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Geometry & Topology 19 (2015) 2767–2799

Finite part of operator K–theory for groups finitely embeddable into Hilbert space and the degree of nonrigidity of manifolds

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In this paper, we study lower bounds on the K–theory of the maximal C^*–algebra of a discrete group based on the amount of torsion it contains. We call this the finite part of the operator K–theory and give a lower bound that is valid for a large class of groups, called the finitely embeddable groups. The class of finitely embeddable groups includes all residually finite groups, amenable groups, Gromov's monster groups, virtually torsion-free groups (eg $\mathrm{Out}(F_n)$), and any group of analytic diffeomorphisms of an analytic connected manifold fixing a given point. We apply this result to measure the degree of nonrigidity for any compact oriented manifold M with dimension $4k-1$ $(k>1)$. In this case, we derive a lower bound on the rank of the structure group $S(M)$, which is roughly defined to be the abelian group of all pairs (M', f), where M' is a compact manifold and $f\colon M'\to M$ is a homotopy equivalence. In many interesting cases, we obtain a lower bound on the reduced structure group $\widetilde{S}(M)$, which measures the size of the collections of compact manifolds homotopic equivalent to but not homeomorphic to M by any homeomorphism at all (not necessary homeomorphism in the homotopy equivalence class). For a compact Riemannian manifold M with dimension greater than or equal to 5 and positive scalar curvature metric, there is an abelian group $P(M)$ that measures the size of the space of all positive scalar curvature metrics on M. We obtain a lower bound on the rank of the abelian group $P(M)$ when the compact smooth spin manifold M has dimension $2k-1$ $(k>2)$ and the fundamental group of M is finitely embeddable.

19K99; 20F99, 58D29

1 Introduction

The purpose of this paper is to find simple lower bounds for operator K–theory of discrete groups of the conceptual simplicity of the dimension of a vector space and use the lower bounds to study the degree of nonrigidity for compact manifolds and the size of the space of positive scalar curvature metrics for compact manifolds. Our measure

Published: 20 October 2015 DOI: 10.2140/gt.2015.19.2767

of simplicity will be that "extending the ring" should not lose information. In other words, we wish to have lower bounds for operator K–theory of discrete groups, that survive any inclusion into larger groups.

The classical Novikov conjecture uses group homology to give lower bounds on K–theory, but the existence of acyclic groups kills lower bounds that survive inclusion into large groups (by Baumslag, Dyer and Miller [6], any finitely presented group can be embedded into an acyclic group). The Baum–Connes conjecture and Farrell–Jones conjecture give conjectural pictures of how K–theory and L–theory of groups should be built out of group homology and K–theory and L–theory of finite groups. The algebraic K–theory of finite groups is typically very complicated, but it seems that very little survives inclusions; for example, the Whitehead group of symmetric groups is trivial (see Oliver [18]).

However, the classical trace map on the group algebra can be used to detect K_0 elements of the group algebra coming from finite subgroups: for each finite-order element g in the group, the map summing up all coefficients of group elements in the conjugacy class of the element g is a trace on the group algebra; see Stallings [28]. This shows that the number of conjugacy classes of the finite-order elements in the group is a lower bound for K_0 of the real group algebra. Our requirement that the K_0 elements survive inclusion into larger groups means that the best we can reasonably hope for would be to have a copy of $\mathbb{Z}$ for every finite order that occurs among the elements of the group; since any two elements of the same order in a group are conjugate in an HNN extension containing that group.

Part of our interest in these type of questions comes from a basic problem in geometry and topology. Suppose one wants to know, for instance, whether or not two manifolds are homeomorphic, then surgery theory tells us to divide this problem in two parts: first decide if they are homotopy equivalent, and then decide if a given homotopy equivalence is homotopic to a homemorphism. The problem with this is that the answer to the second question depends on which homotopy equivalence is chosen. If one has an invariant that survives inclusion into larger groups, then by including the fundamental group π into $\pi \rtimes \mathrm{Aut}(\pi)$, one can remove dependence on identification of fundamental group, and one has a chance of getting invariants that distinguish manifolds (or concordance classes of metrics with positive scalar curvature) that cannot be deduced from surgery theory. The theory we develop here suffices to provide such invariants for many interesting class of groups including all linear groups, mapping class groups and outer automorphism groups of free groups.

It is worth noting that the theory of L^2 eta invariants developed by Cheeger and Gromov [10] gives some nontrivial information about these kinds of problems; see

Chang and Weinberger [9] and Piazza and Schick [19]. However, this theory can only tell the difference between the presence and absence of torsion in the fundamental group, and does not exploit the existence of elements of different orders. If the torsion is central, then one can modify that construction to handle it along the lines we described above without much difficulty, but in general, we have not had much success in pursuing this direction. We shall approach these invariants from the point of view of K–theory of C^*–algebras. In that setting we succeed in somewhat more generality. We introduce a general class of groups for which we can obtain a lower bound for operator K–theory. This class of groups include all residually finite groups, amenable groups, hyperbolic groups, Burnside groups, Gromov's monster groups, virtually torsion-free groups (eg $\mathrm{Out}(F_n)$), and any group of analytic diffeomorphisms of an analytic connected manifold fixing a given point. As explained, these results also have applications to sizes of structure group and the size of concordance classes of metrics of positive scalar curvature.

Now we describe our results in more detail.

Let G be a countable group. An element $g \in G$ is said to have order d if d is the smallest positive integer such that $g^d = e$, where e is the identity element of G. If no such positive integer exists, we say that the order of g is ∞.

If $g \in G$ is an element in G with finite order d, then we can define an idempotent in the group algebra $\mathbb{Q}G$ by

$$p_g = \frac{1}{d}\Big(\sum_{k=1}^{d} g^k\Big).$$

For the rest of this paper, we denote the maximal group C^*–algebra of G by $C^*(G)$. We define $K_0^{\mathrm{fin}}(C^*(G))$, the finite part of $K_0(C^*(G))$, to be the abelian subgroup of $K_0(C^*(G))$ generated by $[p_g]$ for all elements $g \neq e$ in G with finite order.

We remark that rationally all representations of finite groups are induced from finite cyclic groups; see Serre [26]. This explains that the finite part of K–theory, rationally, contains all K–theory elements which can be constructed using finite subgroups, despite being constructed using only cyclic subgroups.

Conjecture 1.1 *Let $\{g_1, \dots, g_n\}$ be a collection of elements in G with distinct finite orders such that $g_i \neq e$ for all $1 \leq i \leq n$.*

(1) *$\{[p_{g_1}], \dots, [p_{g_n}]\}$ generates an abelian subgroup of $K_0^{\mathrm{fin}}(C^*(G))$ of rank n.*

(2) *Any nonzero element in the abelian subgroup of $K_0^{\mathrm{fin}}(C^*(G))$ generated by $\{[p_{g_1}], \dots, [p_{g_n}]\}$ is not in the image of the assembly map $\mu\colon K_0(BG) \simeq K_0^G(EG) \to K_0(C^*(G))$, where EG is the universal space for proper free G–actions.*

In fact, we can state a stronger conjecture in terms of K–theory elements coming from finite subgroups and the number of conjugacy classes of nontrivial finite-order elements. Such a stronger conjecture follows from the strong Novikov conjecture but would not survive inclusion into large groups.

The following concept is due to Gromov.

Definition 1.2 A countable discrete group G is said to be coarsely embeddable into Hilbert space H if there exists a map $f\colon G \to H$ satisfying:

(1) For any finite subset $F \subseteq G$, there exists $R > 0$ such that if $g^{-1}h \in F$, then $d(f(g), f(h)) \leq R$.

(2) For any $S > 0$, there exists a finite subset $E \subseteq G$ such that if $g^{-1}h \in G - E$, then $d(f(g), f(h)) \geq S$.

The class of groups coarsely embeddable into Hilbert space includes amenable groups (Bekka, Cherix and Valette [7]), hyperbolic groups (Sela [25]), and linear groups (Guentner, Higson and Weinberger [15]). However, Gromov's monster groups are not coarsely embeddable into Hilbert space (Gromov [13] and Arzhantseva and Delzant [1]). The importance of the concept of coarse embeddability is due to the theorem that the strong Novikov conjecture holds for groups coarsely embeddable into Hilbert space (Yu [34] and Skandalis, Tu and Yu [27]). Kasparov and Yu [16] introduced a weaker condition, coarse embeddability into Banach spaces with Property H, and proved the strong Novikov conjecture for groups coarsely embeddable into Banach spaces with Property H.

The following concept is more flexible than coarse embeddability into Hilbert space.

Definition 1.3 A countable discrete group G is said to be finitely embeddable into Hilbert space H if for any finite subset $F \subseteq G$, there exists a group G' coarsely embeddable into H such that there is a map $\phi\colon F \to G'$ satisfying:

(1) $\phi(gh) = \phi(g)\phi(h)$ if $g, h \in F$ and $gh \in F$.

(2) If g is a finite-order element in F, then $\operatorname{order}(\phi(g)) = \operatorname{order}(g)$.

We mention that the class of groups finitely embeddable into Hilbert space include all residually finite groups, amenable groups, hyperbolic groups, Burnside groups, Gromov's monster groups, virtually torsion-free groups (eg $\mathrm{Out}(F_n)$), and any group of analytic diffeomorphisms of an analytic connected manifold fixing a given point. Narutaka Ozawa, Denis Osin and Thomas Delzant have independently constructed examples of groups which are not finitely embeddable into Hilbert space. We can similarly define a concept of finite embeddability into Banach spaces with Property H.

The general validity of Conjecture 1.1 is still open. The following result proves Conjecture 1.1 for a large class of groups.

Theorem 1.4 *Conjecture 1.1 holds for groups finitely embeddable into Hilbert space.*

We define $N_{\mathrm{fin}}(G)$ to be the cardinality of the set

$$\{d > 0 \mid \exists g \in G,\ g \neq e,\ \mathrm{order}(g) = d\}.$$

If M is a compact oriented manifold, the structure group $S(M)$ in the topological category is the abelian group of equivalence classes of all pairs (f, M') such that M' is a compact oriented manifold and $f\colon M' \to M$, is an orientation-preserving homotopy equivalence; see Ranicki [20]. The rank of $S(M)$ measures the degree of nonrigidity for M.

The following result explains why it is interesting to study the finite part of $K_0(C^*(G))$.

Theorem 1.5 *Let M be a compact oriented manifold with dimension $4k-1$ $(k>1)$ and $\pi_1(M) = G$. If Conjecture 1.1 holds for G, then the rank of the structure group $S(M)$ is greater than or equal to $N_{\mathrm{fin}}(G)$.*

The following result is a consequence of Theorems 1.4 and 1.5.

Corollary 1.6 *Let M be a compact oriented manifold with dimension $4k-1$ $(k>1)$ and $\pi_1(M) = G$. If G is finitely embeddable into Hilbert space, then the rank of the structure group $S(M)$ is greater than or equal to $N_{\mathrm{fin}}(G)$.*

We conjecture that elements of the structure group distinguished by the method of this paper are actually different manifolds. We shall make this precise in the following few paragraphs.

Let M be a compact oriented manifold. Let $S_0(M)$ be the abelian subgroup of $S(M)$ generated by elements $[(f, M')] - [(\psi \circ f, M')]$, where $f\colon M' \to M$ is an orientation-preserving homotopy equivalence and $\psi\colon M \to M$, is an orientation-preserving self homotopy equivalence. We define the reduced structure group $\widetilde{S}(M)$ to be the quotient group $S(M)/S_0(M)$ (it is the coinvariants of the action of orientation-preserving self homotopy equivalence of M on $S(M)$).

The following conjecture gives a lower bound on the "size" of the set of different manifolds in the structure group.

Conjecture 1.7 *If M is a compact oriented manifold with dimension $4k-1$ $(k>1)$ and $\pi_1(M)=G$, then the rank of the reduced structure group $\widetilde{S}(M)$ is greater than or equal to $N_{\mathrm{fin}}(G)$.*

In an interesting case, we can prove this conjecture.

Theorem 1.8 *If G has a homomorphism ϕ to a residually finite group such that* kernel(ϕ) *is torsion free, then the above conjecture holds.*

Let M be a compact smooth manifold with dimension greater than or equal to 5. If M has a metric with positive scalar curvature, following Stolz alone [30] and with Rosenberg [23], we will introduce an abelian group $P(M)$ of concordance classes of all positive scalar curvature metrics on M (see Section 4 for a more precise definition). The following theorem gives an estimation on the "size" of the space of positive scalar curvature metrics on M when the fundamental group is finitely embeddable into Hilbert space.

Theorem 1.9 (1) *Let M be a compact smooth spin manifold with dimension $2k-1$ $(k>2)$. If M has a metric with positive scalar curvature and $\pi_1(M)=G$ is finitely embeddable into Hilbert space, then the rank of the abelian group $P(M)$ is greater than or equal to $N_{\mathrm{fin}}(G)$;*

(2) *Let M be a compact smooth spin manifold with dimension $4k-1$ $(k>1)$. If M has a metric with positive scalar curvature and $\pi_1(M)=G$ is finitely embeddable into Hilbert space, then the rank of the abelian group $P(M)$ is greater than or equal to $N_{\mathrm{fin}}(G)+1$.*

We remark that the main results in this paper remain to be true under the weaker condition of finite embeddability into Banach spaces with Property H.

In general, we also have the following results on the "size" of the space of positive scalar curvature metrics on M.

Theorem 1.10 (1) *Let M be a compact smooth spin manifold with a positive scalar curvature metric and dimension $2k-1$ $(k>2)$. If $\pi_1(M)$ is not torsion free, then the rank of the abelian group $P(M)$ is greater than or equal to one.*

(2) *Let M be a compact smooth spin manifold with a positive scalar curvature metric and dimension $4k-1$ $(k>1)$. If $\pi_1(M)$ is not torsion free, then the rank of the abelian group $P(M)$ is greater than or equal to two.*

Part (1) of the above theorem is motivated by and is a slight refinement of a theorem of Piazza–Schick stating that if M is a compact smooth spin manifold with a positive scalar curvature metric and dimension $2k-1$ $(k>2)$, and $\pi_1(M)$ is not torsion free, than the moduli space of positive scalar curvature metrics on M has infinitely many connected components; see Piazza and Schick [19]. Gromov and Lawson [14] proved that the space of positive scalar curvature metrics on a compact spin manifold M has infinitely many connected components if the dimension of M is $4k$ $(k>1)$.

This paper is organized as follows. In Section 2, we study the finite part of K–theory for group C^*–algebras. In Section 3, we discuss its application to nonrigidity of compact oriented manifolds. In Section 4, we discuss its application to positive scalar metrics. In Section 5, we study groups finitely embeddable into Hilbert space.

The authors wish to thank Sherry Gong, John Roe, Jonathan Rosenberg, Melissa Liu, Rufus Willett, Zhizhang Xie, and Romain Tessera for very helpful discussions. The authors thank the referee for providing detailed comments which improve the exposition of this article. We also thank Narutaka Ozawa, Denis Osin and Thomas Delzant for informing us their very interesting constructions of groups that do not finitely embed into Hilbert space. This paper was written during the second author's stay at the Shanghai Center for Mathematical Sciences (SCMS). The second author wishes to thank SCMS for providing a wonderful environment. The authors are partially supported by NSF.

2 Finite part of K–theory for group C^*–algebras

In this section, we study the finite part of K–theory for group C^*–algebras. In particular, we introduce a concept of finite representability and prove that Conjecture 1.1 is closed under finite representability.

Definition 2.1 Let $\mathcal{F}$ be a family of countable groups. We say that a countable group G is finitely representable in $\mathcal{F}$ if for any finite subset $F\subseteq G$, there exists a group $G'\in\mathcal{F}$ such that there is a map $\phi\colon F\to G'$ satisfying:

(1) $\phi(gh)=\phi(g)\phi(h)$ if $g,h\in F$ and $gh\in F$.

(2) If g is a finite-order element in F, then $\operatorname{order}(\phi(g))=\operatorname{order}(g)$.

Theorem 2.2 *Let $\mathcal{F}$ be a family of countable groups for which Conjecture 1.1 holds. If G is finitely representable in $\mathcal{F}$, then Conjecture 1.1 holds for G.*

Proof Let $\{F_k\}_{k=1}^{\infty}$ be an increasing family of finite subsets of G such that:

(1) $\bigcup_{k=1}^{\infty} F_k = G$.

(2) For each k, there exists a group G'_k in $\mathcal{F}$ and a map $f_k\colon F_k \to G'_k$ satisfying $f_k(gh) = f_k(g)f_k(h)$ for all $g, h \in F_k$ such that $gh \in F_k$.

(3) If g is a finite-order element in F_k, then $\operatorname{order}(f_k(g)) = \operatorname{order}(g)$.

Let ϕ_k be the linear map $\mathbb{C}F_k \to C^*(G'_k)$ extending f_k. The family of maps $\{\phi_k\}_{k=1}^{\infty}$ induces a $*$–homomorphism

$$\phi\colon \mathbb{C}G \to \prod_{k=1}^{\infty} C^*(G'_k) \Big/ \Big(\bigoplus_{k=1}^{\infty} C^*(G'_k)\Big), \quad \phi(a) = \Big[\prod_{k=n_a}^{\infty} \phi_k(a)\Big]$$

for $a \in \mathbb{C}G$, where n_a is a positive integer such that the support of a is contained in F_k for all $k \geq n_a$.

By the definition of the maximal group C^*–algebra, ϕ can be extended to a $*$–homomorphism (still denoted by ϕ)

$$\phi\colon C^*G \to \prod_{k=1}^{\infty} C^*(G'_k) \Big/ \Big(\bigoplus_{k=1}^{\infty} C^*(G'_k)\Big).$$

We have the following six-term exact sequence:

$$\begin{array}{ccccc}
K_0\Big(\bigoplus_{k=1}^{\infty} C^*(G'_k)\Big) & \xrightarrow{i_*} & K_0\Big(\prod_{k=1}^{\infty} C^*(G'_k)\Big) & \xrightarrow{\pi_*} & K_0\Big(\dfrac{\prod_{k=1}^{\infty} C^*(G'_k)}{\bigoplus_{k=1}^{\infty} C^*(G'_k)}\Big) \\
\uparrow & & & & \downarrow \partial \\
K_1\Big(\dfrac{\prod_{k=1}^{\infty} C^*(G'_k)}{\bigoplus_{k=1}^{\infty} C^*(G'_k)}\Big) & \longleftarrow & K_1\Big(\prod_{k=1}^{\infty} C^*(G'_k)\Big) & \xleftarrow{j_*} & K_1\Big(\bigoplus_{k=1}^{\infty} C^*(G'_k)\Big)
\end{array}$$

Observe that j_* is injective. By exactness, π_* is surjective. If g is an element in G with finite order d, there exists a positive integer $N_g > 0$ such that $g^m \in F_{N_g}$ for all m. Let $p_k = \phi_k(p_g)$ for all $k \geq N_g$. Define

$$p'_g = \prod_{k=N_g}^{\infty} p_k.$$

Let $\{g_1, \ldots, g_n\}$ be a collection of elements in G with $\operatorname{order}(g_l) = d_l$ such that $g_l \neq e$ for each l and $\operatorname{order}(g_{l_1}) < \operatorname{order}(g_{l_2})$ when $1 \leq l_1 < l_2 \leq n$. The rank of the subgroup of $K_0(C^*(G))$ generated by $\{[p_{g_1}], \ldots, [p_{g_n}]\}$ is greater than or equal to the rank of the abelian subgroup of $K_0(\prod_{k=1}^{\infty} C^*(G'_k)/(\bigoplus_{k=1}^{\infty} C^*(G'_k)))$ generated

by $\{[\phi(p_{g_1})],\dots,[\phi(p_{g_n})]\}$. In the same time, the rank of the abelian subgroup of $K_0(\prod_{k=1}^{\infty} C^*(G'_k)/(\bigoplus_{k=1}^{\infty} C^*(G'_k)))$ generated by $\{[\phi(p_{g_1})],\dots,[\phi(p_{g_n})]\}$ is the same as the rank of the abelian subgroup of $K_0(\prod_{k=1}^{\infty} C^*(G'_k))$ generated by $\{[p'_{g_1}],\dots,[p'_{g_n}]\}$. This is because, if z is a nonzero element in the abelian subgroup of $K_0(\prod_{k=1}^{\infty} C^*(G'_k))$ generated by $\{[p'_{g_1}],\dots,[p'_{g_n}]\}$, then z is not in the image of i_*. As a consequence, we have

$$\pi_*(z)\neq 0 \quad \text{in } K_0\Big(\prod_{k=1}^{\infty} C^*(G'_k)\Big/\Big(\bigoplus_{k=1}^{\infty} C^*(G'_k)\Big)\Big).$$

By assumption, the rank of the abelian subgroup of $K_0(\prod_{k=1}^{\infty} C^*(G'_k))$ generated by $\{[p'_{g_1}],\dots,[p'_{g_n}]\}$ is n. Hence the rank of the abelian subgroup of

$$K_0\Big(\prod_{k=1}^{\infty} C^*(G'_k)\Big/\Big(\bigoplus_{k=1}^{\infty} C^*(G'_k)\Big)\Big)$$

generated by $\{[\phi(p_{g_1})],\dots,[\phi(p_{g_n})]\}$ is n. It follows that the rank of the subgroup of $K_0(C^*(G))$ generated by $\{[p_{g_1}],\dots,[p_{g_n}]\}$ is also n.

Let x be a nonzero element in the abelian subgroup of $K_0^{\mathrm{fin}}(C^*(G))$ generated by $\{[p_{g_1}],\dots,[p_{g_n}]\}$. Assume by contradiction that x is in the image of the map $\mu\colon K_0^G(EG)\to K_0(C^*(G))$. Let N be a positive integer such that $g_i^k\in F_N$ for all i and k. This implies that the element

$$\prod_{k=N}^{\infty}\phi_*(x) \quad \text{in} \quad \prod_{k=1}^{\infty} K_0(C^*(G'_k))\Big/\Big(\bigoplus_{k=1}^{\infty} K_0(C^*(G'_k))\Big)$$

is in the image of the map $[\prod_{k=1}^{\infty}\mu]$:

$$\prod_{k=1}^{\infty} K_0^{G'_k}(EG'_k)\Big/\Big(\bigoplus_{k=1}^{\infty} K_0^{G'_k}(EG'_k)\Big)\to\prod_{k=1}^{\infty} K_0(C^*(G'_k))\Big/\Big(\bigoplus_{k=1}^{\infty} K_0(C^*(G'_k))\Big).$$

This implies that $\phi_*(x)$ is in the image of $\mu\colon K_0^{G'_k}(EG'_k)\to K_0(C^*(G'_k)$ for some large $k\geq N$. This contradicts the assumption that G'_k satisfies Conjecture 1.1. □

We remark that the above result is still open for reduced group C^*–algebras.

Theorem 2.3 (1) *If there exists an element $g\neq e$ in G with finite order, then the rank of the abelian subgroup of $K_0(C^*(G))$ generated by $[p_g]$ is one and any nonzero element in the abelian subgroup is not in the image of the map $\mu\colon K_0^G(EG)\to K_0(C^*(G))$.*

(2) *If there exist two elements in G with finite orders such that the order of g_1 is not equal to the order of g_2, then the rank of abelian subgroup of $K_0(C^*(G))$ generated by $[p_{g_1}]$ and $[p_{g_2}]$ is two.*

Proof (1) Let tr: $\mathbb{C}G \to \mathbb{C}$ be the canonical trace defined by

$$\mathrm{tr}\Big(\sum_{g\in G} c_g g\Big) = c_e.$$

It is easy to see that tr extends to a finite trace on both $C^*(G)$ and $C_r^*(G)$. Let d be the order of g. We have

$$\mathrm{tr}(p_g) = \frac{1}{d}.$$

This shows that the abelian subgroup generated by $[p_g]$ in $K_0(C^*(G))$ or $K_0(C_r^*(G))$ has rank one.

Let τ be the trace on $\mathbb{C}G$ defined by

$$\tau\Big(\sum_{g\in G} c_g g\Big) = \sum_{g\in G} c_g.$$

τ is a $*$–homomorphism from $\mathbb{C}G$ to $\mathbb{C}$. By the definition of $C^*(G)$, τ extends to a $*$–homomorphism from $C^*(G)$ to $\mathbb{C}$. It follows that τ is a trace on $C^*(G)$. Note that τ is the character of the trivial representation of G and $\tau(p_g) = 1$.

Assume by contradiction that a nonzero element x in the abelian subgroup is in the image of the map $\mu\colon K_0^G(EG) \to K_0(C^*(G))$. This means that there exists $y \in K_0^G(EG)$ satisfying $\mu(y) = x$. By Atiyah's L^2 index theorem in [2] and Theorem A.1 in the appendix of this paper, we have

$$\mathrm{tr}(x) = \tau(x) = \mathrm{index}(y),$$

where $\mathrm{index}(y)$ is the Fredholm index of K–homology class y. This is a contradiction with the fact that $\mathrm{tr}(p_g) - \tau(p_g) \neq 0$ and x is a nonzero multiple of $[p_g]$.

(2) We have

$$\mathrm{tr}(p_{g_1}) = \frac{1}{d_1}, \quad \mathrm{tr}(p_{g_2}) = \frac{1}{d_2}, \quad \tau(p_{g_1}) = \tau(p_{g_2}) = 1.$$

This proves that the abelian subgroup generated by $[p_{g_1}]$ and $[p_{g_2}]$ in $K_0(C^*(G))$ has rank two. □

We remark that the first part of (1) in the above theorem holds for both maximal and reduced group C^*–algebras. It remains open to prove the reduced group C^*–algebra analogue of second part of (1) in the above theorem and the reduced group C^*–algebra analogue of part (2) in the above theorem. It is also an open question whether any nonzero element in the abelian subgroup of $K_0(C_r^*(G))$ generated by $[p_{g_1}]$ and $[p_{g_2}]$ is not in the image of the map $\mu\colon K_0^G(EG) \to K_0(C_r^*(G))$ for any two nontrivial finite-order elements g_1 and g_2 in G.

3 Applications to degree of nonrigidity of manifolds

In this section, we use the finite part of K–theory for maximal group C^*–algebras to estimate the degree of nonrigidity for compact manifolds. We also estimate the rank of the finite part of K–theory for maximal group C^*–algebras in terms of the size of the set of finite-order elements when the groups are finitely embeddable into Hilbert space. John Roe pointed out that it is possible to prove Lemma 3.2 of this section using the Chern character constructions in [5] and [17]. Our proof of Lemma 3.2 is self-contained and is based on a Lefschetz fixed-point-type argument.

Let G be a countable group. Let X be a locally compact space with a proper and free cocompact action of G.

Let $C_0(X)$ be the algebra of all continuous functions on X vanishing at infinity. If H is a X–module (ie H is a Hilbert space with an action of $C_0(X)$), we define the support of an operator $T\colon H \to H$, support(T), to be the complement of the subset of $X \times X$ consisting of all points $(x, y) \in X \times X$ such that there exists f and g in $C_0(X)$ such that $f(x) \neq 0$, $g(y) \neq 0$, and $fTg = 0$. In this paper, all X–modules are ample in the sense that no nonzero function of $C_0(X)$ acts the X–modules as a compact operator.

We endow X with a G–invariant proper metric d (compatible with the topology of X). An operator on H is said to have finite propagation [21] if

$$\text{propagation}(T) = \sup\{d(x, y) \mid (x, y) \in \text{support}(T)\} < \infty.$$

Recall that an operator T on H is called locally traceable if for any pair of compactly supported functions f and g, fTg is a trace class operator. For each $p \geq 1$, we can similarly define the concept of locally Schatten-p class operators. Let $(\mathcal{S}_1 X)^G$ be the algebra of G–invariant and locally traceable operators on H with finite propagation. We observe that $(\mathcal{S}_1 X)^G$ is isomorphic to $\mathcal{S}_1 G$, the group algebra over the ring $\mathcal{S}_1$ of

all trace class operators. There is a canonical trace tr on $\mathcal{S}_1 G$,

$$\mathrm{tr}\left(\sum_{g\in G} s_g g\right) = \mathrm{trace}(s_e).$$

We use the same notation tr to be the corresponding trace on $(\mathcal{S}_1 X)^G$.

Lemma 3.1 *Let H be as above, let $T \in (\mathcal{S}_1 X)^G$. If there exists $\delta > 0$ such that*

$$\mathrm{support}(T) \subseteq \{(x, y) \in X \times X \mid d(x, y) \geq \delta\},$$

then $\mathrm{tr}(T) = 0$.

Proof Let $Z \subseteq X$ be a Borel fundamental domain of X, ie a domain such that:

(1) $\bigcup_{g\in G} gZ = X$.

(2) $gZ \cap Z = \varnothing$ for any $g \neq e$.

(3) Z is bounded.

Let χ_Z be the characteristic function of Z. We have

$$\mathrm{tr}(T) = \mathrm{trace}(\chi_Z T \chi_Z).$$

Note that the support of $\chi_Z T \chi_Z$ is contained in $\{(x, y) \in X \times X \mid d(x, y) \geq \delta\}$.

We decompose

$$Z = Z_1 \cup \cdots \cup Z_N$$

such that

$$\mathrm{diameter}(Z_i) < \tfrac{1}{2}\delta.$$

Let $H_Z = \chi_Z H$, $H_i = \chi_{Z_i} H$. Let $T_{i,j}$ be the restriction of $\chi_{Z_i} T \chi_{Z_j}$ to H_j and let T_Z be the restriction of $\chi_Z T \chi_Z$ to H_Z.

We have a Hilbert space decomposition

$$H_Z = \bigoplus_{i=1}^{N} H_i.$$

We have a corresponding matrix representation of T_Z,

$$T_Z = (T_{i,j})_{1\leq i,j\leq N}.$$

By the support condition on T_Z, we have $T_{i,i} = 0$ for all i. This implies that $\mathrm{trace}(T_Z) = 0$. □

Let G be a countable group and let $\mathcal{S}$ be the ring of Schatten class operators. Let $\mathcal{S}G$ be the group algebra over the ring $\mathcal{S}$ [33]. Let $j\colon \mathbb{C}G \to \mathcal{S}G$ be the inclusion homomorphism defined by

$$j(a) = p_0 a$$

for all $a \in \mathbb{C}G$, where p_0 is a rank-one projection in $\mathcal{S}$. We define the finite part of $K_0(\mathcal{S}G)$ to be the abelian subgroup of $K_0(\mathcal{S}G)$ generated by $j_*[p_g]$ for all finite-order elements $g \neq e$ in G.

Lemma 3.2 *Any nonzero element in the finite part of $K_0(\mathcal{S}G)$ is not in the image of the assembly map*

$$A\colon H_0^{\mathrm{Or}\,G}(EG, \mathbb{K}(\mathcal{S})^{-\infty}) \to K_0(\mathcal{S}G),$$

where the assembly map A is defined as in [12].

Proof Let g be a finite-order element $g \neq e$ in G. Let $\mathrm{tr}_g\colon \mathbb{C}G \to \mathbb{C}$, be the trace defined by

$$\mathrm{tr}_g\Big(\sum_\gamma c_\gamma \gamma\Big) = \sum_{\gamma \in C(g)} c_\gamma,$$

where $C(g)$ is the conjugacy class of g, ie

$$C(g) = \{\gamma \in G \mid \exists\, h \in G,\ h^{-1}\gamma h = g\}.$$

For any integer $m > 0$, let $\mathcal{S}_m$ be the ring of Schatten-m class operators. Let $n = 2k$ be the smallest even number greater than or equal to m. Define an n–cyclic cocycle $\tau_g^{(n)}$ on $\mathcal{S}_m G$ by

$$\tau_g^{(n)}(a_0, a_1, \dots, a_n) = \sum_{\gamma \in C(g)} \mathrm{tr}(\gamma^{-1} a_0 a_1 \cdots a_n)$$

for all $a_i \in \mathcal{S}_m G$, where $C(g)$ is the conjugacy class of g and $\mathrm{tr}\colon \mathcal{S}_1 G \to \mathbb{C}$, is the trace defined by

$$\mathrm{tr}\Big(\sum_{\gamma \in \Gamma} b_\gamma \gamma\Big) = \mathrm{trace}(b_e).$$

If S is the suspension operator in Connes' theory of cyclic cohomology theory [11], then

$$(S^k \,\mathrm{tr}_g)(a_0, \dots, a_n) = \mathrm{tr}_g(a_0 \cdots a_n)$$

for all $a_i \in \mathbb{C}G$.

We have

$$S^k \,\mathrm{tr}_g = j^* \tau_g^{(n)}$$

as cyclic cocycles over $\mathbb{C}G$. In particular, this implies that the pairing between tr_g and any idempotent p in the matrix algebra of $\mathbb{C}G$ is the same as the pairing between $\tau_g^{(n)}$ and $j(p)$, ie

$$\tau_g^{(n)}(j(p),\dots,j(p)) = \mathrm{tr}_g(p),$$

where the cyclic cocycle $\tau_g^{(n)}$ is naturally extended to a cyclic cocycle over the matrix algebra of $\mathbb{C}G$.

If an element $[z]$ is in the image of the assembly

$$A\colon H_0^{\mathrm{Or}\,G}(EG, \mathbb{K}(\mathcal{S})^{-\infty}) \to K_0(\mathcal{S}G),$$

then there exist $m > 0$ and a locally compact simplicial complex X with a proper and cocompact action of G such that $[z]$ is in the image of the assembly map

$$A\colon H_0^{\mathrm{Or}\,G}(X, \mathbb{K}(\mathcal{S}_m)^{-\infty}) \to K_0((\mathcal{S}_m X)^G) \cong K_0(\mathcal{S}_m G),$$

where X is endowed with a G–invariant proper metric d (compatible with the topology of X) and $(\mathcal{S}_m X)^G$ is the algebra of G–invariant and locally Schatten-m operators on H with finite propagation ($(\mathcal{S}_m X)^G$ is isomorphic to $\mathcal{S}_m G$). By cocompactness of the G–action on X, there exists $\delta > 0$ such that $d(\gamma x, x) \geq 10\delta$ for all $x \in X$ and $\gamma \neq e$ in G. If H is a G–X–module, let $(\mathcal{S}_m X)^G$ be the algebra of G–invariant and locally Schatten-m class operators on H with finite propagation. The class $[z]$ can be represented by

$$[q] - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right),$$

where q is an idempotent in the matrix algebra of $((\mathcal{S}_m X)^G)^+$ ($((\mathcal{S}_m X)^G)^+$ is obtained from $(\mathcal{S}_m X)^G$ by adjoining a unit). We naturally extend the cyclic cocycle $\tau_g^{(n)}$ to a cyclic cocycle over the matrix algebra of $((\mathcal{S}_m X)^G)^+$ (still denoted by $\tau_g^{(n)}$) by setting $\tau_g^{(n)}(a_0,\dots,a_n) = 0$ if a_i is a scalar matrix for some i.

In [4], Bartels, Farrell, Jones and Reich give a different definition of the assembly map and prove that their definition coincides with the assembly map defined in [12] (Corollary 6.3 in [4]). By the definition of the assembly map in [4], we can write

$$q = q_1 + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

such that q_1 is an element in the matrix algebra over the algebra of G–invariant and locally Schatten-m class operators on a G–X–module H with finite propagation satisfying

$$\mathrm{propagation}(q_1) < \delta/(n+1).$$

We have

$$\mathrm{propagation}(q_1^{n+1}) < \delta.$$

This, together with Lemma 3.1, implies

$$\operatorname{tr}(\gamma^{-1}q_1^{n+1}) = 0$$

for all $\gamma \neq e$ in G. It follows that

$$\tau_g^{(n)}(q, \dots, q) = 0.$$

Recall that the pairing between the cyclic cocycle $\tau_g^{(n)}$ and $K_0((\mathcal{S}_m X)^G)$ gives a homomorphism $(\tau_g^{(n)})_*\colon K_0(\mathcal{S}_m G) \to \mathbb{C}$. This implies that

$$(*) \qquad (\tau_g^{(n)})_*(z) = 0$$

for any finite-order element $g \neq e$ in G and any element $[z]$ in the image of the assembly map

$$A\colon H_0^{\mathrm{Or}\,G}(X, \mathbb{K}(\mathcal{S}_m)^{-\infty}) \to K_0((\mathcal{S}_m X)^G).$$

If x is a nonzero element in the finite part of $K_0(\mathcal{S}G)$, then x is a nonzero element in the finite part of $K_0(\mathcal{S}_m G)$ if m is some large natural number. We can write

$$z = \sum_{i=1}^{l} c_i [j(p_{g_i})]$$

for some $c_i \in \mathbb{Z}$ and finite-order elements $g_i \neq e \in G$. We can assume that the conjugacy class of g_{i_1} is different from the conjugacy class of g_{i_2} when $i_1 \neq i_2$ (otherwise $[j(p_{g_{i_1}})] = [j(p_{g_{i_2}})]$ in $K_0(\mathcal{S}_m G)$). Without loss of generality, we can assume that $c_l \neq 0$ and $\operatorname{order}(g_{i_1}) \leq \operatorname{order}(g_{i_2})$ when $i_1 \leq i_2$. Since the conjugacy class of g_{i_1} is different from the conjugacy class of g_{i_2} when $i_1 \neq i_2$, we have

$$\operatorname{tr}_{g_i}(p_{g_i}) \neq 0, \ \operatorname{tr}_{g_{i_2}}(p_{g_{i_1}}) = 0 \quad \text{when } i_1 < i_2.$$

The above facts, together with the identity

$$\tau_{g_l}^{(n)}(j(p_{g_i}), \dots, j(p_{g_i})) = \operatorname{tr}_{g_l}(p_{g_i}),$$

imply that

$$(\tau_{g_l}^{(n)})_*(x) \neq 0.$$

Combining this with the equation $(*)$, we obtain Lemma 3.2. □

Lemma 3.3 *If $\{g_1, \dots, g_n\}$ is a collection of finite-order elements in G with distinct conjugacy classes, then $\{j_*[p_{g_1}], \dots, j_*[p_{g_n}]\}$ generates an abelian subgroup of $K_0(\mathcal{S}G)$ with rank n, where j_* is as in Lemma 3.2. In particular, if $\{g_1, \dots, g_n\}$ is a collection of nontrivial finite-order elements in G with distinct orders, then $\{j_*[p_{g_1}], \dots, j_*[p_{g_n}]\}$ generates an abelian subgroup of $K_0(\mathcal{S}G)$ with rank n.*

Proof Lemma 3.3 essentially follows from the argument in the last paragraph of the proof of Lemma 3.2.

It suffices to prove that $\{j_*[p_{g_1}], \dots, j_*[p_{g_n}]\}$ generates an abelian subgroup of $K_0(\mathcal{S}_m G)$ with rank n for any natural number $m \geq 1$.

Let g be an element in G with finite order. Let tr_g be defined as in the proof of Lemma 3.2. By assumption, $g_1, \dots, g_n$ have distinct conjugacy classes. Without loss of generality, we can assume that $\mathrm{order}(g_{i_1}) \leq \mathrm{order}(g_{i_2})$ when $i_1 \leq i_2$. As a consequence, we have

$$\mathrm{tr}_{g_i}(p_{g_i}) \neq 0, \quad \mathrm{tr}_{g_{i_2}}(p_{g_{i_1}}) = 0 \quad \text{when } i_1 < i_2.$$

As in the proof of Lemma 3.2, tr_{g_i} can be lifted to a cyclic cocycle $\tau_{g_i}^{(n)}$ on $\mathcal{S}_m G$ when $n = 2k \geq m$. Hence for each m, there exist homomorphisms $(\tau_{g_i}^{(n)})_*\colon K_0(\mathcal{S}_m G) \to \mathbb{C}$ such that

$$(\tau_{g_i}^{(n)})_*([p_{g_i}]) \neq 0, \quad (\tau_{g_{i_2}}^{(n)})_*([p_{g_{i_1}}]) = 0 \quad \text{when } i_1 < i_2.$$

Now Lemma 3.3 follows. □

Proof of Theorem 1.4 By Theorem 2.2, it suffices to prove Conjecture 1.1 for groups coarsely embeddable into Hilbert space.

Let $\underline{E}G$ be the universal space for proper G–action. We have the commutative diagram

$$\begin{array}{ccc} H_0^{\mathrm{Or}\,G}(\underline{E}G, \mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{A} & K_0(\mathcal{S}G) \\ \downarrow \psi_* & & \downarrow i_* \\ K_0^G(\underline{E}G) & \xrightarrow{\mu} & K_0(C^*(G)), \end{array}$$

where the right vertical map is induced by the inclusion map i and the left vertical map is induced by a map at the spectra level.

The strong Novikov conjecture holds for groups coarsely embeddable into Hilbert space, ie the bottom horizontal map μ in the above diagram is injective [34; 27]. This, together with the fact that the left vertical map ψ_* in the above diagram is an isomorphism, implies that the right vertical map i_* in the above diagram is an isomorphism from the image of $A\colon H_0^{\mathrm{Or}\,G}(\underline{E}G, \mathbb{K}(\mathcal{S})^{-\infty}) \to K_0(\mathcal{S}G)$, to the image of $\mu\colon K_0^G(\underline{E}G) \to K_0(C^*(G))$. We observe that the finite part of $K_0(\mathcal{S}G)$ is contained in the image of the map $A\colon H_0^{\mathrm{Or}\,G}(\underline{E}G, \mathbb{K}(\mathcal{S})^{-\infty}) \to K_0(\mathcal{S}G)$. The above facts, together with Lemma 3.3, imply that the abelian subgroup of $K_0(C^*(G))$ generated by $\{[p_{g_1}], \dots, [p_{g_n}]\}$ has rank n, where $\{g_1, \dots, g_n\}$ is a collection of finite-order elements in G as in Conjecture 1.1.

We also have the commutative diagram

$$\begin{array}{ccc} H_0^{\mathrm{Or}\,G}(EG,\mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{A} & K_0(\mathcal{S}G) \\ \downarrow \psi_* & & \downarrow i_* \\ K_0^G(EG) & \xrightarrow{\mu} & K_0(C^*(G)), \end{array}$$

where the left vertical map in the above diagram is compatible with the left vertical map in the previous diagram via the natural homomorphism from $H_0^{\mathrm{Or}\,G}(EG,\mathbb{K}(\mathcal{S})^{-\infty})$ to $H_0^{\mathrm{Or}\,G}(\underline{E}G,\mathbb{K}(\mathcal{S})^{-\infty})$ and the natural homomorphism from $K_0^G(EG)$ to $K_0^G(\underline{E}G)$.

Assume that by contradiction there is a nonzero element x in the abelian subgroup of $K_0(C^*(G))$ generated by $\{[p_{g_1}],\dots,[p_{g_n}]\}$ such that $x=\mu(y)$ for some $y\in K_0^G(EG)$. The left vertical map ψ_* in the above diagram is an isomorphism. This implies $y=\psi_*(y')$ for some $y'\in H_0^{\mathrm{Or}\,G}(EG,\mathbb{K}(\mathcal{S})^{-\infty})$. There exists an element x' in the finite part of $K_0(\mathcal{S}G)$ such that $i_*(x')=x$. We have

$$i_*(A(y')-x')=0.$$

We recall that the finite part of $K_0(\mathcal{S}G)$ is contained in the image of the map

$$A\colon H_0^{\mathrm{Or}\,G}(\underline{E}G,\mathbb{K}(\mathcal{S})^{-\infty})\to K_0(\mathcal{S}G).$$

As a consequence, we know that $A(y')-x'$ is in the image of the map

$$A\colon H_0^{\mathrm{Or}\,G}(\underline{E}G,\mathbb{K}(\mathcal{S})^{-\infty})\to K_0(\mathcal{S}G).$$

It follows from the isomorphism statement before the above diagram that $A(y')-x'=0$. Hence $A(y')=x'$. Observe that x' is a nonzero element in the finite part of $K_0(\mathcal{S}G)$. This is a contradiction with Lemma 3.2. □

We remark that the argument in the above proof shows that the strong Novikov conjecture implies Conjecture 1.1.

Theorem 3.4 *Let M be a compact oriented manifold with dimension $4k-1$ $(k>1)$. Suppose $\pi_1(M)=G$ and $g_1,\dots,g_n$ be finite-order elements in G such that $g_i\neq e$ for all i and $\{[p_{g_1}],\dots,[p_{g_n}]\}$ generates an abelian subgroup of $K_0(C^*(G))$ with rank n. Suppose that any nonzero element in the abelian subgroup of $K_0(C^*(G))$ generated by $\{[p_{g_1}],\dots,[p_{g_n}]\}$ is not in the image of the map $\mu\colon K_0^G(EG)\to K_0(C^*(G))$, then the rank of the structure group $S(M)$ is greater or equal to n.*

Proof We recall the surgery exact sequence

$$\cdots\to H_{4k}(M,\mathbb{L})\to L_{4k}(\mathbb{Z}G)\to S(M)\to H_{4k-1}(M,\mathbb{L})\to\cdots.$$

For each finite subgroup H of G, we have the commutative diagram

$$\begin{array}{ccc} H^H_{4k}(\underline{E}H, \mathbb{L}) & \xrightarrow{A} & L_{4k}(\mathbb{Z}H) \\ \downarrow & & \downarrow \\ H^G_{4k}(\underline{E}G, \mathbb{L}) & \xrightarrow{A} & L_{4k}(\mathbb{Z}G), \end{array}$$

where the vertical maps are induced by the inclusion homomorphism from H to G. For each element g in H with finite order, p_g gives an element in $L_0(\mathbb{Q}H)$. Let $[q_g]$ be the corresponding element in $L_{4k}(\mathbb{Q}H)$ given by periodicity. Recall that

$$L_{4k}(\mathbb{Z}H) \otimes \mathbb{Q} \simeq L_{4k}(\mathbb{Q}H) \otimes \mathbb{Q}.$$

For each element g in H with finite order, we use the same notation $[q_g]$ to denote the element in $L_{4k}(\mathbb{Z}H) \otimes \mathbb{Q}$ corresponding to $[q_g] \in L_{4k}(\mathbb{Q}H)$ under the above isomorphism.

We have the commutative diagram

$$\begin{array}{ccc} H^G_{4k}(EG, \mathbb{L}) \otimes \mathbb{Q} & \xrightarrow{A} & L_{4k}(\mathbb{Z}G) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ K^G_0(EG) \otimes \mathbb{Q} & \xrightarrow{\mu} & K_0(C^*(G)) \otimes \mathbb{Q}, \end{array}$$

where the left vertical map is induced by a map at the spectra level and the right vertical map is induced by the inclusion map

$$L_{4k}(\mathbb{Z}G) \to L_{4k}(C^*(G)) \cong K_0(C^*(G))$$

(see [22] for the last identification).

The above commutative diagram, together with the assumption that any nonzero element in the abelian subgroup of $K_0(C^*(G))$ generated by $\{[p_{g_1}], \dots, [p_{g_n}]\}$ is not in the image of the map $\mu\colon K^G_0(EG) \to K_0(C^*(G))$, implies that:

(1) Any nonzero element in the abelian subgroup of $L_{4k}(\mathbb{Z}G) \otimes \mathbb{Q}$ generated by $\{[q_{g_1}], \dots, [q_{g_n}]\}$ is not in the image of the rational assembly map

$$A\colon H^G_{4k}(EG, \mathbb{L}) \otimes \mathbb{Q} \to L_{4k}(\mathbb{Z}G) \otimes \mathbb{Q}.$$

(2) The abelian subgroup of $L_{4k}(\mathbb{Z}G) \otimes \mathbb{Q}$ generated by $\{[q_{g_1}], \dots, [q_{g_n}]\}$ has rank n.

By exactness of the surgery sequence, we know that the map: $L_{4k}(\mathbb{Z}G)\otimes\mathbb{Q}\to S(M)\otimes\mathbb{Q}$, is injective on the abelian subgroup of $L_{4k}(\mathbb{Z}G)\otimes\mathbb{Q}$ generated by $\{[q_{g_1}],\dots,[q_{g_n}]\}$. This, together with the fact that the abelian subgroup of $L_{4k}(\mathbb{Z}G)\otimes\mathbb{Q}$ generated by $\{[q_{g_1}],\dots,[q_{g_n}]\}$ has rank n, implies our theorem. □

Corollary 3.5 *Let G be a countable group. If G is finitely embeddable into Hilbert space and M is a compact oriented manifold with dimension $4k-1$ $(k>1)$ and $\pi_1(M)=G$, then the rank of the structure group $S(M)$ is greater than or equal to $N_{\mathrm{fin}}(G)$.*

Let G be a countable group. We define $r_{\mathrm{fin}}(G)$ to be the rank of the abelian subgroup of $K_0(C^*(G))$ generated by $[p_g]$ for all finite-order elements g in G. We emphasize that here we allow g to be the identity e.

The proof of the following result is similar to that of Theorem 1.4 and is therefore omitted.

Theorem 3.6 *If G is a countable group finitely embeddable into Hilbert space, then $r_{\mathrm{fin}}(G)$ is greater than or equal to $N_{\mathrm{fin}}(G)+1$.*

The following result is a consequence of Theorem 3.4 and Theorem 2.3(1).

Corollary 3.7 *Let M be a compact oriented manifold with dimension $4k-1$ $(k>1)$ and $\pi_1(M)=G$. If there exists an element $g\neq e$ in G with finite order, then the rank of $S(M)$ is greater than or equal to one.*

The above corollary gives a different proof of a theorem of Chang and Weinberger [9].

Remark We mention that if the group satisfies the strong Novikov conjecture, then the same method can be used to prove a stronger statement than Theorem 3.4. In this case, one can show that the number of conjugacy classes of nontrivial finite-order elements in the group is a lower bound for both the rank of the finite part of operator K–theory and the rank of the structure group $S(M)$ when the dimension of M is $4k-1$ $(k>1)$. To prove this result, we need to consider K–theory classes induced by all representations of finite subgroups of the fundamental group. However, our method doesn't yield the same lower bound for finitely embeddable groups.

In surgery theory, if M is a compact oriented manifold, the elements in the structure group $S(M)$ are pairs (f,M'), where M' is a compact oriented manifold and $f\colon M'\to M$, is an orientation-preserving homotopy equivalence. When an element

is nontrivial, it is often the case that M' and M are homeomorphic, but f is not homotopy equivalent to a homeomorphism. In [9], Chang and Weinberger observed that the von Neumann trace can be used to distinguish the manifolds, not just structures.

We conjecture that elements of the structure group distinguished by the method of this paper are actually different manifolds.

Let M be a compact oriented manifold. Let $S_0(M)$ be the abelian subgroup of $S(M)$ generated by elements $[(f, M')]-[(\psi\circ f, M')]$, where $f\colon M'\to M$, is an orientation-preserving homotopy equivalence and $\psi\colon M\to M$, is an orientation-preserving self homotopy equivalence. We define the reduced structure group $\widetilde{S}(M)$ to be the quotient group $S(M)/S_0(M)$ (it is the coinvariants of the action of orientation-preserving self homotopy equivalences of M on $S(M)$).

The following conjecture gives a lower bound on the "size" of the set of different manifolds in the structure group.

Conjecture 3.8 *If M is a compact oriented manifold with dimension $4k-1$ $(k>1)$ and $\pi_1(M)=G$, then the rank of the reduced structure group $\widetilde{S}(M)$ is greater than or equal to $N_{\mathrm{fin}}(G)$.*

In a special case, we can verify this conjecture.

Theorem 3.9 *If G has a homomorphism ϕ to a residually finite group such that* kernel(ϕ) *is torsion free, then the above conjecture holds.*

Proof For each integer $m\geq 1$, let G_m be the intersection of all normal subgroups of G with index at most m. Observe that G_m is a finite-index subgroup of G and G_m is preserved under the action of $\mathrm{Aut}(G)$, the group of all automorphisms of G. As a consequence, the semidirect product $G\rtimes\mathrm{Aut}(G)$ has a homomorphism ϕ_m to the finite group $Q=G/G_m\rtimes\mathrm{Aut}(G/G_m)$.

Let $\{g_1,\dots,g_n\}$ be elements in G with distinct finite orders $\{d_1,\dots,d_n\}$. Without loss of generality, we can assume that $d_i<d_j$ when $i<j$. Let $g_i'=\phi_m(g_i)$ in $Q=G/G_m\rtimes\mathrm{Aut}(G/G_m)$ for each i. By the assumption of our theorem, there exists a sufficiently large integer m such that g_i' has order d_i for all $1\leq i\leq n$. If N is a compact oriented manifold with dimension $4k-1$ and $\pi_1(N)=G$, there is an associated map $\sigma\colon N\to BQ$ obtained as a composition of the classifying map from N to BG with the map induced from the inclusion map: $G\to G\rtimes\mathrm{Aut}(G)$, and the homomorphism $\phi_m\colon G\rtimes\mathrm{Aut}(G)\to Q$. The bordism group of such pairs (N,σ) over BQ is finite since Q is finite and $4k-1$ is odd. Hence there exists a positive integer

l such that lN is the boundary of a compact oriented manifold W equipped with a map $h\colon W \to BQ$ that extends σ. Let W_Q and N_Q be respectively the Q–covers of W and N. We note that W_Q is the pull back of the Q–principal bundle of EQ over BQ by the map h. We define a g_i–torsion invariant of (N,ψ) by

$$\tau_{g_i}(N,\sigma) = \frac{1}{l}\big(\mathrm{tr}(g_i'|_{H_+}) - \mathrm{tr}(g_i'|_{H_-})\big),$$

where

$$H^{2k}(W_Q, N_Q) = H^{2k}_+(W_Q, N_Q) \oplus H^{2k}_-(W_Q, N_Q)$$

is the decomposition corresponding to the positive and negative parts of the symmetric bilinear form associated to the cup product on $H^{2k}(W_Q, N_Q)$, and $H_+ = H^{2k}_+(W_Q, N_Q)$ and $H_- = H^{2k}_-(W_Q, N_Q)$, and tr is the standard trace on the algebra of all linear operators on the finite-dimensional vector space such that the trace of the identity is the dimension of the vector space. It is not difficult to see that this invariant is independent of the choice of (W,σ).

Note that if ψ is an orientation-preserving self homotopy equivalence of N, then

$$\tau_{g_i}(N,\sigma) = \tau_{g_i}(N,\sigma\circ\psi).$$

The above identity can be shown as follows. Let ψ_* be the element in $\mathrm{Aut}(G)$ induced by the map ψ and denote the element $\phi_m(\psi_*)$ in Q by ψ_*'. Observe that $\sigma\circ\psi$ is homotopy equivalent to $\widetilde{\psi}\circ\sigma$, where $\widetilde{\psi}$ is the map from BQ to BQ induced by the homomorphism

$$Q \to Q, \quad q \mapsto \psi_*' q (\psi_*')^{-1}.$$

Let $\widetilde{h}$ be the map from W to BQ defined by $\widetilde{h} = \widetilde{\psi}\circ h$. It is easy to see that $\widetilde{h}$ extends the map $\sigma\circ\psi$ from N to BQ. By the definition of the g_i–torsion invariant of $(N,\sigma\circ\psi)$, we have

$$\tau_{g_i}(N,\sigma\circ\psi) = \frac{1}{l}\big(\mathrm{tr}(\psi_*' g_i' (\psi_*')^{-1}|_{H_+}) - \mathrm{tr}(\psi_*' g_i' (\psi_*')^{-1}|_{H_-})\big).$$

Now our desired identity follows from the above equation and the trace property.

The above identity implies that the homomorphism from $S(M)$ to $\mathbb{R}$ defined by

$$(f, M') \to \tau_{g_i}(M', \theta\circ f)$$

is 0 on $S_0(M)$, where θ is the map from M to BQ obtained by composing the classifying map $M \to BG$ with the map from BG to BQ derived from the inclusion $G \to G\rtimes\mathrm{Aut}(G)$ and the homomorphism ϕ_m from $G\rtimes\mathrm{Aut}(G)$ to Q. As a consequence, this homomorphism from $S(M)$ to $\mathbb{R}$ induces a homomorphism from $\widetilde{S}(M)$ to $\mathbb{R}$.

Let q_{g_i} be the element in $L_{4k}(\mathbb{Z}G)$ as in the proof of Theorem 3.4. There is a cobordism Y_i realizing q_{g_i} such that $\partial Y_i = M \cup (-M_i)$ for some compact manifold M_i and the maps

$$(f_i, \partial_0 f_i, \partial_1 f_i)\colon (Y_i, M, M_i) \to (M \times [0,1], M \times \{0\}, M \times \{1\})$$

satisfy the following conditions:

(1) $\partial_0 f_i$, the restriction of f_i to M, is the identity map from M to M.

(2) $\partial_1 f_i$, the restriction of f_i to M_i, is an orientation-preserving homotopy equivalence from M_i to M.

We abbreviate $\tau_{g_i}(M_j, \theta \circ (\partial_1 f_i))$ by $\tau_{g_i}(M_j)$ and $\tau_{g_i}(M, \theta)$ by $\tau_{g_i}(M)$. We have

$$\tau_{g_i}(M_j) - \tau_{g_i}(M) = c \operatorname{tr}(g_i' p_{g_j'}),$$

where c is a nonzero constant and

$$p_{g_j'} = \frac{1}{d_j} \sum_{k=1}^{d_j} (g_j')^k$$

and tr is the canonical trace on the group algebra of Q. It follows that

$$\tau_{g_i}(M_j) - \tau_{g_i}(M) \neq 0 \quad \text{if } i = j, \quad \tau_{g_i}(M_j) - \tau_{g_i}(M) = 0 \quad \text{if } i > j.$$

This implies the abelian subgroup of $\widetilde{S}(M)$ generated by

$$[(f_1, M_1)] - [(\mathrm{id}_M, M)], \quad \ldots, \quad [(f_n, M_n)] - [(\mathrm{id}_M, M)]$$

has rank n, where id_M is the identity map on M. □

Corollary 3.10 *If G is residually finite, then Conjecture 3.8 holds.*

We remark that the proof of Theorem 3.9 shows that elements of the structure group distinguished by the method of this paper are indeed different manifolds in this special case.

4 Applications to the space of positive scalar curvature metrics

In this section, we apply our result on the finite part of K–theory for group C^*–algebras to estimate the size of the space of positive scalar curvature metrics on a compact smooth spin manifold. This section is influenced by the previous work of Rosenberg

and Stolz [23; 29; 30]. In the finite group case, our result follows from the work of Stolz [29].

If a compact smooth spin manifold M has a positive scalar curvature metric and the dimension of M is greater than or equal to 5, we introduce an abelian group $P(M)$ of equivalence classes of all positive scalar curvature metrics on M. We give a lower bound on the rank of $P(M)$ when the dimension of M is $2k-1$ with $k>2$ and the fundamental group $\pi_1(M)$ is finitely embeddable into Hilbert space. For general group, we show that if $\pi_1(M)$ is not torsion free, then the rank of the abelian group $P(M)$ is at least one when dimension of M is $2k-1$ $(k>2)$ and is at least two when dimension of M is $4k-1$ $(k>1)$. We remark that we don't need the spin condition to define $P(M)$, but we require the spin condition to get a lower bound on its size.

Let M be a compact smooth manifold M with $\pi_1(M)=G$ and dimension greater than or equal to five. Assume that M has a positive scalar curvature metric g_M. Let I be the closed interval. We first form a connected sum $(M\times I)\sharp(M\times I)$, where the connected sum is performed away from the boundary of each copy of $M\times I$. We define the generalized connected sum $(M\times I)\natural(M\times I)$ to be the manifold obtained from $(M\times I)\sharp(M\times I)$ by surgering the kernel of the homomorphism from $\pi_1((M\times I)\sharp(M\times I))=G*G$ to G. $(M\times I)\natural(M\times I)$ has four boundary components, two components being M and the other two being $-M$, where $-M$ is the manifold M with reversed orientation. If g_1 and g_2 are positive scalar curvature metrics on M, we place g_M on one boundary component M and g_1 and g_2 on the other two boundary components $-M$. By the surgery theorem [14; 24], there exists a positive scalar curvature metric on $(M\times I)\natural(M\times I)$ such that it is a product metric near the other boundary component M. We denote by g the positive scalar curvature metric on this boundary component M.

We prove that if g and g' are two positive scalar curvature metrics on M obtained from the same pair of positive scalar curvature metrics g_1 and g_2 by the above process, then g and g' are concordant. We can glue the remaining boundaries of the two generalized connected sums to form a cobordism of M to itself (the metric is a product near the boundary and has positive scalar curvature) with the fundamental group surjecting into G. We surger away the kernel of the surjection. We note that the nullcobordisms of a given n–manifold N with a homomorphism of $\pi_1(N)\to G$ are torsors, ie given one nullcobordism, the remaining nullcobordisms, up to nullcobordism relative to the boundary, form an abelian group $(\cong\Omega_{n+1}(N))$. Here the preferred null cobordism is $N\times I$. We claim that the element given by our cobordism is trivial; this is because we can glue the boundaries of the cylinders $M\times I$ in the cobordism to obtain $(M\times S^1)\natural(M\times S^1)$ which bounds $(M\times D^2)\natural(M\times D^2)$. Consequently we can apply the surgery theorem [14; 24] to this cobordism of (M,g) to (M,g') to

obtain a new positive scalar curvature metric cobordism to these manifolds where the underlying manifold is $M \times I$. Thus g and g' are concordant.

Two positive scalar curvature metrics g and g' on M are defined to be equivalent if and only if they are concordant. We define the equivalence class $[g]$ to be the sum of the equivalence classes of $[g_1]$ and $[g_2]$ with respect to $[g_M]$. By argument similar to that in the previous paragraph, we can show that the set of the concordance classes of all positive scalar curvature metrics on M is an abelian semigroup with respect to this sum operation. We define the abelian group $P(M)$ to be the Grothendieck group of the above abelian semigroup. We point put that our definition of $P(M)$ is closely related to Stolz's group R_n of concordance classes of positive scalar curvature metrics [30].

Recall that $r_{\text{fin}}(G)$ is the rank of the abelian subgroup of $K_0(C^*(G))$ generated by $[p_g]$ for all finite-order elements g in G. Here g is allowed to be the identity element e.

Theorem 4.1 (1) *Let M be a compact smooth spin manifold with a positive scalar curvature metric and dimension $2k-1$ $(k>2)$. The rank of the abelian group $P(M)$ is greater than or equal to $r_{\text{fin}}(G)-1$.*

(2) *Let M be a compact smooth spin manifold with a positive scalar curvature metric and dimension $4k-1$ $(k>1)$. The rank of the abelian group $P(M)$ is greater than or equal to $r_{\text{fin}}(G)$.*

The following result is a consequence of the above theorem and Theorem 3.6.

Corollary 4.2 (1) *Let M be a compact smooth spin manifold with a positive scalar curvature metric and dimension $2k-1$ $(k>2)$. If $\pi_1(M)=G$ is finitely embeddable into Hilbert space, then the rank of the abelian group $P(M)$ is greater than or equal to $N_{\text{fin}}(G)$.*

(2) *Let M be a compact smooth spin manifold with a positive scalar curvature metric and the dimension $4k-1$ $(k>1)$. If $\pi_1(M)=G$ is finitely embeddable into Hilbert space, then the rank of the abelian group $P(M)$ is greater than or equal to $N_{\text{fin}}(G)+1$.*

The following result is a consequence of Theorem 4.1 and Theorem 2.3(2).

Corollary 4.3 (1) *Let M be a compact smooth spin manifold with a positive scalar curvature metric and dimension $2k-1$ $(k>2)$. If $\pi_1(M)$ is not torsion free, then the rank of the abelian group $P(M)$ is greater than or equal to one;*

(2) *Let M be a compact smooth spin manifold with a positive scalar curvature metric and the dimension $4k-1$ $(k>1)$. If $\pi_1(M)$ is not torsion free, then the rank of the abelian group $P(M)$ is greater than or equal to two.*

Piazza and Schick used a different method to prove that the space of positive scalar curvature has infinitely many connected components when M is a compact smooth spin manifold with a positive scalar curvature metric and dimension $2k-1$ $(k>2)$, and the fundamental group $\pi_1(M)$ is not torsion free [19].

We need some preparation to prove the main result in this section. Let F be a finite group and N be a F–manifold. We say that N is F–connected if N/F is connected.

Proposition 4.4 *Let d and k be positive integers.*

(1) *There exist $\mathbb{Z}/d\mathbb{Z}$–connected compact smooth spin $\mathbb{Z}/d\mathbb{Z}$–manifolds $N_1,\dots,N_n$ such that the dimension of each N_i is $2k$, the $\mathbb{Z}/d\mathbb{Z}$–equivariant indices of the Dirac operators on $N_1,\dots,N_n$ rationally generate $\mathrm{RO}(\mathbb{Z}/d\mathbb{Z})\otimes\mathbb{Q}$ modulo rational multiples of the regular representation (here $\mathrm{RO}(\mathbb{Z}/d\mathbb{Z})$ is the real representation ring of $\mathbb{Z}/d\mathbb{Z}$), and the $\mathbb{Z}/d\mathbb{Z}$ action on each N_l $(1\le l\le n)$ is free except for finitely many fixed points of the $\mathbb{Z}/d\mathbb{Z}$ action.*

(2) *There exist $\mathbb{Z}/d\mathbb{Z}$–connected compact smooth spin $\mathbb{Z}/d\mathbb{Z}$–manifolds $N_1,\dots,N_n$ such that the dimension of each N_i is $4k$ and the $\mathbb{Z}/d\mathbb{Z}$–equivariant indices of the Dirac operators on $N_1,\dots,N_n$ rationally generate $\mathrm{RO}(\mathbb{Z}/d\mathbb{Z})\otimes\mathbb{Q}$, and the $\mathbb{Z}/d\mathbb{Z}$ action on each N_l $(1\le l\le n)$ is free except for finitely many fixed points of the $\mathbb{Z}/d\mathbb{Z}$ action.*

Proof (1) Let $\{a_1,\dots,a_d\}$ be distinct points in $\mathbb{C}-\{0\}$. Define a two-dimensional smooth compact surface S by

$$S=\{[x,y,z]\in\mathbb{CP}^2 \mid y^d=(x-a_1z)\cdots(x-a_dz)\}.$$

Observe that S is a surface with genus $\frac{1}{2}(d-1)(d-2)$. We have:

(a) $[0,1,0]$ is not in S.

(b) The map $\pi\colon S\to\mathbb{CP}^1$ sending $[x,y,z]$ to $[x,z]$ is a branched covering with degree d.

(c) The map π has exactly d branch points $[a_1,1],\dots,[a_d,1]$ in $\mathbb{CP}^1$.

For each positive integer $1\le l\le d$, we have a natural action $\mathbb{Z}/d\mathbb{Z}$ on S by

$$\alpha_l([m])[x,y,z]=\big[x,\exp(2\pi iml/d)y,z\big]$$

for any $[m]\in\mathbb{Z}/d\mathbb{Z}$ and $[x,y,z]\in S$.

We shall prove Proposition 4.4(1) by induction. We first deal with the special case that d is 2 or a prime number.

Let N_l be the product of k number of copies of S with the diagonal action of α_l. The assumption that d is 2 or a prime number implies that the action of $\mathbb{Z}/d\mathbb{Z}$ on each N_l is free except for finitely many fixed points.

By Atiyah and Bott [3, Theorem 8.35], we know that the $\mathbb{Z}/d\mathbb{Z}$–equivariant indices of the Dirac operators on $\{N_1, \dots, N_d\}$ generate $\mathrm{RO}(\mathbb{Z}/d\mathbb{Z}) \otimes \mathbb{Q}$ modulo rational multiples of the regular representation.

Assume by induction that the proposition is true for any divisor of d less than d. Let $\{N_1, \dots N_s\}$ be the set of $\mathbb{Z}/d\mathbb{Z}$–connected compact smooth spin $\mathbb{Z}/d\mathbb{Z}$–manifolds whose $\mathbb{Z}/d\mathbb{Z}$–equivariant indices of Dirac operators rationally generate all real representations induced from all proper subgroups modulo rational multiples of the regular representation. It suffices to construct a set of $\mathbb{Z}/d\mathbb{Z}$–connected compact smooth spin $\mathbb{Z}/d\mathbb{Z}$–manifolds with dimension $2k$ whose $\mathbb{Z}/d\mathbb{Z}$–equivariant indices of Dirac operators rationally generate all real representations with characters supported on each generator of the abelian group $\mathbb{Z}/d\mathbb{Z}$ and its inverse.

Let $\{[l_1], \dots, [l_m]\}$ be the set of all generators of the abelian group $\mathbb{Z}/d\mathbb{Z}$. Let N_{s+l_i} be the product of k number of copies of S with the diagonal action of α_{l_i}. The fact that l_i is a generator of the abelian group $\mathbb{Z}/d\mathbb{Z}$ implies that the action of $\mathbb{Z}/d\mathbb{Z}$ on each N_{s+l_i} is free except for finitely many fixed points. By the induction hypothesis and Atiyah and Bott [3, Theorem 8.35], we know that the $\mathbb{Z}/d\mathbb{Z}$–equivariant indices of the Dirac operators on $\{N_{s+l_1}, \dots, N_{s+l_m}\}$ rationally generate all real representations with characters supported on each generator of the abelian group $\mathbb{Z}/d\mathbb{Z}$ and its inverse modulo the abelian subgroup rationally generated by all representations induced from proper subgroups of $\mathbb{Z}/d\mathbb{Z}$. This, together with the induction hypothesis, implies that $\mathbb{Z}/d\mathbb{Z}$–equivariant indices of the Dirac operators on $\{N_1, \dots, N_s, N_{s+l_1}, \dots, N_{s+l_m}\}$ generate $\mathrm{RO}(\mathbb{Z}/d\mathbb{Z}) \otimes \mathbb{Q}$ modulo rational multiples of the regular representation.

(2) For the second part of the proposition, when $d = 1$, $\mathbb{Z}/d\mathbb{Z}$ is a trivial group and we can take a $4k$–dimensional compact smooth spin manifold whose Dirac operator has nonzero index. For example, we can take the product of k copies of the Kummer (or K3) surface

$$\{(z_0, z_1, z_2, z_3) \mid z_0^4 + z_1^4 + z_2^3 + z_3^4 = 0\} \subseteq \mathbb{CP}^3.$$

This proves part (2) for the trivial group case. The rest of the proof goes exactly the same as in the proof of part (1) with the dimension of the manifold changed to $4k$. We point out that in the induction process, the trivial subgroup induces the regular representation. □

We remark that Proposition 4.4 can be generalized to any finite group by using representations induced from its cyclic subgroups.

Now we are ready to prove Theorem 4.1.

Proof Let $\widetilde{M}$ be the universal cover of M. We will prove Theorem 4.1 when the dimension of M is $4k-1$, the proof is completely similar when the dimension is $2k-1$.

For each finite-order element g in G with order d. By Proposition 4.4, there exist $\mathbb{Z}/d\mathbb{Z}$–connected compact smooth spin $\mathbb{Z}/d\mathbb{Z}$–manifolds $\{N_1,\dots,N_n\}$ such that the dimension of each N_i is $4k$ and the sum of the $\mathbb{Z}/d\mathbb{Z}$–equivariant indices of the Dirac operators on $\{N_1,\dots,N_n\}$ is a nonzero multiple of the trivial representation of $\mathbb{Z}/d\mathbb{Z}$.

Let $N_{g,l}=G\times_{\mathbb{Z}/d\mathbb{Z}}N_l$, where $\mathbb{Z}/d\mathbb{Z}$ acts on N_l as in Proposition 4.4 and $\mathbb{Z}/d\mathbb{Z}$ acts on G by $[m]h=hg^m$ for all $h\in G$ and $[m]\in\mathbb{Z}/d\mathbb{Z}$. Observe that $N_{g,l}$ is a G–manifold.

Let $\{g_1,\dots,g_r\}$ be a collection of finite-order elements such that $\{[p_{g_1}],\dots,[p_{g_r}]\}$ generates an abelian subgroup of $K_0(C^*(G))$ with rank r. Let

$$N_{g_i}=\bigsqcup_{l=1}^{j_i}N_{g_i,l}$$

be the disjoint union of all G–manifolds described as above. Let I be the unit interval $[0,1]$. We first form a generalized G–equivariant connected sum $(\widetilde{M}\times I)\natural N_{g_i}$ along a free G–orbit of each $N_{g_i,l}$ and away from the boundary of $\widetilde{M}\times I$ as follows. We first obtain a G^{*j_i}–equivariant connected sum $(\widetilde{M}\times I)\sharp N_{g_i}$ along a free G–orbit of each $N_{g_i,l}$ and away from the boundary of $\widetilde{M}\times I$, where G^{*j_i} is the free product of j_i copies of G. More precisely, we inductively form the G^{*j_i}–equivariant connected sum $(\cdots((\widetilde{M}\times I)\sharp N_{g_i,1})\cdots)\sharp N_{g_i,j_i}$, where the equivariant connected sum is inductively taken along a free orbit and away from the boundary. We denote this space by $(\widetilde{M}\times I)\sharp N_{g_i}$. We then perform surgeries on $(\widetilde{M}\times I)\sharp N_{g_i}$ to obtain a G–equivariant cobordism between two copies of G–manifold $\widetilde{M}$.

For any positive scalar curvature metric h on M, by [24, Theorem 2.4], the above cobordism gives us another positive scalar curvature metric h_i on M. By the assumption on $\{[p_{g_1}],\dots,[p_{g_r}]\}$, the relative higher index theorem [32], the relative higher index of the Dirac operator $M\times\mathbb{R}$ associated to the positive scalar curvature metrics of h_i and g_M is $[p_{g_i}]$ in $K_0(C^*(G))$. As a consequence, we know that $\{[h_1],\dots,[h_r]\}$ generates an abelian subgroup of $P(M)$ with rank r. □

We mention that if the fundamental group satisfies the strong Novikov conjecture, then the same method can be used to prove that the rank of $P(M)$ is greater than or equal to the number of conjugacy classes of nontrivial finite-order elements in the

fundamental group when the dimension of M is $2k-1$ $(k>2)$, and the rank of $P(M)$ is greater than or equal to the number of conjugacy classes of finite-order elements in the fundamental group when the dimension of M is $4k-1$ $(k>1)$. To prove this result, we need to consider K–theory classes induced by all representations of finite subgroups of the fundamental group.

5 Groups finitely embeddable into Hilbert space

In this section, we show that various classes of groups are finitely embeddable into Hilbert space. In particular, we show that this class of groups include all residually finite groups, amenable groups, Gromov's monster groups, all virtually torsion-free groups (eg $\mathrm{Out}(F_n)$), and all groups of analytic diffeomorphisms of an analytic connected manifold fixing a given point.

We first introduce a concept of groups locally embeddable into Hilbert space.

Definition 5.1 A countable discrete group G is said to be locally embeddable into Hilbert space H if for any finite subset $F \subseteq G$, there exists a group G' coarsely embeddable into H such that there is a map $\phi\colon F \to G'$ satisfying:

(1) $\phi(e) = e$ if $e \in F$.

(2) $\phi(gh) = \phi(g)\phi(h)$ if $g, h \in F$ and $gh \in F$.

(3) ϕ is injective.

Clearly groups locally embeddable into Hilbert spaces are finitely embeddable into Hilbert space. Recall that a group G is said to be locally embeddable into finite groups (LEF) if the group G' in the above definition can always to be chosen to be a finite group. Observe that all residually finite groups are LEF.

Proposition 5.2 *Let N be an analytic connected manifold and $x_0 \in N$. If G is a countable group of analytic diffeomorphisms fixing the point x_0, then G is locally embeddable into Hilbert space.*

Proof For any positive integer k, let J_k be the finite-dimensional vector space of all k–jets at x_0. Let $\mathrm{GL}(J_k)$ be the Lie group of all linear isomorphisms from J_k to J_k. Any diffeomorphism of N fixing x_0 induces an isomorphism of J_k. It follows that, for each k, there is a natural homomorphism ψ_k from G to $\mathrm{GL}(J_k)$. Let $G' = \psi_k(G)$. G' is coarsely embeddable into Hilbert space since $\mathrm{GL}(J_k)$ is a Lie group with finitely many connected components [15]. For any finite subset F of G, we restrict ψ_k to F to obtain a map ϕ from F to G'. By analyticity, we can verify that G' and ϕ satisfy the conditions in Definition 5.1 when k is large enough. □

It is an open question whether an arbitrary countable group of analytic diffeomorphisms on an analytic connected manifold is locally or finitely embeddable into Hilbert space. More generally, it is an open question whether an arbitrary countable group of diffeomorphisms on a smooth connected manifold is locally or finitely embeddable into Hilbert space.

The next result states that certain limit groups are locally embeddable into Hilbert space.

Proposition 5.3 *If G is the limit of a sequence of groups $\{G_k\}_{k=1}^{\infty}$ coarsely embeddable into Hilbert space such that, for each finite subset $F \subset G_k$ for some k, there exists $l \geq k$ such that the map from the image of F in G_l to G is injective, then G is locally embeddable into Hilbert space.*

We point out that in the above proposition the homomorphism from G_k to G_l $(k > l)$ is not assumed to be injective. Examples of groups satisfying the above proposition include Burnside groups and Gromov's monster groups [1; 13]. The proof of the above result is straightforward and is therefore omitted.

Proposition 5.4 *If G has a torsion-free normal subgroup G_0 such that G/G_0 is residually finite, then G is finitely embeddable into Hilbert space.*

Proof Let ϕ be the quotient homomorphism from G to $G' = G/G_0$. We have $\text{order}(\phi(g)) = \text{order}(g)$ for all finite-order elements g in G. This can be seen as follows. The order of $\phi(g)$ is a divisor of the order of g. If by contradiction that g is a finite-order element in G whose order is greater than 1 and $\text{order}(\phi(g)) \neq \text{order}(g)$, then $\text{order}(g) = n\,\text{order}(\phi(g))$ for some positive integer $n > 1$. This would imply that g^n is in the kernel of ϕ. Hence g^n is in G_0. But g^n is a nontrivial finite-order element. This is a contradiction with the assumption that G_0 is torsion free. Now our proposition follows from the definition of residual finiteness of G'. □

Corollary 5.5 *If G is virtually torsion free, then G is finitely embeddable into Hilbert space.*

An example of a group satisfying the assumption in the above corollary is $\text{Out}(F_n)$ [8, Proposition 1.2].

Sofic groups have many common examples with finitely embeddable groups. It is an open question if sofic groups are finitely embeddable into Hilbert space.

Appendix

In this appendix, we prove the following analogue of Atiyah's L^2 index theorem for the maximal group C^*–algebra. This result is folklore. For completeness, we include a proof here.

Theorem A.1 *Let G be a countable group. Let X be a locally compact space with a proper, free, and cocompact action of G. Let μ be the assembly map $K_0^G(X) \to K_0(C^*(G))$. If τ is the trace on $C^*(G)$ given by $\tau(\sum c_g g) = \sum c_g$, then*

$$\tau(\mu[(F, H)]) = \operatorname{index}(F)$$

for any K–homology class $[(F, H)]$ in $K_0^G(X)$, where $\operatorname{index}(F)$ is the Fredholm index of F.

Proof In [31], we give a different proof of Atiyah's L^2 index theorem. We shall use a key ingredient in this proof.

Let d be a G–invariant metric on X (compatible with the topology of X). The assumption that the G action on X is proper and free implies that there exists $\delta > 0$ such that $d(x, gx) \geq 10\delta$ for all $x \in X$ and $g \neq G$ in G. By the proof of Atiyah's L^2 index theorem [31, page 1402], we can write

$$\mu[(F, H)] = [p] - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right),$$

where

$$p = p_1 + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

such that p_1 is an element in the matrix algebra over $(\mathcal{S}_1 X)^G$, the algebra of G–invariant and locally traceable operators on a G–X–module H with finite propagation, satisfying

$$\operatorname{propagation}(p_1) < \delta.$$

Note that $(\mathcal{S}_1 X)^G$ is isomorphic to $\mathcal{S}_1 G$, the group algebra of G over the ring $\mathcal{S}_1$. We can extend the trace τ naturally to a trace on $\mathcal{S}_1 G$. We have

$$\tau(\mu[(F, H)]) = \tau(p_1) = \sum_{g \in G} \operatorname{tr}(g^{-1} p_1),$$

where tr is the canonical trace on $\mathcal{S}_1 G$ as in Lemma 3.1. We also note that the sum in the above identity is a finite sum since p_1 has finite propagation. Now by Lemma 3.1 and the choice of δ, we have $\operatorname{tr}(g^{-1} p_1) = 0$ for all $g \neq e$. As a consequence, we have $\tau(p_1) = \operatorname{tr}(p_1)$. By Atiyah's L^2 index theorem [2], we have $\operatorname{tr}(p_1) = \operatorname{index}(F)$. It follows that $\tau(\mu[(F, H)]) = \operatorname{index}(F)$. □

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Proposed: Benson Farb Received: 14 April 2014
Seconded: Steve Ferry, Leonid Polterovich Accepted: 28 December 2014