\mathcal{L}) \cong \mathrm{HT}_u^{i+n_+(L)}(L) \cong \mathrm{HT}_u^{|\mathrm{CR}(L)|-(i+n_+(L))}(\bar{L}) \cong \mathrm{HT}_u^{n_-(L)-i}(\bar{L}) \\ \cong \mathrm{HT}_u^{-i+n_+(\bar{L})}(\bar{L}) \cong \mathrm{HT}^{-i}(\bar{\mathcal{L}}). \end{aligned}$$

This completes the proof. □

Theorem 7.2 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two nonsplit oriented links, and let $\mathcal{L} = \mathcal{L}_1 \# \mathcal{L}_2$, in some manner. Then*

$$\mathrm{HT}^k(\mathcal{L}) \cong \bigoplus_{i+j=k} \mathrm{HT}^i(\mathcal{L}_1) \otimes \mathrm{HT}^j(\mathcal{L}_2),$$

where $\cong$ denotes stable equivalence.

Proof Let L be a standard connect sum diagram in the plane for $\mathcal{L}$, where the portion of L with $x < 0$ is a diagram for $\mathcal{L}_1$ with an arc removed and the portion of L with $x > 0$ is a diagram for $\mathcal{L}_2$. We will choose our marked point for L to lie on one of the two arcs intersecting $x = 0$. For L_1 and L_2, we choose the marked point to lie on the removed arc. Let the regions for L_1 correspond to $x_1, \dots, x_k$ and the regions for L_2 correspond to $y_1, \dots, y_l$. The regions x_1 and y_1 should correspond to the bounded regions abutting the marked points. We will ignore the unbounded region in all diagrams, as it will always abut the marked point, and thus not play a role in the calculations. In the diagram for L, there are $k + l - 1$ regions which we will label $z_1, z_2, \dots, z_k, \dots, z_{l+k-1}$. The region z_1 corresponds to $x_1 + y_1$, while $z_i \sim x_i$ for $2 \le i \le k$ and $z_i \sim y_{i-k+1}$ for $k + 1 \le i \le l + k - 1$ (where $\sim$ means corresponds to the "same" region). These identifications will be implicitly used in the stable equivalence.

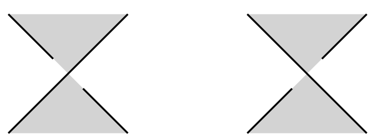
If we color black the bounded region abutting the basepoint in each diagram, then B_L is B_{L_1} and B_{L_2} fused at their basepoint. Every spanning tree in B_L is thus the fusion of a spanning tree for B_{L_1} and B_{L_2}. As δ and n_+ will both add under connect sums, if $T \in O_i(L_1)$ and $T' \in O_j(L_2)$ then $T \# T' \in O_{i+j}(L)$. To compute $\partial(T \# T')$ we need to consider trees in B_L where we have removed one edge of $T \# T'$, and reconnected the resulting pieces with an edge of $B_L \backslash (T \# T')$. If the removal occurs in B_{L_1}, we have a disconnected component in $\{x < 0\}$. There are no edges which cross $x = 0$, so the replacement must also occur with an edge from $B_{L_1} \backslash T$. The same argument applies if the edge removed occurs in $\{x > 0\}$. These are precisely the trees which occur in either $\partial_{L_1}(T)$ or $\partial_{L_2}(T')$. Furthermore, due to the placement of the basepoints, if we measure all coefficients using z_i's, then all the coefficients will also be the same as the connect sum only altered the region abutting the basepoint. Thus $\partial_L(T \# T') = \partial_{L_1}(T) \# T' + T \# \partial_{L_2}(T')$ (where we extend # linearly). Consequently, the chain complex for L is the tensor product of chain complexes for L_1 and L_2. Since we are working over a field, the result follows. □

8 The Euler characteristic

8.1 The Göritz matrix of L (following [9])

This section recalls some results concerning the signature and determinant of a link, which will be useful in the following sections.

Let L be an oriented link diagram and color the elements of $\mathfrak{F}_L$ with the colors black and white, in checkerboard fashion. Let $\mathfrak{W} = \{W_0, \dots, W_n\} \subset \mathfrak{F}_L$ be the $n+1$ faces which are colored white. To each crossing $c \in \mathrm{CR}(L)$ we assign a value $\mu(c)$ according to:



$\mu(c) = +1$ $\mu(c) = -1$

In addition, using the orientation we can assign a chirality: $\chi(c) = +1$ for positive crossings and $\chi(c) = -1$ for negative crossings. Finally, crossings will be called Type I if $\mu(c)\chi(c) = -1$ and Type II if $\mu(c)\chi(c) = +1$.

Let

$$\mu(L) = \sum_{\text{c of Type II}} \mu(c), \quad g(W_i, W_j) = -\sum_{c \in \bar{W}_i \cap \bar{W}_j} \mu(c), \quad g(W_i) = -\sum_{j \neq i} g(W_i, W_j).$$

The Göritz matrix of L is the matrix $G(L)$ with $G_{ij}(L) = g(W_i, W_j)$ for $0 \le i \ne j \le n$ and $G_{ii}(L) = g(W_i)$ for $1 \le i \le n$. The matrix G can be used to compute both the signature, $\sigma(\mathcal{L})$ and determinant $\det(\mathcal{L})$ (where the right-handed trefoil is taken to have $\sigma = -2$) using

(1) the *Gordon–Litherland formula* $\sigma(\mathcal{L}) = \operatorname{sign}(G(L)) - \mu(L)$,

(2) $\det(\mathcal{L}) = |\det(G(L))|$.

If L is a connected, reduced alternating diagram, then there is a simpler formulation. Checkerboard-color the diagram so that every crossing is incompatible with the black regions. Then $\sigma(\mathcal{L}) = n - n_+$, where n is as above.

8.2 Computing the Euler characteristic

Theorem 8.1 *For $\mathcal{L}$, an oriented link in S^3, let*

$$P(\delta) = \sum_{i \in \mathbb{Z}} \operatorname{rk}_{\mathbb{F}_L}(\mathrm{HT}^i(\mathcal{L}))\delta^i.$$

Then $\det(\mathcal{L}) = |P(i)|$, *where* $i = \sqrt{-1}$.

Proof Let L be a diagram for $\mathcal{L}$, checkerboard color the faces of L, and let

$$R(\delta) = \sum_{i \in \mathbb{Z}} \operatorname{rk}_{\mathbb{F}_L} \mathrm{CT}_{i+n_+(L)}(L)\delta^i$$

be the Poincaré polynomial for the (shifted) chain groups. Then $|R(i)| = |P(i)|$ and the Euler characteristic can be determined from the chain groups. This is a sum over all the spanning trees for the white Tait graph W_L.

Let V_L and E_L be the number of vertices and edges in W_L. The Kirchoff matrix–tree theorem (see Kauffman [7, page 129]) can be used to compute $\det(L)$ from the spanning trees of W_L. We will work in the ring $\mathbb{Z}[\delta^{\pm 1}]$. Label each edge in W_L with $\kappa(c) = +1$ if $\mu(c) = +1$ for the corresponding crossing c in L. Label the edge with $\kappa(c) = \delta$ if $\mu(c) = -1$ for the corresponding crossing. To each spanning tree T of W_L, let w_T be the product of the labels attached to those edges in T. Then the matrix–tree theorem asserts that

$$\sum_{T \in \mathrm{trees}(W_L)} w_T = \det[A]_{11},$$

where $[A]_{11}$ is the $(1,1)$–minor of the matrix $V_L \times V_L$ formed by the matrix A using the elements

- $A_{ij} = -\sum_{c \in W_i \cap W_j} \kappa(c)$, the sum of labels of edges between vertices i and j,
- $A_{ii} = -\sum_{j \neq i} A_{ij}$.

Call the resulting polynomial $Q(\delta)$.

If we specify $\delta \to 1$ then the determinant above computes the number of maximal spanning trees for W_L [7]. However, if we set $\delta = -1$, then the matrix A is the Göritz matrix, which has determinant, up to sign, equal to $\det(\mathcal{L})$. Consequently, $|Q(-1)| = \det(L)$. We now relate the polynomial $Q(\delta)$ to the polynomial $R(\delta)$.

First, we gather some statistics for W_L, the white Tait graph for L. We let

- n_0 be the number of crossings where $\mu(c) = -1$,
- $\widetilde{e}(L)$ equal $n_0 - V_L + 1$,
- $\nu(L)$ equal $\widetilde{e}(L) - n_+(L)$.

For a spanning tree T in W_L, let k be the number of edges in T corresponding to $\mu = +1$ crossings. In $Q(\delta)$ the tree T contributes δ^k. We relate this ti the δ–grading of the generator in $\mathrm{CT}^*(L)$. First, these crossings are the ones which must be resolved using a 1–resolution in the single circle resolution corresponding to T. However, to obtain $\delta(T)$ we must also count the number of 1–resolutions on edges not in T. There are $V_L - 1$ edges in T, and thus $(V_L - 1) - k$ edges in T which correspond to $\mu = -1$ crossings. Outside of T there are $n_0 - (V_L - 1) + k$ edges corresponding to $\mu = -1$ crossings. Since these edges are not included in the tree T, the corresponding crossing must use a 1 resolution in the single circle resolution determined by T. Thus, T contributes $\delta^k \cdot \delta^{n_0 - V_L + 1 + k} = \delta^{2k} \cdot \delta^{\widetilde{e}(L)}$ to the polynomial $\sum \mathrm{rk}\, \mathrm{CT}_i(L)\delta^i$ for the unshifted complex. Consequently, this polynomial equals $\delta^{\widetilde{e}(L)} Q(\delta^2)$. Shifting alters the powers of δ by multiplying by $\delta^{-n_+(L)}$, and thus we obtain the polynomial $R(\delta)$ for the shifted complex:

$$R(\delta) = \delta^{\nu(L)} Q(\delta^2).$$

We now plug in $\delta = i$ and take the complex modulus. This produces $|R(i)| = |(i)^{\mu(L)} Q(-1)| = \det(\mathcal{L})$. The conclusion then follows from $|R(i)| = |P(i)|$. □

9 The skein exact sequence

9.1 A long exact sequence

Let $c \in \mathrm{CR}(L)$ be as depicted in Figure 14. We have labeled the regions abutting this crossing x_1, x_2, x_3 and x_4. A priori some of these could be equal: x_1 could equal x_3

or x_2 could equal x_4. If so, one of the resolutions of L at this crossing will result in a disconnected diagram. For the moment, we require that all four regions be distinct, ie that both the 0–resolution and the 1–resolution at c result in connected link diagrams.

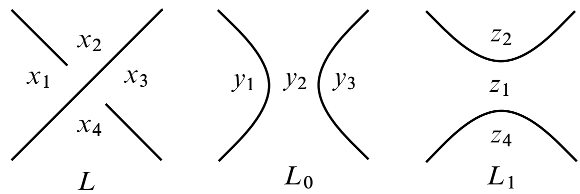


Figure 14

Theorem 9.1 *Let L_0 and L_1 be the diagrams found by resolving L using the $c \notin S$ and $c \in S$ rules, respectively. Then*

$$\mathrm{CT}^*(L) \cong \mathrm{MC}\Big(\mathrm{CT}^*(L_0) \otimes_{\mathbb{F}_{L_0}} \mathbb{F}_L \xrightarrow{\tau_c} \mathrm{CT}^*(L_1) \otimes_{\mathbb{F}_{L_1}} \mathbb{F}_L\Big),$$

where:

- *$\mathbb{F}_{L_0}$ acts on $\mathbb{F}_L$ by $y_1 \to x_1$, $y_3 \to x_3$, and $y_2 \to x_2 + x_4$ and $\mathbb{F}_{L_1}$ acts by $z_1 \to x_1 + x_3$, $z_2 \to x_2$, and $z_4 \to x_4$.*
- *If $T \in O_i(L_0)$ then*

$$\tau_c(T) = \sum_{T' \in O_{i+1}(L_1)} \langle \overline{T}, \overline{T}' \rangle_{\mathbb{F}_L} T',$$

where T' is a single circle resolution of L_1, $\overline{T}$ and $\overline{T}'$ are the corresponding single circle resolutions in $O(L)$, and $\overline{T}'$ differs from $\overline{T}$ at c and one other crossing.

Proof $\mathrm{CT}^*(L)$ can be decomposed along those resolutions $\overline{T}$ with $c \notin \overline{T}$ and $c \in \overline{T}$, $\mathrm{CT}^*(L) \cong C_0^i \oplus C_1^i$. Those $\overline{T}$ with $c \notin \overline{T}$ are in one-to-one correspondence with the single circle resolutions T of L_0. Likewise, those $\overline{T}'$ with $c \in \overline{T}'$ provide single circle resolutions T' for L_1. If we consider ∂_L we can decompose into three parts $\partial_0 \oplus \partial_{01} \oplus \partial_1$, where ∂_0 counts those pairs $\overline{T} \to \overline{T}'$ which have $c \notin \overline{T}, \overline{T}'$, ∂_1 counts those pairs $\overline{T} \to \overline{T}'$ which have $c \in \overline{T}, \overline{T}'$ and ∂_{01} counts pairs $\overline{T} \to \overline{T}'$ which have $c \notin \overline{T}$ but $c \in \overline{T}'$. Then $(C_0, \partial_0) \cong \mathrm{CT}^*(L_0) \otimes_{\mathbb{F}_{L_0}} \mathbb{F}_L$, where the tensor product arises because two of the formal variables for L, x_2, x_4 occur in the same region in L_0. Using the dissection results, we see that this only changes the coefficient field. Likewise, $(C_1, \partial_1) \cong \mathrm{CT}^*(L_1) \otimes_{\mathbb{F}_{L_1}} \mathbb{F}_L$. The map τ_c comes from ∂_{01}, and its form is readily described from that of the differential, ∂_L. Lastly, we verify the shift in gradings: note that $C_0 \cong \mathrm{CT}^*(L_0)$ with no shift, since $c \notin \overline{T}$ has no effect on δ. However, $C_1 \cong \mathrm{CT}^*(L_1)[1]$ since $c \in \overline{T}$ implies that the grading on $\mathrm{CT}^*(L_1)$, where no resolution occurs, will be shifted up when considered in $\mathrm{CT}^*(L)$. Namely, $C_1^i \cong \mathrm{CT}^{i-1}(L_1)$. Since ∂_L increases δ by 2, this is the correct shift for a mapping cone. □

Proposition 9.2 *Given a crossing c in a link diagram L, there is a long exact sequence*

$$(5)\quad \cdots \to \mathrm{HT}_u^{i-1}(L_1)\otimes\mathbb{F}_L \to \mathrm{HT}_u^i(L) \to \mathrm{HT}_u^i(L_0)\otimes\mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}_u^{i+1}(L_1)\otimes\mathbb{F}_L \to \cdots .$$

When L is oriented and c is a positive crossing, then if $e = n_+(L) - n_+(L_1)$ (for any orientation on L_1), then

$$(6)\quad \cdots \to \mathrm{HT}^{i+e-1}(\mathcal{L}_1)\otimes\mathbb{F}_L \to \mathrm{HT}^i(\mathcal{L}) \to \mathrm{HT}^{i+1}(\mathcal{L}_0)\otimes\mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}^{i+e+1}(\mathcal{L}_1)\otimes\mathbb{F}_L \to \cdots .$$

On the other hand, if c is negative, let $f = n_+(L) - n_+(L_0)$. Then

$$(7)\quad \cdots \to \mathrm{HT}^{i-1}(\mathcal{L}_1)\otimes\mathbb{F}_L \to \mathrm{HT}^i(\mathcal{L}) \to \mathrm{HT}^{i+f}(\mathcal{L}_0)\otimes\mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}^{i+1}(\mathcal{L}_1)\otimes\mathbb{F}_L \to \cdots .$$

Proof The exact sequence (5) is an immediate consequence of the description of $\mathrm{CT}^*(L)$ as a mapping cone, using standard homological algebra. To verify (6) assume that c is positive. We will adjust subscripts to account for the shifting of $\mathrm{HT}_u^*(L)$ by $[-n_+(L_+)]$:

$$\cdots \to \mathrm{HT}_u^{i+n_+-1}(L_1)\otimes\mathbb{F}_L \to \mathrm{HT}_u^{i+n_+}(L) \to \mathrm{HT}_u^{i+n_+}(L_0)\otimes\mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}_u^{i+n_++1}(L_1)\otimes\mathbb{F}_L \to \cdots ,$$

$$\cdots \to \mathrm{HT}_u^{i+n_+(L_1)+e-1}(L_1)\otimes\mathbb{F}_L \to \mathrm{HT}^i(\mathcal{L}) \to \mathrm{HT}_u^{i+n_+(L_0)+1}(L_0)\otimes\mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}_u^{i+n_+(L_1)+e+1}(L_1)\otimes\mathbb{F}_L \to \cdots ,$$

$$\cdots \to \mathrm{HT}^{i+e-1}(\mathcal{L}_1)\otimes\mathbb{F}_L \to \mathrm{HT}^i(\mathcal{L}) \to \mathrm{HT}^{i+1}(\mathcal{L}_0)\otimes\mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}^{i+e+1}(\mathcal{L}_1)\otimes\mathbb{F}_L \to \cdots .$$

When c is negative, we proceed as before:

$$\cdots \to \mathrm{HT}_u^{i+n_+-1}(L_1)\otimes\mathbb{F}_L \to \mathrm{HT}_u^{i+n_+}(L) \to \mathrm{HT}_u^{i+n_+}(L_0)\otimes\mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}_u^{i+n_++1}(L_1)\otimes\mathbb{F}_L \to \cdots .$$

But now $n_+(L) = n_+(L_1)$ since we resolve a negative crossing. Furthermore, if we orient L_0 we may compute $f = n_+(L) - n_+(L_0)$ and

$$\cdots \to \mathrm{HT}_u^{i+n_+(L_1)-1}(L_1) \otimes \mathbb{F}_L \to \mathrm{HT}^i(\mathcal{L}) \to \mathrm{HT}_u^{i+n_+(L_0)+f}(L_0) \otimes \mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}_u^{i+n_+(L_1)+1}(L_1) \otimes \mathbb{F}_L \to \cdots,$$

$$\cdots \to \mathrm{HT}^{i-1}(\mathcal{L}_1) \otimes \mathbb{F}_L \to \mathrm{HT}^i(\mathcal{L}) \to \mathrm{HT}^{i+f}(\mathcal{L}_0) \otimes \mathbb{F}_L \xrightarrow{\tau_{c,*}} \mathrm{HT}^{i+1}(\mathcal{L}_1) \otimes \mathbb{F}_L \to \cdots.$$

This concludes the proof. □

9.2 Quasialternating links

The skein exact sequence allows us to compute the spanning tree homology of quasialternating links as in [9]. Recall that a link $\mathcal{L}$ is called quasialternating if it is in the set $\mathcal{Q}$, the smallest set of links such that:

- The unknot is in $\mathcal{Q}$.
- If $\mathcal{L}$ has a diagram L containing a crossing c such that the two resolutions at c, L_0 and L_1 represent links $\mathcal{L}_0, \mathcal{L}_1 \in \mathcal{Q}$ with $\det(\mathcal{L}) = \det(\mathcal{L}_0) + \det(\mathcal{L}_1)$, then $L \in \mathcal{Q}$.

Alternating links are quasialternating, and $\det(\mathcal{L}) > 0$ when $\mathcal{L}$ is quasialternating.

Theorem 9.3 *If $\mathcal{L}$ is a quasialternating link with a connected diagram,* $\mathrm{HT}^i(\mathcal{L}) \cong 0$ *when $i \neq \sigma(L)$ and has rank* $\det(L)$ *when $i = \sigma(L)$.*

Proof In [9], C Manolescu and P Ozsváth prove the identical result for Khovanov homology using the following:

Lemma 9.4 (Manolescu and Ozsváth) *Suppose that* $\det(\mathcal{L}_v)$, $\det(\mathcal{L}_h) > 0$ *and* $\det(\mathcal{L}_+) = \det(\mathcal{L}_v) + \det(\mathcal{L}_h)$. *Then*

$$\sigma(\mathcal{L}_v) - \sigma(\mathcal{L}_+) = 1,$$
$$\sigma(\mathcal{L}_h) - \sigma(\mathcal{L}_+) = e',$$

where $e' = n_-(L_h) - n_-(L_+)$.

Exactly the same argument proves the result for the spanning tree homology. □

This result is especially simple for links admitting alternating, connected diagrams. Pick a nonsplit, reduced diagram L for the link. Checkerboard-color the faces of L so that the white regions are joined at a crossing c in any resolution S, where $c \in S$. Suppose there are $n_W + 1$ white faces. Thus, for each tree T representing a generator for $\mathrm{CT}^*(L)$ we have $\delta(T) = n_W$ since there are $(n_W + 1) - 1$ edges in any spanning tree of W_L, and each edge in T receives a 1 resolution while each edge not in T receives a 0 resolution. Consequently, every tree T occurs in a single grading, and $\partial_L = 0$. Therefore, the homology is supported in only one grading, and has rank equal to the number of spanning trees for W_L. It is well known that the number of spanning trees of W_L is equal to $\det(L)$. Furthermore, by the result of Gordon and Litherland, we may compute the signature of $\mathcal{L}$ by $\sigma(\mathcal{L}) = n_W - n_+(L) = \delta(T) - n_+(L)$. This is the grading for T in $\mathrm{HT}^*(\mathcal{L})$. Consequently, all the homology is in the grading given by $\sigma(\mathcal{L})$.

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