msp

Geometry & Topology 19 (2015) 1493–1521

Alexandrov spaces with maximal number of extremal points

NINA LEBEDEVA

We show that any n–dimensional nonnegatively curved Alexandrov space with the maximal possible number of extremal points is isometric to a quotient space of $\mathbb{R}^n$ by an action of a crystallographic group. We describe all such actions.

53C45, 53C45, 20F55; 51K10

1 Introduction

If the space of directions at a point p in an Alexandrov space has diameter less than or equal to $\frac{\pi}{2}$, this point is called *extremal*. Equivalently, the one-point set $\{p\}$ is an *extremal set* as defined by G Perelman and A Petrunin in [11]. Yet equivalently, p is a *critical point* of every distance function.

It has been proven by G Perelman [10] that every n–dimensional Alexandrov space with nonnegative curvature has at most 2^n extremal points. For completeness, we present this proof in Section 1A. This proof is a slight modification of a proof of the following problem in discrete geometry:

Problem *Assume $x_1, x_2, \dots, x_m$ is a collection of points in the n–dimensional Euclidean space such that $\angle x_i x_j x_k \leqslant \frac{\pi}{2}$ for any distinct i, j and k. Show that $m \leqslant 2^n$ and moreover, if $m = 2^n$ then the x_i form the set of vertices of a right parallelepiped.*

This problem posted by Erdős in [4] was solved by Danzer and Grünbaum in [3].

In this paper we study nonnegatively curved n–dimensional Alexandrov spaces with 2^n extremal points, we call such spaces *n–boxes*.

Classification of n–boxes is a folklore problem. Clearly, right parallelepipeds are boxes. It was suggested [9] that these might be the only examples. Soon it was noticed [5] that the boundary of the 3–dimensional Euclidean tetrahedron whose opposite edges are equal (or equivalently, whose four faces are congruent triangles) is also a 2–box. Later, it was conjectured [12] that all n–boxes have to be isometric to a quotient of a flat torus by an action of a group of isometries which is isomorphic to a product of

Published: 21 May 2015 DOI: 10.2140/gt.2015.19.1493

$\mathbb{Z}_2$–groups. However, it turns out that not all n–boxes can be obtained this way. The first counterexample arises in dimension 3; it is a space $\square'_2$ constructed in this section.

Our main results are Theorems 1.1 and 1.3.

Theorem 1.1 *For any n–box there exists a group Γ and a discrete cocompact isometric action $\Gamma \curvearrowright \mathbb{R}^n$ such that the n–box is isometric to the quotient space $\mathbb{R}^n/\,\Gamma$.*

Theorem 1.3 below describes all possible actions on $\mathbb{R}^n$ which produce n–boxes. Proposition 1.1 implies that it is sufficient to describe the actions $\Gamma \curvearrowright \mathbb{R}^n$ up to affine conjugation. We first need the following:

Definition 1.2 Let $\Gamma \curvearrowright \mathbb{R}^n$ be a group action. We call a point $x \in \mathbb{R}^n$ a *singular point* (for the action $\Gamma \curvearrowright \mathbb{R}^n$) if it is a unique fixed point for some subgroup of Γ.

Proposition 1.1 *For a discrete action $\Gamma \curvearrowright \mathbb{R}^n$ by isometries the quotient space $\mathcal{A} = \mathbb{R}^n/\,\Gamma$ is an Alexandrov space of nonnegative curvature. Moreover, a point $e \in \mathcal{A}$ is extremal if and only if it is an image of a singular point.*

In particular if two such actions are affine conjugate, then the number of extremal points in the corresponding quotient spaces are equal.

Proof The space $\mathcal{A}$ is an Alexandrov space of nonnegative curvature because it is a quotient space by isometric action of the Alexandrov space of nonnegative curvature (see Burago, Gromov and Perelman [2, Corollary of 4.6]). Obviously, the image of any singular point is a vertex in the polyhedron $\mathcal{A}$, and any preimage of any vertex in $\mathcal{A}$ is a singular point. It remains to note that for any discrete isometric action $G \curvearrowright S^{n-1}$, the condition $\operatorname{diam} S^{n-1}/G < \pi$ implies that $\operatorname{diam} S^{n-1}/G \leqslant \pi/2$, hence all vertices in $\mathcal{A}$ are extremal points. $\square$

Recall that the Coxeter group associated with an n–polyhedron is the group generated by reflections in its faces; such a group is defined together with an action on $\mathbb{R}^n$. Let us denote by $\bigoplus^n \curvearrowright \mathbb{R}^n$ the action of the Coxeter group $\bigoplus^n$ of the unit cube.

Theorem 1.3 *Let $\Gamma \curvearrowright \mathbb{R}^n$ be a subaction of $\bigoplus^n \curvearrowright \mathbb{R}^n$ such that any vertex e of the unit cube is an isolated fixed point for some subgroup of Γ. Then $\mathbb{R}^n/\,\Gamma$ is an n–box.*

Moreover, all n–boxes arise from such actions $\Gamma \curvearrowright \mathbb{R}^n$ or their affine conjugate actions.

It is immediate from the theorem that $[\bigoplus^n : \Gamma] = 2^k$ for some $k \in \{0, \dots, n-1\}$. Note that Theorem 1.3 makes possible to list all group actions which produce n–boxes. Let us fix a set S of faces of the n–cube Q with the following property. Any vertex of Q is the only intersection of all faces in S containing this vertex. Then the group action generated by reflections in the elements of S gives an example of an action $\Gamma \curvearrowright \mathbb{R}^n$ described in Theorem 1.3 and any such an action can be obtained in this way. It remains to find all isometric actions that are affine conjugate to the constructed action. This is equivalent to finding all parallel metrics invariant with respect to reflections in the elements of S. Therefore, one can think of any n–box as a space glued from 2^k copies of the cube equipped with a parallel metric g which is invariant under all reflections in the elements of S.

Let us use our construction to classify n–boxes in low dimensions.

- For $n = 1$ there exists only one action, up to affine conjugation group actions, which produces a 1–box. The corresponding quotient space $\mathbb{I} = [0, 1]$ carries a 1–parameter family of metrics.
- For $n = 2$ there exist two spaces (up to a choice of a parallel metric): the square $\square = \mathbb{I} \times \mathbb{I}$ and the double square $\square_2$. The square $\square$ admits a 2–parameter family of metrics; this family gives rise to all possible rectangles. The double square $\square_2$ admits a 3–parameter family of metrics; this family gives rise to surfaces of 3–simplexes whose opposite (nonintersecting) edges are equal. Such simplexes are sometimes called *disphenoids*.
- For $n = 3$ there are five 3–boxes (up to a choice of a parallel metric): the cube $\cube = \mathbb{I} \times \mathbb{I} \times \mathbb{I}$; the double cube $\cube_2$ (obtained by gluing two copies of the cube along their common boundary); doubling of the cube in the 5 faces $\cube'_2$ (obtained by gluing two copies of the cube along 5 faces of their common boundary); the product $\cube''_2 = \mathbb{I} \times \square_2$ and the quotient $\cube_4$ of the standard torus by the central symmetry. The dimensions of the spaces of metrics are respectively 3, 3, 3, 4 and 6.

Structure of the paper In Section 1A, for the sake of completeness, we reproduce the proof that n–dimensional Alexandrov space with nonnegative curvature has at most 2^n extremal points.

The proof of Theorem 1.1 (Sections 2, 3 and 4) is organized into two steps.

In Sections 2 and 3 we show that an n–box $\mathcal{A}$ has to be a polyhedral space (Theorem 3.1). According to Proposition 2.3, it is sufficient to show that each point $p \in \mathcal{A}$ has a conic neighborhood (see Definition 2.4). This is proved in Key lemma 3.2.

In Section 4, we show that $\mathcal{A}$ is a flat orbifold. By Proposition 2.4, it is sufficient to show that an angle around any face of codimension 2 in $\mathcal{A}$ has to be π or 2π. This is proved in Theorem 4.1.

In Sections 5, 6 and 7, we prove Theorem 1.3.

Acknowledgements I want to thank Anton Petrunin for bringing this problem to my attention and for useful discussions, Dmitri Burago for careful reading the paper and valuable notes, and Vasilisa Shramchenko for correcting my English. I am grateful to the referee for helpful remarks and pointing out misprints.

The author thanks the Shapiro visitors program for support and the Mathematics Department of Penn State University where this work was partially done. This paper is also partially supported by RFBR grant 14-01-00062.

1A The upper estimate for the number of extremal points

In this subsection we give the proof (due to Erdős, Danzer, Grünbaum, Perelman) of Theorem 1.4. We also introduce notation which are used further.

Theorem 1.4 *The number of extremal points of an n–dimensional nonnegatively curved Alexandrov space is at most 2^n.*

Denote by A an n–dimensional nonnegatively curved Alexandrov space.

We label the extremal points in A by $e_1, e_2, \dots, e_m$.

For a triangle abc in A we denote by $\tilde{a}\tilde{b}\tilde{c}$ a comparison triangle in $\mathbb{R}^2$ (ie a triangle with the same lengths of sides). We denote by $\tilde{\angle} abc$ the angle at $\tilde{b}$ of the triangle $\tilde{a}\tilde{b}\tilde{c}$.

For a point $a \in A$ we denote by Σ_a the unite tangent space at a.

Proof of Theorem 1.4 First, we have the following lemma:

Lemma 1.5 *Let A be an Alexandrov space with curvature greater than or equal to 0, $x, y \in A$ and $e \in A$ be an extremal point. Assume z is the midpoint of a shortest path $[xy]$ in A. Then*
$$|xz| = |yz| \leqslant |ez|.$$
Moreover, if $|xz| = |ez|$ then
$$\angle xze = \tilde{\angle} xze, \quad \angle ezy = \tilde{\angle} ezy,$$
and there is a unique flat triangle xey in A with a given median $[ez]$ (flat triangle means here the subset of A isometric to the Euclidean triangle).

Proof of Lemma 1.5 Let us assume the contrary, ie $|xz| > |ez|$. Consider a comparison triangle $\tilde{x}\tilde{e}\tilde{z}$ for the chosen triangle xez. Let $\tilde{y}$ be a point on the line extension of $[\tilde{x}\tilde{z}]$ such that $|\tilde{y}\tilde{z}| = |\tilde{z}\tilde{x}|$. Since $|\tilde{x}\tilde{z}| > |\tilde{e}\tilde{z}|$, we have $\angle\tilde{x}\tilde{e}\tilde{y} > \pi/2$. From triangle comparison, we have $|ey| \leqslant |\tilde{e}\tilde{y}|$. It follows that

$$\angle xey \geqslant \tilde{\angle} xey \geqslant \angle\tilde{x}\tilde{e}\tilde{y} > \pi/2.$$

In particular, $\operatorname{diam} \Sigma_e > \pi/2$, which is a contradiction.

In the case of equality $|xz| = |ez|$, by using the same comparison picture as above we have $\angle\tilde{x}\tilde{e}\tilde{y} = \pi/2$. Reasoning by contradiction, assume $\angle xze > \tilde{\angle} xze$. Then from the triangle comparison we obtain $|e_2 x| < |\tilde{e}_2\tilde{x}|$ and hence

$$\angle xey \geqslant \tilde{\angle} xey > \angle\tilde{x}\tilde{e}\tilde{y} = \pi/2,$$

which is a contradiction, proving angle equalities.

Now the existence of a flat triangle follows from Lemma 2.1. □

Now, let us introduce additional notation:

(1) Denote by W_i the set of midpoints of all geodesics $[e_i x]$ with $x \in A$.

(2) Denote by V_i the Voronoi domain of e_i ie

$$V_i = \{x \in A \mid |e_i x| \leqslant |e_j x| \text{ for all } i\}.$$

From Lemma 1.5, we have $W_i \subset V_i$ for all i.

Further, consider a map $\varphi_i \colon W_i \to A$, implicitly defined by the following relation: $x = \varphi(z)$ if z is a midpoint of a geodesic $[e_i x]$.

By triangle comparison, we have

$$|\varphi_i(z)\, \varphi_i(z')| \leqslant 2 \cdot |zz'|$$

for all $z, z' \in W_i$. In particular, the map φ_i is well defined.

Hence

$$\operatorname{vol} V_i \geqslant \operatorname{vol} W_i \geqslant \tfrac{1}{2^n} \cdot \operatorname{vol} A.$$

Since

$$\sum_{i=1}^{m} \operatorname{vol} V_i = \operatorname{vol} A,$$

we get $m \leqslant 2^n$. □

Note that from the proof we immediately get the following:

Corollary 1.1 *Let $\mathcal{A}$, n, m, V_i and W_i be as in the proof of Theorem 1.4. If $m = 2^n$ then $W_i = V_i$ and $\operatorname{vol} V_i = \frac{1}{2^n} \cdot \operatorname{vol} \mathcal{A}$ for all i.*

2 Preliminary statements

In this section we prove a number of technical statements needed in the proof of Theorem 1.1.

2A Flat slices in Alexandrov space

Lemma 2.1 *Let A be an n–dimensional Alexandrov space with nonnegative curvature and $[px_1], [px_2], \dots, [px_k]$ be geodesics in A.*

Assume that

$$\angle x_i p x_j = \widetilde{\angle} x_i p x_j$$

for all i, j and that all directions $\uparrow_{[px_i]}$ lie in a subcone E of $T_p A$ which is isometric to a convex cone in the Euclidean space.

Then all geodesics $[px_i]$ lie in a subset of A which is isometric to a convex polyhedron in the Euclidean space.

Proof Set $\tilde{x}_i = \log_p x_i \in T_p$ and $\tilde{p} = \log_p p$ ($\tilde{p}$ is the vertex of T_p). Clearly

- $\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_k \in E$,
- $|px_i| = |\tilde{p}\tilde{x}_i|$ for each i,
- $|x_i x_j| = |\tilde{x}_i \tilde{x}_j|$ for all i, j.

Since E is Euclidean, by the Kirszbraun theorem, there is a short map $s\colon A \to E$ such that $s(p) = \tilde{p}$ and $s(x_i) = \tilde{x}_i$ for each i.

On the other hand the gradient exponent gexp_p is also a short map. Thus the composition $f = s \circ \mathrm{gexp}_p$ is also short. Clearly f does not move $\tilde{x}_i$ and $\tilde{p}$. It follows that f does not move any point in $Q = \mathrm{Conv}(\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_k)$. Therefore, gexp_p maps Q isometrically into A. □

2B Affine functions

In this section A is an Alexandrov space of nonnegative curvature.

Definition 2.2 Let $\Omega \subset \mathcal{A}$ be an open subset and $\lambda \in \mathbb{R}$. A locally Lipschitz function $f\colon \Omega \to \mathbb{R}$ is called λ–*quasiaffine* if

$$(f \circ \gamma)''(t) \equiv \lambda$$

for any unit speed geodesic γ in Ω. We also call 0–quasiaffine functions *affine functions*.

For an Alexandrov space A, its subset $\Omega \subset A$ and a function $f\colon \Omega \to \mathbb{R}$ we denote by $\bar{A}$ the doubling of A, by $\bar{\Omega} \subset \bar{A}$ the doubling of Ω and by $\bar{f} : \bar{\Omega} \to \mathbb{R}$ the tautological extension of f.

Definition 2.3 We say that a λ–quasiaffine $f\colon \Omega \to \mathbb{R}$ satisfies the boundary condition if $\bar{f} : \bar{\Omega} \to \mathbb{R}$ is λ–quasiaffine.

For $i \in \{1,2\}$, assume $f_i\colon \Omega \to \mathbb{R}$ to be a λ_i–quasiaffine function. Then $f_1 + f_2$ is $(\lambda_1 + \lambda_2)$–quasiaffine. Also for any real constant c, $c \cdot f_1$ is $(c \cdot \lambda_1)$–quasiaffine.

2B.1 Cones and splittings For the proofs of Propositions 2.1 and 2.2 and Lemma 2.5, we refer to Alexander and Bishop [1]. Functions considered in this paper are defined on the whole Alexandrov space, but the proof works also for our local case. It suffices to note that every shortest path between points in $B_{r/4}(p)$ lies inside $B_r(p)$.

Proposition 2.1 *Let $f_1, f_2, \dots, f_k$ be affine functions defined on a ball $B_r(p) \subset A$ such that the functions $1, f_1, f_2, \dots, f_k$ form a linearly independent system. Then the ball $B_{r/4}(p)$ is isometric to an open ball in a product $\mathbb{R}^k \times X$ for some metric space X. Gradients $\nabla f_1, \nabla f_2, \dots, \nabla f_k$ are tangent to $\mathbb{R}^k$ fibers.*

Definition 2.4 A point $p \in A$ admits a *conic neighborhood* if there is an isometry from a neighborhood of p to an open set in a Euclidean cone, which sends p to the vertex of the cone.

Proposition 2.2 *Suppose a ball $B_r(p) \subset A$ admits a 1–affine function f. Then the ball $B_{r/4}$ can be isometrically identified with an open ball in a Euclidean cone. Gradients ∇f are tangent to rays of the cone. If $\nabla_p f = 0$ we have $f = \frac{1}{2}\operatorname{dist}_p^2 + c$ and the ball $B_{r/4}(p)$ is a conic neighborhood of p.*

Lemma 2.5 *Let f be a λ–quasiaffine function defined in some neighborhood $U \ni p$ in A and f satisfies the boundary condition. Then the tangent cone T_pA splits along a line with a direction $\nabla_p f$ and $d_p f = \langle \nabla_p f, \cdot \rangle$.*

2B.2 Dimensions of spaces of affine functions

Definition 2.6 For a set F of affine functions defined in some neighborhood $U \ni p$ in A we denote by $\#_L(F, p)$ the maximal number of functions in F, say $f_1, \dots, f_k$, such that the functions $1, f_1, f_2, \dots, f_k$ form a linear independent system in some small ball $B_r(p) \subset U$. We note that since an affine function on every geodesic is determined by its initial value and its initial derivative then $\#_L(F, p)$ does not depend on r.

For a set F of 1–affine functions defined in some neighborhood $U \ni p$ in A we define a set of affine functions $F^0 = \{\sum \alpha_i f_i \mid f_i \in F, \alpha_i \in \mathbb{R}, \sum \alpha_i = 0\}$ and define $\#_A(F, p)$ to be $\#_L(F^0, p)$.

It follows from Lemma 2.5 that the gradients of functions in F lie in a linear subspace of $T_p A$. Therefore we can define the following numbers: the dimension $\#_L(\nabla F, p)$ of the vector subspace in $T_p\mathcal{A}$, generated by the gradients of functions in F and the dimension $\#_A(\nabla F, p)$ of the affine subspace generated by endpoints of these gradients.

Lemma 2.7 *Let F be a finite set of affine functions defined in a ball $B_r(p)$. Then*

$$\#_L(F, p) = \#_L(\nabla F, p).$$

Let F be a finite set of 1–affine functions defined in a ball $B_r(p)$. Then

$$\#_A(F, p) = \#_A(\nabla F, p).$$

Proof It follows from Lemma 2.5 that the differential of every affine (1–quasiaffine) function is uniquely determined by its gradient and hence every affine (1–quasiaffine) function $f : B_r(p) \to \mathbb{R}$ is determined by $f(p)$ and $\nabla_p f$. Now the proof is straightforward. □

Corollary 2.1 *Let F be a finite set of 1–affine functions defined in a ball $B_r(p)$. Then the ball $B_{r/4}(p)$ can be isometrically identified with an open ball in $\mathbb{R}^{\#_A(\nabla F,p)} \times \mathcal{C}$, where $\mathcal{C}$ is a Euclidean cone. If $\#_L(\nabla F, p) = \#_A(\nabla F, p)$, then the ball $B_{r/4}(p)$ is a conic neighborhood of p. Gradients of functions in F are tangent to products of $\mathbb{R}^{\#_A(\nabla F,p)}$ factors and rays of the cone $\mathcal{C}$.*

Proof Let us consider the set $F^0 = \{\sum \alpha_i f_i \mid f_i \in F, \alpha_i \in \mathbb{R}, \sum \alpha_i = 0\}$. Then F^0 is a set of affine functions and $\#_L(\nabla F_0, p) = \#_A(\nabla F, p)$, hence by Lemma 2.7 and Proposition 2.1 we obtain that the ball $B_{r/4}(p)$ is isometric to an open subset of $\mathbb{R}^{\#_A(\nabla F,p)} \times X$. Applying Proposition 2.2 we obtain that X is isometric to an open subset in Euclidean cone. If $\#_L(\nabla F, p) = \#_A(\nabla F, p)$, there are numbers α_i such that $\sum \alpha_i = 1$ and $\sum \alpha_i \nabla_p f_i = 0$. Then the function $f = \sum \alpha_i f_i$ is 1–affine and $\nabla_p f = 0$. Hence by Proposition 2.2, $f = \frac{1}{2}\mathrm{dist}_p^2 + c$ and the ball $B_{r/4}(p)$ is a conic neighborhood of p. □

2B.3 Moving lemma The next lemma is a technical tool for our proof of Lemma 3.3. The lemma shows how we can move a point in the domain of some collection of 1–affine functions. Corollary 2.1 makes it possible to shift a point in a flat subset so that the distances behave as Euclidean ones.

Moving lemma 2.1 *Let points $x, p_1, \dots, p_k \in A$ and $r > 0$. Suppose that x does not admit a conic neighborhood and the following conditions hold:*

(i) *The functions $f_1 = \frac{1}{2} \cdot \mathrm{dist}^2_{p_1}, \dots, f_k = \frac{1}{2} \cdot \mathrm{dist}^2_{p_k}$ are 1–affine in a neighborhood $B_r(x)$.*

(ii) *We have $|p_1 x| = |p_2 x| = \cdots = |p_k x|$.*

Then there exists a unique unit vector $v \in \mathrm{Span}(\nabla f_1, \dots, \nabla f_k)$ such that $\angle(v, \nabla f_1) = \cdots = \angle(v, \nabla f_k) = \alpha < \pi/2$ and a shortest path $\gamma\colon [0, r/4] \to A$, with $\gamma(0) = x$ and $\gamma'(0) = v$. For every point $y = \gamma(t)$ where $t \in [0, r/4]$ we have the following:

(1) *Some small neighborhoods of x and y are homothetic.*

(2) *We have $f_i(\gamma(t)) = |\nabla f_i| \cos(\alpha) t + \frac{1}{2} t^2$, in particular $|p_1 y| = \cdots = |p_k y| > |p_1 x|$.*

(3) *We have $\angle(\gamma'(t)), \nabla f_1) = \cdots = \angle(\gamma'(t)), \nabla f_k) < \alpha$ and $\#(\nabla\{f_1, \dots, f_k\}, y) = \#(\nabla\{f_1, \dots, f_k\}, x)$.*

(4) *Suppose that for some $p \in A$ the corresponding function $f_p = \frac{1}{2} \cdot \mathrm{dist}^2_p$ is 1–affine in some neighborhood of y, $f_p(y) = f_i(y)$ and $\angle(\nabla_y f_p, \gamma'(t)) \neq \angle(\nabla_y f_i, \gamma'(t))$, then $\#_A(\{f_p, f_1, \dots, f_k\}, y) = \#_A(\{f_1, \dots, f_k\}, x) + 1$.*

Proof We apply Corollary 2.1 and obtain an isometric decomposition of $B_{r/4}(x)$ as a subset of $\mathbb{R}^m \times \mathcal{C}$, where $m = \#_A(\nabla F, p)$ and $\mathcal{C}$ is a Euclidean cone. Vectors $\nabla_x f_1, \dots, \nabla_x f_k$ are tangent to a subset of $\mathbb{R}^m \times \mathbb{R}_+$, namely a product of $\mathbb{R}^m$ and a ray in $\mathcal{C}$. We call this set a flat $(m+1)$–slice.

For any set F of 1–affine functions one of the following equalities holds: $\#_L(\nabla F, p) = \#_A(\nabla F, p) + 1$, or $\#_L(\nabla F, p) = \#_A(\nabla F, p)$. Since x does not have a conic neighborhood by Corollary 2.1 we have

$$\#_L(\nabla\{f_1, \dots, f_k\}, x) = \#_A(\nabla\{f_1, \dots, f_k\}, x) + 1.$$

Hence there exists a unique unit vector $v \in \mathrm{Span}(\nabla f_1, \dots, \nabla f_k)$ such that $\angle(v, \nabla f_1) = \cdots = \angle(v, \nabla f_k) = \alpha < \pi/2$. Then there exists a shortest path $\gamma\colon [0, r/4] \to A$, with $\gamma(0) = x$, $\gamma'(0) = v$ in our flat $(m+1)$–slice. Properties (1)–(3) follow from the Euclidean structure.

We show (4) arguing by contradiction. Suppose the conclusion of (4) does not hold, then

$$\#_A(\nabla\{f_1, \dots, f_k, f_p\}, x) = \#_A(\nabla\{f_1, \dots, f_k\}, x).$$

Hence $\nabla_y f_p$ lies in the affine hull the endpoints of vectors $\nabla_y f_1, \dots, \nabla_y f_k$. We also know that $|\nabla_y f_p| = |\nabla_y f_1| = |\nabla_y f_2| = \cdots = |\nabla_y f_k|$ and $\angle(\gamma'(t)), \nabla_y f_1) = \angle(\gamma'(t)), \nabla_y f_2) = \cdots = \angle(\gamma'(t)), \nabla_y f_k)$. Then $\angle(\nabla_y f_p, \gamma'(t)) = \angle(\nabla_y f_i, \gamma'(t))$; this is a contradiction. □

2B.4 Volume evolution for a gradient flow Given a semiconcave function $f\colon A \to \mathbb{R}$, we denote by $\Phi_f^t\colon A \to A$ the corresponding gradient flow for a time t.

Theorem 2.8 *Let f be a λ–concave function and $\Omega \subset A$ an open set. Then for every $t > 0$, we have*

$$\operatorname{vol}\Phi_f^t(\Omega) \leqslant \exp(n\cdot\lambda\cdot t)\cdot\operatorname{vol}\Omega.$$

Moreover if the equality holds for some $t > 0$, then f is λ–affine in Ω and satisfies the boundary condition.

Proof Here we denote by γ^- and γ^+ the tangent vectors of a curve γ if we go backward or forward correspondingly.

By λ–concavity of f, we mean that

$$d_p f(\gamma^+(a)) + d_q f(\gamma^+(b)) \geqslant -\lambda|pq|$$

for every unit speed shortest path γ in Ω between p and q. To prove that f is λ–affine it suffices to show that this inequality turns into an equality. We consider gradient curves $p(t)$ and $q(t)$ and let l be the distance function $l(t) = |p(t)q(t)|$.

By the first variation formula,

$$l'(t) \leqslant -(\langle\gamma^+(a), \nabla_p f\rangle + \langle\gamma^-(b), \nabla_q f\rangle).$$

By definition of the gradient, for every point x and $v \in T_x\mathcal{A}$ we have $\langle v, \nabla_x f\rangle \geqslant d_x f(w)$. Thus

$$l'(t) \leqslant \lambda|pq|,$$

and applying Proposition 2.5 we obtain the required volume inequality. In the case when this volume inequality becomes an equality, applying Proposition 2.5 we obtain that $l'(t) = \lambda|pq|$. Hence

$$d_p f(\gamma^+(a)) = \langle\gamma^+(a), \nabla_p f\rangle, \quad d_q f(\gamma^-(b)) = \langle\gamma^-(b), \nabla_q f\rangle,$$

and λ–quasiaffinity follows.

To prove the boundary condition it is enough to check the 1–quasiaffinity on every shortest path $\gamma\colon[-h,h] \to \overline{\Omega}$ intersecting $\overline{\partial A}$ only once at a point $x = \gamma(0) \in \overline{\partial A}$. Clearly, it suffices to prove that $d_x\bar{f}(-\gamma^+(0)) = -d_x\bar{f}(\gamma^+(0))$.

By the above, for every $x \in A\cap\Omega$ we have $d_x f = \langle\nabla_x f, \cdot\rangle$ and hence the tangent cone $T_x A$ splits along a line with a direction $\nabla_x f$. Then for every $x \in \partial A\cap\Omega$ both vectors $\nabla_x f, -\nabla_x f$ lie in $\partial T_x A$ and are glued with themselves under doubling. Hence the tangent cone of the doubling $T_x\overline{\Omega}$ also splits along a line with a direction $\nabla_x\bar{f}$. Thus $\angle(-\gamma^+(0), \nabla f) = \pi - \angle(\gamma^+(0), \nabla f)$ and $d_x\bar{f}(\gamma^-(0)) = -d_x\bar{f}(\gamma^+(0))$. □

2C Polyhedral spaces

Definition 2.9 A metric on a simplicial complex $\mathcal{S}$ is called *polyhedral* if each simplex in $\mathcal{S}$ is isometric to a simplex in a Euclidean space.

A metric space $\mathcal{P}$ is said to be *polyhedral space* if it is isometric to a simplicial complex with a polyhedral metric.

For the proof of Proposition 2.4 we need the following definition.

Definition 2.10 A metric on a simplicial complex $\mathcal{S}$ is said to be *spherically polyhedral* if each simplex in $\mathcal{S}$ is isometric to a simplex in the unit sphere in $\mathbb{R}^n$.

A metric space $\mathcal{P}$ is said to be *spherically polyhedral space* if it is isometric to a simplicial complex with a polyhedral metric.

The proof of the following characterization of polyhedral spaces can be found by the author and Petrunin in [7].

Proposition 2.3 *Let X be a compact length space. Assume that each point $x \in X$ has a conic neighborhood. Then X is a polyhedral space.*

2D Orbifolds

It is known that for any orbifold that can be equipped with a metric of constant curvature the universal branched cover is a manifold. The following proposition is colloquially known but we did not find appropriate reference. This proposition characterizes $\mathbb{R}^n$–quotient spaces or equivalently flat orbifolds among all polyhedral spaces.

Proposition 2.4 *A polyhedral space $P = (\mathcal{S}, d)$ is isometric to a quotient space $\mathbb{R}^n/\Gamma$ for a discrete action by isometries $\Gamma \curvearrowright \mathbb{R}^n$ if and only if:*

(1) *The simplicial complex $\mathcal{S}$ of P is an n–dimensional pseudomanifold, ie $\mathcal{S}$ is connected; any simplex in $\mathcal{S}$ is a face of a simplex of dimension n; the link of every simplex of dimension less than or equal to $n-2$ is connected; every simplex of dimension $n-1$ belongs to at most two simplexes of dimension n.*

(2) *For any point x on a face F of codimension 2 in P, the normal cone $N_x F$ of F at x is isometric to a quotient of $\mathbb{R}^2$ by a subgroup of rotations. Namely, $N_x F$ is isometric to a cone over S^1 of length $2 \cdot \pi/k$ or to a cone over an interval of length π/k for some $k \in \mathbb{N}$.*

Proof The "only if" part is obvious. To prove the "if" part it is sufficient to check that P is an orbifold, ie for any point x in P the tangent space is of the form $\mathbb{R}^n/\Gamma$.

It is convenient to prove the same statement as in our Proposition for a spherical polyhedral space in place of polyhedral space by and for S^n/Γ in place of $\mathbb{R}^n/\Gamma$. So let us say that a space is 'good' if it is polyhedral or spherical polyhedral space and possesses (1) and (2).

We prove by inverse induction on dimension that every 'good' space is isometric to $\mathbb{R}^n/\Gamma$ or S^n/Γ. The base $k=2$ follows because of condition (2). Suppose any 'good' space of dimension $k-1$ is isometric to $\mathbb{R}^{k-1}/\Gamma$ or S^{k-1}/Γ. Then for any k–dimensional 'good' space P and any point $x \in P$ the unit tangent space $\Sigma_x P$ is a spherical polyhedral space which inherits properties (1) and (2) and hence is 'good'. Hence by the induction hypothesis $\Sigma_x P = S^k/\Gamma$ and P is an orbifold. This proves the induction step. □

2E Volume preserving + 1–Lipschitz = isometry

The proof of the following fact can be found by Li in [8].

Proposition 2.5 *Let $\mathcal{X}$ and $\mathcal{Y}$ be m–dimensional Alexandrov spaces, $\Omega \subset \mathcal{X} \backslash \partial\mathcal{X}$ an open set and $f\colon \Omega \to Y$ a 1–Lipschitz volume preserving map. Then f is a locally distance preserving; ie for every point $x \in \Omega$ there is a neighborhood $\Omega_x \ni x$ such that the restriction $f|\Omega_x$ is a distance preserving map.*

3 Any n–box is a polyhedral space

In what follows we denote by $\mathcal{A}$ an n–box.

We keep the notation for e_i, V_i, W_i and φ_i for all $i \in \{1, 2, \dots, 2^n\}$ from Section 1A. According to Corollary 1.1, $V_i = W_i$ for all i.

Denote by $\mathfrak{C}_i$ the cutlocus of e_i; ie the set of points $z \in \mathcal{A}\backslash\{e_i\}$ which do not lie in the interior of every shortest path $[e_i x]$.

In this section we prove the following result:

Theorem 3.1 *Every n–box is a polyhedral space.*

Proposition 2.3 implies that it is sufficient to prove the following lemma:

Key lemma 3.2 *Every point $x \in \mathcal{A}$ has a conic neighborhood.*

Proposition 3.1 *Each function $f_i = \frac{1}{2} \cdot \mathrm{dist}^2_{e_i}$ is 1–quasiaffine and satisfies the boundary condition in $\mathcal{A} \setminus \mathfrak{C}_i$.*

Proof It is sufficient to note the restriction $\Phi_{f_i}^{\ln 2}|_{W_i}$ coincides with φ_i. Then from Corollary 1.1 and Theorem 2.8 it follows that f_i is 1–quasiaffine and satisfies the boundary condition. $\square$

Proof of Key lemma 3.2 It follows from Proposition 3.1 that for every point $x \in V_i$ the function f_i is 1–affine in a neighborhood of x. Therefore, we can define for a given point $x \in \mathcal{A}$ an index set $J_x \subset \{1, \dots, n\}$ and a positive integer $\#(x)$ as follows:

$$J_x = \{i \in \{1, \dots, n\} \mid x \in V_i\},$$
$$\#(x) = \#_A\{f_i \mid i \in J_x\}.$$

According to Lemma 2.7 and Corollary 2.1 we have that $\#(x) \leqslant n$ for every $x \in \mathcal{A}$. Moreover if $\#(x) = n$ then x has a flat neighborhood.

The main technical point of the proof of the Key lemma 3.2 is the following:

Lemma 3.3 *Assume a point $x \in \mathcal{A}$ has no conic neighborhood. Then there is a point $x' \in \mathcal{A}$ such that a neighborhood of x' is homothetic to a neighborhood of x and $\#(x') > \#(x)$.*

We prove this lemma in Section 3A.

Now to prove Key lemma 3.2 we argue by contradiction. Let us assume the contrary, ie there is a point $x \in \mathcal{A}$ which has no conic neighborhood.

Applying Lemma 3.3 for $x_0 = x$, we get a point x_1 with a neighborhood homothetic to a neighborhood of x_0 and $\#(x_1) \geqslant \#(x_0) + 1$. In particular, x_1 does not admit a conic neighborhood.

Therefore we can apply Lemma 3.3 $(n+1)$ times to get a point $x_{n+1} \in \mathcal{A}$ such that $\#(x_{n+1}) \geqslant n+1$. We arrive to a contradiction since $\#(z) \leqslant n$ for any $z \in \mathcal{A}$. $\square$

3A Proof of Lemma 3.3

For each i the sets V_i and $\mathfrak{C}_i$ are closed and disjoint. Hence Proposition 3.1 implies that there exists $r_0 > 0$ such that for every i and $x \in V_i$ the function $\frac{1}{2}\operatorname{dist}_{e_i}^2$ is 1–quasiaffine in $B_{4r_0}(x)$.

Now we fix $x \in \mathcal{A}$ and suppose that x does not have a conic neighborhood. We apply Moving lemma 2.1 for x and $\{f_i \mid i \in J_x\}$. We can shift x equidistantly from points e_i for $i \in J_x$ so that the points still lie in all V_i for $i \in J_x$. We continue until we meet a domain V_j for some $j \notin J_x$. Let us formulate the exact statement.

Let $\gamma_0\colon [0,r_0]\to A$ be the shortest path from Moving lemma 2.1. We have a dichotomy:

(1) There exists a minimal value $t_0\in(0,r_0]$, such that $\gamma_0(t_0)\in V_j$ for some $j_0\notin J_x$. Set $y=\gamma_0(t_0)\in V_{j_0}$ and $f_i=\frac{1}{2}\operatorname{dist}^2_{e_i}$, for $i\in J_x\cup\{j_0\}$. We have the angle inequality $\angle(\nabla_y f_{j_0},\gamma'(t))>\angle(\nabla_y f_i,\gamma'(t))$ (indeed, otherwise we would have that $f_{j_0}(t_0-\epsilon)\leqslant f_i(t_0-\epsilon)$ for sufficiently small $\epsilon>0$, this would contradict the choice of t_0). Thus we can apply Moving Moving lemma 2.1(4) with $p:=e_{j_0}$, $f_p=f_{j_0}=\frac{1}{2}\operatorname{dist}^2_{e_{j_0}}$. Then some small neighborhoods of x and y are homothetic and
$$\#(y)\geqslant\#_A(\{f_i\mid i\in J_x\}\cup\{f_{j_0}\},y)=\#_A(\{f_i\mid i\in J_x\},x)+1=\#(x)+1.$$

(2) The shortest path $\gamma_0([0,r_0])$ does not intersect any V_j for $j\notin J_x$.
In this case we apply Moving lemma 2.1 recursively for $x_1=\gamma_0(r_0)$ and so on. After k iteration we have an estimate $f_i(x_k)>(|\nabla_x f_i|\cos(\alpha_0)r_0)\cdot k$, $i\in J_x$ where $\alpha_0=\angle(\nabla_x f_i,\gamma_0'(0))$. The diameter of A is finite, therefore after finitely many steps we arrive at case (1).

4 n–boxes are flat orbifolds

In this section we finish the proof of Theorem 1.1.

Note that according to Theorem 3.1 and Proposition 2.4, it suffices to show the following:

Theorem 4.1 *Let an n–dimensional polyhedral space $\mathcal{A}$ be a box. Then the normal cone for each face of codimension 2 in $\mathcal{A}$ is isometric to one of the following spaces: $\mathbb{R}^2$, $\mathbb{R}_+\times\mathbb{R}$, $\mathbb{R}_+\times\mathbb{R}_+$ or a cone over a circle of length π.*

The proof of this theorem is in Section 4A.

Let $\mathcal{A}$ be an n–box. We keep the same notation as above: e_i denote extremal points of $\mathcal{A}$, V_i the corresponding Voronoi domain, $\mathfrak{C}_i$ the cut locus of e_i; $i\in\{1,2,\dots,2^n\}$. A minimizing geodesic $[e_ie_j]$ between two extremal points is called an *edge*. For any k–dimensional $B\subset\mathcal{A}$ we define $\operatorname{relint}(B)$ to be the subset of all points in B such that some small ball neighborhood in B is isometric to a Euclidean k–dimensional ball.

Let $p\in\mathcal{A}$ be a point which lies on a face of codimension 2; ie $T_p\mathcal{A}=\mathbb{R}^{m-2}\times L$, where L denotes a 2–dimensional cone containing no lines. Take the set of all points in $\mathcal{A}$ with tangent cone isometric to $\mathbb{R}^{m-2}\times L$; we call its closure H *hyperedge* (we name it this way since H has codimension 2 in $\mathcal{A}$).

Here are the simplest properties of hyperedges of n–boxes:

Lemma 4.2 *Let H be a hyperedge. Then:*

(1) *H contains at least one vertex e_i.*

(2) *If a vertex $e_j \notin H$ then $H \subset \mathfrak{C}_j$.*

Proof Let us keep notation L for a 2–dimensional cone from the definition of a hyperedge.

(1) Indeed, take a point $x \in H$ with a tangent cone isometric to $\mathbb{R}^{m-2} \times L$. Then $x \in V_i$ for some $i \in \{1, \dots, 2^n\}$. Then x is a midpoint of some shortest path $[e_i y]$. Hence for all points $z \in (e_i y)$ the tangent cone $T_z\mathcal{A}$ is isometric to $\mathbb{R}^{m-2} \times L$. Thus $e_i \in H$.

(2) Let us assume the contrary. Then there exist a vertex $e_j \notin H$ and a point $x \notin \mathfrak{C}_j$ with a tangent cone isometric to $\mathbb{R}^{m-2} \times L$. Hence the shortest path $[e_j x]$ can be extended to some shortest path $[e_j y]$, then for all points $z \in (e_j y)$ the tangent cone $T_z\mathcal{A}$ is isometric to $\mathbb{R}^{m-2} \times L$. Then $e_j \in H$, this is a contradiction. □

4A Proof of Theorem 4.1

Definition 4.3 Let $\mathcal{A}$ be a box and $H \subset \mathcal{A}$ a hyperedge. We say that a vertex $e_i \in \mathcal{A}$ *pushes H in a vertex $e_j \in H$* if there exists a flat $(n-2)$–dimensional simplex $\Delta \subset H$, such that $e_j \in \Delta \subset \mathfrak{C}_i$.

Definition 4.4 Let $\mathcal{A}$ be a box and $H \subset \mathcal{A}$ a hyperedge. We say that H *separates a vertex $e_i \in \mathcal{A}$ from a vertex $e_j \in H$* if there exists a flat $(n-2)$–dimensional simplex $\Delta \subset H$, such that $e_j \in \Delta \subset \mathfrak{C}_i$ and

$$\operatorname{relint}(\varphi_i^{-1}(\triangle)) \cap V_k = \varnothing \quad \text{for every } k \neq i, j.$$

To prove Theorem 4.1, we need the following lemma:

Lemma 4.5 *Let $\mathcal{A}$ be a box and H a hyperedge. Then there are vertices $e_i \in \mathcal{A}$ and $e_j \in H$ such that H separates e_i from e_j.*

The proof of this lemma is in Section 4B. Now let us show how Theorem 4.1 follows from Lemma 4.5.

Proof of Theorem 4.1 Let us introduce some notation:

- K_i denotes the completion of $\mathcal{A} \setminus \mathfrak{C}_i$ equipped with an intrinsic metric.
- Clearly K_i is isometric to $2{\cdot}V_i$. Denote by $\psi_i\colon g_i^{-1}(V_i) \to K_i$ the homothety centered at e_i and with coefficient 2.
- $g_i\colon K_i \to \mathcal{A}$ is the corresponding gluing map (which is piecewise linear).

Note that in these notation we have $g_i \circ \psi_i \circ g_i^{-1} = \varphi_i$.

To prove Theorem 4.1 we take a hyperedge H containing a given $(n-2)$–dimensional face, apply Lemma 4.5 and obtain that H separates some vertices $e_i \in \mathcal{A}$, $e_j \in H$. Let Δ be from Definition 4.4. It is sufficient now to prove that for some point $x \in \operatorname{relint}(\Delta)$ the normal cone to H at this point is one of the 4 cones described in the theorem. We can assume that Δ is sufficiently small so that $g_i^{-1}(\operatorname{relint}(\Delta))$ are disjoint isometric copies of $\operatorname{relint}(\Delta)$. By $\Delta_1, \dots, \Delta_l$ we denote closures of its preimages and by $e_j^1 \in \Delta_1, \dots, e_j^l \in \Delta_l$ the corresponding preimages of e_j.

The next lemma describes a possible structure of the tangent space of a point in the preimage $g_i^{-1}(\operatorname{int}(\Delta))$.

Lemma 4.6 *Using our notation, let a point $x \in g_i^{-1}(\operatorname{int}(\Delta)) \subset \partial K_i$. Then there are two possibilities:*

(1) *If $\psi_i^{-1}(x) \notin \partial K_i$ then $T_x K_i = \mathbb{R}^{n-1} \times \mathbb{R}_+$.*

(2) *If $\psi_i^{-1}(x) \in \partial K_i$ then $T_x K_i = \mathbb{R}^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$.*

Proof We can assume that $x \in \Delta_1$. For $y = \psi_i^{-1}(x)$ we know, that $T_y K_i$ contains an isometric copy of $\mathbb{R}^{n-2} \times \mathbb{R}$. Hence $T_y K_i = \mathbb{R}^n$ or $T_y K_i = \mathbb{R}^{n-1} \times \mathbb{R}_+$. We know also that $\psi_i^{-1}(\Delta_1)$ is a flat $(n-2)$ simplex equidistant from e_i and e_j^1 with midpoint $\psi_i^{-1}(e_j^1)$ as a vertex. In a small neighborhood U of y we have that

$$g_i^{-1}(V_i) \cap U = \{z \in U \mid |ze_i| \leqslant |ze_j^1|\}.$$

It follows that $T_y(g_i^{-1}(V_i))$ can be presented as one part of perpendicular bisection of $T_y K_i$ with respect to $e_i e_j^1$. Thus we have

- $T_y(g_i^{-1}(V_i)) = \mathbb{R}^{n-1} \times \mathbb{R}_+$ if $T_y K_i = \mathbb{R}^n$;
- $T_y(g_i^{-1}(V_i)) = \mathbb{R}^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$ if $T_y K_i = \mathbb{R}^{n-1} \times \mathbb{R}_+$.

It remains to note that $T_y(g_i^{-1}(V_i))$ is isometric to $T_x K_i$. □

Lemma 4.7 *For a point $x \in g^{-1}(\mathfrak{C}_i) \subset \partial K_i$ the condition $\psi_i^{-1}(x) \in \partial K_i$ implies $g_i(x) \in \partial \mathcal{A}$.*

Proof This follows since our space is polyhedral and $g_i(\partial K_i) \setminus \mathfrak{C}_i \subset \partial \mathcal{A}$. □

We can consider the space K_i as the result of a cutting off the polyhedral space $\mathcal{A}$ along $(n-1)$–polyhedral subspace $\mathfrak{C}_i$. The map g_i glues $\mathcal{A}$ back from K_i. Then if the point $x \in \mathfrak{C}_i$ has l preimages $x_1, \dots, x_l \in K_i$ under g_i, its tangent space T_x can be glued out from the tangent spaces $T_{x_1}, \dots, T_{x_l}$. We write this as

$$T_x = T_{x_1} \sqcup \dots \sqcup T_{x_l},$$

and the gluing maps are $d_{x_1} g_i \colon T_{x_1} \to T_x, \dots, d_{x_l} g_i \colon T_{x_l} \to T_x$.

Fix x and let $g_i^{-1}(x) = \{x_1, \dots, x_l\} \subset K_i$. Then there are two possibilities:

(1) If $\Delta \subset \partial \mathcal{A}$, then:
  (a) For some $1 \leqslant k_0 \leqslant l$ the point $\psi^{-1}(x_{k_0}) \notin \partial K_i$. Then by Lemma 4.6 $T_{x_k} = \mathbb{R}^{n-1} \times \mathbb{R}_+$, $l = 1$ and $T_x = \mathbb{R}^{n-1} \times \mathbb{R}_+$.
  (b) For all $k \in \{1, \dots, l\}$ points $\psi^{-1}(x_k) \in \partial K_i$. Then $T_{x_k} = \mathbb{R}^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$. This is only possible if $l = 1$ and $T_{x_k} = \mathbb{R}^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$ or $l = 2$ and $T_x \mathcal{A} = \mathbb{R}^{n-1} \times \mathbb{R}_+$.

(2) If $\operatorname{int}(\Delta) \cap \partial \mathcal{A} = \varnothing$, then in this case Lemma 4.7 implies that for all $k \in \{1, \dots, l\}$, $\psi^{-1}(x_k) \notin \partial K_i$ and by Lemma 4.6 $T_{x_k} = \mathbb{R}^{n-1} \times \mathbb{R}_+$. This only possible if $l = 1$ and $T_x \mathcal{A} = \mathbb{R}^{m-2} \times L$, where L is a cone over S^1 of length π or $l = 2$ and $T_x \mathcal{A} = \mathbb{R}^n$.

This completes the proof of Theorem 4.1. □

4B Proof of Lemma 4.5

Let us note that if there is a vertex, say $e_1 \notin H$, then the proof would be much simpler. It would be sufficient to take the shortest edge between vertices in H and outside H. So the difficulty is if there is no such a vertex.

To find vertices separated by H we start with Lemma 4.8 to find a pair of vertices e_i, e_j such that e_i presses down H at e_j. Then we can decrease the distance $|e_i e_j|$ between points with the same property using Lemma 4.9 until we find a pair of vertices, such that H separates one from the other.

Lemma 4.8 *Let $\mathcal{A}$ be a box. Then for any hyperedge $H \subset \mathcal{A}$ there are vertices $e_i \in \mathcal{A}$ and $e_j \in H$ such that e_i pushes H in e_j.*

Proof Suppose there exists at least one vertex $e_i \notin H$; then e_i pushes H in every vertex $e_j \in H$. Otherwise consider any flat n–simplex with vertices in $\{e_1, \dots, e_{2^n}\}$ say $\triangle_{e_{i_0},\dots,e_{i_n}}$. The existence of such a simplex can be proved by using the same construction as in the proof of Section 3A: moving out from vertices we can find a point $x \in \mathcal{A}$ with $\#(x) = n$ and from Lemma 2.1 it follows that corresponding $n+1$ vertices form flat n simplex. Since the codimension of H is 2, one of the vertices e_{i_1}, $e_{i_2}, \dots, e_{i_n}$ has to push H in e_{i_0}. □

Lemma 4.9 *Let e_i and e_j be two vertices and H a hyperedge in an n–box $\mathcal{A}$ and $e_j \in H$. Assume e_i pushes H in e_j but H does not separate e_i from e_j. Then there is $k \neq i, j$ such that*

$$\max\{|e_k e_i|, |e_k e_j|\} < |e_i e_j|$$

and one of the following holds:

- *e_k pushes H in e_j.*
- *$e_k \in H$ and e_i pushes H in e_k.*

To prove Lemma 4.9 we need the following:

Sublemma 4.10 *For any vertices e_i, e_k and a point $x \in V_i \cap V_k$ there is a shortest path $[\varphi_i(x) e_k]$ inside $\mathfrak{C}_i$.*

Proof By Lemma 1.5 there is a flat triangle $e_i e_k \varphi_i(x)$ with median $[x e_k]$ and right angle at e_k. If some point of the edge $[\varphi_i(x) e_k]$ of this triangle does not lie in $\mathfrak{C}_i$ then we would have $\operatorname{diam} \Sigma_{e_k} > \pi/2$, contradiction. □

Proof of Lemma 4.9 In conditions of our lemma there exists an $(n-2)$–simplex Δ with a vertex $m \in \varphi_i^{-1}(e_j)$ such that $\varphi_i(\Delta) \subset H$ and $\Delta \subset V_i \cap V_k$ for some $k \neq i, j$. Then by Lemma 1.5 there is a flat triangle $e_i e_j e_k$ with median $[e_k m]$ and right angle in e_k. Then

$$\max\{|e_k e_i|, |e_k e_j|\} < |e_i e_j|.$$

Now if e_k pushes H in e_j the proof is completed. Suppose contrary. We can assume that $\operatorname{relint}(\varphi_i(\Delta)) \subset \mathcal{A} \setminus \mathfrak{C}_k$. By Sublemma 4.10 for every point $y \in \varphi_i(\Delta)$ there is a shortest path $[y e_k]$ inside $\mathfrak{C}_i$, if in addition $y \notin \mathfrak{C}_k$ then $[y e_k] \subset H$. Then points of all such shortest paths for $y \in \operatorname{relint}(\varphi_i(\Delta))$ form an $(n-2)$–dimensional subset of H. In particular $e_k \in H$ and e_j presses down H at e_k. □

5 The structure of the action of the orbifold group of an n–box

Now we are in position to prove Theorem 1.3.

In what follows we assume that $\Gamma \curvearrowright \mathbb{R}^n$ is a discrete cocompact action by isometries and $\Pi : \mathbb{R}^n \to \mathbb{R}^n/\,\Gamma$ denotes the projection. Let us denote by $\mathcal{E}$ the set of singular points for $\Gamma \curvearrowright \mathbb{R}^n$.

It follows from Proposition 1.1 that $\mathbb{R}^n/\,\Gamma$ is a box if and only if the number of Γ–orbits in $\mathcal{E}$ is 2^n. This implies in particular the first part of Theorem 1.3. We reduce the second part of Theorem 1.3 to three propositions below in this section. To formulate the propositions we need some definitions and notation.

For any $x \in \mathcal{E}$ we denote by V_x its Voronoy cell with respect to $\mathcal{E}$, ie

$$V_x = \{z \in \mathbb{R}^n \mid |z-x| \leqslant |z-y| \text{ for every } y \in \mathcal{E}\}.$$

Given $x \in \mathbb{R}^n$, we denote by $\Gamma_x^\# \subset O(n)$ the action of the stabilizer Γ_x on the vector space $\mathbb{R}^n$.

Definition 5.1 We say that an action $\Gamma \curvearrowright \mathbb{R}^n$ has a *reflection property* if $\mathcal{E} \neq \varnothing$ and for any adjacent $x, y \in \mathcal{E}$ (ie $\dim(V_x \cap V_y) = n-1$) the stabilizer Γ_x can only fix or reflect the point y: $\Gamma_x^\#(\{\overrightarrow{xy}\}) = \{\overrightarrow{xy}, -\overrightarrow{xy}\}$.

We say that a discrete subset $E \subset \mathbb{R}^n$ is a *lattice* if there is a finite set of *generating* vectors $\vec{a}_1, \dots, \vec{a}_l$ such that for any point $x \in E$ we have

$$E = \{x + k_1 \cdot \vec{a}_1 + k_2 \cdot \vec{a}_2 + \dots + k_l \cdot \vec{a}_l \mid k_1, k_2, \dots, k_l \in \mathbb{Z}\}.$$

If the dimension of the affine hull of E equals k we say that E is a *k–lattice*.

We say that a group $\Gamma \curvearrowright \mathbb{R}^n$ *reflects generating vectors* $\vec{a}_1, \dots, \vec{a}_l$ if for any $x \in E$ and $i = 1, \dots, l$ we have that $\Gamma_x^\#(\{\vec{a}_i\}) = \{\vec{a}_i, -\vec{a}_i\}$.

The second part of Theorem 1.3 follows from Theorem 1.1 and the next three propositions.

Proposition 5.1 *Assume that the number of Γ–orbits in $\mathcal{E}$ be 2^n. Then $\Gamma \curvearrowright \mathbb{R}^n$ has a reflection property.*

We prove this in Section 6.

Proposition 5.2 *Let an action $\Gamma \curvearrowright \mathbb{R}^n$ have a reflection property. Then $\mathcal{E}$ is an n–lattice. Moreover, there exist n generating vectors for $\mathcal{E}$ and Γ reflects these generating vectors.*

The proof is in Section 7.

Proposition 5.3 *Let the number of Γ–orbits in $\mathcal{E}$ be 2^n. Suppose that $\mathcal{E}$ is a lattice and there exist n generating linearly independent vectors $\vec{a}_1, \dots, \vec{a}_n$ for $\mathcal{E}$ such that Γ reflects this generating vectors.*

Then the action $\Gamma \curvearrowright \mathbb{R}^n$ is affine conjugate to a subaction of the Coxeter group associated to the unit cube.

Proof Let us denote by Γ_* a subgroup of Γ generated by stabilizers of all singular points. Let us denote by $2 \cdot \mathcal{E}$ the set of vectors $\left\{\sum_{i=1}^n 2\alpha_i \vec{a}_i \mid k_1, \dots, k_n \in \mathbb{Z}\right\}$. Since Γ_* reflects the generating set we have that for any $y \in \mathcal{E}$, $\Gamma_*(y) \supset y + 2 \cdot \mathcal{E}$. The set $\mathcal{E}$ is invariant under Γ and the number of orbits equals 2^n. Hence we have that $\Gamma_*(y) = y + 2 \cdot \mathcal{E}$ for any $y \in \mathcal{E}$. By the same arguments we obtain that $\Gamma(y) = y + 2 \cdot \mathcal{E}$ for any $y \in \mathcal{E}$. Therefore $\Gamma(y) = \Gamma_*(y)$, hence $\Gamma = \Gamma_*$.

We fix coordinates in $\mathbb{R}^n$: a point $O \in \mathbb{R}^n$ and an orthonormal basis $e_1, \dots, e_n$. Let $\bigoplus^n \curvearrowright \mathbb{R}^n$ be the corresponding action of the Coxeter group of the unit cube. We define an affine map on the basis: $F(x_0) = O$, for some point $x_0 \in \mathcal{E}$ and $F(a_i) = e_i$. We define an action $G \curvearrowright \mathbb{R}^n$ by $G = F \circ \Gamma \circ F^{-1}$, this action is affine conjugate to the action $\Gamma \curvearrowright \mathbb{R}^n$. We have that the integer lattice $\mathbb{Z}^n$ is the set of singular points of the action $G \curvearrowright \mathbb{R}^n$, the group G is generated by stabilizers of points of $\mathbb{Z}^n$ and G reflects the generating set $e_1, \dots, e_n$ of the lattice $\mathbb{Z}^n$. It follows that $G \leqslant \bigoplus^n$. □

6 Properties of a group action for an n–box

In this section we prove Proposition 5.1. The quotient space $\mathbb{R}^n / \Gamma$ is an n–box, we denote it by $\mathcal{A}$ and keep all notation for n–boxes we used before.

We precede the proof by three lemmas. The first two are technical facts about Voronoy domains, and Lemma 6.3 is the main geometric observation for our proof of Proposition 5.1:

Lemma 6.1 *Let $M = \mathbb{R}^k$ or $M = S^k$ and $G \curvearrowright M$ be a discrete cocompact action by isometries. Let us denote the quotient space M/G by M'. Let p be the projection $M \to M'$. We fix some finite collection of points $s_1, \dots, s_l \in M'$ and consider Voronoy decompositions of M' and of M with respect to the sets $\{s_1, \dots, s_l\}$ and $p^{-1}(\{s_1, \dots, s_l\})$ respectively. Then for every $i \in \{1, \dots, l\}$ and every point $s \in p^{-1}(s_i) \subset M$ the corresponding Voronoy domain $V_s \subset M$ can be characterized by the following property: a point $y \in V_s$ if and only if $(y \in p^{-1}(V_{s_i}))$ and $(|sy| = |s_i p(y)|)$.*

Proof The proof is straightforward and uses just two properties of the projection map: the map p does not increase distances and for any $x, y \in M'$ there exist $x^0 \in p^{-1}(x)$, $y^0 \in p^{-1}(y)$ such that $|x^0 y^0| = |xy|$. $\square$

Lemma 6.2 *For any two points $e_i^0 \in \Pi^{-1}(e_i)$ and $e_j^0 \in \Pi^{-1}(e_j)$ such that $V_{e_i^0} \cap V_{e_j^0} \neq \varnothing$ the projection Π is a distance preserving map, that is*

$$|\Pi(e_i^0)\Pi(e_j^0)| = |e_i^0 e_j^0| = |e_i e_j|.$$

Proof Let a point x lie in $V_{e_i^0} \cap V_{e_j^0}$. By Lemma 6.1 we have that $\Pi(x) \in V_i \cap V_j$. Then by Lemma 1.5 there exists a unique flat totally geodesic triangle $e_i e_j \Pi(x)$ and the triangle $e_i^0 e_j^0 x$ is its isometric lifting. $\square$

Lemma 6.3 *Let S^k be a k–dimensional sphere, G a discrete subgroup of isometries of S^k, $B = S^k/G$ with a projection $p\colon S^k \to B$ and* diam $B \leqslant \pi/2$. *Suppose that for a point $v \in S^k$ the following holds: there exists a $(k-1)$–dimensional subset $F \subset B$ such that $|p(v)x| = \pi/2$ for all $x \in F$ (further we refer to this as "$\pi/2$–property"). Then the orbit of v contains exactly two points: $G(v) = \{v, v^-\}$, where $v^- \in S^k$ is the diametrical point for v.*

Proof For a point $w \in S^k$, we denote by $S_w^\perp$ the equator $\{y \in S^k \mid |yw| = \pi/2\}$. We consider the Voronoy decomposition of S^k with respect to the set $p^{-1}(p(v))$. Lemma 6.1 implies that $p^{-1}(F) \cap V_w \subset S_w^\perp$ for any $w \in p^{-1}(p(v))$. Since for any $w \in p^{-1}(p(v))$ $\operatorname{diam}(V_w) \leqslant \pi/2$ and $\dim(p^{-1}(F)) = k-1$ there are exactly two Voronoy domains, which are semispheres and the set $p^{-1}(p(v)) = G(v)$ consists of two diametric points. $\square$

Proof of Proposition 5.1 Let us fix some adjacent points $e_i^0 \in \Pi^{-1}(e_i)$, $e_j^0 \in \Pi^{-1}(e_j)$. Let m^0 be a midpoint between them. Let $m = \Pi(m^0)$. It follows from Lemma 6.2 that $|e_i^0 e_j^0| = |e_i e_j|$ and $|e_i m| = |e_j m|$, hence by Corollary 1.1 we have that the midpoint m lies in $V_i \cap V_j$. Then applying Lemma 6.1 we obtain that $m^0 \in V_{e_i^0} \cap V_{e_j^0}$. It follows from the definition of adjacent vertices and convexity of Voronoy domains that there exists a flat $(n-1)$–triangle Δ^0 with a vertex m^0 such that $\Delta^0 \subset V_{e_i^0} \cap V_{e_j^0}$. Hence there exists a flat $(n-1)$–triangle Δ with a vertex $m = \Pi(m^0)$ such that $\Delta \subset V_{e_i} \cap V_{e_j}$ (indeed we can take a sufficiently small triangle in $\Pi(\Delta^0)$).

We consider the action of the stabilizer on the unit sphere $\Gamma_{e_j^0} \curvearrowright \Sigma_{e_j^0}\mathbb{R}^n$ and the corresponding quotient space $\Sigma_{e_j^0}\mathbb{R}^n / \Gamma_{e_j^0} = \Sigma_{e_j}\mathcal{A}$, and denote the corresponding projection by $p\colon \Sigma_{e_j^0}\mathbb{R}^n \to \Sigma_{e_j}\mathcal{A}$. To prove the reflection property we apply Lemma 6.3 to this action, the vector

$$v = \overrightarrow{e_j^0 e_i^0}/|e_j^0 e_i^0|$$

(as a point in $\Sigma_{e_j^0}\mathbb{R}^n$) and the set $F = d_m\varphi_i(\Sigma_m\Delta)$. It is not difficult to see that the map $d_m\varphi\colon T_mV_i \to T_{e_j}\mathcal{A}$ does not decrease dimensions (indeed, this map is a gluing map), hence $\dim F = n-2$. To verify conditions of Lemma 6.3 it remains to prove $\pi/2$–property for the vector v and the set F. For any vector $w \in \Sigma_m\Delta$ we have that $w \perp e_ie_j$ because $\Delta \subset V_{e_i} \cap V_{e_j}$. For this vector we can construct a flat totally geodesic triangle e_ie_jx in $\mathcal{A}$ (as in Lemma 1.5) such that $\angle e_ie_jx = \pi/2$ and the vector w is a tangent vector to this triangle at m. By the definition of the map φ_i we have $d\varphi_i(w) = \vec{e}_jx/|e_jx|$, let us note that $p(v) = \vec{e}_jm/|e_jm|$. We obtain that $d\varphi_i(w) \perp p(v)$, then conclusion of Lemma 6.3 implies that the stabilizer $\Gamma_{e_j^0}$ can only fix or reflect e_i^0. □

7 Reflection property gives a lattice of singular points

In this section we prove Proposition 5.2.

For a point $x \in \mathcal{E}$ we denote the set of all adjacent vertices by

$$\mathfrak{S}(x) = \{y \in \mathcal{E} \mid \dim(V_x \cap V_y) = n-1\}.$$

The main technical point of the proof is the following:

Lemma 7.1 *Let an action $\Gamma \curvearrowright \mathbb{R}^n$ have the reflection property. Let $x \in \mathcal{E}$, $y, z \in \mathfrak{S}(x)$, and let us denote $z^* = y + \overrightarrow{xz}$. Then*

$$z^* \in \mathcal{E}.$$

The proof of this lemma is in Section 7A. In Section 7B we finish the proof of Proposition 5.2.

7A Proof of Lemma 7.1

We consider two cases. First, every element of the stabilizer Γ_x may reflect or fix points y and z only simultaneously. The proof of this case is in Section 7A.1 (see Lemma 7.3(2)). The other possibility is if there exists an element in Γ_x that reflects point y and fixes point z, the proof for this case is in Section 7A.2.

For any two points $x, y \in \mathbb{R}^k$ we will denote by $\mathfrak{c}_x\colon \mathbb{R}^k \to \mathbb{R}^k$ the central symmetry with the center x and by $\mathfrak{c}_{xy}\colon \mathbb{R}^k \to \mathbb{R}^k$ the symmetry with an axis xy. For points $x_1, \dots, x_l$ we denote by $\langle x_1, \dots, x_l\rangle$ the affine hull of these points. Then in the first case $\Gamma_x|_{\langle x,y,z\rangle} = \{\mathrm{id}\,|_{\langle x,y,z\rangle}, \mathfrak{c}_x|_{\langle x,y,z\rangle}\}$ and in the second case $\Gamma_x|_{\langle x,y,z\rangle} = \{\mathrm{id}\,|_{\langle x,y,z\rangle}, \mathfrak{c}_x|_{\langle x,y,z\rangle}, \mathfrak{c}_{xy}|_{\langle x,y,z\rangle}, \mathfrak{c}_{xz}|_{\langle x,y,z\rangle}\}$.

7A.1 The order of $\Gamma_x|_{\langle x,y,z\rangle}$ equals 2 For any two points $x, y \in \mathbb{R}^n$ we denote the stabilizer by $\Gamma_{x,y} = \Gamma_x \cap \Gamma_y$ and by $\Gamma^\#_{x,y} \subset O(n)$ the action of this stabilizer on the associate vector space $\mathbb{R}^n$. For any tree points $x, y, z \in \mathbb{R}^n$ we denote the stabilizer of these points by $\Gamma_{x,y,z} = \Gamma_x \cap \Gamma_y \cap \Gamma_z$.

First we prove one auxiliary statement:

Lemma 7.2 *For a point $x \in \mathfrak{S}(y)$ and every vector $v \in \mathbb{R}^n$, if $\Gamma^\#_{x,y}(v) = \{v\}$ then $\Gamma^\#_y(v) = \{v, -v\}$.*

Proof We can find points $x_1, \dots, x_{n-1} \in \mathfrak{S}(y)$, such that vectors $\overrightarrow{yx}, \overrightarrow{yx_1}, \dots, \overrightarrow{yx_{n-1}}$ are linearly independent. Let $P = \{v \in \mathbb{R}^n \mid \Gamma^\#_{x,y}(v) = v\}$. Reordering if necessary we can assume that $x = x_0, x_1, \dots, x_k \in P$ and $x_{k+1}, \dots, x_{n-1} \notin P$. We know that $\Gamma^\#_y(\overrightarrow{yx_i}) = \{\overrightarrow{yx_i}, -\overrightarrow{yx_i}\}$ for every $i = 0, \dots, n-1$. Then considering the group action for the decomposition in our basis $v = v^0\overrightarrow{yx_0} + \cdots + v^{n-1}\overrightarrow{yx_{n-1}}$ we obtain that for every $v \in P$ coordinates $v^{k+1} = \cdots = v^{n-1} = 0$, ie $P = \langle x_0, \dots, x_k\rangle$. Then for every $v \in P$ we have $\Gamma^\#_y(v) = \{v, -v\}$. □

Lemma 7.3 *In conditions of Lemma 7.1 suppose additionally that*

$$\Gamma_x|_{\langle x,y,z\rangle} = \{\mathrm{id}|_{\langle x,y,z\rangle}, \mathfrak{c}_y|_{\langle x,y,z\rangle}\}.$$

Then

(1) $\Gamma_y|_{\langle x,y,z\rangle} = \{\mathrm{id}|_{\langle x,y,z\rangle}, \mathfrak{c}_y|_{\langle x,y,z\rangle}\}$,

(2) $z^* \in \mathcal{E}$.

Proof (1) Conditions of the lemma imply that

$$\Gamma^\#_{x,y}(\{\overrightarrow{xz}\}) = \{\overrightarrow{xz}\} \quad \text{and} \quad \overrightarrow{xz} = \overrightarrow{yz^*},$$

then by Lemma 7.2 $\Gamma_y(\overrightarrow{yz^*}) = \{\overrightarrow{yz^*}, -\overrightarrow{yz^*}\}$. Then for every $\gamma \in \Gamma_y$ we have

$$\gamma(\overrightarrow{xz}) = \overrightarrow{xz} \quad \text{and} \quad \gamma(\overrightarrow{yz^*}) = \overrightarrow{yz^*}$$

or

$$\gamma(\overrightarrow{xz}) = -\overrightarrow{xz} \quad \text{and} \quad \gamma(\overrightarrow{yz^*}) = -\overrightarrow{yz^*}.$$

Then (1) follows.

(2) For every $v \neq 0$ we want to find $\gamma \in \Gamma_{z^*}$ such that $\gamma(v) \neq v$. We consider two possibilities.

First if $\Gamma_{x,y,z}(v) \neq \{v\}$ we can find the required element $\gamma \in \Gamma_{x,y,z} \subset \Gamma_{z^*}$.

If $\Gamma_{x,y,z}(v)=\{v\}$ then for arbitrary three elements $\gamma_x \in \Gamma_x \setminus \Gamma_{x,y,z}$, $\gamma_y \in \Gamma_y \setminus \Gamma_{x,y,z}$, $\gamma_z \in \Gamma_z \setminus \Gamma_{x,y,z}$ by Lemma 7.2 we will have $\gamma_x(v)=-v, \gamma_z(v)=-v, \gamma_z(v)=-v$. Then $\gamma_x \circ \gamma_y \circ \gamma_z(v)=-v$ and $\gamma_x \circ \gamma_y \circ \gamma_z(z^*)=z^*$. □

7A.2 The order of $\Gamma_x|_{\langle x,y,z\rangle}$ equals 4 In this subsection we are in conditions of Lemma 7.1 and assume that the order of the group $\Gamma_x|_{\langle x,y,z\rangle}$ is 4.

Definition 7.4 Let vectors $v_1, v_2 \in \mathbb{R}^n$ and W be a subset of vectors in $\mathbb{R}^n$. The *angle chain between v_1 and v_2 through the set W* is the ordered set of vectors $w_1, \dots, w_k \in W$ with the following property:

$$\angle(v_1,w_1)\neq\frac{\pi}{2}, \quad \angle(w_1,w_2)\neq\frac{\pi}{2}, \quad \dots, \quad \angle(w_{k-1},w_k)\neq\frac{\pi}{2}, \quad \angle(w_k,v_2)\neq\frac{\pi}{2}.$$

We say that two $(n-1)$–faces of a convex polyhedron in $\mathbb{R}^n$ are *adjacent* if they have a common $(n-2)$–face.

First we prove the following lemmas.

Lemma 7.5 *Suppose we are in the conditions of Lemma 7.1 and the order of the group $\Gamma_x|_{\langle x,y,z\rangle}$ is 4. Then there is no angle chain between $\overrightarrow{xy}$ and $\overrightarrow{xz}$ through the set of vectors $\{\overrightarrow{xt} \mid t \in \mathfrak{S}(x)\}$ and $\overrightarrow{xy} \perp \overrightarrow{xz}$.*

Proof It is sufficient to note that if there would be such an angle chain, then the stabilizer Γ_x could fix or reflect vectors $\overrightarrow{xy}$ and $\overrightarrow{xz}$ only simultaneously. This contradicts to the fact that the order of the group $\Gamma_x|_{\langle x,y,z\rangle}$ is 4. □

We introduce the following notation for half spaces and hyperplanes determined by a vector $v \in \mathbb{R}^n$ or by an origin $x \in \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$:

$$H_v^- = \{w \in \mathbb{R}^n | \langle w,v\rangle \leqslant |v|^2\}, \quad H_{x,v}^- = \{y \in \mathbb{R}^n | \langle \overrightarrow{xy},v\rangle \leqslant |v|^2\},$$
$$H_v^0 = \{w \in \mathbb{R}^n | \langle w,v\rangle = |v|^2\}, \quad H_{x.v}^0 = \{y \in \mathbb{R}^n | \langle \overrightarrow{xy},v\rangle = |v|^2\}.$$

We need the following observation in geometry of convex polyhedra.

Lemma 7.6 *Let an n–dimensional convex polyhedron $F \subset \mathbb{R}^n$ be represented as an intersection of half spaces $F=\cap H_{v_i}^-$ for the set of vectors $V=\{v_1,\dots,v_k\}\subset\mathbb{R}^n$ and suppose this set is minimal (or equivalently that all intersections $F_i=H_{v_i}^0\cap F$ are hyperfaces in F). Suppose in addition that there is no angle chain between v_1 and v_2 through the set V and that $v_1 \perp v_2$.*

Then faces $F_1=H_{v_1}^0\cap F$ and $F_2=H_{v_2}^0\cap F$ are adjacent.

Proof We consider a partition of V into 3 subsets. V_1 is a subset of those vectors in V that have an angle chain to v_1 through V; V_2 is a subset of those vectors in V that have an angle chain to v_2 through V and $V_3 = V \setminus (V_1 \cup V_2)$.

Let $L = H^0_{v_1} \cap H^0_{v_2}$. For any vector $w \in \mathbb{R}^n$ we denote by w^L the orthogonal projection of w to L.

Let $h_w = H^-_w \cap L$, then h_w is a halfspace in L with a normal vector w^L. Set

$$I_1 = \bigcap_{w \in V_1} h_w, \quad I_2 = \bigcap_{w \in V_2} h_w, \quad I_3 = \bigcap_{w \in V_3} h_w.$$

We claim that

(i) $\dim I_i = n-2$, for $i = 1, 2, 3$,

(ii) for $w_i \in V_i$ $w_j \in V_j$ we have $w_i^L \perp w_j^L$ if $i \neq j$.

Let us first note, that these two properties imply that $\dim(I_1 \cap I_2 \cap I_3) = n-2$. This would imply the lemma, because $F_1 \cap F_2 = \bigcap_{w \in V} h_w = I_1 \cap I_2 \cap I_3$.

Let us show (i) and (ii).

We define sets $W_1 = \{v \in V \mid \langle v, v_2 \rangle = 0\}$ and $W_2 = \{v \in V \mid \langle v, v_1 \rangle = 0\}$. Let note that: $V = W_1 \cup W_2$, $W_1 \supset V_1$, $W_2 \supset V_2$ and $W_1 \cap W_2 \supset V_3$.

We define

$$J_1 = \bigcap_{w \in W_1} h_w, \quad J_2 = \bigcap_{w \in W_2} h_w.$$

We consider J_1 as an intersection

$$J_1 = \Big(\bigcap_{w \in W_1} H^-_w \cap H^0_{v_1} \Big) \cap L \subset H^0_{v_1}.$$

Let us note that:

(1) L is a hyperplane in $H^0_{v_1}$ with a normal vector v_2.

(2) For every $w \in W_1$ the set $H_w \cap H^0_{v_1}$ is a half space in $H^0_{v_1}$ with a normal vector orthogonal to v_2.

(3) We have $\dim(\bigcap_{w \in W_1} H^-_w \cap H^0_{v_1}) = n-1$ (this follows from the inclusion $\bigcap_{w \in W_1} H^-_w \cap H^0_{v_1} \supset F_1$).

These three properties imply that $\dim J_1 = n-1$. By the same arguments $\dim J_2 = n-1$. We know that $I_1 \supset J_1$, $I_2 \supset J_2$ and $I_3 \supset J_1 \cup J_2$, hence (i) follows.

For any $w_1 \in W_1$ and $w_2 \in W_2$ the condition $w_1 \perp w_2$ implies that $w_1^L \perp w_2^L$; hence (ii) follows. $\square$

For two points $x, y \in \mathcal{E}$ we denote the common $(n-1)$–face of polyhedra V_x and V_y by $V_{xy} = V_x \cap V_y$.

Lemma 7.7 *Suppose that points $x, y, z \in \mathcal{E}$, $xy \perp xz$ and faces V_{xy}, V_{xz} are adjacent. Suppose in addition that for any point $y^* \in \mathfrak{S}(y) \cap \langle x, y, z\rangle$ we have $yy^* \perp yx$. Then the point $z^* = x + \overrightarrow{xy} + \overrightarrow{xz} \in \mathcal{E}$.*

Proof Consider the intersection $V_{xyz} = V_{xy} \cap V_{xz}$, this intersection is an $(n-2)$–face of polyhedron V_x and hence it is an $(n-2)$–face of polyhedron V_y (because of the Voronoy decomposition structure). We can represent V_y as an intersection

$$V_y = \bigcap_{t \in \mathfrak{S}(y)} H^-_{y,(1/2)\overrightarrow{yt}}.$$

Hence there exist two points $y_1, y_2 \in \mathfrak{S}(y) \cap \langle x, y, z\rangle$ such that

$$V_{xyz} \subset H^0_{(1/2)\overrightarrow{yy_1}} \cap H^0_{(1/2)\overrightarrow{yy_2}}.$$

One of these points, say y_1, coincides with x. Then the other point, y_2, coincides with z^*. □

Proof of Lemma 7.1 We can represent V_x as an intersection

$$V_x = \bigcap_{t \in \mathfrak{S}(x)} H^-_{x,(1/2)\overrightarrow{xt}}.$$

Because of Lemma 7.5 we can apply Lemma 7.6 to $F = V_x$, $F_1 = V_{xy}$, $F_2 = V_{xz}$ and obtain that faces $F_1 = V_{xy}$, $F_2 = V_{xz}$ are adjacent. We apply Lemma 7.7 and then Lemma 7.1 follows. □

7B

Here we finish the proof of Proposition 5.2. The proposition follows directly from the next two lemmas.

Lemma 7.8 *The set $\mathcal{E}$ is a lattice with generating vectors $\{\overrightarrow{xy}\}_{y \in \mathfrak{S}(x)}$.*

Proof We know that

$$V_x = \bigcap_{t \in \mathfrak{S}(x)} H^-_{x,(1/2)\overrightarrow{xt}} = \bigcap_{t \in \mathcal{E} \setminus x} H^-_{x,(1/2)\overrightarrow{xt}}. \tag{1}$$

First we show that for every $x, y \in \mathcal{E}$,

$$\mathfrak{S}(y) = \mathfrak{S}(x) + \overrightarrow{xy}. \tag{2}$$

Indeed Lemma 7.1 implies that for any $x, y \in \mathcal{E}$ such that $y \in \mathfrak{S}(x)$ (further we call such points adjacent) we have $\mathfrak{S}(x) + \overrightarrow{xy} \subset \mathcal{E}$ and hence by (1) $V_y \subset V_x + \overrightarrow{xy}$. Then changing x and y we obtain the equality (2) in this case. For arbitrary $x, y \in \mathcal{E}$ the equality (2) can be obtained by joining x and y with a chain of adjacent points.

To show Lemma 7.8 it is sufficient to prove that for any $x, y, z \in \mathcal{E}$ we have $x + \overrightarrow{xy} + \overrightarrow{xz} \in \mathcal{E}$ $x - \overrightarrow{xy} \in \mathcal{E}$. The second inclusion needs the central symmetry of the set $\mathfrak{S}(x)$, that follows from the reflection property. After this both inclusions can be proved by using (2) and joining correspondent points with a chain of adjacent points. □

The next lemma shows that we can reduce our generating set for $\mathcal{E}$ to the n generating vectors with the same property that Γ reflects this vectors.

Lemma 7.9 *Suppose $S \in \mathbb{R}^n$ is an n–lattice with a generating set $a_1, \dots, a_s$, G is a subgroup of isometries of $\mathbb{R}^n$ which reflects this generating set. Then there exists an n–generating set $b_1, \dots, b_n$ for S such that G reflects this generating set.*

Proof It is known that there exists an n–generating set for any n–lattice (sometimes it is called a short basis), so the problem is to find an n–generating set reflected by G.

We define an equivalence relation on the generating set $a_1, \dots, a_s$. We set $a_{i^*} \sim a_{j^*}$ if there is an angle chain connecting a_{i^*} and a_{j^*} through $\{a_1, \dots, a_s\}$. We denote by $q_1, \dots, q_l$ the equivalence classes. Let note that vectors from the different classes are mutually orthogonal, equivalently vectors can be fixed or reflected by stabilizers G_x (where $x \in S$) only simultaneously.

We fix a point $x \in S$. For every equivalence class we consider a lattice

$$S_i = \left\{ x + \sum_{a \in q_i} n_a a, n_a \in \mathbb{Z} \right\},$$

then

$$S = \bigoplus_{i=1}^{l} \mathcal{S}^i.$$

Let $d(i)$ be the dimension of the affine hull of S_i, then we can choose a $d(i)$–basis for S_i. (We note that this basis is independent on the choice of $x \in S$). Vectors of this basis are reflected by G because any stabilizer can act on every S_i only identically or by central symmetry. Then the union of these $d(i)$–bases is an n–basis for S reflected by G. □

8 Comments and open questions

Let us describe here how some problems about the number of extremal subsets can be reformulated algebraically.

Let $\Gamma \curvearrowright \mathbb{R}^n$ be an effective discrete group action by isometries. Then the quotient space $\mathcal{A} = \mathbb{R}^n/\Gamma$ is an Alexandrov space of nonnegative curvature. Extremal subsets in such an Alexandrov space can be described in terms of the group action.

We call a subset $E \subset \mathbb{R}^n$ a *singular subset* (for the action $\Gamma \curvearrowright \mathbb{R}^n$) if it is a set of all fixed point for some subgroup of Γ, (we call a 0–dimensional singular subset a *singular point*). Note that since the action is effective every subgroup that fixes some point is finite. Also, every finite subgroup fixes some nonempty affine subset.

It is not difficult to show that a subset of $\mathcal{A}$ is a primitive extremal subset if and only if it is an image of a singular subset in $\mathbb{R}^n$. Therefore there is a bijection between maximal finite subgroups of Γ up to conjugation and primitive extremal subsets in $\mathcal{A}$.

Denote by $N(\Gamma \curvearrowright \mathbb{R}^n)$ the number of orbits of singular points and by $M(\Gamma)$ the number of maximal finite subgroups in Γ up to conjugation. It follows that $N(\Gamma \curvearrowright \mathbb{R}^n) \leqslant M(\Gamma)$. Some maximal subgroups of Γ might fix affine subspaces of positive dimension, therefore $M(\Gamma)$ might be strictly bigger than $N(\Gamma \curvearrowright \mathbb{R}^n)$. From Theorem 1.4 we have the following:

Corollary 8.1 *For any cocompact effective discrete action by isometries $\Gamma \curvearrowright \mathbb{R}^n$, we have $N(\Gamma \curvearrowright \mathbb{R}^n) \leqslant 2^n$.*

We believe that the following stronger statement is true.

Conjecture *For any cocompact effective discrete action by isometries $\Gamma \curvearrowright \mathbb{R}^n$, we have $M(\Gamma) \leqslant 2^n$.*

There is a discussion on this conjecture; see [6].

References

[1] **S B Alexander**, **R L Bishop**, *A cone splitting theorem for Alexandrov spaces*, Pacific J. Math. 218 (2005) 1–15 MR2224586

[2] **Y Burago**, **M Gromov**, **G Perelman**, *A D Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk 47 (1992) 3–51, 222 MR1185284 In Russian; translated in Russian Math. Surveys 47 (1992) 1–58

[3] **L Danzer**, **B Grünbaum**, *Über zwei Probleme bezüglich konvexer Körper von P Erdős und von V L Klee*, Math. Z. 79 (1962) 95–99 MR0138040

[4] **P Erdős**, *Some unsolved problems*, Michigan Math. J. 4 (1957) 291–300 MR0098702

[5] **V Kapovich**, *Private communication*

[6] **N Lebedeva**, *Number of subgroups in a Bieberbach group* Available at `http://mathoverflow.net/questions/13714`

[7] **N Lebedeva**, **A Petrunin**, *Local characterization of polytopes* `arXiv:1402.6670` to appear in Geom. Dedic.

[8] **N Li**, *Volume and gluing rigidity in Alexandrov geometry* Available at `http://front.math.ucdavis.edu/1110.5498`

[9] **G Perelman**, *Private communication*

[10] **G Perelman**, *Spaces with curvature bounded below*, from: "Proceedings of the International Congress of Mathematicians, Volumes 1, 2", (S D Chatterji, editor), Birkhäuser (1995) 517–525 MR1403952

[11] **G Y Perelman**, **A M Petrunin**, *Extremal subsets in Aleksandrov spaces and the generalized Liberman theorem*, Algebra i Analiz 5 (1993) 242–256 MR1220499 In Russian; translated in St. Petersburg Math. J. 5 (1994) 215–227

[12] **A Petrunin**, *Private communication*

St. Petersburg Department of V A Steklov Institute of Mathematics of RAS
Fontanka 27, St. Petersburg, 191023, Russia

Mathematics Department, St. Petersburg State University
Universitetsky pr., 28, Stary Peterhof, 198504, Russia

`lebed@pdmi.ras.ru`

Proposed: John Lott Received: 26 December 2013
Seconded: Tobias H Colding, Dmitri Burago Revised: 21 May 2014