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Uniqueness of the bowl soliton

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We prove that any translating soliton for the mean curvature flow that is noncollapsed and uniformly 2–convex must be the rotationally symmetric bowl soliton. In particular, this proves a conjecture of White and Wang in the 2–convex case in arbitrary dimension.

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1 Introduction

A (complete, embedded, oriented) hypersurface $M^n \subset \mathbb{R}^{n+1}$ is called a *translating soliton* if its mean curvature H and its normal vector ν are related by the equation

$$H = \langle V, \nu \rangle \tag{1}$$

for some $V \in \mathbb{R}^{n+1}$, $V \neq 0$. Solutions of (1) correspond to translating solutions $\{M_t = M + tV\}_{t \in \mathbb{R}}$ of the mean curvature flow,

$$\partial_t x = H\nu. \tag{2}$$

Translating solitons play a key role in the study of slowly forming singularities (see eg Hamilton [11], Angenent and Velázquez [3], Huisken and Sinestrari [15; 16] and White [27]) and have received a lot of attention in the last 20 years.

It is not hard to see that there exists a unique solution (up to rigid motion) of (1) that is rotationally symmetric and strictly convex; see Altschuler and Wu [1]. For $n = 1$, this is the *grim reaper* (Mullins [20]), given by the explicit formula $y = -\log\cos(x)$, $x \in (-\pi/2, \pi/2)$. For $n \geq 2$, which we assume from now on, the solution roughly looks like a paraboloid and is usually called the *bowl soliton*.

A well-known problem concerns the uniqueness of solutions of (1); see eg White [27, Conjecture 2 and the unnumbered remark below]. A very important contribution was made by X Wang [25], who proved that for $n = 2$ any entire convex solution must be the rotationally symmetric bowl soliton, but that for $n \geq 3$ there exist entire strictly convex solutions that are not rotationally symmetric. However, the most relevant solutions of (1) are of course the ones that arise as singularity models for the mean

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curvature flow, and it is unknown whether or not Wang's solutions can actually arise as singularity models.

Conjecture 1.1 (White [27, page 133], Wang [25, page 1237]) *All translating solitons that arise as a blowup limit of a mean-convex mean curvature flow must be rotationally symmetric.*

The conjecture is motivated by the deep regularity and structure theory for mean-convex mean curvature flow due to White [26; 27]; see also Haslhofer and Kleiner [12]. In particular, it is known that the only *shrinking* solitons that can occur as blowup limits in the mean-convex case are the round shrinking cylinders $S^j \times \mathbb{R}^{n-j}$. Recently, Colding and Minicozzi [10] proved that even the axis of such a cylindrical tangent flow is unique, ie independent of the sequence of rescaling factors.

In general, a very important feature of blowup limits of a mean-convex mean curvature flow is that they are always noncollapsed; see White [26], Sheng and Wang [22], Andrews [2] and Haslhofer and Kleiner [12]. To address Conjecture 1.1 we can thus focus on solutions of (1) that are *α–noncollapsed*, ie solutions with positive mean curvature such that at each $p \in M$ the inscribed radius and the outer radius are at least $\alpha/H(p)$. Indeed, by the references quoted above, see eg [2, Theorem 3] and [12, Theorem 1.14], every blowup limit of a mean-convex flow is α–noncollapsed for some $\alpha > 0$.[1]

While we do not know at the moment how to address Conjecture 1.1 in full generality (see however Remark 1.6), here we manage to prove it in an important special case, namely the uniformly 2–convex case in arbitrary dimension; see Corollary 1.4. We recall that an oriented hypersurface is called *uniformly* 2*–convex* if it is mean-convex and satisfies

$$\lambda_1 + \lambda_2 \geq \beta H \tag{3}$$

for some $\beta > 0$, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the principal curvatures. Uniform 2–convexity is preserved under mean curvature flow, and arises eg in the construction of mean curvature flow with surgery by Huisken and Sinestrari [17]; see also Haslhofer and Kleiner [13] and Brendle and Huisken [6]. In this setting of flows with surgery, there can be some high-curvature regions, eg regions like a degenerate neckpinch, that are modeled on translating solitons that must be α–noncollapsed and uniformly 2–convex. As it turns out, in such a situation one can always find at a controlled distance from the tip an almost round cylindrical region where one can perform the surgery, and

[1] In fact, a more quantitative analysis shows that the optimal constant α is at least 1; see Haslhofer and Kleiner [14, Corollary 1.5].

thus (somewhat surprisingly) one can prove the existence of mean curvature flow with surgery without actually knowing whether or not the translating solitons that occur are rotationally symmetric. However, in addition to its existence, one of course wants to know what the flow with surgery looks like. In particular, one wonders whether all translating solitons that occur as singularity models in this context are actually rotationally symmetric.

Our main theorem is the following:

Theorem 1.2 *Any solution of the translating soliton equation* (1) *that is α–noncollapsed and uniformly 2–convex must be the rotationally symmetric bowl soliton.*

Theorem 1.2 holds in arbitrary dimension.

Remark 1.3 In the special case $n = 2$, the uniform 2–convexity assumption is of course automatic for $\beta = 1$. In particular, this yields a shorter proof of the 2–dimensional uniqueness result of Wang [25, Theorem 1.1], under somewhat different assumptions. (Wang assumes that the solution is convex and that it can be written as an entire graph. We assume that the solution is α–noncollapsed.)

As an immediate consequence of Theorem 1.2, we obtain an affirmative answer to the conjecture of White and Wang (Conjecture 1.1) in the 2–convex case in arbitrary dimension.

Corollary 1.4 *The only translating soliton that can arise as a blowup limit of a mean curvature flow of closed embedded 2–convex hypersurfaces is the rotationally symmetric bowl soliton.*

As another consequence of Theorem 1.2, we obtain a classification of the translating solitons that can arise as models for high-curvature regions in the mean curvature flow with surgery. We state this in the language of the canonical neighborhood theorem [13, Theorem 1.22].

Corollary 1.5 *The only translating soliton that can arise as a canonical neighborhood in the mean curvature flow with surgery is the rotationally symmetric bowl soliton.*

Let us now discuss some related results. In a very important recent paper [4], Brendle proved the uniqueness of translating solitons for the Ricci flow in dimension three under just a κ–noncollapsing assumption, as suggested by Perelman [21]. In [5], Brendle extended his result to higher dimensions, assuming that the soliton has positive sectional curvature and is asymptotically cylindrical. Using similar techniques, Chodosh [7] and Chodosh and Fong [8] obtained uniqueness results for asymptotically conical Ricci expanders. Using ideas centered around the backwards uniqueness of the heat equation, L Wang [23; 24] and Wang and Kotschwar [18] proved uniqueness results for asymptotically conical/cylindrical solitons for the mean curvature flow and Ricci flow. Using Alexandrov's reflection principle, Martin, Savas-Halilaj and Smoczyk proved a uniqueness result for translating solitons for the mean curvature flow, imposing strong asymptotic assumptions [19, Theorem B and the asymptotic assumptions in (3.1)]. The work [9] by Clutterbuck, Schnürer and Schulze also implicitly includes a uniqueness result under related assumptions.

Our proof of Theorem 1.2 follows a scheme inspired by the recent work of Brendle [4] and uses some estimates for α–noncollapsed flows from Haslhofer and Kleiner [12]. The present article seems to be the first one where Brendle's scheme of proof is implemented for a geometric flow other than the Ricci flow. Another feature of our proof is that we incorporate the ambient euclidean space into our setup right from the beginning; this allows us to give a quite short and efficient argument.

Let us now outline the main steps of our proof, pretending $n = 2$ and $V = \partial/\partial z$ for ease of notation. In Section 2, we study the asymptotic geometry of M. Using some estimates from [12], we prove that $H \sim z^{-1/2}$ and that suitable rescalings at infinity are modeled on the round shrinking cylinder $S^1 \times \mathbb{R}$, or $S^{n-1} \times \mathbb{R}$ in arbitrary dimension (Proposition 2.3). In Section 3, we consider the function $f_R = \langle R, \nu\rangle$, where $R = x_1 \partial_{x_2} - x_2 \partial_{x_1}$ is a rotation centered at the origin. The function f_R satisfies the same linear elliptic equation as the mean curvature H, and we prove a weighted estimate for it (Proposition 3.1). In Section 4, we prove a decay estimate for the corresponding linear parabolic equation on the round shrinking cylinder (Proposition 4.1). Finally, in Section 5, we carry out a blowdown and centering argument, which is the key step of our proof. Namely, to prove rotational symmetry we would like to find a point $\bar{x} \in \mathbb{R}^2$ such that the rotation function $f_{R_{\bar{x}}} = \langle (x_1 - \bar{x}_1)\partial_{x_2} - (x_2 - \bar{x}_2)\partial_{x_1}, \nu\rangle$ centered at $\bar{x} \in \mathbb{R}^2$ vanishes identically. To this end, we consider the function

$$B(h) := \inf_{x \in \mathbb{R}^2} \sup_{\{z=h\}} |f_{R_x}| = \inf_{T \in \mathbb{R}^2} \sup_{\{z=h\}} |f_R - \langle T, \nu\rangle|, \tag{4}$$

where we observe that the infimum over different centers can be alternatively written as an infimum where translation functions are subtracted off; this is important to get a better rate of decay in the estimate for the parabolic equation, see Proposition 4.1. We then fix a suitable small constant $\tau > 0$ and assume towards a contradiction that $B(h) > 0$ for h large. On the one hand, since $B(h) \leq O(h^{1/2})$ we can find a sequence

$h_m \to \infty$ such that

$$B(h_m) \leq 2\tau^{-1/2} B(\tau h_m). \tag{5}$$

On the other hand, using the estimates we just described (in particular, the fact that we can find a suitable blowdown that is a shrinking cylinder and Proposition 4.1, which gives a decay estimate on the cylinder) we argue that

$$B(\tau h_m) \leq \tfrac{1}{4}\tau^{1/2} B(h_m) \tag{6}$$

for m large. This gives the desired contradiction and concludes our outline of the proof.

Finally, here is some partial progress towards the general case of Conjecture 1.1.

Remark 1.6 (General case) Let $M \subset \mathbb{R}^{n+1}$ be a translating soliton that arises as a blowup limit of a mean-convex flow. By the convexity estimate in [12, Theorem 1.10], the soliton must be convex. Let $k \in \{2, \dots, n\}$ be the smallest integer such that

$$\inf_M \frac{\lambda_1 + \cdots + \lambda_k}{H} > 0. \tag{7}$$

Let p_j be a sequence of points going to infinity such that

$$\frac{\lambda_1 + \cdots + \lambda_{k-1}}{H}(p_j) \to 0.$$

By the global convergence theorem [12, Theorem 1.12], after rescaling by $H^{-1}(p_j)$ we can pass to a smooth limit, which we call the asymptotic soliton. By the strict maximum principle, this limit must split off $k-1$ lines. Together with (7) and [12, Lemma 3.14] this implies that the asymptotic soliton is a round cylinder $S^{n+1-k} \times \mathbb{R}^{k-1}$. If we knew that the soliton M splits off $k-2$ lines, then by Theorem 1.2 we could conclude that M is isometric to $\mathbb{R}^{k-2} \times \mathrm{Bowl}_{n-k+2}$.

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2 Asymptotic geometry

Throughout this article $M^n \subset \mathbb{R}^{n+1}$ denotes a translating soliton that is α–noncollapsed and uniformly 2–convex. Without loss of generality, we can assume that $V = \partial/\partial z$, where z denotes the last coordinate in $\mathbb{R}^{n+1}$. We recall the soliton equation,

$$H = \langle V, \nu \rangle. \tag{8}$$

We can decompose V into its normal part $V^\perp = \langle V, \nu\rangle \nu$ and its tangential part $\overline{V} = V - V^\perp$. Note that

$$H^2 + |\overline{V}|^2 = 1, \tag{9}$$

in particular $H \leq 1$. We also recall the evolution equation for the mean curvature,

$$-\nabla_{\overline{V}} H = \Delta H + |A|^2 H. \tag{10}$$

Moreover, differentiating the soliton equation we obtain the identity

$$\nabla H = -A(\overline{V}, \cdot). \tag{11}$$

By [12, Corollary 2.15], the soliton can be written in the form $M = \partial K$, where K is a convex domain (in particular, connected). In fact, K must be strictly convex. Indeed, if λ_1 vanished at some point, then by the strict maximum principle M would split off a line. Together with the uniform 2–convexity and [12, Lemma 3.14] this would imply that $M = S^{n-1} \times \mathbb{R}$, which is absurd.

Fix a point $p_0 \in M$. Note that M must be noncompact, by comparison with round spheres.

Lemma 2.1 *We have $H(p) \to 0$ as $|p - p_0| \to \infty$.*

Proof If this is not the case, there is a sequence of points $p_j \in \partial K$ going to infinity such that $\liminf_{j\to\infty} H(p_j) > 0$. After passing to a subsequence, we can assume that $(p_j - p_0)/|p_j - p_0|$ converges to some direction $\omega \in S^n$. Recall that $|H| \leq 1$. Let $\{\widehat{M}_t\}_{t\in\mathbb{R}}$ be the sequence of mean curvature flows obtained from $\{M_t = M + tV\}_{t\in\mathbb{R}}$ by shifting p_j to the origin, and pass to a subsequential limit $\{\widehat{M}_t^\infty\}_{t\in\mathbb{R}}$. Since K is convex and $(p_j - p_0)/|p_j - p_0| \to \omega$, the limit contains a line. Together with the uniform 2–convexity and [12, Lemma 3.14] this implies that $\{\widehat{M}_t^\infty\}_{t\in\mathbb{R}}$ is a family of round shrinking cylinders $S^{n-1} \times \mathbb{R}$. This contradicts the fact that $\{\widehat{M}_t^\infty\}_{t\in\mathbb{R}}$ is an eternal flow with bounded curvature. □

By Lemma 2.1, we can find a point $o \in M$ where H attains its maximum. After translating coordinates, we can assume that o is the origin in $\mathbb{R}^{n+1}$. By (11) and strict convexity, the vector $\overline{V}$ must vanish at o. Thus, $H(o) = 1$ and $T_o M = \mathbb{R}^n \subset \mathbb{R}^{n+1}$ is horizontal. In particular, M is contained in the upper half-plane.

Lemma 2.2 *There exists a constant $c > 0$ such that $\inf_{\{z=h\}} H \geq c h^{-1/2}$ for h large.*

Proof Choose $c=\alpha^2/(8n)$. Assume that for some large h there is a point $p\in\{z=h\}$ with $H(p)<ch^{-1/2}$. By the α–noncollapsing condition, we can find an interior ball of radius at least $\alpha h^{1/2}/c$ tangent at p. By Lemma 2.1 and the soliton equation, the vector $\nu(p)$ is almost horizontal. Thus, it takes time at least

$$T=\frac{1}{2}\frac{\alpha^2}{2nc}h=2h \tag{12}$$

until the interior ball leaves the half-space $\{z\leq h\}$. On the other hand, since M is contained in the upper half-plane and $\{M_t=M+tV\}_{t\in\mathbb{R}}$ moves in the z–direction with unit speed, the ball must leave the half-space $\{z\leq h\}$ in time at most h; this is a contradiction. □

Proposition 2.3 *For any sequence $h_m\to\infty$, the sequence of flows $\{\widehat{M}_t^m\}_{t\in(-\infty,1)}$ obtained by translating $p_m=(0,h_m)\in\mathbb{R}^{n+1}$ to the origin and parabolically rescaling by $\lambda_m=h_m^{-1/2}$, ie*

$$\widehat{M}_t^m=\lambda_m\cdot(M_{\lambda_m^{-2}t}-p_m), \tag{13}$$

converges to a family of round shrinking cylinders

$$C_t=S^{n-1}_{r(t)}\times\mathbb{R}, \tag{14}$$

where $r(t)=\sqrt{2(n-1)(1-t)}$.

Proof Write $M=\partial K$ as before, and let $\widehat{K}_t^m=\lambda_m\cdot(K_{\lambda_m^{-2}t}-p_m)$. Note that $\widehat{K}_t^m$ contains the origin for all $t<1$. By [12, Theorem 1.14] we can thus find a subsequence that converges smoothly to a limit $\{\widehat{K}_t^\infty\}_{t\in(-\infty,1)}$, whose time slices are convex and nonempty. By Lemma 2.2, the limit is nontrivial, ie $\widehat{K}_t^\infty\neq\mathbb{R}^{n+1}$ and $\partial\widehat{K}_t^\infty$ has strictly positive mean curvature for all $t<1$. Note that the vertical line through the origin is contained in $\widehat{K}_t^\infty$ for all $t<1$. Thus, by the uniform 2–convexity and [12, Lemma 3.14], the flow $\{\widehat{K}_t^\infty\}_{t\in(-\infty,1)}$ must be a family of round shrinking cylinders $\{C_t=S^{n-1}_{r(t)}\times\mathbb{R}\}_{t\in(-\infty,1)}$. Since we have $\widehat{K}^m_{1+\varepsilon}\subset\{z\geq\varepsilon h_m^{1/2}\}$ for every $\varepsilon>0$, the cylinders become extinct at time $T=1$, and thus their radius is given by the formula $r(t)=\sqrt{2(n-1)(1-t)}$. Finally, by uniqueness of the limit, the subsequential convergence actually entails convergence of the full sequence. □

Corollary 2.4 *For $z\to\infty$, the mean curvature satisfies the estimate*

$$H=\Big(\frac{n-1}{2}\Big)^{1/2}z^{-1/2}+o(z^{-1/2}). \tag{15}$$

3 Weighted estimate for the rotation functions

For any translation vector field T and any rotation vector field R in $\mathbb{R}^{n+1}$, we consider the functions $f_T = \langle T, \nu\rangle$ and $f_R = \langle R, \nu\rangle$. Since mean curvature flow is invariant under isometries of the ambient space, these functions satisfy the elliptic equation

$$(\Delta_z + |A|^2) f = 0, \tag{16}$$

where $\Delta_z = \Delta + \nabla_{\overline{V}}$ is the drift Laplacian. For example, in the case $T = V$ we have $f_V = \langle V, \nu\rangle = H$, which shows that (16) then reduces to (10) and moreover gives the asymptotics $f_V \sim z^{-1/2}$.

Proposition 3.1 *For all $h > 0$, we have the weighted estimate*

$$\sup_{\{z\leq h\}} \left|\frac{f_R}{H}\right| \leq \sup_{\{z=h\}} \left|\frac{f_R}{H}\right|. \tag{17}$$

Proof Pick the smallest $\theta \geq 0$ such that $u := \theta H - f_R \geq 0$ in $\{z \leq h\}$. If $\theta = 0$, then $f_R \leq 0$. Assume now $\theta > 0$. Since

$$(\Delta_z + |A|^2) u = 0, \tag{18}$$

the minimum of u in $\{z \leq h\}$ is attained at a point $p \in \{z = h\}$. By minimality of θ, the minimum must be zero. Thus,

$$\frac{f_R}{H} \leq \theta = \frac{f_R}{H}\bigg|_p \leq \sup_{\{z=h\}} \left|\frac{f_R}{H}\right|. \tag{19}$$

Repeating the same argument with f_R replaced by $-f_R$ proves the proposition. □

4 Decay estimate on the cylinder

Proposition 4.1 *Consider the family of shrinking cylinders*

$$\{C_t = S^{n-1}_{r(t)} \times \mathbb{R} \subset \mathbb{R}^{n+1}\}_{t\in(0,1)},$$

where $r(t) = \sqrt{2(n-1)(1-t)}$. Let $f = \{f(t)\}_{t\in(0,1)}$ be a family of functions on C_t satisfying the parabolic equation

$$\partial_t f = (\Delta_{C_t} + |A_{C_t}|^2) f. \tag{20}$$

Suppose that f is invariant under translations along the axis of the cylinder, that

$$\int_{S^{n-1}_{r(t)}\times[-1,1]} f(t) = 0 \tag{21}$$

for all $t \in (0,1)$ *and that* $|f(t)| \leq 1$ *for* $t \in (0, \frac{1}{2})$. *Then*

$$\inf_{T \in \mathbb{R}^n} \sup_{C_t} |f(t) - f_T| \leq D(1-t)^{1/2+1/(n-1)} \tag{22}$$

for all $t \in [\frac{1}{2}, 1)$, *where* $D < \infty$ *is a constant.*

Proof Since the family of functions f is invariant under translations, the Laplacian scales like one over distance squared and $|A_{C_t}|^2 = (n-1)/r(t)^2$, we can identify f with a family of functions $\tilde{f} = \{\tilde{f}(t)\}_{t \in (0,1)}$ on the unit sphere S^{n-1} satisfying the parabolic equation

$$\partial_t \tilde{f} = \frac{1}{2(n-1)(1-t)} (\Delta_{\mathbb{S}^{n-1}} + n - 1) \tilde{f}. \tag{23}$$

The other assumptions then read $\int_{S^{n-1}} \tilde{f}(t) = 0$ for all $t \in (0,1)$ and $|\tilde{f}(t)| \leq 1$ for $t \in (0, \frac{1}{2})$.

The eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$ are $\lambda_\ell = \ell(\ell + n - 2)$, so we have $\lambda_0 = 0$, $\lambda_1 = n-1$, $\lambda_2 = 2n, \ldots$. If we write $\tilde{f} = \sum \gamma_j(t)\varphi_j(x)$, then

$$\frac{d}{dt}\gamma_j = \frac{1}{2(n-1)(1-t)}(-\lambda_j + n - 1)\gamma_j, \tag{24}$$

which has the solution

$$\gamma_j(t) = \gamma_j(0)(1-t)^{(\lambda_j - n + 1)/(2(n-1))}. \tag{25}$$

In particular,

$$\gamma_2(t) = \gamma_2(0)(1-t)^{1/2+1/(n-1)}. \tag{26}$$

The assertion follows. □

5 Blowdown analysis

In this final section, we conclude the proof of the main theorem (Theorem 1.2) by performing a blowdown argument and using the results from the previous sections. Let $G(x_1, \ldots, x_{n+1}) := x_1 \partial_{x_2} - x_2 \partial_{x_1}$ and let $\mathcal{R}$ be the set of all $A_* G$, where $A \in \mathrm{SO}_n \subset \mathrm{SO}_{n+1}$. For an arbitrary center $\bar{x} \in \mathbb{R}^n \subset \mathbb{R}^{n+1}$, we define $\mathcal{R}_{\bar{x}} := \{R(\cdot - \bar{x}) \mid R \in \mathcal{R}\}$. Consider the function

$$B(h) := \inf_{x \in \mathbb{R}^n} \sup_{R \in \mathcal{R}_x} \sup_{\{z=h\}} |f_R|. \tag{27}$$

Note that

$$G_{\bar{x}}(x) := G(x-\bar{x}) = (x_1-\bar{x}_1)\partial_{x_2} - (x_2-\bar{x}_2)\partial_{x_1} = G(x) - (\bar{x}_1\partial_{x_2} - \bar{x}_2\partial_{x_1}). \tag{28}$$

More generally, if we let $R_{\bar{x}}(x) := R(x-\bar{x})$ and $A_{\bar{x}}(x) := A(x-\bar{x})+\bar{x}$ for $R \in \mathcal{R}$ and $A \in \mathrm{SO}_n$, then

$$A_{\bar{x}} G_{\bar{x}} = AG - T_{A,\bar{x}}, \tag{29}$$

where $T_{A,\bar{x}}$ represents a translation along the vector $A\bar{x} + AS\bar{x} - \bar{x}$, with $S\bar{x} = (-\bar{x}_2, \bar{x}_1, 0, \dots, 0)^t$. For a dense set of $A \in \mathrm{SO}_n$, the matrix $A(I+S)-I$ is invertible. We conclude that

$$B(h) = \inf_{T\in\mathbb{R}^n} \sup_{R\in\mathcal{R}} \sup_{\{z=h\}} |f_R - f_T|. \tag{30}$$

Case 1 Suppose that there is a sequence $h_m \to \infty$ with $B(h_m) = 0$. For each m, choose x_m such that

$$\sup_{\{z=h_m\}} |f_{R_{x_m}}| = 0 \tag{31}$$

for every $R_{x_m} \in \mathcal{R}_{x_m}$. Proposition 3.1 implies that $f_{R_{x_m}} = 0$ in the region $\{z \le h_m\}$. By (29), we can write $f_{R_{x_m}} = f_R - f_{T_m}$ for some $R \in \mathcal{R}$ and some translation T_m (depending on R_{x_m} and x_m). Then, $f_R - f_{T_m} = 0$ in the region $\{z \le h_m\}$. Thus, x_m is constant and $f_{R_{x_m}} = 0$ everywhere for every $R_{x_m} \in \mathcal{R}_{x_m}$, and hence we have proven rotational symmetry.

Case 2 Suppose now $B(h) > 0$ for h large. Fix $\tau \in (0, \frac{1}{2})$ such that $\tau^{-1/(n-1)} > 8D$, where D is the constant from Proposition 4.1. Since $B(h) \le O(h^{1/2})$,[2] we can then find $h_m \to \infty$ such that

$$B(h_m) \le 2\tau^{-1/2} B(\tau h_m). \tag{32}$$

Choose x_m such that

$$\sup_{R\in\mathcal{R}_{x_m}} \sup_{\{z=h_m\}} |f_R| = B(h_m) \tag{33}$$

and $R_m \in \mathcal{R}_{x_m}$ such that

$$B(\tau h_m) = \inf_{T} \sup_{\{z\in\tau h_m\}} |f_{R_m} - f_T|. \tag{34}$$

[2] Using the maximum principle, we can prove that the tensor $S = (n-1)A - Hg + H dz \otimes dz$ satisfies $|S| \le O(h^{-1+\varepsilon})$ for any $\varepsilon > 0$, which implies the better estimate $B(h) \le O(h^{\varepsilon})$. However, the rough estimate $B(h) \le O(h^{1/2})$, which follows from the soliton being asymptotically cylindrical, turns out to be good enough for our purpose.

The function $\widetilde{f}_m := (1/B(h_m)) f_{R_m}$ satisfies the elliptic equation

$$(\Delta_z + |A|^2)\widetilde{f}_m = 0. \tag{35}$$

Applying Proposition 3.1 and using the asymptotics for H, we obtain

$$\sup_{\{z=h\}} |\widetilde{f}_m| \le 2\left(\frac{h_m}{h}\right)^{1/2} \tag{36}$$

for $h \le h_m$ and m large.

Recall that the family $\{M_t = M_0 + tV\}_{t\in\mathbb{R}}$ moves by mean curvature flow. If we view $\widetilde{f}_m$ as a one-parameter family of functions on M_t, then the drift term in $\Delta_z = \Delta + \nabla_{\overline{V}}$ becomes a time derivative and (35) takes the form

$$\partial_t \widetilde{f}_m = (\Delta + |A|^2)\widetilde{f}_m. \tag{37}$$

Let $p_m = (0, h_m) \in \mathbb{R}^{n+1}$. Consider the parabolic rescaling

$$(x,t) \mapsto (\lambda_m(x - p_m), \lambda_m^2 t),$$

where $\lambda_m = h_m^{-1/2}$. In other words, let

$$\widehat{M}_t^m = \lambda_m (M_{\lambda_m^{-2}t} - p_m), \tag{38}$$

$$\widehat{f}_m(x,t) = \widetilde{f}_m(\lambda_m^{-1}x + p_m, \lambda_m^{-2}t) \quad \text{for } x \in \widehat{M}_t^m. \tag{39}$$

Note that $\widehat{M}_t^m$ moves by mean curvature flow and that $\widehat{f}_m$ satisfies the parabolic equation

$$\partial_t \widehat{f}_m = (\Delta + |A|^2)\widehat{f}_m. \tag{40}$$

By Proposition 2.3, for $m \to \infty$ the mean curvature flows $\widehat{M}_t^m$ converge to the family of shrinking cylinders

$$C_t = S^{n-1}_{r(t)} \times \mathbb{R}, \tag{41}$$

where $r(t) = \sqrt{2(n-1)(1-t)}$. Using the estimate (36) we obtain

$$\limsup_{m\to\infty} \sup_{t\in[\delta,1-\delta]} \sup_{|z|\le\delta^{-1}} |\widehat{f}_m| < \infty \tag{42}$$

for any given $\delta \in (0, \frac{1}{2})$. Hence, the functions $\widehat{f}_m$ converge (subsequentially) to a family of functions $f = \{f(t)\}_{t\in(0,1)}$ on C_t that satisfy

$$\partial_t f = (\Delta_{C_t} + |A_{C_t}|^2) f. \tag{43}$$

The limit f is invariant under translations and the estimate (36) implies that $|f(t)| \leq 4$ for $t \in (0, \frac{1}{2})$. Moreover, since $\operatorname{div}_{\mathbb{R}^{n+1}} R_m = 0$ and $\langle R_m, V \rangle = 0$, the divergence theorem yields that

$$\int_{\{h_1 \leq z \leq h_2\}} \langle R_m, \nu \rangle = 0 \tag{44}$$

on M for any $h_1 < h_2$. Thus, f also satisfies assumption (21). Hence, Proposition 4.1 implies that

$$\inf_T \sup_{C_t} |f(t) - f_T| \leq 4D(1-t)^{1/2+1/(n-1)} \tag{45}$$

for all $t \in [\frac{1}{2}, 1)$. On the other hand, we have the following:

$$\begin{aligned} \inf_T \sup_{\{z=0\}} |\widehat{f}_m(1-\tau) - f_T| &= \inf_T \sup_{\{z=h_m\}} |\widetilde{f}_m(h_m(1-\tau)) - f_T| \\ &= \inf_T \sup_{\{z=\tau h_m\}} |\widetilde{f}_m(0) - f_T| \\ &= \frac{1}{B(h_m)} \inf_T \sup_{\{z=\tau h_m\}} |f_{R_m} - f_T| = \frac{B(\tau h_m)}{B(h_m)} \geq \tfrac{1}{2}\tau^{1/2} \end{aligned} \tag{46}$$

Taking the limit as $m \to \infty$ gives

$$\inf_T \sup_{C_{1-\tau}} |f(1-\tau) - f_T| \geq \tfrac{1}{2}\tau^{1/2}. \tag{47}$$

Since $\tau^{-1/(n-1)} > 8D$, the inequalities (45) and (47) are in contradiction. This completes the proof.

References

[1] **S J Altschuler**, **L F Wu**, *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*, Calc. Var. Partial Differential Equations 2 (1994) 101–111 MR1384396

[2] **B Andrews**, *Noncollapsing in mean-convex mean curvature flow*, Geom. Topol. 16 (2012) 1413–1418 MR2967056

[3] **S B Angenent**, **J J L Velázquez**, *Degenerate neckpinches in mean curvature flow*, J. Reine Angew. Math. 482 (1997) 15–66 MR1427656

[4] **S Brendle**, *Rotational symmetry of self-similar solutions to the Ricci flow*, Invent. Math. 194 (2013) 731–764 MR3127066

[5] **S Brendle**, *Rotational symmetry of Ricci solitons in higher dimensions*, J. Differential Geom. 97 (2014) 191–214 MR3231974

[6] **S Brendle**, **G Huisken**, *Mean curvature flow with surgery of mean convex surfaces in* $\mathbb{R}^3$, Invent. Math. (2015) published online in April

[7] **O Chodosh**, *Expanding Ricci solitons asymptotic to cones*, Calc. Var. Partial Differential Equations 51 (2014) 1–15 MR3247379

[8] **O Chodosh**, **F T-H Fong**, *Rotational symmetry of conical Kähler–Ricci solitons*, Math. Ann. (2015) published online in June

[9] **J Clutterbuck**, **O C Schnürer**, **F Schulze**, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial Differential Equations 29 (2007) 281–293 MR2321890

[10] **T H Colding**, **W P Minicozzi, II**, *Uniqueness of blowups and Łojasiewicz inequalities*, Ann. of Math. 182 (2015) 221–285

[11] **R S Hamilton**, *Harnack estimate for the mean curvature flow*, J. Differential Geom. 41 (1995) 215–226 MR1316556

[12] **R Haslhofer**, **B Kleiner**, *Mean curvature flow of mean convex hypersurfaces* `arXiv:1304.0926`

[13] **R Haslhofer**, **B Kleiner**, *Mean curvature flow with surgery* `arXiv:1404.2332`

[14] **R Haslhofer**, **B Kleiner**, *On Brendle's estimate for the inscribed radius under mean curvature flow*, Int. Math. Res. Not. (2014) published online in September

[15] **G Huisken**, **C Sinestrari**, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. 183 (1999) 45–70 MR1719551

[16] **G Huisken**, **C Sinestrari**, *Mean curvature flow singularities for mean convex surfaces*, Calc. Var. Partial Differential Equations 8 (1999) 1–14 MR1666878

[17] **G Huisken**, **C Sinestrari**, *Mean curvature flow with surgeries of two-convex hypersurfaces*, Invent. Math. 175 (2009) 137–221 MR2461428

[18] **B Kotschwar**, **L Wang**, *Rigidity of asymptotically conical shrinking gradient Ricci solitons*, J. Differential Geom. 100 (2015) 55–108 MR3326574

[19] **F Martin**, **A Savas-Halilaj**, **K Smoczyk**, *On the topology of translating solitons of the mean curvature flow* `arXiv:1404.6703`

[20] **W W Mullins**, *Two-dimensional motion of idealized grain boundaries*, J. Appl. Phys. 27 (1956) 900–904 MR0078836

[21] **G Perelman**, *The entropy formula for the Ricci flow and its geometric applications* `arXiv:math/0211159`

[22] **W Sheng**, **X-J Wang**, *Singularity profile in the mean curvature flow*, Methods Appl. Anal. 16 (2009) 139–155 MR2563745

[23] **L Wang**, *Uniqueness of self-similar shrinkers with asymptotically conical ends*, J. Amer. Math. Soc. 27 (2014) 613–638 MR3194490

[24] **L Wang**, *Uniqueness of self-similar shrinkers with asymptotically cylindrical ends*, J. Reine Angew. Math. (2014) published online in March

[25] **X-J Wang**, *Convex solutions to the mean curvature flow*, Ann. of Math. 173 (2011) 1185–1239 MR2800714

[26] **B White**, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. 13 (2000) 665–695 MR1758759

[27] **B White**, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. 16 (2003) 123–138 MR1937202

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