

Cylinder deformations in orbit closures of translation surfaces

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Let M be a translation surface. We show that certain deformations of M supported on the set of all cylinders in a given direction remain in the $\mathrm{GL}^+(2, \mathbb{R})$ -orbit closure of M . Applications are given concerning complete periodicity, affine field of definition and the number of parallel cylinders which may be found on a translation surface in a given orbit closure.

32G15, 37D40

1 Introduction

Context We demonstrate new and direct connections between the orbit closure of a translation surface and the flat geometry of cylinders on this translation surface.

Several celebrated results already draw connections between the orbit closure of a translation surface and the cylinders on the translation surface:

- The Siegel–Veech formula establishes such a connection asymptotically in counting problems; see Veech [26] and Eskin and Masur [5].
- The Veech dichotomy shows in particular that for translation surfaces in closed orbits, a very strong form of complete periodicity holds [25].
- Work of Calta [4] and also very recent work of Lanneau and Nguyen [13] show that in certain special orbit closures in low genus, every translation surface is completely periodic; see also McMullen [17].

Furthermore, the work of Smillie and Weiss [23] gives that every *horocycle flow* orbit closure contains a periodic translation surface (**Theorem 5.2** below).

Main result The horocycle flow is defined as part of the $\mathrm{GL}^+(2, \mathbb{R})$ -action,

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \subset \mathrm{GL}^+(2, \mathbb{R}).$$

We will also be interested in the vertical stretch,

$$a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \subset \mathrm{GL}^+(2, \mathbb{R}).$$

Given a collection \mathcal{C} of horizontal cylinders on a translation surface M , we define $u_t^{\mathcal{C}}(M)$ to be the translation surface obtained by applying the horocycle flow to the cylinders in \mathcal{C} but not to the rest of M . This deformation of M can be understood very concretely by expressing M as a collection of polygons, including a rectangle for each cylinder in \mathcal{C} , with parallel edge identifications. In this case, $u_t^{\mathcal{C}}(M)$ is obtained by letting u_t act linearly on the rectangles which give cylinders in \mathcal{C} but not on the remaining polygons, and then regluing.

We call $u_t^{\mathcal{C}}(M)$ the *cylinder shear*. Similarly we define the *cylinder stretch* $a_t^{\mathcal{C}}(M)$ by applying a_t only to the cylinders in \mathcal{C} . Both cylinder shear and stretch depend on the choice of a set \mathcal{C} of horizontal cylinders. Our main result is the following theorem.

Theorem 1.1 (The cylinder deformation theorem) *Let M be a translation surface, and let \mathcal{C} be the collection of all horizontal cylinders on M . Then for all $s, t \in \mathbb{R}$, the surface $a_s^{\mathcal{C}}(u_t^{\mathcal{C}}(M))$ remains in the $\mathrm{GL}^+(2, \mathbb{R})$ -orbit closure of M .*

We will in fact show that these *cylinder deformations* remain in any affine invariant submanifold containing M . The key tool is the result of Smillie and Weiss [23] mentioned above. Then we appeal to a recent theorem of Eskin, Mirzakhani and Mohammadi which shows that the $\mathrm{GL}^+(2, \mathbb{R})$ -orbit closure of every translation surface is an *affine invariant submanifold* [6; 7], thus concluding the proof of [Theorem 1.1](#).

A more powerful version of the cylinder deformation theorem, important for applications but more cumbersome to state, is given in [Section 5](#).

The idea of the cylinder deformation theorem was an outgrowth of conversations with Alex Eskin on the conjecture that affine invariant submanifolds are quasiprojective varieties.¹ This conjecture restricts the types of linear equations in period coordinates that may define an affine invariant submanifold in a way consistent with the cylinder deformation theorem. The author is profoundly grateful and deeply indebted to both Alex Eskin and Martin Möller for many helpful conversations on this conjecture.

¹Added in proof: This conjecture has been recently established by Filip [8; 9]

Remark 1.2 In genus 2, it is possible to recover [Theorem 1.1](#) from McMullen's classification of orbit closures [19].

Remark 1.3 Other interesting cylinder deformations can often be shown to remain in the orbit closure (see [Corollary 3.4](#) and [Theorem 5.1](#)). However, if M has several horizontal cylinders, it is not always true that the surface obtained by shearing or stretching just one cylinder, or by shearing or stretching the different cylinders different amounts, will stay in the orbit closure. For example if M is a lattice surface with more than one horizontal cylinder, then the surface obtained by shearing just one horizontal cylinder will not remain in the orbit closure of M (which is equal to the orbit of M in this case).

Remark 1.4 There is no known algorithm to find all the cylinders in a given direction on a translation surface. Nonetheless, Alex Eskin has pointed out that the cylinder deformation theorem might allow rigorous computer-aided proofs that many translation surfaces have large orbit closures. Such a computer program might look for directions where all the cylinders can be determined (for example directions in which the translation surface is composed of cylinders and tori) and then use the cylinder deformation theorem to produce vectors in the tangent space.

Complete periodicity Over any affine invariant submanifold \mathcal{M} , there is a natural bundle H_{rel}^1 whose fiber over a translation surface M is $H^1(M, \Sigma; \mathbb{C})$ (the cohomology of M rel Σ), where Σ is the set of singularities of M . The tangent bundle $T(\mathcal{M})$ of \mathcal{M} is naturally a flat subbundle of H_{rel}^1 . There is a natural map $p: H^1(M, \Sigma; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})$.

A translation surface which is the union of cylinders in some direction is said to be *periodic* in that direction. A translation surface is said to be *completely periodic* if whenever there is a cylinder in some direction, then that direction is periodic.

Theorem 1.5 *If $\dim_{\mathbb{C}} p(T(\mathcal{M})) = 2$, then every translation surface in \mathcal{M} is completely periodic.*

Corollary 1.6 *All translation surfaces in all the Prym eigenform loci are completely periodic.*

Barak Weiss has pointed out to the author that the proof of [Theorem 1.5](#), and hence also [Corollary 1.6](#), presented below does not logically depend on the recent work of Eskin, Mirzakhani and Mohammadi. In the case of Prym eigenform loci of dimension 5, [Corollary 1.6](#) is due to Lanneau and Nguyen [13]; in the genus-2 case it is

due to Calta [4], see also McMullen [17]; and for closed orbits it is part of the Veech dichotomy [25].

Lanneau has communicated to the author that there is a completely periodic translation surface M in $\mathcal{H}(4)^{\text{hyp}}$ which is not a lattice surface. Since this M is not a lattice surface, its orbit closure \mathcal{M} satisfies $\dim_{\mathbb{C}} T(\mathcal{M}) > 2$. In the minimal stratum the map p is an isomorphism (there are no relative periods), so for the example in question $\dim_{\mathbb{C}} p(T(\mathcal{M})) = \dim_{\mathbb{C}} T(\mathcal{M}) > 2$.

Lanneau's example shows that there are completely periodic translation surfaces whose orbit closures have $\dim_{\mathbb{C}} p(T(\mathcal{M})) > 2$. Thus the strongest converse to [Theorem 1.5](#) which may be hoped for is the following:

Theorem 1.7 *If $\dim_{\mathbb{C}} p(T(\mathcal{M})) > 2$, then there exist translation surfaces in \mathcal{M} that are not completely periodic.*

There are many examples of orbit closures \mathcal{M} with $\dim_{\mathbb{C}} p(T(\mathcal{M})) = 2$ coming from Teichmüller curves (closed $\text{SL}(2, \mathbb{R})$ -orbits) and ramified covering constructions. The first examples not of this form were discovered in genus 2 independently by McMullen [15] and, from a different perspective, Calta [4]. McMullen later generalized his approach to provide examples in genus 3, 4 and 5 [18], giving the Prym eigenform loci mentioned above. There are no other currently known affine invariant submanifolds \mathcal{M} with $\dim_{\mathbb{C}} p(T(\mathcal{M})) = 2$.

It is notable that [Theorem 1.5](#) holds in all genera, whereas the methods of Calta, Lanneau and Nguyen, and the work of McMullen on flux [16], exploit low-genus phenomena. (McMullen has pointed out to the author that some of his methods do extend to higher genus, see for example [20, Theorem 1.1].)

On the one hand, this may be an indication that no high-genus affine invariant submanifolds \mathcal{M} with $\dim_{\mathbb{C}} p(T(\mathcal{M})) = 2$ exist that do not arise from covering constructions. On the other hand, Mirzakhani conjectures (see Wright [29]) that any orbit closure \mathcal{M} that does not arise from a covering construction must satisfy $\dim_{\mathbb{C}} p(T(\mathcal{M})) = 2$, and so if any genuinely new orbit closures are found, we expect them to have this property.²

Affine field of definition The affine field of definition of an affine invariant submanifold \mathcal{M} is the smallest subfield of \mathbb{R} such that \mathcal{M} can be defined in local period coordinates by linear equations with coefficients in this field. All linear equations in this paper will be assumed to be homogeneous (that is, have zero constant term). The affine field of definition of \mathcal{M} is denoted by $k(\mathcal{M})$.

²Added in proof: Mirzakhani and the author have found a counterexample to this conjecture.

Theorem 1.8 (Wright [29, Theorem 1.1]) *The affine field of definition of any affine invariant submanifold \mathcal{M} is a number field of degree at most the genus. Moreover, it can be computed as the intersection of the holonomy fields of translation surfaces in \mathcal{M} .*

Here we present a different way of controlling the affine field of definition, in terms of only part of the flat geometry of a single translation surface in \mathcal{M} .

Theorem 1.9 *Suppose that M is a translation surface in an affine invariant submanifold \mathcal{M} , and suppose M has at least one horizontal cylinder. (M need not be horizontally periodic.) Let $\{c_1, \dots, c_n\}$ be the set of circumferences of horizontal cylinders on M . Then if $n > 1$, $k(\mathcal{M}) \subset \mathbb{Q}[c_2/c_1, \dots, c_n/c_1]$, and if $n = 1$, $k(\mathcal{M}) = \mathbb{Q}$.*

We wish to emphasize the $n = 1$ case: if a translation surface has a direction with exactly one cylinder, then the affine field of definition of its orbit closure is \mathbb{Q} .

[Theorem 1.9](#) is a special case of [Theorem 7.1](#), which in particular additionally asserts the equality $k(\mathcal{M}) = \mathbb{Q}[c_2/c_1, \dots, c_n/c_1]$ if the horizontal cylinders of M remain parallel on nearby translation surfaces in \mathcal{M} .

Finding many cylinders Let $T^M(\mathcal{M})$ denote the tangent space to \mathcal{M} at the point $M \in \mathcal{M}$, and denote by $T^M(\mathcal{M})^*$ the space of linear functionals on $T^M(\mathcal{M})$. Any absolute or relative homology class α on M naturally defines a linear functional $\alpha^* \in T^M(\mathcal{M})^*$, since $T^M(\mathcal{M}) \subset H^1(M, \Sigma; \mathbb{C}) = H_1(M, \Sigma; \mathbb{C})^*$. Continuing our effort to understand the connection between orbit closures and flat geometry, we prove:

Theorem 1.10 *Let \mathcal{M} be an affine invariant submanifold, and set*

$$k = \frac{1}{2} \dim_{\mathbb{C}} p(T(\mathcal{M})).$$

Then there is some horizontally periodic translation surface $M \in \mathcal{M}$ whose horizontal core curves span a subspace of $T^M(\mathcal{M})^$ of dimension k . No set of core curves of parallel cylinders on a translation surface $M \in \mathcal{M}$ may span a subspace of $T^M(\mathcal{M})^*$ of dimension greater than k .*

There is a horizontally periodic translation surface in \mathcal{M} with at least

$$\dim_{\mathbb{C}} T(\mathcal{M}) - k \geq k$$

horizontal cylinders.

The implied statement that $\dim_{\mathbb{C}} p(T(\mathcal{M}))$ is even is given by Avila, Eskin and Möller's result that $p(T(\mathcal{M}))$ is symplectic [2]. This is a key tool in the proof of [Theorem 1.10](#).

[Theorem 1.10](#) motivates the following definition.

Definition 1.11 The *cylinder rank* of an affine invariant submanifold \mathcal{M} is given by $\frac{1}{2} \dim_{\mathbb{C}} p(T(\mathcal{M}))$.

We believe cylinder rank may be the most important numerical invariant of an affine invariant submanifold. [Theorem 1.5](#) shows that complete periodicity prevails in rank 1 but not in higher rank.

Background For an introduction to translation surfaces, see the surveys by Masur and Tabachnikov [\[14\]](#) and Zorich [\[30\]](#).

There are many examples of affine invariant submanifolds that arise from ramified covering constructions. Examples not coming from covering constructions however are rare. Besides the examples mentioned above (following [Theorem 1.7](#)), the only additional currently known examples are closed orbits: the Veech–Ward–Bouw–Möller curves (see Veech [\[25\]](#), Ward [\[27\]](#), Bouw and Möller [\[3\]](#) and Wright [\[28\]](#)), as well as two sporadic examples due to Vorobets and to Kenyon and Smillie; see Hubert and Schmidt [\[10\]](#), Kenyon and Smillie [\[12\]](#) and also [\[28\]](#) for a more detailed summary.

Organization [Section 2](#) gives some definitions and describes the effect of cylinder shear in period coordinates. [Section 3](#) describes the orbit closure of a cylinder shear. The key tool in the proof of [Theorem 1.1](#) is introduced in [Section 4](#), and the proof is completed in [Section 5](#). [Sections 6, 7, 8 and 9](#) prove [Theorems 1.5, 1.9, 1.10 and 1.7](#), respectively. Two open problems related to our work are listed in [Section 10](#).

Added in proof The cylinder deformation theorem has recently been used to classify higher rank affine invariant submanifolds of $\mathcal{H}(4)$; see Nguyen and Wright [\[22\]](#) and Aulicino, Nguyen and Wright [\[1\]](#).

Acknowledgements The author thanks his thesis advisor Alex Eskin for inspiring this project, as well as Jon Chaika, Simion Filip, David Aulicino, Barak Weiss and John Smillie for enjoyable and helpful conversations on the results of this paper. The author is very grateful to Martin Möller and Maryam Mirzakhani for sharing their insight into orbit closures with him, and to Erwan Laneeau for very helpful communications about complete periodicity in low genus. The author thanks Curtis McMullen and the referee for helpful comments on an earlier draft.

2 Period coordinates

Suppose $g \geq 1$ and let α be a partition of $2g - 2$. The stratum $\mathcal{H}(\alpha)$ is defined to be the set of (X, ω) , where X is a genus- g closed Riemann surface and ω is a holomorphic

1-form on X whose zeroes have multiplicities given by α . For technical reasons, we prefer to work with a finite cover \mathcal{H} of $\mathcal{H}(\alpha)$ that is a manifold instead of an orbifold. Given a translation surface (X, ω) , let $\Sigma \subset X$ denote the set of zeros of ω . (We will also refer to Σ as the set of singularities of (X, ω) , since it is at these points where the flat metric is singular.) We consider the relative cohomology group $H^1(X, \Sigma; \mathbb{Z})$ and also the bundle H_{rel}^1 of relative cohomology, whose fiber over (X, ω) is $H^1(X, \Sigma; \mathbb{C})$. For any neighborhood $\mathcal{U} \subset \mathcal{H}$ over which the bundle H_{rel}^1 is trivializable, we define the local period coordinate $\Phi: \mathcal{U} \rightarrow H^1(X, \Sigma; \mathbb{C}) \simeq \mathbb{C}^m$ by

$$\Phi(X, \omega) = [\omega].$$

Here $[\omega]$ is the relative cohomology class determined by ω , and m is the dimension of relative cohomology.

Definition 2.1 An affine invariant submanifold of \mathcal{H} is a closed connected subset $\mathcal{M} \subset \mathcal{H}$ for which every point in \mathcal{M} has a neighborhood \mathcal{U} as above, satisfying $\Phi(\mathcal{U} \cap \mathcal{M}) = \Phi(\mathcal{U}) \cap (V \otimes \mathbb{C})$, where V is a subspace of $H^1(X, \Sigma; \mathbb{R})$.

Affine invariant submanifolds are indeed invariant under the natural $\text{GL}^+(2, \mathbb{R})$ -action. (Here $\text{GL}^+(2, \mathbb{R})$ denotes the subgroup of $\text{GL}(2, \mathbb{R})$ with positive determinant.)

More concretely, pick a basis $\gamma_1, \dots, \gamma_m$ for $H_1(X, \Sigma; \mathbb{Z})$. The local period coordinates can be more explicitly written using the isomorphism $H_1(X, \Sigma; \mathbb{C}) \simeq \mathbb{C}^m$ defined by this choice of basis:

$$\Phi(X, \omega) = \left(\int_{\gamma_i} \omega \right)_{i=1}^m.$$

An affine invariant submanifold is a submanifold of a stratum defined locally by homogenous linear equations with real coefficients in the period coordinates $\int_{\gamma_i} \omega$.

Theorem 2.2 (Eskin, Mirzakhani and Mohammadi [7, Theorem 2.1]) *The $\text{GL}^+(2, \mathbb{R})$ orbit closure of any translation surface is an affine invariant submanifold.*

In light of this, to establish [Theorem 1.1](#) it suffices to show that if $M = (X, \omega)$ is contained in an affine invariant submanifold \mathcal{M} , then the cylinder shear and stretch remains in \mathcal{M} , and this is what we will do. It will be necessary to understand the effect of cylinder shear and stretch in period coordinates.

Given a collection \mathcal{C} of horizontal cylinders on a translation surface M , we define $\eta_{\mathcal{C}} \in H^1(M, \Sigma; \mathbb{C})$ to be the derivative of the cylinder shear $u_t^{\mathcal{C}}(M)$ in local period coordinates. (In fact, the $\eta_{\mathcal{C}}$ we define will be real: $\eta_{\mathcal{C}} \in H^1(M, \Sigma; \mathbb{R})$.) That is,

$$\eta_{\mathcal{C}} = \frac{d}{dt} \Big|_{t=0} \Phi(u_t^{\mathcal{C}}(M)),$$

where Φ is a local period coordinate on a neighborhood of M . It is clear from the definition that $\eta_{\mathcal{C}}$ is zero on any relative cycle not intersecting any of the cylinders in \mathcal{C} , as well as on the core curves of the cylinders of \mathcal{C} ; but on any relative cycle joining a zero on the bottom edge of a cylinder in \mathcal{C} to a zero on the top edge of this cylinder, $\eta_{\mathcal{C}}$ is equal to the height of the cylinder. Relative cycles of these three types span $H_1(X, \Sigma; \mathbb{Z})$, so this is a complete description of $\eta_{\mathcal{C}}$.

Lemma 2.3 *The deformation $u_t^{\mathcal{C}}(M)$ is linear in local period coordinates,*

$$\Phi(u_t^{\mathcal{C}}(M)) = \Phi(M) + t\eta_{\mathcal{C}},$$

for t small enough. *The cylinder stretch in local period coordinates is*

$$\Phi(a_t^{\mathcal{C}}(M)) = \Phi(M) + (e^t - 1)i\eta_{\mathcal{C}}.$$

Proof First set $(X_t, \omega_t) = u_t^{\mathcal{C}}(M)$. It suffices to show that

$$\int_{\gamma} \omega_t = \int_{\gamma} \omega + t\eta_{\mathcal{C}}(\gamma)$$

for any set of γ which spans $H_1(M, \Sigma; \mathbb{Z})$. However, this is clear for the three types of relative cycles described above. For γ supported off the cylinders of \mathcal{C} or a core curve of a cylinder in \mathcal{C} , it is clear that $\int_{\gamma} \omega_t$ is constant. For γ a relative cycle joining a zero on the bottom edge of a cylinder in \mathcal{C} to a zero on the top edge of this cylinder, the imaginary part of $\int_{\gamma} \omega_t$ remains constant and the real part increases linearly in proportion to the height of the cylinder.

The proof is extremely similar for the cylinder stretch $a_t^{\mathcal{C}}(M)$. □

Note that our discussion also shows the following, which will be crucial in the sequel:

Lemma 2.4 *Suppose that \mathcal{C} consists of cylinders C_1, \dots, C_r , with core curves $\alpha_1, \dots, \alpha_r$ and heights $h_1, \dots, h_r \in \mathbb{R}$. Let $I_{\alpha_i} \in H^1(M, \Sigma; \mathbb{Z})$ be the cohomology class defined to be zero on all relative homology classes that can be realized disjointly from the interior of C_i , and defined to be one on a relative cycle joining a zero on the bottom edge of C_i to a zero on the top edge of this cylinder. Then*

$$\eta_{\mathcal{C}} = \sum_{i=1}^r h_i I_{\alpha_i}.$$

Proof Again this is verified on the above-described spanning set for the relative homology. □

Remark 2.5 A more coordinate-free approach could be taken above: $\eta_{\mathcal{C}}$ is the cohomology class of a (discontinuous) one-form supported on the union of the cylinders in \mathcal{C} . And I_{α} is dual to α , under the Poincaré duality isomorphism $H_1(X \setminus \Sigma; \mathbb{Z}) \rightarrow H^1(X, \Sigma; \mathbb{Z})$. This approach has significant advantages. We have written things in coordinates because that was how we first thought of them.

3 The orbit closure of a cylinder shear

Part of the intuition for [Theorem 1.1](#) comes from the following. Given two sets of real numbers $S, S' \subset \mathbb{R}$, we say they are *independent* (over \mathbb{Q}) if $\text{span}_{\mathbb{Q}} S \cap \text{span}_{\mathbb{Q}} S' = \{0\}$.

Lemma 3.1 *Let M be a horizontally periodic translation surface, and let \mathcal{C}_1 be a collection of horizontal cylinders on M such that the set of moduli of the cylinders in \mathcal{C}_1 is independent from the set of moduli of the remaining horizontal cylinders on M . Then for all t , the cylinder shear $u_t^{\mathcal{C}_1}(M)$ remains in the u_t -orbit closure of M .*

The proof is standard, but is included here for completeness.

Given a collection of cylinders $\mathcal{C} = \{C_1, \dots, C_r\}$, we can also define the cylinder deformation $u_{t_1, \dots, t_r}^{\mathcal{C}}(M)$ that applies u_{t_i} to C_i for $i = 1, \dots, r$ and does nothing to the rest of the translation surface. Suppose that the height of C_i is h_i and the circumference is c_i .

Lemma 3.2 $u_t^{\mathcal{C}}(M) = u_{t_1, \dots, t_r}^{\mathcal{C}}(M)$, where $t_i = t \bmod c_i / h_i$.

Here we use the notation $x \bmod y$ to denote the unique number $x' \in [0, y)$ that differs from x by an integral multiple of y . The proof is left to the reader, and consists only of the observation that applying u_{c_i/h_i} to a cylinder of circumference c_i and height h_i is equivalent to applying an element of the mapping class group: it is a full Dehn twist, and returns the original translation surface.

Lemma 3.3 *Consider the flow $f_t(v) = v + t(1, \dots, 1)$ on the torus*

$$\mathbb{T}^r = [0, c_1/h_1) \times \dots \times [0, c_r/h_r).$$

The orbit closure of v is the set of all $v + (t_1/m_1, \dots, t_r/m_r)$, where the t_i satisfy all homogeneous linear relations with rational coefficients that are satisfied by the $m_i = h_i/c_i$.

The condition is that whenever $\sum_{i=1}^r q_i m_i = 0$ with all $q_i \in \mathbb{Q}$, then $\sum_{i=1}^r q_i t_i = 0$ also. (We remind the reader that all linear equations in this paper are homogeneous unless otherwise specified.)

Proof Let $\mathbb{T}_0 = [0, 1)^r$ be the standard torus, and define $\Psi: \mathbb{T} \rightarrow \mathbb{T}_0$ by

$$\Psi(v_i)_{i=1}^r = (v_i m_i)_{i=1}^k.$$

Define the flow $g_t(w) = w + t(m_1, \dots, m_r)$ on \mathbb{T}_0 . We see that $\Phi(f_t(v)) = g_t(\Psi(v))$. It is standard that the g_t -orbit closure of any $w \in \mathbb{T}_0$ is the smallest subtorus of \mathbb{T}_0 containing (m_1, \dots, m_r) , translated by w ; see for example [11, page 33]. That is, the orbit closure is the set of all $w + (t_1, \dots, t_r)$, where the t_i satisfy all rational homogeneous linear equations that the m_i do.

Moving back to \mathbb{T} using Ψ^{-1} , we see that the orbit closure of $v \in \mathbb{T}$ is the set of all $v + (t_1/m_1, \dots, t_r/m_r)$, as claimed. \square

Corollary 3.4 *The u_t^C -orbit closure of M is equal to the set of*

$$u_{t_1/m_1, \dots, t_r/m_r}^C(M),$$

where the t_i satisfy all homogeneous linear relations with rational coefficients that the $m_i = h_i/c_i$ do. In particular, if \mathcal{C} is the disjoint union of two collections of cylinders \mathcal{C}_1 and \mathcal{C}_2 whose sets of moduli are independent, then $u_t^{C_j}(M)$ is in the u_t^C -orbit closure of M , for all t and $j = 1, 2$.

Proof The first statement follows directly from the previous two lemmas. For the second statement, assume that $\mathcal{C}_1 = \{C_1, \dots, C_p\}$, and set $t_i = m_i$ for $i \leq p$ and $t_i = 0$ otherwise. This choice of t_i satisfies all rational linear equations that the m_i do, so by the first statement $u_t^{C_1}(M) = u_{tt_1/m_1, \dots, tt_r/m_r}^C(M)$ is in the u_t^C -orbit closure of M . This proves the second statement for $j = 1$, and the $j = 2$ case is identical. \square

Proof of Lemma 3.1 Set \mathcal{C} to be the collection of all horizontal cylinders on M , so $u_t = u_t^C$. Set $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. Then by the previous corollary, $u_t^{C_1}(M)$ is in the u_t -orbit closure of M . \square

4 Real deformations

Let M be a translation surface and \mathcal{U} be a neighborhood of M on which local period coordinates Φ are defined.

Definition 4.1 A real deformation of M is any $M' \in \mathcal{U}$ such that $\Phi(M') - \Phi(M) \in H^1(M, \Sigma; \mathbb{R})$. All real deformations can be obtained as $\Phi^{-1}(\Phi(M) + \zeta)$, where $\zeta \in H^1(M, \Sigma; \mathbb{R})$ is sufficiently small.

Remark 4.2 Fix a translation surface M . There is a small simply connected neighborhood of M in the stratum in which every translation surface is *marked* by a homeomorphism to M rel singularities that is unique up to isotopy rel singularities. Any curve on M which is either closed or joins a pair of singularities can thus be parallel transported to translation surfaces M' in the neighborhood, and the result is unique up to isotopy rel singularities.

Now fix a finite set of saddle connections on M . By decreasing the size of the neighborhood of M , we may assume that on each M' in the neighborhood, the isotopy class of the curve representing each saddle connection in the finite set is still represented by a saddle connection. The same discussion applies to flat regular geodesics on M (ie core curves of cylinders).

In this way we may make precise statements such as “a cylinder on M persists at nearby translation surfaces,” and we may speak of this cylinder on translation surfaces sufficiently close to M . However, having now indicated the relevant standard techniques, we will omit technical details about markings in the remainder of this article. Whenever we speak of deformations of a translation surface M , or surfaces “nearby” M , it is implicit that there exists a small neighborhood of M on which the statements are true.

Lemma 4.3 *The horizontal cylinders on M persist under any real deformation and maintain constant height.*

Since horizontal cylinders persist, given a collection \mathcal{C} of horizontal cylinders on M , we will also speak of this collection of cylinders on any real deformation of M .

Proof Consider a path of real deformations $\Phi^{-1}(\Phi(M) + t\zeta)$, where t is sufficiently small. Along this path, the imaginary part of the holonomy of every saddle connection does not change. Any cylinder on M is bounded by a collection of horizontal saddle connections, and since these saddle connections remain horizontal, the cylinder persists. Furthermore, since the height of a cylinder is the imaginary part of a saddle connection joining the lower and upper boundaries, this also remains constant. \square

Recall that $T^M(\mathcal{M})$ denotes the tangent space to \mathcal{M} at the point $M \in \mathcal{M}$, and that $T^M(\mathcal{M})$ is naturally a subspace of $H^1(M, \Sigma; \mathbb{C})$, where $\Sigma \subset M$ is the set of singularities of M .

Definition 4.4 Suppose $M \in \mathcal{M}$. Two relative homology classes $\alpha, \beta \in H_1(M, \Sigma; \mathbb{Z})$ are called \mathcal{M} -*collinear* if they have collinear images in $T^M(\mathcal{M})^*$. (Two nonzero vectors in a vector space are called collinear if they are scalar multiples. By [Remark 4.5](#) below, we see that the scalars must be real, even if a priori we allow them to be complex numbers.)

For example, if α and 2β are equal in $T^M(\mathcal{M})^*$ then α and β are \mathcal{M} -collinear, and this means exactly that for any translation surface in \mathcal{M} near M , the holonomy along β is equal to twice the holonomy along α . In other words, this equality must be one of the linear equations defining \mathcal{M} in local period coordinates at M .

Remark 4.5 Relative homology classes $\alpha, \beta \in H_1(M, \Sigma; \mathbb{Z})$ also define functionals α^*, β^* on the real part of the tangent space $T_{\mathbb{R}}^M(\mathcal{M}) = T^M(\mathcal{M}) \cap H^1(M, \Sigma; \mathbb{R})$. Since $T^M(\mathcal{M})$ is the complexification of $T_{\mathbb{R}}^M(\mathcal{M})$, we see that α and β are collinear in $T_{\mathbb{R}}^M(\mathcal{M})^*$ if and only if they are collinear in $T^M(\mathcal{M})^*$.

Definition 4.6 Two saddle connections on $M \in \mathcal{M}$ are \mathcal{M} -parallel if they are parallel at M and at every nearby $M' \in \mathcal{M}$. Two cylinders on $M \in \mathcal{M}$ are \mathcal{M} -parallel if they are parallel at M and at every nearby $M' \in \mathcal{M}$.

Lemma 4.7 Two saddle connections are \mathcal{M} -parallel if and only if their relative homology classes are \mathcal{M} -collinear. Two cylinders are \mathcal{M} -parallel if and only if their core curves are \mathcal{M} -collinear.

Proof This follows from the linear algebra fact that if ϕ and ϕ' are nonzero linear functionals on a complex vector space, such that ϕ/ϕ' is always real (when $\phi' \neq 0$), then there exists a real number c such that $\phi = c\phi'$. \square

Being \mathcal{M} -collinear or \mathcal{M} -parallel is an equivalence relation.

Lemma 4.8 Suppose that $M \in \mathcal{M}$ has horizontal cylinders C_1, \dots, C_n with moduli m_1, \dots, m_n . (M is not necessarily horizontally periodic.) Suppose that there is a relation

$$\sum_{i \in S} q_i m_i = 0,$$

where $0 \neq q_i \in \mathbb{Q}$ for all $i \in S$. Suppose furthermore that this relation holds not only at M but also at every real deformation of M in \mathcal{M} , and that S is minimal with this property.

Then the cylinders C_i , $i \in S$, are \mathcal{M} -parallel.

The proof uses the following elementary lemma.

Lemma 4.9 Suppose that V is a finite-dimensional real vector space and $F \subset V^*$ is a collection of linear functionals on V , no two of which are collinear. Then the collection of functions $1/w$ for $w \in F$ are linearly independent over \mathbb{R} . This remains true when the functions are restricted to any nonempty open set in V .

Proof Consider a linear combination $g = \sum q_i w_i^{-1}$, with $q_i \in \mathbb{R}$ and $w_i \in F$ distinct. Suppose some q_i is nonzero. Pick

$$v \in \ker(w_i) \setminus \bigcup_{j \neq i} \ker(w_j).$$

Note that g has a pole at v , and so in particular g is not zero.

It follows that if $\sum q_i w_i^{-1} = 0$, then all $q_i = 0$.

The restriction map from $\text{span } w_i^{-1}$, restricting from V to an open subset of V , is an injection. So the statement remains true when the functions are restricted to any nonempty open set in V . □

Proof of Lemma 4.8 Consider a neighborhood \mathcal{V} of $\text{Re } \Phi(M)$ in

$$V = T_{\mathbb{R}}^M(\mathcal{M}) := H^1(M, \Sigma; \mathbb{R}) \cap T^M(\mathcal{M}),$$

small enough that for all $v \in \mathcal{V}$, $\Phi^{-1}(v + \text{Im } \Phi(M))$ is a valid real deformation of M .

For each i , we define a function m_i on \mathcal{V} by setting $m_i(v)$ to be the modulus of the cylinder C_i on $\Phi^{-1}(v + \text{Im } \Phi(M))$. Over all real deformations, the height h_i of C_i is constant, but the circumference varies according to the formula

$$c_i = \alpha_i(v + \text{Im } \Phi(M)) = \alpha_i(v),$$

where $\alpha_i \in H_1(M, \mathbb{Z})$ is the core curve of the cylinder C_i . Hence we have

$$m_i(v) = \frac{h_i}{\alpha_i(v)}.$$

Now suppose there is some $S \subset \{1, \dots, n\}$ and $0 \neq q_i \in \mathbb{Q}$ such that

$$\sum_{i \in S} q_i \frac{h_i}{\alpha_i(v)} = 0$$

for all $v \in \mathcal{V}$, and suppose S is minimal with this property. Then by the previous lemma, for all $i, j \in S$, we get that $\alpha_i(v)$ and $\alpha_j(v)$ are collinear as functionals on $T_{\mathbb{R}}^M(\mathcal{M})$. Hence α_i and α_j are collinear in $T_{\mathbb{R}}^M(\mathcal{M})^*$, and hence also in $T^M(\mathcal{M})^*$. □

Lemma 4.10 *Suppose $M \in \mathcal{M}$ is horizontally periodic and \mathcal{C} is an equivalence class of \mathcal{M} -parallel horizontal cylinders. Then there is a real deformation $M' \in \mathcal{M}$ of M where the set of moduli of the cylinders in \mathcal{C} becomes independent of the set of moduli of the remaining horizontal cylinders.*

Proof Suppose $\mathcal{C} = \{C_1, \dots, C_r\}$ and the remaining cylinders are C_{r+1}, \dots, C_n . Let m_i be the modulus of the cylinder C_i ; so m_i will be a function of the real deformation. If it is not possible to make $\{m_1, \dots, m_r\}$ independent of $\{m_{r+1}, \dots, m_n\}$ using a real deformation in \mathcal{M} , then there is some rational relation

$$\sum_{i=1}^r q_i m_i = \sum_{j=r+1}^n q_j m_j,$$

where all $q_i \in \mathbb{Q}$, and this relation holds at all real deformations of M , and neither the right-hand side nor the left-hand side is identically zero.

Assume that we are given such a relation where the number of nonzero q_i ($i = 1, \dots, n$) is minimized. Let $S \subset \{1, \dots, n\}$ be the set of i for which $q_i \neq 0$. It is not hard to see that this S is minimal, in that there is no rational relation among the m_i where i runs over any proper subset of S . (This is where minimality is used. Note that minimality is assumed in [Lemma 4.8](#).)

Therefore if such a relation exists, by [Lemma 4.8](#) we see that some C_i for $i \leq r$ is \mathcal{M} -parallel to some C_j for $j > r$, which is a contradiction. \square

Lemma 4.11 *Suppose $M \in \mathcal{M}$ is horizontally periodic and \mathcal{C} is an equivalence class of \mathcal{M} -parallel horizontal cylinders.*

Then $u_t^{\mathcal{C}}(M)$ remains in \mathcal{M} for all t .

Proof It suffices to show that $\eta_{\mathcal{C}} \in T^{\mathcal{M}}(M)$, where $\eta_{\mathcal{C}}$ is described in [Section 2](#).

By [Lemma 4.8](#) there is a real deformation $M' \in \mathcal{M}$ of M where the set of moduli of the cylinders in \mathcal{C} becomes independent of the set of moduli of the remaining horizontal cylinders. It is easy to check that the definition of $\eta_{\mathcal{C}}$ is the same on M and M' , since the heights of cylinders have not changed. (By this we mean that $\eta_{\mathcal{C}}$ at M' is the parallel translate of $\eta_{\mathcal{C}}$ at M .)

By [Lemma 3.1](#), $\eta_{\mathcal{C}}$ is in the tangent space of \mathcal{M} at M' . \square

5 Proof of the cylinder deformation theorem

In this section, we will prove our main result:

Theorem 5.1 (The cylinder deformation theorem, detailed version) *Let \mathcal{M} be an affine invariant submanifold. Suppose that \mathcal{C} is an equivalence class of \mathcal{M} -parallel horizontal cylinders on $M \in \mathcal{M}$. Then for all $s, t \in \mathbb{R}$, the surface $a_s^{\mathcal{C}}(u_t^{\mathcal{C}}(M))$ is in \mathcal{M} .*

The cylinder deformation theorem as stated in the introduction is a corollary of this, since the set of all horizontal cylinders on M is a union of equivalence classes of \mathcal{M} -parallel horizontal cylinders.

Our remaining key tool is the following theorem.

Theorem 5.2 (Smillie and Weiss [23, Corollary 6]) *Every u_t -orbit closure contains a horizontally periodic translation surface.*

Theorem 5.2 relies on the quantitative recurrence of horocycle flow and on the existence of compact minimal sets for the horocycle flow, established by Minsky and Weiss [21].

Proof of Theorem 5.1 There is some horizontally periodic translation surface M' on which the horocycle flow orbit of M accumulates. Each horizontal cylinder of M is “present” on M' , and it has the same height and circumference on both M and M' .

Let \mathcal{C}' be the set of horizontal cylinders of M , so $\mathcal{C} \subset \mathcal{C}'$. We will speak of \mathcal{C}' and \mathcal{C} as being collections of cylinders on M' .

Since the horocycle flow orbit of M accumulates at M' , there are arbitrarily small deformations of M' where the cylinders of \mathcal{C}' stay horizontal but the remaining horizontal cylinders of M' do not. These deformations are exactly $u_t(M)$ for t very large, so $u_t(M)$ is very close to M' . The only horizontal cylinders on $u_t(M)$ are the ones coming from M . So $u_t(M)$ is a deformation of M' (just something close to M') where the cylinders in \mathcal{C}' remain horizontal, but all the other horizontal cylinders on M' are no longer horizontal on $u_t(M)$.

Thus no cylinder in \mathcal{C}' is \mathcal{M} -parallel to a horizontal cylinder on M' not in \mathcal{C}' . By Lemma 4.7, the equivalence classes of \mathcal{M} -parallel cylinders in \mathcal{C}' are the same at M and M' . In particular, \mathcal{C} is an equivalence class of \mathcal{M} -parallel horizontal cylinders on M' .

Hence by Lemma 4.11, $\eta_{\mathcal{C}}$ is in the tangent space $T^{M'}(\mathcal{M})$ at M' . Hence $\eta_{\mathcal{C}}$ is in the tangent space $T^M(\mathcal{M})$ at M . We conclude that the cylinder deformations of M , which result from adding $\eta_{\mathcal{C}}$ or $i\eta_{\mathcal{C}}$ in period coordinates, remain in M . \square

6 Complete periodicity

We now prove Theorem 1.5.

Lemma 6.1 *Let M be a translation surface that is not horizontally periodic. Then there is some curve γ on M which is supported off the union of the horizontal cylinders on M for which the holonomy of γ is not real (that is, not horizontal).*

Proof Let x be some generic point in M not contained in (or on the boundary of) any horizontal cylinder. (By generic, we mean generic with respect to the Poincaré recurrence theorem applied to the horizontal straight line flow in M .) Follow the flow from x in the horizontal direction for a long time, until returning very close to x . Close up this path to get a curve γ with the required property. \square

Lemma 6.2 *Let $M = (X, \omega)$ be a translation surface, and let \mathcal{C} be the collection of all horizontal cylinders on M . Suppose that \mathcal{C} is nonempty but M is not horizontally periodic. Then*

$$p(\eta_{\mathcal{C}}) \notin \text{span}_{\mathbb{C}}(\text{Re } p([\omega]), \text{Im } p([\omega])).$$

Proof Suppose to the contrary that

$$p(\eta_{\mathcal{C}}) = c_1 \text{Re } p([\omega]) + c_2 \text{Im } p([\omega]).$$

Since $\eta_{\mathcal{C}}$ evaluated at any core curve of a cylinder is zero, we see that $c_1 = 0$.

Now consider the curve γ produced by Lemma 6.1. Note that $p(\eta_{\mathcal{C}})(\gamma) = 0$; but $\text{Im } p([\omega])(\gamma) \neq 0$, so we see that $c_2 = 0$ as well.

However, it is not hard to see that $p(\eta_{\mathcal{C}}) \neq 0$, because it pairs nontrivially with homology classes whose intersection number with each horizontal core curve is positive (Lemma 2.4). \square

Proof of Theorem 1.5 Suppose that $M = (X, \omega)$ is not completely periodic. Without loss of generality, rotate M so that there is at least one cylinder in the horizontal direction but M is not horizontally periodic. Let \mathcal{C} be the collection of horizontal cylinders on M . Let \mathcal{M} be any affine invariant submanifold containing M . By Theorem 1.1, we know that $\eta_{\mathcal{C}} \in T^{\mathcal{M}}(\mathcal{M})$. By Lemma 6.2, we see that $p(T^{\mathcal{M}}(\mathcal{M}))$ contains at least three linearly independent vectors, namely $\text{Re } p([\omega])$, $\text{Im } p([\omega])$ and $p(\eta_{\mathcal{C}})$.

We have shown that if \mathcal{M} contains a translation surface that is not completely periodic, then $\dim_{\mathbb{C}} p(T(\mathcal{M})) > 2$. \square

7 Affine field of definition

In this section we prove Theorem 1.9, which is a corollary of the following more precise result.

Theorem 7.1 *Suppose that $M \in \mathcal{M}$ has an equivalence class \mathcal{C} of \mathcal{M} -parallel cylinders consisting of $r \geq 1$ cylinders with circumferences c_1, \dots, c_r . Then the affine field of definition of \mathcal{M} is $\mathbf{k}(\mathcal{M}) = \mathbb{Q}[c_2/c_1, \dots, c_r/c_1]$ if $r > 1$, and $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$ if $r = 1$.*

First we prove the easier direction in [Theorem 7.1](#).

Lemma 7.2 *Let $V \subset \mathbb{C}^m$ be a linear subspace. Let e_i^* ($i = 1, \dots, m$) be the coordinate functions. If the equation $e_i^* - ce_j^* = 0$ holds on V for some $i \neq j$ and $c \in \mathbb{C}$, but the equation $e_i^* = 0$ does not hold on V , then the affine field of definition of V contains $\mathbb{Q}[c]$.*

In this context, the affine field of definition of V is the smallest subfield of \mathbb{C} with the property that V may be described by linear equations in the e_i^* with coefficients in this field.

Proof The projection of V to the (i, j) -coordinate plane is a line of slope c . The affine field of definition of this line is $\mathbb{Q}[c]$. If V can be defined by linear equations with coefficients in some field, then so can its projection to any coordinate hyperplane. So it follows that the affine field of definition of V contains $\mathbb{Q}[c]$. □

Proof that $k(\mathcal{M}) \supset \mathbb{Q}[c_2/c_1, \dots, c_r/c_1]$ Let $\mathcal{C} = \{C_1, \dots, C_r\}$, and let the core curve of the cylinder C_i be denoted by γ_i .

Pick a basis of $H_1(M, \Sigma; \mathbb{Z})$, where $\Sigma \subset M$ is the set of cone points of M , so that this basis includes as many core curves of cylinders in \mathcal{C} as possible. Without loss of generality, assume that $\gamma_1, \dots, \gamma_p$ are part of the basis.

Note that $\mathbb{Q}[c_2/c_1, \dots, c_r/c_1] = \mathbb{Q}[c_2/c_1, \dots, c_p/c_1]$. Indeed, γ_i for $i > p$ is in the \mathbb{Z} -span of $\gamma_1, \dots, \gamma_p$, so c_i for $i > p$ is an integral linear combination of c_1, \dots, c_p .

Assume the rest of the basis is $\alpha_1, \dots, \alpha_w$. For each $\gamma \in H_1(M, \Sigma; \mathbb{Z})$ we set

$$\gamma^*(X, \omega) = \int_{\gamma} \omega,$$

so that

$$(\gamma_1^*, \dots, \gamma_p^*, \alpha_1^*, \dots, \alpha_w^*)$$

give local coordinates for \mathcal{M} near M . The affine field of definition of \mathcal{M} is the smallest subfield of \mathbb{R} such that \mathcal{M} is defined in these local coordinates by linear equations with coefficients in this subfield.

The linear equations $\gamma_i^* - (c_i/c_1)\gamma_1^* = 0$ hold on \mathcal{M} in these local coordinates. It follows from the previous lemma that $k(\mathcal{M}) \supset \mathbb{Q}[c_2/c_1, \dots, c_p/c_1]$. □

To proceed, we require the following technical lemma.

Lemma 7.3 Suppose that $M \in \mathcal{M}$ is a translation surface, and suppose \mathcal{C} is an equivalence class of \mathcal{M} -parallel horizontal cylinders of circumferences c_1, \dots, c_r . There is a tangent vector $\eta \in T^M(\mathcal{M})$ that is defined over $\mathbb{Q}[c_2/c_1, \dots, c_r/c_1]$ and for which $p(\eta) \neq 0$.

When we say that η is defined over some field, we mean that the real and imaginary parts of the periods of η over any integral relative homology class are contained in this field.

Proof Let the cylinders in \mathcal{C} be C_1, \dots, C_r . Let the height, circumference and modulus of C_i be h_i, c_i and $m_i = h_i/c_i$, respectively. By Corollary 3.4, if t_1, \dots, t_r satisfy all rational linear relations that the m_i do, then $u_{tt_1/m_1, \dots, tt_r/m_r}^{\mathcal{C}}(M)$ remains in \mathcal{M} for all $t \in \mathbb{R}$.

Differentiating this at $t = 0$, we find that $\eta \in T^M(\mathcal{M})$, where

$$\eta = \sum_{i=1}^r \frac{t_i}{m_i} h_i J_{\alpha_i} = \sum_{i=1}^r t_i c_i J_{\alpha_i}.$$

Compare to Lemma 2.4.

The restrictions on the t_i form a set of rational linear equations which has nonzero solutions. Therefore this system has a nonzero solution with all $t_i \in \mathbb{Q}$. Using a scaling $t'_i = t_i/c_1$ of this solution gives the result. \square

We also require the following theorem.

Theorem 7.4 (The simplicity theorem [29, Theorem 5.1]) *The only proper flat subbundles of $T(\mathcal{M})$ are contained in $\ker(p)$.*

Proof of Theorem 7.1 It remains only to show that $k(\mathcal{M}) \subset \mathbb{Q}[c_2/c_1, \dots, c_r/c_1]$. Let $\eta \in T^M(\mathcal{M})$ be given by Lemma 7.3. Since $p(\eta) \neq 0$, the simplicity theorem gives that the orbit of η under the monodromy representation of H_{rel}^1 spans $T^M(\mathcal{M})$. Since the monodromy of H_{rel}^1 is integral, this gives that $T^M(\mathcal{M})$ is spanned by vectors defined over $\mathbb{Q}[c_2/c_1, \dots, c_r/c_1]$. Hence $T^M(\mathcal{M})$ is defined over $\mathbb{Q}[c_2/c_1, \dots, c_r/c_1]$, that is, $k(\mathcal{M}) \subset \mathbb{Q}[c_2/c_1, \dots, c_r/c_1]$. (Similar arguments are explained in more detail in [29].) \square

Proof of Theorem 1.9 Without loss of generality, assume that the horizontal cylinders C_1, \dots, C_r form an equivalence class of \mathcal{M} -parallel cylinders, where $r \leq n$. By Theorem 7.1,

$$k(\mathcal{M}) = \mathbb{Q}[c_2/c_1, \dots, c_r/c_1] \subset \mathbb{Q}[c_2/c_1, \dots, c_n/c_1]. \quad \square$$

8 Finding many cylinders

This section is devoted to proving [Theorem 1.10](#). The proof will consider the *twist space* and the *cylinder preserving space* of a horizontally periodic translation surface in some affine invariant submanifold.

Definition 8.1 Let $M \in \mathcal{M}$ be horizontally periodic. The *twist space* of \mathcal{M} at M is the subspace $\text{Twist}(M, \mathcal{M})$ of $T_{\mathbb{R}}^M(\mathcal{M})$ of cohomology classes in $T_{\mathbb{R}}^M(\mathcal{M})$ that are zero on all horizontal saddle connections.

Lemma 8.2 Let \mathcal{H} be a stratum, and let $M \in \mathcal{H}$ be horizontally periodic. Let C_1, \dots, C_n be the horizontal cylinders on M . Then the twist space of \mathcal{H} at M is

$$\text{Twist}(M, \mathcal{H}) = \text{span}_{\mathbb{R}}(\eta_{C_i})_{i=1}^n.$$

Proof Classes in $\text{span}_{\mathbb{R}}(\eta_{C_i})_{i=1}^n$ are zero on horizontal saddle connections, and hence

$$\text{Twist}(M, \mathcal{H}) \supset \text{span}_{\mathbb{R}}(\eta_{C_i})_{i=1}^n.$$

Pick one “cross curve” for each horizontal cylinder; a saddle connection from a zero on the bottom of the cylinder to a zero on the top. These cross curves, together with the horizontal saddle connections, span relative homology.

Consider a class $\eta \in \text{Twist}(M, \mathcal{H})$. By definition, it is zero on all horizontal saddle connections. Suppose that the value of η on the cross curve for C_i is b_i . Then we see that

$$\eta = \sum_{i=1}^n \frac{b_i}{h_i} \eta_{C_i},$$

where h_i is the height of C_i . Indeed, both the left-hand side and the right-hand side are zero on horizontal saddle connections, and by [Lemma 2.4](#), the right-hand side takes value b_i on any cross curve for C_i . Since the right-hand side and the left-hand side agree on a basis for relative homology, they are equal. \square

Corollary 8.3 Let $M \in \mathcal{M}$ be horizontally periodic, and let C_1, \dots, C_n be the horizontal cylinders on M . Then

$$\text{Twist}(M, \mathcal{M}) = T_{\mathbb{R}}^M(\mathcal{M}) \cap \text{span}_{\mathbb{R}}(\eta_{C_i})_{i=1}^n.$$

Proof By definition, $\text{Twist}(M, \mathcal{M})$ is the set of real cohomology classes on M that are in $T_{\mathbb{R}}^M(\mathcal{M})$ and that are zero on all horizontal saddle connections. By the previous lemma, the space of real cohomology classes on M which are zero on all horizontal saddle connections is exactly $\text{span}_{\mathbb{R}}(\eta_{C_i})_{i=1}^n$. \square

Definition 8.4 Let $M \in \mathcal{M}$ be horizontally periodic. The *cylinder preserving space* of \mathcal{M} at M is the subspace $\text{Pres}(M, \mathcal{M})$ of $T_{\mathbb{R}}^M$ of cohomology classes that are zero on the core curves of all horizontal cylinders.

Note that $\text{Twist}(M, \mathcal{M}) \subset \text{Pres}(M, \mathcal{M})$.

Lemma 8.5 Suppose that $\eta \in \text{Pres}(M, \mathcal{M}) \otimes \mathbb{C}$ is in the complexification of the cylinder preserving space of \mathcal{M} at M , and consider the deformation M_t defined by $\Phi(M_t) = \Phi(M) + t\eta$. For t sufficiently small, the horizontal cylinders on M persist on M_t .

$\text{Pres}(M, \mathcal{M}) \otimes \mathbb{C}$ is the subspace of $T^M(\mathcal{M})$ of complex cohomology classes that are zero on the core curves of all horizontal cylinders.

Proof Indeed, the core curve of each horizontal cylinder on M continues to have real holonomy at M_t . \square

We now arrive at the key idea of this section, which will be used to prove [Theorem 1.10](#).

Lemma 8.6 Suppose $M \in \mathcal{M}$ is horizontally periodic and

$$\text{Twist}(M, \mathcal{M}) \subsetneq \text{Pres}(M, \mathcal{M}).$$

Then there is a horizontally periodic translation surface in \mathcal{M} with more horizontal cylinders than M .

Sublemma 8.7 Suppose $M \in \mathcal{M}$ is horizontally periodic and

$$\eta \in \text{Pres}(M, \mathcal{M}) \setminus \text{Twist}(M, \mathcal{M})$$

is sufficiently small. Consider the deformation M' of M defined by $\Phi(M') = \Phi(M) + i\eta$. Then all of the horizontal cylinders from M persist on M' , but their union is not all of M' .

Proof of Sublemma Because $i\eta$ is in the complexification of the cylinder preserving space, [Lemma 8.5](#) gives that the horizontal cylinders from M persist on M' , so it remains only to show that their union is not all of M' .

Because η is not in the twist space, it takes a nonzero value on some horizontal saddle connection. On M' , this saddle connection is no longer horizontal, but instead acquires some very small vertical part, which may be assumed to be vastly smaller than the height of any horizontal cylinder on M .

Thus it can be arranged for M' to have a saddle connection whose vertical part is vastly smaller than the heights on M' of the cylinders that have persisted from M . The union of the cylinders on this M' that have persisted from M is not all of M' ; in particular, the union cannot contain the saddle connection under consideration. \square

Proof of Lemma 8.6 By assumption we may find $\eta \in \text{Pres}(M, \mathcal{M}) \setminus \text{Twist}(M, \mathcal{M})$. The sublemma thus gives a deformation $M' \in \mathcal{M}$ of M where the horizontal cylinders of M persist but do not cover the whole surface. By Theorem 5.2, we may find an M'' in the horocycle flow orbit closure of M' which is horizontally periodic. The horizontal cylinders on M' persist on M'' ; in particular, for each horizontal cylinder on M there is a corresponding horizontal cylinder on M'' , but these horizontal cylinders do not cover M'' . So M'' must have more horizontal cylinders than M . \square

To apply Lemma 8.6 we need to control the size of the twist space and the cylinder preserving space. Controlling the size of the cylinder preserving space is straightforward.

Lemma 8.8 *Suppose that $M \in \mathcal{M}$ is horizontally periodic. Suppose that the core curves of the horizontal cylinders on M span a subspace of $T^M(\mathcal{M})^*$ of dimension d . Then $\text{Pres}(M, \mathcal{M})$ has codimension d in $T_{\mathbb{R}}^M(\mathcal{M})$.*

Proof Suppose that the core curves of the horizontal cylinders on M are $\alpha_1, \dots, \alpha_n$. Then the subspace $\text{Pres}(M, \mathcal{M})$ of $T_{\mathbb{R}}^M(\mathcal{M})$ is defined by the equations $\alpha_i^* = 0$, $i = 1, \dots, n$. The α_i^* span a subspace of $T_{\mathbb{R}}^M(\mathcal{M})^*$ of dimension d by supposition, so we see that $\text{Pres}(M, \mathcal{M})$ has codimension d in $T_{\mathbb{R}}^M(\mathcal{M})$. \square

Controlling the size of the twist space is more difficult. For this we require:

Theorem 8.9 (Avila, Eskin and Möller [2, Theorem 1.4]) *$p(T(\mathcal{M}))$ is symplectic.*

Lemma 8.10 *Let $M \in \mathcal{M}$ be horizontally periodic. Then $p(\text{Twist}(M, \mathcal{M}))$ is isotropic.*

Proof Let C_1, \dots, C_n be the horizontal cylinders on M . By Corollary 8.3, it suffices to prove that

$$\text{span}_{\mathbb{R}}(p(\eta_{C_i}))_{i=1}^n$$

is isotropic. Lemma 2.4 gives that η_{C_i} is, up to a scalar multiple, equal to the intersection number with the core curve of C_i . Since the core curves of the C_i are disjoint, the result follows. \square

Corollary 8.11 *Let $k = \frac{1}{2} \dim_{\mathbb{C}} p(T(\mathcal{M}))$. Then $\text{Twist}(M, \mathcal{M})$ has codimension at least k in $T_{\mathbb{R}}^M(\mathcal{M})$.*

Proof The maximal dimension of an isotropic subspace of $p(T(\mathcal{M}))$ is k . Hence $p(\text{Twist}(M, \mathcal{M}))$ has codimension at least k in $p(T_{\mathbb{R}}^M(\mathcal{M}))$, and it follows that $\text{Twist}(M, \mathcal{M})$ has codimension at least k in $T_{\mathbb{R}}^M(\mathcal{M})$. \square

Corollary 8.12 *Let $M \in \mathcal{M}$ be horizontally periodic, and let $k = \frac{1}{2} \dim_{\mathbb{C}} p(T(\mathcal{M}))$. Suppose that the core curves of the horizontal cylinders on M span a subspace of $T^M(\mathcal{M})^*$ of dimension d . If $d < k$, then there is a translation surface $M' \in \mathcal{M}$ with more horizontal cylinders than M .*

Proof If $d < k$, then by [Lemma 8.8](#) and [Corollary 8.11](#), the cylinder preserving space at M must be strictly larger than the twist space at M . Hence the result follows by [Lemma 8.6](#). \square

Proof of Theorem 1.10 Pick $M \in \mathcal{M}$ with the maximal number of parallel cylinders. Without loss of generality, we may assume that the cylinders are horizontal. In this case M must be horizontally periodic, or else [Theorem 5.2](#) would produce a horizontally periodic translation surface in the u_t -orbit closure, which would necessarily contain more horizontal cylinders.

By [Corollary 8.12](#), the core curves of the horizontal cylinders on M span a subspace of $p(T^M(\mathcal{M}))^*$ of dimension at least k . In particular, there must be at least k equivalence classes of \mathcal{M} -parallel horizontal cylinders on M .

This shows the first statement of [Theorem 1.10](#). The second (converse) statement follows directly from [Theorem 8.9](#): the core curves of parallel cylinders span an isotropic subspace of $p(T^M(\mathcal{M}))^*$, and any isotropic subspace has dimension at most k . Since the core curves of the cylinders are absolute homology classes, their span in $p(T^M(\mathcal{M}))$ is isomorphic to their span in $T^M(\mathcal{M})$.

The final statement follows from [Lemma 8.6](#), which gives that

$$\text{Twist}(M, \mathcal{M}) = \text{Pres}(M, \mathcal{M}).$$

Thus $\text{Twist}(M, \mathcal{M})$ has codimension k in $T_{\mathbb{R}}^M(\mathcal{M})$, and so by [Corollary 8.3](#) we get that M must have at least $\dim_{\mathbb{C}} T(\mathcal{M}) - k$ many horizontal cylinders. \square

9 Proof of Theorem 1.7

From [Theorem 1.10](#), we will need only the following results.

Corollary 9.1 *If $\frac{1}{2} \dim_{\mathbb{C}} p(T(\mathcal{M})) > 1$, then there is a translation surface in \mathcal{M} with at least two equivalence classes of \mathcal{M} -parallel horizontal cylinders.*

Lemma 9.2 *Suppose C_1, \dots, C_s are some equivalence classes of \mathcal{M} -parallel horizontal cylinders on $M \in \mathcal{M}$, whose core curves span a subspace of $T^M(\mathcal{M})^*$ of dimension d . (There might be additional equivalence classes of \mathcal{M} -parallel horizontal cylinders on M .) The set of deformations of M where these s equivalence classes of \mathcal{M} -parallel horizontal cylinders remain horizontal is locally a manifold of real codimension d .*

Proof Let α_i be the core curve of a cylinder in C_i . In local period coordinates, the set of deformations in question is defined by $\text{Im}(\alpha_i^*) = 0$, $i = 1, \dots, s$. By supposition, the α_i^* span a subspace of dimension d , so the manifold is of real codimension d . \square

Corollary 9.3 *The set of translation surfaces in \mathcal{M} with at least two equivalence classes of \mathcal{M} -parallel horizontal cylinders is the countable union of immersed submanifolds of real codimension at least two.*

Proof This follows immediately from Lemma 9.2. \square

Proof of Theorem 1.7 By Corollary 9.1, we may find a translation surface M with at least two equivalence classes of \mathcal{M} -parallel horizontal cylinders. Lemma 9.2 gives that there is an immersed submanifold of real codimension 1 where one of these equivalence classes of \mathcal{M} -parallel cylinders remains horizontal.

By Corollary 9.3, not all of these translation surfaces can have at least two equivalence classes of \mathcal{M} -parallel horizontal cylinders. Therefore there is a deformation of M with exactly one equivalence class of \mathcal{M} -parallel horizontal cylinders. This equivalence class cannot cover all of the deformation, since it did not cover all of M .

This deformation has at least one horizontal cylinder, but is not horizontally periodic. \square

10 Two open problems

The cylinder deformation theorem leads naturally to two problems, both of which were already of interest before our work.

Problem 10.1 Find an algorithm which, given a translation surface, finds all the cylinders in any given direction.

We do not know if such an algorithm exists, however even solutions in special cases would be interesting. For a solution to be useful, the algorithm must be guaranteed to terminate. (It is easy to give algorithms which can find cylinders in a given direction on a translation surface, but it seems hard in general to certify that all the cylinders in a given direction have been found.)

Problem 10.2 Let M be a surface which is *completely parabolic*, which by definition means that M is completely periodic and that any two parallel cylinders on M have rationally related moduli. Must M be a lattice surface?

Problem 10.2 is in fact a problem of Smillie and Weiss [24, Question 5]. For completely parabolic surfaces M , the cylinder deformation theorem yields nothing. Even considering closures of cylinder deformations of M gives absolutely no indication that M has large orbit closure.

Problem 10.2 is an extreme special case of the following more open-ended question: To what extent does the cylinder deformation theorem account for all deformations of a translation surface M which remain in the orbit closure of M ?

References

- [1] **D Aulicino, D-M Nguyen, A Wright**, *Classification of higher rank orbit closures in $\mathcal{H}^{\text{odd}}(4)$* [arXiv:1308.5879](#) to appear in J. Eur. Math. Soc.
- [2] **A Avila, A Eskin, M Möller**, *Symplectic and isometric $\text{SL}(2, \mathbb{R})$ -invariant subbundles of the Hodge bundle* [arXiv:1209.2854](#) to appear in J. Reine Angew. Math.
- [3] **II Bouw, M Möller**, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Ann. of Math. 172 (2010) 139–185 [MR2680418](#)
- [4] **K Calta**, *Veech surfaces and complete periodicity in genus two*, J. Amer. Math. Soc. 17 (2004) 871–908 [MR2083470](#)
- [5] **A Eskin, H Masur**, *Asymptotic formulas on flat surfaces*, Ergodic Theory Dynam. Systems 21 (2001) 443–478 [MR1827113](#)
- [6] **A Eskin, M Mirzakhani**, *Invariant and stationary measures for the $\text{SL}(2, \mathbb{R})$ action on moduli space* [arXiv:1302.3320](#)
- [7] **A Eskin, M Mirzakhani, A Mohammadi**, *Isolation, equidistribution, and orbit closures for the $\text{SL}(2, \mathbb{R})$ action on moduli space* [arXiv:1305.3015](#)
- [8] **S Filip**, *Semisimplicity and rigidity of the Kontsevich–Zorich cocycle* [arXiv:1307.7314](#)
- [9] **S Filip**, *Splitting mixed Hodge structures over affine invariant manifolds* [arXiv:1311.2350](#)
- [10] **P Hubert, T A Schmidt**, *Invariants of translation surfaces*, Ann. Inst. Fourier (Grenoble) 51 (2001) 461–495 [MR1824961](#)
- [11] **A Katok, B Hasselblatt**, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications 54, Cambridge Univ. Press (1995) [MR1326374](#)

- [12] **R Kenyon, J Smillie**, *Billiards on rational-angled triangles*, Comment. Math. Helv. 75 (2000) 65–108 [MR1760496](#)
- [13] **E Lanneau, D-M Nguyen**, *Complete periodicity of Prym eigenforms* [arXiv:1301.0783](#)
- [14] **H Masur, S Tabachnikov**, *Rational billiards and flat structures*, from: “Handbook of dynamical systems, Vol. 1A”, (B Hasselblatt, A Katok, editors), North-Holland, Amsterdam (2002) 1015–1089 [MR1928530](#)
- [15] **C T McMullen**, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. 16 (2003) 857–885 [MR1992827](#)
- [16] **C T McMullen**, *Teichmüller geodesics of infinite complexity*, Acta Math. 191 (2003) 191–223 [MR2051398](#)
- [17] **C T McMullen**, *Teichmüller curves in genus two: The decagon and beyond*, J. Reine Angew. Math. 582 (2005) 173–199 [MR2139715](#)
- [18] **C T McMullen**, *Prym varieties and Teichmüller curves*, Duke Math. J. 133 (2006) 569–590 [MR2228463](#)
- [19] **C T McMullen**, *Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two*, Ann. of Math. 165 (2007) 397–456 [MR2299738](#)
- [20] **C T McMullen**, *Cascades in the dynamics of measured foliations*, preprint (2012) Available at <http://nrs.harvard.edu/urn-3:HUL.InstRepos:11891557>
- [21] **Y Minsky, B Weiss**, *Nondivergence of horocyclic flows on moduli space*, J. Reine Angew. Math. 552 (2002) 131–177 [MR1940435](#)
- [22] **D-M Nguyen, A Wright**, *Non-Veech surfaces in $\mathcal{H}^{\text{hyp}}(4)$ are generic*, Geom. Funct. Anal. 24 (2014) 1316–1335 [MR3248487](#)
- [23] **J Smillie, B Weiss**, *Minimal sets for flows on moduli space*, Israel J. Math. 142 (2004) 249–260 [MR2085718](#)
- [24] **J Smillie, B Weiss**, *Finiteness results for flat surfaces: A survey and problem list*, from: “Partially hyperbolic dynamics, laminations, and Teichmüller flow”, (G Forni, M Lyubich, C Pugh, M Shub, editors), Fields Inst. Commun. 51, Amer. Math. Soc. (2007) 125–137 [MR2388694](#)
- [25] **W A Veech**, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. 97 (1989) 553–583 [MR1005006](#)
- [26] **W A Veech**, *Siegel measures*, Ann. of Math. 148 (1998) 895–944 [MR1670061](#)
- [27] **C C Ward**, *Calculation of Fuchsian groups associated to billiards in a rational triangle*, Ergodic Theory Dynam. Systems 18 (1998) 1019–1042 [MR1645350](#)
- [28] **A Wright**, *Schwarz triangle mappings and Teichmüller curves: the Veech–Ward–Bow–Möller curves*, Geom. Funct. Anal. 23 (2013) 776–809 [MR3053761](#)

- [29] **A Wright**, *The field of definition of affine invariant submanifolds of the moduli space of abelian differentials*, *Geom. Topol.* 18 (2014) 1323–1341 [MR3254934](#)
- [30] **A Zorich**, *Flat surfaces*, from: “Frontiers in number theory, physics, and geometry, I”, (P Cartier, B Julia, P Moussa, P Vanhove, editors), Springer, Berlin (2006) 437–583 [MR2261104](#)

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Received: 11 October 2013

Seconded: Benson Farb, Leonid Polterovich

Accepted: 1 Jul 2014