Approximation theory for nonorientable minimal surfaces and applications

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We prove a version of the classical Runge and Mergelyan uniform approximation theorems for nonorientable minimal surfaces in Euclidean 3–space $\mathbb{R}^3$. Then we obtain some geometric applications. Among them, we emphasize the following ones:

- A Gunning–Narasimhan-type theorem for nonorientable conformal surfaces.
- An existence theorem for nonorientable minimal surfaces in $\mathbb{R}^3$ with arbitrary conformal structure, properly projecting into a plane.
- An existence result for nonorientable minimal surfaces in $\mathbb{R}^3$ with arbitrary conformal structure and Gauss map omitting one projective direction.

1 Introduction

The Runge and Mergelyan theorems are the central results in the theory of uniform approximation by holomorphic functions in one complex variable. The former, which dates back to 1885, asserts that if the complement $\mathbb{C} \setminus K$ of a compact set $K \subset \mathbb{C}$ has no relatively compact connected components, then every holomorphic function in (an open neighborhood of) K can be approximated, uniformly on K, by holomorphic functions on $\mathbb{C}$; see Runge [32]. If $K \subset \mathbb{C}$ is an arbitrary compact set, then Mergelyan's theorem [28], which dates back to 1951, ensures that continuous functions $K \to \mathbb{C}$, holomorphic in the interior K° of K, can be approximated uniformly on K by holomorphic functions in open neighborhoods of K in $\mathbb{C}$. In 1958, Bishop [11] extended these results to Riemann surfaces (see Forster [15] for a modern proof using functional analysis):

Runge–Mergelyan theorem *Let $\mathcal{N}$ be an open Riemann surface (ie noncompact) and let $K \subset \mathcal{N}$ be a compact set such that $\mathcal{N} \setminus K$ has no relatively compact connected components. Then any continuous function $K \to \mathbb{C}$, holomorphic in the interior K° of K, can be uniformly approximated on K by holomorphic functions $\mathcal{N} \to \mathbb{C}$.*

A compact subset $K \subset \mathcal{N}$ satisfying the hypotheses of the above theorem is said to be a *Runge set* in the open Riemann surface $\mathcal{N}$. Runge and Mergelyan's theorems admit plenty of generalizations; the extension of Runge's theorem to functions of several complex variables is known as the Oka–Weil theorem (see eg Hörmander [19]), and in the more general setting, maps $S \supset K \to O$ from a holomorphically convex set K of a Stein manifold S to an Oka manifold O satisfy the Runge property (see Forstnerič [16] for a good reference).

On the other hand, conformal minimal immersions of open Riemann surfaces into Euclidean space are harmonic functions. This basic fact has strongly influenced the theory of minimal surfaces, furnishing it of powerful tools coming from complex analysis. In particular, Runge's theorem (combined with the López–Ros transformation for minimal surfaces [25]) has been the key tool for constructing complete hyperbolic minimal surfaces in $\mathbb{R}^3$ of finite topology; see Jorge and Xavier [20], Nadirashvili [30] and Morales [29] for pioneering papers. However, the direct application of Runge's theorem has a limited reach and it seems to be insufficient for constructing minimal surfaces with more complicated geometry. With the aim of overcoming this constraint, Alarcón and López [6, Theorem 4.9] obtained a Runge–Mergelyan-type theorem for conformal minimal immersions of open Riemann surfaces into $\mathbb{R}^3$. This result has been a versatile tool for constructing both minimal surfaces in $\mathbb{R}^3$ and curves in the complex 3–space $\mathbb{C}^3$; see Alarcón and López [6; 7; 8; 10] and Alarcón, Fernández and López [2] for a number of applications. For instance and in contrast to the severe restrictions imposed by the use of the López–Ros transformation, it allows one to prescribe the conformal structure of the examples.

In the same spirit, a Runge–Mergelyan-type theorem for a large family of directed holomorphic immersions of open Riemann surfaces into $\mathbb{C}^n$ (including null curves), $n \geq 3$, has been recently shown, with different techniques, by Alarcón and Forstnerič [4].

In this paper we focus on nonorientable minimal surfaces in $\mathbb{R}^3$. This subject should not be considered as a minor or secondary one; on the contrary, nonorientable surfaces present themselves quite naturally in the very origin of minimal surface theory (recall for instance that a Möbius minimal strip can be obtained by solving a simple Plateau problem; see Douglas [12] for more information), and they present a rich and interesting geometry. Nonorientable minimal surfaces were first studied systematically by Lie [22] in the third quarter of the 19th century; the development of their global theory was begun by Meeks [27]. A particular issue is that constructing nonorientable surfaces via Weierstrass representation is in general hard, due to the higher subtlety of the period problem. Runge's theorem has been already used in several constructions of complete nonorientable minimal surfaces in $\mathbb{R}^3$ (see López [23], López, Martín and

Morales [24] and Ferrer, Martín and Meeks [14]); however, as in the orientable case, its direct application does not seem to suffice for more involved constructions.

The aim of this paper is to prove a Runge–Mergelyan-type theorem for nonorientable minimal surfaces in $\mathbb{R}^3$. For a precise statement of our main result, the following notation is required. Every nonorientable minimal surface $M \subset \mathbb{R}^3$ can be represented by a triple $(\mathcal{N}, \mathfrak{I}, X)$, where $\mathcal{N}$ is an open Riemann surface, $\mathfrak{I} \colon \mathcal{N} \to \mathcal{N}$ is an antiholomorphic involution without fixed points, and $X \colon \mathcal{N} \to \mathbb{R}^3$ is an $\mathfrak{I}$–*invariant* conformal minimal immersion (that is, satisfying $X \circ \mathfrak{I} = X$) such that $M = X(\mathcal{N})$; see [27] and Section 2.2 for details. We say that a subset S of $\mathcal{N}$ is $\mathfrak{I}$–*admissible* (see Definition 3.2) if it is Runge in $\mathcal{N}$, $\mathfrak{I}(S) = S$ and $S = R_S \cup C_S$, where $R_S := \overline{S^{\circ}}$ consists of a finite collection of pairwise disjoint compact regions in $\mathcal{N}$ with $\mathcal{C}^1$ boundary, and $C_S := \overline{S \setminus R_S}$ is a finite collection of pairwise disjoint analytical Jordan arcs, meeting R_S only in their endpoints, and such that their intersections with the boundary bR_S of R_S are transverse. Finally, we say that a $\mathcal{C}^1$ map $Y \colon S \to \mathbb{R}^3$ is an $\mathfrak{I}$–*invariant generalized minimal immersion* (see Section 3.2) if $Y|_{R_S}$ is a conformal minimal immersion, $Y|_{C_S}$ is regular and $Y \circ \mathfrak{I} = Y$.

Our main result is the following.

Theorem 1.1 *Let $\mathcal{N}$ be an open Riemann surface, let $\mathfrak{I} \colon \mathcal{N} \to \mathcal{N}$ be an antiholomorphic involution without fixed points, and let $S \subset \mathcal{N}$ be an $\mathfrak{I}$–admissible set. Then any $\mathfrak{I}$–invariant generalized minimal immersion $S \to \mathbb{R}^3$ can be $\mathcal{C}^1$ uniformly approximated on S by $\mathfrak{I}$–invariant conformal minimal immersions $\mathcal{N} \to \mathbb{R}^3$.*

Concerning the proof, it is important to point out that the compatibility condition with respect to the antiholomorphic involution and the higher difficulty of the period problem involve a much more careful analysis than in the orientable case; see [6]. In particular, our approach requires several trickier approximation results by meromorphic functions and 1–forms (see Lemmas 4.2 and 4.6).

Theorem 1.1 has many geometric applications. In Theorem 6.6 we show that, for any open Riemann surface $\mathcal{N}$ and antiholomorphic involution $\mathfrak{I} \colon \mathcal{N} \to \mathcal{N}$ without fixed points, there exist $\mathfrak{I}$–invariant conformal minimal immersions $\mathcal{N} \to \mathbb{R}^3$ properly projecting into a plane (see [6] for the analogous result in the orientable case). This links with an old question by Schoen and Yau [33, page 18]; see also [6; 10]. We also prove an existence theorem of complete conformal $\mathfrak{I}$–invariant minimal immersions $\mathcal{N} \to \mathbb{R}^3$ with a prescribed coordinate function; see Theorem 6.8. As a consequence, in Corollary 6.10 we exhibit complete nonorientable minimal surfaces in $\mathbb{R}^3$ whose Gauss map omits one point of the projective plane $\mathbb{RP}^2$ (see [2] for the orientable

framework). Other geometric applications of Theorem 1.1 can be found; see Alarcón and López [9].

Theorem 1.1 follows from the more general Theorem 5.6, which also deals with the flux map of the approximating surfaces. In particular, Theorem 5.6 implies the analogous result of Theorem 1.1 for curves $F\colon \mathcal{N} \to \mathbb{C}^3$ enjoying the symmetry $F \circ \mathfrak{I} = \overline{F}$; see Corollary 6.1. We also derive a Runge–Mergelyan-type theorem for harmonic functions $h\colon \mathcal{N} \to \mathbb{R}$ satisfying $h \circ \mathfrak{I} = h$ (see Theorem 6.3).

Finally, in a different line of applications, we prove an extension of the classical Gunning–Narasimhan theorem [18] (see also Kusunoki and Sainouchi [21]); more specifically, we show that, for any open Riemann surface $\mathcal{N}$ and any antiholomorphic involution $\mathfrak{I}\colon \mathcal{N} \to \mathcal{N}$ without fixed points, there exist holomorphic 1–forms ϑ on $\mathcal{N}$ with $\mathfrak{I}^*\vartheta = \overline{\vartheta}$ and prescribed periods and canonical divisor (see Theorem 6.4).

Outline of the paper The necessary notation and background on nonorientable minimal surfaces in $\mathbb{R}^3$ is introduced in Section 2. In Section 3 we describe the compact subsets involved in the Mergelyan-type approximation, and define the notion of conformal nonorientable minimal immersion from such a subset into $\mathbb{R}^3$. In Section 4 we prove several preliminary approximation results that flatten the way to the proof of the main theorem in Section 5. Finally, the applications are derived in Section 6.

2 Preliminaries

Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Given a compact topological space K and a continuous map $f\colon K \to \mathbb{K}^n$, we denote by

$$\|f\|_{0,K} := \max_K \{\|f(p)\| : p \in K\}$$

the maximum norm of f on K. The corresponding space of continuous functions on K will be endowed with the $\mathcal{C}^0$ topology associated to $\|\cdot\|_{0,K}$.

Given a topological surface N, we denote by bN the (possibly nonconnected) 1–dimensional topological manifold determined by its boundary points. Given a subset $A \subset N$, we denote by A° and $\overline{A}$ the interior and the closure of A in N, respectively. Open connected subsets of $N \setminus bN$ will be called *domains* of N, and those proper connected topological subspaces of N being compact surfaces with boundary will said to be *regions* of N.

2.1 Riemann surfaces and nonorientability

It is well known that any Riemann surface is orientable; in fact, the conformal structure of the surface induces a (positive) orientation on it. In this subsection, we describe the notion of *nonorientable Riemann surface*.

A Riemann surface $\mathcal{N}$ is said to be *open* if it is noncompact and $b\mathcal{N} = \varnothing$. We denote by ∂ the global complex operator given by $\partial|_U = \frac{\partial}{\partial z}dz$ for any conformal chart (U, z) on $\mathcal{N}$.

Definition 2.1 Let $\underline{\mathcal{N}}$ be a smooth nonorientable surface with empty boundary. A system of coordinates $\mathscr{C}$ on $\underline{\mathcal{N}}$ is said to be a *conformal structure* on $\underline{\mathcal{N}}$ if the change of coordinates is conformal or anticonformal. The couple $(\underline{\mathcal{N}}, \mathscr{C})$ is said to be a *nonorientable Riemann surface*. If there is no place for ambiguity, we simply write $\underline{\mathcal{N}}$ instead of $(\underline{\mathcal{N}}, \mathscr{C})$.

Definition 2.2 Let $\underline{\mathcal{N}} \equiv (\underline{\mathcal{N}}, \mathscr{C})$ be a nonorientable Riemann surface. Denote by $\pi\colon \mathcal{N} \to \underline{\mathcal{N}}$ the oriented 2–sheeted covering of $\underline{\mathcal{N}}$, and call $\mathfrak{I}\colon \mathcal{N} \to \mathcal{N}$ the deck transformation of π. Call $\pi^*(\mathscr{C})$ the holomorphic system of coordinates in $\mathcal{N}$ determined by the positively oriented lifts by π of the charts in $\mathscr{C}$.

Notice that the couple $\mathcal{N} \equiv (\mathcal{N}, \pi^*(\mathscr{C}))$ is a (connected) open Riemann surface and $\mathfrak{I}$ is an antiholomorphic involution in $\mathcal{N}$ without fixed points. The conformal map $\pi\colon \mathcal{N} \to \underline{\mathcal{N}}$ is said to be the conformal orientable two-sheeted covering of $\underline{\mathcal{N}}$.

Objects related to $\underline{\mathcal{N}}$ will be denoted with underlined text (for instance $\underline{S}$, $\underline{X}$ etc), whereas those related to $\mathcal{N}$ will be not.

As a consequence of Definition 2.2, the nonorientable Riemann surface $\underline{\mathcal{N}}$ can be naturally identified with the orbit space $\mathcal{N}/\mathfrak{I}$, and the covering map π with the natural projection $\mathcal{N} \to \mathcal{N}/\mathfrak{I}$. In other words, a nonorientable Riemann surface $\underline{\mathcal{N}}$ is nothing but a connected open Riemann surface $\mathcal{N}$ equipped with an antiholomorphic involution $\mathfrak{I}$ without fixed points.

From now on in Section 2, let $\underline{\mathcal{N}}$, $\mathcal{N}$, π and $\mathfrak{I}$ be as in Definition 2.2.

Definition 2.3 A subset $A \subset \mathcal{N}$ is said to by $\mathfrak{I}$–invariant if $\mathfrak{I}(A) = A$, or equivalently, $\pi^{-1}(\pi(A)) = A$. If A is $\mathfrak{I}$–invariant, we write $\underline{A} = \pi(A)$. Likewise, given $\underline{B} \subset \underline{\mathcal{N}}$, we write B for the $\mathfrak{I}$–invariant set $\pi^{-1}(\underline{B})$.

Definition 2.4   Let A be an $\mathfrak{I}$–invariant subset in $\mathcal{N}$ and let $f\colon A \to \mathbb{R}^n$ be a map ($n \in \mathbb{N}$). The map f is said to be $\mathfrak{I}$–*invariant* if

$$f \circ (\mathfrak{I}|_A) = \overline{f}.$$

In this case, we denote by $\underline{f}$ the only map $\underline{f}\colon \underline{A} \to \mathbb{R}^n$ satisfying $f = \underline{f} \circ (\pi|_A)$. Likewise, given a map $\underline{f}\colon \underline{A} \to \mathbb{R}^n$ we denote by f the $\mathfrak{I}$–invariant map $f = \underline{f} \circ (\pi|_A)\colon A \to \mathbb{R}^n$.

For any set $A \subset \mathcal{N}$, we denote by $\mathfrak{Div}(A)$ the free commutative group of divisors with support on A with multiplicative notation. If $D = \prod_{i=1}^{n} Q_i^{n_i} \in \mathfrak{Div}(A)$, where $n_i \in \mathbb{Z} \setminus \{0\}$ for all $i \in \{1, \ldots, n\}$, we set $\mathrm{supp}(D) := \{Q_1, \ldots, Q_n\}$ to be the support of D. A divisor $D \in \mathfrak{Div}(A)$ is said to be integral if $D = \prod_{i=1}^{n} Q_i^{n_i}$ and $n_i \geq 0$ for all i. Given D_1, $D_2 \in \mathfrak{Div}(A)$, $D_1 \geq D_2$ means that $D_1 D_2^{-1}$ is integral. If A is $\mathfrak{I}$–invariant, then we denote by $\mathfrak{Div}_{\mathfrak{I}}(A)$ the group of $\mathfrak{I}$–invariant divisors of A; that is to say, satisfying $\mathfrak{I}(D) = D$.

In the sequel, W will denote an $\mathfrak{I}$–invariant open subset of $\mathcal{N}$.

We denote by

- $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(W)$ the real vector space of holomorphic functions f on W such that $f \circ (\mathfrak{I}|_W) = \overline{f}$,

- $\mathcal{F}_{\mathfrak{m},\mathfrak{I}}(W)$ the real vector space of meromorphic functions f on W such that $f \circ (\mathfrak{I}|_W) = \overline{f}$,

- $\Omega_{\mathfrak{h},\mathfrak{I}}(W)$ the real vector space of holomorphic 1–forms θ on W such that $\mathfrak{I}^*(\theta) = \overline{\theta}$,

- $\Omega_{\mathfrak{m},\mathfrak{I}}(W)$ the real vector space of meromorphic 1–forms θ on W such that $\mathfrak{I}^*(\theta) = \overline{\theta}$.

Here, and from now on, $\overline{\cdot}$ means complex conjugation. We also denote by

- $\mathcal{G}_{\mathfrak{I}}(W)$ the family of meromorphic functions g on W satisfying $g \circ (\mathfrak{I}|_W) = -\dfrac{1}{\overline{g}}$.

By elementary symmetrization arguments, it is easy to check $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(W)$, $\Omega_{\mathfrak{h},\mathfrak{I}}(W) \neq \varnothing$. It is also known that $\mathcal{G}_{\mathfrak{I}}(\mathcal{N}) \neq \varnothing$ when $\mathcal{N}$ is a compact Riemann surface (see Martens [26]). As an application of Theorem 5.6, we will prove that in fact every open nonorientable Riemann surface $(\mathcal{N}, \mathfrak{I})$ carries conformal maps into the projective plane omitting one point (see Corollary 6.10); in particular $\mathcal{G}_{\mathfrak{I}}(\mathcal{N}) \neq \varnothing$.

Let us recall some well-known topological facts regarding nonorientable surfaces.

In the remainder of this subsection, we will assume that W is a domain of finite topology. Then $(W, \mathfrak{I}|_W)$ is topologically equivalent to

$$(\mathcal{S} \setminus \{P_1, \ldots, P_{k+1}, \tilde{\mathfrak{J}}(P_1), \ldots, \tilde{\mathfrak{J}}(P_{k+1})\}, \tilde{\mathfrak{J}}),$$

where $\mathcal{S}$ is a compact surface of genus v, $\tilde{\mathfrak{J}} \colon \mathcal{S} \to \mathcal{S}$ is an orientation reversing involution without fixed points, and $\{P_1, \ldots, P_{k+1}\} \subset \mathcal{S}$, $k \in \mathbb{N} \cup \{0\}$.

As a consequence, the first homology groups $\mathcal{H}_1(W, \mathbb{Z})$ and $\mathcal{H}_1(W, \mathbb{R})$ of W are *free* abelian groups of dimension $2v_0 + 1$, where $v_0 := v + k$. Furthermore, elementary topological arguments give that $\mathcal{H}_1(W, \mathbb{R})$ admits an $\mathfrak{I}$–basis accordingly to the following definition:

Definition 2.5 A basis $\mathcal{B} = \{c_0, c_1, \ldots, c_{v_0}, d_1, \ldots, d_{v_0}\}$ of $\mathcal{H}_1(W, \mathbb{R})$ is said to be an $\mathfrak{I}$–basis if

- $c_j := \gamma_j - \mathfrak{I}_*(\gamma_j), \quad j = 0, 1, \ldots, v_0,$
- $d_j := \gamma_j + \mathfrak{I}_*(\gamma_j), \quad j = 1, \ldots, v_0,$

for some closed curves $\{\gamma_j : j = 0, \ldots, v_0\} \subset \mathcal{H}_1(W, \mathbb{Z})$. Observe that

(1) $$\mathfrak{I}_*(c_j) = -c_j \quad \text{and} \quad \mathfrak{I}_*(d_j) = d_j \quad \text{for all } j.$$

Let $\mathcal{H}^1_{\mathrm{hol}, \mathfrak{I}}(W)$ be the first real de Rham cohomology group $\Omega_{\mathfrak{h}, \mathfrak{I}}(W)/\sim$, where as usual $\sim$ denotes the equivalence relation "*the difference is exact*". Recall that, on an open Riemann surface, the first de Rham cohomology group is representable by holomorphic functions (this is in fact a special case of the Cartan–Serre theorem on Stein manifolds and affine algebraic manifolds). Notice that (1) gives that $\mathrm{Re} \int_{c_j} \tau = 0$ and $\mathrm{Im} \int_{d_j} \tau = 0$ for all j and $\tau \in \Omega_{\mathfrak{h}, \mathfrak{I}}(W)$. Further, basic cohomology theory gives that the map

(2) $$\mathcal{H}^1_{\mathrm{hol}, \mathfrak{I}}(W) \to \mathbb{R}^{v_0+1} \times \mathbb{R}^{v_0}, \quad [\tau] \mapsto \left(\left(-\imath \int_{c_j} \tau \right)_{j=0,\ldots,v_0}, \left(\int_{d_j} \tau \right)_{j=1,\ldots,v_0} \right),$$

is a (real) linear isomorphism for any $\mathfrak{I}$–basis $\{c_0, c_1, \ldots, c_{v_0}, d_1, \ldots, d_{v_0}\}$ of $\mathcal{H}_1(W, \mathbb{R})$.

2.2 Nonorientable minimal surfaces

In this subsection we describe the Weierstrass representation formula for nonorientable minimal surfaces, and introduce some notation.

Definition 2.6 A map $\underline{X}\colon \underline{\mathcal{N}} \to \mathbb{R}^3$ is said to be a *conformal nonorientable minimal immersion* if the $\mathfrak{I}$–invariant map

$$X := \underline{X} \circ \pi \colon \mathcal{N} \to \mathbb{R}^3$$

(see Definition 2.4) is a conformal minimal immersion. In this case, $X(\mathcal{N})=\underline{X}(\underline{\mathcal{N}})\subset\mathbb{R}^3$ is a nonorientable minimal surface.

For any $\mathfrak{I}$–invariant subset $A \subset \mathcal{N}$, we denote by $\mathcal{M}_\mathfrak{I}(A)$ the space of $\mathfrak{I}$–invariant conformal minimal immersions of $\mathfrak{I}$–invariant open subsets of $\mathcal{N}$ containing A into $\mathbb{R}^3$.

Let $A \subset \mathcal{N}$ be an $\mathfrak{I}$–invariant subset, and let $X \in \mathcal{M}_\mathfrak{I}(A)$. We use the notation $\phi_j = \partial X_j$, $j = 1, 2, 3$, and $\Phi = \partial X \equiv (\phi_j)_{j=1,2,3}$. The 1–forms ϕ_j are holomorphic (on an open neighborhood of A), have no real periods, and satisfy

$$\sum_{j=1}^{3} \phi_j^2 = 0, \tag{3}$$

$$\mathfrak{I}^*\Phi = \bar{\Phi}; \tag{4}$$

see [27]. The intrinsic metric in (an open neighborhood of) A is given by $ds^2 = \sum_{j=1}^{3} |\phi_j|^2$; hence

$$\sum_{j=1}^{3} |\phi_j|^2 \quad \text{does not vanish anywhere on } A. \tag{5}$$

By definition, the triple Φ is said to be the *Weierstrass representation* of X. The meromorphic function

$$g = \frac{\phi_3}{\phi_1 - \iota\phi_2} \tag{6}$$

(here, and from now on, we denote $\iota = \sqrt{-1}$) corresponds to the Gauss map of X up to the stereographic projection, and

$$\Phi = \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{\iota}{2}\left(\frac{1}{g} + g\right), 1\right)\phi_3 \tag{7}$$

(see Osserman [31]). It follows from (4) and (6) that the complex Gauss map $g\colon A \to \bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ of X satisfies

$$g \circ (\mathfrak{I}|_A) = -\frac{1}{\bar{g}}. \tag{8}$$

Remark 2.7 We denote by $\mathfrak{A}\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the antipodal map $\mathfrak{A}(z) = -1/\overline{z}$, by $\mathbb{RP}^2 = \overline{\mathbb{C}}/\mathfrak{A}$ the projective plane, and by $\pi_{\mathfrak{A}}\colon \overline{\mathbb{C}} \to \mathbb{RP}^2$ the orientable 2–sheeted covering of $\mathbb{RP}^2$.

Every meromorphic g in (an open neighborhood of) A satisfying (8) induces a unique conformal map $G\colon \underline{A} \to \mathbb{RP}^2$ such that $G \circ (\pi|_A) = \pi_{\mathfrak{A}} \circ g$.

Definition 2.8 The conformal map $G\colon \underline{A} \to \mathbb{RP}^2$ induced by the complex Gauss map g of X is said to be the *complex Gauss map* of the conformal nonorientable minimal immersion $\underline{X}$.

Conversely, any vectorial holomorphic 1–form $\Phi = (\phi_j)_{j=1,2,3}$ on (an open neighborhood of) A *without real periods*, satisfying (3), (4) and (5), determines an $\mathfrak{I}$–invariant conformal minimal immersion $X\colon A \to \mathbb{R}^3$ by the expression

$$X = \mathrm{Re} \int \Phi,$$

hence, a conformal nonorientable minimal immersion $\underline{X}\colon \underline{A} \to \mathbb{R}^3$; see [27]. By definition, the couple $(\Phi, \mathfrak{I})$ is said to be the *Weierstrass representation* of $\underline{X}$.

Remark 2.9 A vectorial holomorphic 1–form Φ on (an open neighborhood of) A satisfying (4) has no real periods if and only if

$$\int_{\gamma} \Phi = 0 \quad \text{for any } \gamma \in \mathcal{H}_1(A, \mathbb{Z}) \text{ with } \mathfrak{I}_*(\gamma) = \gamma.$$

To finish this subsection we present the *flux* of a conformal minimal immersion.

Definition 2.10 Let A be an $\mathfrak{I}$–invariant subset in $\mathcal{N}$, $X \in \mathcal{M}_{\mathfrak{I}}(A)$, (see Definition 2.6) and $\gamma(s)$ be an arc-length parameterized curve in A. The *conormal vector field* of X along γ is the unique unitary tangent vector field μ of X along γ such that $\{dX(\gamma'(s)), \mu(s)\}$ is a positive basis for all s.

If γ is closed, $\mathfrak{p}_X(\gamma) := \int_{\gamma} \mu(s)\, ds$ is said to be the *flux* of X along γ.

Given an $\mathfrak{I}$–invariant subset A in $\mathcal{N}$, and $X \in \mathcal{M}_{\mathfrak{I}}(A)$, it is easy to check that $\mathfrak{p}_X(\gamma) = \mathrm{Im} \int_{\gamma} \partial X$ and that the flux map

$$\mathfrak{p}_X\colon \mathcal{H}_1(A, \mathbb{Z}) \to \mathbb{R}^3$$

is a group morphism. Furthermore, since X is $\mathfrak{I}$–invariant and $\mathfrak{I}$ reverses the orientation, the flux map $\mathfrak{p}_X\colon \mathcal{H}_1(A, \mathbb{Z}) \to \mathbb{R}^3$ of X satisfies

$$(9) \qquad \mathfrak{p}_X(\mathfrak{I}_*(\gamma)) = -\mathfrak{p}_X(\gamma) \quad \text{for all } \gamma \in \mathcal{H}_1(A, \mathbb{Z}).$$

By definition, the couple $(\mathfrak{p}_X, \mathfrak{I})$ is said to be the flux map of $\underline{X}$.

3   Admissible subsets for the Mergelyan approximation

We begin this section by describing the subsets involved in the Mergelyan approximation theorem in Section 5. Although there is room for generalizations, the sets considered in Definition 3.2 are sufficient for our geometric applications.

Remark 3.1 From now on in the paper, $\underline{\mathcal{N}}$, $\mathcal{N}$, π and $\mathfrak{I}$ will be as in Definition 2.2. Also, $\sigma_{\mathcal{N}}^2$ will denote a conformal Riemannian metric on $\mathcal{N}$ such that $\mathfrak{I}^*(\sigma_{\mathcal{N}}^2) = \sigma_{\mathcal{N}}^2$.

First of all, recall that a subset $A \subset \mathcal{N}$ is said to be *Runge* (in $\mathcal{N}$) if $\mathcal{N} \setminus A$ has no relatively compact connected components.

A compact Jordan arc in $\mathcal{N}$ is said to be analytical (smooth, continuous etc) if it is contained in an open analytical (smooth, continuous etc) Jordan arc in $\mathcal{N}$.
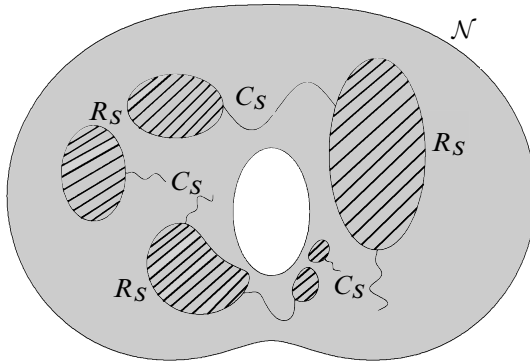

Figure 1: An $\mathfrak{I}$–admissible set $S \subset \mathcal{N}$

Definition 3.2 A (possibly nonconnected) $\mathfrak{I}$–invariant compact subset $S \subset \mathcal{N}$ is said to be $\mathfrak{I}$–admissible in $\mathcal{N}$ if and only if (see Figure 1):

(i) S is Runge.

(ii) $R_S := \overline{S^\circ}$ is nonempty and consists of a finite collection of pairwise disjoint compact regions in $\mathcal{N}$ with $\mathcal{C}^0$ boundary.

(iii) $C_S := \overline{S \setminus R_S}$ consists of a finite collection of pairwise disjoint analytical Jordan arcs.

(iv) Any component α of C_S with an endpoint $P \in R_S$ admits an analytical extension β in $\mathcal{N}$ such that the unique component of $\beta \setminus \alpha$ with endpoint P lies in R_S.

An $\mathfrak{I}$–invariant compact subset $S \subset \mathcal{N}$ satisfying (ii), (iii) and (iv) is Runge (hence $\mathfrak{I}$–admissible) if and only if $i_* \colon \mathcal{H}_1(S, \mathbb{Z}) \to \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$ is a monomorphism, where $\mathcal{H}_1(\cdot, \mathbb{Z})$ means the first homology group, $i \colon S \to \mathcal{N}$ is the inclusion map, and i_* is the induced group morphism. If $S \subset \mathcal{N}$ is an $\mathfrak{I}$–invariant compact Runge subset consisting of a finite collection of pairwise disjoint compact regions with $\mathcal{C}^0$ boundary, then S is $\mathfrak{I}$–admissible; that is to say, we allow C_S to be empty. The most typical $\mathfrak{I}$–admissible subsets S in $\mathcal{N}$ consist of a finite collection of pairwise disjoint compact regions R_S with $\mathcal{C}^1$ boundary, and a finite collection of Jordan analytical arcs C_S meeting bR_S transversally.

3.1 Functions on $\mathfrak{I}$–admissible subsets

From now on in this section, S will denote an $\mathfrak{I}$–admissible subset in $\mathcal{N}$, in the sense of Definition 3.2.

Definition 3.3 We denote by

- $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ the real vector space of continuous functions $f \colon S \to \mathbb{C}$, holomorphic on an open neighborhood of R_S in $\mathcal{N}$, such that $f \circ \mathfrak{I}|_S = \overline{f}$,
- $\mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$ the real vector space of continuous functions $f \colon S \to \overline{\mathbb{C}}$, meromorphic on an open neighborhood of R_S in $\mathcal{N}$, satisfying that $f \circ \mathfrak{I}|_S = \overline{f}$ and $f^{-1}(\infty) \subset S° = R_S \setminus bR_S$.

Likewise, we denote by

- $\mathcal{G}_{\mathfrak{I}}(S)$ the family of continuous functions $g \colon S \to \overline{\mathbb{C}}$, meromorphic on an open neighborhood of R_S in $\mathcal{N}$, satisfying that $g \circ \mathfrak{I}|_S = -1/\overline{g}$ and $g^{-1}(\{0, \infty\}) \subset S° = R_S \setminus bR_S$.

A 1–form θ on S is said to be of type $(1, 0)$ if for any conformal chart (U, z) in $\mathcal{N}$, $\theta|_{U \cap S} = h(z)\, dz$ for some function $h \colon U \cap S \to \overline{\mathbb{C}}$. Finite sequences $\Theta = (\theta_1, \ldots, \theta_n)$, where θ_j is a $(1, 0)$–type 1–form for all $j \in \{1, \ldots, n\}$, are said to be n–dimensional vectorial $(1, 0)$–forms on S. The space of continuous n–dimensional $(1, 0)$–forms on S will be endowed with the $\mathcal{C}^0$ topology induced by the norm

$$(10) \qquad \|\Theta\|_{0,S} := \left\| \frac{\Theta}{\sigma_\mathcal{N}} \right\|_{0,S} = \max_S \left\{ \left(\sum_{j=1}^{n} \left| \frac{\theta_j}{\sigma_\mathcal{N}} \right|^2 \right)^{1/2} \right\};$$

see Remark 3.1.

Definition 3.4 For any $f \in \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$ we write $(f)_0$, $(f)_\infty$ and (f) for the zero divisor $(f|_{R_S})_0$, the polar divisor $(f|_{R_S})_\infty$ and the divisor $(f|_{R_S})$, respectively; see Farkas and Kra [13].

Notice that all these divisors lie in $\mathfrak{Div}_{\mathfrak{J}}(R_S)$. Obviously, we have that $\mathrm{supp}((f)_{\infty}) = f^{-1}(\infty) \subset R_S$ and $\mathrm{supp}((f)_0) = f^{-1}(0) \cap R_S$. Likewise we define the corresponding divisors for functions $g \in \mathcal{G}_{\mathfrak{J}}(S)$, but in this case they do not lie in $\mathfrak{Div}_{\mathfrak{J}}(R_S)$ unless $(g) = 1$. In fact, $\mathfrak{J}((g)_0) = (g)_{\infty}$.

The following Gunning–Narasimhan-type result for relatively compact $\mathfrak{J}$–invariant domains is required for later purposes. A general theorem in this line for nonorientable Riemann surfaces will be shown later in Section 6; see Theorem 6.4.

Proposition 3.5 *Let W be a relatively compact $\mathfrak{J}$–invariant open subset in $\mathcal{N}$. Then there exists a nowhere-vanishing holomorphic 1–form τ on W such that $\mathfrak{J}^*(\tau) = \overline{\tau}$.*

Proof Since the same argument applies separately to each connected component, we may assume that W is a domain.

Take a nowhere-vanishing holomorphic 1–form τ_0 on $\mathcal{N}$ (see [18]). If $\tau_0 + \overline{\mathfrak{J}^*(\tau_0)}$ vanishes everywhere on W, then it suffices to set $\tau := \iota \tau_0|_W$. Otherwise, consider $\tau_1 := (\tau_0 + \overline{\mathfrak{J}^*(\tau_0)})|_W$. Since $\mathfrak{J}$ is an antiholomorphic involution, τ_1 is holomorphic and $\mathfrak{J}^*(\tau_1) = \overline{\tau}_1$. By the identity principle τ_1 has finitely many zeros on the compact set $\overline{W}$; hence on W. Denote by D the divisor associated to $\tau_1|_{\overline{W}}$. Since $\tau_1 \in \Omega_{\mathfrak{h},\mathfrak{J}}(W)$ we can write $D = D_1 \mathfrak{J}(D_1)$, where $\mathrm{supp}(D_1) \cap \mathrm{supp}(\mathfrak{J}(D_1)) = \varnothing$. Since $\mathcal{N} \setminus \overline{W}$ is a nonempty open set, the Weierstrass theorem furnishes a meromorphic function h on $\mathcal{N}$ such that $h|_{\overline{W}}$ is holomorphic and $(h|_{\overline{W}})_0 = D_1$. Set $H := h \cdot \overline{h \circ \mathfrak{J}} \in \mathcal{F}_{\mathfrak{h},\mathfrak{J}}(\mathcal{N})$ and observe that $(H|_{\overline{W}})_0 = D$. We finish by setting $\tau := \tau_1/(H|_W)$. □

From now on in this section, let W and τ be as in Proposition 3.5 such that $S \subset W$. The following notions do not depend on the chosen W and τ.

Definition 3.6 We denote by

- $\Omega_{\mathfrak{h},\mathfrak{J}}(S)$ the real vector space of 1–forms θ of type $(1, 0)$ on S such that $\theta/\tau \in \mathcal{F}_{\mathfrak{h},\mathfrak{J}}(S)$,

- $\Omega_{\mathfrak{m},\mathfrak{J}}(S)$ the real vector space of 1–forms θ of type $(1, 0)$ on S such that $\theta/\tau \in \mathcal{F}_{\mathfrak{m},\mathfrak{J}}(S)$.

Define as above the associated divisors $(\theta)_0$ and $(\theta)_{\infty}$ of zeros and poles, respectively, for any $\theta \in \Omega_{\mathfrak{m},\mathfrak{J}}(S)$. Likewise, denote by $(\theta) = (\theta)_0/(\theta)_{\infty}$ the divisor of θ, and notice that all these divisors lie in $\mathfrak{Div}_{\mathfrak{J}}(S)$.

Definition 3.7 We shall say that:

- A function $f \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ can be approximated in the $\mathcal{C}^0$ topology on S by functions in $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(W)$ if there exists $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(W)$ such that $\{\|f_n|_S - f\|_{0,S}\}_{n \in \mathbb{N}} \to 0$.

- A function $f \in \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$ can be approximated in the $\mathcal{C}^0$ topology on S by functions in $\mathcal{F}_{\mathfrak{m},\mathfrak{I}}(W)$ if there exists $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(W)$ such that $f_n|_S - f \in \mathcal{F}_{\mathfrak{h}}(S)$ for all n and $\{\|f_n|_S - f\|_{0,S}\}_{n \in \mathbb{N}} \to 0$ (in particular, $(f_n)_\infty = (f)_\infty$ on S° for all n).

- A 1–form $\theta \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)$ can be approximated in the $\mathcal{C}^0$ topology on S by 1–forms in $\Omega_{\mathfrak{h},\mathfrak{I}}(W)$ if there exists $\{\theta_n\}_{n \in \mathbb{N}} \subset \Omega_{\mathfrak{h},\mathfrak{I}}(W)$ such that $\{\|\theta_n|_S - \theta\|_{0,S}\}_{n \in \mathbb{N}} \to 0$.

- A 1–form $\theta \in \Omega_{\mathfrak{m},\mathfrak{I}}(S)$ can be approximated in the $\mathcal{C}^0$ topology on S by 1–forms in $\Omega_{\mathfrak{m},\mathfrak{I}}(W)$ if there exists $\{\theta_n\}_{n \in \mathbb{N}} \subset \Omega_{\mathfrak{m},\mathfrak{I}}(W)$ such that $\theta_n|_S - \theta \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)$ for all n and $\{\|\theta_n|_S - \theta\|_{0,S}\}_{n \in \mathbb{N}} \to 0$ (in particular $(\theta_n)_\infty = (\theta)_\infty$ on S° for all n).

- A function $g \in \mathcal{G}_{\mathfrak{I}}(S)$ can be approximated in the $\mathcal{C}^0$ topology on S by functions in $\mathcal{G}_{\mathfrak{I}}(W)$ if there exists $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{G}_{\mathfrak{I}}(W)$ such that $g_n - g$ is holomorphic on (a neighborhood of) R_S and $\{\|g_n|_S - g\|_{0,S}\}_{n \in \mathbb{N}} \to 0$.

We define the notions of approximation in the $\mathcal{C}^0$ topology of vectorial functions and 1–forms in a similar way.

Definition 3.8 A function $f\colon S \to \mathbb{C}^n$, holomorphic on an open neighborhood of R_S, is said to be *smooth* if there exists a holomorphic extension f_0 of $f|_{R_S}$ to an open domain V in $\mathcal{N}$ containing R_S such that, for any component α of C_S and any open analytical Jordan arc β in $\mathcal{N}$ containing α, $f|_\alpha$ admits a smooth extension f_β to β satisfying that $f_\beta|_{V \cap \beta} = f_0|_{V \cap \beta}$.

Likewise we define the notion of smoothness for functions $S \to \bar{\mathbb{C}}^n$, holomorphic on an open neighborhood of R_S, and for functions $S \to \mathbb{R}^n$, harmonic on an open neighborhood of R_S.

Definition 3.9 A vectorial 1–form Θ of type $(1,0)$ on S, meromorphic on an open neighborhood of R_S, is said to be *smooth* if $\Theta/\tau\colon S \to \bar{\mathbb{C}}^n$ is smooth in the sense of Definition 3.8.

Definition 3.10 Given a smooth function (in the sense of Definition 3.8) $f \in \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S) \cup \mathcal{G}_{\mathfrak{I}}(S)$, we denote by df the 1–form of type $(1,0)$ given by

$$df|_{R_S} = \partial(f|_{R_S}) \quad \text{and} \quad df|_{\alpha \cap U} = (f \circ \alpha)'(x) dz|_{\alpha \cap U}$$

for any component α of C_S, where $(U, z = x + \imath y)$ is any conformal chart on $\mathcal{N}$ satisfying that $z(\alpha \cap U) \subset \mathbb{R} \equiv \{y = 0\}$ (the existence of such a conformal chart is guaranteed by the analyticity of α). Notice that df is well defined and smooth in the sense of Definition 3.9. Furthermore, $df|_\alpha(t) = (f \circ \alpha)'(t) \, dt$ for any component α of C_S, where t is any smooth parameter along α.

If $f \in \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$ is a smooth function, then df belongs to $\Omega_{\mathfrak{m},\mathfrak{I}}(S)$ (to $\Omega_{\mathfrak{h},\mathfrak{I}}(S)$ if $f \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$).

A smooth 1–form $\theta \in \Omega_{\mathfrak{m},\mathfrak{I}}(S)$ is said to be *exact* if $\theta = df$ for some smooth $f \in \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$, or equivalently if $\int_\gamma \theta = 0$ for all $\gamma \in \mathcal{H}_1(S, \mathbb{Z})$. The exactness of vectorial 1–forms in $\Omega_{\mathfrak{m},\mathfrak{I}}(S)^n$, $n \in \mathbb{N}$, is defined in the same way.

3.2   Conformal minimal immersions on $\mathfrak{I}$–admissible subsets

Let us begin this subsection by generalizing the notion of conformal minimal immersion to maps defined on $\mathfrak{I}$–admissible sets; see Definition 3.2, and also Definition 2.3, for notation.

Definition 3.11   A map $\underline{X}: \underline{S} \to \mathbb{R}^3$ is said to be a *generalized nonorientable minimal immersion* if the $\mathfrak{I}$–invariant map

$$X := \underline{X} \circ \pi|_S \colon S \to \mathbb{R}^3$$

(see Definition 2.4) satisfies the following properties:

- $X|_{R_S} \in \mathcal{M}_\mathfrak{I}(R_S)$ (see Definition 2.6).
- X is smooth in the sense of Definition 3.8 (observe that X is harmonic on an open neighborhood of R_S since $X|_{R_S} \in \mathcal{M}_\mathfrak{I}(R_S)$).
- $X|_{C_S}$ is regular, ie $X|_\alpha$ is a regular curve for all $\alpha \subset C_S$.

In this case, we also say that X is an $\mathfrak{I}$–*invariant generalized minimal immersion*, and write $X \in \mathcal{M}_{\mathfrak{g},\mathfrak{I}}(S)$.

Notice that $X|_S \in \mathcal{M}_{\mathfrak{g},\mathfrak{I}}(S)$ for all $X \in \mathcal{M}_\mathfrak{I}(S)$.

Let $X \in \mathcal{M}_{\mathfrak{g},\mathfrak{I}}(S)$, and let ϖ be a smooth 3–dimensional real 1–form on C_S. This means that $\varpi = (\varpi_j)_{j=1,2,3}$, where ϖ_j is a real smooth 1–form on C_S, $j = 1, 2, 3$. For any $\alpha \subset C_S$ we write $\varpi|_\alpha = \varpi(\alpha(s)) \, ds$, where s is the arc-length parameter of $X \circ \alpha$. By definition, ϖ is said to be a *mark* along C_S with respect to X if for any arc $\alpha \subset C_S$ the following conditions hold:

- $\varpi(\alpha(s)) \in \mathbb{R}^3$ is a smooth unitary vector field along α orthogonal to $(X \circ \alpha)'(s)$.

- ϖ extends smoothly to any open analytical arc β in $\mathcal{N}$ containing α.

- $\varpi(\beta(s))$ is unitary, orthogonal to $(X \circ \beta)'(s)$, and tangent to $X(R_S)$ at $\beta(s)$ for any $\beta(s) \in \beta \cap R_S$, where as above s is the arc-length parameter of $(X \circ \beta)(s)$.

Let $\mathfrak{n} \colon R_S \to \mathbb{S}^2$ denote the Gauss map of the (oriented) conformal minimal immersion $X|_{R_S}$. The mark ϖ is said to be *orientable* with respect to X if the orientations at the two endpoints of each arc in C_S agree, that is to say, if there exists $\delta \in \{-1, 1\}$ such that for any regular embedded curve $\alpha \subset S$ and arc-length parametrization $(X \circ \alpha)(s)$ of $X \circ \alpha$,

$$(X \circ \alpha)'(s_0) \times \varpi(\alpha(s_0)) = \delta \mathfrak{n}(\alpha(s_0)) \quad \text{for all } s_0 \in \alpha^{-1}(C_S \cap R_S).$$

Orientable marks along C_S with respect to X always exist since $\mathcal{N}$ is orientable. An orientable mark ϖ with respect to X is said to be *positively oriented* if $\delta = 1$. Obviously, if ϖ is orientable with respect to X then either ϖ or $-\varpi$ is positively oriented.

In the sequel we will only consider orientable marks.

If ϖ is a positively oriented mark along C_S with respect to X, we denote by $\mathfrak{n}_{\varpi} \colon S \to \mathbb{S}^2 \subset \mathbb{R}^3$ the map given by $\mathfrak{n}_{\varpi}|_{R_S} = \mathfrak{n}$ and $(\mathfrak{n}_{\varpi} \circ \alpha)(s) := (X \circ \alpha)'(s) \times \varpi(\alpha(s))$, where α is any component of C_S and s is any arc-length parameter of $X \circ \alpha$. By definition, $\mathfrak{n}_{\varpi}$ is said to be the *(generalized) Gauss map* of X associated to the orientable mark ϖ.

Definition 3.12 We denote by $\mathcal{M}^*_{\mathfrak{g}, \mathfrak{J}}(S)$ the space of marked immersions $X_{\varpi} := (X, \varpi)$, where $X \in \mathcal{M}_{\mathfrak{g}, \mathfrak{J}}(S)$ and ϖ is a positively oriented mark along C_S with respect to X satisfying the following properties:

- $\mathfrak{J}^*(\varpi) = -\varpi$, or equivalently,

(11) $$\mathfrak{n}_{\varpi} \circ \mathfrak{J} = -\mathfrak{n}_{\varpi}.$$

- If $\mathrm{st} \colon \mathbb{S}^2 \to \overline{\mathbb{C}}$ is the stereographic projection, the function $\mathrm{st} \circ \mathfrak{n}_{\varpi} \colon S \to \overline{\mathbb{C}}$, which is holomorphic on an open neighborhood of R_S, is smooth in the sense of Definition 3.8.

Remark 3.13 Marked minimal immersions play the role of $\mathfrak{J}$–invariant conformal minimal immersions of $\mathfrak{J}$–admissible subsets into $\mathbb{R}^3$. They will be the natural initial conditions for the Mergelyan approximation theorem in Section 5.

Let $X_\varpi \in \mathcal{M}^*_{\mathfrak{g},\mathfrak{I}}(S)$, and let $\partial X_\varpi = (\widehat{\phi}_j)_{j=1,2,3}$ be the complex vectorial 1–form of type $(1,0)$ on S determined by

$$\partial X_\varpi|_{R_S} = \partial(X|_{R_S}), \quad \partial X_\varpi(\alpha'(s)) = dX(\alpha'(s)) + \imath\varpi(\alpha(s)),$$

where α is a component of C_S and s is an arc-length parameter of $X \circ \alpha$. If $(U, z = x + \imath y)$ is a conformal chart on $\mathcal{N}$ such that $\alpha \cap U = z^{-1}(\mathbb{R} \cap z(U))$, it is clear that $(\partial X_\varpi)|_{\alpha \cap U} = [dX(\alpha'(s)) + \imath\varpi(\alpha(s))]s'(x)dz|_{\alpha \cap U}$, hence $\partial X_\varpi \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)^3$.

Furthermore, the smooth function $\widehat{g} := \mathrm{st} \circ \mathfrak{n}_\varpi \colon S \to \overline{\mathbb{C}}$ (see Definition 3.12) satisfies

$$\widehat{g} = \frac{\widehat{\phi}_3}{\widehat{\phi}_1 - \imath\widehat{\phi}_2}$$

(see [31]) and formally meets (8); hence $\widehat{g} \in \mathcal{G}_{\mathfrak{I}}(S)$ provided that $\widehat{g}^{-1}(\{0, \infty\}) \subset R_S \setminus bR_S$.

Obviously, $\widehat{\phi}_j$ is smooth on S, $j = 1, 2, 3$, and the same occurs for $\widehat{g}$ (see Definition 3.8 and Definition 3.9). In addition, ∂X_ϖ formally satisfies (3), (4), (5), and $\mathrm{Re}\,\widehat{\phi}_j$ is an exact real 1–form on S, $j = 1, 2, 3$; hence we also have

$$X(P) = X(Q) + \mathrm{Re} \int_Q^P (\widehat{\phi}_j)_{j=1,2,3}, \quad P, Q \in S.$$

For these reasons, $(\widehat{g}, \widehat{\phi}_3)$ will be said to be the *generalized Weierstrass data* of X_ϖ. Since ∂X_ϖ and $\widehat{g}$ formally satisfy (4) and (8), one can introduce the *generalized complex Gauss map* $\widehat{G} \colon \underline{S} \to \mathbb{RP}^2$ of $\underline{X}$ associated to ϖ; see Definition 2.8.

Notice that $X|_{R_S} \in \mathcal{M}_{\mathfrak{I}}(R_S)$, hence $(\phi_j)_{j=1,2,3} := (\widehat{\phi}_j|_{R_S})_{j=1,2,3}$ and $g := \widehat{g}|_{R_S}$ are obviously the Weierstrass data and the complex Gauss map of $X|_{R_S}$, respectively.

The space $\mathcal{M}^*_{\mathfrak{g},\mathfrak{I}}(S)$ is naturally endowed with the following $\mathcal{C}^1$ topology.

Definition 3.14 Given X_ϖ, $Y_\xi \in \mathcal{M}^*_{\mathfrak{g},\mathfrak{I}}(S)$, we set

$$\|X_\varpi - Y_\xi\|_{1,S} := \|X - Y\|_{0,S} + \|\partial X_\varpi - \partial Y_\xi\|_{0,S};$$

see (10). Given $F \in \mathcal{M}_{\mathfrak{I}}(S)$, we denote by ϖ_F the conormal mark of F along C_S, that is, the only positively oriented mark along C_S with respect to F satisfying $(\mathfrak{n} \circ \alpha)(s) := (F \circ \alpha)'(s) \times \varpi_F(\alpha(s))$, where $\mathfrak{n} \colon S \to \mathbb{S}^2$ is the Gauss map of F, α is any component of C_S, and s is the arc-length parameter of $F \circ \alpha$. Notice that ϖ_F satisfies (11) and $\partial F|_S = \partial F_{\varpi_F}$, where $F_{\varpi_F} := (F|_S, \varpi_F) \in \mathcal{M}^*_{\mathfrak{g},\mathfrak{I}}(S)$.

Given F, $G \in \mathcal{M}_{\mathfrak{I}}(S)$, we set

$$\|F - X_\varpi\|_{1,S} := \|F_{\varpi_F} - X_\varpi\|_{1,S} \quad \text{and} \quad \|F - G\|_{1,S} := \|F_{\varpi_F} - G_{\varpi_G}\|_{1,S}.$$

Definition 3.15 We shall say that $X_\varpi \in \mathcal{M}^*_{\mathfrak{g},\mathfrak{I}}(S)$ can be approximated in the $\mathcal{C}^1$ topology on S by $\mathfrak{I}$–invariant conformal minimal immersions in $\mathcal{M}_\mathfrak{I}(W)$ if for any $\epsilon > 0$ there exists $Y \in \mathcal{M}_\mathfrak{I}(W)$ such that $\|Y - X_\varpi\|_{1,S} < \epsilon$.

If $X_\varpi \in \mathcal{M}^*_{\mathfrak{g},\mathfrak{I}}(S)$, then the group homomorphism

$$\mathfrak{p}_{X_\varpi} \colon \mathcal{H}_1(S, \mathbb{Z}) \to \mathbb{R}^3, \quad \mathfrak{p}_{X_\varpi}(\gamma) = \mathrm{Im} \int_\gamma \partial X_\varpi,$$

is said to be the *generalized flux map* of X_ϖ. Notice that $\mathfrak{p}_{X_\varpi}$ satisfies (9). Obviously, $\mathfrak{p}_{X_{\varpi_Y}} = \mathfrak{p}_Y|_{\mathcal{H}_1(S,\mathbb{Z})}$ provided that $X = Y|_S$ for some $Y \in \mathcal{M}_\mathfrak{I}(S)$.

4 Approximation results

Throughout this section, $S \subset \mathcal{N}$ will denote an $\mathfrak{I}$–admissible subset, and W a relatively compact $\mathfrak{I}$–invariant open subset of $\mathcal{N}$ containing S.

In this section we state and prove several preliminary approximation results that will be key in the proof of the main theorem, in Section 5. In particular, Lemmas 4.2 and 4.6 deal with functions in $\mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$ and $\mathcal{G}_\mathfrak{I}(S)$, respectively. We begin with the following:

Lemma 4.1 For any $f \in \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$ and integral (or effective) divisor $D \in \mathfrak{Div}_\mathfrak{I}(S)$ with $\mathrm{supp}(D) \subset S^\circ$, there exists $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(W)$ such that $f_n|_S - f \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ and $(f_n|_S - f) \geq D$ for all $n \in \mathbb{N}$, and $\{\|f_n|_S - f\|_{0,S}\}_{n\in\mathbb{N}} \to 0$.

Proof By classical approximation results (see [6, Theorem 4.1] for details), there exists a sequence of meromorphic functions $\{h_n \colon W \to \overline{\mathbb{C}}\}_{n\in\mathbb{N}}$ such that $h_n|_S - f \colon S \to \mathbb{C}$ is continuous on S and holomorphic in a neighborhood of R_S, $(h_n|_S - f) \geq D$ for all $n \in \mathbb{N}$ (see Definition 3.4), and $\{\|h_n|_S - f\|_{0,S}\}_{n\in\mathbb{N}} \to 0$. Since $f \circ \mathfrak{I} = \overline{f}$ and D is $\mathfrak{I}$–invariant, the sequence $\{\overline{h_n \circ \mathfrak{I}} \colon W \to \mathbb{C}\}_{n\in\mathbb{N}}$ meets the same properties. Therefore, it suffices to set $f_n := \frac{1}{2}(h_n + \overline{h_n \circ \mathfrak{I}})$ for all $n \in \mathbb{N}$. □

Let us prove the following deeper approximation result for functions in $\mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$:

Lemma 4.2 Let $f \in \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$ vanishing nowhere on $S \setminus S^\circ = (bR_S) \cup C_S$, and let $D \in \mathfrak{Div}_\mathfrak{I}(S)$ an integral divisor with $\mathrm{supp}(D) \subset S^\circ$.

Then there exists $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(W)$ satisfying $f_n|_S - f \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$, $\mathrm{supp}((f_n)) \subset S^\circ$, $(f_n) = (f)$, $(f_n|_S - f) \geq D$ for all $n \in \mathbb{N}$, and $\{\|f_n|_S - f\|_{0,S}\}_{n\in\mathbb{N}} \to 0$. In particular, f_n is holomorphic and does not vanish anywhere on $W \setminus S^\circ$ for all $n \in \mathbb{N}$.

Proof Since the space of smooth functions (in the sense of Definition 3.8) is dense in $\mathcal{F}_{\mathfrak{m},\mathfrak{J}}(S)$ under the $\mathcal{C}^0$ topology, we can assume that f is smooth. Furthermore, since the same argument applies separately to each connected component, we may assume that W is a domain.

Let us begin the proof with the following reduction:

Claim 4.3 *It suffices to prove the lemma for nowhere-vanishing functions in $\mathcal{F}_{\mathfrak{h},\mathfrak{J}}(S)$.*

Proof Assume that the lemma holds for nowhere-vanishing functions in $\mathcal{F}_{\mathfrak{h},\mathfrak{J}}(S)$.

Take any $f \in \mathcal{F}_{\mathfrak{m},\mathfrak{J}}(S)$ and $D \in \mathfrak{Div}_{\mathfrak{J}}(S)$ as in the statement of the lemma, and write $(f) = D_1 \mathfrak{J}(D_1)$, where $\mathrm{supp}(D_1) \cap \mathrm{supp}(\mathfrak{J}(D_1)) = \varnothing$. Since W is relatively compact in the open Riemann surface $\mathcal{N}$, the Riemann–Roch theorem furnishes a meromorphic function h_0 on $\mathcal{N}$ with $(h_0|_{\overline{W}}) = D_1$. The function $F := f/(h_0(\overline{h_0 \circ \mathfrak{J}}))$ lies in $\mathcal{F}_{\mathfrak{h},\mathfrak{J}}(S)$ and does not vanish anywhere on S. By our assumption, there exists a sequence of nowhere vanishing functions $\{F_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}_{\mathfrak{h},\mathfrak{J}}(W)$ approximating F on S and satisfying $(F - F_n) \geq D_2$, where $D_2 \in \mathfrak{Div}_{\mathfrak{J}}(S)$ is any given integral divisor with $\mathrm{supp}(D_2) \subset S^\circ$ and $D_2 \geq DD_1^{-1}\mathfrak{J}(D_1)^{-1}$.

If we choose $f_n := F_n h_0 (\overline{h_0 \circ \mathfrak{J}}) \in \mathcal{F}_{\mathfrak{m},\mathfrak{J}}(W)$, then $f_n|_S - f \in \mathcal{F}_{\mathfrak{h},\mathfrak{J}}(S)$, $(f_n|_S - f) \geq D$ and $(f_n) = (f)$ for all $n \in \mathbb{N}$, and $\{f_n\}_{n \in \mathbb{N}}$ approximates f on S. $\qquad\square$

In the sequel we will assume that $f \in \mathcal{F}_{\mathfrak{h},\mathfrak{J}}(S)$ and has no zeros.

By the isomorphism (2), there exists $\tau \in \Omega_{\mathfrak{h},\mathfrak{J}}(W)$ such that

$$(12) \qquad\qquad \int_\gamma df/f = \int_\gamma \tau \quad \text{for all } \gamma \in \mathcal{H}_1(S, \mathbb{Z}),$$

$$(13) \qquad\qquad \frac{1}{2\pi\iota} \int_\gamma \tau \in \mathbb{Z} \quad \text{for all } \gamma \in \mathcal{H}_1(W, \mathbb{Z}).$$

Here we have taken into account that f is assumed to be smooth, $\mathcal{H}_1(S, \mathbb{Z})$ is a natural subgroup of $\mathcal{H}_1(W, \mathbb{Z})$ (recall that S is Runge in W) and $\frac{1}{2\pi\iota} \int_\gamma df/f \in \mathbb{Z}$ for all $\gamma \in \mathcal{H}_1(S, \mathbb{Z})$.

We need the following:

Claim 4.4 *There exist a nowhere vanishing function $v \in \mathcal{F}_{\mathfrak{h},\mathfrak{J}}(W)$ and a function $u \in \mathcal{F}_{\mathfrak{h},\mathfrak{J}}(S)$ such that $d\log(v) = \tau$, $du = f/df - \tau|_S$ and $f = ve^u$.*

Proof To construct v, fix $P_0 \in W$ and notice that $\mathrm{Re} \int_{P_0}^{\mathfrak{I}(P_0)} \tau = 0$ independently on the arc connecting P_0 and $\mathfrak{I}(P_0)$. Indeed, take any oriented Jordan arc $\gamma \subset W$ with initial point P_0 and final point $\mathfrak{I}(P_0)$, and simply observe that

$$\overline{\int_\gamma \tau} = \int_\gamma \overline{\tau} = \int_\gamma \mathfrak{I}^*(\tau) = \int_{\mathfrak{I}_*(\gamma)} \tau = -\int_\gamma \tau + 2k\pi\iota, \quad k \in \mathbb{Z}.$$

For the last equality, take into account that $\gamma + \mathfrak{I}_*(\gamma) \in \mathcal{H}_1(W, \mathbb{Z})$ and (13).

Therefore, the well-defined function

$$v := \exp\left(\int_{P_0} \tau - \frac{1}{2} \int_{P_0}^{\mathfrak{I}(P_0)} \tau \right)$$

lies in $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(W)$, and obviously satisfies $d \log(v) = \tau$.

To construct u, recall that $df/f - \tau|_S$ lies in $\Omega_{\mathfrak{h},\mathfrak{I}}(S)$ and is exact; see (12). For each connected component C of S, fix $P_C \in C$ and set $u|_C := A_C + \int_{P_C} (f/df - \tau)$, where the constant $A_C \in \mathbb{C}$ is chosen so that $(f - ve^{u}|_C)(P_C) = 0$. Since the function $\kappa := f/(e^u v)$ is locally constant on S (just observe that $d \log(\kappa) = 0$) and $\kappa(P_C) = 1$ for any connected component C of S, we infer that $\kappa = 1$, that is to say, $f = ve^u$ on S.

Since $f, v|_S \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$, we have that $u \circ \mathfrak{I} = \bar{u} + 2m\pi\iota$ for some $m \in \mathbb{Z}$. Up to replacing u and v for $u - m\pi\iota$ and $e^{m\pi\iota}v$, respectively, we get that $u \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ and the functions u and v solve the claim. $\square$

Let $u \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ and $v \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(W)$ like in the previous claim. By Lemma 4.1, there exists $\{h_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(W)$ such that $(h_n - u) \geq D$ for all n, $\{\|h_n|_S - u\|_{0,S}\}_{n \in \mathbb{N}} \to 0$. To conclude, it suffices to set $f_n := e^{h_n} v$ for all n. $\square$

We now derive the analogous approximation result for 1–forms in $\Omega_{\mathfrak{m},\mathfrak{I}}(S)$.

Lemma 4.5 *Let $\theta \in \Omega_{\mathfrak{m},\mathfrak{I}}(S)$ be nowhere vanishing on $S \setminus S^\circ$, and consider an integral divisor $D \in \mathfrak{Div}_{\mathfrak{I}}(S)$ with $\mathrm{supp}(D) \subset S^\circ$.*

Then there exists $\{\theta_n\}_{n \in \mathbb{N}} \in \Omega_{\mathfrak{m},\mathfrak{I}}(W)$ satisfying that $\theta_n - \theta \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)$, $(\theta_n - \theta) \geq D$, $\mathrm{supp}((\theta_n)) \subset S^\circ$, $(\theta_n) = (\theta)$ for all $n \in \mathbb{N}$, and $\{\|\theta_n|_S - \theta\|_{0,S}\}_{n \in \mathbb{N}} \to 0$. In particular, θ_n is holomorphic and does not vanish anywhere on $W \setminus S^\circ$ for all $n \in \mathbb{N}$.

Proof Let $\tau \in \Omega_{\mathfrak{h},\mathfrak{I}}(W)$ having no zeros (see Proposition 3.5). Set $f := \theta/\tau \in \mathcal{F}_{\mathfrak{m},\mathfrak{I}}(S)$, and notice that $(f) = (\theta)$; in particular f has no zeros on $S \setminus S^\circ$. By Lemma 4.2, there exists $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}_{\mathfrak{m},\mathfrak{I}}(W)$ such that $\{\|f_n|_S - f\|_{0,S}\}_{n \in \mathbb{N}} \to 0$ and $(f_n) = (f)$ and $(f_n - f) \geq D$ on W for all $n \in \mathbb{N}$. It suffices to set $\theta_n := f_n \tau \in \Omega_{\mathfrak{m},\mathfrak{I}}(W)$ for all $n \in \mathbb{N}$. $\square$

We finish this section by proving a similar approximation result for functions in $\mathcal{G}_{\mathfrak{I}}(S)$.

Lemma 4.6 Let $g \in \mathcal{G}_{\mathfrak{I}}(S)$ be nowhere vanishing on $S \setminus S^{\circ}$, and let $D \in \mathfrak{Div}_{\mathfrak{I}}(S)$ be an integral divisor with $\mathrm{supp}(D) \subset S^{\circ}$.

Then there exists $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{G}_{\mathfrak{I}}(W)$ satisfying that $g_n - g$ is holomorphic on R_S, $(g_n - g) \geq D$, $\mathrm{supp}((g_n)) \subset S^{\circ}$, $(g_n) = (g)$ for all $n \in \mathbb{N}$, and $\{\|g_n|_S - g\|_{0,S}\}_{n \in \mathbb{N}} \to 0$. In particular, g_n is holomorphic and does not vanish anywhere on $W \setminus S^{\circ}$ for all $n \in \mathbb{N}$.

Proof Since smooth functions are dense in $\mathcal{G}_{\mathfrak{m},\mathfrak{I}}(S)$ with respect to the $\mathcal{C}^0$ topology, we can suppose without loss of generality that g is smooth. Furthermore, since the same argument applies separately to each connected component, we may assume W is a domain.

Claim 4.7 It suffices to prove the lemma for nowhere-vanishing functions in $\mathcal{G}_{\mathfrak{I}}(S)$.

Proof Assume that $(g) \neq 1$. Then, consider a nonconstant meromorphic function h on $\mathcal{N}$ satisfying $h \circ \mathfrak{I} = 1/\overline{h}$ and $(h|_{\overline{W}}) = (g)$. To construct h, write $(g) = D_1 \mathfrak{I}(D_1)^{-1}$, where D_1 is an integral divisor and $\mathrm{supp}(D_1) \cap \mathrm{supp}(\mathfrak{I}(D_1)) = \varnothing$. Since $\mathcal{N} \setminus \overline{W}$ is open, the Riemann–Roch theorem provides a meromorphic function H on $\mathcal{N}$ with $(H|_{\overline{W}}) = D_1$. Setting $h = H/\overline{H \circ \mathfrak{I}}$ we are done.

The function $f = g/h \colon S \to \overline{\mathbb{C}}$ lies in $\mathcal{G}_{\mathfrak{I}}(S)$ and is nowhere vanishing. By our assumption, there exists a sequence of nowhere vanishing functions $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{G}_{\mathfrak{I}}(W)$ approximating f on S, and satisfying that $f_n|_S - f$ is holomorphic on R_S and $(f_n - f) \geq D(g)^{-1}$ for all n.

Choosing $g_n := f_n h \in \mathcal{G}_{\mathfrak{m},\mathfrak{I}}(W)$, one has that $g_n|_S - g$ is holomorphic on R_S, $(g_n|_S - g) \geq D$ and $(g_n) = (g)$ for all $n \in \mathbb{N}$, and $\{g_n\}_{n \in \mathbb{N}}$ approximates g on S. □

In the sequel we will suppose that g is nowhere vanishing.

Notice that $\imath\, dg/g \in \Omega_{\mathfrak{h},\mathfrak{I}}(W)$. Reasoning as in the proof of Lemma 4.2, there exists a holomorphic 1–form $\tau \in \Omega_{\mathfrak{h},\mathfrak{I}}(W)$ such that

$$(14) \qquad \int_{\gamma} \imath\, dg/g = \int_{\gamma} \tau \quad \text{for all } \gamma \in \mathcal{H}_1(S, \mathbb{Z}),$$

$$(15) \qquad \frac{1}{2\pi} \int_{\gamma} \tau \in \mathbb{Z} \quad \text{for all } \gamma \in \mathcal{H}_1(W, \mathbb{Z}).$$

We need the following:

Claim 4.8 There exist a nowhere vanishing holomorphic $v \in \mathcal{G}_{\mathfrak{I}}(W)$ and a function $u \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ such that $d\log(v) = -\imath\tau$, $du = \imath\, dg/g - \tau|_S$ and $g = ve^{-\imath u}$.

Proof To construct v, fix $P_0 \in W$ and reasoning as in Claim 4.4 observe that $\frac{1}{\pi} \operatorname{Re} \int_{P_0}^{\Im(P_0)} \tau \in \mathbb{Z}$ independently on the arc connecting P_0 and $\Im(P_0)$. The function

$$v := \exp\left(-\iota \int_{P_0}^{\Im(P_0)} \tau + \frac{\iota}{2} \int_{P_0}^{\Im(P_0)} \tau\right)$$

is well defined (see (15)), nowhere vanishing and satisfies $d\log(v) = -\iota\tau$. Further,

$$(16) \qquad v(\overline{v \circ \Im}) = \pm 1 \text{ depending on whether } \frac{1}{\pi} \operatorname{Re} \int_{P_0}^{\Im(P_0)} \tau \text{ is even or odd.}$$

To construct u we proceed as in Claim 4.4. Take into account that $\iota\, dg/g - \tau|_S$ lies in $\Omega_{\mathfrak{h},\Im}(S)$ and is exact; see (14). For each connected component C of S, fix $P_C \in C$ and set $u|_C := A_C + \int_{P_C}(\iota\, dg/g - \tau)$, where $A_C \in \mathbb{C}$ is chosen so that $(g - ve^{-\iota u})(P_C) = 0$. Since $\kappa := g/(ve^{-\iota u})$ is locally constant on S and $\kappa(P_C) = 1$ for any connected component C of S, we infer that $\kappa = 1$ and $g = ve^{-\iota u}$ on S.

The facts $g \in \mathcal{G}_{\Im}(S)$ and (16) imply that $u \circ \Im = \bar{u} + m\pi$ for some $m \in \mathbb{Z}$. Since $\Im$ is an involution, we infer that $m = 0$, $u \in \mathcal{F}_{\mathfrak{h},\Im}(S)$, and $v(\overline{v \circ \Im}) = -1$; see (16).

The functions u and v solve the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By Lemma 4.1, there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_{\mathfrak{h},\Im}(W)$ such that $\{\|h_n|_S - u\|_{0,S}\}_{n\in\mathbb{N}} \to 0$ and $(h_n - u)_0 \geq D$ for all $n \in \mathbb{N}$. The sequence of nowhere vanishing functions $\{g_n := e^{-\iota h_n}v\}_{n\in\mathbb{N}} \subset \mathcal{G}_{\mathfrak{h},\Im}(W)$ proves the lemma. $\qquad\square$

5 Runge–Mergelyan theorem for nonorientable minimal surfaces

In this section we prove the main result of the paper (Theorem 5.6). Most of the technical computations are contained in Lemma 5.1; Theorem 5.6 will follow by a recursive application of it. In particular, Lemma 5.1 deals with the problem of controlling the periods in the approximation procedure.

Lemma 5.1 *Let $S \subset \mathcal{N}$ be an $\Im$–admissible subset, let W be a relatively compact $\Im$–invariant domain in $\mathcal{N}$ with finite topology, containing S and let $\Phi = (\phi_j)_{j=1,2,3}$ be a smooth triple in $\Omega_{\mathfrak{h},\Im}(S)^3$ such that $\sum_{j=1}^{3} \phi_j^2 = 0$ and $\sum_{j=1}^{3} |\phi_j|^2$ does not vanish anywhere on S.*

Then Φ can be approximated in the $\mathcal{C}^0$ topology on S by a sequence

$$\{\Phi_n = (\phi_{j,n})_{j=1,2,3}\}_{n\in\mathbb{N}} \subset \Omega_{\mathfrak{h},\Im}(W)^3$$

meeting the following requirements:

(i) $\phi_{3,n}$ is nowhere vanishing on $W \setminus R_S$. Furthermore, $(\phi_{3,n}|_E) = (\phi_3|_E) \in \mathfrak{Div}(E)$ for any connected component E of R_S such that ϕ_3 does not vanish everywhere on E.

(ii) $\sum_{j=1}^{3} \phi_{j,n}^2 = 0$ and $\sum_{j=1}^{3} |\phi_{j,n}|^2$ does not vanishes anywhere on W.

(iii) $\Phi_n - \Phi$ is exact on S for all $n \in \mathbb{N}$.

Proof We use the notation

$$g = \frac{\phi_3}{\phi_1 - \iota \phi_2},$$

and recall that $g \circ \mathfrak{I} = -1/\overline{g}$; see Section 3.2. This implies in particular that $g|_{R_S}$ is not constant, but it could be locally constant. We rule out this possibility in the following claim.

Claim 5.2 *Without loss of generality, it can be assumed that $g|_E$ is not constant for any connected component E of R_S.*

Proof Assume that the lemma holds when g is nonconstant on every connected component of R_S, and let us show that it also holds in the general case.

We discuss first the following particular case.

Case 1 Assume g does not vanish identically on any connected component of R_S.

Let E be a connected component of R_S such that $g|_E$ is constant. By our assumption, we have that

(17) $g|_E \neq 0, \infty$ and $\phi_3|_E$ is nowhere vanishing.

Since $g \circ \mathfrak{I} = -1/\overline{g}$, $E \cap \mathfrak{I}(E) = \varnothing$. Label $E_1, \mathfrak{I}(E_1), \ldots, E_k, \mathfrak{I}(E_k)$ the family of connected components E of R_S such that $g|_E$ is constant.

We use the notation $\Lambda_1 = \bigcup_{j=1}^{k} E_j$ and by $\Lambda_2 = R_S \setminus (\Lambda_1 \cup \mathfrak{I}(\Lambda_1))$. Let $\mathcal{B}_1$ be a homology basis of $\mathcal{H}_1(\Lambda_1, \mathbb{Z})$ and denote by $\nu_1 \in \mathbb{N}$ the number of elements in $\mathcal{B}_1$. Denote by $\mathcal{O}(\Lambda_1)$ the space of holomorphic functions $\Lambda_1 \to \mathbb{C}$.

For each $h \in \mathcal{O}(\Lambda_1)$, consider the holomorphic function and 1–form on Λ_1 given by

(18) $g(h) := (g + h)$ and $\phi_3(h) := \dfrac{g + h}{g} \phi_3.$

Let $\mathcal{P} \colon \mathcal{O}(\Lambda_1) \to \mathbb{C}^{2\nu_1}$ be the *period map* given by

$$\mathcal{P}(h) = \left(\int_c (g(h)\phi_3(h) - g\phi_3, \ \phi_3(h) - \phi_3) \right)_{c \in \mathcal{B}_1}.$$

Notice that $\mathcal{P}(\lambda h) = 0$ for any $\lambda \in \mathbb{C}$ and any $h \in \mathcal{O}(\Lambda_1)$ with $\mathcal{P}(h) = 0$. Since $\mathcal{O}(\Lambda_1)$ has infinite dimension, there exists a nonconstant function $h \in \mathcal{P}^{-1}(0) \subset \mathcal{O}(\Lambda_1)$. Let $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \setminus \{0\}$ be any sequence converging to zero, and define

$$(19) \qquad\qquad h_n := \lambda_n h \in \mathcal{P}^{-1}(0) \quad \text{for all } n \in \mathbb{N}.$$

Obviously, $\{h_n\}_{n \in \mathbb{N}} \to 0$ in the $\mathcal{C}^0$ topology on Λ_1; hence without loss of generality we can assume that

$$(20) \qquad\qquad g(h_n) \text{ and } \phi_3(h_n) \text{ are nowhere vanishing on } \Lambda_1 \text{ for all } n.$$

Choose smooth $g_n \in \mathcal{G}_{\mathfrak{J}}(S)$ and $\phi_{3,n} \in \Omega_{\mathfrak{h},\mathfrak{J}}(S)$ such that

$$g_n|_{\Lambda_1} = g(h_n), \quad g_n|_{\Lambda_2} = g, \quad \phi_{3,n}|_{\Lambda_1} = \phi_3(h_n), \quad \phi_{3,n}|_{\Lambda_2} = \phi_3,$$

and $\{\Psi_n\}_{n \in \mathbb{N}} \subset \Omega_{\mathfrak{h},\mathfrak{J}}(S)$ converges to Φ in the $\mathcal{C}^0$ topology on S, where Ψ_n are the Weierstrass data associated to $(g_n, \phi_{3,n})$ via (7). Notice that $(\phi_{3,n}) = (\phi_3)$ for all n; see (17) and (20). Observe that (18) and (19) imply that $\Psi_n - \Phi$ is exact on $R_S = \Lambda_1 \cup \mathfrak{J}(\Lambda_1) \cup \Lambda_2$. Furthermore, up to a slight smooth deformation of g_n and $\phi_{3,n}$ over C_S, we can also assume that $\Psi_n - \Phi$ is exact on S for all $n \in \mathbb{N}$.

By our assumptions, the lemma holds for the triple Ψ_n, for all $n \in \mathbb{N}$. To finish, use a standard diagonal argument.

Case 2 Assume that g vanishes everywhere on some connected components of R_S.

Call $\Lambda_0 \neq \varnothing$ the union of those connected components of R_S on which g is identically 0 or ∞, and notice that $\phi_3|_{\Lambda_0}$ vanishes everywhere and $\Phi|_{\Lambda_0}$ is exact. Take a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{O}(3, \mathbb{R})$ converging to the identity matrix such that the third coordinate of $(\Phi|_{\Lambda_0}) \cdot A_n$ vanishes everywhere on no connected component of Λ_0. Choose a smooth $\Theta_n \in \Omega_{\mathfrak{h},\mathfrak{J}}(S)^3$ such that $\Theta_n|_{\Lambda_0} := (\Phi|_{\Lambda_0}) \cdot A_n$, $\Theta_n|_{R_S \setminus \Lambda_0} := \Phi|_{R_S \setminus \Lambda_0}$, $\Theta_n - \Phi$ is exact on S and $\{\Theta_n\}_{n \in \mathbb{N}} \to \Phi$ in the $\mathcal{C}^0$ topology on S. Since the third coordinate of Θ_n vanishes everywhere on no connected component of R_S, the conclusion of Lemma 5.1 holds for each Θ_n; take into account Case 1. By a diagonal argument, it also holds for Φ and we are done.

This proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

From now on, we assume that g is nonconstant on every connected component of R_S. Let us check the following:

Claim 5.3 *Without loss of generality, it can be assumed that ϕ_j and $d(\phi_i / \phi_j)$ are nowhere vanishing on $(bR_S) \cup C_S$, for all $i, j \in \{1, 2, 3\}$, $i \neq j$. In particular, g does not vanish anywhere on C_S; hence $g \in \mathcal{G}_{\mathfrak{J}}(S)$.*

Proof Let $M_1 \supset M_2 \supset \cdots$ be a sequence of $\mathfrak{I}$–invariant compact regions in W such that M_n° is a tubular neighborhood of R_S in W for all $n \in \mathbb{N}$, $M_n \subset M_{n-1}^\circ$ for all $n \in \mathbb{N}$, $\bigcap_{n \in \mathbb{N}} M_n = R_S$, Φ holomorphically extends to M_1, $\sum_{j=1}^3 |\phi_j|^2$ is nowhere vanishing on M_1, and ϕ_j and $d(\phi_i/\phi_j)$ are nowhere vanishing on bM_n, for all $i, j \in \{1, 2, 3\}$, $i \neq j$, $n \in \mathbb{N}$ (recall that g is constant over no connected component of R_S by Claim 5.2). In addition, choose M_n so that $S_n := M_n \cup C_S \subset W$ is an $\mathfrak{I}$–admissible set in $\mathcal{N}$ and $\gamma \setminus M_n^\circ$ is a nonempty Jordan arc, for any component γ of C_S. Observe that $C_{S_n} = C_S \setminus M_n^\circ$, for all $n \in \mathbb{N}$.

Let $(h_n, \psi_{3,n}) \in \mathcal{G}_{\mathfrak{I}}(S_n) \times \Omega_{\mathfrak{h},\mathfrak{I}}(S_n)$ be any smooth couple meeting the following requirements:

- $(h_n, \psi_{3,n})|_{R_{S_n}} = (g, \phi_3)|_{R_{S_n}}$ and $\sum_{j=1}^3 |\psi_{j,n}|^2$ does not vanish anywhere on S_n, where $\Psi_n = (\psi_{j,n})_{j=1,2,3} \in \Omega_{\mathfrak{h},\mathfrak{I}}(S_n)^3$ are the Weierstrass data associated to $(h_n, \psi_{3,n})$ via (7).

- $\psi_{j,n}$ and $d(\psi_{i,n}/\psi_{j,n})$ are nowhere vanishing on $(bR_{S_n}) \cup C_{S_n}$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$.

- $\Psi_n|_S - \Phi$ is exact on S.

- The sequence $\{\Psi_n|_S\}_{n \in \mathbb{N}} \subset \Omega_{\mathfrak{h},\mathfrak{I}}(S)^3$ converges to Φ in the $\mathcal{C}^0$ topology on S.

The existence of such sequence follows from similar arguments as those in Claim 5.2; just use classical approximation results by smooth functions to suitably extend the pair $(g, \phi_3)|_{R_{S_n}}$ to C_{S_n}.

If we assume that the lemma holds for the triple Ψ_n and the $\mathfrak{I}$–admissible set S_n, for all $n \in \mathbb{N}$, then, using again a standard diagonal argument, we conclude that it also holds for the triple Φ. □

From now on, we assume that ϕ_j and $d(\phi_i/\phi_j)$ vanish nowhere on $(bR_S) \cup C_S$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$.

Let $W_S \subset W$ be a domain of finite topology such that $S \subset W_S$ and $i_*: \mathcal{H}_1(S, \mathbb{Z}) \to \mathcal{H}_1(W_S, \mathbb{Z})$ is an isomorphism, where as usual $i: S \to W_S$ denotes the inclusion map. Denote by $\nu = 2\nu_0 + 1$, $\nu_0 \in \mathbb{N}$, the number of generators of $\mathcal{H}_1(S, \mathbb{Z})$ of S. Take an $\mathfrak{I}$–basis $\mathcal{B}_S = \{c_0, c_1, \ldots, c_{\nu_0}, d_1, \ldots, d_{\nu_0}\}$ of the homology group with real coefficients $\mathcal{H}_1(S, \mathbb{R}) \overset{i_*}{\equiv} \mathcal{H}_1(W_S, \mathbb{R})$; see Definition 2.5.

Recall that

$$(21) \qquad \mathfrak{I}_*(c_j) = -c_j \quad \text{and} \quad \mathfrak{I}_*(d_j) = d_j \quad \text{for all } j.$$

For any pair of functions $(h_1, h_2) \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2$, denote by $\Phi(h_1, h_2)$ the Weierstrass data on S associated to the pair $(e^{\imath h_1} g, e^{h_2} \phi_3)$ by (7); that is,

$$\Phi(h_1, h_2) = \left(\frac{1}{2}\left(\frac{1}{e^{\imath h_1} g} - e^{\imath h_1} g \right), \frac{\imath}{2}\left(\frac{1}{e^{\imath h_1} g} + e^{\imath h_1} g \right), 1 \right) e^{h_2} \phi_3.$$

Observe that $\Phi(h_1, h_2) \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)^3$, and so, by (21),

$$(22) \quad \int_{c_j} \Phi(h_1, h_2) \in \imath \mathbb{R}^3 \quad \text{and} \quad \int_{d_j} \Phi(h_1, h_2) \in \mathbb{R}^3 \quad \text{for all } j \text{ and } (h_1, h_2).$$

The same happens in particular to the triple $\Phi = \Phi(0, 0)$.

Denote by $\mathcal{P} \colon \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2 \to \mathbb{R}^{6\nu_0+3} \equiv \mathbb{R}^{3\nu_0+3} \times \mathbb{R}^{3\nu_0}$ the *period map* given by

$$(23) \quad \mathcal{P}(h_1, h_2)$$

$$= \left(\left[\operatorname{Im} \int_{c_j} \Phi(h_1, h_2) - \Phi \right]_{j \in \{0,\ldots,\nu_0\}}, \left[\int_{d_j} \Phi(h_1, h_2) - \Phi \right]_{j \in \{1,\ldots,\nu_0\}} \right).$$

Notice that $\Phi(h_1, h_2)$ satisfies items (i) and (ii) in the lemma; if in addition $\mathcal{P}(h_1, h_2) = 0$, then it also meets item (iii) (take into account (22)). On the other hand, endowing the real space $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2$ with the maximum norm, one has that the period map $\mathcal{P}$ above is Fréchet differentiable.

The key to the proof of Lemma 5.1 is to show that the Fréchet derivative of $\mathcal{P}$ has maximal rank equal to $6\nu_0 + 3$ at $(0, 0)$.

Claim 5.4 *The Fréchet derivative* $\mathcal{A} \colon \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2 \to \mathbb{R}^{6\nu_0+3}$ *of* $\mathcal{P}$ *at* $(0, 0)$ *has maximal rank.*

Proof We proceed by contradiction. Assume that $\mathcal{A}(\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2)$ is contained in a linear subspace

$$\mathcal{U} = \Bigg\{ ([(x_{c_j,k})_{k=1,2,3}]_{j=0,\ldots,\nu_0}, [(x_{d_j,k})_{k=1,2,3}]_{j=1,\ldots,\nu_0}) \in \mathbb{R}^{6\nu_0+3} :$$

$$\sum_{k=1}^{3}\left(\sum_{j=0}^{\nu_0} \lambda_{c_j,k} x_{c_j,k} + \sum_{j=1}^{\nu_0} \lambda_{d_j,k} x_{d_j,k} \right) = 0 \Bigg\} \subset \mathbb{R}^{6\nu_0+3},$$

where $\lambda_{c_j,k}$ and $\lambda_{d_j,k}$ are real numbers for all j and k, not all of them equal to zero. Let Γ_k be the element of the homology group with complex coefficients $\mathcal{H}_1(S, \mathbb{C})$ given by

$$(24) \quad \Gamma_k = -\imath \sum_{j=0}^{\nu_0} \lambda_{c_j,k} c_j + \sum_{j=1}^{\nu_0} \lambda_{d_j,k} d_j, \quad k = 1, 2, 3.$$

Since $\mathcal{A}(\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2) \subset \mathcal{U}$,

$$(25) \qquad -\int_{\Gamma_1} h\phi_2 + \int_{\Gamma_2} h\phi_1 = 0 \quad \text{for all } h \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W}),$$

$$(26) \qquad \int_{\Gamma_1} h\phi_1 + \int_{\Gamma_2} h\phi_2 + \int_{\Gamma_3} h\phi_3 = 0 \quad \text{for all } h \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W}).$$

Let us show that $\Gamma_1 = 0$.

Indeed, reason by contradiction and assume that $\Gamma_1 \neq 0$. We use the notation $\Sigma_1 = \{f \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W}) : (f) \geq (\phi_1)^2\}$. By Claim 5.3, the function df/ϕ_1 lies in $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$. Therefore, for any $f \in \Sigma_1$, Lemma 4.1 applies and ensures that df/ϕ_1 can be approximated in the $\mathcal{C}^0$ topology on $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ by functions in $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})$. As a consequence, equation (25) can be applied formally to $h = df/\phi_1$, implying that

$$\int_{\Gamma_1} \frac{\phi_2}{\phi_1} df = 0 \quad \text{for all } f \in \Sigma_1.$$

By Claim 5.3 one can integrate by parts in the above equation and obtain that

$$(27) \qquad \int_{\Gamma_1} f d\left(\frac{\phi_2}{\phi_1}\right) = 0 \quad \text{for all } f \in \Sigma_1.$$

Since $\Gamma_1 \neq 0$, the isomorphism (2) gives a holomorphic 1–form $\tau \in \Omega_{\mathfrak{h},\mathfrak{I}}(\overline{W})$ such that

$$(28) \qquad \int_{\Gamma_1} \tau \in \mathbb{R} \setminus \{0\};$$

taking into account (24) and (21).

On the other hand, since W is open and relatively compact in $\mathcal{N}$, there exists $u \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})$ such that $(\tau + du)_0 \geq (\phi_1)^2(d(\phi_2/\phi_1))$ by an application of the Riemann–Roch theorem. Set

$$f_0 := \frac{\tau + du}{d(\phi_2/\phi_1)} \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$$

(see Claim 5.3) and note that $(f_0) \geq (\phi_1)^2$. By Lemma 4.1, f_0 can be approximated in the $\mathcal{C}^0$ topology on $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ by functions in Σ_1; hence equation (27) can be applied formally to $f = f_0$, giving that $0 = \int_{\Gamma_1} (\tau + du) = \int_{\Gamma_1} \tau$; contradicting (28).

Therefore $\Gamma_1 = 0$ and equation (25) becomes

$$(29) \qquad \int_{\Gamma_2} h\phi_1 = 0 \quad \text{for all } h \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W}).$$

Next we show that $\Gamma_2 = 0$. As above, reasoning by contradiction, we can find a 1–form $\tau \in \Omega_{\mathfrak{h},\mathfrak{I}}(\overline{W})$ and a function $u \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})$ such that

$$\text{(30)} \qquad \int_{\Gamma_2} \tau \neq 0$$

and $(\tau + du)_0 \geq (\phi_2)$. In this case, we set

$$h = \frac{\tau + du}{\phi_2} \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S);$$

see Claim 5.3. By Lemma 4.1, one can approximate h in the $\mathcal{C}^0$ topology on $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)$ by functions in $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})$; hence (29) formally applies to h giving that $0 = \int_{\Gamma_2}(\tau + du) = \int_{\Gamma_2} \tau$; which contradicts (30).

Finally, since $\Gamma_1 = \Gamma_2 = 0$, (26) becomes $\int_{\Gamma_3} h\phi_3 = 0$ for all $h \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})$. The same argument as that in the previous paragraph gives that $\Gamma_3 = 0$ as well.

Since $\Gamma_k = 0$ for all $k = 1, 2, 3$, (24) implies that $\lambda_{c_j,k} = 0 = \lambda_{d_j,k}$ for all j and k. This contradiction finishes the proof. $\qquad \square$

Let $\{e_1, \ldots, e_{6\nu_0+3}\}$ be a basis of $\mathbb{R}^{6\nu_0+3}$. For any $j \in \{1, \ldots, 6\nu_0 + 3\}$ choose $H_j = (h_{1,j}, h_{2,j}) \in \mathcal{A}^{-1}(e_j) \subset \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2$, and denote by $\mathcal{Q}\colon \mathbb{R}^{6\nu_0+3} \to \mathbb{R}^{6\nu_0+3}$ the analytic map given by

$$\mathcal{Q}((x_j)_{j=1,\ldots,6\nu_0+3}) = \mathcal{P}\left(\sum_{j=1}^{6\nu_0+3} x_j H_j\right),$$

where $\mathcal{P}$ is the period map (23). Claim 5.4 guarantees that the differential of $\mathcal{Q}$ at $0 \in \mathbb{R}^{6\nu_0+3}$ is an isomorphism; hence there exists a closed Euclidean ball $U \subset \mathbb{R}^{6\nu_0+3}$ centered at the origin, satisfying that $\mathcal{Q}\colon U \to \mathcal{Q}(U)$ is an analytical diffeomorphism. In particular, $0 = \mathcal{Q}(0)$ lies in the interior of $\mathcal{Q}(U)$.

On the other hand, taking into account Claim 5.3, Lemmas 4.5 and 4.6 furnish a sequence $\{(\sigma_n, \psi_{3,n})\}_{n \in \mathbb{N}} \subset \mathcal{G}_{\mathfrak{I}}(\overline{W}) \times \Omega_{\mathfrak{h},\mathfrak{I}}(\overline{W})$ such that

$$\text{(31)} \qquad (\sigma_n) = (g) \quad \text{and} \quad (\psi_{3,n}) = (\phi_3) \in \mathfrak{Div}_{\mathfrak{I}}(R_S) \quad \text{for all } n \in \mathbb{N},$$

$$\text{(32)} \qquad \{(\sigma_n, \psi_{3,n})|_S\}_{n \in \mathbb{N}} \to (g, \phi_3) \quad \text{in the } \mathcal{C}^0 \text{ topology on } S.$$

For any pair of functions $(h_1, h_2) \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2$, we denote by $\Psi_n(h_1, h_2) \in \Omega_{\mathfrak{h},\mathfrak{I}}(\overline{W})^3$ the Weierstrass data associated to the pair $(e^{ih_1}\sigma_n, e^{h_2}\psi_{3,n})$ by (7); that is to say,

$$\Psi_n(h_1, h_2) = \left(\frac{1}{2}\left(\frac{1}{e^{ih_1}\sigma_n} - e^{ih_1}\sigma_n\right), \frac{i}{2}\left(\frac{1}{e^{ih_1}\sigma_n} + e^{ih_1}\sigma_n\right), 1\right)e^{h_2}\psi_{3,n}.$$

By (21), one has that $\int_{c_j} \Psi_n(h_1, h_2) \in \iota\mathbb{R}^3$ and $\int_{d_j} \Psi_n(h_1, h_2) \in \mathbb{R}^3$ for all j and $(h_1, h_2) \in \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2$. Denote by $\mathcal{P}_n \colon \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2 \to \mathbb{R}^{6v_0+3} \equiv \mathbb{R}^{3v_0+3} \times \mathbb{R}^{3v_0}$ the period map given by

$$(33) \quad \mathcal{P}_n(h_1, h_2)$$
$$= \left(\left[\operatorname{Im} \int_{c_j} \Psi_n(h_1, h_2) - \Phi \right]_{j \in \{0,\dots,v_0\}}, \left[\int_{d_j} \Psi_n(h_1, h_2) - \Phi \right]_{j \in \{1,\dots,v_0\}} \right),$$

and notice that $\mathcal{P}_n$ is Fréchet differentiable if we endow the real space $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})^2$ with the maximum norm.

Denote by $\mathcal{Q}_n \colon \mathbb{R}^{6v_0+3} \to \mathbb{R}^{6v_0+3}$ the analytic map given by

$$\mathcal{Q}_n((x_j)_{j=1,\dots,6v_0+3}) = \mathcal{P}_n\left(\sum_{j=1}^{6v_0+3} x_j H_j \right) \quad \text{for all } n \in \mathbb{N}.$$

Since $\{\mathcal{Q}_n\}_{n\in\mathbb{N}} \to \mathcal{Q}$ uniformly on compact subsets of $\mathbb{R}^{6v_0+3}$, $\mathcal{Q}_n \colon U \to \mathcal{Q}_n(U)$ is an analytical diffeomorphism and $0 \in \mathcal{Q}_n(U)$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Denote by $y_n = (y_{j,n})_{j=1,\dots,6v_0+3}$ the unique point in U mapped to 0 by $\mathcal{Q}_n$, $n \geq n_0$. Since $\mathcal{Q}(0) = 0$,

$$(34) \qquad \qquad \text{the sequence } \{y_n\}_{n\geq n_0} \text{ converges to } 0.$$

Set

$$g_n := e^{\sum_{j=1}^{6v_0+3} y_{j,n} h_{1,j}} \sigma_n \in \mathcal{G}_{\mathfrak{I}}(\overline{W}) \quad \text{and} \quad \phi_{3,n} := e^{\sum_{j=1}^{6v_0+3} y_{j,n} h_{2,j}} \psi_{3,n} \in \Omega_{\mathfrak{h},\mathfrak{I}}(\overline{W})$$

for all $n \geq n_0$. Denote by Φ_n the Weierstrass data on W associated to the pair $(g_n, \phi_{3,n})$ by (7), $n \geq n_0$, and let us check that the sequence $\{\Phi_n\}_{n\geq n_0}$ solves the lemma. Indeed, $\{\Phi_n\}_{n\geq n_0}$ converges to Φ in the $\mathcal{C}^0$ topology on S by (32) and (34). Since $\mathcal{Q}_n(y_n) = 0$, $\Phi_n - \Phi$ is exact on S. Finally, conditions Lemma 5.1(i) and (ii) are ensured by (31). $\qquad \square$

By a minor modification of the proof of Lemma 5.1, we can obtain the analogous approximation result for Weierstrass data with a fixed component 1–form. This will be very useful for applications; see Section 6.

Lemma 5.5 *In Lemma 5.1 one can choose $\phi_{3,n} = \phi_3$ for all $n \in \mathbb{N}$, provided that ϕ_3 extends holomorphically to W, vanishes everywhere on no connected component of R_S, and does not vanish anywhere on C_S.*

Proof Reasoning as in the proof of Claim 5.3, it can be assumed without loss of generality that ϕ_j and $d(\phi_i/\phi_j)$ are nowhere vanishing on $(bR_S) \cup C_S$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$. In this case, we take $\mathfrak{I}$–admissible sets S_n as those in the proof of Claim 5.3, and replace the Weierstrass data (g, ϕ_3) on S_n by suitable (h_n, ϕ_3) for all $n \in \mathbb{N}$.

As in Claim 5.4, one can now check that the Fréchet derivative $\hat{\mathcal{A}} : \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W}) \to \mathbb{R}^{4v_0+2}$ of the period map $\hat{\mathcal{P}} : \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W}) \to \mathbb{R}^{4v_0+2} \equiv \mathbb{R}^{4v_0+2} \times \{0\} \subset \mathbb{R}^{6v_0+3}$, $\hat{\mathcal{P}}(h) := \mathcal{P}(h, 0)$, at $h = 0$ has maximal rank, where $\mathcal{P}$ is the map (23). Then fix a basis $\{e_1, \ldots, e_{4v_0+2}\}$ of $\mathbb{R}^{4v_0+2}$, and for any $j \in \{1, \ldots, 4v_0 + 2\}$ choose a function $\hat{H}_j \in \hat{\mathcal{A}}^{-1}(e_j) \subset \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W})$. Denote by $\hat{\mathcal{Q}} : \mathbb{R}^{4v_0+2} \to \mathbb{R}^{4v_0+2}$ the analytic map $\hat{\mathcal{Q}}((x_j)_{j=1,\ldots,4v_0+2}) = \hat{\mathcal{P}}(\sum_{j=1,\ldots,4v_0+2} x_j \hat{H}_j)$.

Write

$$(35) \qquad (\phi_3|_{W \setminus S}) = D_1 \mathfrak{I}(D_1),$$

where $\operatorname{supp}(D_1) \cap \operatorname{supp}(\mathfrak{I}(D_1)) = \varnothing$. Since W is relatively compact in $\mathcal{N}$, the Riemann–Roch theorem provides a holomorphic function $H_1 : \overline{W} \to \mathbb{C}$ such that $(H_1) = D_1$. Set $H := H_1/\overline{\mathfrak{I}(H_1)}$, and notice that H is a meromorphic function on $\overline{W}$, $(H) = D_1 \mathfrak{I}(D_1)^{-1}$ and $H \circ \mathfrak{I} = 1/\overline{H}$. Since g does not vanish anywhere on $S \setminus S^\circ$, $g/H \in \mathcal{G}_{\mathfrak{I}}(S)$ does; hence Lemma 4.6 furnishes a sequence $\{\rho_n\}_{n \in \mathbb{N}} \subset \mathcal{G}_{\mathfrak{I}}(\overline{W})$ such that $(\rho_n) = (g) \in \mathfrak{Div}(S^\circ)$ for all $n \in \mathbb{N}$ and $\{\rho_n|_S\}_{n \in \mathbb{N}} \to g/H$ in the $\mathcal{C}^0$ topology on S. Set $\sigma_n := \rho_n H \in \mathcal{G}_{\mathfrak{I}}(W)$ and notice that

$$(36) \qquad (\sigma_n) = (g) D_1 \mathfrak{I}(D_1)^{-1} \quad \text{for all } n \in \mathbb{N}$$

and $\{\sigma_n|_S\}_{n \in \mathbb{N}} \to g$ in the $\mathcal{C}^0$ topology on S. Observe that (35) and (36) ensure that three 1–forms of the Weierstrass data associated by (7) to the pair (σ_n, ϕ_3) are holomorphic and have no common zeros.

Denote by $\hat{\mathcal{P}}_n : \mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\overline{W}) \to \mathbb{R}^{4v_0+2} \equiv \mathbb{R}^{4v_0+2} \times \{0\} \subset \mathbb{R}^{6v_0+3}$ the period map given by $\hat{\mathcal{P}}_n(h) = \mathcal{P}_n(h, 0)$, where $\mathcal{P}_n$ is the map (33), and denote by $\hat{\mathcal{Q}}_n : \mathbb{R}^{4v_0+2} \to \mathbb{R}^{4v_0+2}$ the analytical map $\hat{\mathcal{Q}}_n((x_j)_{j=1,\ldots,4v_0+2}) = \hat{\mathcal{P}}_n(\sum_{j=1}^{4v_0+2} x_j \hat{H}_j)$ for all $n \in \mathbb{N}$. To conclude the proof, we argue as in the proof of Lemma 5.1. $\qquad \square$

We are now ready to state and prove the main result of this paper. Theorem 1.1 is a particular instance of the following.

Theorem 5.6 (Runge–Mergelyan theorem for nonorientable minimal surfaces) *Let $S \subset \mathcal{N}$ be an $\mathfrak{I}$–admissible subset (see Definition 3.2), let $X_{\varpi} \in \mathcal{M}^*_{\mathfrak{g},\mathfrak{I}}(S)$ (see Definition 3.12) and let $\mathfrak{p} : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^3$ be a group homomorphism such that $\mathfrak{p}(\mathfrak{I}_*(\gamma)) = -\mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$, and $\mathfrak{p}|_{\mathcal{H}_1(S,\mathbb{Z})}$ is the generalized flux*

map $\mathfrak{p}_{X_{\varpi}}$ of X_{ϖ}. Write $X_{\varpi} = (X = (X_j)_{j=1,2,3}, \varpi)$, $\partial X_{\varpi} = (\phi_j)_{j=1,2,3}$ and $\mathfrak{p} = (\mathfrak{p}_j)_{j=1,2,3}$.

Then the following assertions hold:

(i) X_{ϖ} can be approximated in the $\mathcal{C}^1$ topology on S by $\mathfrak{I}$–invariant conformal minimal immersions $Y = (Y_j)_{j=1,2,3} \colon \mathcal{N} \to \mathbb{R}^3$ such that $\mathfrak{p}_Y = \mathfrak{p}$ and ∂Y_3 does not vanish anywhere on $\mathcal{N} \setminus R_S$. Furthermore, Y can be chosen so that $(\partial Y_3|_E) = (\phi_3|_E) \in \mathfrak{Div}(E)$ for any connected component E of R_S such that ϕ_3 does not vanish everywhere on E.

(ii) If ϕ_3 is not identically zero and extends to $\mathcal{N}$ as a holomorphic 1–form without real periods, vanishing nowhere on C_S, and satisfying $\mathfrak{p}_3(\gamma) = \mathrm{Im} \int_\gamma \phi_3$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$, then X_{ϖ} can be approximated in the $\mathcal{C}^1$ topology on S by $\mathfrak{I}$–invariant conformal minimal immersions $Y = (Y_j)_{j=1,2,3} \colon \mathcal{N} \to \mathbb{R}^3$ with flux map $\mathfrak{p}_Y = \mathfrak{p}$ and third coordinate function $Y_3 = X_3$.

Proof We begin with the following assertion.

Claim 5.7 There exists a connected $\mathfrak{I}$–admissible subset $\widehat{S} \subset \mathcal{N}$ such that $R_{\widehat{S}} = R_S$ and $C_{\widehat{S}} \supset C_S$; ie $\widehat{S}$ is constructed by adding a finite family of Jordan arcs to S.

Proof If S is connected choose $\widehat{S} = S$.

Assume that S is not connected. We distinguish the following two cases. (See Remark 3.1 and Definitions 2.2 and 2.3 for notation.)

Case 1: $\underline{S}$ is a connected subset of the nonorientable Riemann surface $\underline{\mathcal{N}}$ In this situation, any tubular neighborhood of $\underline{S}$ is an orientable surface. Then take any Jordan arc $\gamma \subset \underline{\mathcal{N}}$ in bR_S and otherwise disjoint from $\underline{S}$ such that any tubular neighborhood of $\underline{\widehat{S}} := \underline{S} \cup \gamma$ is nonorientable. Since $\underline{\mathcal{N}} \setminus \widehat{S}$ has no relatively compact connected components, $\widehat{S}$ meets the requirements of the claim.

Case 2: $\underline{S}$ is not connected Then consider a finite family $\underline{\Gamma}$ of pairwise disjoint Jordan arcs in $\underline{\mathcal{N}}$ such that

- γ has endpoints in $\underline{bR_S}$ and is otherwise disjoint from $\underline{S}$ for all $\gamma \in \underline{\Gamma}$,
- $\underline{S'} := \underline{S} \cup \underline{\Gamma}$ is connected,
- $\underline{\mathcal{N}} \setminus S'$ has no relatively compact connected components.

This reduces the proof of the claim to Case 1 applied to S'. □

Let $\widehat{S}$ be as in Claim 5.7. Let us prove assertion (i).

Fix $\epsilon > 0$. Fix $P_0 \in S$ and let $\Phi_0 = (\phi_{0,j})_{j=1,2,3}$ be any extension of ∂X_{ϖ} to $\widehat{S}$ such that

(i) $\phi_{0,j} \in \Omega_{\mathfrak{h},\mathfrak{I}}(\widehat{S})$ and is smooth for all $j = 1, 2, 3$,

(ii) Φ_0 has no real periods, $\sum_{j=1}^{3} \phi_{0,j}^2 = 0$ and $\sum_{j=1}^{3} |\phi_{0,j}|^2$ does not vanish anywhere on $\widehat{S}$,

(iii) $X(P) = X(P_0) + \mathrm{Re} \int_{P_0}^{P} \Phi_0$ for all $P \in S$,

(iv) $\mathrm{Im} \int_{\gamma} \Phi_0 = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(\widehat{S}, \mathbb{Z})$.

To construct Φ_0, just define Φ_0 on the arcs $C_{\widehat{S}} \setminus C_S$ in a suitable way. Denote by $F_0 \in \mathcal{M}_{\mathfrak{g},\mathfrak{I}}^{*}(M_0)$ the marked immersion with generalized Weierstrass data Φ_0 and $F_0(P_0) = X_{\varpi}(P_0)$.

Set $M_0 := \widehat{S}$ and M_1 a tubular neighborhood of M_0. Let $\{M_n\}_{n \in \mathbb{N}}$ be an exhaustion of $\mathcal{N}$ by Runge connected $\mathfrak{I}$–invariant compact regions such that the Euler characteristic $\chi(M_n^{\circ} \setminus M_{n-1}) \in \{0, -2\}$ for all $n \geq 2$. Existence of such an exhaustion is well known; one can choose an exhaustion of $\underline{\mathcal{N}}$ by Runge connected compact regions such that the difference of two consecutive ones has Euler characteristic in $\{0, -1\}$, and pull it back to $\mathcal{N}$ by the covering projection. To be more precise, $\{M_n\}_{n \in \mathbb{N}}$ meets the following topological description:

Remark 5.8 The region M_n is obtained from M_{n-1}, $n \geq 2$, by one of the following four procedures:

(i) M_n is a tubular neighborhood of M_{n-1}. In this case $\chi(M_n^{\circ} \setminus M_{n-1}) = 0$.

(ii) M_n is a tubular neighborhood of $M_{n-1} \cup \gamma \cup \mathfrak{I}(\gamma)$, where γ is a Jordan arc in $\mathcal{N}$ with endpoints in a connected component c of bM_{n-1} and otherwise disjoint from M_{n-1}, such that $\gamma \cap \mathfrak{I}(\gamma) = \varnothing$ and $M_{n-1} \cup \gamma \cup \mathfrak{I}(\gamma)$ is an $\mathfrak{I}$–admissible subset in $\mathcal{N}$. In this case, M_n has the same genus as M_{n-1} and two more boundary components; hence $\chi(M_n^{\circ} \setminus M_{n-1}) = -2$. (See Figure 2.)

(iii) M_n is a tubular neighborhood of $M_{n-1} \cup \gamma \cup \mathfrak{I}(\gamma)$, where γ is a Jordan arc in $\mathcal{N}$ with an endpoint in a connected component c of bM_{n-1}, the other endpoint in $\mathfrak{I}(c)$, and otherwise disjoint from M_{n-1}, such that $\gamma \cap \mathfrak{I}(\gamma) = \varnothing$ and $M_{n-1} \cup \gamma \cup \mathfrak{I}(\gamma)$ is an $\mathfrak{I}$–admissible subset in $\mathcal{N}$. In this case, M_n has the same number of boundary components as M_{n-1} and one more handle; hence $\chi(M_n^{\circ} \setminus M_{n-1}) = -2$. (See Figure 3.)
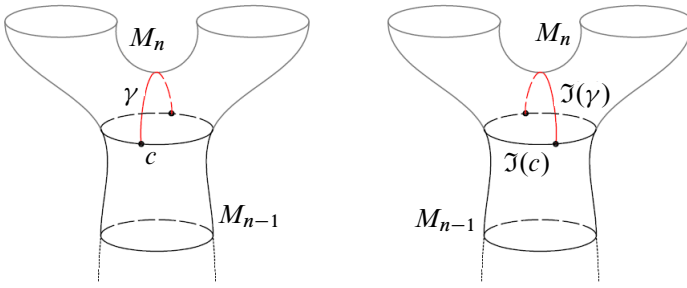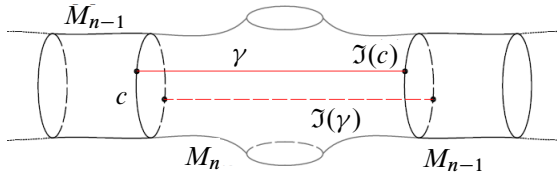
Figure 2: M_n; procedure 2

Figure 3: M_n; procedure 3

(iv) M_n is a tubular neighborhood of $M_{n-1} \cup \gamma \cup \mathfrak{I}(\gamma)$, where γ is a Jordan arc in $\mathcal{N}$ with an endpoint in a connected component c_1 of bM_{n-1}, the other endpoint in a connected component $c_2 \neq c_1$ of bM_{n-1}, and otherwise disjoint from M_{n-1}, such that $c_2 \cap \mathfrak{I}(c_1) = \varnothing$, $\gamma \cap \mathfrak{I}(\gamma) = \varnothing$ and $M_{n-1} \cup \gamma \cup \mathfrak{I}(\gamma)$ is an $\mathfrak{I}$–admissible subset in $\mathcal{N}$. In this case, M_n has two less boundary components than M_{n-1} and two more handles; hence $\chi(M_n^\circ \setminus M_{n-1}) = -2$. (See Figure 4.)
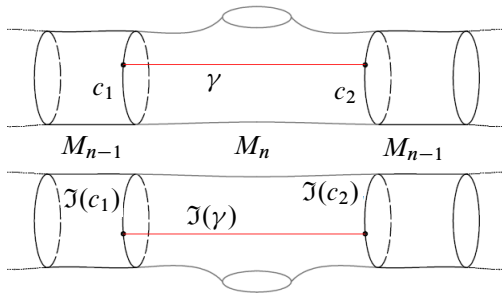

Figure 4: M_n; procedure 4

Let $0 < \xi < \epsilon$ be specified later and let us construct a sequence $\{F_n = (F_{n,j})_{j=1,2,3}\}_{n\in\mathbb{N}}$, where $F_n \in \mathcal{M}_{\mathfrak{I}}(M_n)$, such that:

(A$_n$) $\|F_n - F_{n-1}\|_{1,M_{n-1}} < \xi/2^{n+1}$.

(B$_n$) $\partial F_{n,3}$ does not vanish anywhere on $M_n \setminus S$, $(\partial F_{n,3}|_E) = (\phi_3|_E) \in \mathfrak{Div}_{\mathfrak{I}}(R_S)$ for any connected component E of R_S such that ϕ_3 does not vanish everywhere on E.

(C$_n$) $\mathfrak{p}_{F_n} = \mathfrak{p}|_{\mathcal{H}_1(M_n, \mathbb{Z})}$ for all $n \in \mathbb{N}$.

We follow a recursive process. Set $F_1 := F_0(P_0) + \mathrm{Re} \int_{P_0} \Phi_1$, where $\Phi_1 \in \Omega_{\mathfrak{h}, \mathfrak{I}}(M_1)^3$ is a triple resulting from an application of Lemma 5.1 to the data

$$S = M_0, \quad W \text{ a tubular neighborhood of } M_1, \quad \Phi = \partial F_0 = \Phi_0,$$

close enough to ∂F_0 in the $\mathcal{C}^0$ topology on M_0 to ensure (A$_1$). Recall that M_1 is a tubular neighborhood of M_0, hence Φ_1 has no real periods and F_1 is well-defined. Properties (B$_1$) and (C$_1$) follow from (i)–(iv).

Let $n \geq 2$, assume that we have constructed $F_1, \ldots, F_{n-1}$, and let us construct F_n. We distinguish the following two cases:

Case 1 Assume that $\chi(M_n^\circ \setminus M_{n-1}) = 0$. Then $M_n^\circ \setminus M_{n-1}$ consists of a finite family of pairwise disjoint open annuli and $\mathcal{H}_1(M_n, \mathbb{Z}) = \mathcal{H}_1(M_{n-1}, \mathbb{Z})$; see Remark 5.8(1). Applying Lemma 5.1 to the data

$$S = M_{n-1}, \quad W \text{ a tubular neighborhood of } M_n, \quad \Phi = \partial F_{n-1},$$

one obtains a triple $\Phi_n \in \Omega_{\mathfrak{h}, \mathfrak{I}}(M_n)^3$ arbitrarily close to ∂F_{n-1} in the $\mathcal{C}^0$ topology on M_{n-1}. If this approximation is close enough, then $F_n := F_{n-1}(P_0) + \mathrm{Re} \int_{P_0} \Phi_n$ clearly satisfies (A$_n$). Further, since M_n is a tubular neighborhood of M_{n-1}, Φ_n has no real periods and F_n is well-defined. Properties (B$_n$) and (C$_n$) follow straightforwardly from (B$_{n-1}$), (C$_{n-1}$) and Lemma 5.1.

Case 2 Assume that $\chi(M_n^\circ \setminus M_{n-1}) = -2$. In this case, there exists an analytic Jordan arc $\gamma \subset M_n^\circ \setminus M_{n-1}^\circ$, attached to bM_{n-1} at its endpoints and otherwise disjoint to M_{n-1}, such that $\gamma \cap \mathfrak{I}(\gamma) = \varnothing$, $\widetilde{S} := M_{n-1} \cup \gamma \cup \mathfrak{I}(\gamma)$ is an $\mathfrak{I}$–admissible set in $\mathcal{N}$, and $\chi(M_n^\circ \setminus \widetilde{S}) = 0$; see Remark 5.8(2–4). Extend F_{n-1} to a generalized marked immersion (in the sense of Definition 3.12) $\widetilde{F} \in \mathcal{M}_{\mathfrak{g}, \mathfrak{I}}^*(\widetilde{S})$ such that $\mathfrak{p}_{\widetilde{F}} = \mathfrak{p}|_{\mathcal{H}_1(\widetilde{S}, \mathbb{Z})}$. Up to approximating $\widetilde{F}$ by a minimal immersion in $\mathcal{M}_{\mathfrak{I}}(\widetilde{M}_{n-1})$ via Lemma 5.1, where $\widetilde{M}_{n-1} \subset M_n^\circ$ is a tubular neighborhood of $\widetilde{S}$, one can reduce the proof to the previous case.

This concludes the construction of the sequence $\{F_n\}_{n \in \mathbb{N}}$.

By properties (A$_n$), $n \in \mathbb{N}$, the sequence $\{F_n\}_{n \in \mathbb{N}}$ converges in the $\mathcal{C}^1$ topology on compact sets of $\mathcal{N}$ to an $\mathfrak{I}$–invariant conformal harmonic map $Y = (Y_j)_{j=1,2,3} \colon \mathcal{N} \to \mathbb{R}^3$ such that $\|Y - X_\varpi\|_{1,S} < \xi < \epsilon$; take also (iii) into account. From (ii), (B$_n$), $n \in \mathbb{N}$,

and Hurwitz's theorem, it follows that ∂Y_3 does not vanish anywhere on $\mathcal{N} \setminus S$ and $(\partial Y_3|_E) = (\phi_3|_E) \in \mathfrak{Div}_{\mathfrak{I}}(R_S)$ for any connected component E of R_S such that ϕ_3 does not vanish everywhere on E. On the other hand, if ξ is taken small enough from the beginning, then Y is a conformal minimal immersion; indeed, it has branch points neither on S (since ∂Y is close to ∂X_{ϖ} on S) nor on $\mathcal{N} \setminus S$ (since ∂Y_3 does not vanish anywhere on $\mathcal{N} \setminus S$). Finally, (iv) and (C_n), $n \in \mathbb{N}$, give $\mathfrak{p}_Y = \mathfrak{p}$. This proves Theorem 5.6(i).

In order to prove Theorem 5.6(ii) we reason analogously but use Lemma 5.5 instead of Lemma 5.1. $\square$

Notice that Theorem 5.6 is a general existence result for nonorientable minimal surfaces in $\mathbb{R}^3$ with arbitrary conformal structure. In fact, in the next section we construct such surfaces with additional geometrical properties; see Theorems 6.6 and 6.8.

6 Applications

We conclude the paper with some applications of the results in the previous section. In Section 6.1 we will derive approximation theorems of Runge–Mergelyan type for other objects than nonorientable minimal surfaces; see Corollary 6.1 and Theorem 6.3. In Section 6.2 we will prove an existence theorem of Gunning–Narasimhan type on nonorientable Riemann surfaces (see Theorem 6.4). Finally, in Sections 6.3 and 6.4 we show general existence results of nonorientable minimal surfaces in $\mathbb{R}^3$ with given underlying conformal structure and additional topological or geometric properties.

6.1 Some Runge–Mergelyan-type results

We say a holomorphic immersion $(F_j)_{j=1,2,3} \colon \mathcal{N} \to \mathbb{C}^3$ is a *null curve* if $\sum_{j=1}^{3} (dF_j)^2$ vanishes everywhere on the open Riemann surface $\mathcal{N}$. Minimal surfaces in $\mathbb{R}^3$ are locally the real part of null curves in $\mathbb{C}^3$. (See [31] for a good reference.)

We can now derive the analogous result to Theorem 5.6 for $\mathfrak{I}$–*symmetric* null curves.

Corollary 6.1 (Runge–Mergelyan theorem for $\mathfrak{I}$–symmetric null curves in $\mathbb{C}^3$) *Let $S \subset \mathcal{N}$ be an $\mathfrak{I}$–admissible subset and let $F = (F_j)_{j=1,2,3} \colon S \to \mathbb{C}^3$ be a smooth function in $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(S)^3$ such that $\sum_{j=1}^{3} (dF_j)^2$ vanishes everywhere on S and $\sum_{j=1}^{3} |dF_j|^2$ does not vanish anywhere on S.*

The following assertions hold:

- F can be uniformly approximated in the $\mathcal{C}^1$ topology on S by null curves $H = (H_j)_{j=1,2,3}\colon \mathcal{N} \to \mathbb{C}^3$ in $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\mathcal{N})^3$ such that dH_3 does not vanish anywhere on $\mathcal{N} \setminus R_S$. Furthermore, H can be chosen so that $(dH_3|_E) = (dF_3|_E) \in \mathfrak{Div}(E)$ for any connected component E of R_S such that dF_3 does not vanish everywhere on E.

- If F_3 is nonconstant and extends to $\mathcal{N}$ as a holomorphic function whose differential does not vanish anywhere on C_S, then F can be approximated in the $\mathcal{C}^1$ topology on S by null curves $H = (H_j)_{j=1,2,3}\colon \mathcal{N} \to \mathbb{C}^3$ in $\mathcal{F}_{\mathfrak{h},\mathfrak{I}}(\mathcal{N})^3$, where $H_3 = F_3$.

Proof Up to suitably extending F to a connected $\mathfrak{I}$–admissible subset of $\mathcal{N}$ containing S (see Claim 5.7), we assume without loss of generality that S is connected.

By Theorem 5.6, there exists a sequence $\{Y_n = (Y_{n,j})_{j=1,2,3}\}_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathfrak{I}}(\mathcal{N})$ with $\mathfrak{p}_{Y_n} = 0$ for all n, approximating $X_{\varpi} \equiv (X,\varpi) = (\operatorname{Re} F, \operatorname{Im} dF)$ in the $\mathcal{C}^1$ topology on S, and whose third coordinates $Y_{n,3}$, $n \in \mathbb{N}$, satisfy the required properties. If we fix $P_0 \in S$, the sequence of null curves $\{F(P_0) + \int_{P_0} \partial Y_n\}_{n\in\mathbb{N}}$ on $\mathcal{N}$ proves the corollary. $\qquad\square$

We next point out that Theorem 5.6 is also valid for marked harmonic functions in the following sense:

Definition 6.2 Let $S \subset \mathcal{N}$ be an $\mathfrak{I}$–admissible subset. By a marked $\mathfrak{I}$–invariant harmonic function on S we mean a pair $h_\theta \equiv (h,\theta)$, where $h\colon S \to \mathbb{R}^3$ is an $\mathfrak{I}$–invariant function, harmonic on an open neighborhood of R_S, and smooth in the sense of Definition 3.8, and $\theta \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)$ is a 1–form such that $\theta|_{R_S}$ equals the complex derivative $\partial(h|_{R_S})$ of $h|_{R_S}$, θ has no real periods, and $h = \operatorname{Re} \int^P \theta$.

If h_θ is a marked $\mathfrak{I}$–invariant harmonic function we set $\partial h_\theta := \theta$. Analogously to Definition 3.15, the space of marked $\mathfrak{I}$–invariant harmonic functions on S is endowed with a natural $\mathcal{C}^1$ topology.

If $\theta \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)$, $\gamma \in \mathcal{H}(S,\mathbb{Z})$, and $\mathfrak{I}_*(\gamma) = \gamma$, then $\int_\gamma \theta \in \mathbb{R}$. Likewise, if $\mathfrak{I}_*(\gamma) = -\gamma$ then $\int_\gamma \theta \in \iota\mathbb{R}$. In particular, θ has no real periods if and only if $\int_\gamma \theta = 0$ for all $\gamma \in \mathcal{H}(S,\mathbb{Z})$ with $\mathfrak{I}_*(\gamma) = \gamma$, and in this case $\int_{\mathfrak{I}_*(\gamma)} \theta = -\int_\gamma \theta \in \iota\mathbb{R}$ for all $\gamma \in \mathcal{H}(S,\mathbb{Z})$.

Theorem 6.3 (Runge–Mergelyan theorem for harmonic functions of nonorientable Riemann surfaces) *Let $S \subset \mathcal{N}$ be an $\mathfrak{I}$–admissible subset. Let h_θ be a marked $\mathfrak{I}$–invariant harmonic function on S. Let $\mathfrak{p}\colon \mathcal{H}_1(\mathcal{N},\mathbb{Z}) \to \mathbb{R}$ be a group homomorphism*

such that $\mathfrak{p}(\mathfrak{I}_*(\gamma)) = -\mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$, and $\mathfrak{p}(\gamma) = \mathrm{Im} \int_\gamma \partial h_\theta$ for all $\gamma \in \mathcal{H}_1(S, \mathbb{Z})$.

Then h_θ can be approximated in the $\mathcal{C}^1$ topology on S by $\mathfrak{I}$–invariant harmonic functions $\widehat{h}$ on $\mathcal{N}$, such that $\partial\widehat{h}$ does not vanish anywhere on $\mathcal{N} \setminus R_S$ and $\mathfrak{p}(\gamma) = \mathrm{Im} \int_\gamma \partial\widehat{h}$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$. Furthermore, $\widehat{h}$ can be chosen so that $(\partial\widehat{h}|_E) = (\partial h_\theta|_E) \in \mathfrak{Div}_\mathfrak{I}(E)$ for any connected component E of R_S such that $h|_E$ is nonconstant.

Proof We assume without loss of generality that ∂h_θ does not vanish anywhere on C_S (see Claim 5.3) and that S is connected (see Claim 5.7 and the proof of Theorem 5.6). We use the notation $\phi_3 = \partial h_\theta \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)$. Let $U \subset \mathcal{N}$ be an $\mathfrak{I}$–invariant relatively compact domain with finite topology, containing S. Reasoning as in the proof of Lemma 4.6, the Riemann–Roch theorem gives $g_0 \in \mathcal{G}_\mathfrak{I}(U)$ such that $(\phi_3|_{R_S}) = (g_0)_0(g_0)_\infty$. Then one can easily extend $g_0|_{R_S}$ to a function $g \in \mathcal{G}_\mathfrak{I}(S)$ such that the triple $\Phi = (\phi_j)_{j=1,2,3}$, obtained from the pair (g, ϕ_3) via (7), satisfies the requirements in Lemma 5.1.

Let $M_0 := S$ and let $\{M_n\}_{n \in \mathbb{N}}$ be an exhaustion of $\mathcal{N}$ by Runge connected $\mathfrak{I}$–invariant compact regions such that the Euler characteristic $\chi(M_n^\circ \setminus M_{n-1}) \in \{0, -2\}$ for all $n \in \mathbb{N}$; see Remark 5.8.

Let $\epsilon > 0$. Fix $\xi > 0$ to be specified later. Arguing as in the proof of Theorem 5.6, a recursive application of Lemma 5.1 gives a sequence $\{\Psi_n = (\psi_{n,j})_{j=1,2,3}\}_{n \in \mathbb{N}} \subset \Omega_{\mathfrak{h},\mathfrak{I}}(M_n)^3$ such that $\psi_{n,3}$ does not vanish anywhere on $M_n \setminus R_S$, $\int_\gamma \psi_{n,3} = \iota\mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(M_n, \mathbb{Z})$, and $\{\Psi_n\}_{n \in \mathbb{N}}$ converges in the $\mathcal{C}^0$ topology on compact sets of $\mathcal{N}$ to a triple $\Psi = (\psi_j)_{j=1,2,3} \in \Omega_{\mathfrak{h},\mathfrak{I}}(\mathcal{N})^3$ with

$$(37) \qquad\qquad \|\Psi - \Phi\|_{0,M_0} < \xi.$$

Therefore ψ_3 does not vanish anywhere on $\mathcal{N} \setminus R_S$ (by Hurwitz's theorem) and $\int_\gamma \psi_3 = \iota\mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$. Furthermore, $\psi_{n,3}$ can be chosen so that $(\psi_{n,3}|_E) = (\phi_3|_E) \in \mathfrak{Div}_\mathfrak{I}(E)$ for any connected component E of R_S such that ϕ_3 does not vanish everywhere on E, $n \in \mathbb{N}$. Therefore, Hurwitz's theorem gives that $(\psi_3|_E) = (\phi_3|_E) \in \mathfrak{Div}_\mathfrak{I}(E)$ as well for any such E.

By (37), the $\mathfrak{I}$–invariant harmonic function $\widehat{h} := h(P_0) + \mathrm{Re} \int_{P_0} \psi_3$, where $P_0 \in S$, satisfies $\|\widehat{h} - h_\theta\|_{1,S} < \epsilon$ if ξ is chosen small enough. This proves the theorem. $\qquad\square$

6.2 An application to Riemann surface theory

Gunning and Narasimhan [18] showed that every open Riemann surface carries exact nowhere vanishing holomorphic 1–forms. This result was extended to the existence

of holomorphic 1–forms with prescribed periods and divisor by Kusunoki and Sainouchi [21]. Let us show the analogous result for nonorientable Riemann surfaces.

Theorem 6.4 *Let D' be an integral divisor on $\mathcal{N}$, possibly with countably infinite support, such that $\operatorname{supp}(D') \cap \operatorname{supp}(\mathfrak{I}(D')) = \varnothing$ and $\operatorname{supp}(D') \cap K$ is finite for any compact $K \subset \mathcal{N}$. Call $D = D' \cup \mathfrak{I}(D')$. Let $\mathfrak{p} \colon \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{C}^3$ be a group homomorphism such that $\mathfrak{p}(\mathfrak{I}_*(\gamma)) = \overline{\mathfrak{p}(\gamma)}$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$.*

Then there exists $\vartheta \in \Omega_{\mathfrak{h},\mathfrak{I}}(\mathcal{N})$ such that $(\vartheta) = D$ and $\int_\gamma \vartheta = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$.

Proof Let $M_0 \subset \mathcal{N}$ be a connected $\mathfrak{I}$–admissible set such that $R_{M_0} = U \cup \mathfrak{I}(U)$, where U is a closed disc in $\mathcal{N}$ and $U \cap \mathfrak{I}(U) = \varnothing$, and $\operatorname{supp}(D) \cap (M_0 \setminus M_0^\circ) = \varnothing$. Denote by D_0 the restriction of D to M_0 (that is, the unique integer divisor in M_0 such that $\operatorname{supp}(D/D_0) \cap M_0 = \varnothing$), and recall that $\operatorname{supp}(D_0)$ consists of finitely many points. Take $\phi_{0,3} \in \Omega_{\mathfrak{h},\mathfrak{I}}(R_{M_0})$ vanishing everywhere on no connected component of R_{M_0} and with $(\phi_{0,3}) = D_0$. As in the proof of Theorem 5.6, one can extend $\phi_{0,3}$ to a smooth 1–form $\psi_0 \in \Omega_{\mathfrak{h},\mathfrak{I}}(M_0)$ vanishing nowhere on $M_0 \setminus M_0^\circ$ and satisfying that $\int_\gamma \psi_0 = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(M_0, \mathbb{Z})$.

Let $\{M_n\}_{n \in \mathbb{N}}$ be an exhaustion of $\mathcal{N}$ by Runge connected $\mathfrak{I}$–invariant compact regions such that the Euler characteristic $\chi(M_n^\circ \setminus M_{n-1}) \in \{0, -2\}$ and $\operatorname{supp}(D) \cap bM_n = \varnothing$ for all $n \in \mathbb{N}$; see Remark 5.8. Denote by D_n the restriction of D to M_n, and recall that $\operatorname{supp}(D_n)$ consists of finitely many points, $n \in \mathbb{N}$.

Assume that $\chi(M_1^\circ \setminus M_0) = 0$. Let $N_0 \subset M_1^\circ \setminus M_0$ be a Runge $\mathfrak{I}$–invariant compact region containing $\operatorname{supp}(D_1) \setminus \operatorname{supp}(D_0)$, and consisting of a finite collection of pairwise disjoint closed discs. Notice that $M_0 \cup N_0$ is $\mathfrak{I}$–admissible. Take $\phi_{1,3} \in \Omega_{\mathfrak{h},\mathfrak{I}}(M_0 \cup N_0)$ vanishing everywhere on no connected component of $R_{M_0} \cup N_0$, and satisfying that $\phi_{1,3}|_{M_0} = \psi_0$, $(\phi_{1,3}) = D_1$, and $\int_\gamma \phi_{1,3} = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(M_1, \mathbb{Z}) = \mathcal{H}_1(M_0, \mathbb{Z})$. As in the proof of Theorem 6.3, one can easily find a triple $\Phi_1 = (\phi_{1,j})_{j=1,2,3} \in \Omega_{\mathfrak{h},\mathfrak{I}}(M_0 \cup N_0)^3$ meeting the requirements of Lemma 5.1. Therefore, this lemma furnishes $\Psi_1 = (\psi_{1,j})_{j=1,2,3} \in \Omega_{\mathfrak{h},\mathfrak{I}}(M_1)^3$ as close as desired to Φ_1 in the $\mathcal{C}^0$ topology on $M_0 \cup N_0$, satisfying that $\Psi_1 - \Phi_1$ is exact on $M_0 \cup N_0$ and $(\psi_{1,3}) = (\phi_{1,3})$. Call $\psi_1 := \psi_{1,3} \in \Omega_{\mathfrak{h},\mathfrak{I}}(M_1)$ and notice that $(\psi_1) = D_1$, $\int_\gamma \psi_1 = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(M_1, \mathbb{Z})$, and ψ_1 is as close as desired to ψ_0 in the $\mathcal{C}^0$ topology on M_0.

Assume that, on the contrary, $\chi(M_1^\circ \setminus M_0) = -2$. Then take a Jordan arc α in $M_1^\circ \setminus (M_0^\circ \cup \operatorname{supp}(D))$, with endpoints in R_{M_0} and otherwise disjoint from M_0, such that $S := M_0 \cup \alpha \cup \mathfrak{I}(\alpha)$ is $\mathfrak{I}$–admissible and $\chi(M_1^\circ \setminus S) = 0$; see Remark 5.8. Extend (with the same name) $\phi_{3,0}$ to a smooth 1–form $\phi_{3,0} \in \Omega_{\mathfrak{h},\mathfrak{I}}(S)$ vanishing nowhere

on α and satisfying that $\int_{\gamma} \phi_{3,0} = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(M_1, \mathbb{Z}) = \mathcal{H}_1(S, \mathbb{Z})$. Then, one can follow the argument in the above paragraph, replacing M_0 by S, and obtain as above $\psi_1 \in \Omega_{\mathfrak{h},\mathfrak{I}}(M_1)$ as close as desired to ψ_0 in the $\mathcal{C}^0$ topology on M_0, with $(\psi_1) = D_1$ and $\int_{\gamma} \psi_1 = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(M_1, \mathbb{Z})$.

Repeating this argument inductively, one constructs a sequence $\{\psi_n\}_{n \in \mathbb{N}} \subset \Omega_{\mathfrak{h},\mathfrak{I}}(M_n)$ such that ψ_n is as close as desired to ψ_{n-1} in the $\mathcal{C}^0$ topology on M_{n-1} for all $n > 1$, $(\psi_n) = D_n$ and $\int_{\gamma} \psi_n = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(M_n, \mathbb{Z})$ for all $n \in \mathbb{N}$. Furthermore, one can assume that $\{\psi_n\}_{n \in \mathbb{N}}$ converges uniformly on compact subsets of $\mathcal{N}$ to a holomorphic 1–form $\vartheta \in \Omega_{\mathfrak{h},\mathfrak{I}}(\mathcal{N})$. Since ψ_0 is not identically zero and ϑ can be constructed as close as desired to ψ_0 on M_0, ϑ may be assumed to be nonidentically zero as well. Obviously $\int_{\gamma} \vartheta = \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$. By Hurwitz's theorem, $(\vartheta) = D$ and we are done. $\qquad \square$

Theorem 6.4 is of particular interest in the special case of the trivial divisor and zero periods. We can prove the following Gunning–Narasimhan-type result.

Corollary 6.5 *Every open Riemann surface $\mathcal{N}$ carrying an antiholomorphic involution $\mathfrak{I}\colon \mathcal{N} \to \mathcal{N}$ without fixed points admits an $\mathfrak{I}$–invariant holomorphic function $f\colon \mathcal{N} \to \mathbb{C}$ with no critical points.*

Proof Let $\vartheta \in \Omega_{\mathfrak{h},\mathfrak{I}}(\mathcal{N})$ be given by Theorem 6.4 applied to $D = \varnothing$ and $\mathfrak{p}$ identically zero. Choose $P_0 \in \mathcal{N}$. The function $f\colon \mathcal{N} \to \mathbb{C}$, $f(P) = \int_{P_0}^{P} \vartheta$, satisfies the conclusion of the corollary. $\qquad \square$

6.3 Nonorientable minimal surfaces in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$

In this subsection we show that any open Riemann surface $\mathcal{N}$ endowed with an antiholomorphic involution $\mathfrak{I}\colon \mathcal{N} \to \mathcal{N}$ without fixed points, is furnished with an $\mathfrak{I}$–invariant conformal minimal immersion $\mathcal{N} \to \mathbb{R}^3$ whose image surface is a nonorientable minimal surface properly projecting into a plane, contained in a wedge in $\mathbb{R}^3$ of any given angle greater than π. Furthermore, the flux map of such surface can be prescribed under the compatibility condition (9). This existence theorem links with a classical question by Schoen and Yau [33, page 18]; see [6; 10] for a good setting on this problem.

Theorem 6.6 *Let $\mathfrak{p}\colon \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^3$ be a group homomorphism satisfying*

$$\mathfrak{p}(\mathfrak{I}_*(\gamma)) = -\mathfrak{p}(\gamma) \quad \text{for all } \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z}),$$

and let θ be a real number in $(0, \pi/2)$.

Let $M \subset \mathcal{N}$ be a Runge $\mathfrak{I}$–invariant compact region, and consider $Y \in \mathcal{M}_{\mathfrak{I}}(M)$ with flux map $\mathfrak{p}_Y = \mathfrak{p}|_{\mathcal{H}_1(M,\mathbb{Z})}$, satisfying

$$(x_3 + \tan(\theta)|x_1|) \circ Y > 1 \quad \text{everywhere on } M.$$

Then for any $\epsilon > 0$ there exists a conformal minimal immersion $X: \mathcal{N} \to \mathbb{R}^3$ satisfying $X \circ \mathfrak{I} = X$ and the following properties:

- $\mathfrak{p}_X = \mathfrak{p}$.
- $(x_3 + \tan(\theta)|x_1|) \circ X: \mathcal{N} \to \mathbb{R}$ is a positive proper function.
- $\|X - Y\|_{1,M} < \epsilon$.

The corresponding theorem for orientable minimal surfaces was obtained by Alarcón and López in [6, Theorem 5.6], as application of the Runge–Mergelyan approximation result [6, Theorem 4.9]. We adapt the proof in [6] to the nonorientable framework, sketching the necessary modifications. In this case our main tool is Theorem 5.6. The complete details could easily be filled in by an interested reader.

We denote by $x_k: \mathbb{R}^3 \to \mathbb{R}$ the kth coordinate function, $k = 1, 2, 3$. Given numbers $\theta \in (-\pi/2, \pi/2)$ and $\delta \in \mathbb{R}$, we denote by

$$\Pi_\delta(\theta) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 + \tan(\theta)x_1 > \delta\}.$$

Theorem 6.6 will follow from a standard recursive application of the following approximation result.

Lemma 6.7 Let M, $V \subset \mathcal{N}$ be two Runge $\mathfrak{I}$–invariant compact regions with analytical boundary such that $M \subset V^\circ$ and the Euler characteristic $\chi(V \setminus M^\circ) \in \{-2, 0\}$.

Let $X \in \mathcal{M}_{\mathfrak{I}}(M)$ and let $\mathfrak{p}: \mathcal{H}_1(V, \mathbb{Z}) \to \mathbb{R}$ be any homomorphism extension of the flux map $\mathfrak{p}_X$ of X, satisfying

$$\mathfrak{p}(\mathfrak{I}_*(\gamma)) = -\mathfrak{p}(\gamma) \quad \text{for all } \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z}).$$

Let $\theta \in (0, \pi/4)$ and $\delta > 0$, and assume that

$$(38) \qquad\qquad X(bM) \subset \Pi_\delta(\theta) \cup \Pi_\delta(-\theta).$$

Then, for any $\epsilon > 0$ there exists $Y \in \mathcal{M}_{\mathfrak{I}}(V)$ enjoying the following properties:

(i)   The flux map $\mathfrak{p}_Y$ of Y equals $\mathfrak{p}$.

(ii) $\|Y - X\|_{1,M} < \epsilon$.

(iii) $Y(bV) \subset \Pi_{\delta+1}(\theta) \cup \Pi_{\delta+1}(-\theta)$.

(iv) $Y(V \setminus M) \subset \Pi_\delta(\theta) \cup \Pi_\delta(-\theta)$.

Proof of Lemma 6.7 in the case $\chi(V \setminus M^\circ) = 0$ Since $M \subset V^\circ$ and $V^\circ \setminus M$ has no relatively compact connected components in V°, then $V \setminus M^\circ = \bigcup_{j=1}^{\mathfrak{j}}(A_j \cup \mathfrak{I}(A_j))$, where $\mathfrak{j} \in \mathbb{N}$ denotes the number of boundary components of V (hence, of M) and $A_1, \mathfrak{I}(A_1), \ldots, A_{\mathfrak{j}}, \mathfrak{I}(A_{\mathfrak{j}})$ are pairwise disjoint compact annuli.

Write $bA_j = \alpha_j \cup \beta_j$, where $\alpha_j \subset bM$ and $\beta_j \subset bV$ for all $j = 1, \ldots, \mathfrak{j}$. Obviously, $\mathfrak{I}(\alpha_j) \subset bM$, $\mathfrak{I}(\beta_j) \subset bV$, and $b\mathfrak{I}(A_j) = \mathfrak{I}(\alpha_j) \cup \mathfrak{I}(\beta_j)$ for all $j = 1, \ldots, \mathfrak{j}$.

From inclusion (38), it follows the existence of a natural number $\mathfrak{i} \geq 2$, a collection of subarcs $\{\alpha_j^i : (i, j) \in I = \mathbb{Z}_{\mathfrak{i}} \times \{1, \ldots, \mathfrak{j}\}\}$, where $\alpha_j^i \subset \alpha_j$ for all $(i, j) \in I$ and $\mathbb{Z}_{\mathfrak{i}} = \{0, \ldots, \mathfrak{i} - 1\}$ denotes the additive cyclic group of integers modulus $\mathfrak{i} \in \mathbb{N}$, and subsets I_+ and I_- of I, such that

- $I_+ \cap I_- = \varnothing$, $I_+ \cup I_- = I$,
- α_j^i and α_j^{i+1} have a common endpoint Q_j^{i+1} and are otherwise disjoint,
- $X(\alpha_j^i) \subset \Pi_\delta(\pm\theta)$ for all $(i, j) \in I_\pm$.

In particular, $\alpha_j = \bigcup_{i \in \mathbb{Z}_{\mathfrak{i}}} \alpha_j^i$ for all $j = 1, \ldots, \mathfrak{j}$, and $X(\mathfrak{I}(\alpha_j^i)) \subset \Pi_\delta(\pm\theta)$ for all $(i, j) \in I_\pm$; recall that X is $\mathfrak{I}$–invariant.

Choose a family $\{r_j^i : (i, j) \in I\}$ of pairwise disjoint analytical compact Jordan arcs such that r_j^i is contained in A_j, has initial point $Q_j^i \in \alpha_j$, final point $P_j^i \in \beta_j$, is otherwise disjoint from bA_j, and meets transversally α_j^i at the point Q_j^i for all $(i, j) \in I$. The set

$$S := M \cup \left(\bigcup_{(i,j) \in I} r_j^i \cup \mathfrak{I}(r_j^i) \right)$$

is $\mathfrak{I}$–admissible in the sense of Definition 3.2.

In a first step, we construct an $\mathfrak{I}$–invariant conformal minimal immersion $H \in \mathcal{M}_{\mathfrak{I}}(V)$ meeting the theses of the lemma on points of S; more specifically, satisfying

(1_H) $\|H - X\|_{1,S} < \epsilon/3$,

(2_H) $H(r_j^i \cup \alpha_j^i \cup r_j^{i+1}) \subset \Pi_\delta(\pm\theta)$ for all $(i, j) \in I_\pm$,

(3_H) $H(\{P_j^i, P_j^{i+1}\}) \subset \Pi_{\delta+1}(\pm\theta)$ for all $(i, j) \in I_\pm$,

(4_H) $\mathfrak{p}_H = \mathfrak{p}$.

Such an H is furnished by Theorem 5.6(i) applied to a suitable $\mathfrak{I}$–invariant extension $\hat{X}$ of X to S, formally meeting properties (2_H) and (3_H) (cf [6, Subsection 5.1]). To construct such an extension, we first define $\hat{X}$ over

$$\bigcup_{(i,j)\in I} r_j^i$$

and then we extend it to S (that is to say; we define $\hat{X}$ over $\bigcup_{(i,j)\in I}\mathfrak{I}(r_j^i)$) to be $\mathfrak{I}$–invariant.

Denote by Ω_j^i the closed disc in A_j bounded by $\alpha_j^i \cup r_j^i \cup r_j^{i+1}$ and the compact Jordan arc $\beta_j^i \subset \beta_j$ connecting P_j^i and P_j^{i+1}, and containing no P_j^k for $k \neq i, i+1$, $(i,j) \in I$. Since H is continuous, properties (2_H) and (3_H) extend to small open neighborhoods of $r_j^i \cup \alpha_j^i \cup r_j^{i+1}$ and $\{P_j^i, P_j^{i+1}\}$, respectively; hence there exists a closed disc

$$K_j^i \subset \Omega_j^i \setminus (r_j^i \cup \alpha_j^i \cup r_j^{i+1}),$$

intersecting β_j^i in a compact Jordan arc, such that

- $H(\overline{\Omega_j^i \setminus K_j^i}) \subset \Pi_\delta(\pm\theta)$ for all $(i,j) \in I_\pm$,

- $H(\overline{\beta_j^i \setminus K_j^i}) \subset \Pi_{\delta+1}(\pm\theta)$ for all $(i,j) \in I_\pm$.

Assume without loss of generality that $I_+ \neq \varnothing$; otherwise $I_- = I \neq \varnothing$ and we would reason in a symmetric way. Consider the $\mathfrak{I}$–admissible set

$$S_+ := M \cup \left(\bigcup_{(i,j)\in I_-} \Omega_j^i \cup \mathfrak{I}(\Omega_j^i)\right) \cup \left(\bigcup_{(i,j)\in I_+} K_j^i \cup \mathfrak{I}(K_j^i)\right).$$

In a second step, we construct an $\mathfrak{I}$–invariant conformal minimal immersion $Z \in \mathcal{M}_\mathfrak{I}(V)$ meeting the theses of the lemma on points of S_+; more concretely, satisfying

(1_Z) $\|Z - H\|_{1, M \cup (\bigcup_{(i,j)\in I_-} \Omega_j^i \cup \mathfrak{I}(\Omega_j^i))} < \epsilon/3$,

(2_Z) $Z(\overline{\Omega_j^i \setminus K_j^i}) \subset \Pi_\delta(\pm\theta)$ for all $(i,j) \in I_\pm$,

(3_Z) $Z(\overline{\beta_j^i \setminus K_j^i}) \subset \Pi_{\delta+1}(\pm\theta)$ for all $(i,j) \in I_\pm$,

(4_Z) $Z(K_j^i) \subset \Pi_{\delta+1}(-\theta)$ for all $(i,j) \in I_+ = I \setminus I_-$,

(5_Z) $\mathfrak{p}_Z = \mathfrak{p}$.

The immersion Z is furnished by Theorem 5.6(ii) applied to a suitable $\mathfrak{I}$–invariant extension $\hat{H}$ of $H|_{M\cup(\bigcup_{(i,j)\in I_-}\Omega_j^i\cup\mathfrak{I}(\Omega_j^i))}$ to S_+. The key point is to ensure that

$$(39)\qquad (x_3+\tan(\theta)x_1)\circ Z=(x_3+\tan(\theta)x_1)\circ H \quad\text{everywhere on } V,$$

which is possible by Theorem 5.6(ii) up to suitably rotating H; see [6, Subsection 5.1]. As above, in order to construct $\hat{H}$, we first define it over $\bigcup_{(i,j)\in I_+}K_j^i$, formally meeting (4_Z) and (39), and then we extend it to S_+ (that is; we define $\hat{H}$ over $\bigcup_{(i,j)\in I_+}\mathfrak{I}(K_j^i)$) as an $\mathfrak{I}$–invariant map.

If $I_-=\varnothing$ the proof is already done; otherwise we consider the $\mathfrak{I}$–admissible set

$$S_-:=M\cup\left(\bigcup_{(i,j)\in I_+}\Omega_j^i\cup\mathfrak{I}(\Omega_j^i)\right)\cup\left(\bigcup_{(i,j)\in I_-}K_j^i\cup\mathfrak{I}(K_j^i)\right).$$

To finish the proof, we construct an $\mathfrak{I}$–invariant conformal minimal immersion $Y\in\mathcal{M}_{\mathfrak{I}}(V)$ satisfying the following properties:

(1_Y) $\|Y-Z\|_{1,M\cup(\bigcup_{(i,j)\in I_+}\Omega_j^i\cup\mathfrak{I}(\Omega_j^i))}<\epsilon/3.$

(2_Y) $Y(\overline{\Omega_j^i\setminus K_j^i})\subset\Pi_\delta(\pm\theta)$ for all $(i,j)\in I_\pm.$

(3_Y) $Y(\overline{\beta_j^i\setminus K_j^i})\subset\Pi_{\delta+1}(\pm\theta)$ for all $(i,j)\in I_\pm.$

(4_Y) $Y(K_j^i)\subset\Pi_{\delta+1}(\pm\theta)$ for all $(i,j)\in I\setminus I_\pm.$

(5_Y) $\mathfrak{p}_Y=\mathfrak{p}.$

Such Y is furnished by Theorem 5.6(ii) applied to a suitable $\mathfrak{I}$–invariant extension of $Z|_{M\cup(\bigcup_{(i,j)\in I_+}\Omega_j^i\cup\mathfrak{I}(\Omega_j^i))}$ to S_-, in a symmetric way to the previous step; see again [6, Subsection 5.1].

The immersion Y meets all the requirements in Lemma 6.7. $\qquad\square$

Proof of Lemma 6.7 in the case $\chi(V\setminus M°)=-2$ In this case, there exists an analytical Jordan arc $\gamma\subset V°\setminus M°$, attached to bM at its endpoints and otherwise disjoint to M, such that $\gamma\cap\mathfrak{I}(\gamma)=\varnothing$, $S:=M\cup\gamma\cup\mathfrak{I}(\gamma)$ is an $\mathfrak{I}$–admissible set in $\mathcal{N}$, and $\chi(V°\setminus S)=0$; see Remark 5.8(2–4). Extend X to a suitable generalized marked immersion $\widetilde{X}_\varpi=(\widetilde{X},\varpi)\in\mathcal{M}_{\mathfrak{g},\mathfrak{I}}^*(S)$, satisfying

$$\widetilde{X}(S\setminus S°)\subset\Pi_\delta(\theta)\cup\Pi_\delta(-\theta)$$

and $\mathfrak{p}_{\widetilde{X}_\varpi}=\mathfrak{p}|_{\mathcal{H}_1(S,\mathbb{Z})}$. Applying Theorem 5.6 to $\widetilde{X}_\varpi$ we then reduce the proof to the case when $\chi(V\setminus M°)=0$, and are done. $\qquad\square$

Proof of Theorem 6.6 Reasoning as in the proof of Theorem 5.6, we assume without loss of generality that M is connected. Set $M_0 := M$ and let $\{M_n\}_{n \in \mathbb{N}}$ be an exhaustion of $\mathcal{N}$ by Runge connected $\mathfrak{I}$–invariant compact regions such that the Euler characteristic $\chi(M_n^\circ \setminus M_{n-1}) \in \{0, -2\}$ for all $n \in \mathbb{N}$; see Remark 5.8. To prove the theorem we follow the argument that shows [6, Theorem 5.6], using Lemma 6.7 instead of [6, Lemma 5.1]. $\qquad\square$

6.4 Nonorientable minimal surfaces in $\mathbb{R}^3$ and harmonic functions

Let $\mathcal{N}$ be an open Riemann surface endowed with an antiholomorphic involution $\mathfrak{I}\colon \mathcal{N} \to \mathcal{N}$ without fixed points. In this subsection we show that every nonconstant $\mathfrak{I}$–invariant harmonic function $h\colon \mathcal{N} \to \mathbb{R}$ is a coordinate function of a complete conformal $\mathfrak{I}$–invariant minimal immersion $\mathcal{N} \to \mathbb{R}^3$; see Theorem 6.8. We then derive existence of complete nonorientable minimal surfaces in $\mathbb{R}^3$, with *arbitrary conformal structure*, whose Gauss map (see Definition 2.8) omits one point of the projective plane $\mathbb{RP}^2$; see Corollary 6.10. Recall that, by Fujimoto [17], the Gauss map of complete nonorientable minimal surfaces in $\mathbb{R}^3$ misses at most two points of $\mathbb{RP}^2$. Furthermore, there exist nonorientable Riemann surfaces which do not carry complete conformal minimal immersions into $\mathbb{R}^3$ with Gauss map omitting two points of $\mathbb{RP}^2$; for instance, by Picard's great theorem, those being parabolic and of finite topology.

The analogous results in the orientable framework were obtained by Alarcón, Fernández and López in [2] (see also Alarcón and Fernández [1] for a partial result). Again, we only sketch here the necessary modifications to adapt the proof in [2] to the nonorientable setting, by using Theorem 5.6.

Theorem 6.8 *Let $h\colon \mathcal{N} \to \mathbb{R}^3$ be a nonconstant $\mathfrak{I}$–invariant harmonic function, and let $\mathfrak{p}\colon \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^3$ be a group homomorphism such that $\mathfrak{p}(\mathfrak{I}_*(\gamma)) = -\mathfrak{p}(\gamma)$ and the third coordinate of $\mathfrak{p}(\gamma)$ equals $\operatorname{Im} \int_\gamma \partial h$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$.*

Then there exists a complete conformal $\mathfrak{I}$–invariant minimal immersion $X = (X_1, X_2, X_3)\colon \mathcal{N} \to \mathbb{R}^3$ with $X_3 = h$ and $\mathfrak{p}_X = \mathfrak{p}$.

The proof of Theorem 6.8 relies on a recursive application of the following.

Lemma 6.9 *Let M, $V \subset \mathcal{N}$ be two Runge $\mathfrak{I}$–invariant compact regions with analytical boundary such that $M \subset V^\circ$ and the Euler characteristic $\chi(V \setminus M^\circ) \in \{-2, 0\}$.*

Let $h\colon V \to \mathbb{R}$ be a nonconstant $\mathfrak{I}$–invariant harmonic function, let $X = (X_1, X_2, X_3) \in \mathcal{M}_{\mathfrak{I}}(M)$, and let $\mathfrak{p}\colon \mathcal{H}_1(V, \mathbb{Z}) \to \mathbb{R}$ be a group homomorphism, satisfying $X_3 = h|_M$,

$\mathfrak{p}_X = \mathfrak{p}|_{\mathcal{H}_1(M,\mathbb{Z})}$, $\mathfrak{p}(\mathfrak{I}_*(\gamma)) = -\mathfrak{p}(\gamma)$, and the third coordinate of $\mathfrak{p}(\gamma)$ equals $\mathrm{Im} \int_\gamma \partial h$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$.

Then, for any $P_0 \in M$ and $\epsilon > 0$, there exists $Y = (Y_1, Y_2, Y_3) \in \mathcal{M}_\mathfrak{I}(V)$ enjoying the following properties:

(i)   The flux map $\mathfrak{p}_Y$ of Y equals $\mathfrak{p}$.

(ii) $\|Y - X\|_{1,M} < \epsilon$.

(iii) $Y_3 = h$.

(iv) $\mathrm{dist}_Y(P_0, bV)) > 1/\epsilon$, where dist_Y denotes the distance on V in the intrinsic metric of the immersion Y.

Proof of Lemma 6.9 in the case $\chi(V \setminus M^\circ) = 0$ As in the proof of Lemma 6.7, write $V \setminus M^\circ = \bigcup_{j=1}^{\mathfrak{j}}(A_j \cup \mathfrak{I}(A_j))$, where $\mathfrak{j} \in \mathbb{N}$ denotes the number of boundary components of V and $A_1, \mathfrak{I}(A_1), \ldots, A_\mathfrak{j}, \mathfrak{I}(A_\mathfrak{j})$ are pairwise disjoint compact annuli.

On the interior of each annuli A_j, we define a labyrinth of compact sets $\mathcal{K}_j$ *adapted to dh* as that in the proof of [2, Claim 3.2] (this follows the spirit of Jorge and Xavier's original construction of a complete minimal surface in a slab of $\mathbb{R}^3$ [20]). We use the notation $\mathcal{K} = \bigcup_{j=1}^{\mathfrak{j}} \mathcal{K}_j \cup \mathfrak{I}(\mathcal{K}_j)$ and denote by $S \subset \mathcal{N}$ the $\mathfrak{I}$–admissible set

$$S = M \cup \mathcal{K}.$$

To finish, we reason as in the proof of [2, Claim 3.2]. In a first step we extend X to S as an $\mathfrak{I}$–invariant conformal minimal immersion $\widehat{X} = (\widehat{X}_1, \widehat{X}_2, \widehat{X}_3): S \to \mathbb{R}^3$, such that $\widehat{X}_3 = h|_S$ and whose intrinsic metric is sufficiently large over $\mathcal{K}$. In order to find a suitable $\widehat{X}$ we first argue as in [2, Claim 3.2] to extend X to $\bigcup_{j=1}^{\mathfrak{j}} \mathcal{K}_j$, and then we define $\widehat{X}$ over $\bigcup_{j=1}^{\mathfrak{j}} \mathfrak{I}(\mathcal{K}_j)$ to be $\mathfrak{I}$–invariant. The proof now can be concluded by applying Theorem 5.6(ii) to $\widehat{X}$; see again the proof of [2, Claim 3.2]. □

Proof of Lemma 6.9 in the case $\chi(V \setminus M^\circ) = -2$ Let $\gamma \subset V^\circ \setminus M^\circ$ be an analytical Jordan arc attached to bM at its endpoints and otherwise disjoint to M, such that $\gamma \cap \mathfrak{I}(\gamma) = \varnothing$, $S := M \cup \gamma \cup \mathfrak{I}(\gamma)$ is an $\mathfrak{I}$–admissible set in $\mathcal{N}$, and $\chi(V^\circ \setminus S) = 0$; see Remark 5.8(2–4). Extend X to a generalized marked immersion $\widetilde{X}_\varpi = (\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3), \varpi) \in \mathcal{M}_{\mathfrak{g},\mathfrak{I}}^*(S)$, satisfying $\widetilde{X}_3 = h|_S$ and $\mathfrak{p}_{\widetilde{X}_\varpi} = \mathfrak{p}|_{\mathcal{H}_1(S,\mathbb{Z})}$. We then reduce the proof to the case when $\chi(V \setminus M^\circ) = 0$, by using Theorem 5.6(ii). □

Proof of Theorem 6.8   Reasoning as in the proof of Theorem 5.6, we assume without loss of generality M is connected. Set $M_0 := M$ and let $\{M_n\}_{n\in\mathbb{N}}$ be an exhaustion of $\mathcal{N}$ by Runge connected $\mathfrak{I}$–invariant compact regions such that the Euler characteristic $\chi(M_n^\circ \setminus M_{n-1}) \in \{0, -2\}$ for all $n \in \mathbb{N}$; see Remark 5.8. To finish we follow the argument in the proof of [2, Theorem 4.1], replacing [2, Lemma 3.1] by Lemma 6.9. □

Corollary 6.10 Let $\mathfrak{p}\colon \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^3$ be a group homomorphism such that $\mathfrak{p}(\mathfrak{I}_*(\gamma)) = -\mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$.

Then there exists a complete conformal $\mathfrak{I}$–invariant minimal immersion $X\colon \mathcal{N} \to \mathbb{R}^3$ such that $\mathfrak{p}_X = \mathfrak{p}$, and the complex Gauss map of $\underline{X}\colon \underline{\mathcal{N}} \to \mathbb{R}^3$ (see Definition 2.8) omits one point of $\mathbb{RP}^2$.

Proof By Theorem 6.4, there exists a nowhere-vanishing holomorphic 1–form ϑ on $\mathcal{N}$ such that $\mathfrak{I}^*\vartheta = \overline{\vartheta}$ and $\int_\gamma \vartheta = \iota \mathfrak{p}(\gamma)$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$. Applying Theorem 6.8 to $h := \mathrm{Re} \int^P \vartheta$, we get a complete conformal $\mathfrak{I}$–invariant minimal immersion $X\colon \mathcal{N} \to \mathbb{R}^3$ such that $\mathfrak{p}_X = \mathfrak{p}$, and whose complex Gauss map has neither zeros nor poles. Therefore, the Gauss map of $\underline{X}\colon \underline{\mathcal{N}} \to \mathbb{R}^3$ omits one point of $\mathbb{RP}^2$ (see Remark 2.7). This concludes the proof. $\square$

In [3], Alarcón, Fernández and López extended the results in [2] to minimal surfaces in $\mathbb{R}^n$, $n \geq 3$. The key tool was a Runge–Mergelyan-type theorem for minimal surfaces in $\mathbb{R}^n$. In the forthcoming paper [5], we will show the analogous result in the nonorientable framework; this will allow us to generalize the results in this subsection to nonorientable minimal surfaces in $\mathbb{R}^n$.

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