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Infinite-time singularities of the Kähler–Ricci flow

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We study the long-time behavior of the Kähler–Ricci flow on compact Kähler manifolds. We give an almost complete classification of the singularity type of the flow at infinity, depending only on the underlying complex structure. If the manifold is of intermediate Kodaira dimension and has semiample canonical bundle, so it is fibered by Calabi–Yau varieties, we show that parabolic rescalings around any point on a smooth fiber converge smoothly to a unique limit, which is the product of a Ricci-flat metric on the fiber and a flat metric on Euclidean space. An analogous result holds for collapsing limits of Ricci-flat Kähler metrics.

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1 Introduction

This paper has two main goals. The first goal is to prove some results about collapsing limits of either Ricci-flat Kähler metrics or certain solutions of the Kähler–Ricci flow, which complement and complete the very recent results proved by Tosatti, Weinkove and Yang in [27]. In a nutshell, we prove higher-order a priori estimates for certain rescalings of the solutions restricted to the fibers of a holomorphic map, and we then classify the possible blowup limit spaces. As it turns out, these limits are unique and are given by the product of a Ricci-flat Kähler metric on a compact Calabi–Yau manifold times a Euclidean metric on $\mathbb{C}^m$.

Our second goal is to study infinite-time singularities of the Kähler–Ricci flow. As we recall below, these are divided into two classes called "type IIb" and "type III". (The other cases, "type I" and "type IIa", are reserved for finite-time singularities.) Assuming that the canonical bundle is semiample (which conjecturally is always true whenever we have a long-time solution), we give an almost complete classification (which is complete in complex dimension 2) of which singularity type arises depending on the complex structure of the underlying manifold. As a consequence, in all the cases that we cover, the singularity type does not depend on the initial Kähler metric.

We now discuss the first goal in detail. As we said above, we consider two different setups. In the first setup (which is the same as in Tosatti [24], Gross, Tosatti and

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Zhang [8; 9], Hein and Tosatti [11] and Tosatti, Weinkove and Yang [27]) we let (X, ω_X) be a compact Ricci-flat Calabi–Yau $(n+m)$–manifold, $n > 0$, which admits a holomorphic map $\pi\colon X \to Z$, where (Z, ω_Z) is a compact Kähler manifold. Denote by $B = \pi(X)$ the image of π, and assume that B is an irreducible normal subvariety of Z with dimension $m > 0$ and that the map $\pi\colon X \to B$ has connected fibers. Denote by $\chi = \pi^*\omega_Z$ a smooth nonnegative real $(1,1)$–form on X whose cohomology class lies on the boundary of the Kähler cone of X, and denote also by χ the restriction of ω_Z to the regular part of B. In practice we can take $Z = \mathbb{CP}^N$ if B is a projective variety, or $Z = B$ if B is smooth. In general, given a map $\pi\colon X \to B$ as above, there is a proper analytic subvariety $S' \subset B$, which consists of the singular points of B together with the critical values of π, such that if we let $S := \pi^{-1}(S')$ we obtain an analytic subvariety of X and $\pi\colon X \setminus S \to B \setminus S'$ is a smooth submersion. For any $y \in B \setminus S'$, the fiber $X_y = \pi^{-1}(y)$ is a smooth Calabi–Yau manifold of dimension n, and it is equipped with the Kähler metric $\omega_X|_{X_y}$. Consider the Kähler metrics on X given by $\tilde{\omega} = \tilde{\omega}(t) = \chi + e^{-t}\omega_X$ with $t \geqslant 0$; Yau's theorem [30] guarantees that (for each t) there exists a unique Ricci-flat Kähler metric on X cohomologous to $\tilde{\omega}$, which we call ω. Then the metrics $\omega = \omega(t) = \tilde{\omega} + \sqrt{-1}\,\partial\bar{\partial}\varphi$, with potentials normalized by $\sup_X \varphi = 0$, satisfy a family of complex Monge–Ampère equations

$$\omega^{n+m} = (\tilde{\omega} + \sqrt{-1}\,\partial\bar{\partial}\varphi)^{n+m} = c_t e^{-nt}\omega_X^{n+m}, \tag{1-1}$$

where c_t is a constant that has a positive limit as $t \to \infty$. Thanks to the work of Tosatti [24], Song and Tian [21], Gross, Tosatti and Zhang [8; 9], Tosatti, Weinkove and Yang [27] and Zhang [31], we know that in this case the Ricci-flat metrics $\omega(t)$ have bounded diameter and collapse locally uniformly on $X \setminus S$ to a canonical Kähler metric on $B \setminus S'$, and when the fibers X_y are tori, the collapse is smooth and with locally bounded curvature on $X \setminus S$. Also, for general smooth fibers, the rescaled metrics along the fibers, $e^t\omega|_{X_y}$, converge in C^α to a Ricci-flat metric on X_y.

The second setup is as follows (see Song and Tian [20; 22; 21], Fong and Zhang [5] and Tosatti, Weinkove and Yang [27]). Now (X, ω_X) is a compact Kähler $(n+m)$–manifold, $n > 0$, with semiample canonical bundle and Kodaira dimension equal to $m > 0$. As explained for example in [27], sections of K_X^ℓ, for ℓ large, give rise to a fiber space $\pi\colon X \to B$ called the *Iitaka fibration* of X, with B a normal projective variety of dimension m, and the smooth fibers $X_y = \pi^{-1}(y)$, $y \in B \setminus S'$, also Calabi–Yau n–manifolds. We let χ be the restriction of ω_{FS}/ℓ to B, as well as its pullback to X. This time we consider the solution $\omega = \omega(t)$ of the *normalized* Kähler–Ricci flow

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega) - \omega, \quad \omega(0) = \omega_X, \tag{1-2}$$

which exists for all $t \geqslant 0$. Thanks to the works cited above, we know that the evolving metrics have uniformly bounded scalar curvature and collapse locally uniformly on $X \setminus S$ to a canonical Kähler metric on $B \setminus S'$, and again we have smooth collapse when the smooth fibers are tori. Again, the rescaled metrics along the fibers, $e^t \omega|_{X_y}$, converge in C^α to a Ricci-flat metric on X_y.

Our first goal is to improve this last statement by showing the following higher-order estimates:

Theorem 1.1 *Assume we are in either the first or the second setup. Given a compact subset $K \subset B \setminus S'$ and $k \geqslant 0$, there is a constant C_k such that*

$$\|e^t \omega|_{X_y}\|_{C^k(X_y, \omega_X|_{X_y})} \leqslant C_k, \quad e^t \omega|_{X_y} \geqslant C_0^{-1} \omega_X|_{X_y} \tag{1-3}$$

holds for all $t \geqslant 0$ and for all $y \in K$.

The case $k = 1$ of Theorem 1.1 was proved recently in [27], where it was also shown that as $t \to \infty$, the metrics $e^t \omega|_{X_y}$ converge in $C^\alpha(X_y, \omega_X|_{X_y})$, $0 < \alpha < 1$, to the unique Ricci-flat Kähler metric on X_y cohomologous to $\omega_X|_{X_y}$. Combining this and Theorem 1.1, we immediately conclude the following:

Corollary 1.2 *Assume we are in either the first or the second setup. Given $y \in B \setminus S'$, we have*

$$e^t \omega|_{X_y} \to \omega_{SRF,y} \tag{1-4}$$

in the smooth topology on X_y as $t \to \infty$, where $\omega_{SRF,y}$ is the unique Ricci-flat metric on X_y cohomologous to $\omega_X|_{X_y}$.

This solves affirmatively a problem raised by Tosatti [25, Question 4.1; 26, Question 3].

In the second setup, we investigate in more detail the nature of the singularity of the Kähler–Ricci flow as $t \to \infty$, and more precisely we want to determine the possible limits (or "tangent flows") that one obtains by parabolically rescaling the flow around the "singularity at infinity". In general, determining whether tangents to solutions of geometric PDEs are unique or not is a very challenging problem. In the above setup, we prove that such parabolic rescalings (centered at a point on a smooth fiber) converge to a unique limit. More precisely we have:

Theorem 1.3 *Assume the same setup for the Kähler–Ricci flow (1-2) as above. Given $t_k \to \infty$ and $x \in X \setminus S$, let V be the preimage of a sufficiently small neighborhood of $\pi(x)$ and let*

$$\omega_k(t) = e^{t_k} \omega(te^{-t_k} + t_k)$$

be the parabolically rescaled flows. Then, after passing to a subsequence, the flows $(V, \omega_k(t), x)$, $t \in [-1, 0]$, *converge in the smooth Cheeger–Gromov sense to*

$$(X_y \times \mathbb{C}^m, \omega_\infty, (x, 0)),$$

where $y = \pi(x)$ *and* ω_∞ *is the product of the unique Ricci-flat Kähler metric on* X_y *in the class* $[\omega_X|_{X_y}]$ *and a flat metric on* $\mathbb{C}^m$, *viewed as a static solution of the flow. In particular, there is a unique such limit up to holomorphic isometry.*

In fact, analogous results to those in Theorem 1.3 also hold in the first setup of collapse of Calabi–Yau manifolds, with essentially the same proofs. We briefly discuss this at the end of Section 3.

We now come to our second goal; to characterize infinite-time singularities of the Kähler–Ricci flow. Recall that a long-time solution of the *unnormalized* Kähler–Ricci flow

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \quad \omega(0) = \omega_X, \tag{1-5}$$

is said to be of type IIb if

$$\sup_{X \times [0,\infty)} t|\mathrm{Rm}(\omega(t))|_{\omega(t)} = +\infty$$

and of type III if

$$\sup_{X \times [0,\infty)} t|\mathrm{Rm}(\omega(t))|_{\omega(t)} < +\infty.$$

If the flow is instead normalized as in (1-2), then one has to remove the factor t from these conditions.

When $\dim X = 1$, so X is a compact Riemann surface, it follows easily from the work of Hamilton [10] that all long-time solutions of the flow are of type III, and these are exactly the Kähler–Ricci flow solutions on compact Riemann surfaces of genus $g \geqslant 1$. Recent work of Bamler [1] shows that the same statement holds for the (not Kähler) Ricci flow on compact Riemannian 3–manifolds. Homogeneous type-III solutions were studied by Lott [15]. The simplest example of a type-IIb solution on a compact Riemannian 4–manifold is a nonflat Ricci-flat Kähler metric on a $K3$ surface, which exists thanks to Yau [30] and provides a static solution of the Kähler–Ricci flow that is of type IIb. As we will see below, there are also nonstatic type-IIb solutions in dimension 4. In the case of higher-dimensional Kähler–Ricci flows, the only general result that we are aware of is due to Fong and Zhang [5], who proved that if $\pi\colon X \to B$ is a holomorphic submersion with fibers equal to complex tori, with $c_1(B) < 0$ and with X projective and initial Kähler class rational, then the flow is of type III. This used key ideas of Gross, Tosatti and Zhang [8] in the case of collapse of Ricci-flat metrics, and the projectivity/rationality assumptions were recently removed by Hein and Tosatti [11]. See also Gill [7] for the case when X is a product.

Our goal is to have a more detailed understanding of which long-time solutions of the Kähler–Ricci flows are of type IIb or type III. Our main tool is the following observation:

Proposition 1.4 *Let X be a compact Kähler manifold with K_X nef and which contains a possibly singular rational curve $C \subset X$ (ie C is the birational image of a nonconstant holomorphic map $f\colon \mathbb{CP}^1 \to X$) such that $\int_C c_1(X) = 0$. Then any solution of the unnormalized Kähler–Ricci flow (1-5) on X must be of type IIb.*

For example if X is a minimal Kähler surface of general type which contains a (-2)–curve (and there are many such examples), then this result applies and we get examples of nonstatic long-time Kähler–Ricci flow solutions in 4 real dimensions which are of type IIb.

Now in general, if a Kähler–Ricci flow solution on a compact Kähler manifold X exists for all $t \geqslant 0$, then necessarily the canonical bundle K_X is nef. (The converse is also true; see Tian and Zhang [23].) The abundance conjecture in algebraic geometry (or rather its generalization to Kähler manifolds) predicts that if K_X is nef then in fact it is semiample, namely K_X^ℓ is base-point-free for some $\ell \geqslant 1$. We will assume that this is the case, and so the sections of K_X^ℓ, for some large ℓ, define a fiber space $\pi\colon X \to B$; the Iitaka fibration of X. As before, we denote by $S' \subset B$ the singular set of B together with the critical values of π and let $S = \pi^{-1}(S')$, so that $S = \varnothing$ precisely when B is smooth and π is a submersion. Somewhat imprecisely, we will refer to S as the set of singular fibers of π. We will also write $X_y = \pi^{-1}(y)$ for one of the smooth fibers, $y \in B \setminus S'$. We then have the following result:

Theorem 1.5 *Let X be a compact Kähler n–manifold with K_X semiample, and consider a solution of the Kähler–Ricci flow (1-5).*

Suppose $\kappa(X) = 0$.

- *If X is not a finite quotient of a torus, then the solution is of type IIb.*
- *If X is a finite quotient of a torus, then the solution is of type III.*

Suppose $\kappa(X) = n$.

- *If K_X is ample, then the solution is of type III.*
- *If K_X is not ample, then the solution is of type IIb.*

Suppose $0 < \kappa(X) < n$.

- *If* X_y *is not a finite quotient of a torus, then the solution is of type IIb.*
- *If* X_y *is a finite quotient of a torus and* $S = \varnothing$, *then the solution is of type III.*

In particular, in these cases the type of singularity does not depend on the initial metric.

This result leaves out only the case when $0 < \kappa(X) < n$ and the general fiber X_y is a finite quotient of a torus and there are singular fibers. In this case, sometimes one can find a component of a singular fiber which is uniruled, and then the flow is of type IIb by Proposition 1.4, but other times there is no such component, and then it seems highly nontrivial to determine whether the flow is of type III or IIb. In any case, we expect the singularity type to be always independent of the initial metric.

For $n = 2$, we are able to solve this question and complete the singularity classification. (Note that since the abundance conjecture holds for surfaces, it is enough to assume that K_X is nef.) In this case, $\pi\colon X \to B$ is an elliptic fibration with X a minimal properly elliptic Kähler surface and with some singular fibers. Recall that in Kodaira's terminology (see Barth, Hulek, Peters and Van de Ven [2]) a fiber of type mI_0, $m > 1$, is a smooth elliptic curve with multiplicity m. We can then complete Theorem 1.5 in dimension 2 as follows:

Theorem 1.6 *Let* X *be a minimal Kähler surface with* $\kappa(X) = 1$, *and let* $\pi\colon X \to B$ *be an elliptic fibration which is not a submersion everywhere. Then the flow is of type III if and only if the only singular fibers of* π *are of type* mI_0, $m > 1$.

In the minimal model program, we sometimes have many different minimal models in one birational equivalence class. In [13], Kawamata showed that different minimal models can be connected by a sequence of flops. A corollary of Proposition 1.4 is the relationship between this multiminimal model phenomenon and the singularity type of the Kähler–Ricci flow.

Corollary 1.7 *Let* X *be a projective* n*–manifold with* K_X *nef. If* X *has a different minimal model* Y, *ie a* $\mathbb{Q}$*–factorial terminal variety* Y *with* K_Y *nef and with a birational map* $\alpha\colon Y \dashrightarrow X$, *which is not isomorphic to* X, *then any solution of the Kähler–Ricci flow (1-5) on* X *is of type IIb. As a consequence, if there is a flow solution of type III, then* X *is the unique minimal model in its birational equivalence class.*

This paper is organized as follows. In Section 2 we prove Theorem 1.1. Theorem 1.3 is proved in Section 3, and finally in Section 4 we give the proofs of Proposition 1.4, Theorems 1.5 and 1.6 and Corollary 1.7.

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2 Estimates along the fibers

In this section we prove Theorem 1.1 by deriving a priori C^∞ estimates for the rescaled metrics restricted to a smooth fiber.

We first consider the setup of collapse of Ricci-flat Kähler metrics, as described in the Introduction. We need two preliminary results. The first one follows from work of Tosatti [24]; see also [8, Lemma 4.1].

Lemma 2.1 [24; 8] *Given any compact set $K \subset X \setminus S$, there is a constant C such that on K the Ricci–flat metrics ω satisfy*

$$C^{-1}(\chi + e^{-t}\omega_X) \leqslant \omega \leqslant C(\chi + e^{-t}\omega_X) \tag{2-1}$$

for all $t > 0$.

The second ingredient is the following local estimate for Ricci-flat Kähler metrics, which is contained in [11, Sections 3.2 and 3.3] and is an adaptation (and in fact a special case) of a similar result from [19] for the Kähler–Ricci flow.

Lemma 2.2 [11] *Let $B_1(0)$ be the unit ball in $\mathbb{C}^n$ and let ω_E be the Euclidean metric. Assume that ω is a Ricci-flat Kähler metric on $B_1(0)$ which satisfies*

$$A^{-1}\omega_E \leqslant \omega \leqslant A\,\omega_E \tag{2-2}$$

for some positive constant A. Then for any $k \geqslant 1$ there is a constant C_k that depends only on k, n, m, A such that on $B_{1/2}(0)$ we have

$$\|\omega\|_{C^k(B_{1/2}(0),\omega_E)} \leqslant C_k. \tag{2-3}$$

Proof of Theorem 1.1 for Ricci-flat Kähler metrics Fix a point $y_0 \in B \setminus S'$, a point $x \in X_{y_0}$ and a small chart $U \subset X$ with coordinates $(y, z) = (y_1, \dots, y_m, z_1, \dots, z_n)$ centered at x, where $(y_1, \dots, y_m)$ are the pullback via π of coordinates on the image $\pi(U) \subset B$, and such that the projection π in this coordinates is just $(y, z) \mapsto y$. Such a chart exists because π is a submersion near x; see [14, page 60]. We can assume that U equals the polydisc where $|y_i| < 1$ for $1 \leqslant i \leqslant m$ and $|z_\alpha| < 1$ for $1 \leqslant \alpha \leqslant n$.

For each $t \geqslant 0$, consider the polydiscs $B_t = B_{e^{t/2}}(0) \subset \mathbb{C}^m$, let D be the unit polydisc in $\mathbb{C}^n$ and define maps

$$F_t\colon B_t \times D \to U, \quad F_t(y,z) = (ye^{-t/2}, z).$$

Note that the stretching F_t is the identity when restricted to $\{0\}\times D$, and as t approaches zero the polydiscs $B_t \times D$ exhaust $\mathbb{C}^m \times D$. On U we can write

$$\omega_X(y,z) = \sqrt{-1}\sum_{i,j=1}^{m} g_{i\bar{j}}(y,z)\, dy_i \wedge d\bar{y}_j + 2\,\mathrm{Re}\,\sqrt{-1}\sum_{i=1}^{m}\sum_{\alpha=1}^{n} g_{i\bar{\alpha}}(y,z)\, dy_i \wedge d\bar{z}_\alpha \\ + \sqrt{-1}\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}(y,z)\, dz_\alpha \wedge d\bar{z}_\beta,$$

so that

$$F_t^*\omega_X(y,z) = e^{-t}\sqrt{-1}\sum_{i,j=1}^{m} g_{i\bar{j}}(ye^{-t/2},z)\, dy_i \wedge d\bar{y}_j \\ + 2e^{-t/2}\,\mathrm{Re}\,\sqrt{-1}\sum_{i=1}^{m}\sum_{\alpha=1}^{n} g_{i\bar{\alpha}}(ye^{-t/2},z)\, dy_i \wedge d\bar{z}_\alpha \\ + \sqrt{-1}\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}(ye^{-t/2},z)\, dz_\alpha \wedge d\bar{z}_\beta.$$

Clearly, as t goes to infinity these metrics converge smoothly on compact sets of $\mathbb{C}^m \times D$ to the nonnegative form $\eta = \sqrt{-1}\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}(0,z)\, dz_\alpha \wedge d\bar{z}_\beta$, which is constant in the y directions and is just equal to the restriction of ω_X on the fiber $X_{y_0} \cap U$, under the identification $X_{y_0} \cap U = \{0\} \times D$. The rescaled pullback metrics $e^t F_t^*\omega(t)$ are Ricci-flat Kähler on $B_t \times D$. Next, we pull back (2-1) and multiply it by e^t to get

$$C^{-1}(e^t F_t^*\chi + F_t^*\omega_X) \leqslant e^t F_t^*\omega \leqslant C(e^t F_t^*\chi + F_t^*\omega_X), \tag{2-4}$$

which holds on $B_t \times D$. But χ is the pullback of a metric

$$\sqrt{-1}\sum_{i,j=1}^{m} \chi_{i\bar{j}}(y)\, dy_i \wedge d\bar{y}_j$$

on B, and so we have that

$$e^t(F_t^*\chi)(y,z) = \sqrt{-1}\sum_{i,j=1}^{m} \chi_{i\bar{j}}(ye^{-t/2})\, dy_i \wedge d\bar{y}_j,$$

which as t goes to infinity converge smoothly on compact sets of $\mathbb{C}^m \times D$ to the nonnegative form $\eta' = \sqrt{-1}\sum_{i,j=1}^m \chi_{i\bar{j}}(0)\, dy_i \wedge d\bar{y}_j$. In particular, $\eta+\eta'$ is a Kähler metric on $\mathbb{C}^m \times D$ which is uniformly equivalent to ω_E on the whole of $\mathbb{C}^m \times D$. Since $\eta+\eta'$ is the limit as t goes to infinity of $e^t F_t^*\chi + F_t^*\omega_X$, we see that given any compact set K of $B_t \times D$, there is a constant C independent of t such that on K we have

$$C^{-1}\omega_E \leqslant e^t F_t^*\omega \leqslant C\omega_E. \tag{2-5}$$

Using (2-5), we can then apply Lemma 2.2 to get that the Ricci-flat metrics $e^t F_t^*\omega$ have uniform C^∞ bounds on any compact subset of $B_t \times D$. If we now restrict to $\{0\}\times D$, which is identified with $X_{y_0} \cap U$, then the maps F_t are just the identity, and by covering X_{y_0} by finitely many such charts this shows that the rescaled metrics along any smooth fiber $e^t\omega|_{X_{y_0}}$ have uniform $C^\infty(X_{y_0}, \omega_X|_{X_{y_0}})$ bounds. The uniform lower bound for $e^t\omega|_{X_{y_0}}$ follows at once from (2-1).

Finally, the fact that the estimates are uniform as we vary y_0 in a compact subset of $B\setminus S'$ follows from the fact that all the constants in the proof we just finished vary continuously as we vary y_0. □

Next, we consider the second setup from the Introduction, namely collapse of the Kähler–Ricci flow. In this case the same estimate as in (2-1) was proved in [5] (see also [21]), and the local estimates that replace Lemma 2.2 are given in [19].

Proof of Theorem 1.1 for the Kähler–Ricci flow Given $x \in X\setminus S$, $y_0 = \pi(x)$ and $t_k \to \infty$, we will show that there are constant C_ℓ, $\ell \geqslant 0$, such that

$$\left\| e^{t_k}\omega(t_k)|_{X_{y_0}} \right\|_{C^\ell(X_{y_0}, \omega_X|_{X_{y_0}})} \leqslant C_\ell, \quad e^{t_k}\omega(t_k)|_{X_{y_0}} \geqslant C_0^{-1}\omega_X|_{X_{y_0}}, \tag{2-6}$$

for all $k, \ell \geqslant 0$, and that these estimates are uniform as y_0 varies in a compact set of $B\setminus S'$. Once this is proved, the estimates stated in Theorem 1.1 follow easily from an argument by contradiction, using compactness.

As in the case of Ricci-flat Kähler metrics, we choose a chart U centered at x with local product coordinates (y,z) as before, with y in the unit polydisc in $\mathbb{C}^m$ and z in the unit polydisc D in $\mathbb{C}^n$, and define stretching maps

$$F_k\colon B_k \times D \to U, \quad F_k(y,z) = (ye^{-t_k/2}, z),$$

where $B_k = B_{e^{t_k/2}}(0) \subset \mathbb{C}^m$. Thanks to the analog of (2-1) in [5], on U we have

$$C^{-1}(\chi + e^{-t}\omega_X) \leqslant \omega(t) \leqslant C(\chi + e^{-t}\omega_X).$$

We consider the parabolically rescaled and stretched metrics

$$\tilde{\omega}_k(t) = e^{t_k} F_k^*\omega(te^{-t_k} + t_k), \quad t \in [-1, 0].$$

On U we have that

$$C^{-1}(e^{t_k}\chi + e^{-te^{-t_k}}\omega_X) \leqslant e^{t_k}\omega(te^{-t_k}+t_k) \leqslant C(e^{t_k}\chi + e^{-te^{-t_k}}\omega_X),$$

and since we assume that $t\in[-1,0]$, we obtain

$$C^{-1}(e^{t_k}\chi + \omega_X) \leqslant e^{t_k}\omega(te^{-t_k}+t_k) \leqslant C(e^{t_k}\chi + \omega_X),$$

and so

$$C^{-1}F_k^*(e^{t_k}\chi+\omega_X) \leqslant \widetilde{\omega}_k(t) \leqslant CF_k^*(e^{t_k}\chi+\omega_X). \tag{2-7}$$

The same calculation as in the case of Ricci-flat metrics shows that the metrics $F_k^*(e^{t_k}\chi+\omega_X)$ converge smoothly on compact sets of $\mathbb{C}^m\times D$ to the product of a flat metric on $\mathbb{C}^m$ and the metric $\omega_X|_{X_{y_0}\cap D}$; in particular, given any compact subset K of $B_k\times D$, there is a constant C such that on K we have

$$C^{-1}\omega_E \leqslant F_k^*(e^{t_k}\chi+\omega_X) \leqslant C\omega_E,$$

where ω_E is the Euclidean metric on $\mathbb{C}^{n+m}$. Therefore on $K\times[-1,0]$ we have

$$C^{-1}\omega_E \leqslant \widetilde{\omega}_k(t) \leqslant C\omega_E.$$

The metrics $\widetilde{\omega}_k(t)$ for $t\in[-1,0]$ satisfy

$$\frac{\partial}{\partial t}\widetilde{\omega}_k(t) = -\operatorname{Ric}(\widetilde{\omega}_k(t)) - e^{-t_k}\widetilde{\omega}_k(t). \tag{2-8}$$

Note that the coefficient e^{-t_k} is uniformly bounded (and in fact goes to zero). Then the interior estimates of [19] give us that, up to shrinking K slightly,

$$\|\widetilde{\omega}_k(t)\|_{C^\ell(K,\omega_E)} \leqslant C_\ell, \quad \widetilde{\omega}_k(t) \geqslant C_0^{-1}\omega_E,$$

for $t\in[-\frac{1}{2},0]$ and for all k,ℓ, and for some uniform constants C_ℓ. Setting $t=0$ we obtain local C^∞ bounds for the metrics $e^{t_k}F_k^*\omega(t_k)$. If we now restrict to $\{0\}\times D$, which is identified with $X_{y_0}\cap U$, then the maps F_k are just the identity, and by covering X_{y_0} by finitely many such charts this proves (2-6). Again, the fact that the estimates are uniform as y_0 varies in a compact subset of $B\setminus S'$ follows from the proof we just finished. □

3 Blowup limits of the Kähler–Ricci flow

In this section, we describe the possible blowup limits at time infinity of the Kähler–Ricci flow in the same setup as in the Introduction, thus proving Theorem 1.3.

Proof of Theorem 1.3 Given a point $x \in X \setminus S$, $y_0 = \pi(x)$ and $t_k \to \infty$, choose a chart U centered at x with local product coordinates (y, z) as before, where z varies in the unit polydisc $D \subset \mathbb{C}^n$ and y in the unit polydisc in $\mathbb{C}^m$. We let

$$\omega_k(t) = e^{t_k} \omega(te^{-t_k} + t_k)$$

be the parabolic rescalings of the metrics along the flow, with $t \in [-2, 1]$. As in the proof of Theorem 1.1, we can define stretchings

$$F_k\colon B_k \times D \to U, \quad F_k(y, z) = (ye^{-t_k/2}, z),$$

where $B_k = B_{e^{t_k/2}}(0) \subset \mathbb{C}^m$, and we have that the metrics $F_k^*\omega_k(t)$, $t \in [-1, 0]$, have uniform C^∞ bounds on compact sets of $\mathbb{C}^m \times D$, and therefore up to passing to a subsequence they converge smoothly to a solution $\omega_\infty(t)$, $t \in [-1, 0]$, of the Kähler–Ricci flow

$$\frac{\partial}{\partial t}\omega_\infty(t) = -\operatorname{Ric}(\omega_\infty(t))$$

on $\mathbb{C}^m \times D$. This limit flow is unnormalized because the coefficient e^{-t_k} in (2-8) converges to zero. Also, passing to the limit in (2-7) we see that

$$C^{-1}(\omega_X|_{X_{y_0} \cap D} + \omega_E) \leqslant \omega_\infty(t) \leqslant C(\omega_X|_{X_{y_0} \cap D} + \omega_E) \tag{3-1}$$

on the whole of $\mathbb{C}^m \times D$ and for all $t \geqslant 0$, where ω_E is a flat metric on $\mathbb{C}^m$. However, the original metrics $\omega(t)$ have a uniform bound on their scalar curvature [20], so after rescaling, the scalar curvature of $\widetilde{\omega}_k(t)$ goes to zero uniformly as k goes to infinity. Therefore $\omega_\infty(t)$ is scalar flat, and from the pointwise evolution equation for its scalar curvature, we see that $\omega_\infty(t)$ is Ricci-flat, and hence independent of t.

Our next task is to glue together these local limits to obtain a global Cheeger–Gromov limit on $X_{y_0} \times \mathbb{C}^m$. To do this, we cover X_{y_0} by finitely many charts $\{U_\alpha\}$ as above and let $V = \bigcup_\alpha U_\alpha$. On each U_α we have local product coordinates (y, z^α), with y and z^α belonging to the unit polydiscs in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, and π is given by $(y, z^\alpha) \mapsto y$. If $U_\alpha \cap U_\beta \neq \varnothing$, then on this overlap the two local coordinates are related by holomorphic transformation maps

$$z^\beta = h^{\alpha\beta}(y, z^\alpha).$$

Fix a radius $R > 0$ and let $\widetilde{B}_k = B_{e^{-t_k/2}R}(0)$ be the polydisc of radius $e^{-t_k/2}R$, where k is large enough that $e^{-t_k/2}R < 1$. We define a holomorphic coordinate change on $\widetilde{B}_k$ by $y_k = e^{t_k/2}y$, so that $\widetilde{B}_k$ becomes the polydisc $B_R(0)$ of radius R in these new coordinates. This is the same as applying the stretching maps F_k as before. Then the complex manifold $V_k = \pi^{-1}(\widetilde{B}_k)$ (here we are viewing $\widetilde{B}_k$ as a subset of B) is obtained by gluing the polydiscs

$$\{(w,z^\alpha)\mid w\in B_R(0), z^\alpha\in D\},\quad \{(w,z^\beta)\mid w\in B_R(0), z^\beta\in D\}$$

via the transformation maps

$$z^\beta = h^{\alpha\beta}(we^{-t_k/2}, z^\alpha).$$

As $k\to\infty$, these converge smoothly on compact sets to the transformation maps

$$z^\beta = h^{\alpha\beta}(0, z^\alpha),$$

which give the product manifold $B_R(0)\times X_{y_0}$. Also, after changing coordinates the metrics $e^{t_k}(\chi+e^{-t_k}\omega_X)$ on V_k converge smoothly on compact sets as $k\to\infty$ to the product metric $\omega_E+\omega_X|_{X_{y_0}}$, where ω_E is a Euclidean metric on $\mathbb{C}^m$, as in the proof of Theorem 1.1. On the other hand, as discussed above, after a coordinate change the metrics $\omega_k(t)$ live on V_k, and after passing to a subsequence they converge smoothly on compact sets to a Ricci-flat Kähler metric ω_∞ (independent of t) on $B_R(0)\times X_{y_0}$. By making R larger and larger, we obtain the desired Cheeger–Gromov convergence of the flow on V to a Ricci-flat Kähler metric ω_∞ on $X_{y_0}\times\mathbb{C}^m$. Also, thanks to (3-1), we have that ω_∞ is uniformly equivalent to $\omega_E+\omega_X|_{X_{y_0}}$ on $X_{y_0}\times\mathbb{C}^m$.

It remains to show that ω_∞ is the product of a Ricci-flat Kähler metric on X_{y_0} and a flat metric on $\mathbb{C}^m$. Thanks to the recent results in [27, (1.8)], we know that the restrictions $\omega_\infty|_{X_y}$, $y\in\mathbb{C}^m$, are all equal to the same Ricci-flat Kähler metric $\omega_F:=\omega_{SRF,y_0}$ on X_{y_0} in the class $[\omega_X|_{X_{y_0}}]$. For simplicity of notation, let us also write $F=X_{y_0}$.

We claim that there is a smooth function u on $F\times\mathbb{C}^m$ such that

$$\omega_\infty=\omega_F+\omega_E+\sqrt{-1}\,\partial\bar\partial u \tag{3-2}$$

on $F\times\mathbb{C}^m$, where $\omega_F+\omega_E$ is the product Ricci-flat metric. To prove (3-2), first observe that $[\omega_\infty]=[\omega_F+\omega_E]$ in $H^2(F\times\mathbb{C}^m)$, and so there is a real 1–form ζ on $X\times\mathbb{C}^m$ such that

$$\omega_\infty=\omega_F+\omega_E+d\zeta=\omega_F+\omega_E+\partial\zeta^{0,1}+\overline{\partial\zeta^{0,1}},\quad \bar\partial\zeta^{0,1}=0, \tag{3-3}$$

where $\zeta=\zeta^{0,1}+\overline{\zeta^{0,1}}$. Since $c_1(F)=0$, the Bogomolov–Calabi decomposition theorem shows that there is a finite unramified cover $\pi\colon T\times\widetilde F\times\mathbb{C}^m\to F\times\mathbb{C}^m$, where T is a torus (possibly a point) and $\widetilde F$ is simply connected, with $c_1(\widetilde F)=0$ (also possibly a point). Projection to the T factor induces an isomorphism

$$H^{0,1}(T\times\widetilde F\times\mathbb{C}^m)\cong H^{0,1}(T),$$

where we used that $H^{0,1}(\mathbb{C}^m)=0$ by the $\overline{\partial}$–Poincaré lemma. If we write $T=\mathbb{C}^k/\Lambda$ and let $\{z_i\}$ be the standard coordinates on $\mathbb{C}^k$, then $H^{0,1}(T)$ is generated by the constant coefficient forms $\{d\overline{z}_i\}$, and so on $T\times\widetilde{F}\times\mathbb{C}^m$ we can write

$$\pi^*\zeta^{0,1}=\sum_{i=1}^k\sigma_i\,d\overline{z}_i+\overline{\partial}h$$

for some $\sigma_i\in\mathbb{C}$ and a complex-valued function h, where here $d\overline{z}_i$ also denote the pullbacks to $T\times\widetilde{F}\times\mathbb{C}^m$. Therefore

$$\pi^*\partial\zeta^{0,1}=\partial\overline{\partial}h.$$

If we let $\widetilde{h}$ be the average of h under the action of the (finite) deck transformation group of π, then $\widetilde{h}$ descends to a complex-valued function on $F\times\mathbb{C}^m$ and we have

$$\partial\zeta^{0,1}=\partial\overline{\partial}\widetilde{h},$$

which together with (3-3) proves (3-2), with $u=2\operatorname{Im}\widetilde{h}$. This argument is a very simple special case of [8, Proposition 3.1].

Now we restrict (3-2) to any fiber $F\times\{y\}$, and since we have that $\omega_\infty|_{F\times\{y\}}=\omega_F$ for all y, we conclude that $(\sqrt{-1}\,\partial\overline{\partial}u)|_{F\times\{y\}}=0$, ie $u|_{F\times\{y\}}$ is a constant (which depends on y). In other words, u is the pullback of a smooth function on $\mathbb{C}^m$. Therefore (3-2) now says that ω_∞ is the product of the Ricci-flat Kähler metric ω_F on F times the Kähler metric $\hat{\omega}:=\omega_E+\sqrt{-1}\,\partial\overline{\partial}u$ on $\mathbb{C}^m$. Since ω_∞ is Ricci-flat, we have that $\hat{\omega}$ is Ricci-flat as well, and the only thing left to prove is that $\hat{\omega}$ is flat. But earlier we proved that

$$C^{-1}(\omega_F+\omega_E)\leqslant\omega_\infty\leqslant C(\omega_F+\omega_E)$$

on the whole of $F\times\mathbb{C}^m$, and so we conclude that

$$C^{-1}\omega_E\leqslant\hat{\omega}\leqslant C\omega_E$$

on $\mathbb{C}^m$. We can write $\hat{\omega}=\sqrt{-1}\,\partial\overline{\partial}\varphi$ on $\mathbb{C}^m$, and the function $\log\det(\varphi_{i\overline{j}})$ is harmonic and bounded on $\mathbb{C}^m$, hence constant; we can then apply [18, Theorem 2] to conclude that $\hat{\omega}$ is flat. □

In the end of the proof we used a Liouville-type theorem asserting that a Ricci-flat Kähler metric on $\mathbb{C}^m$ which is uniformly equivalent to the Euclidean metric must be flat. The proof in [18] uses integral estimates, but in fact this result can also be easily proved using a local Calabi-type C^3 estimate as in [30]. (We leave the details to the interested reader.) Another related Liouville theorem was proved recently in [28].

Lastly, we mention the analogous result to Theorem 1.3 for the case of Ricci-flat Kähler metrics (the first setup in Section 1).

Theorem 3.1 *Assume the first setup in the Introduction, as in (1-1). For $x \in X \setminus S$, let V be the preimage of a sufficiently small neighborhood of $\pi(x)$. Then $(V, e^t\omega(t), x)$ converge in the smooth Cheeger–Gromov sense as $t \to \infty$ to $(X_y \times \mathbb{C}^m, \omega_\infty, (x, 0))$, where $y = \pi(x)$ and ω_∞ is the product of the unique Ricci-flat Kähler metric on X_y in the class $[\omega_X|_{X_y}]$ and a flat metric on $\mathbb{C}^m$.*

Proof Since the proof is almost identical to the one of Theorem 1.3, we will be very brief. First we work on a small polydisc centered at x with local product coordinates and, as in the proof of Theorem 1.1, we show that after stretching the coordinates the metrics $e^t\omega(t)$ have uniform C^∞ bounds, and therefore we obtain sequential limits which are Ricci-flat Kähler metrics. As in the proof of Theorem 1.3, limits on different charts glue together to give a Ricci-flat Kähler metric on $X_y \times \mathbb{C}^m$, which is uniformly equivalent to a product metric. The same argument as before shows that such a metric is unique up to holomorphic isometry and is the product of a Ricci-flat metric on X_y and a flat metric on $\mathbb{C}^m$. It follows that this is the Cheeger–Gromov limit of the whole family $e^t\omega(t)$, without passing to subsequences. □

4 Infinite-time singularities of the Kähler–Ricci flow

Now we study the singularity types of long-time solutions of the Kähler–Ricci flow.

We wish to prove the criterion stated in Proposition 1.4, which ensures that a long-time solution of the unnormalized Kähler–Ricci flow (1-5) is of type IIb. First, we make an elementary observation: if there are points $x_k \in X$, times $t_k \to \infty$, 2–planes $\pi_k \subset T_{x_k}X$ and a constant $\kappa > 0$ such that

$$\mathrm{Sec}_{\omega(t_k)}(\pi_k) \geqslant \kappa \tag{4-1}$$

for all k, then in particular $\sup_X |\mathrm{Rm}(\omega(t_k))|_{\omega(t_k)} \geqslant \kappa$, and so

$$\sup_{X\times[0,\infty)} t|\mathrm{Rm}(\omega(t))|_{\omega(t)} = +\infty,$$

and the flow is of type IIb.

We now consider Proposition 1.4. To get an intuition for why such a result should hold, let us first assume that the map $f\colon \mathbb{CP}^1 \to X$ is a smooth embedding. Then we can apply the Gauss–Bonnet theorem to get

$$4\pi = \int_{\mathbb{CP}^1} K(f^*\omega(t)) f^*\omega(t) \leqslant \sup_{\mathbb{CP}^1} K(f^*\omega(t)) \int_C \omega(t) \leqslant \sup_C K(\omega(t)) \int_C \omega_X,$$

where $K(f^*\omega(t))$ denotes the Gauss curvature of the pullback metric and $\sup_C K(\omega(t))$ is the maximum of the bisectional curvatures of $\omega(t)$ at points of $C \subset X$. Here we used that the bisectional curvature decreases in submanifolds and that $\int_C c_1(X) = 0$. We can therefore apply the observation above and conclude that the flow is of type IIb. We now give the proof in the general case.

Proof of Proposition 1.4 Let $\omega(t)$ be any solution of the unnormalized Kähler–Ricci flow (1-5). Since K_X is nef, we know that $\omega(t)$ exists for all positive t. Note that the existence of a rational curve C in X implies that X does not admit any Kähler metric with nonpositive bisectional curvature, by Yau's Schwarz lemma [29]. In particular, $K(t) = \sup_X \mathrm{Bisec}_{\omega(t)}$, the largest bisectional curvature of $\omega(t)$, satisfies $K(t) > 0$ for all t. Our goal is to give an effective uniform positive lower bound for $K(t)$. We now work on $\mathbb{C} \subset \mathbb{CP}^1$, so that we have a nonconstant entire holomorphic map $f\colon \mathbb{C} \to X$. Consider the time-dependent smooth function $e(t)$ on $\mathbb{C}$ (the "energy density") given by

$$e(t) = \mathrm{tr}_{\omega_E}(f^*\omega(t)),$$

where ω_E is the Euclidean metric on $\mathbb{C}$. The usual Schwarz lemma calculation [29] gives

$$\Delta_E e(t) \geqslant -K(t)e(t)^2.$$

Since the map f is nonconstant, there is one point in $\mathbb{C}$ where $e(t)$ is nonzero for all t, and we may assume that this is the origin. A standard ε–regularity argument (see eg [16, Lemma 4.3.2]) shows that for each fixed t, if

$$\int_{B_r} e(t) \leqslant \frac{\pi}{8K(t)}, \tag{4-2}$$

then we have

$$e(t)(0) \leqslant \frac{8}{\pi r^2}\int_{B_r} e(t), \tag{4-3}$$

where B_r is the Euclidean disc of radius r centered at the origin. Thanks to the assumption that $\int_C c_1(X) = 0$, we have

$$\lim_{r\to\infty}\int_{B_r} e(t) = \int_C \omega(t) = \int_C \omega_X,$$

which is a positive constant. For a fixed t, if (4-2) was true for all $r > 0$, then we could let $r \to \infty$ in (4-3) and obtain $e(t)(0) = 0$, a contradiction. Therefore, for each t there is some $r(t) > 0$ with

$$\frac{\pi}{8K(t)} \leqslant \int_{B_{r(t)}} e(t) \leqslant \int_C \omega_X,$$

and so $K(t) \geqslant \kappa > 0$. Since each bisectional curvature is the sum of two sectional curvatures, we conclude that $\omega(t)$ has some sectional curvature which is larger than $\kappa/2$, and we can thus apply the observation in (4-1) to conclude that the flow is of type IIb. □

Remark 4.1 The argument above shows the following general conclusion about bisectional curvature. If (X, ω) is a compact Kähler manifold containing a rational curve C, then

$$\sup_X \mathrm{Bisec}_\omega \geqslant \frac{\pi}{8\int_C \omega}.$$

In many special cases we know the existence of rational curves, for example on $K3$ surfaces and quintic Calabi–Yau 3–folds. More generally, the existence of rational curves plays an important role in the classification of projective varieties, which is one of the ingredients in the proof of Theorem 1.5, Theorem 1.6 and Corollary 1.7. Another method to obtain rational curves is to use the Gromov–Witten invariant. For a class $A \in H_2(X, \mathbb{Z})$, if the Gromov–Witten invariant $\mathrm{GW}^X_{A,\omega,0}$ of genus 0 is defined, for instance when X is a Calabi–Yau 3–fold, and does not vanish (see [16]), then there is a rational curve $C \subset X$ with $\int_C \omega \leqslant \int_A \omega$, by the following argument. Let J_k be a sequence of regular almost-complex structures compatible with ω and which converge to the complex structure J of X. The assumption that $\mathrm{GW}^X_{A,\omega,0} \neq 0$ implies that for every k there is a J_k–holomorphic curve $f_k\colon \mathbb{CP}^1 \to X$ representing A. If the f_k converge to a limit when $k \to \infty$, then we obtain a rational curve in X, and if not, we still have one by the bubbling process; see [16]. Because of the importance of Ricci-flat Kähler metrics on Calabi–Yau 3–folds, this lower bound for the supremum of the bisectional curvatures may have some independent interest.

Proof of Theorem 1.5 (A) Assume $\kappa(X) = 0$.

Since K_X is semiample and $\kappa(X) = 0$, we conclude that K_X^ℓ is trivial for some $\ell \geqslant 1$, ie X is Calabi–Yau.

Case 1 X is not a finite quotient of a torus, which is equivalent to the fact that none of the Ricci-flat Kähler metrics constructed by Yau [30] are flat. Let ω_∞ be the unique Ricci-flat Kähler metric in the class $[\omega_X]$, and let $x \in X$ be a point and $\pi \subset T_x X$ a 2–plane with $\mathrm{Sec}_{\omega_\infty}(\pi) \geqslant \kappa > 0$ for some constant $\kappa > 0$. Indeed we may choose x to be any point where ω_∞ is not flat, and since the Ricci curvature vanishes, there must be a positive sectional curvature at x. Thanks to [3], we know that the solution $\omega(t)$ of the unnormalized flow (1-5) converges smoothly to ω_∞ as $t \to \infty$. In particular $\mathrm{Sec}_{\omega(t)}(\pi) \geqslant \kappa/2 > 0$ for all t large. Thanks to the observation in (4-1), the flow is of type IIb.

Case 2 X is a finite quotient of a torus. In this case the unnormalized flow (1-5) converges smoothly to a flat metric ω_∞ [3]. In fact, it is easy to see that this convergence is exponentially fast; see [17]. Briefly, one considers the Mabuchi energy

$$Y(t) = \int_X |\nabla\dot{\varphi}|^2_{g(t)}\omega(t)^n,$$

where $\omega(t) = \omega_X + \sqrt{-1}\,\partial\bar{\partial}\varphi(t)$ and $\dot{\varphi} = \partial\varphi/\partial t$, and using the smooth convergence of $\omega(t)$ to ω_∞ one easily shows that $dY/dt \leqslant -\eta Y$ for some $\eta > 0$ and all $t \geqslant 0$, hence $Y \leqslant Ce^{-\eta t}$. Similarly, for $k \geqslant 2$ one considers

$$Y_k(t) = \int_X |\nabla^k_{\mathbb{R}}\,\dot{\varphi}|^2_{g(t)}\omega(t)^n,$$

and using interpolation inequalities and induction on k one easily proves that $dY_k/dt \leqslant -2\eta Y_k + C_k e^{-\eta t}$, hence $Y_k \leqslant C_k e^{-\eta t}$. Using the Poincaré and Sobolev–Morrey inequalities, one deduces that $\|\dot{\varphi}\|_{C^k(X,\omega_X)} \leqslant C_k e^{-\eta t}$, and the exponential smooth convergence of $\omega(t)$ to ω_∞ follows immediately. Therefore there are constant $C, \eta > 0$ such that $|\mathrm{Rm}(\omega(t))|_{\omega(t)} \leqslant Ce^{-\eta t}$, and it follows that the flow is of type III.

(B) Assume $\kappa(X) = n$.

Case 1 K_X is ample. In this case, [3] shows that the normalized Kähler–Ricci flow (1-2) converges smoothly to the unique Kähler–Einstein metric ω_∞ with $\mathrm{Ric}(\omega_\infty) = -\omega_\infty$. In particular, the sectional curvatures along the normalized flow remain uniformly bounded for all positive time. Translating this back to the unnormalized flow, we see that the flow is of type III.

Case 2 K_X is not ample. By assumption we have that K_X is big and semiample (so, in particular, nef). It follows that X is Moishezon and Kähler, hence projective. Take ℓ sufficiently large and divisible so that sections of K_X^ℓ give a holomorphic map $f\colon X \to \mathbb{CP}^N$ with image a normal projective variety with at worst canonical singularities and ample canonical divisor. Since K_X is not ample, f is not an isomorphism with its image, and by Zariski's main theorem there is a fiber $F \subset X$ of f which is positive dimensional. Each irreducible component of F is uniruled by a result of Kawamata [12, Theorem 2], and if C is a rational curve contained in F then $f(C)$ is a point, and so $\int_C c_1(X) = 0$. The criterion in Proposition 1.4 then shows that the flow is of type IIb.

(C) Assume $0 < \kappa(X) < n$.

Case 1 The generic fiber X_y of the Iitaka fibration $\pi\colon X \to B$ is not a finite quotient of a torus. Let $y \in B \setminus S'$ and fix a point $x \in X_y = \pi^{-1}(y)$. If $\omega(t)$ is the solution of the normalized Kähler–Ricci flow (1-2) and $\widetilde{\omega}(s)$ is the solution of the unnormalized flow (1-5), then we have that

$$\widetilde{\omega}(s) = e^t \omega(t), \quad s = e^t - 1.$$

We have proved in Theorem 1.3 that if $U \supset X_y$ is the preimage of a sufficiently small neighborhood of y, then there is a sequence $t_k \to \infty$ such that $(U, e^{t_k}\omega(t_k), x)$ converge smoothly in the sense of Cheeger–Gromov to $(X_y \times \mathbb{C}^m, \omega_\infty, (x,0))$, and ω_∞ is the product of a Ricci-flat Kähler metric on X_y and a flat metric on $\mathbb{C}^m$. Since X_y is not a finite quotient of a torus, we conclude that ω_∞ is not flat, and so there is some point $x' \in X_y$ and a 2–plane $\pi \subset T_{x'}(X_y \times \{0\})$ with $\mathrm{Sec}_{\omega_\infty}(\pi) \geqslant \kappa > 0$ for some constant $\kappa > 0$. Because of the smooth Cheeger–Gromov convergence, we conclude that (up to renaming the sequence t_k) there are 2–planes $\pi_k \subset T_{x'}X$ with $\mathrm{Sec}_{\widetilde{\omega}(t_k)}(\pi_k) \geqslant \kappa/2$. By the observation in (4-1), the flow is of type IIb.

Case 2 The generic fiber X_y is a finite quotient of a torus, and $S = \varnothing$. We assume first that the Iitaka fibration $\pi\colon X \to B$ is a smooth submersion whose fibers are complex tori (not just finite quotients of tori). In this case, when X is furthermore projective and $[\omega_X]$ is rational, Fong and Zhang [5] adapted the estimates of Gross, Tosatti and Zhang [8] to prove that the solution $\omega(t)$ of the normalized Kähler–Ricci flow (1-2) has uniformly bounded curvature for all time. The projectivity and rationality assumptions were recently removed in [11]. Translating this back to the unnormalized flow, we see that the flow is of type III.

Next, we treat the general case when the fibers are finite quotients of tori and still $S = \varnothing$. Since π is a proper submersion, it is a smooth fiber bundle, so given any $y_0 \in B$ we can find a small coordinate ball $U \ni y_0$ such that there is a diffeomorphism $f\colon U \times F \to \pi^{-1}(U)$ compatible with the projections to U, with F diffeomorphic to a finite quotient of a torus. We pull back the complex structure from $\pi^{-1}(U)$, so we obtain a (in general nonproduct) complex structure on $U \times F$ which makes the map f biholomorphic. Let $\widetilde{F} \to F$ be a finite unramified covering with $\widetilde{F}$ diffeomorphic to a torus, and put the pullback complex structure on $U \times \widetilde{F}$ (again, in general not the product complex structure) so that the projection $p\colon U \times \widetilde{F} \to U \times F$ is holomorphic. The projection $\pi_U\colon U \times \widetilde{F} \to U$ equals $\pi \circ f \circ p$ and so is holomorphic, and therefore every fiber $\pi_U^{-1}(y)$, $y \in U$, is a compact Kähler manifold diffeomorphic to a torus, hence it is biholomorphic to a torus; see eg [4, Proposition 2.9]. Therefore $f \circ p\colon U \times \widetilde{F} \to \pi^{-1}(U)$ is a holomorphic finite unramified covering compatible with the projections to U, and $\pi_U\colon U \times \widetilde{F} \to U$ is a holomorphic submersion with fibers biholomorphic to complex tori and with total space Kähler. We can then use the local estimates in [11; 5; 8] to conclude that the pullback of the normalized flow to $U \times \widetilde{F}$ has bounded curvature. But the metrics along this flow are all invariant under the group of holomorphic deck transformations of $f \circ p$ and therefore descend to the flow on $\pi^{-1}(U)$, which has also bounded curvature. Since y_0 was arbitrary, we conclude that the flow on X is of type III. $\square$

Lastly, we give the proof of Theorem 1.6.

Proof of Theorem 1.6 Let X be a minimal Kähler surface with $\kappa(X) = 1$ and $\pi\colon X \to B$ an elliptic fibration which is not a submersion everywhere. Thanks to Kodaira's classification of the singular fibers of elliptic surfaces [2, V.7], we see that either some singular fiber of π contains a rational curve C or otherwise the only singular fibers are of type mI_0, $m > 1$, ie smooth elliptic curves with nontrivial multiplicity.

In the first case, Kodaira's canonical bundle formula [2] gives

$$K_X = \pi^*(K_B \otimes L) \otimes \mathcal{O}\Big(\sum_i (m_i - 1) F_i\Big) \tag{4-4}$$

for some line bundle L on B, where m_i is the multiplicity of the component F_i, ie $\pi^*(P_i) = m_i F_i$ as Weil divisors, where $S' = \{P_i\} \subset B$ are all the critical values of π. Then if ℓ is sufficiently large that $\ell(1 - 1/m_i) \in \mathbb{N}$ for all i, we have

$$K_X^\ell = \pi^*\bigg((K_B \otimes L)^\ell \otimes \mathcal{O}\Big(\sum_i \frac{\ell(m_i-1)}{m_i} P_i\Big)\bigg),$$

and since $\pi(C)$ is a point, this shows that $\int_C c_1(X) = 0$, and so we conclude that the flow is of type IIb by Proposition 1.4.

It remains to show that if the only singular fibers are multiples of a smooth elliptic curve, then the flow is of type III, ie that along the normalized flow (1-2) the curvature remains uniformly bounded for all time.

On compact sets away from the singular fibers, this is true thanks to [5]; see [8; 11; 7]. Let then $\Delta \subset B$ be the unit disc in some coordinate chart such that $S' \cap \Delta = \{y_0\}$, the center of the disc, so the fiber X_{y_0} is of type mI_0 for some $m > 1$ (which could depend on the point y_0), and let $U = \pi^{-1}(\Delta)$. Then, thanks to the local description of such singular fibers [6, Proposition 1.6.2], we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{p} & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Delta & \xrightarrow{q} & \Delta \end{array} \tag{4-5}$$

Here $q(z)=z^m$, the map $\tilde{\pi}\colon \tilde{U}\to\Delta$ is a holomorphic submersion whose fibers are elliptic curves and the map $p\colon \tilde{U}\to U$ is a holomorphic finite unramified covering. If we can show that the pullback of the normalized flow to $\tilde{U}$ has bounded curvature, then the same is true for the flow on U, and since $y_0\in S'$ was arbitrary, we will conclude that the flow on X is of type III.

As before, let χ be the restriction of ω_{FS}/ℓ to $\Delta\subset B\subset \mathbb{P}H^0(K_X^{\ell})$. This is a smooth semipositive form on Δ, although χ may not be positive definite at the center of the disc. Let $v=|z|^{2/m}-\psi$ on Δ, where ψ is a potential for χ on Δ, and define $\omega_B=\chi+\sqrt{-1}\,\partial\bar\partial v$, which is an orbifold flat Kähler metric on Δ, so that $q^*\omega_B$ is the Euclidean metric on Δ (and so q^*v is smooth). For simplicity we will also denote π^*v by v and $\pi^*\omega_B$ by ω_B, so that with this notation we have that p^*v is smooth on $\tilde{U}$. Up to shrinking Δ, we may also assume that v is defined in a neighborhood of $\bar\Delta$. Thanks to the estimate (2-1) for the normalized flow, which was proved in [5] (see also [21]), and since on ∂U the semipositive form χ is uniformly equivalent to $\pi^*\omega_B$, we conclude that there is a constant C such that on ∂U we have

$$C^{-1}(\pi^*\omega_B+e^{-t}\omega_X)\leqslant\omega(t)\leqslant C(\pi^*\omega_B+e^{-t}\omega_X)$$

for all $t>0$. Therefore on $\partial\tilde{U}$ we get

$$C^{-1}(\tilde{\pi}^*q^*\omega_B+e^{-t}p^*\omega_X)\leqslant p^*\omega(t)\leqslant C(\tilde{\pi}^*q^*\omega_B+e^{-t}p^*\omega_X). \tag{4-6}$$

Our goal is to prove the same estimate on the whole of $\tilde{U}$. We follow the same strategy as in [5]; see [24; 21]. Recall from [22] that the Kähler–Ricci flow is of the form

$$\omega(t)=(1-e^{-t})\chi+e^{-t}\omega_X+\sqrt{-1}\,\partial\bar\partial\varphi(t)$$

and the potentials φ satisfy a uniform L^∞ bound $|\varphi(t)|\leqslant C$ for all $t\geqslant 0$. If we let

$$\tilde\varphi(t)=\varphi(t)-(1-e^{-t})v,$$

then $\tilde\varphi$ is still uniformly bounded and $\omega(t)=(1-e^{-t})\omega_B+e^{-t}\omega_X+\sqrt{-1}\,\partial\bar\partial\tilde\varphi(t)$ holds on $U\setminus S$. The function $\tilde\varphi$ may not be smooth on $U\cap S$, but after pulling back to $\tilde{U}$ it becomes $p^*\tilde\varphi=p^*\varphi-(1-e^{-t})\tilde\pi^*q^*v$, which is smooth everywhere, and we have

$$p^*\omega(t)=(1-e^{-t})\tilde\pi^*q^*\omega_B+e^{-t}p^*\omega_X+\sqrt{-1}\,\partial\bar\partial p^*\tilde\varphi(t).$$

The parabolic Schwarz lemma calculation (see [21; 29]) applied to the map

$$\tilde\pi\colon (\tilde{U},p^*\omega(t))\to(\Delta,q^*\omega_B)$$

gives

$$\left(\frac{\partial}{\partial t}-\Delta_{p^*\omega(t)}\right)\left(\mathrm{tr}_{p^*\omega(t)}(\tilde\pi^*q^*\omega_B)-2p^*\tilde\varphi(t)\right)\leqslant-\mathrm{tr}_{p^*\omega(t)}(\tilde\pi^*q^*\omega_B)+4$$

on $\widetilde{U}$, for t large. Since the quantity $\mathrm{tr}_{p^*\omega(t)}(\widetilde{\pi}^*q^*\omega_B) - 2p^*\widetilde{\varphi}(t)$ is uniformly bounded on $\partial\widetilde{U}$ thanks to (4-6) and the bound on $\widetilde{\varphi}$, the maximum principle gives

$$\mathrm{tr}_{p^*\omega(t)}(\widetilde{\pi}^*q^*\omega_B) \leqslant C \tag{4-7}$$

on $\widetilde{U}\times[0,\infty)$. Let now y be any point in Δ and consider the fiber $\widetilde{X}_y = \widetilde{\pi}^{-1}(y)$, which is a smooth elliptic curve. Restricting to $\widetilde{X}_y$ and using $\widetilde{\pi}^*q^*\omega_B \leqslant Cp^*\omega(t)$, as in [21, Corollary 5.2] or [24, (3.9)], we easily see that

$$\mathrm{osc}_{\widetilde{X}_y}(e^t p^*\widetilde{\varphi}) \leqslant C, \tag{4-8}$$

independent of $y\in\Delta$ and $t \geqslant 0$. We then let $\widehat{\varphi}_y(t)$ to be the average of $p^*\widetilde{\varphi}(t)$ on $\widetilde{X}_y$ with respect to the volume form $p^*\omega_X|_{\widetilde{X}_y}$. This defines a smooth function on Δ, uniformly bounded for all $t\geqslant 0$, and we denote its pullback to $\widetilde{U}$ by $\widehat{\varphi}(t)$. The bound (4-8) gives $\sup_{\widetilde{U}}|e^t(p^*\widetilde{\varphi}-\widehat{\varphi})| \leqslant C$, independent of $t\geqslant 0$. Then a calculation as in [5] (see also [21; 24]), gives

$$\left(\frac{\partial}{\partial t}-\Delta_{p^*\omega(t)}\right)\bigl(\log\mathrm{tr}_{p^*\omega(t)}(e^{-t}p^*\omega_X)-Ae^t(p^*\widetilde{\varphi}-\widehat{\varphi})\bigr) \leqslant -\mathrm{tr}_{p^*\omega(t)}(p^*\omega_X)+CAe^t,$$

if A is large enough. Since the quantity $\log\mathrm{tr}_{p^*\omega(t)}(e^{-t}p^*\omega_X) - Ae^t(p^*\widetilde{\varphi}-\widehat{\varphi})$ is uniformly bounded on $\partial\widetilde{U}$ thanks to (4-6) and the bound on $e^t(p^*\widetilde{\varphi}-\widehat{\varphi})$, the maximum principle gives $\mathrm{tr}_{p^*\omega(t)}(e^{-t}p^*\omega_X) \leqslant C$ on $\widetilde{U}\times[0,\infty)$. Adding this estimate to (4-7) we conclude that

$$\widetilde{\pi}^*q^*\omega_B + e^{-t}p^*\omega_X \leqslant Cp^*\omega(t) \tag{4-9}$$

on $\widetilde{U}\times[0,\infty)$. Now the main theorem of [20] shows that $|\dot{\varphi}|\leqslant C$ holds on $X\times[0,\infty)$, where $\dot{\varphi} = \partial\varphi/\partial t$. We write the flow on X as the parabolic complex Monge–Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log\frac{e^t\bigl((1-e^{-t})\chi + e^{-t}\omega_X + \sqrt{-1}\,\partial\overline{\partial}\varphi\bigr)^2}{\Omega} - \varphi,$$

where Ω is a smooth volume form on X with $\sqrt{-1}\,\partial\overline{\partial}\log\Omega = \chi$. Pulling back to $\widetilde{U}$, we may write it as

$$\frac{\partial}{\partial t}p^*\widetilde{\varphi} = \log\frac{e^t\bigl((1-e^{-t})\widetilde{\pi}^*q^*\omega_B + e^{-t}p^*\omega_X + \sqrt{-1}\,\partial\overline{\partial}p^*\widetilde{\varphi}\bigr)^2}{p^*\Omega} - p^*\widetilde{\varphi}$$

and still have $|\partial p^*\widetilde{\varphi}/\partial t| \leqslant C$. (Recall that $p^*\widetilde{\varphi}$ is smooth on $\widetilde{U}$.) Then we see that on $\widetilde{U}$ the metrics $p^*\omega(t)$ and $\widetilde{\pi}^*q^*\omega_B + e^{-t}p^*\omega_X$ have uniformly equivalent volume forms, both comparable to $e^{-t}p^*\Omega$, and this together with (4-9) finally proves that (4-6) holds on all of $\widetilde{U}\times[0,\infty)$.

Now that we have established (4-6), we can use the same procedure as in [11; 5; 8] to conclude that the pullback of the normalized flow to $\widetilde{U}$ has bounded curvature, and we are done. □

For example, if X is a minimal properly elliptic Kähler surface such that the sections of K_X already give rise to the Iitaka fibration $\pi\colon X \to B$, then it is easy to see using Kodaira's canonical bundle formula (4-4) that there can be no multiple fibers of π, so in this case if $S \neq \varnothing$ then the flow is always of type IIb. It is also easy to construct examples of minimal properly elliptic Kähler surfaces with $S \neq \varnothing$ and all singular fibers of type mI_0, in which case the flow is of type III. For example, we can take a compact Riemann surface Σ of genus $g \geqslant 2$ with an automorphism $f\colon \Sigma \to \Sigma$ of order m with finitely many fixed points, then take the elliptic curve $E = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ and consider the free $\mathbb{Z}_m$–action on $\Sigma \times E$ generated by $(x, z) \mapsto (f(x), z + 1/m)$, $x \in \Sigma$, $z \in E$. Then $\pi\colon X = (\Sigma \times E)/\mathbb{Z}_m \to B = \Sigma/\mathbb{Z}_m$ is a minimal properly elliptic Kähler surface with singular fibers of type mI_0 above the fixed points of f, and π is the Iitaka fibration of X.

Proof of Corollary 1.7 It is shown in [13] that X and Y can be connected by a nontrivial sequence of finitely many flops. In particular, by [13, Lemma 2], there is a rational curve C in X such that $\int_C c_1(X) = 0$. The conclusion then follows from Proposition 1.4. □

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