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A thick subcategory theorem for modules over certain ring spectra

AKHIL MATHEW

We classify thick subcategories of the ∞–categories of perfect modules over ring spectra which arise as functions on even periodic derived stacks satisfying affineness and regularity conditions. For example, we show that the thick subcategories of perfect modules over TMF are in natural bijection with the subsets of the underlying space of the moduli stack of elliptic curves which are closed under specialization.

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1 Introduction

1.1 Generalities

Let $\mathcal{C}$ be a stable ∞–category. Recall the following definition.

Definition 1.1 A full subcategory $\mathcal{C}' \subset \mathcal{C}$ is *thick* if $\mathcal{C}'$ is a stable subcategory (ie it is closed under finite limits and colimits), and $\mathcal{C}'$ is closed under retracts.

Note that this definition only depends on the underlying *homotopy category* of $\mathcal{C}$ and its triangulated structure, and can be studied without the language of ∞–categories. Since many properties of objects in $\mathcal{C}$ are controlled by thick subcategories, it is generally very useful to have classifications of the possible thick subcategories of $\mathcal{C}$.

In the setting of the stable ∞–category $\mathrm{Sp}^{\omega}_{(p)}$ of finite p–local spectra, the following fundamental result was proved by Hopkins and Smith:

Theorem 1.2 (Hopkins and Smith [14]) *There is a descending sequence of thick subcategories*

$$\mathrm{Sp}^{\omega}_{(p)} = \mathcal{C}_0 \supsetneq \mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \cdots$$

such that every nonzero thick subcategory of $\mathrm{Sp}^{\omega}_{(p)}$ *is one of the* $\mathcal{C}_i$.

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The subcategories $\mathcal{C}_i$ can be described in terms of the geometry of the moduli stack M_{FG} of formal groups. Namely, recall that any finite spectrum defines a $\mathbb{Z}/2$–graded coherent sheaf on M_{FG} (via its complex bordism). The stack M_{FG} localized at p has a filtration by *height*

$$M_{\mathrm{FG}} \supset M_{\mathrm{FG}}^{\geq 1} \supset M_{\mathrm{FG}}^{\geq 2} \supset \cdots$$

such that the successive "differences" are quotients of a point by an ind-étale group scheme (and in particular, this tower cannot be refined); for a discussion of this, see Goerss [11]. The moduli stack $M_{\mathrm{FG}}^{\geq i}$ parametrizes formal groups of height at least i, and $M_{\mathrm{FG}}^{\geq 1}$ parametrizes formal groups over an $\mathbb{F}_p$–algebra. A spectrum $X \in \mathrm{Sp}_{(p)}^{\omega}$ belongs to $\mathcal{C}_i$ precisely when its associated quasicoherent sheaf is set-theoretically supported on $M_{\mathrm{FG}}^{\geq i}$. For instance, $\mathcal{C}_1$ consists of the p–torsion spectra.

In stable homotopy theory, Theorem 1.2 is extremely useful as a sort of "Tauberian" theorem. If one wishes to prove a property of all finite (p–local) spectra, eg of the sphere, then it suffices to show that the property is thick and that a single finite spectrum with nontrivial rational homology satisfies it.

After the results of [14], thick subcategories have been studied in a number of other settings. For instance, given a commutative ring R, one may consider the ∞–category $D_{\mathrm{perf}}(R)$ of perfect complexes of R–modules, and one may consider the thick subcategories of $D_{\mathrm{perf}}(R)$. Given a subset $Z \subset \operatorname{Spec} R$ closed under specialization (equivalently, a union of closed subsets), one may define a thick subcategory of $D_{\mathrm{perf}}(R)$ consisting of complexes whose cohomologies are set-theoretically supported on Z. When R is noetherian, one has the following theorem:

Theorem 1.3 (Hopkins [12] and Neeman [26]) *The above construction establishes a bijection between thick subcategories of $D_{\mathrm{perf}}(R)$ and subsets of* $\operatorname{Spec} R$ *closed under specialization.*

Given a symmetric monoidal stable ∞–category $(\mathcal{C}, \otimes, \mathbf{1})$, one may also consider *thick tensor ideals*: these are thick subcategories $\mathcal{D} \subset \mathcal{C}$ such that if $X \in \mathcal{D}, Y \in \mathcal{C}$, then the tensor product $X \otimes Y$ belongs to $\mathcal{D}$ as well. Thick tensor ideals have been extensively investigated (see Balmer [5]), and one can, for instance, arrange the "prime" ones into a topological space.

For example, for the perfect derived category of a noetherian scheme, thick tensor-ideals were classified in work of Thomason [32] generalizing Theorem 1.3. In this case, again thick tensor-ideals are classified in terms of the *supports* of objects. The notion of "support" has been axiomatized in work of Benson, Iyengar and Krause [7] with a view towards diverse applications, including stable module categories for finite groups, for triangulated categories with an action of a commutative ring as endomorphisms of the identity. In our setting, we will be working with symmetric monoidal stable ∞–categories (such as $D_{\mathrm{perf}}(R)$), where the unit object generates $\mathcal{C}$ under finite colimits, so thick subcategories are automatically tensor ideals. Moreover, the "base ring" will be somewhat complicated, so it will be convenient to work locally, using techniques of the author and Meier [24].

1.2 Methods

The goal of this paper is to classify thick subcategories in a different setting: that is, for ∞–categories of perfect modules over structured ring spectra which arise as global sections of the structure sheaf on even periodic derived stacks. Such ring spectra play an important role in stable homotopy theory: for instance, TMF arises as the ring of functions on a derived version of the moduli stack of elliptic curves.

Our goal is to understand the structure of the ∞–category $\mathrm{Mod}^{\omega}(\mathrm{TMF})$ of perfect TMF–modules, for instance. One difficulty in doing so is that the algebraic structure of $\pi_* \mathrm{TMF}$ is extremely complicated, while the analysis of ring spectra and modules over them is greatly simplified when one has nice (eg regular) homotopy rings. Nonetheless, we know that TMF is obtained as the homotopy inverse limit of a diagram of *elliptic spectra*, which are even periodic E_∞–rings whose formal group is associated to an elliptic curve classified by an étale map to M_{ell}. More precisely, there is a *derived Deligne–Mumford stack* $(M_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}})$, whose underlying ordinary stack is the moduli stack of elliptic curves, such that the global sections of the structure sheaf $\mathcal{O}^{\mathrm{top}}$ gives TMF. These elliptic spectra obtained by evaluating $\mathcal{O}^{\mathrm{top}}$ on an affine scheme étale over M_{ell} are much better behaved: not only are they even periodic, but their homotopy rings are regular noetherian. The classification of thick subcategories for perfect modules over *them* is significantly simpler by the following result which will be proved in Theorem 2.13 below.

Theorem 1.4 *Suppose R is an even periodic E_∞–ring with $\pi_0(R)$ regular noetherian. Then there is a canonical bijection between thick subcategories of the ∞–category of perfect R–modules and subsets of $\mathrm{Spec}\, \pi_0(R)$ closed under specialization.*

Therefore, we shall approach the classification of thick subcategories of $\mathrm{Mod}^{\omega}(\mathrm{TMF})$ by relating $\mathrm{Mod}^{\omega}(\mathrm{TMF})$ to the ∞–categories of perfect modules over elliptic spectra. The essential ingredients for doing this are in [24]. In that paper, Meier and the author prove:

Theorem 1.5 [24] *The ∞–category of* TMF*–modules* Mod(TMF) *is equivalent the ∞–category of quasicoherent sheaves on the derived moduli stack of elliptic curves.*

We refer to Lurie [20, Section 2.3] for generalities on quasicoherent sheaves on derived stacks. By [20, Proposition 2.3.12], we can rewrite this statement by saying that

$$\mathrm{Mod}(\mathrm{TMF}) \simeq \varprojlim_{\mathrm{Spec}\, R \to M_{\mathrm{ell}}} \mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\, R))$$

as $\mathrm{Spec}\, R \to M_{\mathrm{ell}}$ ranges over the étale maps from affine schemes.

More generally, the main result of [24] gives a criterion for this phenomenon of "affineness", which had been first explored by Meier in [25].

We briefly review the setup. Let X be a Deligne–Mumford stack equipped with a flat map $X \to M_{\mathrm{FG}}$. Given any affine scheme $\mathrm{Spec}\, R$ and an étale map $\mathrm{Spec}\, R \to X$, the composite $\mathrm{Spec}\, R \to X \to M_{\mathrm{FG}}$ classifies a formal group over R which yields an associated (weakly) even periodic, Landweber-exact homology theory and even a homotopy commutative ring spectrum. In particular, we get a functor

$$\mathrm{Aff}^{\mathrm{et}}_{/X} \to \text{Homotopy commutative ring spectra}$$

from the category of affine schemes étale over X to the category of homotopy commutative ring spectra and homotopy classes of maps between them. We refer to Lurie [19, Lecture 18] for the fundamentals of even periodic Landweber-exact ring spectra.

In certain important cases, one has a lifting

$$\begin{tikzcd} & \mathrm{CAlg} \arrow[d] \\ \mathrm{Aff}^{\mathrm{et}}_{/X} \arrow[ur, "\mathcal{O}^{\mathrm{top}}"] \arrow[r] & \text{homotopy commutative ring spectra} \end{tikzcd}$$

where CAlg is the ∞–category of E_∞–rings. Such a lift is called an *even periodic refinement* $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ of $X \to M_{\mathrm{FG}}$. Such even periodic refinements exist, for instance, for X the moduli stack of elliptic curves, and yield important examples of derived stacks.

Given an even periodic refinement $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$, we can consider the E_∞–ring $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ of "functions" on the derived stack $\mathfrak{X}$ and its ∞–category of modules. We can also consider a related ∞–category, the ∞–category $\mathrm{QCoh}(\mathfrak{X})$ of *quasicoherent sheaves* on $\mathfrak{X}$, obtained (by [20, Proposition 2.3.12]) as the homotopy limit of the module categories $\mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\, R))$ as $\mathrm{Spec}\, R \to X$ ranges over all étale maps from affines.

The main result of [24] now runs as follows.

Theorem 1.6 [24] *Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ be an even periodic refinement of a Deligne–Mumford stack X equipped with a flat map $X \to M_{\mathrm{FG}}$. If $X \to M_{\mathrm{FG}}$ is quasi-affine, then the global sections functor establishes an equivalence of symmetric monoidal ∞–categories*

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})).$$

The hypotheses of Theorem 1.6 apply in several important cases, such as the derived version of the moduli stack of elliptic curves as well as its Deligne–Mumford compactification.

1.3 Results

In this paper, we will use Theorem 1.6 to illuminate the structure of the ∞–category of *perfect* $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$–modules. In particular, we will classify the thick subcategories of perfect $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$–modules, under some further constraints.

In classifying thick subcategories, it will generally be more convenient to work with $\mathrm{QCoh}(\mathfrak{X})$ (at least implicitly), and in particular with the *homotopy group sheaves* π_0, π_1 of a given object in $\mathrm{QCoh}(\mathfrak{X})$, which in the perfect case will define a $\mathbb{Z}/2$–graded coherent sheaf on X. Given a subset of the underlying space of X closed under specialization, it follows that we can define a thick subcategory of $\mathrm{QCoh}(\mathfrak{X})$ consisting of objects whose homotopy group sheaves are supported on that subset.

The main result of this paper is:

Theorem 1.7 *Suppose X is regular and affine flat over M_{FG}. Then the thick subcategories of perfect modules over $\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})) \simeq \mathrm{QCoh}(\mathfrak{X})$ are in bijection (as indicated above) with the subsets of X closed under specialization.*

Theorem 1.7 in particular applies to the derived moduli stack of elliptic curves, and one has:

Corollary 1.8 *There is a bijection between thick subcategories of perfect* TMF*–modules and subsets of M_{ell} which are closed under specialization.*

We can also apply this to the classification of thick subcategories of perfect modules over the Hopkins–Miller EO_n–spectra (in view of [24, Section 6.2]).

Corollary 1.9 *Let G be a finite subgroup of the n^{th} Morava stabilizer group, so that G acts on Morava E–theory E_n. Then thick subcategories of perfect E_n^{hG}–modules are in bijection with G–invariant subsets of $\operatorname{Spec} \pi_0 E_n$ closed under specialization.*

The proof of Theorem 1.7 follows the outline of Theorem 1.2, although since we already have the nilpotence technology of Devinatz, Hopkins and Smith [8] and [14] and can in particular use results such as Theorem 1.2, many steps are much simpler. The proof that all possible thick subcategories come from subsets of X closed under specialization is essentially formal once certain "residue fields" are constructed, using the techniques of Baker and Richter [3; 4]. (The analog in Theorem 1.2 is given by the Morava K–theories.) The harder step is to show that all the different subsets closed under specialization are realized, which requires in addition some algebraic preliminaries about the structure of M_{FG} and topological preliminaries of vanishing lines in spectral sequences, which are of interest in themselves.

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2 The affine case

2.1 Setup

Let R be an even periodic E_∞–ring such that $\pi_0 R$ is *regular noetherian*. We do not need to assume that R is Landweber-exact. Consider the stable ∞–category $\mathrm{Mod}^\omega(R)$ of perfect R–modules. The goal of this section is to classify the thick subcategories of $\mathrm{Mod}^\omega(R)$ in Theorem 2.13 below.

Proposition 2.1 *Let R be as above, and let M be a perfect R–module. Then we have the following.*

1. *$\pi_0 M \oplus \pi_1 M$ is a finitely generated $\pi_0 R$–module.*
2. *Assume that $\pi_0 R$ has finite global (ie Krull) dimension. In this case, an R–module M is in fact perfect* if and only if *$\pi_0 M \oplus \pi_1 M$ is a finitely generated R–module.*

Proof The first assertion is equivalent to the assertion that $\pi_*(M)$ is a finitely generated $\pi_*(R)$–module. Consider the collection of $M \in \mathrm{Mod}^\omega(R)$ for which this holds. It clearly contains R. Since $\pi_*(R)$ is noetherian, it follows from the long exact sequence of a cofibering that this collection is a stable subcategory of $\mathrm{Mod}^\omega(R)$, and it is also closed under retracts. Therefore, it is all of $\mathrm{Mod}^\omega(R)$.

Suppose conversely that $\pi_0(R)$ has finite global dimension and $\pi_*(M)$ is a finitely generated $\pi_*(R)$–module. We claim that M is perfect. For this, we use descending induction on the (finite) projective dimension of $\pi_0(M) \oplus \pi_1(M)$ over $\pi_0(R)$. If $\pi_0(M), \pi_1(M)$ are projective R–modules, then it is well known that M is perfect: in fact, M is a retract of a sum of copies of R and ΣR. Suppose the projective dimension of $\pi_0(M) \oplus \pi_1(M)$ is equal to $n > 0$. Choose a map $R^s \vee \Sigma R^t \to M$ which induces an epimorphism on π_*, which we may do as $\pi_*(M)$ is a finitely generated $\pi_*(R)$–module, and let F be the fiber of this map. Clearly, $M \in \mathrm{Mod}^\omega(R)$ if and only if $F \in \mathrm{Mod}^\omega(R)$. But $\pi_*(F)$ is finitely generated, and $\pi_0(F) \oplus \pi_1(F)$ has projective dimension less than or equal to $n-1$. By induction, we are done. $\square$

Definition 2.2 If $M \in \mathrm{Mod}^\omega(R)$, the *support* of $\pi_0 M \oplus \pi_1 M$ thus defines a closed subset of $\mathrm{Spec}\, \pi_0 R$, which we will simply write as $\mathrm{Supp}\, M$.

Given a cofiber sequence

$$M' \to M \to M'' \to M'[1]$$

we have

$$\mathrm{Supp}\, M \subset \mathrm{Supp}\, M' \cup \mathrm{Supp}\, M''$$

and, furthermore, the support only shrinks under taking retracts. This enables us to define thick subcategories of $\mathrm{Mod}^\omega(R)$:

Definition 2.3 Given a closed subset $Z \subset \mathrm{Spec}\, \pi_0 R$, we let $\mathrm{Mod}^\omega_Z(R) \subset \mathrm{Mod}^\omega(R)$ be the full subcategory of perfect R–modules M such that $\mathrm{Supp}\, M \subset Z$. It follows from the preceding paragraph that $\mathrm{Mod}^\omega_Z(R)$ is a thick subcategory of $\mathrm{Mod}(R)$.

In general, we cannot expect every thick subcategory of $\mathrm{Mod}^\omega(R)$ to come from a closed subset $Z \subset R$. We might choose instead to work with a *family* of possible choices of Z. If R is a domain, we could, for example, consider the collection of all $M \in \mathrm{Mod}^\omega(R)$ which are *torsion*, ie those whose support does not contain the generic point. This is a thick subcategory, but it is not associated to any single closed subset, but rather to the collection of all closed subsets of $\mathrm{Spec}\, R$ that do not contain the generic point. Rather than working with "families of closed subsets", it is simpler to work with subsets of $\mathrm{Spec}\, R$ closed under specialization.

Definition 2.4 Let $Z \subset \mathrm{Spec}\, \pi_0 R$ be a subset of $\mathrm{Spec}\, \pi_0 R$ closed under specialization. In this case, we define $\mathrm{Mod}^\omega_Z(R) \subset \mathrm{Mod}^\omega(R)$ analogously to be the collection of $M \in \mathrm{Mod}^\omega(R)$ with $\mathrm{Supp}\, M \subset Z$. We thus get a *map* from the collection of specialization-closed subsets of $\mathrm{Spec}\, \pi_0 R$ to the collection of thick subcategories of $\mathrm{Mod}^\omega(R)$.

The goal of this section is to prove that the $\mathrm{Mod}^{\omega}_{Z}(R)$ are *precisely* the thick subcategories of $\mathrm{Mod}^{\omega}(R)$, as Z ranges over the specialization-closed subsets of $\operatorname{Spec} \pi_0 R$. The proof will follow the argument for finite spectra in [14]. The key step, as in [14], is a version of the nilpotence theorem, which is much easier in the present setting.

2.2 Residue fields

The first step is to define "residue fields". Consider the regular ring $\pi_0 R$. For each prime ideal $\mathfrak{p} \subset \pi_0 R$, the localization $(\pi_0 R)_{\mathfrak{p}}$ is a regular local ring, whose maximal ideal $\mathfrak{p}(\pi_0 R)_{\mathfrak{p}}$ is generated by a system of parameters $x_1, \ldots, x_n \in (\pi_0 R)_{\mathfrak{p}}$ such that $(\pi_0 R)_{\mathfrak{p}}/(x_1, \ldots, x_n)$ is a field $k(\mathfrak{p})$.

Definition 2.5 Given $\mathfrak{p}$, consider first $R_{\mathfrak{p}}$, the arithmetic localization of R at $\mathfrak{p}$, as an E_∞–R–algebra using Elmendorf, Kriz, Mandell and May [10, Theorem 2.2] or Lurie [22, Section 8.2.4]. Then consider the R–module $K(\mathfrak{p})$ defined as

$$K(\mathfrak{p}) = R_{\mathfrak{p}}/(x_1, \ldots, x_n) \stackrel{\text{def}}{=} R_{\mathfrak{p}}/x_1 \wedge_R R_{\mathfrak{p}}/x_2 \wedge_R \cdots \wedge_R R_{\mathfrak{p}}/x_n,$$

where R/x for $x \in \pi_0 R$ denotes the cofiber of $x \colon R \to R$, so that

$$K(\mathfrak{p})_* \simeq k(\mathfrak{p})[t^{\pm 1}], \quad |t| = 2.$$

By the results of Angeltveit [2], it follows that $K(\mathfrak{p})$ admits the structure of an A_∞–algebra internal to $\mathrm{Mod}(R)$.

Definition 2.6 For any R–module M, we can form the new R–module $K(\mathfrak{p}) \wedge_R M$ and in particular we obtain (by applying π_*) a *homology theory* $K(\mathfrak{p})_*$ on the category of R–modules, taking value in the category of graded $k(\mathfrak{p})[t^{\pm 1}]$–modules.

This homology theory $K(\mathfrak{p})$ is multiplicative and satisfies a Künneth isomorphism. Strictly speaking, the notation is abusive, since the multiplicative structure $K(\mathfrak{p})_*$ was not constructed only from $\mathfrak{p}$, but also used some other choices; any of those choices will be fine for our purposes.

The thick subcategories $\mathrm{Mod}^{\omega}_{Z}(R)$ can also be defined in terms of the homology theories $K(\mathfrak{p})_*$.

Proposition 2.7 *Given a* perfect *R–module M, we have that $M \in \mathrm{Mod}^{\omega}_{Z}(R)$ if and only if for every $\mathfrak{p} \notin Z$, $K(\mathfrak{p})_* M = 0$.*

Proof It suffices to show that $M_{\mathfrak{p}} = 0$ if and only if $K(\mathfrak{p})_* M = 0$, for any $M \in \mathrm{Mod}^{\omega}(R)$ and $\mathfrak{p} \in \operatorname{Spec} \pi_0 R$. This is a consequence of the fact that if $M_{\mathfrak{p}}/(x_1, \dots, x_n)$ is contractible and M is perfect, so is the Bousfield localization of M at $K(\mathfrak{p})$, ie the smash product $M \wedge_R \widehat{R}_{\mathfrak{p}}$ where the completion $\widehat{R}_{\mathfrak{p}}$ is given by

$$\widehat{R}_{\mathfrak{p}} \simeq \varprojlim_N R_{\mathfrak{p}}/(x_1^N, x_2^N, \dots, x_n^N).$$

We refer to Lurie [21, Section 4] for generalities on completions of E_∞–rings and modules. By classical commutative algebra, $\widehat{R}_{\mathfrak{p}}$ is faithfully flat over $R_{\mathfrak{p}}$, so that $M_{\mathfrak{p}}$ is itself contractible. □

The basic step towards the thick subcategory theorem is given by the following Bousfield decomposition in $\mathrm{Mod}(R)$:

Proposition 2.8 *If M is an R–module, then $M \simeq 0$ if and only if $K(\mathfrak{p})_* M = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$.*

Note that we make no compactness assumptions on M in this proposition.

Proof One direction is obvious, so suppose M is acyclic with respect to all the homology theories $K(\mathfrak{p})_*$. We would like to show that M is contractible. Suppose the contrary.

Since a module over $\pi_0 R$ vanishes if and only if all its localizations at prime ideals vanish, we can assume that $\pi_0 R$ is a (regular) local ring. We may use induction on the dimension of $\pi_0 R$, and thus assume that the localization of M at any nonmaximal prime ideal $\mathfrak{p} \subset \pi_0 R$ is trivial. It follows that if x belongs to the maximal ideal $\mathfrak{m} \subset \pi_0 R$, then the $R[x^{-1}]$–module $M[x^{-1}]$ has the property that its localization at any prime ideal is trivial, so $M[x^{-1}] \simeq 0$ for any $x \in \mathfrak{m}$.

Now we use [27, Lemma 1.34] of Ravenel: if N is a nontrivial R–module, then, for any $a \in \pi_0 R$, at least one of $N[x^{-1}]$ and N/xN has to be nontrivial. (This assertion would be false in ordinary algebra.) Then M/xM for each $x \in \mathfrak{m}$ is nontrivial. Repeating this, and applying the same argument, it follows that $M/(x_1, \dots, x_n)M$ is nontrivial for any system of parameters $(x_1, \dots, x_n)$ for $\mathfrak{m}$. It follows that $K(\mathfrak{m})_* M \neq 0$. □

Given the Bousfield decomposition, the rest of the proof of Theorem 2.13 (which follows [14]) is now completely formal, except for the last statement, and has been axiomatized in Hovey, Palmieri and Strickland [15, Section 6]. For completeness, we give a quick review of the argument.

Corollary 2.9 *If $\phi\colon \Sigma^k M \to M$ is a self-map in $\mathrm{Mod}^\omega(R)$, then ϕ is nilpotent if and only if $K(\mathfrak{p})_*(\phi)$ is nilpotent for each $\mathfrak{p} \in \operatorname{Spec} \pi_0 R$.*

Proof This is a formal consequence of the Bousfield decomposition. Namely, since M is compact, it follows that ϕ is nilpotent if and only if the colimit of

$$M \xrightarrow{\phi} \Sigma^{-k} M \xrightarrow{\phi} \Sigma^{-2k} M \to \cdots$$

is contractible, which by Proposition 2.8 holds if and only if, for each $\mathfrak{p}$, the diagram of finitely generated $k(\mathfrak{p})[t^{\pm 1}]$–modules induced by applying $K(\mathfrak{p})_*$ has zero colimit. This implies the result. □

Corollary 2.10 *Let R' be an R–ring spectrum, that is, a monoid object in the homotopy category of R–modules.*

(1) *Suppose $\alpha \in \pi_* R$ maps to a nilpotent element under the Hurewicz map $\pi_* R \to K(\mathfrak{p})_* R$ for each $\mathfrak{p}$; then α is nilpotent.*
(2) *Suppose a class $\alpha\colon R \to R' \wedge_R F$, for F an R–module, is zero in $K(\mathfrak{p})_*$ for each $\mathfrak{p}$. Then it has the property that $\alpha^k\colon R \to R' \wedge_R F^{\wedge k}$ is null for $k \gg 0$.*

Proof The first claim follows using a similar telescope; it implies the second claim by considering $R' \wedge JF$ for $JF = R \oplus F \oplus F^{\wedge 2} \oplus \cdots$ the free A_∞–algebra (in $\mathrm{Mod}(R)$) on F. □

2.3 Proof of the main result

We are now ready to prove a thick subcategory theorem in the affine case. We will split this into two pieces.

Proposition 2.11 *Let $M, N \in \mathrm{Mod}^\omega(R)$. Suppose $\operatorname{Supp} N \subset \operatorname{Supp} M$. Then the thick subcategory generated by M contains N.*

Proof The first claim is a consequence of Corollary 2.9. Hypotheses as in the claim, we consider a fiber sequence

$$F \xrightarrow{\phi} R \to M \wedge_R \mathbb{D} M,$$

where $\mathbb{D} M = \operatorname{Hom}_R(M, R)$ is the Spanier–Whitehead dual (in R–modules). By hypothesis, ϕ has the property that it is the zero map in $K(\mathfrak{p})_*$–homology for $\mathfrak{p} \in \operatorname{Supp} M$ (but not otherwise).

Moreover, the cofiber of ϕ belongs to the thick subcategory $\mathcal{C}_M \subset \mathrm{Mod}^{\omega}(R)$ generated by M. Thus the cofiber of each iterated tensor power $\phi^r \colon F^{\wedge r} \to R$ belongs to $\mathcal{C}_M$ too, and similarly for the cofiber of

$$1_N \wedge \phi^r \colon N \wedge F^{\wedge r} \to N.$$

But for large $r \gg 0$, these maps are zero. In fact, adjointing over, ϕ gives a map $\psi \colon R \to \mathrm{End}(N) \wedge \mathbb{D}F$ and we equivalently need to show that the sufficiently highly iterated tensor powers $\psi^r \colon R \to \mathrm{End}(N) \wedge \mathbb{D}F^{\wedge r}$ are zero. Since ψ is zero on $K(\mathfrak{p})_*$–homology for *all* $\mathfrak{p}$, this follows from Corollary 2.10.

If $1_N \wedge \phi^r$ is nullhomotopic for $r \gg 0$, it follows that the cofiber (which we saw belongs to $\mathcal{C}_M$) contains N as a direct summand, and we are done with the first claim. □

Proposition 2.12 *Let $Z \subset \mathrm{Spec}\, \pi_0 R$. Then there exists $M \in \mathrm{Mod}^{\omega}(R)$ such that* $\mathrm{Supp}\, M = Z$.

Proof Let $Z \subset \mathrm{Spec}\, \pi_0 R$ be a closed subset corresponding to the radical ideal I. Choose generators $x_1, \dots, x_n \in I$ and take as the desired module

$$M = R/(x_1, \dots, x_n) \simeq R/x_1 R \wedge_R \cdots \wedge_R R/x_n R.$$

Clearly this vanishes when any of the x_i are inverted. Conversely, if $\mathfrak{p} \supset I$, then $R/x_i R$ has nontrivial $K(\mathfrak{p})_*$–homology (since multiplication by x_i induces the zero map on $K(\mathfrak{p})_*$), and therefore the smash power that defines M has nontrivial $K(\mathfrak{p})_*$–homology as well. □

Theorem 2.13 *The thick subcategories of $\mathrm{Mod}^{\omega}(R)$ are precisely the $\{\mathrm{Mod}^{\omega}_Z(R)\}$ for $Z \subset \mathrm{Spec}\, \pi_0 R$ closed under specialization, and these are all distinct.*

We note that this result has been independently obtained in forthcoming work of Benjamin Antieau, Tobias Barthel and David Gepner, and is likely known to others as well.

Proof Given a thick subcategory $\mathcal{C} \subset \mathrm{Mod}^{\omega}(R)$, we associate to it the union Z' (in $\mathrm{Spec}\, R$) of the supports of all the objects in $\mathcal{C}$. Note that if Z_0, Z_1 occur as supports of objects in $\mathcal{C}$, then so does $Z_0 \cup Z_1$, by taking the direct sum. Proposition 2.11 will imply that this subset Z', which is closed under specialization, determines $\mathcal{C}$ in turn. Proposition 2.12 will imply that we can obtain any subset $Z' \subset \mathrm{Spec}\, R$ closed under specialization in this manner, by taking these modules for each closed subset of Z' and the thick subcategory they (together) generate. □

3 Vanishing lines in the descent spectral sequence

In the previous section, we classified the thick subcategories of $\mathrm{Mod}^{\omega}(R)$ when R is an even periodic E_∞–ring with $\pi_0 R$ regular noetherian. The classification relied on the construction of certain "residue fields" of R to detect nilpotence, and then the verification that all the possible thick subcategories one could thus hope for actually exist. The latter part was very easy in the affine case, but it is somewhat trickier in the general case: we actually will need to produce elements in the homotopy groups of $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ for an even periodic derived stack $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$.

We will do this by producing such elements in the descent spectral sequence, and by establishing a general horizontal vanishing line for such descent spectral sequences (based on nilpotence technology). The latter is the goal of this section, and may be of independent interest. We note that in the setting of Theorem 1.2, one is not working in an E_n–localized setting, so the corresponding vanishing line arguments are significantly more difficult than they are for us.

3.1 Towers

Let $\mathcal{C}$ be a stable ∞–category.

Definition 3.1 Let $\mathrm{Tow}(\mathcal{C})$ be the ∞–category of *towers* of objects of $\mathcal{C}$, that is, $\mathrm{Tow}(\mathcal{C}) \simeq \mathrm{Fun}((\mathbb{Z}_{\geq 0})^{\mathrm{op}}, \mathcal{C})$.

To any element of $\mathrm{Tow}(\mathcal{C})$ and object $P \in \mathcal{C}$, there is associated a spectral sequence converging to the homotopy groups of the space (or spectrum) of maps from P into the homotopy inverse limit. For instance, given a cosimplicial spectrum, the spectral sequence associated to the Tot tower is the Bousfield–Kan homotopy spectral sequence.

Let us single out a certain subcategory $\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C})$ of $\mathrm{Tow}(\mathcal{C})$ consisting of objects whose spectral sequences have extremely good convergence properties.

Definition 3.2 The subcategory $\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C}) \subset \mathrm{Tow}(\mathcal{C})$ consists of towers $\cdots \to X_n \to X_{n-1} \to \cdots \to X_0$ with the property that there exists an integer $r > 0$, such that all r–fold composites in the tower are nullhomotopic. The subcategory $\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C})$ is the union of an ascending sequence

$$\mathrm{Tow}^{\mathrm{nil}}_1(\mathcal{C}) \subset \mathrm{Tow}^{\mathrm{nil}}_2(\mathcal{C}) \subset \cdots ,$$

where $\mathrm{Tow}^{\mathrm{nil}}_{r'} \subset \mathrm{Tow}^{\mathrm{nil}}$ consists of towers such that every r'–fold composite is null.

Definition 3.3 More generally, given a collection of objects $\mathfrak{U} \subset \mathcal{C}$, we define $\mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C})$ as the collection of towers $\cdots \to X_n \to X_{n-1} \to \cdots$ such that there exists $r \in \mathbb{Z}_{>0}$ such that every r–fold composite $X_n \to X_{n-r}$ induces the zero map

$$[U, X_n]_* \to [U, X_{n-r}]_* \quad \text{for each } U \in \mathfrak{U}.$$

Taking $\mathfrak{U} = \mathcal{C}$ gives $\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C})$.

We also need a generalization of $\mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C})$ to handle towers that are "very close to constant", but not necessarily at zero.

Definition 3.4 We define $\mathrm{Tow}^{\mathrm{fast}}_{\mathfrak{U}}(\mathcal{C})$ to be the collection of those towers $\{X_n\}$ such that $\varprojlim_i X_i$ exists and such that the cofiber of the map of towers

$$\{\varprojlim_i X_i\} \to \{X_n\},$$

where the first term is the constant tower, belongs to $\mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C})$.

The basic permanency property of these subcategories is given by:

Proposition 3.5 (Hopkins, Palmieri and Smith [13]) *For each $\mathfrak{U} \subset \mathcal{C}$, we have that $\mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C}), \mathrm{Tow}^{\mathrm{fast}}_{\mathfrak{U}}(\mathcal{C}) \subset \mathrm{Tow}(\mathcal{C})$ are thick subcategories.*

Proof The assertion about $\mathrm{Tow}^{\mathrm{fast}}_{\mathfrak{U}}(\mathcal{C})$ follows from the assertion about $\mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C})$, so we need only handle this case. Proposition 3.5 is essentially contained in [13, Corollary 2.3], but we give the proof. It is easy to see that $\mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C}) \subset \mathrm{Tow}(\mathcal{C})$ is closed under retracts and under suspensions and desuspensions. It suffices to show that given a cofiber sequence in $\mathrm{Tow}(\mathcal{C})$

$$\{X_n\} \to \{Y_n\} \to \{Z_n\},$$

if $\{X_n\}, \{Z_n\} \in \mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C})$, then $\{Y_n\} \in \mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C})$.

To see this, suppose every r–fold composite in $\{X_n\}$ induces the zero map on $[U, \cdot]_*$ for each $U \in \mathfrak{U}$, and every r'–fold composite in $\{Z_n\}$ induces the zero map on $[U, \cdot]_*$ for each $U \in \mathfrak{U}$. We claim that every $(r + r')$–fold composite in $\{Y_n\}$ induces the zero map on $[U, \cdot]_*$ for each $U \in \mathfrak{U}$. Fix $U \in \mathfrak{U}$ and let $n \geq 0$. Choose any map

$f\colon \Sigma^k U \to Y_{n+r+r'}$ for $k \in \mathbb{Z}$ arbitrary. Then we have a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd}
X_{n+r+r'} \arrow[r] \arrow[d] & Y_{n+r+r'} \arrow[r] \arrow[d] & Z_{n+r+r'} \arrow[d] \\
X_{n+r} \arrow[r] \arrow[d] & Y_{n+r} \arrow[r] \arrow[d] & Z_{n+r} \arrow[d] \\
X_n \arrow[r] & Y_n \arrow[r] & Z_n
\end{tikzcd}$$

A diagram chase now shows that the composite of f with $Y_{n+r+r'} \to Y_n$ is null. In fact, the composite of f with $Y_{n+r+r'} \to Y_{n+r} \to Z_{n+r}$ is null, and so factors through X_{n+r}. But the composite of any map from $\Sigma^k U \to X_{n+r}$ with $X_{n+r} \to X_n$ is null. □

Remark 3.6 Given a tower $\cdots \to X_n \to X_{n-1} \to \cdots \to X_0$ in $\mathrm{Tow}(\mathcal{C})$, it can belong to $\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C})$ only if it is *pro-zero*, that is, if the associated pro-object in $\mathrm{Pro}(\mathcal{C})$ is equivalent to the zero pro-object, as its inverse limit is contractible and this property is preserved under any exact functor. The converse is false: if a cofinal subset of the X_i are contractible, the tower is automatically pro-zero, but such a tower need not belong to $\mathrm{Tow}^{\mathrm{nil}}$. The associated spectral sequence may support arbitrarily long differentials.

3.2 Vanishing lines

We keep the notation of the previous subsection. We can give an interpretation of $\mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C})$ in terms of vanishing lines. Before it, we make the following definitions.

Definition 3.7 Given an inverse system $\cdots \to A_n \to A_{n-1} \to \cdots \to A_0$ of abelian groups, we define the r^{th} *derived system* to be the inverse system of abelian groups $\{\mathrm{im}(A_{n+r} \to A_n)\}_{n \in \mathbb{Z}_{\geq 0}}$.

Definition 3.8 An inverse system of abelian groups $\cdots \to A_n \to A_{n-1} \to \cdots \to A_0$ is *eventually constant* if the maps $A_n \to A_{n-1}$ are isomorphisms for $n \gg 0$.

Since $\mathcal{C}$ is a stable ∞–category, there is a natural *spectrum* of maps between any two objects $X, Y \in \mathcal{C}$, which for the next result we write simply as $\mathrm{Hom}(X, Y)$.

Proposition 3.9 *The tower $\cdots \to X_n \to X_{n-1} \to \cdots$ in $\mathcal{C}$ belongs to $\mathrm{Tow}^{\mathrm{nil}}_{\mathfrak{U}}(\mathcal{C})$ if and only if, for each $U \in \mathfrak{U}$, the spectral sequence associated to the tower of spectra*

$$\cdots \to \mathrm{Hom}(U, X_n) \to \mathrm{Hom}(U, X_{n-1}) \to \cdots \to \mathrm{Hom}(U, X_0)$$

collapses to zero at a finite stage independent of U (that is, the r^{th} page is identically zero for some r, independent of U).

Proof We consider first the case $\mathcal{C} = \mathrm{Sp}$ and $\mathfrak{U} = \{S^0\}$, and assume that we have a tower of spectra $\{X_n\}_{n\in\mathbb{Z}_{\geq 0}}$, which we extend to $n<0$ by taking $X_{-1} = X_{-2} = \cdots = 0$. In this case, we recall the definition of the spectral sequence associated to the tower. Let F_i be the fiber of $X_i \to X_{i-1}$. One has an exact couple

$$\begin{tikzcd} \bigoplus_i \pi_* X_i \arrow[rr, "\phi"] & & \bigoplus_i \pi_* X_{i-1} \arrow[dl] \\ & \bigoplus_i \pi_* F_i \arrow[ul] & \end{tikzcd}$$

where ϕ is obtained as the direct sums of the maps $\pi_* X_i \to \pi_* X_{i-1}$. The spectral sequence for the homotopy groups of $\varprojlim X_i$ is obtained by repeatedly deriving this exact couple. In particular, the successive derived couples are of the form

$$\begin{tikzcd} \mathrm{Im}\,\phi^{r-1} \arrow[rr, "\phi"] & & \mathrm{Im}\,\phi^{r-1} \arrow[dl] \\ & E_r^{*,*} \arrow[ul] & \end{tikzcd}$$

so that if $E_r^{*,*}$ is identically zero for some r, then ϕ necessarily (by exactness) induces an automorphism of $\mathrm{Im}\,\phi^{r-1}$. In other words, if we consider the inverse system $\{\pi_* X_i\}_{i\in\mathbb{Z}}$, it follows that the r^{th} derived system is constant, and thus necessarily zero, by considering indices below zero.

For the converse, if $\{X_n\} \in \mathrm{Tow}^{\mathrm{nil}}_{\{S^0\}}(\mathrm{Sp})$, say all r–fold composites in the inverse system are zero, then reversing the above argument shows that the exact couple degenerates to zero at the $r+1^{\text{st}}$ stage: the $\mathrm{Im}\,\phi^r$ terms are zero.

The case of a general $(\mathcal{C}, \mathfrak{U})$ now easily reduces to this, because (as in the above argument), the point at which the spectral sequence collapses and the number of successive composites needed to make all the maps nullhomotopic are functions of each other. □

Similarly, we can give a criterion for belonging to $\mathrm{Tow}^{\mathrm{fast}}_{\mathfrak{U}}(\mathcal{C})$. First, however, we need to prove some lemmas about pro-systems of abelian groups.

Lemma 3.10 *Consider a half-exact sequence of inverse systems of abelian groups*

$$\{X_i\} \to \{Y_i\} \to \{Z_i\}.$$

(1) *If $\{Y_i\}, \{Z_i\}$ have eventually constant r^{th} derived systems and $0 \to \{X_i\} \to \{Y_i\}$ is exact, then $\{X_i\}$ has an eventually constant r^{th} derived system.*

(2) *If $\{X_i\}, \{Y_i\}$ have eventually constant r^{th} derived systems and $\{Y_i\} \to \{Z_i\} \to 0$ is exact, then $\{Z_i\}$ has an eventually constant r^{th} derived system.*

(3) *If*

$$0 \to \{X_i\} \to \{Y_i\} \to \{Z_i\} \to 0$$

is exact, and $\{X_i\}$ and $\{Z_i\}$ have eventually constant r^{th} derived systems, then $\{Y_i\}$ has an eventually constant $2r^{th}$ derived system.

Proof We will prove the first and third of these assertions. The second is dual to the first and can be proved similarly (or via an "opposite category" argument, since everything here works in an arbitrary abelian category). The three assertions state that the subcategory of the category of towers of abelian groups consisting of those towers with an eventually constant r^{th} derived system for $r \gg 0$ is closed under finite limits and colimits, and extensions too.

(1) Suppose $\{Y_i\}, \{Z_i\}$ have eventually constant r^{th} derived systems and $\{X_i\}$ is the kernel inverse system of $\{Y_i\} \to \{Z_i\}$. In this case, we want to show that the r^{th} derived system of $\ker(Y_i \to Z_i)$ is eventually constant.

For $i \gg 0$, it follows that

$$\phi\colon \mathrm{Im}(\phi^r\colon X_i \to X_{i-r}) \to \mathrm{Im}(\phi^r\colon X_{i-1} \to X_{i-r-1})$$

is injective, because the analog is true for $\{Y_i\}$.

The harder step is to show that the map is surjective. Equivalently, for $j \gg 0$, we must show that any element of X_j that can be lifted up to X_{j+r} can also be lifted up to X_{j+r+1} (not necessarily in a compatible manner). Fix $x_j \in X_j$ admitting a lift $x_{j+r} \in X_{j+r}$. Then the image y_{j+r} of x_{j+r} in Y_{j+r} lifts the image y_j of x_j in Y_j. It follows that y_j, since it lifts r times, is actually the image of an element $y'_{j+r+1} \in Y_{j+r+1}$ which is "permanent", ie which lifts arbitrarily. The image $z'_{j+r+1} \in Z_{j+r}$ of y'_{j+r+1} maps to zero in Z_j, but z'_{j+r+1} is also "permanent", so it must itself vanish by assumption. Thus y'_{j+r+1} is the image of $x'_{j+r+1} \in X_{j+r+1}$ and this lifts x_j.

(3) Suppose $\{X_i\}, \{Z_i\}$ have eventually constant r^{th} derived systems. Suppose $j \gg 0$ and suppose $y_j \in Y_j$ is in the image of Y_{j+2r}. We need to show two things.

- If $y_j \neq 0$, then the image of y_j in Y_{j-1} is not zero.
- y_j can be lifted to Y_{j+2r+1}.

For the first item, if the image z_j of y_j in Z_j is not zero, then the image z_{j-1} of z_j in Z_{j-1} is also nonzero, because z_j is permanent. If $z_j = 0$, then y_j comes from X_j, and we can apply the same argument to X_j.

Namely, choose $y_{j+2r} \in Y_{j+2r}$ lifting y_j and let $y_{j+r} \in Y_{j+r}$ be the image. The image $z_{j+r} \in Z_{j+r}$ is a permanent element which, by hypothesis, projects to zero in Z_j, so must be zero. In particular, $y_{j+r} \in Y_{j+r}$ comes from X_{j+r}, which means that y_j not only comes from X_j but also that it is the image of a (nonzero) permanent element in X_j. The image of this in X_{j-1} thus cannot be zero. This completes the proof of the first item.

For the second item, the image $z_j \in Z_j$ of y_j is a permanent element, so it lifts uniquely to a permanent element z_{j+2r+1}. Choose a lift $y'_{j+2r+1} \in Y_{j+2r+1}$ of z_{j+2r+1}. Let y'_{j+2r} be the image of y'_{j+2r+1} in Y_{j+2r}. Then $y_{j+2r} - y'_{j+2r}$ maps to an element in Z_{j+2r} which maps to zero in Z_j, and thus maps to zero in Z_{j+r}. In particular, $y_{j+r} - y'_{j+r}$ belongs to X_{j+r}; call it $\bar{x}_{j+r}$. Taking images in Y_j, we get

$$y_j = \bar{x}_j + y'_j,$$

where $\bar{x}_j$ is the image of $\bar{x}_{j+r}$ and y'_j is the image of y'_{j+2r+1}. But $\bar{x}_j$ is permanent, and y'_j is the image of something in Y_{j+2r+1}, so this completes the proof. □

Lemma 3.11 *Let $\{A_i\}, \{B_i\}, \{C_i\}$ be three pro-systems of graded abelian groups. Suppose that there is a long exact sequence of pro-systems*

$$\begin{tikzcd} \{A_i\} \arrow[rr] & & \{B_i\} \arrow[dl] \\ & \{C_i\} \arrow[ul] & \end{tikzcd}$$

where the map $\{C_i\} \to \{A_i\}$ lowers grading by 1. Suppose the r^{th} derived system of $\{A_i\}$ and $\{B_i\}$ are eventually constant. Then the $2r^{th}$ derived system of $\{C_i\}$ is eventually constant.

Proof More generally, suppose given an exact sequence of inverse systems of abelian groups

$$\{M_i\} \to \{N_i\} \to \{P_i\} \to \{Q_i\} \to \{R_i\},$$

and suppose that the r^{th} derived systems of the inverse systems $\{M_i\}, \{N_i\}, \{Q_i\}, \{R_i\}$ are eventually constant. Then we claim that the $2r^{\text{th}}$ derived system of $\{P_i\}$ is eventually constant. In fact, this follows by applying Lemma 3.10 three times, and implies the present result. □

Proposition 3.12 *The tower* $\cdots \to X_n \to X_{n-1} \to \cdots$ *in* $\mathcal{C}$ *belongs to* $\mathrm{Tow}^{\mathrm{fast}}_{\mathfrak{U}}(\mathcal{C})$ *if and only if, for each* $U \in \mathfrak{U}$*, the spectral sequence associated to the tower of spectra*

$$\cdots \to \mathrm{Hom}(U, X_n) \to \mathrm{Hom}(U, X_{n-1}) \to \cdots \to \mathrm{Hom}(U, X_0)$$

collapses at a finite stage with a horizontal vanishing line independent of U*. In other words, there should exist* r, N *such that* $E_r^{s,*} = 0$ *for* $s > N$*, in the spectral sequence associated to the above tower for each* U*.*

Proof Without loss of generality, suppose that $\mathfrak{U} = \{S^0\}$ and $\mathcal{C} = \mathrm{Sp}$. Analysis with exact couples as in the proof of Proposition 3.9 shows that the spectral sequence condition of the proposition holds if and only if there exists r, N such that the map

$$\phi\colon (\mathrm{Im}\,\phi^r\colon \pi_*(X_s) \to \pi_*(X_{s-r})) \to (\mathrm{Im}\,\phi^r\colon \pi_*(X_{s-1}) \to \pi_*(X_{s-r-1}))$$

is an isomorphism for all $s > N$. In other words, the r^{th} derived system of the pro-system $\{\pi_* X_n\}$ is *eventually constant* (rather than constant at zero as in the proof of Proposition 3.9). Necessarily, the stable value of the pro-system must be $\pi_* \varprojlim X_i$.

By Lemma 3.11, it follows that if the condition of the proposition holds for the tower $\{X_i\}_{i \in \mathbb{Z}_{\geq 0}}$, then it also holds for the tower $\{\mathrm{cofib}(\varprojlim X_j \to X_i)\}_{i \in \mathbb{Z}_{\geq 0}}$, since it clearly holds for the constant tower with value $\varprojlim X_j$, and conversely. It follows that we can reduce to the case where $\varprojlim X_j$ is contractible, which is precisely Proposition 3.9. □

Observe that any object in $\mathrm{Tow}^{\mathrm{fast}}_{\mathcal{C}}(\mathcal{C})$ defines a tower yielding a *constant* pro-object: this follows from the analogous assertion about $\mathrm{Tow}^{\mathrm{nil}}_{\mathcal{C}}(\mathcal{C})$.

Remark 3.13 Suppose $\mathfrak{U} = \mathcal{C}$ and $\mathcal{C}$ is presentable. Then if T is any spectrum and $\{X_n\} \in \mathrm{Tow}^{\mathrm{fast}}_{\mathcal{C}}(\mathcal{C})$, the tower $\{T \wedge X_n\}$ also belongs to $\mathrm{Tow}^{\mathrm{fast}}_{\mathcal{C}}(\mathcal{C})$, where we recall that $\mathcal{C}$ is tensored over Sp. This is a consequence of the fact that $\{X_n\}$ defines a constant pro-object in $\mathcal{C}$, so that the natural map

$$T \wedge \varprojlim X_i \to \varprojlim (T \wedge X_i)$$

is an equivalence.

3.3 Vanishing in the descent spectral sequence

Our goal is to show that given $(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$, then for any $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, the Tot tower associated to the given cosimplicial spectrum that "computes" $\Gamma(\mathfrak{X}, \mathcal{F})$ (ie based on a choice of affine, étale hypercovering of X) has a horizontal vanishing line in the homotopy spectral sequence.

Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ be an even periodic refinement of a Deligne–Mumford stack X equipped with a flat map $X \to M_{\mathrm{FG}}$.

Theorem 3.14 *Suppose $X \to M_{\mathrm{FG}}$ is tame.*[1] *There exists $s, N \in \mathbb{Z}_{>0}$ such that for any quasicoherent sheaf $\mathcal{F}$ on $\mathfrak{X}$, the descent spectral sequence for $\pi_*\Gamma(\mathfrak{X}, \mathcal{F})$ has a horizontal vanishing line of height N at the s^{th} page.*

Proof Note that it suffices to *fix* $\mathcal{F}$ and then prove the result for $\mathcal{F}$ specifically: if s and N could not be taken independently of $\mathcal{F}$, taking appropriate wedges would provide a counterexample.

Let M be the least common multiple of the orders of all the automorphism group schemes of geometric points of X. Then, the stack $X[M^{-1}]$ is itself tame, and so (as in [24, Proposition 2.24]) has bounded cohomology, and thus the E_2–page of the descent spectral sequence has a horizontal vanishing line at E_2 itself after inverting some M.

We now work p–locally for a fixed "bad" prime p. In this case, all the spectra $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$ for étale maps $\operatorname{Spec} R \to X$ are L_n–local for some n; see the discussion at the beginning of [24, Section 4.2] and in particular [24, Lemma 4.9]. The collection $\mathcal{C}_{\mathcal{F}} \subset \mathrm{Sp}$ of spectra T such that the Tot tower for $\Gamma(\mathfrak{X}, \mathcal{F} \wedge T)$ belongs to $\mathrm{Tow}^{\mathrm{fast}}_{\Gamma(\mathfrak{X},\mathcal{O}^{\mathrm{top}})}(\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$ is a thick subcategory. Any thick subcategory of $L_n\,\mathrm{Sp}$ that contains the smash powers of E_n contains $L_n S^0$ by the Hopkins–Ravenel smash product theorem. It thus suffices to show that the smash powers of E_n belong to $\mathcal{C}_{\mathcal{F}}$. Since $\mathcal{F}$ was arbitrary, we need to show that the Tot tower for $\Gamma(\mathfrak{X}, \mathcal{F} \wedge E_n)$ belongs to $\mathrm{Tow}^{\mathrm{fast}}_{\Gamma(\mathfrak{X},\mathcal{O}^{\mathrm{top}})}(\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})))$.

However, as in [24, Proposition 4.10], since the stack $X \times_{M_{\mathrm{FG}}} \operatorname{Spec} \pi_0 E_n$ is tame, the descent spectral sequence $\Gamma(\mathfrak{X}, \mathcal{F} \wedge E_n)$ already has a horizontal vanishing line at E_2 (and therefore degenerates at a finite stage), because the fiber product $X \times_{M_{\mathrm{FG}}} \operatorname{Spec} \pi_0 E_n$ is a (quasicompact, separated) *tame* stack. It follows that the statement of the proposition holds p–locally. □

Remark 3.15 The descent spectral sequence for $\pi_*\Gamma(\mathfrak{X}, \mathcal{F})$ will generally be very infinite at the E_2–page because of "stackiness". The descent spectral sequence for KO–theory, which is also the homotopy fixed point spectral sequence for $\mathrm{KO} \simeq \mathrm{KU}^{h\mathbb{Z}/2}$, is displayed in Figure 1. The class η is not nilpotent in the E_2–page, until a d_3 kills η^3. The same phenomenon occurs in the TMF spectral sequence, which is computed in Bauer [6] and Konter [17]. It is a finiteness property of the E_n–local stable homotopy

[1] This is equivalent to the condition that for every geometric point $\operatorname{Spec} k \to X$, the kernel of the automorphism group to the automorphism group of the associated formal group has order prime to the characteristic of k. We refer to Abramovich, Olsson and Vistoli [1] for the general theory. In [24, Proposition 4.10], it is shown that this hypothesis implies that the global sections functor on the derived stack $\mathfrak{X}$ commutes with filtered colimits. For example, representable maps are tame.

category: the E_n–local Adams–Novikov spectral sequence exhibits the same property; see Hovey and Strickland [16].

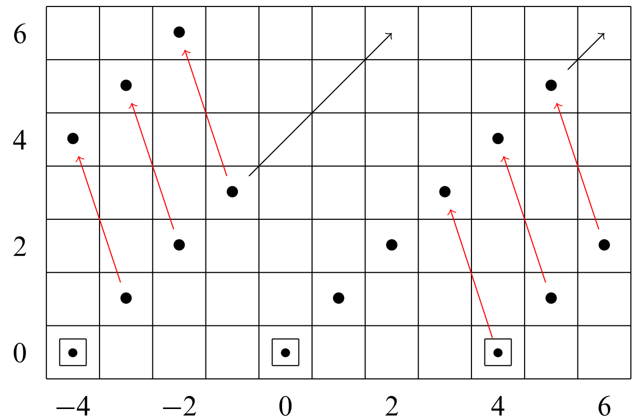

Figure 1: The descent (or homotopy fixed point) spectral sequence for $\mathrm{KO} \simeq K^{h\mathbb{Z}/2}$: arrows in red indicate differentials, while arrows in black indicate recurring patterns. Dots indicate copies of $\mathbb{Z}/2$, while squares indicate copies of $\mathbb{Z}$.

4 Extension to stacks

Fix a *regular* Deligne–Mumford stack X. Let $X \to M_{\mathrm{FG}}$ be a flat morphism and consider an even periodic refinement $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ of this data. We have considered the ∞–category $\mathrm{QCoh}(\mathfrak{X})$ of quasicoherent sheaves on $\mathfrak{X}$, which has good finiteness properties if $X \to M_{\mathrm{FG}}$ is tame, as explored in [24, Section 4].

In this section, we will define thick subcategories of $\mathrm{QCoh}(\mathfrak{X})$ associated to every closed substack of X and prove one half of the thick subcategory theorem. We will also illustrate how the other, more difficult, half can be deduced in the special case of a quotient stack by a finite group.

4.1 Definitions

We keep the previous notation and use the following hypotheses.

Hypotheses (1) X is a regular, separated, noetherian Deligne–Mumford stack.

(2) $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ is an even periodic refinement of a flat map $X \to M_{\mathrm{FG}}$.

(3) The derived stack $\mathfrak{X}$ has the property that the global sections functor establishes an equivalence $\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})) \simeq \mathrm{QCoh}(\mathfrak{X})$. The main result of [24] implies that this holds if $X \to M_{\mathrm{FG}}$ is quasi-affine.

Definition 4.1 Let $\mathrm{QCoh}^{\omega}(\mathfrak{X})$ be the subcategory of quasicoherent sheaves $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ such that for each étale map $\operatorname{Spec} R \to X$, the $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$–module $\mathcal{F}(\operatorname{Spec} R)$ is perfect. Thus, $\mathrm{QCoh}^{\omega}(\mathfrak{X})$ is the ∞–category of *dualizable objects* in $\mathrm{QCoh}(\mathfrak{X})$. Given $\mathcal{F} \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$, the homotopy group sheaves $\pi_i \mathcal{F}$, $i \in \mathbb{Z}$, define *coherent* sheaves on X.

Remark 4.2 Under taking global sections, $\mathrm{QCoh}^{\omega}(\mathfrak{X})$ then corresponds to the dualizable, or equivalently perfect, $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$–modules. This follows because dualizability is a local condition (we refer to [22, Section 4.2.5] for the theory of duality in ∞–categories), and since the dualizable objects in a module category are precisely the perfect modules.

Recall that there is a *topological space* associated to X in the sense of Laumon and Moret-Bailly [18, Chapter 5]. Fix a closed *subset* $Z \subset X$: equivalently, this is an equivalence class of closed substacks of X, where two closed substacks $Z, Z' \subset X$ are equivalent if and only if there is a third substack $Z'' \subset X$ containing Z, Z' as a nilpotent thickening.

Definition 4.3 Given a coherent sheaf $\mathcal{G}$ on X, we say that it is *supported* on Z if there exists a closed substack of X with Z as its underlying space on which $\mathcal{G}$ is supported. (This is a condition on the fibers of $\mathcal{G}$ at field-valued points.)

If we have a particular closed substack $Z \subset X$ in mind (and not simply an equivalence class), we will also say that $\mathcal{G}$ is *scheme-theoretically supported on* Z if $\mathcal{G}$ is an $\mathcal{O}_Z$–module.

Definition 4.4 Let Z be a subset of X closed under specialization. We define a thick subcategory $\mathrm{QCoh}^{\omega}_Z(\mathfrak{X}) \subset \mathrm{QCoh}^{\omega}(\mathfrak{X})$ consisting of those $\mathcal{F} \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ such that the homotopy group sheaves $\pi_i \mathcal{F}$ are supported set-theoretically on Z. We thus get a map from specialization-closed subsets of X to thick subcategories of $\mathrm{Mod}^{\omega}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$.

Using exact sequences, one sees that $\mathrm{QCoh}^{\omega}_Z(\mathfrak{X})$ is in fact thick. Our first goal is to show that every thick subcategory of $\mathrm{QCoh}^{\omega}(X)$ is of this form. As in Section 2, it suffices to construct a sufficient collection of "residue fields". To do this, choose an étale surjection $\operatorname{Spec} R \to X$. The E_∞–ring $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$ fits into the setting of the previous subsection, and we can construct homology theories $K(\mathfrak{p})_*$ on $\mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R))$ for each $\mathfrak{p} \in \operatorname{Spec} R$.

Via pullback, we can define these homology theories on $\mathrm{QCoh}(\mathfrak{X})$. In particular, it follows that given $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, we can define

$$K(\mathfrak{p})_* \mathcal{F} \stackrel{\mathrm{def}}{=} \pi_*(\mathcal{F}(\operatorname{Spec} R)_{\mathfrak{p}}/(x_1, \dots, x_r)),$$

where $x_1, \dots, x_r \in R_{\mathfrak{p}}$ is a system of parameters. Every point of X can be represented by a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ (the topological space of X is the quotient of $\operatorname{Spec} R$ under the two maps $\operatorname{Spec} R \times_X \operatorname{Spec} R \rightrightarrows \operatorname{Spec} R$), and as in the previous section, it follows that the support of $\mathcal{F} \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ contains the point corresponding to $\mathfrak{p} \in \operatorname{Spec} R$ if and only if $K(\mathfrak{p})_* \mathcal{F} \neq 0$.

Proposition 4.5 *Given $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, $\mathcal{F} \simeq 0$ if and only if $K(\mathfrak{p})_* \mathcal{F} = 0$ for each $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof This is a consequence of Proposition 2.8, since $\mathcal{F}$ is contractible if and only if $\mathcal{F}(\operatorname{Spec} R)$ is. □

Now the entire thick subcategory argument (reviewed in the previous section for R–modules) can be carried out in $\mathrm{QCoh}^{\omega}(\mathfrak{X})$, or, one can appeal to the axiomatic framework in [15]. In particular, just as in Proposition 2.11, one concludes:

Proposition 4.6 *If $\mathcal{F}, \mathcal{F}' \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ and $\operatorname{Supp} \mathcal{F} \subset \operatorname{Supp} \mathcal{F}'$, then the thick subcategory generated by $\mathcal{F}'$ contains $\mathcal{F}$.*

Corollary 4.7 *Every thick subcategory of $\mathrm{QCoh}^{\omega}(\mathfrak{X})$ is of the form $\mathrm{QCoh}^{\omega}_Z(\mathfrak{X})$ for some subset $Z \subset X$ closed under specialization.*

The primary goal of the rest of this paper will be to study when the different $\mathrm{QCoh}^{\omega}_Z(\mathfrak{X})$ are distinct: that is, we would like to know when we can realize a given closed subset as the support of a sheaf on $\mathfrak{X}$. We will answer this question, via Theorem 1.7, when $X \to M_{\mathrm{FG}}$ is *affine*.

4.2 The case of a quotient stack

In this subsection, we consider a case of Theorem 1.7. Suppose the Deligne–Mumford stack X is given by the quotient of an affine scheme by a finite group, so

$$X = (\operatorname{Spec} R_0)/G,$$

where $|G| < \infty$, and R_0 is a regular noetherian ring. Consider a flat, affine map $X \to M_{\mathrm{FG}}$ and an even periodic refinement $(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ of this map. In this case, we have a G–action on the E_∞–ring $R = \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R_0)$, and

$$\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}) \simeq R^{hG}.$$

This case was discussed in [24], and it is quite general (for instance, it includes M_{ell} once any prime is inverted). By the main result of [24], one has:

Proposition 4.8 *We have an equivalence*

$$\mathrm{Mod}(R^{hG}) \simeq \mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{Mod}(R)^{hG}.$$

Our goal is to show, as claimed in Theorem 1.7, that every thick subcategory of $\mathrm{Mod}(R^{hG})$ should arise from a G–invariant subset of $\operatorname{Spec} R_0$ closed under specialization (ie a union of closed substacks of X). We begin with some lemmas.

Lemma 4.9 *Let $E_r^{s,t}$ be an upper half-plane spectral sequence of (not necessarily commutative) algebras for $s \geq 0, t \in \mathbb{Z}$. Suppose that $x \in E_2^{0,r}$ is an element. Suppose moreover:*

(1) *x is central in $E_2^{*,*}$.*
(2) *$E_2^{s,t}$ is torsion for $s > 0$ and t arbitrary.*
(3) *The spectral sequence degenerates at a finite stage.*

Then a power of x survives to $E_\infty^{0,r}$.

Proof In fact, suppose $d_2(x)$ is N–torsion; then this implies that x^N survives to E_3. Repeating, it follows that a sufficiently divisible enough power of x survives to E_4, E_5, and so forth to any finite stage. Since the spectral sequence stops at a finite stage, it follows that a high power of x survives the spectral sequence. □

Lemma 4.10 *Let $x \in R_0^G$. Then a sufficiently divisible power of x is in the image of $\pi_0 R^{hG} \to \pi_0 R$.*

Proof To see this, consider the homotopy fixed point spectral sequence

$$H^i(G; \pi_j R) \Longrightarrow \pi_{j-i} R^{hG}.$$

By assumption, x defines an element of $E_2^{0,0}$. This spectral sequence has the two key properties of Lemma 4.9. Above the $s = 0$ line, everything is torsion: in fact, annihilated by $|G|$. Moreover, the spectral sequence degenerates at a finite stage (in fact, with a horizontal vanishing line), thanks to Theorem 3.14. This is enough to imply the lemma by Lemma 4.9. □

Remark 4.11 If x is invertible, one may prove this using the *norm map* $\mathfrak{gl}_1(R) \to \mathfrak{gl}_1(R^{hG}) \simeq \tau_{\geq 0}\mathfrak{gl}_1(R)^{hG}$.

Proposition 4.12 *The thick subcategories of* $\mathrm{Mod}^\omega(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$ *are in bijection with the G–invariant subsets of* $\operatorname{Spec} R_0$ *closed under specialization.*

Proof By Corollary 4.7, the remaining step is to show that given a G–invariant closed subset $Z \subset \operatorname{Spec} R_0$, we can find a perfect $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$–module with support exactly Z. We would like to imitate the construction used to prove Theorem 2.13, although the problem is that the x_i need not live inside $\pi_* \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}) = \pi_* R^{hG}$. We can get around this as follows.

Let Z correspond to the G–invariant radical ideal $I \subset R_0$. We can find an ideal $J \subset I$ such that $\operatorname{rad}(J) = I$ and such that J is itself generated by G–invariant elements. Indeed, for each prime $\mathfrak{p}$ that fails to contain I, we observe that $g\mathfrak{p}$ also fails to contain I for each $g \in G$. Therefore, by prime avoidance (see Eisenbud [9, Lemma 3.3]), choose an element $x \in I \setminus \bigcup_{g \in G} (g\mathfrak{p})$. Then consider its *norm* $N_G x = \prod_{g \in G} gx$, which is in I and not in $\mathfrak{p}$. Taking norms such as these, we can choose G–invariant elements of I such that any prime that fails to contain I fails to contain one of these elements.

Therefore, we choose G–invariant elements $\{x_1, \dots, x_n\} \subset R_0^G$ such that they cut out the closed *subset* $Z \subset \operatorname{Spec} R_0$. We would like to take as our $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$–module M the iterated quotient $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})/(x_1, \dots, x_n)$, except that the x_i do not necessarily belong to $\pi_0 \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$. However, by Lemma 4.10, after raising the x_i to a suitable power, we can arrange that they do belong to $\pi_0 \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$. Then, the quotient $R^{hG}/(x_1, \dots, x_n)$ is the desired R^{hG}–module. □

Corollary 4.13 *Suppose R is an E_∞–ring with an action of a finite group G. Suppose that:*

(1) *R is even periodic with $\pi_0 R$ regular.*

(2) *$R^{hG} \to R$ is a faithful G–Galois extension (in the sense of Rognes [29]).*

Then the thick subcategories of $\operatorname{Mod}^\omega(R^{hG})$ are in bijection with the G–invariant subsets of $\operatorname{Spec} \pi_0 R$ closed under specialization.

Proof This result is not a corollary of the previous ones, but rather of the proof. Since $R^{hG} \to R$ is faithful G–Galois, we have an equivalence of ∞–categories $\operatorname{Mod}(R^{hG}) \simeq \operatorname{Mod}(R)^{hG}$. This is essentially Galois descent, and has been observed independently by Gepner–Lawson and Meier; see for example the author [23, Theorem 9.5].

Moreover, since R is even periodic with $\pi_0 R$ regular, we can use the residue field construction of Definition 2.5 to produce multiplicative homology theories on $\operatorname{Mod}(R^{hG})$ satisfying Künneth isomorphisms that detect all nonzero objects. This gives one half of the classification of thick subcategories.

Using the same construction as above, the argument proving Proposition 4.12 applies here too, provided that Lemma 4.10 applies. But in fact Lemma 4.10 is valid for any faithful G–Galois extension. As shown in [23], the map $R^{hG} \to R$ is "descendable" in the sense of Sections 3–4 of that paper. One can use the analogous properties of the homotopy fixed point (or descent) spectral sequence (degeneration at a finite stage with a horizontal vanishing line, so that Lemma 4.9 applies) which are valid for the homotopy fixed point spectral sequence for any faithful G–Galois extension; see [23, Section 4]. □

5 The general case

In this section, we complete the proof of Theorem 1.7. Throughout this section, we fix the even periodic refinement $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ of the affine map $X \to M_{\mathrm{FG}}$, where X is a regular, noetherian and separated Deligne–Mumford stack. Our goal is now to produce sufficiently many perfect $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$–modules to see that any closed substack of X can be realized as the support. Since X need not be a quotient stack by a finite group, we will need a different approach from the previous section. We will use the theory of graded Hopf algebroids (see, eg Ravenel [28]).

5.1 An abstract periodicity theorem

The basic step in producing $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$–modules is the following analog of the periodicity theorem in our setting.

Theorem 5.1 *Let $\mathcal{F} \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ have the property that the homotopy group sheaves of $\mathcal{F} \wedge \mathbb{D}\mathcal{F}$ are scheme-theoretically supported on a closed substack $Z \subset X$. Given a section $s \in H^0(Z, \omega^k)$, there exists a self-map*

$$\Sigma^{nk}\mathcal{F} \to \mathcal{F}$$

for some n, whose map on homotopy group sheaves is given by multiplication by $s^{\wedge n}$.

Proof In fact, we consider the endomorphism ring $\mathrm{End}(\mathcal{F})$, which is an A_∞–algebra internal to the category $\mathrm{QCoh}^{\omega}(\mathfrak{X})$. Note first that $\mathrm{End}(\mathcal{F})$ has its homotopy group sheaves supported on the closed substack Z. The homotopy group sheaves $\mathrm{End}_*(\mathcal{F})$ of $\mathrm{End}(\mathcal{F})$ form a sheaf of graded associative algebras on Z, and we get a map of quasicoherent sheaves on Z (or X),

$$\bigoplus_{k=-\infty}^{\infty} \omega_Z^{\otimes k} \to \mathrm{End}_*(\mathcal{F}).$$

This is obtained from the natural map

$$\bigoplus_{k=-\infty}^{\infty} \omega^{\otimes k} \to \mathrm{End}_*(\mathcal{F}),$$

which has the property that it factors through the base-change to Z. Note in particular that this map is central. In particular, the section $s \in H^0(Z, \omega^k)$ defines a central element of $H^0(X, \pi_k \mathrm{End}(\mathcal{F}))$: more precisely, a *central* element in the E_2–page of the spectral sequence

$$E_2^{s,t} = H^i(X, \pi_j \mathrm{End}(\mathcal{F})) \Longrightarrow \pi_{j-i}\Gamma(\mathfrak{X}, \mathrm{End}(\mathcal{F})).$$

In this spectral sequence, everything above the horizontal line $s = 0$ is torsion, as the rationalization $X_{\mathbb{Q}}$ is the quotient of an affine scheme by $\mathbb{G}_m$ in view of the affine map $X_{\mathbb{Q}} \to (M_{\mathrm{FG}})_{\mathbb{Q}} \simeq B\mathbb{G}_m$. In particular, the presentation $X_{\mathbb{Q}} = (\text{affine})/\mathbb{G}_m$ implies that $X_{\mathbb{Q}}$ has no higher sheaf cohomology. Therefore, it follows by Theorem 3.14 and Lemma 4.9 that a power of s survives the spectral sequence and defines a global endomorphism of $\mathcal{F}$ as desired. $\square$

This result *almost* reduces our work to pure algebra. The situation becomes slightly tricky, though, because while the set-theoretic support is well behaved in cofiber sequences, the *scheme-theoretic* support (which is what intervenes in Theorem 5.1) is less so. We now note further consequences of Theorem 5.1 that will be used in the sequel.

Lemma 5.2 *Let $Y \subset Y'$ be a nilpotent thickening of Artin stacks and $\mathcal{L} \in \mathrm{Pic}(Y')$. Then any* torsion *section $s \in H^0(Y, \mathcal{L})$ has the property that some power of s extends over Y'.*

Proof It suffices to consider a square-zero thickening $Y \subset Y'$, defined by a square-zero sheaf of ideals $\mathcal{I}$ on Y'. Then we have an exact sequence of sheaves on Y'

$$0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_Y \to 0,$$

and we consider a section s of the last term. The obstruction to its lifting is given by the coboundary $\delta s \in H^1(\mathcal{I}\mathcal{L})$. The coboundary has the property $\delta(s^N) = N s^{N-1} \delta(s)$ (in a natural sense, given the operation of $\mathcal{O}_{Y'}/\mathcal{I}$ on $\mathcal{I}$), which vanishes for N highly divisible by assumption. $\square$

Corollary 5.3 *Let $\mathcal{F} \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$. Suppose $\mathcal{F}$ is supported (scheme-theoretically) on a closed substack $Z \subset X$. Suppose $Z' \subset X$ is another closed substack with the same underlying set as Z and $s \in H^0(Z', \omega^i)$ is a torsion section. Let Z'' be the closed substack of Z' cut out by s. Then there exists $\mathcal{F}' \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ whose set-theoretic support is precisely Z''.*

Proof By Lemma 5.2, we may assume, after raising s to an appropriate power, that $Z' = Z$ and s actually is a section over Z. In this case, we use Theorem 5.1 to produce a self-map of $\mathcal{F}$ which induces multiplication by some tensor power of s on homotopy group sheaves. The cofiber of this map can be taken to be $\mathcal{F}'$. □

5.2 The algebraic setup; the rational piece

Let $Z \subset X$ be a closed substack. In this subsection, we will begin the algebraic preliminaries in showing that there exists an object in $\mathrm{QCoh}^{\omega}(\mathfrak{X})$ set-theoretically supported on Z.

Definition 5.4 Recall (eg from [11]) the covers

$$M_{\mathrm{FG}}^{\mathrm{coord},n} \to M_{\mathrm{FG}},$$

where $M_{\mathrm{FG}}^{\mathrm{coord},n}$ is the moduli stack of formal groups together with a coordinate to degree n. Each of these covers is a torsor for the group G that acts on coordinates to degree n (ie automorphisms of $\operatorname{Spec} \mathbb{Z}[x]/x^{n+1}$). The group G has a map (which admits a splitting)

$$G \twoheadrightarrow \mathbb{G}_m$$

by contemplating the action on the Lie algebra, and the kernel $H \subset G$ is an iterated extension of copies of $\mathbb{G}_a$, This property of the group G will become crucial below.

The inverse limit of the moduli stacks $M_{\mathrm{FG}}^{\mathrm{coord},n}$ parametrizes formal groups together with a coordinate: equivalently, formal group laws. The inverse limit is thus the spectrum of the Lazard ring L. Since $X \times_{M_{\mathrm{FG}}} \operatorname{Spec} L$ is affine by hypothesis, one gets:

Proposition 5.5 *We have $X \times_{M_{\mathrm{FG}}} M_{\mathrm{FG}}^{\mathrm{coord},n}$ is affine for $n \gg 0$.*

Proof Consider the tower of Deligne–Mumford stacks $X^{(n)} = X \times_{M_{\mathrm{FG}}^{\mathrm{coord},n}} M_{\mathrm{FG}}$ as n varies. For $n > 0$, the successive maps in the tower are $\mathbb{G}_a$–torsors and in particular are affine morphisms.

Moreover, the inverse limit of this tower, given by $X^{(\infty)} \stackrel{\mathrm{def}}{=} X \times_{M_{\mathrm{FG}}} \operatorname{Spec} L$ is an affine scheme, by hypothesis, so we want to claim that some term in the tower is itself an affine scheme. For this, we argue first that for $n \gg 0$, the Deligne–Mumford stacks $X^{(n)}$ are *algebraic spaces,* or equivalently, by [18, Corollary 8.1.1], that they have no nontrivial automorphisms. In fact, consider the diagonal maps

$$X^{(n)} \to X^{(n)} \times X^{(n)}$$

for each n. These fit into a tower of cartesian squares as $n \to \infty$, since for any morphism of stacks $Y_1 \to Y_2$, one has a cartesian square:

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y_1 \times Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_2 \times Y_2 \end{tikzcd}$$

Since in the inverse limit, the map $X^{(\infty)} \to X^{(\infty)} \times X^{(\infty)}$ is a closed immersion, it follows by Rydh [30, Proposition B.3] that $X^{(n)} \to X^{(n)} \times X^{(n)}$ is a closed immersion for $n \gg 0$. Thus, $X^{(n)}$ is an algebraic space for $n \gg 0$.

Now, we can apply to the general theory of inverse limits of towers of algebraic spaces under affine morphisms: by the Stacks project [31, Tag 07SQ] if the inverse limit is affine, then some term in the tower (and thus everything above it) must be affine, to conclude. $\square$

Fix one such n. Then we get a quotient stack presentation for X as the quotient of some affine scheme $\operatorname{Spec} R = X \times_{M_{\mathrm{FG}}} M_{\mathrm{FG}}^{\mathrm{coord},n}$ by an action of the algebraic group G. In particular, if $\mathcal{O}(G)$ denotes the ring of functions on G, then we get a presentation for X via a *Hopf algebroid*,

$$\Gamma\colon \quad R \rightrightarrows R \otimes \mathcal{O}(G) \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} \cdots . \tag{1}$$

Remark 5.6 Although we do not need this, these covers arise from certain ring spectra $X(n)$. This Hopf algebroid can be realized in homotopy via the cobar construction

$$\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}) \wedge X(n) \rightrightarrows \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}) \wedge X(n) \wedge X(n) \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} \cdots .$$

Consider the setup above. If we forget the $\mathbb{G}_m$–action but remember the associated grading, the result is a *graded* Hopf algebroid which presents the stack X. A given closed substack of $X \simeq \operatorname{Stack}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$ corresponds to an invariant homogeneous ideal $I \subset R_*$.

Our strategy will be, first, to choose *globally invariant* elements $x_1, \dots, x_r$ that generate I *rationally*, and which exist in homotopy in a very strong sense. Here we use the fact that the stack X is (up to $\mathbb{G}_m$–action) already affine once we rationalize. After we do this, we need to add in more generators to avoid introducing unnecessary irreducible components of the support. In the torsion case, however, the distinction between the set-theoretic and scheme-theoretic support simplifies thanks to the Frobenius.

Lemma 5.7 *In the above setup, there exist invariant homogeneous elements $x_1, \dots, x_r$ in the Hopf algebroid Γ (from (1)) that generate I rationally. In the language of stacks, there are sections $x_i \in H^0(X, \omega^{k_i})$ which cut out the closed substack Z rationally.*

Proof We will prove this using a *different* presentation from (1). On rationalizations, $X_{\mathbb{Q}} \to (M_{\mathrm{FG}})_{\mathbb{Q}} \simeq B\mathbb{G}_m$ is affine, so $X_{\mathbb{Q}}$ is the $\mathbb{G}_m$–quotient of an affine scheme $\operatorname{Spec} C$ with a $\mathbb{G}_m$–action (ie grading). Now a closed substack of $X_{\mathbb{Q}} \simeq (\operatorname{Spec} C)/\mathbb{G}_m$ is defined by a $\mathbb{G}_m$–invariant ideal of C, or a homogeneous ideal of C. We can take $x_1, \dots, x_r$ as homogeneous elements of C (ie sections of $H^0(X_{\mathbb{Q}}, \omega^{k_i})$) which generate this homogeneous ideal. Multiplying by a highly divisible integer, we may assume they extend to sections over X. ◻

Proposition 5.8 *Given the closed substack $Z \subset X$, there exists $\mathcal{F}' \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ such that $\operatorname{Supp} \mathcal{F}' \supset Z$ and $(\operatorname{Supp} \mathcal{F}')_{\mathbb{Q}} = Z_{\mathbb{Q}}$.*

Proof Choose sections $x_1, \dots, x_r \in H^0(X, \omega^{\otimes \bullet})$ such that the closed substack of X cut out by the $\{x_i\}$ is equal, rationally, to Z. After multiplying the x_i by a sufficiently divisible integer, we may assume the x_i vanish along Z as well. After raising the x_i to a sufficiently divisible power, we may assume (by Theorem 3.14 and Lemma 4.9) that each x_i survives to an element in $\pi_*(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$. Then, we can take $\mathcal{F}' = \mathcal{O}^{\mathrm{top}}/x_1 \wedge \cdots \wedge \mathcal{O}^{\mathrm{top}}/x_r$. ◻

5.3 The torsion piece

In this subsection, we prove some algebraic lemmas needed to handle the torsion. Throughout, let B be a base ring, assumed noetherian. Let G be an algebraic group over B with the property that G fits into an exact sequence of group schemes

$$1 \to H \to G \to \mathbb{G}_m \to 1,$$

where the map $G \to \mathbb{G}_m$ has a section (so that G is a semidirect product). Suppose H has a finite filtration with successive quotients isomorphic to $\mathbb{G}_a$. Observe that on any G–quotient, there is a natural line bundle ω obtained from the map $G \to \mathbb{G}_m$, the standard one-dimensional representation of $\mathbb{G}_m$ and the Borel construction. Throughout, a *representation* of an algebraic group (always over the base ring B) will refer to a B–module together with a coaction of the associated Hopf algebra. Given a representation, the *fixed points* consist of the primitive vectors under the coaction map. See Waterhouse [33] for a discussion over fields.

Lemma 5.9 *Let M be a $\mathbb{G}_a$–representation. Then if $M \neq 0$, $M^{\mathbb{G}_a} \neq 0$.*

Proof Fix $m \neq 0$ in M. Consider the coaction map $\psi\colon M \to M \otimes_B B[t]$, and suppose

(2) $$\psi(m) = \sum_{i=0}^{\infty} t^i \otimes m_i, \quad m_i = 0 \text{ for } i \gg 0.$$

Recall that $m_0 = m$, and in particular ψ is injective. Let n be maximal such that $m_n \neq 0$. Then m_n is $\mathbb{G}_a$–invariant. In fact, we get an equality from coassociativity,

$$\sum_{i=0}^{n} t^i \otimes \psi(m_i) = \sum_{i=0}^{n}\sum_{j=0}^{i} \binom{i}{j} t^j \otimes t^{i-j} \otimes m_i,$$

and comparing terms of $t^n \otimes (\cdot)$ shows that $\psi(m_n) = 1 \otimes m_n$. $\square$

Lemma 5.10 *Let A be a B–algebra with an action of the algebraic group H. Let $I \subset A$ be a H–invariant torsion ideal. Suppose I^H consists of nilpotent elements. Then I is nilpotent.*

Proof We first consider the case of $H = \mathbb{G}_a$. Let $\psi\colon A \to A \otimes_B B[t]$ be the coaction map. Choose $x \in I$, and write $\psi(x) = \sum_{i=0}^{n} t^i \otimes x_i$ for the $x_i \in I$ and for some n. By the proof of (2), it follows that x_n is H–invariant and therefore, by assumption, nilpotent. Choose N highly divisible and so large that $x_n^N = 0$.

In this case, it follows that

$$\psi(x^N) = \left(\sum_{i=0}^{n} t^i \otimes x_i\right)^N = \left(\sum_{i=0}^{n-1} t^i \otimes x_i\right)^N,$$

because N is highly divisible and x_n is torsion and nilpotent. (In general, if a is torsion and nilpotent, then $(a+b)^N = b^N$ for N sufficiently highly divisible.) Therefore, when one expands

$$\psi(x^N) = \sum_{j=0}^{\infty} t^j \otimes n_j,$$

the largest j that appears is $j = N(n-1)$, and that term is $t^{N(n-1)} \otimes x_{n-1}^N$. It follows that x_{n-1}^N is H–invariant and therefore nilpotent. Continuing in this way, we can work our way down to conclude that all the x_i, and in particular $x_0 = x$, are nilpotent.

In general, if H is not assumed to be isomorphic to $\mathbb{G}_a$, choose an exact sequence

$$1 \to H' \to H \to \mathbb{G}_a \to 1,$$

and assume inductively that the lemma is valid for H'. In particular, if I is not nilpotent, it follows that $I^{H'} \subset A^{H'}$ contains a nonnilpotent element x. The group $\mathbb{G}_a$ acts on $A^{H'}$ and $I^{H'}$, and in particular $(I^{H'})^{\mathbb{G}_a}$ must contain a nonnilpotent element by the case of the lemma already proved. But then I^H contains a nonnilpotent element, a contradiction. □

Proposition 5.11 *Let Y be an Artin stack obtained as $Y \simeq \operatorname{Spec} R/G$, where R is a noetherian ring. Then given a closed substack $T \subset Y$ such that $T_{\mathbb{Q}} = Y_{\mathbb{Q}}$, there exists a sequence of closed substacks*

$$Y \supset Y_1 \supset Y_2 \supset \cdots \supset Y_r \supset T$$

such that:

1. *There exists an element $y_i \in H^0(Y_i, \omega^{k_i})$ such that Y_{i+1} is the zero locus of y_i.*
2. *Y_r is a nilpotent thickening of T.*

Proof The stack Y is represented by a Hopf algebroid obtained by the G–action on $\operatorname{Spec} R$. The closed substack $T \subset Y$ corresponds to a G–invariant ideal $I \subset R$ with the property that $I \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. It suffices to show that there exist homogeneous elements $y_1, \ldots, y_r \in R$ such that:

1. The image of y_i in H–invariant in $R/(y_1, \ldots, y_{i-1})$ (which is inductively a G–representation by this assumption).
2. $(y_1, \ldots, y_r)$ contains a power of I.

To do this, note first that we may assume I nonnilpotent. In this case, Lemma 5.10 gives us a nonnilpotent H–invariant element $y_1 \in I$, which we can assume homogeneous. We can now form the quotient $R/(y_1)$, which defines a proper closed G–invariant subscheme of $\operatorname{Spec} R$, or equivalently a proper closed substack $Y_1 \subset Y$, which contains T. Now, apply Lemma 5.10 again to the H–action on $R/(y_1)$ and the image of I in here, and continue to get the descending sequence of substacks and the y_i. The process stops once the image of I in $R/(y_1, \ldots, y_r)$ is nilpotent, at which point we have gotten down to a nilpotent thickening of T. □

5.4 Proof of the main theorem

We now restate and prove the main theorem from the introduction.

Theorem 5.12 *The construction $Z \mapsto \mathrm{QCoh}^{\omega}_Z(\mathfrak{X})$ establishes a bijection between specialization-closed subsets of the underlying space of X and thick subcategories of $\mathrm{Mod}^{\omega}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})) \simeq \mathrm{QCoh}^{\omega}(\mathfrak{X})$.*

Proof It thus suffices to show that, given a closed substack $Z \subset X$, there exists $\mathcal{F} \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ such that the (set-theoretic) support of $\mathcal{F}$ is precisely Z.

By Proposition 5.8, there exists $\mathcal{F}' \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ such that the homotopy group sheaves of $\mathcal{F}'$ are supported scheme-theoretically on a closed substack Z' of X with $Z' \supset Z$ and $Z'_{\mathbb{Q}} = Z_{\mathbb{Q}}$. Moreover, by Proposition 5.11 (which is applicable in view of the discussion in Section 5.2), there exists a descending sequence of closed substacks

$$Z' = Z_1 \supset Z_2 \supset \cdots \supset Z_m \supset Z$$

and torsion sections $\overline{x}_i \in H^0(Z_i, \omega^{k_i})$ for $1 \le i \le m$, such that:

- Z_{i+1} is the zero locus of $\overline{x}_i$ on Z_i.
- Z_m is a nilpotent thickening of Z.

We claim that, for each $i = 1, 2, \dots, m$, there exists $\mathcal{F}_i \in \mathrm{QCoh}^{\omega}(\mathfrak{X})$ such that the set-theoretic support of $\mathcal{F}_i$ is precisely Z_i. We prove this by induction on i. For $i = 1$, we can take $\mathcal{F}'$. If we have proved the assertion for i, then the assertion follows for $i + 1$ by Corollary 5.3. Taking $i = m - 1$, we have proved our result. □

Question Does X need to be regular for the results of this paper to hold?

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Department of Mathematics, University of California, Berkeley
Office 854, Berkeley, CA 94720, USA

`amathew@math.berkeley.edu`

`http://math.berkeley.edu/~amathew`

Proposed: Paul Goerss
Seconded: Haynes Miller, Mark Behrens

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