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Virtual domination of 3–manifolds

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For any closed oriented hyperbolic 3–manifold M and any closed oriented 3–manifold N, we show that M admits a finite cover M' such that there exists a degree-2 map $f\colon M' \to N$, ie M virtually 2–dominates N.

57M10; 30F40, 57M50

1 Introduction

1.1 Background and the main result

Traditionally, essential codimension-1 objects in 3–manifolds, eg incompressible surfaces, taut foliations and essential laminations, are very important and interesting objects in 3–manifold topology. There are various methods to construct such essential codimension-1 objects, and most of the constructions use topological methods.

Recently, Kahn and Markovic [8] used hyperbolic geometry and dynamical systems to prove the following surface subgroup theorem: For any closed hyperbolic 3–manifold M, there exists a closed hyperbolic surface S such that there is a π_1–injective almost totally geodesic immersion $S \looparrowright M$. Building on Wise's work [18], Agol showed that the groups of hyperbolic 3–manifolds are virtually special and LERF [1]. Agol's result allows us to find a finite cover of M such that S lifts to an embedded incompressible surface, which solves Thurston's virtual Haken conjecture [16]. Actually, it is the surface subgroup theorem that makes Wise's machine on geometric group theory available for studying closed hyperbolic 3–manifolds.

For any closed hyperbolic 3–manifold M, one can use Kahn–Markovic surfaces to construct an immersed π_1–injective 2–complex $X_n \looparrowright M$ [14]. Here the 2–complex X_n is a local model of homological $\mathbb{Z}_n$–torsion. Then the results of Agol [1] and Haglund and Wise [5] can be applied to $X_n \looparrowright M$, and it was shown in [14] that, for any finite abelian group A and any closed hyperbolic 3–manifold M, M admits a finite cover M' such that A is a direct summand of $\mathrm{Tor}(H_1(M';\mathbb{Z}))$.

The proof of this result suggests that we construct some other types of immersed π_1–injective 2–complexes in closed hyperbolic 3–manifolds. Then LERF or other

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virtual properties of hyperbolic 3–manifolds will imply some other nice results. In this paper we will give another application of this idea, and show the following result:

Theorem 1.1 *For any closed oriented hyperbolic 3–manifold M and any closed oriented 3–manifold N, M admits a finite cover M' such that there exists a degree-2 map $f\colon M' \to N$, ie M virtually 2–dominates N.*

For two n–dimensional closed oriented manifolds M and N, if there exists a degree-d map from M to N, then we say that M *d–dominates* N. If there exists a degree-nonzero map M to N, then we say that M *dominates* N.

Remark A In a previous version of this paper we used results of Gaifullin [4], and could only show that any closed oriented hyperbolic 3–manifold virtually dominates any closed oriented 3–manifold, but with no bound on the degree of the map. The author wants to thank Ian Agol for introducing him to the result of Hilden, Lozano, Montesinos and Whitten [6] which is used to prove the virtual 2–domination result.

Theorem 1.1 answers a question asked by Agol, whether any closed hyperbolic 3–manifold virtually dominates any closed 3–manifold, which was a possible approach to proving the virtual Haken conjecture. In some sense, the existence of a degree-nonzero map from one 3–manifold M to another 3–manifold N implies that M is (topologically) more complicated than N. So Theorem 1.1 implies that any closed hyperbolic 3–manifold is virtually more complicated than any closed 3–manifold. For more information about the history, results and questions on degree-nonzero maps between 3–manifolds, see the survey paper by Wang [17].

Since closed hyperbolic 3–manifolds have virtually positive first Betti number (see Agol [1]), we can suppose that M satisfies $b_1(M) > 0$. Then the following immediate corollary holds:

Corollary 1.2 *For any even number $2d$, any closed oriented hyperbolic 3–manifold M and any closed oriented 3–manifold N, M admits a finite cover M'' such that there exists a degree-$2d$ map $g\colon M'' \to N$, ie M virtually $2d$–dominates N.*

Theorem 1.1 also answers Question 8.2 in Derbez, Liu and Wang [3] for closed hyperbolic 3–manifolds, by taking N to be any closed 3–manifold which supports $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ geometry. Since N has positive Seifert volume (see Brooks and Goldman [2] for the definition), so does a finite cover of M.

Corollary 1.3 *Any closed hyperbolic 3–manifold M admits a finite cover M''' such that M''' has positive Seifert volume ($\mathrm{Iso}_e\widetilde{\mathrm{SL}}_2(\mathbb{R})$–representation volume).*

Remark B Theorem 1.1 also implies a weaker version of the main result in [14]: for any closed hyperbolic 3–manifold M and any finite abelian group A, M admits a finite cover M' such that A is embedded into $\mathrm{Tor}(H_1(M';\mathbb{Z}))$.

Suppose $f\colon M' \to N$ is a degree-d map (degree-2 in our case); then an elementary fact in algebraic topology implies that the restriction of $f^*\colon H^1(N;\mathbb{Z}) \to H^1(M';\mathbb{Z})$ to $\ker(H^1(N;\mathbb{Z}) \xrightarrow{\times d} H^1(N;\mathbb{Z}))$ is injective. So, given the finite abelian group A, we need only take N to be the connected sum of a few lens spaces.

Since closed 3–manifolds with vanishing simplicial volume do not virtually dominate any closed hyperbolic 3–manifold (by considering the simplicial volume), it is natural to ask the following question.

Question 1.4 For any closed oriented 3–manifold M with positive simplicial volume, does M virtually dominate any closed oriented 3–manifold?

To give a positive answer to this question, it suffices to show that any irreducible closed oriented 3–manifold with a hyperbolic piece in its JSJ decomposition virtually dominates some closed oriented hyperbolic 3–manifold. Derbez, Liu and Wang [3, Theorem 1.6(1)] give some evidence for Question 1.4. If one can extend Kahn and Markovic's and Liu and Markovic's theories to cusped hyperbolic 3–manifolds, then Question 1.4 can be confirmed.

The following natural question asks about a quantitative strengthening of Theorem 1.1.

Question 1.5 For any closed oriented hyperbolic 3–manifold M (or closed oriented 3–manifold with positive simplicial volume), does M virtually 1–dominate any closed oriented 3–manifold?

We can only prove that virtual 2–domination exists, but could not promote it to virtual 1–domination, basically because of the $\mathbb{Z}_2$–valued invariant σ (see Corollary 2.11) which was introduced by Liu and Markovic [9, Theorem 1.4].

In Section 2, we will give a quick review of the results in Kahn and Markovic [7; 8] and Liu and Markovic [9] which are necessary for this paper. In Section 3 we will give a topological part of the proof of Theorem 1.1. In Section 3.1 we will endow a closed oriented 3–manifold N with a nice handle structure, by using a result in Hilden, Lozano, Montesinos and Whitten [6]. In Section 3.2, for any closed hyperbolic 3–manifold M, we will construct a 2–complex Z (hinted at by the handle structure of N) and a π_1–injective immersion $j\colon Z \looparrowright M$. In Section 3.3 we will describe how the existence of this immersed π_1–injective 2–complex implies Theorem 1.1. The proof of the π_1–injectivity of $j\colon Z \looparrowright M$ will be delayed to Section 4.

1.2 Sketch of the proof

Here we give a brief sketch of the proof of Theorem 1.1.

Thurston [15] described a hyperbolic 3–orbifold M_0 whose underlying space is S^3 and whose singular set is the Borromean rings with indices 4. Moreover, Hilden, Lozano, Montesinos and Whitten showed that M_0 has the following universal property:

Theorem 1.6 [6] *For any closed oriented 3–manifold N, there is a finite-index subgroup $\Gamma \subset \pi_1(M_0) \subset \mathrm{PSL}_2(\mathbb{C})$ such that N is homeomorphic to $\mathbb{H}^3/\Gamma$ with respect to their orientations. Here we ignore the orbifold structure of $\mathbb{H}^3/\Gamma$, and just think of it as a 3–manifold.*

In Section 3.1 we will construct an orbifold handle structure (see Definition 3.1) for M_0 by following the geometry of the regular dodecahedron (or, equivalently, the regular icosahedron). Then this orbifold handle structure of M_0 is pulled back to an orbifold handle structure of $\mathbb{H}^3/\Gamma$ by the finite-sheeted cover provided by Theorem 1.6. This gives a nice handle structure for N, which is related to the geometry of the regular icosahedron. Note that the relation between this handle structure and the geometry of the regular icosahedron will play a crucial role in our construction of the immersed π_1–injective 2–complex.

For a closed 3–manifold (orbifold) P endowed with an (orbifold) handle structure, we will use $P^{(1)}$ to denote the union of 0– and 1–handles, and use $P^{(2)}$ to denote the union of 0–, 1– and 2–handles.

We will also construct a 2–subcomplex $X \subset M_0$ which is a deformation retract of $M_0^{(2)}$ such that X pulls back to a 2–subcomplex $Y \subset N$ which is a deformation retract of $N^{(2)}$.

Now we begin the construction of the immersed 2–complex $Z \looparrowright M$.

For any closed oriented hyperbolic 3–manifold M and any point $p \in M$, choose twelve unit vectors in $T_p^1 M$ which correspond to the normal vectors of the twelve faces of the regular dodecahedron. By using the exponential mixing property of the frame flow (see Moore [10] and Pollicott [11]), we can construct an immersion of the 1–skeleton $X^{(1)}$ of X into M, denoted by $j'\colon X^{(1)} \looparrowright M$. This construction is hinted at by the geometry of the regular dodecahedron, and satisfies the following conditions:

- The unique 0–cell of $X^{(1)}$ is mapped to p.
- The six 1–cells are mapped to geodesic arcs in M based at p, and their tangent vectors at p are very close to two of those twelve unit vectors (corresponding to the 1–handles of the orbifold handle structure of M_0).

- The image of 1–cells of $X^{(1)}$ are homologous to 0 in $H_1(M;\mathbb{Z})$.
- There exists a large real number $R>0$ such that the image in M of the boundary of any 2–cell of X is homotopic to a closed geodesic whose complex length is very close to $4R$ or R (depending on whether this 2–cell intersects the singular set of M_0 or not).

Since the 1–skeleton $Y^{(1)}$ of Y is a finite cover of $X^{(1)}$, $j'\colon X^{(1)} \looparrowright M$ induces an immersion $Y^{(1)} \looparrowright M$. We take two copies of the immersed 1–complex $Y^{(1)} \looparrowright M$, and denote them by $Y_1^{(1)} \looparrowright M$ and $Y_2^{(1)} \looparrowright M$. For any 2–cell c_i in Y (with an arbitrary orientation), let γ_i be the oriented closed geodesic homotopic to the image of ∂c_i in M. Take two copies of γ_i and denote them by γ_i^1 and γ_i^2; then a recent result of Liu and Markovic [9] (see Corollary 2.11) implies that $\gamma_i^1 \sqcup \gamma_i^2$ bounds an immersed oriented almost totally geodesic subsurface S_i in M (possibly disconnected).

By pasting the immersed 1–complexes $Y_1^{(1)}$ and $Y_2^{(1)}$, almost totally geodesic surfaces $\{S_i\}$, and almost totally geodesic annuli connecting γ_i^j with $Y_j^{(1)}$ for $j=1,2$, we get an immersed 2–complex $j\colon Z \looparrowright M$. The 2–complex Z is connected and locally almost totally geodesic in M except along $Y_1^{(1)} \cup Y_2^{(1)} \subset Z$. Moreover, if the surfaces S_i are complicated enough, then $j_*\colon \pi_1(Z) \to \pi_1(M)$ is injective.

Since Agol [1] showed that the groups of hyperbolic 3–manifolds are LERF, M admits a finite cover M' such that a geometric neighborhood Z is embedded into M', and we denote this neighborhood by K. Let A_i be the annulus on $\partial N^{(1)}$ where the i^{th} 2–handle is attached. Then K is homeomorphic to the quotient space of the disjoint union of $\{S_i \times I\}$ and two copies of $N^{(1)}$ by pasting $(\partial S_i)\times I$ to the two copies of A_i.

Then we can construct a proper degree-2 map $h\colon (K,\partial K) \to (N^{(2)}, \partial N^{(2)})$, which maps the two copies of $N^{(1)}$ in K to $N^{(1)} \subset N^{(2)}$ by the identity map, and maps $S_i \times I$ to the corresponding 2–handle of N. Then by pinching the components of $M' \setminus K$ to wedge of 3–balls, $h\colon K \to N^{(2)}$ can be extended to a degree-2 map $f\colon M' \to N$, as desired.

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2 Kahn and Markovic's and Liu and Markovic's works on constructing almost totally geodesic subsurfaces

In this section, we give a quick review of Kahn and Markovic's and Liu and Markovic's works on constructing almost totally geodesic subsurfaces in closed hyperbolic 3–manifolds. All the material in this chapter can be found in [7; 8; 9], and we only state those results that are necessary for our applications.

2.1 The surface subgroup theorem

Kahn and Markovic proved the following surface subgroup theorem, which is the first step to prove Thurston's virtual Haken and virtual fibered conjectures. (The conjectures were raised in [16], and settled in [1].)

Theorem 2.1 [8] *For any closed hyperbolic 3–manifold M, there exists an immersed closed hyperbolic surface $f\colon S \looparrowright M$ such that $f_*\colon \pi_1(S) \to \pi_1(M)$ is an injective map.*

Actually, the surfaces constructed in Theorem 2.1 are almost totally geodesic surfaces, which are constructed by pasting oriented *good pants* together along oriented *good curves* in an almost totally geodesic way. In the following, we will describe Kahn and Markovic's construction in more detail.

First we need to give some geometric definitions.

Given an oriented geodesic arc α in a closed hyperbolic 3–manifold with initial point p and terminal point q, for two unit vectors $\vec{v}$ and $\vec{w}$ at p and q, respectively, which are normal to α, we define $\boldsymbol{d}_\alpha(\vec{v},\vec{w})$ in the following way. Let $\vec{v}'$ be the parallel transport of $\vec{v}$ to q along α, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be the oriented angle between $\vec{v}'$ and $\vec{w}$ (with respect to the orientation of α), and the length of α be $l > 0$. Then the complex distance between $\vec{v}$ and $\vec{w}$ along α is defined to be $\boldsymbol{d}_\alpha(\vec{v},\vec{w}) = l + \theta i$.

For an oriented closed geodesic γ in a hyperbolic 3–manifold, we define its complex length in a similar way. Choose an arbitrary point p on γ and choose a unit tangent vector $\vec{v}$ at p which is normal to γ. Then we can think of γ as an oriented geodesic arc from p to p. Then the complex length of γ is defined to be $\boldsymbol{l}(\gamma) = \boldsymbol{d}_\gamma(\vec{v}, \vec{v})$. This complex length not only measures the length of γ in the usual sense, but also measures the rotation angle of the corresponding hyperbolic isometry. Note that the complex length of an oriented closed geodesic does not depend on its orientation or the choices we made.

In the following, we will use Π^0 to denote the oriented pair of pants.

Definition 2.2 For a closed hyperbolic 3–manifold M, a map $f\colon \Pi^0 \to M$ is called a *skew pair of pants* if $f_*\colon \pi_1(\Pi^0) \to \pi_1(M)$ is injective and $f(\partial \Pi^0)$ is a union of three closed geodesics.

We will always think about homotopic skew pair of pants as the same object, and we will use Π to denote a skew pair of pants $f\colon \Pi^0 \to M$ when it does not cause any confusion.

Let C_1, C_2 and C_3 be the three oriented boundary components of Π^0 and let γ_1, γ_2 and γ_3 be the three oriented closed geodesics $f(C_1)$, $f(C_2)$ and $f(C_3)$, respectively. Let a_i be the simple arc in Π^0 which connects C_{i-1} and C_{i+1}, such that a_1, a_2 and a_3 are disjoint from each other on Π^0. Then we can assume that $f(a_i)$ is a geodesic arc perpendicular to both γ_{i-1} and γ_{i+1}, and denote $f(a_i)$ by η_i.

Now we fix γ_i, and give orientations for η_{i-1} and η_{i+1} such that they are both pointing away from γ_i. Then η_{i-1} and η_{i+1} divide γ_i into two oriented geodesic arcs γ_i^1 and γ_i^2, such that the orientation on γ_i^1 goes from $\eta_{i-1} \cap \gamma_i$ to $\eta_{i+1} \cap \gamma_i$. Let $\vec{v}_{i-1}$ and $\vec{v}_{i+1}$ be the tangent vectors of η_{i-1} and η_{i+1} at $\eta_{i-1} \cap \gamma_i$ and $\eta_{i+1} \cap \gamma_i$, respectively. Then the pair of vectors $(\vec{v}_{i-1}, \vec{v}_{i+1})$ are called the *pair of feet* of Π on γ_i. The hyperbolic geometry of right-angled hexagons in $\mathbb{H}^3$ implies that

$$\boldsymbol{d}_{\gamma_i^1}(\vec{v}_{i-1}, \vec{v}_{i+1}) = \boldsymbol{d}_{\gamma_i^2}(\vec{v}_{i+1}, \vec{v}_{i-1}).$$

Now we can define the half-length of γ_i with respect to Π as

$$\boldsymbol{hl}_\Pi(C_i) = \boldsymbol{d}_{\gamma_i^1}(\vec{v}_{i-1}, \vec{v}_{i+1}) = \boldsymbol{d}_{\gamma_i^2}(\vec{v}_{i+1}, \vec{v}_{i-1}).$$

Now we are ready to define *good curves* and *good pants*.

Definition 2.3 Fix a small number $\epsilon > 0$ and a large number $R > 0$. For an oriented closed geodesic γ in M, we say γ is an (R, ϵ)*–good curve* if $|\boldsymbol{l}(\gamma) - R| < 2\epsilon$. The set of (R, ϵ)–good curves is denoted by $\boldsymbol{\Gamma}_{R,\epsilon}$.

For a skew pair of pants $f\colon \Pi^0 \to M$, we say it is an (R,ϵ)–*good pants* if

$$|\boldsymbol{hl}_{\Pi}(C) - R/2| < \epsilon$$

for all the three cuffs (boundary components) of Π^0. The set of (R,ϵ)–good pants is denoted by $\mathbf{\Pi}_{R,\epsilon}$.

In the following part of this paper, we will work with a very small positive number $\epsilon > 0$ and a very large number $R > 0$, and the precise value of ϵ and R will be determined later. When R and ϵ are fixed, we will only talk about good curves and good pants instead of (R,ϵ)–good curves and (R,ϵ)–good pants. Note that the oriented boundary components of (R,ϵ)–good pants are always (R,ϵ)–good curves.

For a good curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, the normal bundle $N^1(\gamma)$ of γ is naturally identified with $\mathbb{C}/\boldsymbol{l}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z}$. If γ is the oriented boundary of a good pants Π, we can define the half normal bundle with respect to Π by $N^1(\sqrt{\gamma}) = \mathbb{C}/\boldsymbol{hl}_{\Pi}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z}$. Then the pair of feet of Π on γ are identified to one point in $N^1(\sqrt{\gamma})$, which is called the *foot* of Π on γ and denoted by $\mathrm{foot}_{\gamma}(\Pi)$.

Now we are ready to talk about maps from surfaces to closed hyperbolic 3–manifolds. Suppose S is a compact oriented surface (possibly with boundary) with negative Euler characteristic, equipped with a pants decomposition $\mathcal{C}$ (boundary components of S are also included in $\mathcal{C}$). Then the closure of each component of $S \setminus \mathcal{C}$ is an oriented pair of pants, and we call such a component a pants in S.

Definition 2.4 A map $f\colon S \to M$ is called *viable* if the following conditions hold:

- For each pants Π in S, $f|_{\Pi}\colon \Pi \to M$ is a skew pair of pants.
- For any two pants Π and Π' in S sharing a curve $C \in \mathcal{C}$, $\boldsymbol{hl}_{\Pi}(C) = \boldsymbol{hl}_{\Pi'}(C)$.

So for a viable map $f\colon S \to M$, we will use $\boldsymbol{hl}(C)$ to denote $\boldsymbol{hl}_{\Pi}(C)$ for each $C \in \mathcal{C}$. For two pants Π and Π' in S sharing a curve $C \in \mathcal{C}$, we give C an orientation such that Π lies to the left of C on S, and Π' lies to the right. Let $\gamma = f(C)$ and let γ' be the same closed geodesic with the opposite orientation. Then we can compare the feet of Π and Π' on $N^1(\sqrt{\gamma})$ by the shearing parameter

$$s(C) = \mathrm{foot}_{\gamma}(f|_{\Pi}) - \mathrm{foot}_{\gamma'}(f|_{\Pi'}) - \pi i \in N^1(\sqrt{\gamma}) = \mathbb{C}/\boldsymbol{hl}(C)\mathbb{Z} + 2\pi i\mathbb{Z}.$$

Now we can precisely describe the immersed almost totally geodesic surface constructed in the proof of Theorem 2.1. The closed oriented surface S is equipped with a pants decomposition $\mathcal{C}$, and the map $f\colon S \looparrowright M$ is a viable map satisfying the following inequalities for any $C \in \mathcal{C}$:

$$\begin{cases} |\boldsymbol{hl}(C) - R/2| < \epsilon, \\ |s(C) - 1| < \epsilon/R. \end{cases}$$

We will call a viable map $f\colon S \looparrowright M$ an (R,ϵ)*–almost totally geodesic surface* (S may have boundary) if the first inequality holds for each $C \in \mathcal{C}$ and the second inequality holds for each $C \in \mathcal{C}$ shared by two pants. Note that this definition is more general than the surfaces constructed in Theorem 2.1, since these surfaces may have boundary. In [8], it is shown that (R,ϵ)–almost totally geodesic closed surfaces are π_1–injective if $\epsilon > 0$ is sufficiently small and $R > 0$ is sufficiently large.

The existence of such almost totally geodesic closed surfaces is shown by the following strategy. For any good curve γ, one can consider all the good pants in M with γ as one of its boundary components, then consider all the feet as $\mathrm{foot}_\gamma(\Pi)$ on $N^1(\sqrt{\gamma})$. Kahn and Markovic showed that these feet on $N^1(\sqrt{\gamma})$ are very equidistributed, so they can paste all the good pants together in a proper way such that $|s(C) - 1| < \epsilon/R$ holds.

In the proof of the existence of good pants (curves) and the equidistribution result, the following exponential mixing property of the frame flow played a crucial role.

Theorem 2.5 [10; 11] *Let M be a closed hyperbolic 3–manifold, $\mathcal{F}(M^3)$ be the frame bundle of M, Λ be the Liouville measure on $\mathcal{F}(M^3)$ which is invariant under the frame flow $g_t\colon \mathcal{F}(M^3) \to \mathcal{F}(M^3)$.*

Then there exists a constant $q > 0$ that depends only on M^3 such that the following statement holds. Let $\psi, \phi\colon \mathcal{F}(M^3) \to \mathbb{R}$ be two C^1 functions. Then, for any $r \in \mathbb{R}$,

$$\left| \Lambda(\mathcal{F}(M^3)) \int_{\mathcal{F}(M^3)} (g_r^* \psi)(x) \phi(x) \, d\Lambda(x) - \int_{\mathcal{F}(M^3)} \psi(x) \, d\Lambda(x) \int_{\mathcal{F}(M^3)} \phi(x) \, d\Lambda(x) \right| \le C e^{-q|r|}.$$

Here $C > 0$ only depends on the C^1 norms of ψ and ϕ.

2.2 The Ehrenpreis conjecture

After proving the surface subgroup theorem, Kahn and Markovic worked one dimension lower and proved the following Ehrenpreis conjecture.

Theorem 2.6 [7] *Let S and T be two closed Riemann surfaces with negative Euler characteristics. Then, for any $k > 1$, S and T admit finite covers S_1 and T_1, respectively, such that there exists a k–quasiconformal map $f\colon S_1 \to T_1$.*

To prove the Ehrenpreis conjecture, Kahn and Markovic showed the following theorem.

Theorem 2.7 [7] *Let S be a closed hyperbolic Riemann surface. Then for any $k > 1$ there exists $R_0(K,S) > 0$ such that, for any $R > R_0(K,S)$, the following statement holds. There exists a closed hyperbolic Riemann surface O with pants decomposition $\mathcal{C}$, satisfying $\boldsymbol{l}(C) = R$ and $s(C) = 1$ for any $C \in \mathcal{C}$, such that there exists a k–quasiconformal map $g\colon O \to S_1$ from O to some finite cover S_1 of S.*

As in the proof of Theorem 2.1, one can try to prove Theorem 2.7 by pasting immersed good pants in S along good curves to obtain a finite cover of S which satisfies $|\boldsymbol{l}(C) - R| < 2\epsilon$ and $|s(C) - 1| < \epsilon/R$. Then this finite cover gives the desired k–quasiconformal map. To make the pasting construction work one needs to make sure that, for any good curve γ in S, the number of good pants to the left of γ should exactly equal the number of good pants to the right of γ. However, the equidistribution result only claims that these two numbers are very close to each other, but may not be equal. In dimension 3, since the unit normal bundle of a closed geodesic is connected (a topological torus), such a problem does not appear. However, one does need to take care of this imbalance problem in dimension 2.

To deal with the problem, Kahn and Markovic studied *good pants homology*. Two elements c_1 and c_2 in $\mathbb{R}\boldsymbol{\Gamma}_{R,\epsilon}$ are considered to be equivalent in the (R,ϵ)–good pants homology group if there exists $w \in \mathbb{R}\boldsymbol{\Pi}_{R,\epsilon}$ such that $\partial w = c_1 - c_2$. Then the (R,ϵ)–good pants homology group is defined to be $\mathbb{R}\boldsymbol{\Gamma}_{R,\epsilon}/\partial\mathbb{R}\boldsymbol{\Pi}_{R,\epsilon}$, and Kahn and Markovic proved the following result:

Theorem 2.8 [7] *Given a closed hyperbolic surface S, for small enough $\epsilon > 0$ depending on S and large enough $R > 0$ depending on ϵ and S, the (R,ϵ)–good pants homology group of S is naturally isomorphic to $H_1(S;\mathbb{R})$. Moreover, the same statement holds if real coefficients are replaced by rational coefficients.*

By using Theorem 2.8 and a randomization technique, Kahn and Markovic took care of the imbalance problem, and proved Theorem 2.7.

2.3 Construction of almost totally geodesic subsurfaces with boundary

In [9], Liu and Markovic pushed the theory of good pants homology from dimension 2 to dimension 3. Instead of working on real (or rational) coefficients as in Theorem 2.8, they worked on homology with integer coefficients. They proved the following theorems. (Note that Theorem 2.9 is very similar to Theorem 2.8.)

Theorem 2.9 [9] *Given a closed oriented hyperbolic 3–manifold M, for small enough $\epsilon > 0$ depending on M and large enough $R > 0$ depending on ϵ and M, let $\Omega_{R,\epsilon}(M) = \mathbb{Z}\mathbf{\Gamma}_{R,\epsilon}/\partial\mathbb{Z}\mathbf{\Pi}_{R,\epsilon}$ be the pants cobordism group of M. Then there is a natural isomorphism $\Phi\colon \Omega_{R,\epsilon}(M) \to H_1(\mathrm{SO}(M);\mathbb{Z})$. Here $\mathrm{SO}(M)$ denotes the bundle of orthonormal frames of M which gives the orientation of M.*

Theorem 2.10 [9] *Given a closed hyperbolic 3–manifold M, let $L \subset M$ be a multicurve such that each component of L is noncontractible in M. Then, for any relative homology class $\alpha \in H_2(M, L;\mathbb{Z})$, there exists an oriented connected (R,ϵ)–almost totally geodesic surface $(F, \partial F) \looparrowright (M, L)$ which realizes a nonzero positive integer multiple of α.*

Since Theorem 2.9 plays a crucial role in our proof of the main theorem, we give more explanation about it here. We first fix the constant $\epsilon > 0$ such that 2ϵ is smaller than the injectivity radius of M.

For any $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, a map $\hat{\gamma}\colon S^1 \to \mathrm{SO}(M)$ is defined in [9] which is called the *canonical lifting* of γ and gives the definition of Φ. To define $\hat{\gamma}$, choose a point $p \in \gamma$ and take an orthonormal frame $e_p = (\vec{t}, \vec{n}, \vec{t} \times \vec{n})$ in T_pM such that $\vec{t}$ is tangent to γ and follows the orientation of γ. Then $\hat{\gamma}$ is defined by first flowing e_p once around γ by parallel transportation, then doing a counterclockwise 2π–rotation about $\vec{n}$, and finally traveling back to e_p along a 2ϵ–short path. Then the isomorphism $\Phi\colon \Omega_{R,\epsilon}(M) \to H_1(\mathrm{SO}(M);\mathbb{Z})$ is defined by $\gamma \mapsto [\hat{\gamma}] \in H_1(\mathrm{SO}(M);\mathbb{Z})$.

Note that the concept of (R,ϵ)–almost totally geodesic surface (possibly with boundary) was also implicitly given in [9]. In [9, Lemma 3.8] it is shown that, for any small enough $\epsilon > 0$ and large enough $R > 0$, immersed (R,ϵ)–almost totally geodesic surfaces (possibly with boundary) are π_1–injective.

As a corollary of Theorem 2.9, we have the following result. The statement of Corollary 2.11 is very similar to [9, Theorem 1.4], and the idea of the proof is scattered in a few proofs therein. We reorganize the material there and give the following statement, which is convenient for our application. We also give a brief proof here, by following the idea of [9].

Corollary 2.11 *Let M be a closed hyperbolic 3–manifold. Then, for any small enough $\epsilon > 0$ depending on M and any large enough $R > 0$ depending on ϵ and M, the following statement holds. For any null-homologous oriented (R,ϵ)–multicurve $L \in \mathbb{Z}\mathbf{\Gamma}_{R,\epsilon}$, there is a nontrivial invariant $\sigma(L) \in \mathbb{Z}_2$ defined such that the following properties hold:*

- $\sigma(L_1 \sqcup L_2) = \sigma(L_1) + \sigma(L_2)$.

- *$\sigma(L)=\bar{0}$ if and only if L bounds an immersed oriented (R,ϵ)–almost totally geodesic subsurface S in M. Moreover, if we associate each component l_i ($i=1,\dots,n$) of L with a normal vector $\vec{v}_i\in N^1(\sqrt{l_i})$, then the surface S can be constructed to satisfy the following condition. Let C_i be the boundary component of S which is mapped to l_i, and Π_i be the pants in S with C_i as one of its cuffs. Then the foot of Π_i on l_i is ϵ/R–close to $\vec{v}_i$.*

Proof Since L is null-homologous, by Theorem 2.9,

$$\Phi(L)\in H_1(\mathrm{SO}(3);\mathbb{Z})\subset H_1(\mathrm{SO}(M);\mathbb{Z}).$$

So $\sigma(L)$ is simply defined to be $\Phi(L)\in H_1(\mathrm{SO}(3);\mathbb{Z})\cong\mathbb{Z}_2$. All the statements are clearly true by Theorem 2.9, except the statement that $\sigma(L)=\bar{0}$ implies that L bounds an immersed oriented (R,ϵ)–almost totally geodesic subsurface in M, and the "Moreover" part.

To follow the terminology in [9], an integer-valued measure μ on $\mathbf{\Pi}_{R,\epsilon}$ is defined to be an element in $\mathbb{Z}\mathbf{\Pi}_{R,\epsilon}$ that is nonnegative on each component in $\mathbf{\Pi}_{R,\epsilon}$. We say μ is ubiquitous if μ is positive on each component in $\mathbf{\Pi}_{R,\epsilon}$. We say μ is (R,ϵ')–nearly even-footed if, for each good curve γ, the feet of good pants on $N^1(\sqrt{\gamma})$ (weighted by μ) is ϵ'/R–equivalent to some scaling of the standard Lebesgue measure on the torus [8, Definition 3.5]. Theorem 2.10 of [9] (essentially given by [8]) implies that there exists a ubiquitous $(R,\epsilon/2)$–nearly even-footed integer-valued measure μ_0 on $\mathbf{\Pi}_{R,\epsilon}$ such that $\partial\mu_0=0\in\mathbb{Z}\mathbf{\Gamma}_{R,\epsilon}$. Such a nice measure μ_0 on $\mathbf{\Pi}_{R,\epsilon}$ allows us to construct an immersed (R,ϵ)–almost totally geodesic closed surface in M, which is the desired surface in Theorem 2.1.

Since $\sigma(L)=\bar{0}$, Theorem 2.9 implies that there exists $w\in\mathbb{Z}\mathbf{\Pi}_{R,\epsilon}$ such that $\partial w=L$, and w defines an integer-valued measure on $\mathbf{\Pi}_{R,\epsilon}$. By adding with the ubiquitous $(R,\epsilon/2)$–nearly even-footed integer-valued measure μ_0 on $\mathbf{\Pi}_{R,\epsilon}$, $w+n\mu_0$ is still ubiquitous, (R,ϵ)–nearly even-footed for a sufficiently large positive integer n, and $\partial(w+n\mu_0)=L$. So $w+n\mu_0$ admits an (R,ϵ)–nearly unit shearing gluing (this gluing satisfies $|s(C)-1|<\epsilon/R$), which gives an immersed (R,ϵ)–almost totally geodesic surface S in M with $\partial S=L$. By [9, Lemma 3.8], each component of S is π_1–injective.

For the "Moreover" part, we will construct an immersed surface $S'\looparrowright M$ which contains the surface $S\looparrowright M$ we have constructed as a subsurface. Let C_i be the boundary component of S which is mapped to l_i; we use Π_i to denote the pair of pants in S with C_i as a cuff. Let $\vec{u}_i$ be the foot of Π_i on l_i. For the ubiquitous $(R,\epsilon/2)$–nearly even-footed integer-valued measure μ_0 on $\mathbf{\Pi}_{R,\epsilon}$, we will not use the gluing satisfying the $(R,\epsilon/2)$–nearly unit shearing condition which glues all the cuffs and gives a closed surface. Here we will use some alternative gluing to get the desired subsurface of S'.

To be precise, let $\Pi_{i,+}$ and $\Pi_{i,-}$ be two pants in $\mathbf{\Pi}_{R,\epsilon}$ such that l_i and $\bar{l}_i$ are oriented boundary components of $\Pi_{i,+}$ and $\Pi_{i,-}$, respectively. Moreover, the foot of $\Pi_{i,+}$ on l_i is ϵ/R–close to $\vec{v}_i$, and the foot of $\Pi_{i,-}$ on $\bar{l}_i$ is ϵ/R–close to the $1+\pi i$ shearing of $\vec{u}_i$. Then there exists an (R,ϵ)–nearly unit shearing gluing of μ_0, which glues all the cuffs of the pants given by μ_0, except the two cuffs of $\Pi_{i,+}$ and $\Pi_{i,-}$ corresponding to l_i and $\bar{l}_i$, respectively. This gluing gives an immersed (R,ϵ)–almost totally geodesic subsurface $S_i \looparrowright M$, with two boundary components.

Now we paste S and $\{S_i\}_{i=1}^n$ together, by gluing Π_i with $\Pi_{i,-}$ along l_i, to get a new immersed surface $S' \looparrowright M$. By the construction, we know that S' is a oriented π_1–injective (R,ϵ)–almost totally geodesic subsurface in M, such that L is its oriented boundary, and the foot on l_i is ϵ/R–close to v_i. By throwing away all the closed surface components of S', we can suppose all the components of S' are surfaces with boundary. □

Remark C Liu and Markovic [9] were interested in proving Theorem 2.10, ie in realizing a second homology class α by an immersed connected (R,ϵ)–almost totally geodesic surface. For technical reasons they needed to take a non-zero integer multiple of α. In this paper we only need to find an immersed (R,ϵ)–almost totally geodesic surface bounded by L, but do not care much about its homology class. So taking an integer multiple of L is not necessary.

Remark D For all the statements in this section, the condition

$$\begin{cases} |\boldsymbol{hl}(C) - R/2| < \epsilon, \\ |s(C) - 1| < \epsilon/R, \end{cases}$$

can always be replaced by

$$\begin{cases} |\boldsymbol{hl}(C) - R/2| < \epsilon/R, \\ |s(C) - 1| < \epsilon/R^2, \end{cases}$$

while $\mathbf{\Gamma}_{R,\epsilon}$ and $\mathbf{\Pi}_{R,\epsilon}$ can also be replace by $\mathbf{\Gamma}_{R,\epsilon/R}$ and $\mathbf{\Pi}_{R,\epsilon/R}$ respectively, as we did in [14] (pointed out in [12]). So, when we apply the results in this section, we always suppose such an $1/R$ factor has been multiplied to ϵ. The main reason that such a refinement is applicable is that the exponential mixing property of frame flow [10; 11] gives an exponential mixing rate, which beats any polynomial rate.

3 Construction of the immersed π_1–injective 2–complex and virtual domination

In this section we will prove Theorem 1.1. The main step is to construct an immersed π_1–injective 2–complex in any closed hyperbolic 3–manifold (Section 3.2). The topology and geometry of this 2–complex is suggested by some nice (orbifold) handle structures of M_0 and N, which are described in Section 3.1. To highlight the geometrical and topological idea, the proof of two technical results on geometric estimations are delayed to Section 4.

3.1 (Orbifold) handle structures of M_0 and N

Thurston [15] described a hyperbolic 3–orbifold M_0 whose underlying space is S^3 and whose singular set is the Borromean rings with indices 4 (cone angle $\pi/2$). M_0 can be obtained as a quotient space of the cube in the following way. Draw a dark segment on each face of the cube, as in Figure 1. Then the quotient relation on this face is given by the reflection along the arc, and S^3 is the quotient space of the cube by six such reflections. The six dark segments in the figure correspond to the Borromean rings in S^3.

The rectangles in Figure 1 are actually combinatorial pentagons, and the combinatorial structure of the boundary of the cube in the figure is isomorphic to the combinatorial structure of the boundary of the regular dodecahedron. Since the hyperbolic right-angled regular dodecahedron exists, it is easy to see that M_0 is a hyperbolic 3–orbifold and $\pi_1(M_0)$ is commensurable with the reflection group of the hyperbolic right-angled regular dodecahedron.

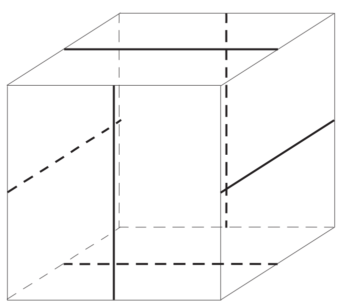

Figure 1: Pasting a cube to get the orbifold M_0

Now we define the concept of orbifold handle structure.

Definition 3.1 Suppose that P is a 3–orbifold whose underlying space is a closed 3–manifold, and that its singular set is a union of disjoint embedded circles. An *orbifold handle structure* of P is a handle structure of the underlying space of P such that each 0– and 1–handle does not intersect with the singular set, and each 2– and 3–handle intersects with the singular set at most along one arc.

Note that if P' is a finite orbifold-cover of P then an orbifold handle structure of P pulls back to an orbifold handle structure of P'.

In Figure 2 we show the 0– and 1–handles of our preferred orbifold handle structure of M_0; here all the 0– and 1–handles do not intersect with the singular set. The 2–handles of M_0 are dual to the edges on the boundary of the dodecahedron, as shown in Figure 2. The 3–handles are dual to the vertices of the dodecahedron.

In Figure 3, the union of the red arcs defines a 1–dimensional subcomplex T of $M_0^{(1)}$. T consists of one 0–cell and six 1–cells, and gives a generating set for $\pi_1(M_0)$. The generators correspond to the six oriented arcs a, b, c, d, e, f in Figure 3, and it is easy to get a presentation of $\pi_1(M_0)$:

$$\begin{aligned}(1)\quad \pi_1(M_0) = \langle a,b,c,d,e,f \mid\, & adb^{-1}d^{-1}, acb^{-1}c^{-1}, eaf^{-1}a^{-1}, ebf^{-1}b^{-1}, \\ & ced^{-1}e^{-1}, cfd^{-1}f^{-1}, a^4, b^4, c^4, d^4, e^4, f^4\rangle.\end{aligned}$$

Each relator in this group presentation corresponds to a 2–handle in the orbifold handle structure of M_0. The first six relators correspond to six 2–handles that do not intersect with the singular set (dual to the edges of the cube). Each of the remaining six relators corresponds to a 2–handle intersecting with the singular set at one arc (dual to the edges of the dark segments of the cube as in Figure 1). Using the hint of the group presentation (1), the reader can try to figure out the position of the 2–handles more explicitly in Figure 2. We do not draw the whole picture here since it will be very messy if everything is shown together.

It is also easy to see that the orbifold handle structure has seven 3–handles. One of the 3–handles does not intersect with the singular set (dual to the vertices of the cube). Each of the remaining six 3–handles intersects with the singular set at one arc.

Figure 4 shows the Kirby diagram of the orbifold handle structure of M_0 as a handle structure of S^3. The six letters A, B, C, D, E, F and their reflections show how the corresponding 1–handles are attached, or, equivalently, how the pairs of disks are identified with each other (by reflection about horizontal or vertical lines). The letters A, B, C, D, E, F correspond to the six generators in the group presentation (1) of $\pi_1(M_0)$.

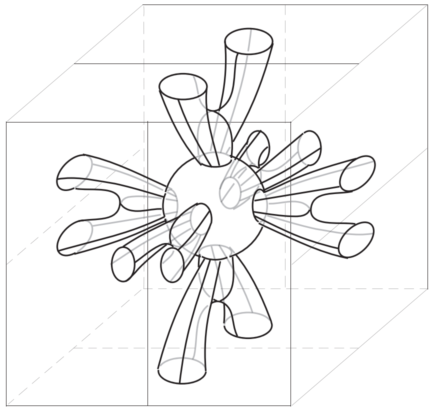

Figure 2: An (orbifold) handle structure of M_0

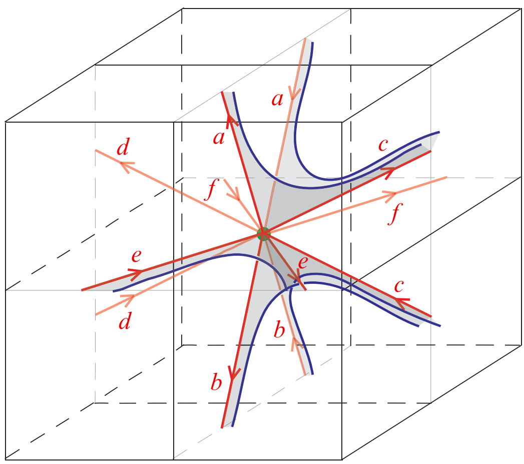

Figure 3: (Part of) the 2–complex X in M_0

From Figure 4, we can see that the orbifold handle structure (group presentation (1)) of M_0 is closely related to the geometry of the regular icosahedron. The 0–handle corresponds to the center of the regular icosahedron. The six 1–handles give twelve vectors at the center, which correspond to the twelve vertices of the regular icosahedron. The twelve 2–handles correspond to the thirty edges on the boundary of the regular icosahedron. The first six 2–handles (corresponding to the first six relators in the group presentation (1)) each correspond to four edges. The remaining six 2–handles each correspond to one edge, since the corresponding 2–handle intersects with the singular set (index 4) along one arc.

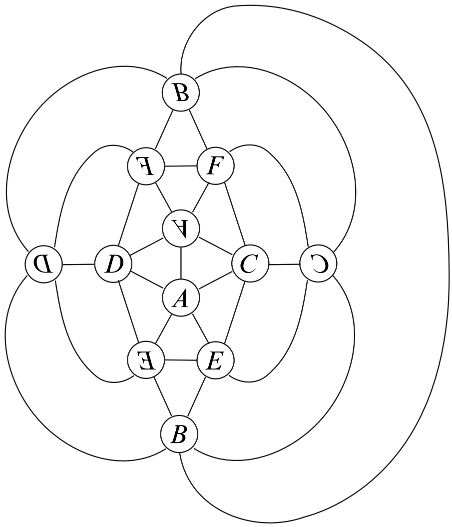

Figure 4: The Kirby diagram of the (orbifold) handle structure of M_0

Now we construct a 2–dimensional subcomplex X in M_0 which is a deformation retract of $M_0^{(2)}$. It is the union of T and twelve topological discs which are the cores of the corresponding twelve 2–handles of M_0 (with boundary as concatenation of arcs in T). Six of these discs are 4–gons that do not intersect with the singular set. The remaining six are monogons, and each of them intersects with the singular set at one point. For the convenience of our further construction, we subdivide each topological disc into the union of a cornered annulus and a round disc such that the cornered annulus lies in $M_0^{(1)}$ and the round disc lies in a 2–handle of M_0. In Figure 3, the shaded part corresponds to two such cornered annuli in M_0. One of them is a four-cornered annulus which corresponds to the relator $acb^{-1}c^{-1}$, and the other component is a one-cornered annulus which corresponds to the relator e^4. We still use X to denote this 2–complex with the refined combinatorial structure, and use X' to denote the intersection of X and $M_0^{(1)}$ (which excludes all the round discs). Although X is not a genuine 2–complex, it is easy to subdivide it to get a 2–complex, so we will simply call X a 2–complex. The 0–cell and 1–cells in $T \subset X$ will still be called 0–cell and 1–cells. The round discs will be called 2–cells, and the intersection of 2–cells and cornered annuli will be called circles.

In summary, $X \subset M_0$ is a 2–complex consisting of one 0–cell, six 1–cells, twelve circles, twelve 2–cells (six of them intersect with the singular set), six four-cornered annuli and six one-cornered annuli. In X', those twelve 2–cells are excluded. In Figure 5, we show the picture of the four-cornered annulus and one-cornered annulus, for the reader's convenience.

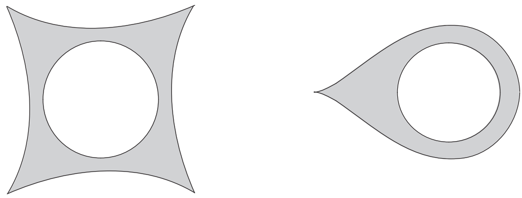

Figure 5: Four–cornered annulus and one–cornered annulus

Theorem 1.6 can be rephrased as the following statement. For any closed oriented 3–manifold N, M_0 admits a finite orbifold cover M_N whose underlying space is homeomorphic to N with respect to their orientations. So the orbifold handle structure of M_0 pulls back to an orbifold handle structure of M_N, which induces a handle structure of N. We may abuse the notational distinction between M_N and N when the orbifold structure is not important in the context.

Suppose $M_N \to M_0$ is a d–sheet orbifold cover. Then the induced handle structure of N has d 0–handles, $6d$ 1–handles and k 2–handles. Here the relation between d and k depends on the geometric behavior of the orbifold covering map $M_N \to M_0$ along the singular set. Actually, $\frac{15}{2}d \leq k \leq 12d$. Those six 2–handles of M_0 that do not intersect with the singular set pull back to $6d$ 2–handles of N. For each of those six 2–handles intersecting with the singular set, the number of the components of its preimage is between $d/4$ and d.

Let Y be the preimage of X in N. Then the finite branched cover $Y \to X$ also has degree d. Y has d 0–cells (corresponding to 0–handles of N), $6d$ 1–cells (corresponding to 1–handles of N), k circles, k 2–cells and k cornered annuli (corresponding to 2–handles of N). Here the cornered annuli might be four-cornered, two-cornered or one-cornered, depending on the branched index. Moreover, all three possibilities do happen [6, Theorem 1.1].

Here Y is a deformation retract of $N^{(2)}$. Let Y' be the intersection of Y with $N^{(1)}$, which excludes all the 2–cells in Y. Then $Y' \cap \partial N^{(1)}$ is the disjoint union of k circles where the 2–handles of N are attached.

3.2 Construction of immersed π_1–injective 2–complex

In this section, for any closed hyperbolic 3–manifold M, we will construct an immersed π_1–injective 2–complex $j\colon Z \looparrowright M$. Here the 2–complex Z contains two copies of Y' as a subcomplex.

In the remainder of this paper, all tangent vectors will be unit vectors if there is no specific description. For any point $p \in M$, we use $T_p^1 M$ to denote the set of all unit tangent vectors at p. For any two vectors $\vec{v}_1, \vec{v}_2 \in T_p^1 M$, we use $\Theta(\vec{v}_1, \vec{v}_2) \in [0, \pi]$ to denote the angle between $\vec{v}_1$ and $\vec{v}_2$.

First we need to introduce the *connection principle* (see [9, Lemma 4.15]). For technical reasons we slightly modify the statement of that result. It follows easily from the exponential mixing property of frame flow [10; 11] and the argument in [12].

Lemma 3.2 (Connection principle) *For any closed hyperbolic 3–manifold M and any small number $0 < \delta < 1$, there exists a constant $L_0(\delta, M) > 0$ such that the following statement holds. Let $\vec{t}_p, \vec{n}_p \in T_p^1 M$ and $\vec{t}_q, \vec{n}_q \in T_q^1 M$ be two pairs of unit orthonormal vectors at $p, q \in M$, respectively. For any real number $L > L_0(\delta, M)$, there exists an oriented geodesic arc γ from p to q, such that the following conditions hold:*

- *The initial and terminal tangent vectors of γ at p and q are δ/L–close to $\vec{t}_p$ and $\vec{t}_q$ respectively.*
- *The length of γ is δ/L–close to L.*
- *The angle between $\vec{n}_q$ and the parallel transportation of $\vec{n}_p$ to q along γ is δ/L–close to 0.* □

Remark E For the convenience of further estimates, we also assume that

$$10^4 L_0(\delta, M) e^{-L_0(\delta, M)/16} < \delta \quad \text{and} \quad L_0(\delta, M) > 10^4.$$

Now we can construct null-homologous closed geodesic arcs with a base point in any hyperbolic 3–manifold, which is a basic step of our construction.

Lemma 3.3 *For any small number $0 < \delta < 1$ there exists $L_1(\delta, M) > 0$ such that, for any real number $L > L_1(\delta, M)$, the following statement holds. Let p be a point in M, $\vec{v}_1, \vec{v}_2$ be two unit vectors in $T_p^1 M$, and $\vec{n} \in T_p^1 M$ be another unit vector orthogonal to both $\vec{v}_1$ and $\vec{v}_2$. Then there exists an oriented geodesic arc γ based at p such that the following conditions hold:*

- *The initial and terminal tangent vectors of γ at p are δ/L–close to $\vec{v}_1$ and $\vec{v}_2$, respectively.*
- *The length of γ is δ/L–close to L.*
- *The angle between $\vec{n}$ and the parallel transport of $\vec{n}$ to p along γ is δ/L–close to 0.*
- *As a closed curve in M, $[\gamma] = 0 \in H_1(M; \mathbb{Z})$.*

Proof Choose two auxiliary unit vectors $\vec{v}_1', \vec{v}_2' \in T_p^1 M$ orthogonal to $\vec{n}$ such that the angles $\theta_1 = \Theta(\vec{v}_1, \vec{v}_2')$, $\theta_2 = \Theta(\vec{v}_2', -\vec{v}_1')$ and $\theta_3 = \Theta(-\vec{v}_1', \vec{v}_2)$ are all in $[0, \pi/3]$.

Now we take $L_1(\delta, M) = 4L_0(\delta/100, M)$. By applying the connection principle (Lemma 3.2), we get two oriented geodesic arcs α and β in M based at p such that the following conditions hold:

- The initial and terminal tangent vectors of α are $(\delta/100)/(L/4) = \delta/25L$–close to $\vec{v}_1$ and $\vec{v}_1'$, respectively, while the initial and terminal tangent vectors of β are $\delta/25L$–close to $-\vec{v}_2$ and $\vec{v}_2'$, respectively.
- The lengths of α and β are $\delta/25L$–close to $L/4$ and $L/4 + \sum_{i=1}^{3} \log\sec(\theta_i/2)$, respectively.
- The angle between $\vec{n}$ and the parallel transport of $\vec{n}$ to p along α is $\delta/25L$–close to 0, and so is the angle between $\vec{n}$ and the parallel transport of $\vec{n}$ to p along β.

Here $\log\sec(\theta/2)$ shows up since $I(\theta) = 2\log\sec(\theta/2)$ is the inefficiency constant given by the exterior angle θ (see [7, Section 4.1]).

Let γ be the geodesic arc homotopic to $\alpha\beta\bar{\alpha}\bar{\beta}$ with respect to the base point p. By applying the estimates in [9, Lemma 4.8] and Remark E on page 2295, we have that γ satisfies the first three conditions in the statement. The fourth condition clearly holds since γ is a commutator in $\pi_1(M, p)$. $\square$

Remark F By Remark E, it is easy to check that

$$10^4 L_1(\delta, M) e^{-L_1(\delta, M)/64} < \delta \quad \text{and} \quad L_1(\delta, M) > 10^4.$$

Let

$$\theta_0 = \arcsin\sqrt{\tfrac{5-\sqrt{5}}{10}} \approx 0.1762\pi \approx 31.7175^\circ.$$

Let $\vec{n}_1$ and $\vec{n}_2$ be the normal vectors of two adjacent faces of the Euclidean regular dodecahedron (pointing outside); the angle between $\vec{n}_1$ and $\vec{n}_2$ equals $2\theta_0$.

Now we can do the first step of our construction. For any closed oriented hyperbolic 3–manifold M, take an arbitrary point $p \in M$ and choose an orthonormal frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $T_p M$ such that $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ coincides with the orientation of M (ie $\vec{e}_3 = \vec{e}_1 \times \vec{e}_2$). Then, for any $\vec{v} \in T_p M$, $\vec{v}$ can be written as a linear combination of $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, as $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$. In this case, we will use $\vec{v} = (v_1, v_2, v_3)$ to denote the coordinate of $\vec{v}$.

Let

$$\vec{v}_a = (\sin\theta_0, 0, \cos\theta_0), \quad \vec{v}_b = (\sin\theta_0, 0, -\cos\theta_0),$$
$$\vec{v}_c = (0, \cos\theta_0, \sin\theta_0), \quad \vec{v}_d = (0, -\cos\theta_0, \sin\theta_0),$$
$$\vec{v}_e = (\cos\theta_0, \sin\theta_0, 0), \quad \vec{v}_f = (-\cos\theta_0, \sin\theta_0, 0),$$

be six unit vectors in $T^1_p M$. These six vectors together with their negatives form the twelve normal vectors of the twelve faces of a Euclidean regular dodecahedron in $T_p M$. Figure 3 shows a picture of these vectors, with $\vec{e}_1$ normal to the front face of the cube, $\vec{e}_2$ normal to the right face, and $\vec{e}_3$ normal to the top face. All these three vectors are pointing outside of the cube.

Now we apply Lemma 3.3 to construct six closed oriented geodesic arcs a, b, c, d, e, f based at p, which correspond to the six generators in the group presentation (1) (we abuse the notation between generators and geodesic arcs based at p). For our application of Lemma 3.3, the constant $\delta > 0$ will be some very small number and $L > L_1(\delta, M)$ will be some very large number which will be determined later. These six geodesic arcs are constructed according to the data in Table 1. Here $\vec{v}_1$, $\vec{v}_2$ and $\vec{n}$ in Table 1 are the input data of Lemma 3.3, and geodesic arcs a, b, c, d, e, f are the output data. $\vec{n}$ is called an *almost-normal vector* of the corresponding geodesic arc at its initial and terminal points.

geodesic arc	$\vec{v}_1$	$\vec{v}_2$	$\vec{n}$
a	$\vec{v}_a$	$\vec{v}_b$	$(0,1,0)$
b	$\vec{v}_b$	$\vec{v}_a$	$(0,1,0)$
c	$\vec{v}_c$	$\vec{v}_d$	$(1,0,0)$
d	$\vec{v}_d$	$\vec{v}_c$	$(1,0,0)$
e	$\vec{v}_e$	$\vec{v}_f$	$(0,0,1)$
f	$\vec{v}_f$	$\vec{v}_e$	$(0,0,1)$

Table 1: Construction of geodesic arcs

Given this construction, we have the following estimate for closed geodesics in M. These closed geodesics are related to the words in the group presentation (1).

Lemma 3.4 *For any positive integer n, the concatenation of n geodesic arcs corresponding to a^n is homotopic to a null-homologous closed geodesic in M, with complex length $25n\delta/L$–close to $n(L - 2\log\csc\theta_0)$. The same statement also holds for b, c, d, e, f.*

For each of the twelve relators in the group presentation (1), the concatenation of the four corresponding geodesic arcs is homotopic to a null-homologous closed geodesic in M, with complex length $100\delta/L$–close to $4(L-2\log\csc\theta_0)$.

Proof We will only prove the lemma for a^n and $acb^{-1}c^{-1}$; the proofs for the other words are exactly the same. These closed geodesics are clearly null-homologous in M, since all the geodesic arcs a,b,c,d,e,f based at p are null-homologous, which is proved in Lemma 3.3.

(1) Proof for a^n By the construction of the geodesic arc a, the initial and terminal tangent vectors of a at p are δ/L–close to $\vec{v}_a$ and $\vec{v}_b$, respectively, and the parallel transportation of $(0,1,0)$ along a is δ/L–close to $(0,1,0)$.

Then [9, Lemma 4.8] and Remark F on page 2296 imply the desired estimate.

(2) Proof for $acb^{-1}c^{-1}$ In this case, the almost-normal vectors in Table 1 of adjacent oriented geodesic arcs, say a and c, do not coincide. So we need to choose new almost-normal vectors at the initial and terminal points of a, c, b^{-1} and c^{-1}, respectively, such that adjacent geodesic arcs share the same almost-normal vector at their intersection point.

Since the parallel transport of $\vec{v}_a=(\sin\theta_0,0,\cos\theta_0)$ and $(0,1,0)$ to p along a are δ/L–close to $\vec{v}_b=(\sin\theta_0,0,-\cos\theta_0)$ and $(0,1,0)$, respectively, the parallel transport of

$$\vec{x}_a := \left(\tfrac{\sqrt{5}+1}{4},\tfrac{\sqrt{5}-1}{4},-\tfrac{1}{2}\right)$$

to p along a is $4\delta/L$–close to

$$\vec{x}_b := \left(-\tfrac{\sqrt{5}+1}{4},\tfrac{\sqrt{5}-1}{4},-\tfrac{1}{2}\right).$$

This estimate holds because

$$\vec{x}_a=\sqrt{\tfrac{5+\sqrt{5}}{8}}(0,1,0)\times\vec{v}_a+\tfrac{\sqrt{5}-1}{4}(0,1,0)$$

and

$$\vec{x}_b=\sqrt{\tfrac{5+\sqrt{5}}{8}}(0,1,0)\times\vec{v}_b+\tfrac{\sqrt{5}-1}{4}(0,1,0).$$

Here $\vec{x}_a$ is orthogonal to both $-\vec{v}_c$ and $\vec{v}_a$, while $\vec{x}_b$ is orthogonal to both $\vec{v}_b$ and $\vec{v}_c$.

The same argument applied to c, b^{-1} and c^{-1}, respectively, gives these estimates:

- The parallel transport of $(-\frac{1}{4}(\sqrt{5}+1),\frac{1}{4}(\sqrt{5}-1),-\frac{1}{2})$ to p along c is $4\delta/L$–close to $(-\frac{1}{4}(\sqrt{5}+1),\frac{1}{4}(\sqrt{5}-1),\frac{1}{2})$.
- The parallel transport of $(-\frac{1}{4}(\sqrt{5}+1),\frac{1}{4}(\sqrt{5}-1),\frac{1}{2})$ to p along $b^{-1)}$ is $4\delta/L$–close to $(\frac{1}{4}(\sqrt{5}+1),\frac{1}{4}(\sqrt{5}-1),\frac{1}{2})$.

- The parallel transport of $(\frac{1}{4}(\sqrt{5}+1), \frac{1}{4}(\sqrt{5}-1), \frac{1}{2})$ to p along $c^{-1})$ is $4\delta/L$–close to $(\frac{1}{4}(\sqrt{5}+1), \frac{1}{4}(\sqrt{5}-1), -\frac{1}{2})$.

Now each geodesic arc of the piecewise geodesic path $acb^{-1}c^{-1}$ is equipped with new almost-normal vectors at the initial and terminal points such that adjacent oriented arcs share the same almost-normal vector at their intersection point. So we can apply [9, Lemma 4.8] and Remark F on page 2296 again to get the desired estimate. □

From the proof of Lemma 3.4 and some elementary computation, we have the following simple but important observation.

Lemma 3.5 *For any two adjacent oriented geodesic arcs in $acb^{-1}c^{-1}$, the tangent vector of the second geodesic arc at its initial point is $2\delta/L$–close to the $\pi-2\theta_0$ clockwise rotation (about the new common almost-normal vector given in Lemma 3.4) of the tangent vector of the first geodesic arc at its terminal point. The same statement also holds for the other relators: $adb^{-1}d^{-1}, eaf^{-1}a^{-1}, ebf^{-1}b^{-1}, ced^{-1}e^{-1}, cfd^{-1}f^{-1}$.* □

This property will play an important role in our construction, since it guarantees that the corresponding four-cornered annulus in X' can be immersed into M with an almost totally geodesic four-cornered annulus as its image, such that one boundary component of the annulus is mapped to the piecewise-geodesic closed path $acb^{-1}c^{-1}$, and the other boundary component is mapped to the corresponding closed geodesic in M.

The reader can imagine the case when the picture appears exactly on a hyperbolic surface (totally geodesic subsurface). If those four "turns" are not all left turns (or all right turns), the four-cornered annulus cannot be immersed into the hyperbolic surface, since the piecewise-geodesic closed path and the closed geodesic may intersect each other.

Now let

$$\delta = \frac{\epsilon}{400} \quad \text{and} \quad L = \cosh^{-1}\frac{\cosh R + \cos^2\theta_0}{\sin^2\theta_0} = R + 2\log\csc\theta_0 + O(e^{-R})$$

be our input data. Here $\epsilon > 0$ is small enough and $R > 0$ is large enough so that Corollary 2.11, Proposition 4.1 and Theorem 4.4 hold for (R,ϵ). Moreover, we also require that Lemma 3.3 and Remark F hold for (L,δ). Then the second part of Lemma 3.4 implies that the twelve closed geodesics in M corresponding to the twelve relators in the group presentation (1) lie in $\mathbf{\Gamma}_{4R,\epsilon/4R}$.

By our choice of L, if the four geodesic arcs lie on a totally geodesic subsurface and the turning angles are all exactly $\pi-2\theta_0$, then the corresponding closed geodesic has complex length exactly $4R$.

Now we are ready to construct an immersed π_1–injective 2–complex $j\colon Z \looparrowright M$, using the hint of the handle structure of N. We divide the construction into a few steps.

Step I Recall that, by the end of Section 3.1, we have constructed a 2–complex X' in M_0 which consists of one 0–cell, six 1–cells, twelve circles, six four-cornered annuli and six one-cornered annuli. X' pulls back to a 2–complex Y' in N; here Y' a d–sheeted cover of X' and is a deformation retract of $N^{(1)}$. Moreover, $Y' \cap \partial N^{(1)}$ is exactly the union of k disjoint circles on $\partial N^{(1)}$ where the 2–handles of N are attached.

We can construct an immersion $j'\colon X' \looparrowright M$ for any closed hyperbolic 3–manifold M in the following way.

Let the 0–cell of X' be mapped to the point $p \in M$ which we have already chosen. Let the six oriented 1–cells of X' (corresponding to the six generators a, b, c, d, e, f in the group presentation (1)) be mapped to the six corresponding oriented geodesic arcs in M based at p, by applying Lemma 3.3 to the data in Table 1. Let the twelve circles in X' be mapped to twelve closed geodesics in M, which are homotopic to the corresponding concatenation of geodesic arcs. Six of them correspond to the first six relators in the group presentation (1), and the remaining six of them correspond to a quarter of the remaining six relators. For example, one of them is mapped to the closed geodesic in M which is homotopic to the closed geodesic arc e, instead of e^4. Lemma 3.4 implies that the first six closed geodesics lie in $\mathbf{\Gamma}_{4R,\epsilon/4R}$, and the remaining six lie in $\mathbf{\Gamma}_{R,\epsilon/R}$.

The four-cornered annuli are mapped to almost totally geodesic subsurfaces in M (by Lemma 3.5). More precisely, the four-cornered annuli (one-cornered annuli) in X' may be subdivided in the following way. First add an arc from each corner point to its opposite circle, to divide the annuli into four (one) 4–gons. Then add a diagonal to each 4–gon, which divides the 4–gon into two triangles.

Then the arcs from the corner points to the opposite circle are mapped to the geodesic arcs in the right homotopic class with initial point p and perpendicular to the corresponding closed geodesics in M. For the remaining part of the cornered annuli, j' maps arcs to geodesic arcs and map triangles to totally geodesic triangles.

Step II The immersion $j'\colon X' \looparrowright M$ induces an immersion $j''\colon Y' \looparrowright M$ by composition with the finite-sheeted cover $Y' \to X'$; here Y' consists of d 0–cells, $6d$ 1–cells, k circles and k cornered annuli. Moreover, Theorem 1.6 and Lemma 3.4 imply that all the circles in Y' are mapped to closed geodesics in $\mathbf{\Gamma}_{4R,\epsilon/4R}$, $\mathbf{\Gamma}_{2R,\epsilon/2R}$ or $\mathbf{\Gamma}_{R,\epsilon/R}$. We denote these circles in Y' by $C_1, \ldots, C_k$ and their images under j'' by $\gamma_1, \ldots, \gamma_k$. We give each C_i an arbitrary orientation, which induces an orientation on γ_i. Lemma 3.4 also implies that γ_i is null-homologous in M for $i = 1, \ldots, k$.

Step III For each circle $C_i \subset Y'$ which lies in a cornered annulus, in the subdivision of Y' induced by the subdivision of X', there are a few (one, two or four) arcs from the corner points to C_i. By the construction of $j'\colon X' \looparrowright M$, j'' maps these arcs to geodesic arcs perpendicular to γ_i, and we choose one such geodesic arc for each γ_i and denote it by α_i. We denote the intersection between $j''(\alpha_i)$ and γ_i by p_i, and denote the unit tangent vector of $j''(\alpha_i)$ at p_i by $\vec{u}_i$ (pointing to p_i). Let $\vec{v}_i$ be the unit normal vector of γ_i which is the $1+\pi i$ shearing of $\vec{u}_i$ along γ_i.

Now we apply Corollary 2.11 to $L_i = \gamma_i \sqcup \gamma_i$, the union of two copies of γ_i. Here $L_i \in \mathbb{Z}\mathbf{\Gamma}_{r_i R, \epsilon/r_i R}$ for $r_i \in \{1,2,4\}$ when $C_i \subset Y'$ is one boundary component of an r_i–cornered annulus. Since $\sigma(\gamma_i) \in \mathbb{Z}_2$, we have $\sigma(L_i) = \bar{0} \in \mathbb{Z}_2$. So Corollary 2.11 implies that L_i bounds an immersed oriented $(r_i R, \epsilon/r_i R)$–almost totally geodesic subsurface $f_i\colon S_i \looparrowright M$, such that the two pairs of pants containing the two boundary components of S_i both have a foot $\epsilon/(r_i R)^2$–close to $\vec{v}_i$. Since we can suppose S_i does not have closed surfaces as its connected components, there are two possibilities: either S_i is a connected surface with two boundary components or the union of two connected surfaces, each with one boundary component. Actually, the second possibility does not always happen. For example, for the closed geodesic γ_j corresponding to e^4 (which exists by [6, Theorem 1.1]), its canonical lifting goes around some closed curve in $\mathrm{SO}(M)$ four times, then does a 2π–counterclockwise rotation about some vector. So $\sigma(\gamma_j) = \bar{1}$, and Corollary 2.11 implies that γ_j itself does not bound an immersed $(4R, \epsilon/4R)$–almost totally geodesic subsurface in M. So S_j is a connected surface in this case.

Step IV By applying the surgery argument as in [14, Section 3.1], we can suppose each S_i satisfies the following property: if we endow S_i with the hyperbolic metric such that each curve in the pants decomposition of S_i has length exactly $r_i R$, and the shearing parameter is exactly 1, then any essential arc $(I, \partial I) \to (S_i, \partial S_i)$ has length greater than $R/2$.

Now we can define our desired 2–complex Z, which is the quotient space of the union of two copies of Y' (denoted by Y_1' and Y_2') and $\{S_i\}_{i=1}^k$. For each S_i, the two boundary components of S_i are pasted to the two copies of C_i in Y_1' and Y_2', respectively, by orientation-preserving homeomorphisms. Then $j''\colon Y' \looparrowright M$ and $f_i\colon S_i \looparrowright M$ give the desired immersion $j\colon Z \looparrowright M$, and we will show that the induced map $j_*\colon \pi_1(Z) \to \pi_1(M)$ is injective.

In the construction of the desired degree-2 map, the union of Y_1' and Y_2' will correspond to $N^{(1)}$, and the S_i will correspond to the 2–handles of N. Our construction of $j\colon Z \looparrowright M$ is complete now.

Let $\rho_0\colon \pi_1(Z)\to \mathrm{PSL}_2(\mathbb{C})$ be the representation such that all the ϵ/R– and ϵ/R^2–closeness statements regarding $j_*\colon \pi_1(Z)\to\pi_1(M)\subset \mathrm{PSL}_2(\mathbb{C})$ are replaced by exact (0–closeness) statements. As an algebraic object, $\rho_0\colon \pi_1(Z)\to \mathrm{PSL}_2(\mathbb{C})$ is accompanied by a map $i_0\colon \widetilde{Z}\to\mathbb{H}^3$ from the universal cover $p\colon \widetilde{Z}\to Z$ of Z to $\mathbb{H}^3$. The map $i_0\colon \widetilde{Z}\to\mathbb{H}^3$ is a geometric object which carries the algebraic information of $\rho_0\colon \pi_1(Z)\to \mathrm{PSL}_2(\mathbb{C})$. The image $i_0(\widetilde{Z})$ is a union of totally geodesic subsurfaces in $\mathbb{H}^3$, pasted along a union of geodesic arcs. More precisely, $i_0\colon \widetilde{Z}\to\mathbb{H}^3$ is defined by the following conditions, which also gives the precise definition of $\rho_0\colon \pi_1(Z)\to \mathrm{PSL}_2(\mathbb{C})$.

Construction 3.6 (a) Each 0–cell x in $\widetilde{Z}$ is associated to an orthonormal frame $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ at $i_0(x)$ with respect to the orientation of $\mathbb{H}^3$.

(b) Each 1–cell t in $\widetilde{Z}$ is endowed with an orientation such that the oriented 1–cell $p(t)$ in Z corresponds to one of the oriented 1–cells a,b,c,d,e,f in X'. Suppose t travels from one 0–cell x to another 0–cell y in $\widetilde{Z}$, and let $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ and $\{\vec{e}_1{}',\vec{e}_2{}',\vec{e}_3{}'\}$ be the two orthonormal frames at $i_0(x)$ and $i_0(y)$, respectively. Then the following conditions hold:

- $i_0(t)$ is a geodesic arc from $i_0(x)$ to $i_0(y)$ with length equal to L.
- The tangent vectors of $i_0(t)$ at $i_0(x)$ and $i_0(y)$ are exactly the corresponding vectors $\vec{v}_1$ and $\vec{v}_2$ in Table 1, respectively (under the coordinate given by orthonormal frames $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ and $\{\vec{e}_1{}',\vec{e}_2{}',\vec{e}_3{}'\}$).
- The parallel transport of $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ to $i_0(y)$ along $i_0(t)$ is equal to the counter-clockwise $\pi-2\theta_0$ rotation of $\{\vec{e}_1{}',\vec{e}_2{}',\vec{e}_3{}'\}$ about a vector $\vec{n}\in T^1_{i_0(q)}(M)$. Here

$$\vec{n}=\begin{cases} \vec{e}_2{}' & \text{if } p(t) \text{ corresponds to } a \text{ in } X',\\ -\vec{e}_2{}' & \text{if } p(t) \text{ corresponds to } b \text{ in } X',\\ \vec{e}_1{}' & \text{if } p(t) \text{ corresponds to } c \text{ in } X',\\ -\vec{e}_1{}' & \text{if } p(t) \text{ corresponds to } d \text{ in } X',\\ \vec{e}_3{}' & \text{if } p(t) \text{ corresponds to } e \text{ in } X',\\ -\vec{e}_3{}' & \text{if } p(t) \text{ corresponds to } f \text{ in } X'.\end{cases} \tag{2}$$

(c) Any component of the preimage of a circle in the pants decomposition of some S_i is mapped to a bi-infinite geodesic in $\mathbb{H}^3$.

(d) Any component t of the preimage of a four-cornered annulus in Z is mapped to a totally geodesic subsurface in $\mathbb{H}^3$ with two boundary components. One of its boundary component is a concatenation of geodesic arcs of length L, and the other boundary component is a bi-infinite geodesic. Moreover, these two boundary components share the same limit points on $\partial\mathbb{H}^3=S^2_\infty$.

(e) For any arc in $\widetilde{Z}$ which is a component of the preimage of either a seam in a pair of pants, or an arc in a cornered annulus from a corner point to the opposite circle, its image under i_0 is a geodesic arc perpendicular to the corresponding bi-infinite geodesics.

(f) For any circle C in the oriented surface S_i which is the boundary of a pair of pants Π, we have $\boldsymbol{hl}_{\Pi}(C) = r_i R/2$, which means that the following statement holds. Take any bi-infinite line $\beta \subset \widetilde{Z}$ which is a component of $p^{-1}(C)$, and any two arcs α and α' in $\widetilde{Z}$ that intersect with β and project to two seams of $\Pi \subset S_i$ such that there is no other such arc between α and α' along β. Then the tangent vector of $i_0(\alpha)$ is the $r_i R/2$–translation of the tangent vector of $i_0(\alpha')$ along $i_0(\beta)$ (with the proper orientation).

(g) For any oriented circle C shared by two pants Π and Π' in S_i such that Π lies to the left of C, we have $s(C) = 1$, which means that the following statement holds. Take any bi-infinite line $\beta \subset \widetilde{Z}$ which is a component of $p^{-1}(C)$, and any arc α in $\widetilde{Z}$ that intersects with β and projects to a seam of Π. Let α' be the arc that intersects with β and projects to a seam of Π' such that $i_0(\alpha')$ is the nearest such arc from $i_0(\alpha)$. Then the tangent vector of $i_0(\alpha')$ is the $(1+\pi i)$–translation of the tangent vector of $i_0(\alpha)$ along $i_0(\beta)$.

(h) For any oriented boundary component C_i of S_i which is the cuff of a pair of pants $\Pi \subset S_i$, in Step III of the construction of $j\colon Z \to M$, we have chosen an arc-α_i which travels from some 0–cell of Z to C_i. Take any bi-infinite line $\beta \subset \widetilde{Z}$ which is a component of $p^{-1}(C_i)$, and any arc α in $\widetilde{Z}$ that intersects with β and projects to α_i. Let α' be the arc that intersects with β and projects to a seam of Π such that $i_0(\alpha')$ is the nearest such arc from $i_0(\alpha)$. Then the tangent vector of $i_0(\alpha')$ is the $(1+\pi i)$–translation of the tangent vector of $i_0(\alpha)$ along $i_0(\beta)$.

(i) Each component of the preimage of a pair of pants in S_i is mapped to a totally geodesic subsurface in $\mathbb{H}^3$.

The existence of such a map $i_0\colon \widetilde{Z} \to \mathbb{H}^3$ can be shown by a developing argument. We can start with any 0–cell x of $\widetilde{Z}$ and let $i_0(x)$ be an arbitrary point in $\mathbb{H}^3$, and we take an arbitrary orthonormal frame at $i_0(x)$, which is the frame as in condition (a). Then the image of the 1–cells adjacent to x can be determined by condition (b). Now we can do this construction inductively to define i_0 on the component of the union of 0–cells and 1–cells of $\widetilde{Z}$ containing x, and we denote this component by Z_0. Then conditions (c) and (d) give bi-infinite geodesics adjacent to $i_0(Z_0)$, which are the images under i_0 of the components of the preimage of circles in Z adjacent to Z_0. Conditions (f), (g) and (h) give the images of other components of the preimage of circles in Z inductively (from bi-infinite lines which are near x to farther lines). Then we can do this construction inductively to further 0–cells, 1–cells, etc, to construct the desired map $i_0\colon \widetilde{Z} \to \mathbb{H}^3$.

Note that Lemma 3.5 is important here to guarantee that condition (d) holds here. If the "all left/right turns" condition does not hold, then the image of the preimage of a four-cornered annulus under i_0 will be a concatenation of infinitely many long and thin totally geodesic polygons, whose boundary has different geometric behavior.

Then $\rho_0\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$ can be defined by $\rho_0(g)(i_0(z)) = i_0(g(z))$ for any $z \in \widetilde{Z}$ and any $g \in \pi_1(Z)$. In this case, the frames at the images of 0–cells are also $\pi_1(Z)$–equivariant.

Although this construction looks complicated, it is simply describing a 2–complex $\widetilde{Z}$ in $\mathbb{H}^3$ which is the union of totally geodesic subsurfaces along a 1–dimensional complex consisting of long geodesic arcs. The map $i_0\colon \widetilde{Z} \to \mathbb{H}^3$ will serve as our standard model of its algebraic counterpart $\rho_0\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$. We will study geometric properties of the deformations of i_0, which gives us algebraic information about small deformations of ρ_0, in particular $j_*\colon \pi_1(Z) \to \pi_1(M) \subset \mathrm{PSL}_2(\mathbb{C})$.

The map $i_0\colon \widetilde{Z} \to \mathbb{H}^3$ induces a path metric on $\widetilde{Z}$, which is denoted by d_0. Then an argument in hyperbolic geometry gives the following result, which will be shown in Section 4.

Proposition 3.7 *When R is large enough, $i_0\colon (\widetilde{Z}, d_0) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ is a embedding and also a quasi-isometric embedding. In particular, $\rho_0\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$ is an injective map.*

The following result claims that a small deformation of ρ_0 is still π_1–injective. It is shown by geometric estimates on the associated map $i\colon \widetilde{Z} \to \mathbb{H}^3$ of $j_*\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$.

Theorem 3.8 *When $\epsilon > 0$ is small enough and $R > 0$ is large enough, the map $j\colon Z \looparrowright M$ induces an injective map $j_*\colon \pi_1(Z) \hookrightarrow \pi_1(M)$, and $j_*(\pi_1(Z)) \subset \pi_1(M)$ is a geometrically finite subgroup. Moreover, $\mathbb{H}^3/j_*(\pi_1(Z))$ is homeomorphic to $\mathbb{H}^3/\rho_0(\pi_1(Z))$ with respect to their orientations.*

The intuitive explanation for Theorem 3.8 is quite simple. Since j_* is a small deformation of ρ_0, $j_*(\pi_1(Z))$ should share geometrical and topological properties with $\rho_0(\pi_1(Z))$ (as in [8] and [14]). We will give a more precise description of the deformations of $\rho_0\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$ in Section 4, and the proof of Theorem 3.8 will also be delayed to Section 4. The main reason is that, although this result is very intuitive, the geometric estimation is technical. We do not want these technicalities to distract the readers from the main idea of the proof of Theorem 1.1.

3.3 Construction of domination

In this section we will prove Theorem 1.1 by assuming Proposition 3.7 and Theorem 3.8. Theorem 1.1 claims that, for any closed oriented hyperbolic 3–manifold M and any closed oriented 3–manifold N, M admits a finite cover M' such that there exists a degree-2 map $f\colon M' \to N$.

To prove Theorem 1.1, we first need to figure out the topology of $\mathbb{H}^3/\rho_0(\pi_1(Z))$.

Let H be the oriented handlebody which is homeomorphic to $N^{(1)}$ (the union of 0– and 1–handles of N), and we fix such an orientation-preserving identification. There are k disjoint annuli $A_1,\dots,A_k$ in $\partial N^{(1)}$ where the k 2–handles of N are attached. We abuse notation and still use $A_1,\dots,A_k$ to denote the corresponding k disjoint annuli on ∂H. Now we take two copies of H and denote them by H_1 and H_2. We also denote the corresponding annuli of A_i on ∂H_1 and ∂H_2 by $A_{i,1}$ and $A_{i,2}$, respectively. Here H_j and $A_{i,j}$ are all endowed with the identical orientations as H and A_i.

Now we take k I–bundles $S_1\times I,\dots,S_k\times I$ over surfaces; these S_i are the oriented surfaces constructed in Step III of the construction of Z. Since the boundary of S_i has two oriented components $C_{i,1}$ and $C_{i,2}$, the boundary of $S_i\times I$ contains two disjoint annuli $C_{i,1}\times I$ and $C_{i,2}\times I$, such that the product orientations on $C_{i,1}\times I$ and $C_{i,2}\times I$ coincide with their induced orientations from the orientation of $S_i\times I$.

Let $j_i\colon A_{i,1}\to A_{i,2}$ be the orientation-preserving homeomorphism of the two annuli give by the identification between H_1 and H_2. Let $k_i\colon C_{i,1}\times I\to C_{i,2}\times I$ be an orientation-preserving homeomorphism which also preserves the orientation of the I–factor. For each $i\in\{1,\dots,k\}$, we take an orientation-reversing homeomorphism $\phi_i\colon C_{i,1}\times I\cup C_{i,2}\times I\to A_{i,1}\cup A_{i,2}$ which maps $C_{i,1}\times I$ to $A_{i,1}$ and maps $C_{i,2}\times I$ to $A_{i,2}$, such that
$$j_i\circ\phi_i|_{C_{i,1}\times I}=\phi_i|_{C_{i,2}\times I}\circ k_i.$$
Now let
$$K=(H_1\cup H_2)\cup_{\{\phi_i\}_{i=1}^k}\Big(\bigcup_{i=1}^k S_i\times I\Big).$$

Then by the construction of $i_0\colon \widetilde{Z}\to\mathbb{H}^3$ in Construction 3.6, it is easy to see that $\mathbb{H}^3/\rho_0(\pi_1(Z))$ is homeomorphic to $\operatorname{int}(L)$ with respect to their orientations. Actually, each boundary component of K is incompressible.

Let $Z^{(1)}$ be the union of 0–cells and 1–cells in Z, let $\widetilde{Z}^{(1)}$ be the preimage of $Z^{(1)}$ in $\widetilde{Z}$, and let $N(\widetilde{Z}^{(1)})$ be a small neighborhood of $i_0(\widetilde{Z}^{(1)})$ in $\mathbb{H}^3$. Then an important point is that $N(\widetilde{Z}^{(1)})/\rho_0(\pi_1(Z))$ is homeomorphic to two copies of $N^{(1)}$, and

$(\partial N(\widetilde{Z}^{(1)})) \cap (i_0(\widetilde{Z}))/\rho_0(\pi_1(Z))$ corresponds to the disjoint union of circles on the two copies of $\partial N^{(1)}$ where the 2–handles of N are attached. This observation holds by the geometry of the regular icosahedron, and may not hold if we start with an arbitrary handle structure of N.

Here is a simple lemma on constructing degree-nonzero maps.

Lemma 3.9 *If M' is a closed oriented 3–manifold which contains a codimension-0 submanifold K such that there is a proper degree-d map $h\colon (K, \partial K) \to (N^{(2)}, \partial N^{(2)})$, then there is a degree-d map $f\colon M' \to N$, which is an extension of h.*

Proof Since each component of $N \setminus N^{(2)}$ is a 3–handle, each has the 2–sphere as its boundary. So, for each component Q of $M' \setminus \mathrm{int}(K)$, $h|_{\partial Q}$ maps ∂Q to the disjoint union of a few 2–spheres. Suppose Q has q boundary components. Let $[q]$ be the topological space consists of q points with the discrete topology, and let $\bigcup_{i=1}^{q} B_i^3$ be the disjoint union of q 3–balls. Then we use E_q to denote the mapping cone of $[q] \to \bigcup_{i=1}^{q} B_i^3$ which maps the i^{th} point $i \in [q]$ to the center of B_i^3.

Now we can define a map $e\colon Q \to E_q$. On a collar neighborhood of ∂Q, $e|_{\partial Q \times I}$ is defined by

$$\partial Q \times I \xrightarrow{h|_{\partial Q \times I}} \bigcup_{i=1}^{q} S^2 \times I \to \bigcup_{i=1}^{q} B^3.$$

For the remaining part of Q, e maps it to the graph in E_q, which is the cone over q points. Then there is an obvious map from E_q to N, which maps the i^{th} 3–ball in E_q to the corresponding 3–handle in N (whose boundary 2–sphere is the image of the i^{th} boundary component of Q) by homeomorphism, and maps the $(q+1)$–vertex graph in E_q to a graph in N which is the union of q paths.

The composition of these two maps gives $f|_Q\colon Q \to N$, and $f|_K$ is defined to be equal to h. So we get a map $f\colon (M', K) \to (N, N^{(2)})$ such that $f|_K\colon K \to N^{(2)}$ is a proper degree-d map and $f(M' \setminus K) \cap N^{(2)}$ is a 1–dimensional subcomplex in $N^{(2)}$. So, for any generic point $m \in \mathrm{int}(N^{(2)})$, $f^{-1}(m) = h^{-1}(m)$, and the sums of local degrees are both equal to $\deg(h) = d$. So $f\colon M' \to N$ is a degree-d map. □

Now we are ready to prove Theorem 1.1:

Proof For any closed hyperbolic 3–manifold M, we have already constructed an immersed π_1–injective 2–complex $j\colon Z \looparrowright M$ such that $\mathbb{H}^3/j_*(\pi_1(Z))$ is homeomorphic to $\mathbb{H}^3/\rho_0(\pi_1(Z))$, which is homeomorphic to $\mathrm{int}(K)$, with respect to their orientations. Let $q\colon \overline{M} \to M$ be the infinite cover of M such that $q_*(\pi_1(\overline{M})) = j_*(\pi_1(Z))$.

Then $\overline{M}$ is homeomorphic to $\mathrm{int}(K)$ with respect to their orientations. By shrinking the boundary of K into its interior a little bit, we get an embedding $\iota\colon K \hookrightarrow \overline{M}$, and $\iota(K)$ is a compact subset of $\overline{M}$.

Building on Wise's work [18], Agol showed that the groups of hyperbolic 3–manifolds are LERF [1]. By Scott's equivalent formulation of LERF [13], there is an intermediate finite cover $M' \to M$ of $\overline{M} \to M$ such that $\iota(K) \subset \overline{M}$ projects into M' by an embedding. We still use K to denote the projection of $\iota(K)$ in M', since it is homeomorphic to K.

Now we need only to construct a proper degree-2 map $h\colon (K, \partial K) \to (N^{(2)}, \partial N^{(2)})$; then Lemma 3.9 implies that M' 2–dominates N.

Now $h|_{H_1 \cup H_2}$ maps H_1 and H_2 to $N^{(1)}$ by the identification between H and $N^{(1)}$, which is an orientation-preserving trivial 2–sheet cover. Moreover, h maps $A_{i,1}$ and $A_{i,2}$ to A_i by an orientation-preserving homeomorphism.

For $K \setminus \mathrm{int}(H_1 \cup H_2) = \bigcup (S_i \times I)$, each $S_i \times I$ is mapped to the i^{th} 2–handle of $N^{(2)}$. Recall that $S_i \times I$ is pasted to $H_1 \cup H_2$ by an orientation-reversing homeomorphism

$$\phi_i\colon C_{i,1} \times I \cup C_{i,2} \times I \to A_{i,1} \cup A_{i,2},$$

and the i^{th} 2–handle $D_i \times I$ of N is pasted to $N^{(1)}$ by an orientation-reversing homeomorphism $(\partial D_i) \times I \to A_i$. Here we can assume that the orientation of the image of ∂D_i and the orientation of the image of $C_{i,1}$ and $C_{i,2}$ on $\partial N^{(1)}$ coincide. Then there is a proper degree-2 map $p_i\colon S_i \to D_i$ such that $p_i|_{C_{i,1}}\colon C_{i,1} \to \partial D_i$ and $p_i|_{C_{i,2}}\colon C_{i,2} \to \partial D_i$ are orientation-preserving homeomorphisms. Here we can first pinch S_i to the wedge of two discs, then an obvious map to the disc can be defined. Now we define $h|_{S_i \times I}\colon S_i \times I \to D_i \times I$ by $p_i \times I$, which extends the definition of h on $H_1 \cup H_2$ to K. Then $h\colon (K, \partial K) \to (N^{(2)}, \partial N^{(2)})$ is a proper degree-2 map, by considering any point in $\mathrm{int}(H_1 \cup H_2)$. $\square$

4 π_1–injectivity of the immersed 2–complex

In this section we will prove two technical results which are stated in Section 3.2: Proposition 3.7 and Theorem 3.8. These two results are both proved by estimates in hyperbolic geometry. To prove the first result, we need only to do estimations on the standard model (union of totally geodesic subsurfaces in $\mathbb{H}^3$). To prove the second result, we need to do estimations on small deformations of the standard model, which is more complicated. The estimations we did in [14] will be helpful for our proof.

Actually, Proposition 3.7 and Theorem 3.8 also hold for other nicely constructed immersed 2–complexes in closed hyperbolic 3–manifolds, by pasting the immersed almost totally geodesic subsurfaces constructed by Kahn and Markovic and by Liu and Markovic. However, it is difficult to give a clean formulation for the general case, so we only deal with the special case which is necessary for this paper.

4.1 Modified paths and estimates for the standard model

In this subsection we will first introduce the concept of *modified path*, then prove Proposition 3.7.

Let $Z^{(0)}$ be the union of 0–cells in Z, and let $Z^{(1)}$ be the union of 0– and 1–cells in Z. We use $\widetilde{Z}^{(0)}$ and $\widetilde{Z}^{(1)}$ to denote the preimages of $Z^{(0)}$ and $Z^{(1)}$ in $\widetilde{Z}$, respectively. The closure of a component of $\widetilde{Z} \setminus \widetilde{Z}^{(1)}$ will be called a *piece* of $\widetilde{Z}$.

For a path in $\widetilde{Z}$ which lies in a piece of $\widetilde{Z}$, if i_0 maps this path to a geodesic arc in $\mathbb{H}^3$, we simply call it a geodesic arc in $\widetilde{Z}$. For two geodesic arcs in $\widetilde{Z}$ sharing a common point, we measure the angle between these two geodesic arcs by measuring the angle between their images in $\mathbb{H}^3$.

For any two points $x, y \in \widetilde{Z}$, let γ be the shortest path in $(\widetilde{Z}, d_0)$ from x to y. Then γ is a concatenation of geodesic arcs, and each geodesic arc lies in a piece of $\widetilde{Z}$. In the concatenation, γ may contain some relatively short geodesic arcs, which are not convenient for our geometric estimation. So we first introduce the concept of *modified path*, which eliminates such short geodesic arcs, but the length of the modified path does not differ from the length of γ very much.

Let γ be the shortest oriented path in $(\widetilde{Z}, d_0)$ from x to y. If γ does not contain any geodesic arc in $\widetilde{Z}^{(1)}$, let $x_1, x_2, \dots, x_n$ be the sequence of intersection points in $\widetilde{Z}^{(1)} \cap (\gamma \setminus \{x, y\})$, which follows the orientation of γ. If γ contains geodesic arcs in $\widetilde{Z}^{(1)}$, we record only their initial and terminal points in $\widetilde{Z}^{(0)}$. In this case, let $x_1, x_2, \dots, x_n$ be the sequence of transverse intersection points in $\widetilde{Z}^{(1)} \cap (\gamma \setminus \{x, y\})$ and the initial and terminal points of geodesic arcs in $\widetilde{Z}^{(1)} \cap \gamma$, such that this sequence follows the orientation of γ. The sequence $x_1, x_2, \dots, x_n$ is called the *intersection sequence* of γ.

For the intersection sequence $x_1, x_2, \dots, x_n$, we define the *modified sequence* via the following inductive process, beginning with x_1. If $d_0(x_1, x_2) \geq R/64$, then we put x_1 at the first position of the modified sequence, and denote it by y_1. If $d_0(x_1, x_2) < R/64$, let $x_1, x_2, \dots, x_j$ be the maximal consecutive subsequence of $x_1, x_2, \dots, x_n$ such that $d_0(x_i, x_{i+1}) < R/64$ holds for $i = 1, 2, \dots, j-1$. Then the 1–cells containing $x_1, x_2, \dots, x_j$ share a common 0–cell, which we denote by y_1. We put y_1 at the first position of the modified sequence, and call it a *modified point* in this case. Then we turn to consider x_2 in the first case ($d_0(x_1, x_2) \geq R/64$), and consider x_{j+1} in the second case ($d_0(x_1, x_2) < R/64$). By applying the same process inductively, we get the modified sequence $y_1, y_2, \dots, y_m$.

The maximal consecutive subsequence $x_i, \dots, x_{i+j}$ satisfying $d_0(x_{i+k}, x_{i+k+1}) < R/64$ for $k = 0, \dots, j-1$ has length at most 3, since the angle between two 1–cells sharing a 0–cell in $\widetilde{Z}$ is $2\theta_0 \approx 0.3524\pi$, and $3 \cdot 0.3524\pi > \pi$.

By the definition of modified sequence, two adjacent points y_i and y_{i+1} in the modified sequence lie in the same piece of $\widetilde{Z}$. Then the *modified path* γ' of γ is defined to be the concatenation of geodesic arcs connecting x to y_1, y_1 to y_2, ..., y_m to y, and each such geodesic arc lies in a piece of $\widetilde{Z}$.

Now we are ready to prove Proposition 3.7, and we restate the result here. In the following part of this section, we will use d to denote the metric $d_{\mathbb{H}^3}$ on $\mathbb{H}^3$ when it does not cause any confusion.

Proposition 4.1 *When $R > 0$ is large enough, $i_0\colon (\widetilde{Z}, d_0) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ is an embedding, and also a quasi-isometric embedding. In particular, $\rho_0\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$ is an injective map.*

Proof For any $x, y \in \widetilde{Z}$ with $d_0(x, y) \geq R/4$, let γ be the shortest path in $(\widetilde{Z}, d_0)$ from x to y, with the intersection sequence $x_1, x_2, \dots, x_n$. Let γ' be the modified path of γ, and let $y_1, y_2, \dots, y_m$ be the modified sequence.

For any two adjacent points y_i and y_{i+1} in the modified sequence, if the geodesic arc in $\widetilde{Z}$ from y_i to y_{i+1} intersects with the preimage of some S_i, then $d_0(y_i, y_{i+1}) \geq R/2$ by Step IV in the construction of Z (Section 3.2).

Now we suppose the geodesic arc from y_i to y_{i+1} does not intersect with the preimage of any S_i. If both y_i and y_{i+1} are modified points, then they are two distinct 0–cells in $\widetilde{Z}$, so $d_0(y_i, y_{i+1}) \geq R$. If neither of them are modified points, we have $d_0(y_i, y_{i+1}) \geq R/64$. If one of them is a modified point and the other one is not, then $d_0(y_i, y_{i+1}) \geq R/128$ holds, by the hyperbolic geometry on $\mathbb{H}^2$ and since $2\theta_0 \approx 0.3524\pi$. So $d_0(y_i, y_{i+1}) \geq R/128$ always holds.

Now we claim that $\angle y_{i-1} y_i y_{i+1}$ in $\mathbb{H}^3$ has a positive lower bound of, say, $\pi/36$ in $\mathbb{H}^3$. Here "$\angle y_{i-1} y_i y_{i+1}$ in $\mathbb{H}^3$" actually means $\angle i_0(y_{i-1}) i_0(y_i) i_0(y_{i+1})$, but we use this notation to simplify the expression.

Proof of the claim:

Case I When y_i is not a modified point, we have $y_i = x_j$. Since $\angle x_{j-1} x_j x_{j+1} \geq 2\pi/5$ in $\mathbb{H}^3$, we need only to show that $\angle x_{j+1} y_i y_{i+1}$ and $\angle x_{j-1} y_i y_{i-1}$ are both very small in $\mathbb{H}^3$. Here we only show that $\angle x_{j+1} y_i y_{i+1}$ is very small in $\mathbb{H}^3$; the proof for $\angle x_{j-1} y_i y_{i-1}$ is exactly the same.

If y_{i+1} is not a modified point, then $x_{j+1} = y_{i+1}$, so $\angle x_{j+1} y_i y_{i+1} = 0$.

So we suppose y_{i+1} is a modified point, and y_{i+1} corresponds to x_{j+1}. If the shortest path from y_i to y_{i+1} intersects the preimage of some S_i, then $d_0(y_i, y_{i+1}) \geq R/2$. By hyperbolic geometry, we know that $d_0(x_{j+1}, y_{i+1}) \leq R/64 + 1$. So

$$\angle x_{j+1} y_i y_{i+1} \leq e^{-R/4} \leq 8e^{-R/64}$$

in $\mathbb{H}^3$.

If the shortest path from y_i to y_{i+1} does not intersect the preimage of any S_i, then the flattened picture (put two adjacent pieces of $\widetilde{Z}$ in the same hyperbolic plane) is shown in Figure 6 (left). We only draw an Euclidean picture to show the position of these points and geodesics, although the real picture lies in $\mathbb{H}^2$. The following computation in the proof of Case I are done in $\mathbb{H}^2$.

By Construction 3.6, we have $\angle y_i y_{i+1} x_{j+1} \leq \angle x_{j+1} y_{i+1} x_{j+2} = 2\theta_0$. Since y_i is not a modified point while y_{i+1} is, $d(y_i, x_{j+1}) \geq R/64$ and $d(x_{j+1}, x_{j+2}) \leq R/64$. Then by hyperbolic geometry in $\mathbb{H}^2$, we have

$$\frac{\sinh d(x_{j+1}, x_{j+2})}{\sin 2\theta_0} \geq \sinh d(y_{i+1}, x_{j+2}) \tag{3}$$

and

$$\frac{\sinh d(y_i, x_{j+2})}{\sin \angle y_i y_{i+1} x_{j+2}} = \frac{\sinh d(y_{i+1}, x_{j+2})}{\sin \angle x_{j+1} y_i y_{i+1}}. \tag{4}$$

Here $2\theta_0 \leq \angle y_i y_{i+1} x_{j+2} \leq 4\theta_0$.

So

$$\begin{aligned}
\sin \angle x_{j+1} y_i y_{i+1} &= \frac{\sinh d(y_{i+1}, x_{j+2}) \cdot \sin \angle y_i y_{i+1} x_{j+2}}{\sinh d(y_i, x_{j+2})} \\
&\leq \frac{\sinh d(y_{i+1}, x_{j+2})}{2 \sinh d(y_i, x_{j+1}) \cdot \sinh d(x_{j+1}, x_{j+2})} \\
&\leq \frac{\sinh d(x_{j+1}, x_{j+2})}{2 \sinh d(y_i, x_{j+1}) \cdot \sinh d(x_{j+1}, x_{j+2}) \cdot \sin 2\theta_0} \\
&= \frac{1}{\sinh d(y_i, x_{j+1}) \cdot 2 \sin 2\theta_0} \leq 4e^{-R/64}.
\end{aligned} \tag{5}$$

In this case $\angle x_{j+1} y_i y_{i+1} \leq 8e^{-R/64}$ in $\mathbb{H}^3$, and the same estimate also holds for $\angle x_{j-1} y_i y_{i-1}$.

Since the angle between two adjacent pieces of $\widetilde{Z}$ in $\mathbb{H}^3$ is at least $2\pi/5$, and γ is the shortest path in $(\widetilde{Z}, d_0)$, $\angle x_{j-1}x_jx_{j+1} \geq 2\pi/5$ in $\mathbb{H}^3$. So, by the above estimate, when y_i is not a modified point, $\angle y_{i-1}y_iy_{i+1} \geq \pi/5$ in $\mathbb{H}^3$.

Case II When y_i is a modified point, y_i corresponds to a consecutive subsequence x_j, x_{j+1} or x_j, x_{j+1}, x_{j+2} of $x_1, x_2, \dots, x_n$.

We first suppose that both y_{i-1} and y_{i+1} are not modified points; the flattened picture for the first subcase (when y_i corresponds to x_j, x_{j+1}) is shown in Figure 6 (right). So, in this flattened picture, $\angle x_{j-1}y_ix_{j+2} \leq \pi$. Since a path starting at some point in $\widetilde{Z}$ needs to go through at least two other pieces of $\widetilde{Z}$ to return to the original piece and $2\theta_0 \approx 0.3524\pi$, the geometry gives us

$$\angle y_{i-1}y_iy_{i+1} = \angle x_{j-1}y_ix_{j+2} \geq \tfrac{1}{2}(3 \times 0.3524\pi - \pi) = 0.0286\pi > \frac{\pi}{36} \quad \text{in } \mathbb{H}^3.$$

The proof for the second subcase (when y_i corresponds to x_j, x_{j+1}, x_{j+2}) is exactly the same.

If y_{i-1} is a modified point, then $d_0(y_{i-1}, x_{j-1}) \leq R/64 + 1$. Moreover, since y_{i-1} and y_i are two different points in $\widetilde{Z}^{(0)}$, $d_0(y_{i-1}, y_i) \geq R$ holds, and hence $\angle x_{j-1}y_iy_{i-1} \leq e^{-R/2}$. The same argument shows that $\angle x_{j+2}y_iy_{i+1} \leq e^{-R/2}$ when y_{i+1} is a modified point. So in this case

$$\angle y_{i-1}y_iy_{i+1} \geq 0.0286\pi - 2e^{-R/2} > \frac{\pi}{36} \quad \text{in } \mathbb{H}^3,$$

and the proof of the claim is done.

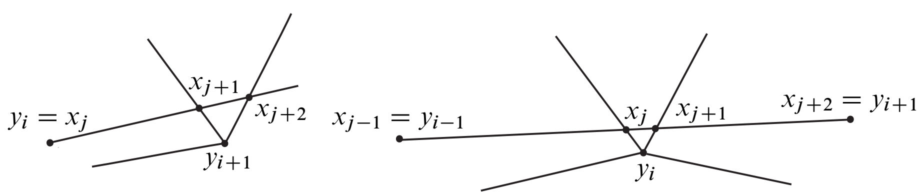

Figure 6: Flattened picture of $\widetilde{Z}$ near modified points

So we know that $d_0(y_i, y_{i+1}) \geq R/128$ and $\angle y_{i-1}y_iy_{i+1} \geq \pi/36$ in $\mathbb{H}^3$. Actually, the same argument implies that $\angle xy_1y_2 \geq \pi/36$ in $\mathbb{H}^3$ when $d_0(x, x_1) \geq R/32$, and similarly for $\angle y_{m-1}y_my$.

Since d_0 is the path metric induced by the hyperbolic metric $d_{\mathbb{H}^3}$,

$$d_{\mathbb{H}^3}(i_0(x), i_0(y)) \leq d_0(x, y). \tag{6}$$

On the other hand, since γ' is a path from x to y,

$$d_0(x,y) \le d_0(x,y_1) + d_0(y_m,y) + \sum_{j=1}^{m-1} d_0(y_j,y_{j+1}). \tag{7}$$

[9, Lemma 4.8(1)] implies that, when $d_0(x,x_1), d_0(y,x_n) \ge R/32$,

$$\begin{aligned} d_{\mathbb{H}^3}(i_0(x),i_0(y)) &\\ &\ge d_0(x,y_1) + d_0(y_m,y) + \sum_{j=1}^{m-1} d_0(y_j,y_{j+1}) - m\big(2\log\big(\sec\tfrac{35\pi}{72}\big)+1\big) \\ &\ge \tfrac{9}{10}\Big(d_0(x,y_1) + d_0(y_m,y) + \sum_{j=1}^{m-1} d_0(y_j,y_{j+1})\Big) \ge \tfrac{9}{10} d_0(x,y). \end{aligned} \tag{8}$$

If $d_0(x,x_1)$ or $d_0(y,x_n)$ is less than $R/32$, the same argument gives

$$d_{\mathbb{H}^3}(i_0(x),i_0(y)) \ge \tfrac{9}{10} d_0(x,y) - \tfrac{1}{8}R. \tag{9}$$

Now we have that $i_0\colon (\widetilde{Z},d_0) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ is a quasi-isometric embedding, so $\rho_0\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$ is an injective map. Actually, the above argument shows that $d(i_0(x),i_0(y)) \ge R/10$ if $d_0(x,y) \ge R/4$. Moreover, $i_0(x) \ne i_0(y)$ holds if $0 < d_0(x,y) < R/4$, by the local geometry of $i_0(\widetilde{Z})$. So we have that $i_0\colon \widetilde{Z} \to \mathbb{H}^3$ is an embedding. □

4.2 Estimates of the deformations of i_0

In this subsection we will prove Theorem 3.8. Actually, $j_*\colon \pi_1(Z) \to \pi_1(M) \subset \mathrm{PSL}_2(\mathbb{C})$ lies in a continuous family of small deformations of $\rho_0\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$, and we will show that all the representations in this family satisfy Theorem 3.8. Each such representation $\rho\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$ is associated to a $\pi_1(Z)$–equivariant partially defined map $i\colon \widetilde{Z} \to \mathbb{H}^3$, which serves as the geometric realization of ρ. We will show that $i\colon \widetilde{Z} \to \mathbb{H}^3$ is a quasi-isometric embedding, which implies that $\rho\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$ is injective.

Now we define a 1–dimensional subcomplex W of Z. The 0–cells and 1–cells of Z and all the circles in the pants decomposition of S_i $(i = 1,2,\dots,k)$ are contained in W. Besides these parts, for each pair of pants in S_i, the three seams are contained in W; for each cornered annulus in Z, a few (one, two or four, depending on the number of corners of this cornered annulus) disjoint arcs from the corners to the circle boundary component are also contained in W. Moreover, the positions of the endpoints of these arcs on the circles are given by the following assignment. When considering

$i_0\colon \widetilde{Z} \to \mathbb{H}^3$, each of the preimages of these arcs is mapped to a geodesic arc which is perpendicular to the corresponding bi-infinite geodesics intersecting it.

By the definition of W, each component of $Z \setminus W$ is a topological disc. Let $\widetilde{W}$ be the preimage of W in $\widetilde{Z}$, and consider the embedding $i_0|_{\widetilde{W}}\colon \widetilde{W} \to \mathbb{H}^3$. Any two intersecting geodesic arcs in $i_0(\widetilde{W})$ are perpendicular to each other, except in the case when the intersection point lies in $i_0(\widetilde{Z}^{(0)})$.

Now let us define a map $i\colon \widetilde{W} \to \mathbb{H}^3$ which realizes some representation $\rho\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$. Before we construct i, we first need to choose some parameters. Recall that, for each S_i, $j(\partial S_i)$ is a multicurve in $\mathbb{Z}\mathbf{\Gamma}_{r_i R, \epsilon/r_i R}$ for $r_i \in \{1,2,4\}$.

Parameters 4.2

- Each 1–cell t in Z is associated with the following parameters:
  - A real number δ_t, such that $|\delta_t| < \epsilon/R$.
  - Two vectors $\vec{u}_{1,t}, \vec{u}_{2,t} \in S^2$, such that the angles between $\vec{u}_{1,t}, \vec{u}_{2,t}$ and the corresponding vectors $\vec{v}_1, \vec{v}_2$ in Table 1 are less than ϵ/R, respectively.
  - An element $S_t \in \mathrm{SO}(3)$, such that $\Theta(\vec{v}, S_t\vec{v}) < \epsilon/R$ for any $\vec{v} \in S^2$.
- Each circle C in the pants decomposition of S_i which does not lie on the boundary of S_i is associated with two complex numbers ξ_C and η_C, such that $|\xi_C| < \epsilon/r_i R$ and $|\eta_C| < \epsilon/(r_i R)^2$.
- Each boundary component C of S_i is associated with a complex number η_C such that $|\eta_C| < \epsilon/(r_i R)^2$.

The map $i\colon \widetilde{W} \to \mathbb{H}^3$ is defined by the following conditions, which are similar to Construction 3.6. Note that the parameters are given for 1–cells and circles in Z, but not in $\widetilde{Z}$. So $i\colon \widetilde{W} \to \mathbb{H}^3$ is $\pi_1(Z)$–equivariant for some map $\rho\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$, which is defined by the following conditions.

Construction 4.3 (a) Each bi-infinite line in $\widetilde{W}$ is subdivided into a concatenation of compact arcs by its intersection with other arcs in $\widetilde{W}$. Then all the arcs in $\widetilde{W}$ are mapped to geodesic arcs in $\mathbb{H}^3$ (under the arc-length parametrization induced by $(\widetilde{Z}, d_0)$).

(b) Each 0–cell $x \in \widetilde{W}$ which is also a 0–cell in $\widetilde{Z}$ is associated with an orthonormal frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ at $i(x)$ with respect to the orientation of $\mathbb{H}^3$.

(c) Each 1–cell $t' \in \widetilde{W}$ which is also a 1–cell in $\widetilde{Z}$ is endowed with an orientation such that the oriented 1–cell $t = p(t')$ in Z corresponds to one of the oriented 1–cells a, b, c, d, e, f in X'. Suppose that t' travels from one 0–cell x to another 0–cell y in $\widetilde{Z}$, and let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{\vec{e}\,'_1, \vec{e}\,'_2, \vec{e}\,'_3\}$ be the two orthonormal frames at $i(x)$ and $i(y)$, respectively. Then the following conditions hold:

- $i(t')$ is a geodesic arc from $i(x)$ to $i(y)$ with length equal to $L+\delta_t$.
- The tangent vectors of $i(t')$ at $i(x)$ and $i(y)$ are equal to $\vec{u}_{1,t}$ and $\vec{u}_{2,t}$, respectively (with respect to the coordinates given by the frames $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ and $\{\vec{e}_1{}',\vec{e}_2{}',\vec{e}_3{}'\}$).
- The parallel transport of $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ to $i(y)$ along $i(t')$ is equal to the composition of S_t and the counterclockwise rotation by $\pi-2\theta_0$ of $\{\vec{e}_1{}',\vec{e}_2{}',\vec{e}_3{}'\}$ about the vector $\vec{n}\in T^1_{i(q)}(M)$ in (2).

(d) For any circle C in the pants decomposition of some S_i, the image of each component of $p^{-1}(C)$ under i is a bi-infinite geodesic in $\mathbb{H}^3$. For any circle C which lies in ∂S_i, the image of each component of $p^{-1}(C)$ under i shares the same limit points with the corresponding concatenation of geodesic arcs, on $\partial\mathbb{H}^3=S^2_\infty$.

(e) Any arc in $\widetilde{W}$ which corresponds to a seam in a pair of pants, or goes from a corner point to the opposite circle in a cornered annulus of Z, is mapped to a geodesic arc which is perpendicular to the corresponding bi-infinite geodesic.

(f) For any circle C in S_i shared by two pair of pants Π_1 and Π_2 in the oriented surface S_i, the equations

$$\begin{cases} \boldsymbol{hl}_{\Pi_1}(C)=\boldsymbol{hl}_{\Pi_2}(C)=r_iR/2+\xi_C, \\ s(C)=1+\eta_C \end{cases}$$

hold. For the definition of $\boldsymbol{hl}_\Pi(C)$ and $s(C)$ in this context, see the explanation in Construction 3.6.

(g) For any oriented boundary component C_i of S_i which is the cuff of a pair of pants $\Pi\subset S_i$, in Step III of the construction of $j\colon Z\to M$ we have chosen an arc $\alpha_i\subset W$ which goes from some 0–cell of Z to C_i. Take an arbitrary bi-infinite line $\beta\subset\widetilde{W}$ which is a component of $p^{-1}(C_i)$, and any arc α in $\widetilde{W}$ that intersects with β and projects to α_i. Let α' be the arc that intersects with β and projects to a seam of Π such that $i_0(\alpha')$ is the nearest such arc to $i_0(\alpha)$. Then the tangent vector of $i(\alpha')$ is the translation by $1+\pi i+\eta_{C_i}$ of the tangent vector of $i(\alpha)$ along $i(\beta)$. Moreover, $|\boldsymbol{hl}_\Pi(C_i)-r_iR/2|<\epsilon/R$ holds; here the value of $\boldsymbol{hl}_\Pi(C_i)$ is determined by the geometry of $i|_{\widetilde{Z}^{(1)}}$.

The existence of $i\colon\widetilde{W}\to\mathbb{H}^3$ can be shown by the same developing argument as in Construction 3.6 (the existence of $i_0\colon\widetilde{Z}\to\mathbb{H}^3$).

Note that when $\delta_t=0$, $\vec{u}_{1,t}$ and $\vec{u}_{2,t}$ are equal to the corresponding vectors $\vec{v}_1$ and $\vec{v}_2$, respectively, $S_t=\mathrm{id}_{SO(3)}$ for any 1–cell t, $\xi_C=0$ and $\eta_C=0$ for any circle C, and the map $i\colon\widetilde{W}\to\mathbb{H}^3$ is exactly the restriction of our standard model $i_0\colon\widetilde{Z}\to\mathbb{H}^3$ on $\widetilde{W}$

(Construction 3.6). By the construction of $j\colon Z \to M$ in Section 3.2, we can choose proper parameters in Parameters 4.2 such that $i\colon \widetilde{W} \to \mathbb{H}^3$ equals $\tilde{j}|_{\widetilde{W}}\colon \widetilde{W} \to \widetilde{M} = \mathbb{H}^3$.

With small parameters in Parameters 4.2, $i\colon (\widetilde{W}, d_0) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ has the following nice property.

Theorem 4.4 *There exist constants $\hat{\epsilon} > 0$ and $\hat{R} > 0$ such that, for any positive numbers $\epsilon < \hat{\epsilon}$ and $R > \hat{R}$, the following statement holds. For any parameters in Parameters 4.2 with ϵ and R as above, the map $i\colon (\widetilde{W}, d_0|_{\widetilde{W}}) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ given by Construction 4.3 is a quasi-isometric embedding.*

Theorem 4.4 implies Theorem 3.8, by the following argument.

Proof of Theorem 3.8 The fact that $i\colon (\widetilde{W}, d_0|_{\widetilde{W}}) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ is a quasi-isometric embedding clearly implies that $j_*\colon \pi_1(Z) \to \pi_1(M) \subset \mathrm{PSL}_2(\mathbb{C})$ is an injective map, when the parameters are properly chosen. We can choose a continuous family of parameters satisfying the conditions in Parameters 4.2, such that the associated family of maps $i_s\colon \widetilde{W} \to \mathbb{H}^3$ $(s \in [0,1])$ connects $i_0|_{\widetilde{W}}\colon \widetilde{W} \to \mathbb{H}^3$ to $\tilde{j}|_{\widetilde{W}}\colon \widetilde{W} \to \mathbb{H}^3$. So there exists a continuous family of representations $\rho_s\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C})$ such that ρ_s lies in $\mathrm{int}(AH(\pi_1(Z)))$ for any $s \in [0,1]$, with ρ_0 given by the standard model $i_0\colon \widetilde{Z} \to \mathbb{H}^3$ and $\rho_1 = j_*$. Here

$$AH(\pi_1(Z)) = \{\rho\colon \pi_1(Z) \to \mathrm{PSL}_2(\mathbb{C}) \mid \rho \text{ is a discrete, faithful representation}\}/\sim.$$

So $\mathbb{H}^3/j_*(\pi_1(Z))$ is homeomorphic to $\mathbb{H}^3/\rho_0(\pi_1(Z))$ with respect to the induced orientations from $\mathbb{H}^3$, which completes the proof of Theorem 3.8. $\square$

So it remains to prove Theorem 4.4.

In [14, Theorems 4.8 and 4.10] we have given estimates for the quasi-isometric constant and angle change on each component of $\widetilde{W} \cap p^{-1}(S_i)$. To state these results, we need the following coordinate system.

For any two points $x, y \in \mathbb{H}^3$, we will use $\overline{xy}$ to denote the oriented geodesic arc in $\mathbb{H}^3$ from x to y.

Let V be a component of $\widetilde{W} \cap p^{-1}(S_i)$. Take two points $x, y \in V$ such that $d_0(x,y) \geq R/4$ and x lies on some component $p^{-1}(\partial S_i) \cap V$, which is denoted by β. We will give coordinates for the tangent vectors of $\overline{i_0(x)i_0(y)}$ and $\overline{i(x)i(y)}$ at $i_0(x)$ and $i(x)$, respectively, with respect to some properly chosen frames. More precisely, endow β with an arbitrary orientation and let $\alpha \subset V$ be an arc that intersects with β and projects to a seam in S_i, such that α is the closest such arc from x.

Let $\vec{e}_1 \in T^1_{i_0(x)}(\mathbb{H}^3)$ be the tangent vector of $i_0(\beta)$ at $i_0(x)$, let $\vec{e}_2 \in T^1_{i_0(x)}(\mathbb{H}^3)$ be the parallel transport of the tangent vector of $i_0(\alpha)$ to $i_0(x)$ along $i_0(\beta)$, and let $\vec{e}_3 \in T^1_{i_0(x)}(\mathbb{H}^3)$ be such that the orthonormal frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ gives the orientation of $\mathbb{H}^3$. Let θ be the angle between $\vec{e}_1$ and the tangent vector of $\overline{i_0(x)i_0(y)}$ at $i_0(x)$, and let ϕ be the angle between $\vec{e}_3$ and the tangent vector of $\overline{i_0(x)i_0(y)}$ at $i_0(x)$. Then we define $\Theta(i_0(\beta), i_0(\alpha), \overline{i_0(x)i_0(y)}) = (\theta, \phi)$. We also define θ', ϕ' and $\Theta(i(\beta), i(\alpha), \overline{i(x)i(y)}) = (\theta', \phi')$ in the same way, with i_0 replaced by i. Note that $\phi = \pi/2$ since $i_0(V)$ lies in a totally geodesic plane in $\mathbb{H}^3$.

Theorem 4.8 and Theorem 4.10 of [14] give the following statement.

Theorem 4.5 *For any $0 < \delta < 1$, there exist constants $\hat{\epsilon} > 0$ and $\hat{R} > 0$ such that, for any positive numbers $\epsilon < \hat{\epsilon}$ and $R > \hat{R}$, the following estimates hold:*

(1) $i|_V\colon (V, d_0|_V) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ *is a $(1 + k\epsilon/R, k(\epsilon + 1/R)^{1/5})$–quasi-isometric embedding for a universal constant k.*

(2) $$\left|\Theta\big(i_0(\beta), i_0(\alpha), \overline{i_0(x)i_0(y)}\big) - \Theta\big(i(\beta), i(\alpha), \overline{i(x)i(y)}\big)\right| = |(\theta - \theta', \phi - \phi')| < (\delta/300)^2.$$

Here we need some estimates about the quasi-isometric constant and angle change on pieces of $\widetilde{Z}$. Each piece of $\widetilde{Z}$ is the union of some component of $p^{-1}(S_i)$ and components of the preimages of cornered annuli. To extend the estimate in Theorem 4.5 to pieces of $\widetilde{Z}$, we first need to give some estimates on the preimages of cornered annuli under the map i.

The following estimate is intuitive and elementary, so we leave it as an exercise for the reader.

Lemma 4.6 *Let x_1, x_2, y_1, y_2 be four points in $\mathbb{H}^2$, and let Q be the union of the four geodesic arcs $\overline{x_1x_2}, \overline{y_1y_2}, \overline{x_1y_1}, \overline{x_2y_2}$ as in Figure 7 (left). Then the following geometric conditions hold:*

- $\overline{y_1y_2}$ *is perpendicular to both $\overline{x_1y_1}$ and $\overline{x_2y_2}$.*
- $\angle y_1x_1x_2 = \angle y_2x_2x_1 = \theta_0$.
- $d(x_1, x_2) = L$.

Let x_1', x_2', y_1', y_2' be another four points in $\mathbb{H}^3$, and let Q' be the union of the four geodesic arcs $\overline{x_1'x_2'}, \overline{y_1'y_2'}, \overline{x_1'y_1'}, \overline{x_2'y_2'}$ as in Figure 7 (right).

- *$\overline{y_1' y_2'}$ is perpendicular to both $\overline{x_1' y_1'}$ and $\overline{x_2' y_2'}$.*
- *$|d(x_1, y_1) - d(x_1', y_1')|, |d(x_2, y_2) - d(x_2', y_2')| < 10\epsilon/R$.*
- *$|d(x_1', x_2') - L| < 10\epsilon/R$.*

Moreover, the angle between the tangent vector of $\overline{y_1' x_1'}$ (at y_1') and the parallel transport of the tangent vector of $\overline{y_2' x_2'}$ (at y_2') to y_1' along $\overline{y_2' y_1'}$ is less than $10\epsilon/R$.

Let $p\colon \overline{x_1 x_2} \to \overline{y_1 y_2}$ and $p'\colon \overline{x_1' x_2'} \to \overline{y_1' y_2'}$ be the nearest-point projections, and let $r\colon Q \to Q'$ be the piecewise-linear homeomorphism sending x_1, x_2, y_1, y_2 to x_1', x_2', y_1', y_2', respectively.

Then, for any $x \in \overline{x_1 x_2}$, the following estimates hold:

- *$|d(p(x), x) - d(p'(r(x)), r(x))| < 80\sqrt{\epsilon/R}$.*
- *$d(r(p(x)), p'(r(x))) < 80\sqrt{\epsilon/R}$.*
- *$|\angle x_1 x p(x) - \angle x_1' r(x) p'(r(x))| < 80\sqrt{\epsilon/R}$.* □

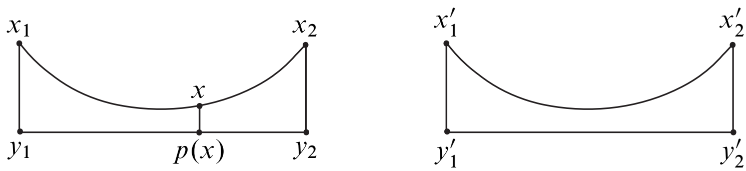

Figure 7: Pictures of 4–gons with two right angles

For three points $p, q, r \in \mathbb{H}^3$ not lying on a geodesic, we use P_{pqr} to denote the hyperbolic plane containing them. For another such hyperbolic plane $P_{p'q'r'}$ intersecting P_{pqr}, we use $\Theta(P_{pqr}, P_{p'q'r'}) \in [0, \pi/2]$ to denote the angle between these two planes.

In $\widetilde{Z}$, each component of the preimage of a cornered annulus is subdivided into a union of 4–gons by $\widetilde{W}$. Let R be the intersection of such a 4–gon with $\widetilde{W}$; then $i_0(R) \subset \mathbb{H}^2 \subset \mathbb{H}^3$ has exactly the same geometry as Q in Lemma 4.6.

To apply Lemma 4.6 to compare the geometry of $i_0(R)$ and $i(R)$, we need the following lemma, which shows that $i(R) \subset \mathbb{H}^3$ satisfies the conditions of Q' in Lemma 4.6. Lemma 4.7 also gives some further estimates, and its proof follows from Construction 4.3 and elementary hyperbolic geometry.

Lemma 4.7 *Let x_1, x_2, y_1, y_2 be the four vertices of $R \subset \widetilde{W}$, with positions as shown in Figure 7 (left). Then the following estimates hold:*

- *$|d(i(x_1), i(x_2)) - L| < \epsilon/R$.*

- $|d(i(x_1),i(y_1))-\cosh^{-1}(\csc\theta_0)|, |d(i(x_2),i(y_2))-\cosh^{-1}(\csc\theta_0)| < 9\epsilon/R$.
- *The angle between the tangent vector of $\overline{i(y_1)i(x_1)}$ (at $i(y_1)$) and the parallel transport of the tangent vector of $\overline{i(y_2)i(x_2)}$ (at $i(y_2)$) to $i(y_1)$ along $\overline{i(y_2)i(y_1)}$ is smaller than $10\epsilon/R$.*
- $|d(i(y_1),i(y_2)) - R| < 30\epsilon/R$.
- *For any $x \in \overline{i(x_1)i(x_2)}$, let x' be the nearest-point projection of $i(x)$ on $\overline{i(y_1)i(y_2)}$. Then*

$$\Theta(P_{i(x_1)i(x_2)x'}, P_{i(y_1)i(y_2)i(x)}), \Theta(P_{i(x_1)i(y_1)i(y_2)}, P_{i(x)i(y_1)i(y_2)}) < 10\epsilon/R. \square$$

Remark G Note that, since $\widetilde{W}$ may contains other 1–cells that intersect R, the definition of $i|_R\colon R \to i(R)$ does not exactly coincide with $r\colon Q \to Q'$. However, since there are at most two such 1–cells intersecting R, and they only intersect $\overline{y_1y_2}$, these two maps only differ on $\overline{y_1y_2}$ by an error of at most $100\epsilon/R$. So the estimates in Lemma 4.6 still hold in this case, with $80\sqrt{\epsilon/R}$ replaced by $100\sqrt{\epsilon/R}$.

Take two points $x, y \in \widetilde{Z}^{(1)}$, lying in the same piece of $\widetilde{Z}$, such that $d_0(x,y) \geq R/128$. Let U be the piece of $\widetilde{Z}$ that contains x and y, and let γ be the shortest path from x to y in $(\widetilde{Z}, d_0)$. Then γ lies in U.

Now we need to give estimates for the length and angle change for $\overline{i(x)i(y)}$ under certain coordinates. To formulate the estimate, we need to give the following angular coordinate system, which is similar to the formulation before Theorem 4.5.

Let t be the 1–cell in $\widetilde{Z}$ which contains x. If x lies in $\widetilde{Z}^{(0)}$, choose any such t which lies in U; there are two such choices. We give an orientation of t such that x is closer to the initial point than the terminal point of t. Let the other oriented 1–cell which lies in U and shares the initial point with t be t', and denote the intersection point of t and t' by z.

Let $\vec{e}_1 \in T^1_{i_0(x)}(\mathbb{H}^3)$ be the tangent vector of $i_0(t)$ at $i_0(x)$. We use $P_{i_0(t)i_0(t')}$ to denote the hyperbolic plane containing $i_0(t)$ and $i_0(t')$, and let $\vec{e}_3$ be the unit normal vector of $P_{i_0(t)i_0(t')}$ at $i_0(x)$. Then we have an orthonormal frame $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ at $i_0(x)$, with $\vec{e}_2 = \vec{e}_3 \times \vec{e}_1$.

Let $\vec{v} \in T^1_{i_0(x)}(\mathbb{H}^3)$ be the tangent vector of $\overline{i_0(x)i_0(y)}$ at $i_0(x)$, let θ be the angle between $\vec{v}$ and $\vec{e}_1$, and let ϕ be the angle between $\vec{v}$ and $\vec{e}_3$. Then we define $\Theta(i_0(t), i_0(t'), \overline{i_0(x)i_0(y)}) = (\theta,\phi)$, and note that $\phi = \pi/2$ here.

In the same way, we define $\Theta(i(t), i(t'), \overline{i(x)i(y)}) = (\theta',\phi')$, with i_0 replaced by i. We also have an orthonormal frame $\{\vec{e}_1',\vec{e}_2',\vec{e}_3'\}$ at $i(x)$ (similar to the definition of the frame $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ at $i_0(x)$). Then we have the following estimate.

Proposition 4.8 *For any $0 < \delta < 1$, there exist constants $\hat{\epsilon} > 0$ and $\hat{R} > 0$ such that, for any positive numbers $\epsilon < \hat{\epsilon}$ and $R > \hat{R}$, the following statement holds. For two points $x, y \in \widetilde{Z}^{(1)}$ as above, with corresponding 1–cells t and t', the following estimates hold:*

- $\frac{1}{2}d(i_0(x), i_0(y)) < d(i(x), i(y)) < 2d(i_0(x), i_0(y))$.
- $|\Theta(i_0(t), i_0(t'), \overline{i_0(x)i_0(y)}) - \Theta(i(t), i(t'), \overline{i(x)i(y)})| = |(\theta - \theta', \frac{\pi}{2} - \phi')| < \delta$.

Proof Let γ be the shortest path in $(\widetilde{Z}, d_0)$ from x to y. Then γ lies in a piece of $\widetilde{Z}$, which is denoted by U. Let $\widetilde{S}$ be the component of the preimage of S_i which is contained in U, and let β be the component of $\partial\widetilde{S}$ that is the closest one to x.

The estimates clearly hold if $\gamma \subset \widetilde{Z}^{(1)}$. So there are two cases to consider: either γ does not intersect $\widetilde{S}$ (but does not lie in $\widetilde{Z}^{(1)}$), or γ intersects $\widetilde{S}$.

In the following, we suppose that $\epsilon > 0$ is small enough and $R > 0$ is large enough that $\epsilon/R < \delta^{10}/10^{40}$ holds.

Case I If γ does not intersect with $\widetilde{S}$, then the picture near γ looks like Figure 8, with $d_0(x, y) > R/128$.

Let ω be the concatenation of geodesic arcs in ∂U from x to y as in Figure 8 (part of ω is drawn with dashed lines). For each geodesic arc in ω, the length of its image under i_0 and under i differ by at most ϵ/R, and the angles between the images of two adjacent arcs under i_0 (which is $2\theta_0$) and under i differ by at most $2\epsilon/R$. Since $d_0(x, y) > R/128$, an exercise in hyperbolic geometry gives the first estimate:

$$\frac{1}{2}d(i_0(x), i_0(y)) < d(i(x), i(y)) < 2d(i_0(x), i_0(y)). \tag{10}$$

Actually, the constant 2 can be replaced by $1 + \epsilon'$ for some small positive constant ϵ'.

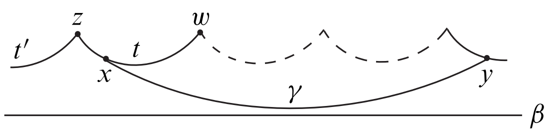

Figure 8: The case when γ does not intersect $\widetilde{S}$

Let x' and y' be the nearest-point projections of $i_0(x)$ and $i_0(y)$ on $i_0(\beta)$, respectively, and let x'' and y'' be the nearest-point projections of $i(x)$ and $i(y)$ on $i(\beta)$, respectively. Since

$$\sin\theta_0 = \sqrt{\tfrac{5-\sqrt{5}}{10}},$$

we have

$$d(x', i_0(x)), d(y', i_0(y)), d(x'', i(x)), d(y'', i(y)) < \cosh^{-1}(\csc\theta_0) + 10\epsilon/R < 2.$$

By the definition of θ and θ', $\theta = \angle i_0(w)i_0(x)i_0(y)$ and $\theta' = \angle i(w)i(x)i(y)$. Since

$$d(i_0(x), i_0(y)), d(i(x), i(y)) > R/256 \quad \text{and} \quad d(i_0(y), y'), d(i(y), y'') < 2,$$

we have

$$\angle i_0(y)i_0(x)y', \angle i(y)i(x)y'' < 10e^{-R/256}.$$

Moreover, since

$$d(i_0(x), y'), d(i(x), y'') > R/256 - 2,$$

and $|d(i_0(x), x') - d(i(x), x'')| < 100\sqrt{\epsilon/R}$ (Lemma 4.6), we have

$$|\angle x' i_0(x)y' - \angle x'' i(x)y''| < 50(\epsilon/R)^{1/4}.$$

Note that the fifth estimate in Lemma 4.7 implies that

$$\Theta(P_{i(z)i(w)x''}, P_{i(x)x''y''}) < 10\epsilon/R,$$

and the third estimate in Lemma 4.6 gives

$$|\angle i_0(z)i_0(x)x' - \angle i(z)i(x)x''| < 100\sqrt{\epsilon/R}.$$

So

$$\begin{aligned}
(11) \qquad |\theta - \theta'| &= |\angle i_0(w)i_0(x)i_0(y) - \angle i(w)i(x)i(y)| \\
&\le |\angle i_0(w)i_0(x)y' - \angle i(w)i(x)y''| + \angle i_0(y)i_0(x)y' + \angle i(y)i(x)y'' \\
&\le |\angle i_0(z)i_0(x)y' - \angle i(z)i(x)y''| + 20e^{-R/256} \\
&\le |\angle i_0(z)i_0(x)x' - \angle i(z)i(x)x''| + |\angle x' i_0(x)y' - \angle x'' i(x)y''| \\
&\qquad + \Theta(P_{i(z)i(w)x''}, P_{i(x)x''y''}) + 20e^{-R/256} \\
&\le 100\sqrt{\epsilon/R} + 50(\epsilon/R)^{1/4} + 10\epsilon/R + 20e^{-R/256} < \delta/2.
\end{aligned}$$

We will use $P_{i(t)i(t')}$ to denote the hyperbolic plane containing $i(t)$ and $i(t')$. Let $\vec{v}$ be the tangent vector of $\overline{i(x)i(y)}$ at $i(x)$ and let $\vec{n}$ be the normal vector of $P_{i(t)i(t')}$ at $i(x)$. Then $\phi' = \Theta(\vec{v}, \vec{n})$, so we need to estimate $|\Theta(\vec{v}, \vec{n}) - \pi/2|$.

Let $\vec{v}'$ be the tangent vector of $\overline{i(x)y''}$ at $i(x)$. Then we know that

$$\Theta(\vec{v}, \vec{v}') = \angle i(y)i(x)y'' < 10e^{-R/256}.$$

Let z'' be the nearest-point projection of $i(z)$ on $i(\beta)$. Then the fifth estimate of Lemma 4.7 implies

$$\Theta(P_{i(z)z''i(w)}, P_{i(z)z''y''}), \Theta(P_{i(z)z''y''}, P_{i(x)z''y''}) < 10\epsilon/R. \tag{12}$$

A similar estimate gives

$$\Theta(P_{i(t)i(t')}, P_{i(z)z''y''}) < 20\epsilon/R. \tag{13}$$

So we have

$$\begin{aligned} \left|\Theta(\vec{v}, \vec{n}) - \tfrac{\pi}{2}\right| &\le \Theta(\vec{v}, \vec{v}') + \left|\Theta(\vec{v}', \vec{n}) - \tfrac{\pi}{2}\right| \\ &\le 10e^{-R/256} + \Theta(P_{i(t)i(t')}, P_{i(x)z''y''}) \\ &\le 10e^{-R/256} + \Theta(P_{i(t)i(t')}, P_{i(z)z''i(w)}) \\ &\qquad\qquad + \Theta(P_{i(z)z''i(w)}, P_{i(x)z''y''}) \\ &\le 10e^{-R/256} + \Theta(P_{i(t)i(t')}, P_{i(z)z''y''}) \\ &\qquad + 2\Theta(P_{i(z)z''i(w)}, P_{i(z)z''y''}) + \Theta(P_{i(z)z''y''}, P_{i(x)z''y''}) \\ &\le 10e^{-R/256} + 50\epsilon/R < \delta/2. \end{aligned} \tag{14}$$

So $|(\theta - \theta', \frac{\pi}{2} - \phi')| < \delta$.

Case II If γ does intersect $\widetilde{S}$, then $d_0(x, y) > R/2$ by Step IV of the construction in Section 3.2. Let $\bar{x}$ be the intersection point of β and γ, and let $\bar{y}$ be the other intersection point in $\partial\widetilde{S} \cap \gamma$ which is close to y. Let α be the component of the preimage of a seam in U that intersects with β and is the closest such arc to $\bar{x}$. Then the picture near x is as shown in Figure 9. Give β an orientation which points to the left in Figure 9, and note that the orientation of t also points to the left.

Let m be the middle point of $\overline{i(\bar{x})i(\bar{y})}$ in $\mathbb{H}^3$. Since $d_0(\bar{x}, \bar{y}) \ge R/2$ and since $i|_{\widetilde{S}\cap\widetilde{W}}\colon \widetilde{S} \cap \widetilde{W} \to \mathbb{H}^3$ is an $(1 + K\epsilon/R, 1)$–quasi-isometric embedding, we have $d(i(\bar{x}), i(\bar{y})) > R/3$. So $d(i(\bar{x}), m), d(i(\bar{y}), m) > R/6$.

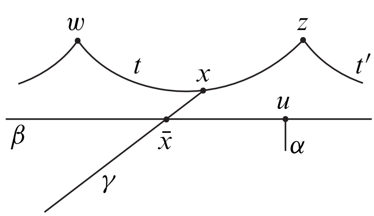

Figure 9: The case when γ intersects $\widetilde{S}$

Now we make the following claim for x and $\bar{x}$, and the same estimates for y and $\bar{y}$ also hold:

- $\frac{2}{3}d(m,i(x)) < d_0(x,\bar{x}) + \frac{1}{2}d_0(\bar{x},\bar{y}) < \frac{3}{2}d(m,i(x))$.
- $\angle i(x)mi(\bar{x}) < \delta/10$.
- $\left|\Theta\big(i_0(t), i_0(t'), \overline{i_0(x)i_0(y)}\big) - \Theta\big(i(t), i(t'), \overline{i(x)m}\big)\right| < \delta/5$.

This claim implies the statement of Proposition 4.8 by the following argument. Since $\angle i(x)mi(\bar{x}) < \delta/10$ and $\angle i(y)mi(\bar{y}) < \delta/10$, $\angle i(x)mi(y) > \pi - \delta/5$ holds. By the first estimate in the claim, we have

$$\begin{aligned} d(i(x),i(y)) &\le d(i(x),m) + d(m,i(y)) \\ &< \tfrac{3}{2}(d_0(x,\bar{x}) + d_0(\bar{x},\bar{y}) + d_0(y,\bar{y})) < 2d_0(x,y), \end{aligned} \tag{15}$$

and

$$d(i(x),i(y)) \ge d(m,i(x)) + d(m,i(y)) - 1 \ge \tfrac{2}{3}d_0(x,y) - 1 \ge \tfrac{1}{2}d_0(x,y). \tag{16}$$

Moreover, $\angle i(x)mi(y) > \pi - \delta/5$ implies that $\angle mi(x)i(y) < \delta/5$. Then the third estimate in the claim implies that

$$\left|\Theta\big(i_0(t), i_0(t'), \overline{i_0(x)i_0(y)}\big) - \Theta\big(i(t), i(t'), \overline{i(x)i(y)}\big)\right| < \delta.$$

Now we need only to prove the claim. There are two subcases to consider: $d_0(x,\bar{x}) < \delta/1000$ and $d_0(x,\bar{x}) \ge \delta/1000$.

Subcase (i) $d_0(x,\bar{x}) < \delta/1000$

In this subcase, there might be a big difference between $\angle i_0(w)i_0(x)i_0(\bar{x})$ and $\angle i(w)i(x)i(\bar{x})$. However, since $d_0(x,\bar{x}) < \delta/1000$ is very small, it will not affect the estimate very much.

By the first estimate of Theorem 4.5, we have

$$d(m,i(\bar{x})) > R/6 \quad \text{and} \quad \tfrac{3}{4}d(m,i(\bar{x})) < \tfrac{1}{2}d_0(\bar{x},\bar{y}) < \tfrac{4}{3}d(m,i(\bar{x})).$$

By the first and the second estimates of Lemma 4.6, we also have $d(i(x),i(\bar{x})) < \delta/250$. So

$$\begin{aligned} \tfrac{2}{3}d(m,i(x)) &< \tfrac{2}{3}(d(m,i(\bar{x})) + d(i(\bar{x}),i(x))) \\ &< \tfrac{3}{4}d(m,i(\bar{x})) < \tfrac{1}{2}d_0(\bar{x},\bar{y}) < d_0(x,\bar{x}) + \tfrac{1}{2}d_0(\bar{x},\bar{y}) \end{aligned} \tag{17}$$

and

$$(18)\quad d_0(x,\bar{x})+\tfrac{1}{2}d_0(\bar{x},\bar{y})<\tfrac{\delta}{1000}+\tfrac{4}{3}d(m,i(\bar{x}))$$
$$\le \tfrac{\delta}{1000}+\tfrac{4}{3}(d(m,i(x))+d(i(x),i(\bar{x})))<\tfrac{3}{2}d(m,i(x))$$

hold, thus the first estimate in the claim is true.

Moreover, since

$$d(i(x),i(\bar{x}))<\delta/250 \quad\text{and}\quad d(m,i(\bar{x}))>R/6,$$

$\angle i(x)mi(\bar{x})<\delta/10$ clearly holds.

Now it remains to show the third estimate in the claim.

Let w'' be the nearest-point projection of $i(w)$ on $i(\beta)$. By the choice of the orientation of t, $d_0(w,x)\ge d_0(z,x)$ holds, so we have $d(i(w),i(x))\ge R/4$. Since $d(i(w),w'')\le 2$, we have $\angle i(w)i(x)w''<10e^{-R/4}$. Moreover, since

$$d(w'',i(\bar{x}))\ge R/4-3,\quad d(m,i(\bar{x}))\ge R/6\quad\text{and}\quad d(i(x),i(\bar{x}))<\delta/250,$$

we have

$$\angle i(x)w''i(\bar{x}),\angle i(x)mi(\bar{x})<\delta\cdot e^{-R/6}.$$

Let $\vec{v}_1$ be the tangent vector of $i(t)$ at $i(x)$, $\vec{v}_2$ be the tangent vector of $\overline{i(x)m}$ at $i(x)$, and $\vec{v}_3$ be the tangent vector of $\overline{i(x)w''}$ at $i(x)$. Let $\vec{u}_1$ be the tangent vector of $i(\beta)$ at $i(\bar{x})$ and $\vec{u}_2$ be the tangent vector of $\overline{i(\bar{x})m}$ at $i(\bar{x})$. For two points $p,q\in\mathbb{H}^3$ and $\vec{v}\in T_p\mathbb{H}^3$, we will use $\vec{v}@q$ to denote the parallel transport of $\vec{v}$ to q along $\overline{pq}$.

Then we have

$$\begin{aligned}
(19)\quad &|\angle i(w)i(x)m-\angle w''i(\bar{x})m|\\
&=|\Theta(\vec{v}_1,\vec{v}_2)-\Theta(\vec{u}_1,\vec{u}_2)|\\
&\le\Theta(\vec{v}_1,\vec{v}_3)+\Theta(\vec{v}_3,\vec{u}_1@i(x))+\Theta(\vec{v}_2,\vec{u}_2@i(x))\\
&\le\angle i(w)i(x)w''+\Theta(\vec{v}_3@w'',\vec{u}_1@w'')+\Theta(\vec{u}_1@w'',\vec{u}_1@i(x)@w'')\\
&\qquad+\Theta(\vec{v}_2@m,\vec{u}_2@m)+\Theta(\vec{u}_2@m,\vec{u}_2@i(x)@m)\\
&\le\angle i(w)i(x)w''+\angle i(x)w''i(\bar{x})+d(i(x),i(\bar{x}))\\
&\qquad+\angle i(x)mi(\bar{x})+d(i(x),i(\bar{x}))\\
&\le 10e^{-R/4}+\delta\cdot e^{-R/6}+\delta/250+\delta\cdot e^{-R/6}+\delta/250<\delta/100.
\end{aligned}$$

Here $\Theta(\vec{u}_1@w'',\vec{u}_1@i(x)@w'')<d(i(x),i(\bar{x}))$ holds by [8, Proposition 4.1].

Let w' be the nearest-point projection of $i_0(w)$ on $i_0(\beta)$. Then a similar (actually easier) argument implies that

$$|\angle i_0(w)i_0(x)i_0(y) - \angle w' i_0(\bar{x}) i_0(y)| < \delta/100. \tag{20}$$

Note that the first coordinate of

$$\left|\Theta\big(i_0(t), i_0(t'), \overline{i_0(x)i_0(y)}\big) - \Theta\big(i(t), i(t'), \overline{i(x)m}\big)\right|$$

equals $|\angle i_0(w)i_0(x)i_0(y) - \angle i(w)i(x)m|$, while the second estimate of Theorem 4.5 implies that $|\angle w' i_0(\bar{x})i_0(\bar{y}) - \angle w'' i(\bar{x})i(\bar{y})| < \delta/100$. So we have

$$\begin{aligned} &|\angle i_0(w)i_0(x)i_0(y) - \angle i(w)i(x)m| \\ &\quad \le |\angle i_0(w)i_0(x)i_0(y) - \angle w' i_0(\bar{x})i_0(\bar{y})| + |\angle i(w)i(x)m - \angle w'' i(\bar{x})m| \\ &\qquad\qquad + |\angle w' i_0(\bar{x})i_0(\bar{y}) - \angle w'' i(\bar{x})m| \\ &\quad \le \delta/100 + \delta/100 + \delta/100 < \delta/10. \end{aligned} \tag{21}$$

Let m' be the nearest-point projection of m on $P_{i(t)i(t')}$ and let $\bar{x}'$ be the nearest-point projection of $i(\bar{x})$ on $P_{i(t)i(t')}$. Then $d(\bar{x}', i(\bar{x})) \le \delta/250$. Note that the second coordinate of

$$\left|\Theta\big(i_0(t), i_0(t'), \overline{i_0(x)i_0(y)}\big) - \Theta\big(i(t), i(t'), \overline{i(x)m}\big)\right|$$

equals $\angle m i(x) m'$.

Let $\vec{n}$ be the normal vector of $P_{i(t)i(t')}$ at $i(x)$ and let $\vec{n}'$ be the normal vector of $P_{i(x)i(\beta)}$ (the hyperbolic plane containing $i(x)$ and $i(\beta)$) at $i(\bar{x})$ such that its parallel transport to $i(x)$ is close to $\vec{n}$. Then, by the estimate of $\Theta(P_{i(t)i(t')}, P_{i(x)i(\beta)})$ in (14), we have

$$\Theta(\vec{n}'@i(x), \vec{n}) = \Theta(P_{i(t)i(t')}, P_{i(x)i(\beta)}) \le 50\epsilon/R. \tag{22}$$

So

$$\begin{aligned} \angle(\vec{n}'@\bar{x}', \vec{n}@\bar{x}') &\le \angle(\vec{n}'@\bar{x}', \vec{n}'@i(x)@\bar{x}') + \angle(\vec{n}'@i(x), \vec{n}) \\ &\le \delta/250 + 50\epsilon/R \le \delta/200. \end{aligned} \tag{23}$$

Then we have

$$\begin{aligned} \sinh d(m, m') &\le \sinh d(i(\bar{x}), \bar{x}') \cosh d(i(\bar{x}), m) \\ &\qquad + \cosh d(i(\bar{x}), \bar{x}') \sinh d(i(\bar{x}), m) \sin \angle(\vec{n}'@\bar{x}', \vec{n}@\bar{x}') \\ &\le \tfrac{\delta}{100} \cosh d(i(\bar{x}), m) + \tfrac{\delta}{100} \sinh d(i(\bar{x}), m). \end{aligned} \tag{24}$$

This inequality implies

$$\sin \angle m i(x) m' = \frac{\sinh d(m,m')}{\sinh d(i(x),m)} \le \frac{\frac{\delta}{100}\cosh d(i(\bar{x}),m) + \frac{\delta}{100}\sinh d(i(\bar{x}),m)}{\sinh\big(d(i(\bar{x}),m) - \frac{\delta}{250}\big)} \le \delta/20. \tag{25}$$

So $\angle m i(x) m' \le \delta/10$, and we finish the proof in the first subcase.

Subcase (ii) $\quad d_0(x,\bar{x}) \ge \delta/1000$

By the first two estimates of Lemma 4.6 and the inequality $d_0(x,\bar{x}) \ge \delta/1000$, we have

$$\tfrac{3}{4} d(i(x), i(\bar{x})) < d_0(x,\bar{x}) < \tfrac{4}{3} d(i(x), i(\bar{x})), \tag{26}$$

while the first estimate of Theorem 4.5 implies that

$$\tfrac{3}{4} d(m, i(\bar{x})) < \tfrac{1}{2} d_0(x,y) < \tfrac{4}{3} d(m, i(\bar{x})). \tag{27}$$

Now we can estimate $\angle m i(\bar{x}) i(x)$. Let x' be the nearest-point projection of $i_0(x)$ on $i_0(\beta)$ and let x'' be the nearest-point projection of $i(x)$ on $i(\beta)$.

If $d(x', i_0(\bar{x})) \ge R$, then

$$d(i_0(x), i_0(\bar{x})) \ge R \quad \text{and} \quad d(i(x), i(\bar{x})) \ge R/2.$$

Since $d(i_0(x), x'), d_0(i(x), x'') < 2$, we have

$$\angle i_0(x) i_0(\bar{x}) x', \angle i(x) i(\bar{x}) x'' \le 10 e^{-R/2}.$$

By the second estimate of Theorem 4.5, $\angle i_0(x) i_0(\bar{x}) x' \le 10 e^{-R/2}$ implies that

$$\angle m i(\bar{x}) x'' \ge \pi - 10 e^{-R/2} - (\delta/300)^2.$$

So

$$\angle m i(\bar{x}) i(x) \ge \pi - 20 e^{-R/2} - (\delta/300)^2 \ge \pi - \delta/25.$$

If $d(x', i_0(\bar{x})) < R$, then x' and $i_0(\bar{x})$ lie in the image of at most two adjacent 1–cells in $\widetilde{Z}$, under the nearest-point projection to $i_0(\beta)$. Then the first two estimates in Lemma 4.6 imply

$$\Big|\Theta\big(i_0(\beta), i_0(\alpha), \overline{i_0(\bar{x}) i_0(x)}\big) - \Theta\big(i(\beta), i(\alpha), \overline{i(\bar{x}) i(x)}\big)\Big| < 10^4 (\epsilon/R)^{1/4} \delta^{-1/2}. \tag{28}$$

The second estimate of Theorem 4.5 gives

$$\Big|\Theta\big(i_0(\beta), i_0(\alpha), \overline{i_0(\bar{x}) i_0(\bar{y})}\big) - \Theta\big(i(\beta), i(\alpha), \overline{i(\bar{x}) m}\big)\Big| < (\delta/300)^2. \tag{29}$$

Then an elementary computation in spherical geometry gives

$$\angle mi(\bar{x})i(x) \geq \pi - 3\sqrt{(\delta/300)^2} - 3\sqrt{10^4(\epsilon/R)^{1/4}\delta^{-1/2}} \geq \pi - \delta/25. \tag{30}$$

So $\angle mi(\bar{x})i(x) \geq \pi - \delta/25$ always holds.

By (26), (27) and (30), we have

$$\begin{aligned} \tfrac{2}{3}d(m,i(x)) &< \tfrac{2}{3}\big(d(m,i(\bar{x})) + d(i(\bar{x}),i(x))\big) \\ &< \tfrac{1}{2}d_0(x,y) + d_0(x,\bar{x}) \\ &< \tfrac{4}{3}\big(d(m,i(\bar{x})) + d(i(x),i(\bar{x}))\big) \\ &< \tfrac{4}{3}(d(m,i(x)) + 1) < \tfrac{3}{2}d(m,i(x)). \end{aligned} \tag{31}$$

Moreover, $\angle mi(\bar{x})i(x) \geq \pi - \delta/25$ implies that

$$\angle i(x)mi(\bar{x}) < \delta/25 < \delta/10 \quad \text{and} \quad \angle mi(x)i(\bar{x}) < \delta/25.$$

The estimates in Lemma 4.6 also imply the following estimate (by considering the cases that $d_0(x', i_0(\bar{x})) \geq R$ and $d_0(x', i_0(\bar{x})) < R$):

$$\Big|\Theta\big(i_0(t), i_0(t'), \overline{i_0(x)i_0(\bar{x})}\big) - \Theta\big(i(t), i(t'), \overline{i(x)i(\bar{x})}\big)\Big| < 10^4(\epsilon/R)^{1/4}\delta^{-1} < \delta/25. \tag{32}$$

Then (32) and the inequality $\angle mi(x)i(\bar{x}) < \delta/25$ together give the desired estimate

$$\Big|\Theta\big(i_0(t), i_0(t'), \overline{i_0(x)i_0(y)}\big) - \Theta\big(i(t), i(t'), \overline{i(x)m}\big)\Big| < \delta/5. \tag{33}$$

So the claim is proved, and the proof of this lemma is complete. □

Now we are ready to prove Theorem 4.4, which finishes the proof of our main theorem (Theorem 1.1).

Proof Choose constants $\hat{\epsilon} > 0$ and $\hat{R} > 0$ such that Proposition 4.8 holds for $\delta = (\pi/360)^2$.

For any two points $x, y \in \widetilde{Z}^{(0)}$, let γ be the shortest path in $(\widetilde{Z}, d_0)$ from x to y, let γ' be the modified path of γ, and let $y_1, y_2, \dots, y_m$ be the corresponding modified sequence.

Note that since $x, y \in \widetilde{Z}^{(0)}$, we have $d_0(x, y_1), d_0(y_m, y) \geq R/2$. Here γ' is a concatenation of geodesic arcs $\gamma_0, \gamma_1, \dots, \gamma_m$ in $(\widetilde{Z}, d_0)$, which connect the sequence of points $y_0 = x, y_1, y_2, \dots, y_m, y_{m+1} = y$. Here γ_i lies in a piece of $\widetilde{Z}$ and has length greater than $R/128$, for $i = 0, 1, \dots, m$. Moreover, in the proof of Proposition 4.1, we have shown that the angle between adjacent geodesic arcs $i_0(\gamma_i)$ and $i_0(\gamma_{i+1})$ is greater than $\pi/36$.

In the first estimate of Proposition 4.8, we showed that

$$d(i(y_i), i(y_{i+1})) < 2d_0(y_i, y_{i+1}),$$

while Equation (8) in the proof of Proposition 4.1 implies that

$$\sum_{i=0}^{m} d(i_0(y_i), i_0(y_{i+1})) \leq \tfrac{10}{9} d(i_0(x), i_0(y)) \leq \tfrac{10}{9} d_0(x, y).$$

So

$$d(i(x), i(y)) \leq \sum_{i=0}^{m} d(i(y_i), i(y_{i+1})) \leq 2 \sum_{i=0}^{m} d_0(y_i, y_{i+1}) \leq 3d_0(x, y). \tag{34}$$

On the other hand, the second estimate in Proposition 4.8 implies the following statement. Let $\vec{u}_1 \in S^2$ be the vector that corresponds to the tangent vector of $i_0(\gamma_i)$ at $i_0(y_{i+1})$ (with respect to the frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ at $i_0(y_{i+1})$), and let $\vec{u}_2 \in S^2$ be the vector that corresponds to the tangent vector of $\overline{i(y_{i+1})i(y_i)}$ at $i(y_{i+1})$ (with respect to the frame $\{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ at $i(y_{i+1})$). Then $\Theta(\vec{u}_1, \vec{u}_2) \leq \pi/90$. Since we have shown that the angle between $i_0(\gamma_i)$ and $i_0(\gamma_{i+1})$ is greater than $\pi/36$ in the proof of Proposition 4.1, the angle between $\overline{i(y_{i+1})i(y_i)}$ and $\overline{i(y_{i+1})i(y_{i+2})}$ is greater than $\pi/180$.

Since the length of $\overline{i(y_{i+1})i(y_i)}$ equals $d(i(y_i), i(y_{i+1}))$, which is greater than $R/256$, then [9, Lemma 4.8] implies that

$$\begin{aligned} d(i(x), i(y)) &\geq \sum_{i=0}^{m} d(i(y_i), i(y_{i+1})) - 2m\big(\log\big(\csc \tfrac{\pi}{360}\big) + 1\big) \\ &\geq \frac{1}{2} \sum_{i=0}^{m} d(i(y_i), i(y_{i+1})) \geq \frac{1}{4} \sum_{i=0}^{m} d_0(y_i, y_{i+1}) \geq \tfrac{1}{4} d_0(x, y). \end{aligned} \tag{35}$$

So $i\colon (\widetilde{W}, d_0|_{\widetilde{W}}) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ is a quasi-isometric embedding. □

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