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Some differentials on Khovanov–Rozansky homology

JACOB RASMUSSEN

We study the relationship between the HOMFLY and $sl(N)$ knot homologies introduced by Khovanov and Rozansky. For each $N > 0$, we show there is a spectral sequence which starts at the HOMFLY homology and converges to the $sl(N)$ homology. As an application, we determine the KR–homology of knots with 9 crossings or fewer.

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1 Introduction

In [12; 13], Khovanov and Rozansky introduced a new class of homological knot invariants which generalize the original construction of the Khovanov homology [9]. In this paper, we investigate these *KR–homologies* and the relations between them. Our motivation was to give some substance to the conjectures made in Dunfield, Gukov and Rasmussen [3] about the behavior of these theories and their relation to the knot Floer homology. Although we are unable to say anything about the latter problem, we hope that we can at least shed some light on the structure of KR–homology.

In order to state our results, we briefly recall the form of these homologies, restricting for the moment to the case of a knot $K \subset S^3$. To such K, the theory of Khovanov and Rozansky [13] assigns a triply graded homology group $\overline{H}^{i,j,k}(K)$ whose graded Euler characteristic is the HOMFLY polynomial. To be precise, we denote by $P_K(a,q)$ the HOMFLY polynomial of K normalized to satisfy the skein relation

$$aP(\diagup\!\!\!\!\!\diagdown) - a^{-1}P(\diagdown\!\!\!\!\!\diagup) = (q-q^{-1})P(\,)(\,),$$

and so that P of the unknot is equal to 1. Then, with an appropriate choice of gradings,

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \overline{H}^{i,j,k}(K) = P_K(a,q).$$

The definition of $\overline{H}$ is closely related to that of another family of homology theories $\overline{H}_N^{I,J}(K)$ $(N > 0)$ introduced by Khovanov and Rozansky [12]. Their graded Euler characteristics give the $sl(N)$ polynomials:

$$\sum_{I,J} (-1)^J q^I \dim \overline{H}_N^{I,J}(K) = P_K(q^N, q).$$

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The large-N behavior of these theories was studied first in Gukov, Schwartz, and Vafa [5], and then later in Dunfield, Gukov and Rasmussen [3], where it was conjectured that the limit of $\overline{H}_N(K)$ as $N \to \infty$ should be a triply graded homology theory $\mathcal{H}(K)$ with Euler characteristic $P_K(a,q)$. In fact, the limiting theory is a regraded version of $\overline{H}(K)$.

Theorem 1 *For all sufficiently large* N,

$$\overline{H}_N^{I,J}(K) \cong \bigoplus_{\substack{i+Nj=I \\ (k-j)/2=J}} \overline{H}^{i,j,k}(K).$$

We remark that $\overline{H}(K)$ is finite-dimensional, so when N is large there will be at most one nontrivial summand on the right-hand side. The exact value of N needed for the theorem to hold depends on K, but it need not be especially big. There are many knots for which $N > 1$ is enough.

Theorem 1 is a special case of the following more general relation between $\overline{H}$ and $\overline{H}_N$.

Theorem 2 *For each* $N > 0$, *there is a spectral sequence* $E_k(N)$ *which starts at* $\overline{H}(K)$ *and converges to* $\overline{H}_N(K)$. *The higher terms in this sequence are invariants of* K.

In some sense, these sequences are all generalizations of Lee's original spectral sequence [16] for the Khovanov homology. As described in Dunfield, Gukov and Rasmussen [3], the idea that they should exist arose from Gornik's work on the $sl(N)$ homology [4]. The exact method by which they are constructed is rather different from that envisioned in [3], but we expect that their content is the same.

Strictly speaking, the statement of the theorem is weaker than the conjecture made in [3], which says that $\overline{H}_N(K)$ should be the homology of a differential $d_N\colon \mathcal{H}(K) \to \mathcal{H}(K)$. The first differential in the sequence $E_k(N)$ provides us with a map $d\colon \overline{H}(K) \to \overline{H}(K)$ whose behavior with respect to the triple grading on $\overline{H}$ matches that predicted for d_N. Thus, if we knew that the spectral sequence converged after the first differential, this part of the conjecture would hold. In all the examples we have considered, $E_k(N)$ does indeed converge after the first differential, but we see no *a priori* reason why this should always be the case.

More generally, [3] conjectured that there should be differentials $d_N\colon \mathcal{H}(K) \to \mathcal{H}(K)$ not just for $N > 0$, but for all $N \in \mathbb{Z}$. Furthermore, d_N and d_{-N} should be interchanged by an involution $\phi\colon \mathcal{H}(K) \to \mathcal{H}(K)$ which generalizes the well-known symmetry of the HOMFLY polynomial: $P_K(a,q) = P_K(a,q^{-1})$. So far, we are unable to explain either this symmetry or the differentials d_N $(N \le 0)$ in terms of $\overline{H}$.

However, there is one surprising exception. The symmetry ϕ should exchange d_1 and d_{-1}, so the conjecture implies that

$$H(\mathcal{H}(K), d_{-1}) \cong H_*(\mathcal{H}(K), d_1) \cong \overline{H}_1(K).$$

The latter group is always isomorphic to $\mathbb{Q}$, so we expect that $H(\mathcal{H}(K), d_{-1}) \cong \mathbb{Q}$ as well. In fact, we have the following result:

Theorem 3 *There is a spectral sequence $E_k(-1)$ that starts at $\overline{H}(K)$ and converges to $\mathbb{Q}$.*

The grading behavior of the first differential $d\colon \overline{H}(K) \to \overline{H}(K)$ matches the expected behavior of d_{-1}, so again, if the sequence converged after this differential, we would be in the situation of the conjecture. The construction of the sequence $E_k(-1)$, while simple, is unlike anything familiar from Khovanov homology. It certainly behaves as if it should be dual to $E_k(1)$ under the symmetry ϕ, but it is not clear how this duality might be realized.

Although the KR–homologies are entirely combinatorial in nature, they have been surprisingly difficult to compute. As an application of the theorems above, we determine the KR–homology of some simple knots. For example, combining Theorem 1 with the main result of Rasmussen [21] gives:

Corollary 1 *If K is a two-bridge knot, then $\overline{H}^{i,j,k}(K) = 0$ unless $i + j + k = \sigma(K)$.*

This condition is similar to the usual notion of thinness in Khovanov homology; see Bar-Natan [1] and Khovanov [10]. We call knots which satisfy it *KR–thin*. The KR–homology of such a knot is completely determined by its HOMFLY polynomial and signature. Many other small knots are KR–thin, and Theorems 2 and 3 provide strong constraints on the homology of those which are not. Using them, it is not difficult to determine the KR–homology of all knots with 9 crossings or fewer.

The rest of the paper is organized as follows. In the first three sections we review (and in some cases, sharpen) various notions introduced by Khovanov and Rozansky, starting with the definitions of the different KR–homologies in Section 2. Section 3 contains material related to the theory of matrix factorizations, while Section 4 describes the relation between KR–homology and the Murakami–Ohtsuki–Yamada state model. In Sections 5 and 6 we construct the spectral sequences of Theorems 2 and 3, respectively. Finally, in Section 7, we explain how these sequences can be applied to the problem of computing the KR–homology.

In writing, we have aimed to give a reasonably self-contained treatment of the KR–homology. In particular, we do not assume that the reader is familiar with Khovanov and Rozansky [12; 13], and much of the first three sections is devoted to a review of those papers. The reasons for this are both technical and expository. On the technical side, the proof of Theorem 2 rests on results which are very similar, but unfortunately not quite identical, to those in [12; 13]. In order to give a complete treatment of these facts, it seemed best to begin at the beginning. From the expository point of view, we hope that readers who are unfamiliar with KR–homology will find it convenient to have the definitions and normalization conventions for the different theories housed under one roof.

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2 Definitions

Our goal in this section is to give a concise (but still self-contained) definition of the various Khovanov–Rozansky homologies. The material here is all drawn from Khovanov and Rozansky [12; 13] and Gornik [4], but we have slightly modified some of the definitions. In particular, the reader should be aware that our grading conventions for the HOMFLY homology are different from the ones introduced in [13].

2.1 Matrix factorizations

We begin by describing a class of algebraic objects known as matrix factorizations. These objects first appeared in the context of algebraic geometry. Their application to knot theory was one of the seminal advances of [12].

Definition 2.1 Suppose R is a commutative ring and that $w \in R$. A *$\mathbb{Z}$–graded matrix factorization with potential w* consists of a free graded R–module C^* $(* \in \mathbb{Z})$ together with a pair of *differentials* $d_\pm\colon C^* \to C^{*\pm 1}$ with the property that $(d_+ + d_-)^2 = w \cdot \mathrm{Id}_C$.

Remark We have included the phrase $\mathbb{Z}$–graded to distinguish this definition from the one used in [12] and [13], where matrix factorizations are $\mathbb{Z}/2$–graded. Unless we're trying to emphasize the distinction, we'll generally be careless and call a $\mathbb{Z}$–graded matrix factorization a matrix factorization.

The $\mathbb{Z}$–grading implies that the condition $(d_+ + d_-)^2 = w \cdot \mathrm{Id}_C$ is equivalent to

$$d_+^2 = d_-^2 = 0 \quad \text{and} \quad d_+ d_- + d_- d_+ = w \cdot \mathrm{Id}_C .$$

Thus a $\mathbb{Z}$–graded matrix factorization gives rise to two different chain complexes $C^*_\pm$ with underlying group C^* and differentials $d_\pm$. If it happens that $w = 0$, we get a third ($\mathbb{Z}/2$–graded) chain complex structure C^*_{tot} on C^*, with differential $d_{\mathrm{tot}} = d_+ + d_-$.

A morphism between two matrix factorizations C^* and D^* is a homomorphism of graded modules $f\colon C^* \to D^*$ which commutes with both differentials. We denote the category of matrix factorizations over a fixed ring R by $\mathrm{GMF}(R)$ and the subcategory of factorizations with fixed potential w by $\mathrm{GMF}_w(R)$.

The tensor product construction plays an important role in the definition of the KR–homology. If C^* and D^* are two matrix factorizations over R, we endow the graded group $C^* \otimes_R D^*$ with differentials $d_\pm$ defined by the requirement that d_- is the differential on the chain complex $C^*_- \otimes D^*_-$, and similarly for d_+. The reader can easily verify

Lemma 2.2 *If C^* and D^* are matrix factorizations with potentials w_1 and w_2, then $C^* \otimes D^*$ is a matrix factorization with potential $w_1 + w_2$.*

The final notion we need is that of a *complex of matrix factorizations with potential w*. This is a $\mathbb{Z}$–graded chain complex defined over the category $\mathrm{GMF}_w(R)$. (Recall that the definition of a chain complex makes sense over any additive category.) More prosaically, such a complex consists of a doubly graded group $C^{*,*}$ equipped with differentials

$$d_\pm\colon C^{i,j} \to C^{i\pm 1,j} \quad \text{and} \quad d_v\colon C^{i,j} \to C^{i,j+1}$$

such that $(d_+ + d_-)^2 = w \cdot \mathrm{Id}_C$, $d_v^2 = 0$, and d_v commutes with both d_+ and d_-. Often, it is more convenient to have d_v anticommute with $d_\pm$. This can be arranged by replacing d_v with $(-1)^i d_v$.

It is helpful to think of $C^{*,*}$ as being a sort of generalized double complex. We envision the group $C^{i,j}$ as sitting over the point (i, j) in the xy–plane, so that the differentials $d_\pm$ carry us one unit to the right and left, respectively, and d_v carries us one unit up. In keeping with this picture, we refer to i and j as the *horizontal* and *vertical* gradings on $C^{*,*}$, and denote them by gr_h and gr_v, respectively. In addition to these gradings, it is also natural to consider the quantities $\mathrm{gr}_\pm = \mathrm{gr}_v \pm \mathrm{gr}_h$, which are the total gradings on the double complexes $C^{*,*}_\pm$.

In the sequel, we will frequently take the tensor product of complexes of matrix factorizations. Since we know how to take tensor products of chain complexes and of

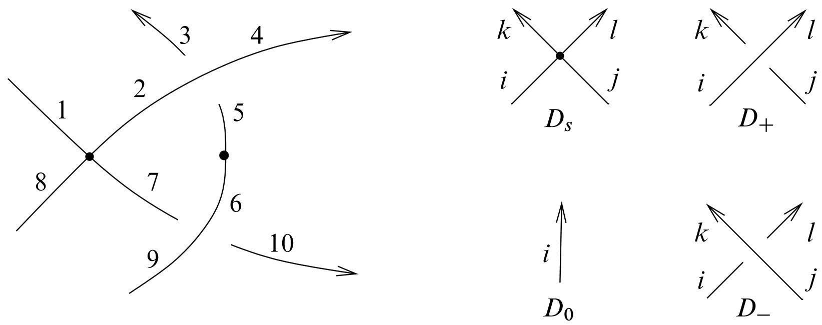


Figure 1: Some singular tangles. Left: a labeled singular tangle, including a mark and a crossing of each type. Right: diagrams of the four elementary tangles.

matrix factorizations, it's clear how this is to be done. From Lemma 2.2, we see that the tensor product of a complex of matrix factorizations with potential w_1 with a complex of matrix factorizations with potential w_2 is a complex of matrix factorizations with potential $w_1 + w_2$.

2.2 Tangle diagrams

KR–homology is most naturally defined in the context of *singular oriented planar tangles*. These are oriented planar diagrams which in addition to the usual over- and undercrossings may also contain some singular points, as illustrated in Figure 1. (In the notation of [12] and [13], singular points correspond to wide edges.) From now on, we will just refer to them as *tangle diagrams.*

More formally, a tangle diagram is an oriented planar graph, all of whose vertices have valence 1, 2, or 4. The 4–valent vertices are *crossings*, and come with an additional decoration indicating whether they are *positive*, *negative*, or *singular*, as represented by the diagrams D_+, D_-, and D_s shown in the figure. Bivalent vertices are called *marks*, and must have one incoming and one outgoing edge. Univalent vertices are called *free ends*. An edge adjacent to such an end is called *external*; all other edges are *internal*. The *connected components* of a diagram are the connected components of the underlying graph (not the connected components of the associated tangle). A component with no free ends is *closed*; other components are *open*. We keep track of the edges in a tangle diagram by labeling them by integers $1, 2, \dots, n$, where n is the number of edges in the diagram. A free end is identified by the label of its adjacent edge.

We now describe some operations for building new tangle diagrams out of old ones. First, if D_1 and D_2 are two tangle diagrams, we can take their disjoint union $D_1 \sqcup D_2$. Second, if i is an edge of D, we can form an new diagram $D(i)$ by inserting a bivalent vertex into i. We can also perform the inverse operation, which is known as *mark removal*. Finally, suppose that D is a tangle diagram with incoming and outgoing free ends labeled i and j, and i is *adjacent* to j in the sense that they can be isotoped onto each other without hitting the rest of the graph. Then we can form a new diagram $D|_{i=j}$ by identifying i and j to form a single bivalent vertex. Any tangle diagram can be built up from the elementary diagrams D_+, D_-, D_s, and D_0 shown in the figure by the operations of disjoint union, identifying free ends, and mark removal.

2.3 Edge rings

Suppose D is a tangle diagram with edges labeled $1, \dots, n$, and let $R'(D)$ be the ring $\mathbb{Q}[X_1, \dots, X_n]$. To an internal vertex v of D, we assign a linear relator $\rho(v)$ in $R'(D)$, given by the sum of the variables corresponding to outgoing edges of v minus the sum of the variables corresponding to ingoing edges. In other words, if v is a mark with incoming edge i and outgoing edge j, $\rho(v) = X_j - X_i$, and if v is a crossing with incoming edges i and j and outgoing edges k and l, $\rho(v) = X_k + X_l - X_i - X_j$.

Definition 2.3 The *edge ring* $R(D)$ is the graded ring $R'(D)/(\rho(v_j))$ where j runs over all internal vertices of D. The grading on $R(D)$ is denoted by q; it is determined by the requirement that $q(X_i) = 2$ for all i.

The edge ring behaves nicely under the operations of disjoint union, mark removal, and identifying free ends. The reader can easily verify that

$$R(D_1 \sqcup D_2) \cong R(D_1) \otimes_{\mathbb{Q}} R(D_2),$$
$$R(D(i)) \cong R(D),$$
$$R(D|_{i=j}) \cong R(D)|_{X_i = X_j}.$$

More generally, suppose that D is obtained from diagrams D_1 and D_2 by first taking their disjoint union and then identifying ends $(i_1, i_2, \dots, i_m)$ of D_1 with ends $(j_1, j_2, \dots, j_m)$ of D_2. Applying the relations above, we see that

$$R(D) \cong R(D_1) \otimes_{\mathbb{Q}[y_1, \dots, y_m]} R(D_2),$$

where y_k acts as X_{i_k} on $R(D_1)$ and X_{j_k} on $R(D_2)$.

Lemma 2.4 *Suppose D is a tangle diagram with V internal vertices, E edges, and C closed components. Then $R(D)$ is isomorphic to a polynomial ring on $E - V + C$ variables.*

Proof It is enough to prove the claim when D is connected; the general result then follows from the tensor product formula for disjoint unions. If D is open and connected, the statement amounts to saying that the relations $\rho(v_j)$ are linearly independent in the vector space spanned by the X_i. Suppose $\sum_j \alpha_j \rho(v_j) = 0$. Then for any internal edge i the coefficient of X_i in the sum vanishes, which means that the values of α on its two ends must be equal. Since D is connected, it follows that all of the α_j are equal. If D is open, considering an external edge shows that $\alpha_j \equiv 0$, while if D is closed and connected, there is a unique linear relation between the $\rho(v_j)$. □

We will also use two subrings of the edge ring. These are the *external ring* $R_e(D)$, which is the subring generated by the X_j, where j runs over the external edges of D, and the *reduced ring* $R_r(D)$ which is generated by the differences $U_{ij} = X_i - X_j$, where i and j run over all edges of D.

More explicitly, the external ring can be described as follows. We assign a sign ϵ_j to each free end of D according to the rule that $\epsilon_j = 1$ if j is an outgoing end, and $\epsilon_j = -1$ if it is incoming. If C is a connected component of D, we assign to it the polynomial $\rho(C) = \sum \epsilon_j X_j$, where j runs over the free ends of C. (Note that if C is an elementary diagram, this reduces to the previous definition.) Then we have:

Lemma 2.5 $R_e(D) \cong \mathbb{Q}[X_j]/(\rho(C))$, *where j runs over the external edges of D and C runs over the set of connected components of D.*

Proof Consider the vector space $V = \langle X_i \mid i \text{ is an edge of } D\rangle$, along with its subspaces $V_e = \langle X_j \mid j \text{ is an exterior edge of } D\rangle$ and $V_c = \langle \rho(c) \mid c \text{ is a crossing of } D\rangle$. Let $I \subset \mathbb{Q}[X_j]$ be the ideal generated by $V_e \cap V_c$. Then $R_e(D) \cong \mathbb{Q}[X_j]/I$, so it suffices to show that $V_e \cap V_c$ is generated by the elements $\rho(C)$, where C runs over the components of D. Now if $\rho = \sum_c \alpha_c \rho(c) \in V_e$, the component of ρ along each internal edge must vanish, which means that α_c has the same value at the two ends of the edge. Thus α_c is constant on connected components, and the claim is proved. □

The edge ring and the reduced ring are related by the following lemma:

Lemma 2.6
$$R(D) \cong R_r(D)[x]$$

Proof Let $R'_r(D)$ be the subring of $R'(D)$ generated by the $X_i - X_j$. Then the map which sends x to X_1 defines an isomorphism from $R'_r(D)[x]$ to $R'(D)$. Since the relations $\rho(v_j)$ are all contained in $R'_r(D)$, this descends to an isomorphism $R_r(D)[x] \cong R(D)$. □

2.4 The KR–complex

The key step in the definition of KR–homology is a process which assigns to a tangle a triply graded complex of matrix factorizations. More precisely, let D be a tangle diagram, and fix as an auxiliary parameter a polynomial $p(x) \in \mathbb{Q}[x]$. Then the *KR–complex* $C_p(D)$ associated to the pair (D, p) is a complex of matrix factorizations over the ring $R(D)$ with potential

$$w_p(D) = \sum_j \epsilon_j \, p(X_j),$$

where the sum runs over the external edges of D.

$C_p(D)$ is a graded module over the graded ring $R(D)$. This grading corresponds to the power of q in the HOMFLY polynomial, and will be referred to as the *q–grading*. The other two gradings on $C_p(D)$ are the homological gradings gr_h and gr_v coming from its structure as a complex of matrix factorizations. The differentials on $C_p(D)$ interact with the q–grading as follows: d_v preserves the q–grading, while d_+ increases it by 2. The differential d_- is usually not homogenous with respect to the q–grading, but if $p(x) = x^n$, d_- raises the q–grading by $2n-2$. We summarize our conventions regarding the various gradings in the following:

Definition 2.7 We say that $x \in C_p^{i,j,k}(D)$ if x is homogenous with respect to all three gradings, and $(i, j, k) = (q(x), 2\,\mathrm{gr}_h(x), 2\,\mathrm{gr}_v(x))$. With respect to this grading, d_v is homogenous of degree $(0, 0, 2)$ and d_+ is homogenous of degree $(2, 2, 0)$. If $p(x) = x^n$, then d_- is homogenous of degree $(2n-2, -2, 0)$.

Remark At first sight, the fact that we have chosen to double the homological gradings may seem rather strange. In fact, there are two good reasons for this choice of normalization. First, as we will explain in Section 4, the quantity $2\,\mathrm{gr}_h$ is naturally related to the power of a in the HOMFLY polynomial. Second, with this normalization, i, j and k all have the same parity when D is an ordinary diagram (ie one with no singular crossings).

2.5 Elementary tangles

Before defining the KR–complex in general, we describe it for the elementary diagrams D_s, D_+ and D_- shown in Figure 1. In each case, $C_p(D)$ will be a complex of matrix factorizations over the ring

$$R = \mathbb{Q}[X_i, X_j, X_k, X_l]/(X_k + X_l - X_i - X_j) \cong \mathbb{Q}[X_i, X_j, X_k]$$

with potential

$$\begin{aligned} W_p(X_i, X_j, X_k, X_l) &= p(X_k) + p(X_l) - p(X_i) - p(X_j) \\ &= p(X_k) + p(X_i + X_j - X_k) - p(X_i) - p(X_j). \end{aligned}$$

The complex $C_p(D_s)$ is a free R–module of rank 2. Since the potential is nonvanishing, the map d_+ must take one copy of R to the other. It is given by multiplication by $X_k X_l - X_i X_j$, which is equal (in R) to $-(X_k - X_i)(X_k - X_j)$. The map d_- takes the first copy of R back to the second, and must be given by multiplication by

$$p_{ij} = -W_p/(X_k - X_i)(X_k - X_j).$$

Note that if we substitute either $X_k = X_i$ or $X_k = X_j$ into W_p, the result vanishes, so the quotient p_{ij} really is an element of R. Finally, the two copies of R have the same vertical grading, so d_v is necessarily trivial. More succinctly, we can represent $C_p(D_s)$ by the diagram

$$C_p(D_s) = R\{1, -2, 0\} \underset{p_{ij}}{\overset{-(X_k - X_i)(X_k - X_j)}{\rightleftarrows}} R\{-1, 0, 0\}.$$

Following [12; 13], we use the notation $R\{i, j, k\}$ to indicate a free R–module of rank one so that the generator $1 \in R^{i,j,k}$ has grading (i, j, k).

Using the same notation, the complexes $C_p(D_+)$ and $C_p(D_-)$ are given by diagrams

$$C_p(D_+) = \begin{array}{ccc} R\{0,-2,0\} & \underset{p_i}{\overset{(X_k - X_i)}{\rightleftarrows}} & R\{0,0,0\} \\ {\scriptstyle (X_j - X_k)}\uparrow & & \uparrow{\scriptstyle 1} \\ R\{2,-2,-2\} & \underset{p_{ij}}{\overset{-(X_k - X_i)(X_k - X_j)}{\rightleftarrows}} & R\{0,0,-2\} \end{array}$$

and

$$C_p(D_-) = \begin{array}{ccc} R\{0,-2,2\} & \underset{p_{ij}}{\overset{-(X_k - X_i)(X_k - X_j)}{\rightleftarrows}} & R\{-2,0,2\} \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle (X_j - X_k)} \\ R\{0,-2,0\} & \underset{p_i}{\overset{(X_k - X_i)}{\rightleftarrows}} & R\{0,0,0\}. \end{array}$$

Here $p_i = W_p/(X_k - X_i)$, and the vertical arrows represent components of the map d_v. The reader can easily verify that in all three complexes, d_+ and d_v are homogenous of degree $(2,2,0)$ and $(0,0,2)$, respectively.

2.6 General tangles

For an arbitrary tangle diagram D, $C_p(D)$ is defined to be a tensor product of smaller complexes, one for each crossing in D. More precisely, if c is a crossing of D, let D_c be the subdiagram composed of the four edges of D adjacent to c. D_c is an elementary diagram, so the complex $C_p(D_c)$ was defined in the previous section. It is a complex of matrix factorizations over the ring $R_c = \mathbb{Q}[X_i, X_j, X_k, X_l]/(X_k + X_l - X_i - X_j)$ with potential $w_p(c) = p(X_k) + p(X_l) - p(X_i) - p(X_j)$.

Next, we consider the complex $C_p(D_c) \otimes_{R_c} R(D)$, which is obtained by replacing each copy of R_c in $C_p(D_c)$ with a copy of $R(D)$. It is a complex of matrix factorizations over $R(D)$. The global complex $C_p(D)$ is defined to be the tensor product

$$C_p(D) = \bigotimes_c (C_p(D_c) \otimes_{R_c} R(D)),$$

over the ring $R(D)$, where the product runs over all crossings of D. In particular, if there are no crossings, $C_p(D) = R(D)$.

We can now verify that $C_p(D)$ has the properties advertised in Section 2.4. First, it is clearly defined over the ring $R(D)$. Second, it is easy to see that $\sum_c w_p(c) = w_p(D)$, so it follows from Lemma 2.2 that $C_p(D)$ has potential $w_p(D)$. Third, the differentials on each individual factor satisfy the grading conventions established in Definition 2.7, so the same is true for $C_p(D)$.

An important (indeed, the defining) property of $C_p(D)$ is that it is *local* in the following sense:

Lemma 2.8 *Suppose D is obtained from diagrams D_1 and D_2 by first taking their disjoint union and then identifying ends $(i_1, i_2, \dots, i_m)$ of D_1 with ends $(j_1, j_2, \dots, j_m)$ of D_2. Then*

$$C_p(D) \cong C_p(D_1) \otimes_{\mathbb{Q}[y_1,\dots,y_m]} C_p(D_2),$$

where y_k acts as X_{i_k} on $C_p(D_1)$ and X_{j_k} on $C_p(D_2)$.

Proof This follows from the fact that the set of crossings for D is the union of the sets of crossings for D_1 and D_2, together with the relation

$$R(D) \cong R(D_1) \otimes_{\mathbb{Q}[y_1,\dots,y_m]} R(D_2)$$

observed in Section 2.3. □

2.7 The HOMFLY homology

We now define the various KR–homologies, starting with the HOMFLY homology of [13]. There are several ways of normalizing this invariant, all of which contain the same information. In addition to the *reduced theory* used in the introduction, there is also an *unreduced theory* which appears naturally in the context of the $sl(N)$ homology. We start with a third variant, which interpolates between these two and is closest to the version of the theory described in [13].

For the next few sections, we assume that L is an oriented link in S^3, and that L is represented by a connected tangle diagram D which is the closure of a braid. (The restriction that D be connected is simply for ease of exposition. The necessary modifications for disconnected diagrams are described in Section 2.10.)

Definition 2.9 The middle HOMFLY homology of L is the group

$$H(L) = H(H(C_p(D), d_+), d_v^*)\{-w+b, w+b-1, w-b+1\},$$

where w and b are the writhe and number of strands of the braid diagram D.

Remarks There are several aspects of this definition which are worth pointing out. First, observe that we have taken homology *twice*: first with respect to d_+, and then with respect to d_v^*, which is the map induced on $H(C_p(D), d_+)$ by d_v. Second, note that d_+ and d_v are homogenous with respect to all three gradings, so the triple grading on $C_p(D)$ descends to a triple grading on $H(L)$. Finally, since d_- does not appear in the definition, $H(L)$ is independent of the parameter p.

Khovanov and Rozansky proved:

Theorem 2.10 [13] *$H(L)$ is an invariant of L.*

A priori, nothing stops us from considering the homology $H(H(C_p(D), d_+), d_v^*)$ for an arbitrary diagram D representing L, but the restriction to diagrams which are braid closures plays an important role in the proof of Theorem 2.10. Indeed, Khovanov and Rozansky prove the invariance of $H(L)$ under braidlike Reidemeister moves and then use the fact that any two braid diagrams of L are related by such moves to conclude that $H(L)$ is a link invariant.

The second major result of [13] is the relation between $H(L)$ and the HOMFLY polynomial:

Theorem 2.11 [13] *For any $L \subset S^3$, we have*

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim H^{i,j,k}(L) = -\frac{P(L)}{q-q^{-1}}.$$

Here, both sides of the equation should be interpreted as Laurent series in q.

2.8 Reduced and unreduced complexes

If i is an edge of D, we define the reduced KR–complex $\overline{C}_p(D,i)$ to be the quotient $C_p(D)/(X_i)$. In [13], Khovanov and Rozansky observe that when $p=0$ this definition is actually independent of i. To see this, recall that $C_0(D)$ is a direct sum of copies of $R(D)$. Define $C_r(D) \subset C_0(D)$ to be the subgroup obtained by replacing each copy of $R(D)$ with a copy of the reduced ring $R_r(D)$. Inspecting the coefficients of d_+ and d_v in $C_0(D_s)$, $C_0(D_+)$ and $C_0(D_-)$, we see that they are all contained in $R_r(D)$. It follows that $C_r(D)$ is a subcomplex of $C_0(D)$.

Lemma 2.12 *In the category* $\mathrm{GMF}(\mathbb{Q})$, *there are isomorphisms*

$$C_0(D) \cong C_r(D) \otimes_{\mathbb{Q}} \mathbb{Q}[x] \quad \textit{and} \quad \overline{C}_0(D,i) \cong C_r(D).$$

Proof The first claim follows immediately from Lemma 2.6. For the second, consider the map $\phi\colon R_r(D) \to R(D)/(X_i)$ which is the composition of the inclusion $R_r(D) \to R(D)$ and the projection $R(D) \to R(D)/(X_i)$. It's easy to see that ϕ is an isomorphism of vector spaces. Since $C_0(D)$ is free over $R(D)$, the induced map $\phi\colon C_r(D) \to C_0(D,i)$ is also an isomorphism. □

Definition 2.13 The reduced HOMFLY homology $\overline{H}(L)$ is defined to be

$$\overline{H}(L) = H(H(C_r(D), d_+), d_v^*)\{-w+b-1, w+b-1, w-b+1\},$$

where, as before, w and b are the writhe and number of strands in the braid diagram D.

From the first part of Lemma 2.12 we see that $H(L) \cong \overline{H}(L) \otimes_{\mathbb{Q}} \mathbb{Q}[x]$. It follows that the graded Euler characteristic of $\overline{H}(L)$ is given by the HOMFLY polynomial:

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \overline{H}^{i,j,k}(L) = P(L).$$

There is also an unreduced version of the KR–complex. If i is an edge of D, we let $U_p(i)$ be the matrix factorization

$$C_p(D_s) = \mathbb{Q}[X_i]\{0,-2,0\} \underset{p'(X_i)}{\overset{0}{\rightleftarrows}} \mathbb{Q}[X_i]\{0,0,0\}.$$

The unreduced complex $\widetilde{C}_p(D,i)$ is defined to be $C_p(D) \otimes_{\mathbb{Q}[X_i]} U_p(i)$.

Definition 2.14 The unreduced HOMFLY homology $\widetilde{H}(L)$ is given by

$$\widetilde{H}(L) = H(H(\widetilde{C}_p(D,i), d_+), d_v^*)\{-w+b, w+b, w-b\},$$

where w and b are the writhe and number of strands in the braid diagram D.

Since both d_+ and d_v are trivial on $U_p(i)$, we see that $\widetilde{H}(L) \cong H(L) \otimes H^*(S^1)$. Its graded Euler characteristic is the unnormalized HOMFLY polynomial of L:

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \widetilde{H}^{i,j,k}(L) = \frac{a-a^{-1}}{q-q^{-1}} P(L) = \widetilde{P}(L).$$

Remark The quantity $w+b$ always has the same parity as the number of components of L. Since all the grading shifts in the complexes $C_p(D_+)$ and $C_p(D_-)$ are even, it follows that all three gradings of $\widetilde{H}(L)$ have the same parity as the number of components of L, and all three gradings of $\overline{H}(L)$ have the opposite parity.

As an example, we describe H, $\overline{H}$, and $\widetilde{H}$ for the unknot. The unknot can be represented by a braid diagram D consisting of a single edge (labeled 1), a single mark, and no crossings. The relation associated to the mark is $X_1 - X_1 = 0$, so $R(D) = \mathbb{Q}[X_1]/(0) \cong \mathbb{Q}[X_1]$, and $R_r(D) \cong \mathbb{Q}$. Since there are no crossings, $C_p(D) \cong R(D)$. It follows that $H(U) \cong \mathbb{Q}[X]$, where $1 \in \mathbb{Q}[X]$ has triple grading $(1,0,0)$; $\overline{H}(U) \cong \mathbb{Q}$, with triple grading $(0,0,0)$; and $\widetilde{H}(U) \cong \mathbb{Q}[X] \oplus \mathbb{Q}[X]$, where the generators have gradings $(1,1,-1)$ and $(1,-1,-1)$.

2.9 The $sl(N)$ homologies

To define the KR–homologies corresponding to the $sl(N)$ polynomial, we add the differential d_- into the mix. Suppose that D is a connected tangle diagram — not necessarily in braid form — representing the link L. Then D is closed, so the potential $w_p(D) = 0$, and the differential $d_{\text{tot}} = d_+ + d_-$ makes $\overline{C}_p(D,i)$ and $\widetilde{C}_p(D,i)$ into chain complexes.

Definition 2.15 For $p(x) \in \mathbb{Q}[x]$, the reduced and unreduced p–homologies are defined by

$$\overline{H}_p(L,i) = H(H(\overline{C}_p(D,i), d_{\text{tot}}), d_v^*),$$
$$\widetilde{H}_p(L) = H(H(\widetilde{C}_p(D,i), d_{\text{tot}}), d_v^*).$$

When $p(x) = x^{N+1}$, this definition was introduced by Khovanov and Rozansky in [12]. The fact that it is interesting for other values of p was observed by Gornik [4].

For the definition to make sense, we should check that $\widetilde{H}_p(L)$ depends only on L, and not on the choice of the diagram D or the marked edge i. This is done in Section 5. Similarly, the reduced homology $\overline{H}_p(L,i)$ depends only on L and the component of L containing i. Unlike the HOMFLY homology, $\overline{H}_p(L,i)$ really does depend on the marked component. However, in the special case when $L = K$ is a knot, there is only one component to choose from, so it makes sense to talk about the reduced homology $\overline{H}_p(K)$.

Next, we consider the grading on these homology groups. For a general polynomial p, d_{tot} will not be homogenous with respect to any linear combination of the gradings q and gr_h on $C_p(D)$, so $\overline{H}_p$ and $\widetilde{H}_p$ will have only the single grading coming from gr_v. However, when $p(x) = X^{N+1}$, d_{tot} is homogenous with respect to the grading

$$\mathrm{gr}_N = q + (N-1)\,\mathrm{gr}_h = i + \frac{N-1}{2} j,$$

so we can view $\overline{H}_p(L,i)$ and $\widetilde{H}_p(L)$ as being doubly graded, with gradings $(\mathrm{gr}_N, \mathrm{gr}_v)$. An additional global shift is needed to make the first grading into a link invariant. We put

$$\overline{H}_N(L,i) = \overline{H}_{x^{N+1}}(D,i)\{(N-1)w, 0\},$$
$$\widetilde{H}_N(L) = \widetilde{H}_{x^{N+1}}(D)\{(N-1)w, 0\},$$

where w is the writhe of the diagram D. In Section 3.4, we verify that $\overline{H}_N$ and $\widetilde{H}_N$ are the $sl(N)$ homology groups defined by Khovanov and Rozansky in [12]. Their graded Euler characteristic is given by the $sl(N)$ polynomial:

Theorem 2.16 [12] *$\widetilde{H}_N(L)$ is an invariant of the link L, while $\overline{H}(L,i)$ is an invariant of the link L and the marked component i. They satisfy*

$$\sum_{I,J} (-1)^J q^I \dim \overline{H}_N^{I,J}(L,i) = P_L(q^N, q),$$
$$\sum_{I,J} (-1)^J q^I \dim \widetilde{H}_N^{I,J}(L) = \widetilde{P}_L(q^N, q).$$

As an example, we again consider the homology of the unknot. The reduced complex satisfies $\overline{C}_P(U) \cong \mathbb{Q}[X_1]/(X_1) \cong \mathbb{Q}$, so $\overline{H}_p(U) \cong \mathbb{Q}$, for any p. The complex $\widetilde{C}_p(U)$ is more complicated. It is composed of two copies of $\mathbb{Q}[X]$, situated in gradings $(0,0,0)$ and $(0,-2,0)$. The differential d_- takes a generator of the first summand to $p'(X)$ times the generator of the second. Thus $\widetilde{H}_p(U) \cong \mathbb{Q}[X]/(p'(X))$, supported in homological grading 0. When $p(X) = X^{N+1}$, we see that $\widetilde{H}_N(U) \cong \mathbb{Q}[X]/(X^N)$. The generator $1 \in \mathbb{Q}[X]/(X^N)$ has polynomial grading $\mathrm{gr}_N = 1 - N$.

Remark It is natural to ask whether the groups above can be defined over $\mathbb{Z}$, rather than just $\mathbb{Q}$. An integral version of the HOMFLY homology can be obtained by simply replacing the edge ring R with an analogous polynomial ring over $\mathbb{Z}$. However, constructing a integral version of the $sl(N)$ homology is more difficult: we'd like to enforce the relation $p'(x) = x^N = 0$, but this forces us to take $p(x) = x^{N+1}/(N+1)$, which is not an element of $\mathbb{Z}[x]$. (It's not hard to see that $\overline{H}_{cp}(L,i) \simeq \overline{H}_p(L,i)$ for any $c \in \mathbb{Q}^*$, so if we work over $\mathbb{Q}$, we're justified in using $p(x) = x^{N+1}$.) Subsequent developments in the field have produced alternate definitions of the $sl(N)$ homology which work over $\mathbb{Z}$, but in this paper we will only consider homologies over $\mathbb{Q}$.

2.10 Disconnected diagrams

We conclude our discussion of KR–homology by describing what happens when the diagram D is disconnected. In this case, we must modify the definition of the complexes $\widetilde{C}_p(D)$ and $\overline{C}_p(D)$. The unreduced complex $\widetilde{C}_p(D)$ is the tensor product

$$\widetilde{C}_p(D) = \bigotimes_j \widetilde{C}_p(D_j, i_j),$$

where j runs over the connected components of D. The definition requires that we specify a collection of edges i_j — one for each component of D. In Section 3.4, we will show that $\widetilde{C}_p(D)$ is essentially independent of the choice of i_j. From Lemma 2.8, we see that

$$C_p(D) = \bigotimes_j C_p(D_j),$$

so from the point of view of the HOMFLY homology, the extra factors $\bigotimes U_p(i_j)$ just add a factor of $H^*(S^1)$ for each component of D.

To define the reduced KR–complex, assume that the special marked edge i is in the component D_1. Then

$$\overline{C}_p(D) = \overline{C}_p(D_1, i) \otimes \bigotimes_{j>1} \widetilde{C}_p(D_j).$$

The definitions of the various KR–homologies now proceed exactly as they did in the case when D had only one component.

3 Matrix factorizations

In this section, we develop some ideas about $\mathbb{Z}$–graded matrix factorizations which will be needed in the rest of the paper. The main difficulty with such factorizations, as compared to the $\mathbb{Z}/2$–graded factorizations used in [12] and [13], is that they lack a good notion of homotopy equivalence. Our first task is to develop an appropriate substitute — the notion of a quasi-isomorphism. After that, we discuss the class of Koszul factorizations introduced by Khovanov and Rozansky in [12] and adapt some of their results to the $\mathbb{Z}$–graded context. We conclude by verifying that the definitions of the various KR–groups given in Section 2 coincide with the original definitions in [12] and [13].

3.1 Positive homology

Given a $\mathbb{Z}$–graded matrix factorization C^*, we define its *positive homology* to be the group

$$H^+(C^*) = H(C^*, d_+).$$

If it happens that $C^* = C_p(D)$, we abbreviate still further and write $H^+(D)$ in place of $H^+(C_p(D))$, and similarly for $\widetilde{H}^+(D) = H^+(\widetilde{C}_p(D))$ and $\overline{H}^+(D) = H^+(\overline{C}_p(D))$. The operation of taking the positive homology gives a covariant functor $\mathcal{H}^+$ from the category GMF(R) to the category of graded R–modules. This naturally extends to a functor from Kom(GMF$_w(R)$) to Kom(R). For example, Definition 2.14 can be rewritten as

$$\widetilde{H}(L) = H(\widetilde{H}^+(D), d_v^*)\{-w+b, w+b, w-b\}$$

in this notation.

When the factorization has potential 0, we can say more:

Lemma 3.1 *There are functors*

$$\mathcal{H}^+\colon \mathrm{GMF}_0(R) \to \mathrm{Kom}(R),$$
$$\mathcal{H}^+\colon \mathrm{Kom}(\mathrm{GMF}_0(R)) \to \mathrm{Kom}(\mathrm{Kom}(R)).$$

Proof If C^* has zero potential, the differentials d_- and d_+ anticommute. The induced map $d_-^*\colon H^+(C^*) \to H^+(C^*)$ makes $H^+(C^*)$ into a chain complex. □

3.2 Quasi-isomorphisms

Roughly speaking, we want to think of two matrix factorizations as being equivalent if their positive homologies are isomorphic as chain complexes. When the potential is nonzero, however, the positive homology isn't a chain complex. To get around this problem, we adopt the following definition:

Definition 3.2 Suppose C^*, D^* are objects of $\mathrm{GMF}_w(R)$ and that $f\colon C^*_+ \to D^*_+$ is a chain map. We say that f is a *quasi-isomorphism* if for every object E^* of $\mathrm{GMF}_{-w}(R)$ the induced map $(f \otimes 1)^*\colon H^+(C^* \otimes E^*) \to H^+(D^* \otimes E^*)$ is an isomorphism which commutes with d_-^*. More generally, we say that C^* and D^* are *quasi-isomorphic* and write $C^* \sim D^*$ if they can be joined by a chain of quasi-isomorphisms.

Note that f is not required to be a morphism of matrix factorizations, but only a map on the positive chain complexes which "looks like" such a morphism when we pass to homology, in the sense that it commutes with d_-^*.

In practice, many of the quasi-isomorphisms we will consider do arise as morphisms.

Definition 3.3 Suppose C^*, D^* are objects of $\mathrm{GMF}_w(R)$ and that $f\colon C^* \to D^*$ is a morphism. We say that f is a *weak equivalence* if $f\colon C^*_+ \to D^*_+$ is a homotopy equivalence.

Lemma 3.4 *A weak equivalence is a quasi-isomorphism.*

Proof Suppose E^* is an object of $\mathrm{GMF}_{-w}(R)$. Then $f \otimes 1\colon C^* \otimes E^* \to D^* \otimes E^*$ is a morphism of $\mathrm{GMF}_0(R)$, so the induced map $(f \otimes 1)^*$ commutes with d_-^*. On the other hand, the map $(f \otimes 1)\colon C^*_+ \otimes E^*_+ \to D^* \otimes E^*_+$ is a homotopy equivalence, so $(f \otimes 1)^*$ is an isomorphism. □

A second source of quasi-isomorphisms is provided by a process we refer to as *twisting*. Suppose that C^* is a matrix factorization of length 3, so that C^i is trivial for $i \neq 0, 1, 2$. Given a homomorphism $H\colon C^2 \to C^0$, we define a deformed version of d_- by the equation

$$d_-(H) = d_- + d_+ H - H d_+.$$

The twisted factorization $C^*(H)$ is the triple $(C^*, d_+, d_-(H))$.

Lemma 3.5 *$C^*(H)$ is a graded matrix factorization with the same potential as C^*.*

Proof Suppose that C has potential w. It is enough to check that

$$\begin{aligned} d_+ d_-(H) + d_-(H) d_+ &= d_+ d_- + d_+^2 H - d_+ H d_+ + d_- d_+ + d_+ H d_+ - H d_+^2 \\ &= d_- d_+ + d_+ d_- = w \end{aligned}$$

and $d_-(H)^2 = 0$. The latter expression contains nine terms. Five of these ($-d_- H d_+$, $d_+ H d_-$, $d_+ H d_+ H$, $H d_+ H d_+$ and $-d_+ H^2 d_+$) vanish for dimensional reasons. Two others (d_-^2 and $-H d_+ d_+ H$) vanish because C^* is a matrix factorization. The remaining two terms d_-d_+H and $-Hd_+d_-$ represent nontrivial maps $C^2 \to C^0$. On C^2, d_-d_+ vanishes for dimensional reasons, so $d_+d_- = w \cdot \mathrm{Id}$. Similarly, on C^0, $d_-d_+ = w \cdot \mathrm{Id}$. Thus the final two terms cancel each other. □

Lemma 3.6 *If C^* and H are as above, the obvious identification $C^*_+ \cong C^*(H)_+$ is a quasi-isomorphism.*

Proof Suppose E^* is a matrix factorization with potential $-w$. Viewed as endomorphisms of the complex $C^*_+ \otimes E^*_+$, the negative differentials on $C^* \otimes E^*$ and $C^*(H) \otimes E^*$ have the form $d_- \otimes 1 \pm 1 \otimes d_-$ and $d_-(H) \otimes 1 \pm 1 \otimes d_-$. Their difference $(d_-(H) - d_-) \otimes 1 = (d_+H - Hd_+) \otimes 1$ is null-homotopic. □

3.3 Koszul factorizations

Suppose R is a ring and that $a, b \in R$. The *short matrix factorization* $\{a, b\}$ is the rank two factorization given by the diagram

$$R \underset{a}{\overset{b}{\rightleftarrows}} R.$$

It has potential ab.

Definition 3.7 [12] Suppose $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ are elements of R^n. The *Koszul factorization* $\{\boldsymbol{a}, \boldsymbol{b}\}$ is the tensor product of the short factorizations $\{a_i, b_i\}$:

$$\{\boldsymbol{a}, \boldsymbol{b}\} = \bigotimes_{i=1}^{n} \{a_i, b_i\}.$$

It is a $\mathbb{Z}$–graded matrix factorization over R, with potential $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_i a_i b_i$. We say that the *order* of the factorization is n.

When we want to explicitly record the values of a_i and b_i, we represent $\{\boldsymbol{a}, \boldsymbol{b}\}$ by the *Koszul matrix*

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}.$$

More intrinsically, we can view the underlying module of $\{\boldsymbol{a}, \boldsymbol{b}\}$ as the exterior algebra $\Lambda^* R^n$, where $\boldsymbol{b}$ is an element of R^n and $\boldsymbol{a}$ is an element of the dual module $(R^n)^*$. The differentials are given by

$$d_+(\boldsymbol{x}) = \boldsymbol{x} \wedge \boldsymbol{b} \quad \text{and} \quad d_-(\boldsymbol{x}) = \boldsymbol{x} \neg \boldsymbol{a}.$$

From this perspective, it's clear that if we express $\boldsymbol{b}$ and $\boldsymbol{a}$ in terms of a new basis for R^n and its dual basis, the resulting Koszul factorization will be isomorphic to $\{\boldsymbol{a}, \boldsymbol{b}\}$. In particular, consider the change-of-basis operation which replaces the standard basis element $\boldsymbol{e}_i$ of R^n with $\boldsymbol{e}_i + c\boldsymbol{e}_j$. At the level of Koszul matrices, this corresponds to the *row operation* which modifies the i^{th} and j^{th} rows as

$$\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \mapsto \begin{pmatrix} a_i + ca_j & b_i \\ a_j & b_j - cb_i \end{pmatrix}$$

and leaves the remaining rows of the Koszul matrix unchanged.

We now recall an important technical tool introduced in [12]. This is the process of "excluding a variable". Suppose that $C = \{\boldsymbol{a}, \boldsymbol{b}\}$ is a Koszul factorization over the ring $R[x]$ with potential w which happens to be contained in R, and that $b_1 = f(x)$ is a monic polynomial of positive degree in x. Let $\boldsymbol{a}', \boldsymbol{b}' \in R[x]^{n-1}$ be the vectors obtained from $\boldsymbol{a}$ and $\boldsymbol{b}$ by omitting the first component, and put $C' = \{\boldsymbol{a}', \boldsymbol{b}'\}$. At the level of modules, $C \cong C' \oplus C'$, and with respect to this decomposition the differentials on C are given by

$$d_+(u, v) = (d'_+ u, d'_+ v + f(x)u),$$
$$d_-(u, v) = (d'_- u + a_1 v, d'_- v).$$

Next, we form the quotient ring $R_1 = R[x]/(f(x))$, and let $\pi\colon R[x] \to R_1$ be the projection. The factorization $C'' = \{\pi(\boldsymbol{a}'), \pi(\boldsymbol{b}')\}$ is a Koszul factorization over R_1 with potential $\pi(w) = w \in R$.

Lemma 3.8 *The map $\phi\colon C \to C''$ defined by $\phi(u, v) = \pi(v)$ is a weak equivalence in the category* $\mathrm{GMF}_w(R)$.

Proof Using the formulas above, it is easy to see that ϕ defines a morphism of matrix factorizations. Thus we need only verify that ϕ has a homotopy inverse with respect to d_+. Since f is monic, every $r \in R_1$ may be written uniquely in the form $r = r_0 + r_1 x + \cdots + r_{k-1} x^{k-1}$, where $r_i \in R$ and $k = \deg f$. The map which sends $r \in R_1$ to this representative defines an R–module homomorphism $\iota\colon R_1 \to R[x]$. C' and C'' are free over $R[x]$ and R_1, respectively, so ι can be used to define an R–module homomorphism $\iota\colon C'' \to C'$. We define a map $\psi\colon C'' \to C$ by

$$\psi(y) = \left(\frac{\iota(d''_+ y) - d'_+ \iota(y)}{f(x)}, \iota(y) \right).$$

It is easy to see that ψ commutes with d_+ and that $\phi\psi = \mathrm{Id}_{C''}$. Finally, we define $H\colon C \to C$ by

$$H(u,v) = \left(\frac{v - \iota(\pi(v))}{f(x)}, 0\right).$$

We leave it as an exercise to the reader to check that H is a homotopy between $\psi\phi$ and Id_C. □

Corollary 3.9 *Suppose $C^* \sim D^*$ as objects of $\mathrm{GMF}_w(R[X])$, where $w \in R$. Then the quotients $C^*/(X)$ and $D^*/(X)$ are quasi-isomorphic as objects of $\mathrm{GMF}_w(R)$.*

Proof By Lemma 3.8, the quotient $C^*/(X)$ is quasi-isomorphic to $C^* \otimes \{0, x\}$. This, in turn, is quasi-isomorphic to $D^* \otimes \{0, x\}$. □

Now suppose that R is a polynomial ring, that $w \in R$, and that $\boldsymbol{b} \in R^n$. It is clear that we can choose $\boldsymbol{a} \in (R^n)^*$ so that $\{\boldsymbol{a}, \boldsymbol{b}\}$ has potential w if and only if w is in the ideal generated by the b_i. To what extent is the choice of $\boldsymbol{a}$ unique? When $n = 1$, we have $a_1 b_1 = w$, so a_1 is uniquely determined unless $b_1 = w = 0$. For $n = 2$, we have the following result.

Lemma 3.10 *Suppose R is a UFD and that $\{\boldsymbol{a}, \boldsymbol{b}\}$ and $\{\boldsymbol{a}', \boldsymbol{b}'\}$ are two order-two Koszul factorizations over R with potential w. If b_1 and b_2 are relatively prime, the factorizations $\{\boldsymbol{a}, \boldsymbol{b}\}$ and $\{\boldsymbol{a}', \boldsymbol{b}'\}$ are related by a twist.*

Proof We have

$$a_1 b_1 + a_2 b_2 = w = a_1' b_1 + a_2' b_2,$$

which implies that $(a_1 - a_1')b_1 + (a_2 - a_2')b_2 = 0$. Since b_1 and b_2 are relatively prime, our two factorizations must be represented by Koszul matrices of the form

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 - kb_2 & b_1 \\ a_2 + kb_1 & b_2 \end{pmatrix}.$$

The second factorization is a twist of the first one, via the map $H\colon R \to R$ which sends x to kx. □

Remark In fact, it is not difficult to see that $\{\boldsymbol{a}, \boldsymbol{b}\}$ and $\{\boldsymbol{a}', \boldsymbol{b}'\}$ are isomorphic as $\mathbb{Z}/2$–graded matrix factorizations.

3.4 Equivalence of definitions

The ideas described above can be used to verify that the definitions of the various KR–homologies given in Section 2 agree with those in [12] and [13]. We assume the reader is already somewhat familiar with these papers, and only briefly recall their content.

To a planar diagram D, we associate the ring $R'(D) = \mathbb{Q}[X_i]$, where i runs over the edges of D. In [12], Khovanov and Rozansky assign to D a complex of matrix factorizations $C_p'(D)$ defined over $R'(D)$ and with potential $w_p(D)$. (Although the definition in [12] is only stated for $p(x) = x^{N+1}$, it works equally well for any p, as implicitly noted by Gornik [4].) $C_p'(D)$ is a tensor product of factors, one for each internal vertex of D. These factors are as follows. To a mark with incoming and outgoing edges labeled i and j, Khovanov and Rozansky associate the short factorization

$$\left\{\frac{p(X_j) - p(X_i)}{X_j - X_i}, X_j - X_i\right\}.$$

To the singular diagram D_s, they associate an order-two Koszul factorization given by the Koszul matrix

$$C_p'(D_s) = \begin{pmatrix} * & X_k + X_l - X_i - X_j \\ * & X_k X_l - X_i X_j \end{pmatrix}.$$

According to the remark following Lemma 3.10, any two such factorizations are isomorphic as $\mathbb{Z}/2$–graded factorizations. Thus, the entries in the left-hand column are more or less immaterial, and we will simply mark them by $*$'s.

Finally, the positive and negative crossings are associated to short complexes of order-two Koszul factorizations, as follows:

$$C_p'(D_+) = \begin{pmatrix} * & X_k + X_l - X_i - X_j \\ * & X_k X_l - X_i X_j \end{pmatrix} \xrightarrow{\chi_1} \begin{pmatrix} * & X_l - X_j \\ * & X_k - X_i \end{pmatrix},$$

$$C_p'(D_-) = \begin{pmatrix} * & X_l - X_j \\ * & X_k - X_i \end{pmatrix} \xrightarrow{\chi_0} \begin{pmatrix} * & X_k + X_l - X_i - X_j \\ * & X_k X_l - X_i X_j \end{pmatrix}.$$

The composition $\chi_0 \chi_1$ is given by multiplication by $X_k - X_j$. Applying a row operation, we see that these complexes are isomorphic to

$$C_p'(D_+) \cong \begin{pmatrix} * & X_k + X_l - X_i - X_j \\ * & X_k X_l - X_i X_j \end{pmatrix} \xrightarrow{\chi_1} \begin{pmatrix} * & X_k + X_l - X_i - X_j \\ * & X_k - X_i \end{pmatrix},$$

$$C_p'(D_-) \cong \begin{pmatrix} * & X_k + X_l - X_i - X_j \\ * & X_k - X_i \end{pmatrix} \xrightarrow{\chi_0} \begin{pmatrix} * & X_k + X_l - X_i - X_j \\ * & X_k X_l - X_i X_j \end{pmatrix}.$$

The matrix factorizations used by Khovanov and Rozansky are $\mathbb{Z}/2$–graded, rather than the $\mathbb{Z}$–graded factorizations that we have been considering. One advantage of this approach is that there is a good notion of homotopy equivalence for such factorizations. This enables them to work in the homotopy category $\mathrm{hmf}_w(R)$ of $\mathbb{Z}/2$–graded matrix factorizations with potential w. There is an obvious forgetful functor from $\mathrm{Kom}(\mathrm{GMF}_w(R))$ to $\mathrm{Kom}(\mathrm{hmf}_w(R))$, so we can view both $C'_p(D)$ and $\widetilde{C}_p(D)$ as objects of the latter category.

Lemma 3.11 *If D is a closed diagram, $C'_p(D) \cong \widetilde{C}_p(D)$ in $\mathrm{Kom}(\mathrm{hmf}_0(R(D)))$.*

Proof In both cases, the complex associated to a disconnected diagram is the tensor product of the complexes associated to its components. Thus we may assume that D is connected. We fix an edge i of D and consider the diagram $D(i)$ obtained by inserting a bivalent vertex v_0 into i. For each vertex v of $D(i)$, the linear relation $\rho(v)$ appears as a matrix entry of every Koszul factorization in the complex $C'_p(D(i))$. By Lemma 2.4, the relations $\{\rho(v) \mid v \neq v_0\}$ are all linearly independent. Thus we can apply [12, Proposition 10] to exclude them. The result is an isomorphic complex C_1 defined over the ring $R'(D_0)/(\rho(v)) \cong R(D)$.

It is shown in [12] that $C'_p(D) \cong C'_p(D(i))$, so to prove the lemma it is enough to show that $C_1 \cong \widetilde{C}_p(D,i)$. To see this, we examine each factor in the complex individually. For example, consider the factor associated to a singular crossing. We have

$$X_k X_l - X_i X_j = -(X_k - X_i)(X_k - X_j)$$

in $R(D)$, so $C'_p(D_s)$ reduces to a short factorization of the form

$$\{\beta, -(X_k - X_i)(X_k - X_j)\},$$

where β is the image of some $\beta' \in R'(D)$ which satisfies

$$\alpha'(X_k + X_l - X_i - X_j) + \beta'(X_k X_l - X_i X_j) = w_p(D_s) = W_p(X_i, X_j, X_k, X_l).$$

It follows that $\beta = -W_p/(X_k - X_i)(X_k - X_j) = p_{ij}$ in $R(D)$. This is the factorization assigned to D_s in Section 2.5.

A similar argument shows that $C'_p(D_+)$ and $C'_p(D_-)$ reduce to complexes of the form

$$\{p_{ij}, -(X_k - X_i)(X_k - X_j)\} \xrightarrow{\chi_1} \{p_i, X_k - X_i\},$$
$$\{p_i, X_k - X_i\} \xrightarrow{\chi_0} \{p_{ij}, -(X_k - X_i)(X_k - X_j)\}.$$

Since the composition $\chi_0 \chi_1$ is given by multiplication by $X_k - X_j$, it is not difficult to see that χ_0 and χ_1 agree with the corresponding maps defined in Section 2.5.

Finally, we consider the short factorization

$$\left\{\frac{p(X_{i_+})-p(X_{i_-})}{X_{i_+}-X_{i_-}}, X_{i_+}-X_{i_-}\right\}$$

coming from the vertex v_0. Since $X_{i_+}=X_{i_-}$ in $R(D)$, this reduces to the short factorization $\{p'(X_i), 0\}$ which appears in the definition of $\widetilde{C}_p(D,i)$. □

Proposition 3.12 *$\overline{H}_N(L,i)$ and $\widetilde{H}_N(L)$ are isomorphic (as doubly graded groups) to the reduced and unreduced $sl(N)$ homology of* [12].

Proof Suppose L is represented by a planar diagram D. The unreduced $sl(N)$ homology of L is defined to be $H(H(C'_p(D), d_{\mathrm{tot}}), d_v^*)$, where $p(x)=x^{N+1}$. From Lemma 3.11 it follows that this is isomorphic to the group $H(H(\widetilde{C}_p(D,i), d_{\mathrm{tot}}), d_v^*)$ which appears in Definition 2.14.

The argument for reduced homology is slightly more involved. In [12], the reduced homology of L with respect to an edge i is defined to be $H(H(C'_p(D), d_{\mathrm{tot}})/X_i, d_v^*)$. Comparing with Definition 2.13, we see that we must show that

$$H(C'_p(D), d_{\mathrm{tot}})/X_i \cong H(C_p(D)/X_i, d_{\mathrm{tot}}).$$

The complex $C'_p(D)$ is free over $\mathbb{Q}[X_i]$, but it is shown in [12] that $H(C'_p(D), d_{\mathrm{tot}})$ is a torsion module over $\mathbb{Q}[X_i]$. Applying the universal coefficient theorem, we see that

$$H(C'_p(D)/X_i, d_{\mathrm{tot}}) \cong H(C'_p(D), d_{\mathrm{tot}})/X_i \otimes H^*(S^1).$$

On the other hand, Lemma 3.11 tells us that the quotient $C'_p(D)/X_i$ is homotopy equivalent to

$$\widetilde{C}_p(D,i)/X_i \cong C_p(D)/X_i \otimes U_p(i)/X_i.$$

$U_p(i)/X_i$ is a rank-two factorization with trivial differentials, so

$$H(\widetilde{C}'_p(D)/X_i, d_{\mathrm{tot}}) \cong H(C_p(D)/X_i, d_{\mathrm{tot}}) \otimes H^*(S^1).$$

Canceling out the extra factors of $H^*(S^1)$, we obtain the desired isomorphism.

It remains to check that the bigradings agree. For the second (homological) grading, this is clearly the case; it is given by gr_v in both cases. To see that $\mathrm{gr}_N = i+(N-1)j/2$ coincides with the q–grading of [12], first note that the complex $C'_p(D)$ is set up so that the right-hand group in each linear factor is unshifted with respect to the q–grading. If we exclude the linear term appearing in such a factor, the q–grading is unaffected. Thus it suffices to check that the gradings agree on quadratic factors. Consider the factorization $C_p(D_s)$ associated to a singular point. According to Section 2.5, the two copies of $R(D)$ used to define this factorization have (i,j) grading shifts of $\{1,-2\}$

and $\{-1,0\}$. These correspond to shifts of $\{2-n\}$ and $\{-1\}$ in gr_N, which precisely match the shifts in the q–grading which appear in the definition of $C'_p(D_s)$ in [12, p. 48]. The calculation for $C'_p(D_\pm)$ is similar, except it also uses the grading shifts in [12, p. 81], part of which goes into the shifts in $C_p(D_\pm)$, and part into the overall shift by $w(N-1)$ which appears in the definition of $\widetilde{H}_N(L)$ and $\overline{H}_N(L,i)$. □

Proposition 3.13 *The middle HOMFLY homology $H(L)$ is isomorphic to the HOMFLY homology of* [13]*. The identification is such that an element with grading (i,j,k) in our notation corresponds to an element with grading $(j/2, i-j/2, k/2)$ in the notation of* [13].

Proof The homology of [13] is defined to be $H(H(C'_a(D), d_{\mathrm{tot}}), d_v^*)$, where $C'_a(D)$ is a certain complex of matrix factorizations defined over the ring $R'(D)[a]$. If we substitute $a=0$, $C'_a(D)$ reduces to $C'_0(D)$. On the other hand, it is proved in [13] that a acts by 0 on $H(C_a(D), d_{\mathrm{tot}})$. Applying the universal coefficient theorem, we find that

$$H(C'_0(D), d_{\mathrm{tot}}) \cong H(C'_a(D), d_{\mathrm{tot}}) \otimes H^*(S^1).$$

On the other hand, Lemma 3.11 implies that

$$H(C'_0(D), d_{\mathrm{tot}}) \cong H(\widetilde{C}_0(D), d_{\mathrm{tot}}) \cong H(C_0(D) \otimes U_0(i), d_{\mathrm{tot}}).$$

Since $U_0(i)$ has trivial differential, the last group is isomorphic to $H(C_0(D), d_{\mathrm{tot}}) \otimes H^*(S^1)$. Canceling the factors of $H^*(S^1)$, we see that

$$H(H(C'_a(D), d_{\mathrm{tot}}), d_v^*) \cong H(H(C_0(D), d_{\mathrm{tot}}), d_v^*) \cong H(L).$$

It remains to compare the triple grading on the two theories. The ring $R'(D)[a]$ is bigraded, with an additional grading corresponding to the power of a as well as the usual q–grading. The first grading in [13] is nominally given by the power of a. Since a acts by 0 on homology, however, any class is homologous to one represented by elements of $R'(D)$. The a–grading of such a class comes entirely from the grading shifts introduced in the definition of $C'_a(D)$. It is easily verified that these shifts are the same as those for gr_h, so the first grading is $\mathrm{gr}_h = j/2$. The second grading in [13] corresponds to the usual q–grading on the ring $R(D)$, but the grading shifts in $C'_a(D)$ differ from ours. Up to an overall shift, the grading shift in [13] corresponds to the difference between our shift in q and gr_h. Thus, the second grading is given by $i-j/2$. Finally, the third grading in [13] is given by $\mathrm{gr}_v = k/2$. □

As a further application of these techniques, we can now make good on our claim that the unreduced complex is independent of the choice of the marked edge used to define it.

Proposition 3.14 *If i and j are two edges of a connected diagram D, the unreduced complexes $\widetilde{C}_p(D,i)$ and $\widetilde{C}_p(D,j)$ are quasi-isomorphic.*

Proof Let $D(i,j)$ be the diagram obtained by inserting bivalent vertices v_i and v_j in edges i and j. Consider the complex $C'_p(D(i,j))$ as an element of the category $\mathrm{GMF}_0(R(D(i,j))$. Arguing as in the proof of Lemma 3.11, we use Lemma 3.8 to exclude the linear relations $\{\rho(v) \mid v \neq v_i\}$. The result is a new complex of matrix factorizations C_i which is quasi-isomorphic to $C'_p(D(i,j))$. The same argument as in the proof of Lemma 3.11 shows that $C_i \cong \widetilde{C}_p(D,i)$. Thus $C'_p(D(i,j))$ is quasi-isomorphic to $\widetilde{C}_p(D,i)$. Similarly, $C'_p(D(i,j))$ is quasi-isomorphic to $\widetilde{C}_p(D,j)$. This proves the claim. □

4 Braid graphs and MOY relations

A tangle diagram all of whose crossings are singular is called a *graph*; a braid diagram all of whose crossings are singular is a *braid graph.* In [19], Murakami, Ohtsuhki and Yamada explain how to assign a HOMFLY polynomial $\widetilde{P}(D)$ to a closed graph D. This assignment can be used to give a state model definition of the HOMFLY polynomial similar to the Kauffman state model [8] for the Jones polynomial. (See [7; 23] for related constructions.) Murakami, Ohtsuki and Yamada also show that the HOMFLY polynomial of a graph satisfies certain relations, which we refer to as *MOY relations*.

In this section, we briefly review these results and describe their generalizations to KR–homology. In [12; 13], Khovanov and Rozansky show that $C_p(D)$ satisfies relations analogous to the MOY relations for the HOMFLY polynomial. The main technical result of this section is that these relations continue to hold in the context of $\mathbb{Z}$–graded matrix factorizations. As an application, we show that the HOMFLY homology of a braid graph is determined by its HOMFLY polynomial. We then use the MOY state model to give a proof of Theorem 2.11 along the lines of the proof for the $sl(N)$ homology given in [12].

4.1 The MOY state model

We begin by recalling the state model of Murakami, Ohtsuki and Yamada [19]. Although their paper is phrased in terms of the $sl(N)$ polynomials, the results we want are easily translated into the language of the HOMFLY polynomial, and we will state them in this form.

Suppose D is a diagram representing an oriented link L. We can "resolve" each crossing of D in one of two ways: either into a pair of arcs (the oriented resolution) or into a singular crossing. To each such resolution, we assign a weight $\mu \in \mathbb{Z}$ depending on whether the crossing is positive or negative and on which resolution it receives. The possible resolutions and their weights are illustrated in Figure 2.

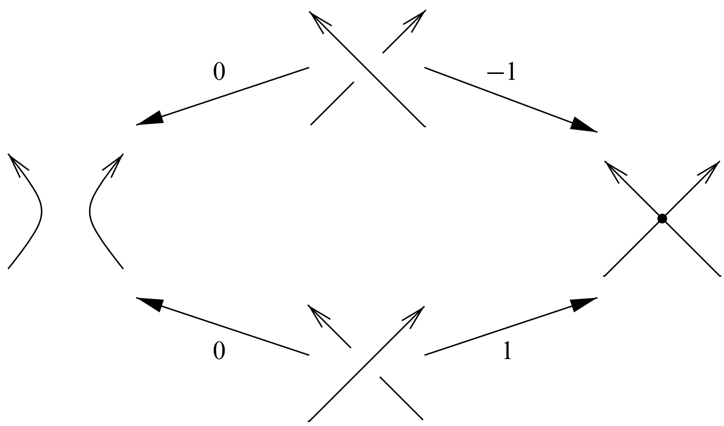


Figure 2: Resolutions and their weights

A *state* of the diagram D is a choice of resolution for each crossing of D. If D has n crossings, it will have 2^n different states. To a state σ we assign a weight $\mu(\sigma)$ given by the sum of the local weights at each crossing. In addition, each σ gives rise to a graph D_σ. In [19] it is shown that the unnormalized HOMFLY polynomial of L is given by the formula

$$\widetilde{P}(L) = (aq^{-1})^{w(D)} \sum_{\sigma} (-q)^{\mu(\sigma)} \widetilde{P}(D_\sigma), \tag{1}$$

where the quantity $\widetilde{P}(D_\sigma)$ is an invariant of the graph D_σ. We think of $\widetilde{P}(D_\sigma)$ as the HOMFLY polynomial of D_σ, and view formula (1) as generalizing the definition of $\widetilde{P}$ to closed tangle diagrams with an arbitrary number of singular crossings.

4.2 Polynomials of braid graphs

In order to use formula (1), we need some way to determine $\widetilde{P}(D)$ when D is a graph. In [19], the authors give a direct geometric procedure for finding these polynomials, or rather, their specializations to $a = q^N$. For our purposes, however, it is more convenient to characterize $\widetilde{P}(D)$ in terms of certain relations given in [19].

Suppose D_{O}, D_{I}, D_{II}, D_{IIIa} and D_{IIIb} are braid graphs containing regions like those shown on the left-hand sides of Figures 3 and 4, and let D'_{O}, D'_{I}, D'_{II}, D'_{IIIa} and D'_{IIIb} be the graphs obtained by replacing these regions with the corresponding ones on the right-hand sides of the figures. It is shown in [19] that $\widetilde{P}$ satisfies the following *MOY relations*:

$$\begin{aligned}
&\text{(O)} \quad \widetilde{P}(D_{\mathrm{O}}) = \frac{a-a^{-1}}{q-q^{-1}}\widetilde{P}(D'_{\mathrm{O}}),\\
&\text{(I)} \quad \widetilde{P}(D_{\mathrm{I}}) = \frac{aq^{-1}-a^{-1}q}{q-q^{-1}}\widetilde{P}(D'_{\mathrm{I}}),\\
&\text{(II)} \quad \widetilde{P}(D_{\mathrm{II}}) = (q+q^{-1})\widetilde{P}(D'_{\mathrm{II}}),\\
&\text{(III)} \quad \widetilde{P}(D_{\mathrm{IIIa}}) + \widetilde{P}(D_{\mathrm{IIIb}}) = \widetilde{P}(D'_{\mathrm{IIIa}}) + \widetilde{P}(D'_{\mathrm{IIIb}}).
\end{aligned}$$

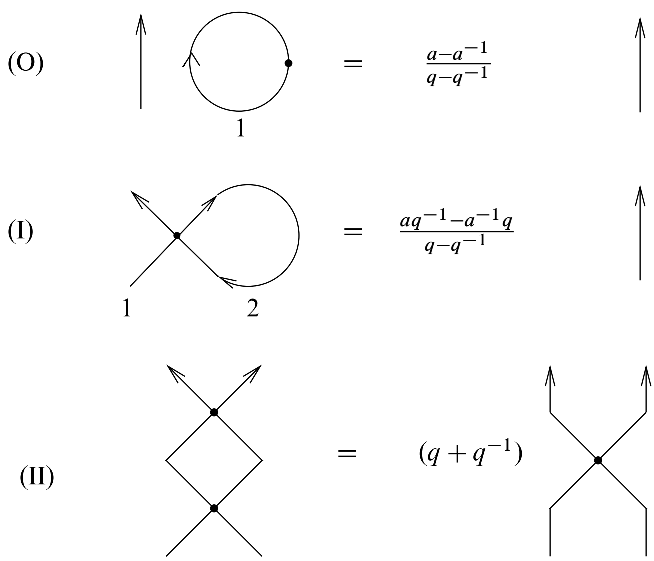


Figure 3: MOY relations O, I, and II

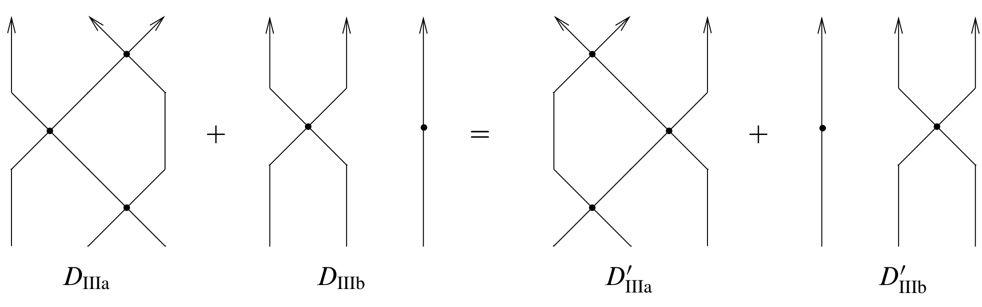


Figure 4: MOY relation III

The HOMFLY polynomial of a braid graph is completely determined by these relations. To see this, we use an induction scheme introduced by Wu [25]. Suppose D is a braid graph on b strands. The crossings of D are naturally arranged into $b-1$ columns, which we number $1,\dots,b-1$ going from left to right. If c is a crossing of D, let $i(c)$ be the number of the column containing it. Following Wu, we define the *complexity* of D to be the sum

$$i(D) = b + \sum_c i(c).$$

The complexity of a diagram on the left-hand side of Figure 3 is strictly greater than the complexity of the corresponding diagram on the right. Similarly, the complexity of diagram D_{IIIa} is greater than that of the other three diagrams in Figure 4.

Lemma 4.1 [25] *Suppose D is a nonempty braid graph which is the closure of an open braid graph D_o. Then either D contains a region of the form D_{O} or D_{I}, or D_o contains a region of the form D_{II} or D_{IIIa}.*

In other words, D can be related to braid graphs of lesser complexity by one of MOY moves O–III. Moreover, we may assume that moves of type II and III take place in the open braid D_o.

Corollary 4.2 *If D is a braid graph, $\widetilde{P}(D)$ is determined by MOY relations* O–III *and the fact that $\widetilde{P}$ of the empty graph is* 1*.*

4.3 Homology of braid graphs

The KR–complex of a braid graph satisfies decomposition rules analogous to the MOY relations O–III. In the context of $\mathbb{Z}/2$–graded matrix factorizations, such rules were introduced in [12] and later applied to the HOMFLY homology in [13; 25]. Similar MOY decompositions also hold in the derived category of $\mathbb{Z}$–graded matrix factorizations. We collect their statements here, but postpone the proofs to the end of this section. Note that, although $\widetilde{C}_p(D)$ is generally a complex of matrix factorizations, when D is a graph, the complex is supported in a single vertical grading. Thus $\widetilde{C}_P(D)$ is most naturally viewed as an object of the $\mathrm{GMF}(R(D))$. It is doubly graded, with gradings $(q, 2\,\mathrm{gr}_h)$.

Proposition 4.3 *Let D_{O} and D'_{O} be two braid graphs related as in the first line of Figure 3, and let $C_p(\mathrm{O})$ be the matrix factorization*

$$\mathbb{Q}[X_1]\{0,-2\} \underset{p'(X_1)}{\overset{0}{\rightleftarrows}} \mathbb{Q}[X_1]\{0,0\}.$$

Then $C_p(D_{\mathrm{O}}) \cong C_p(D'_{\mathrm{O}}) \otimes_{\mathbb{Q}} C_p(\mathrm{O})$ in $\mathrm{GMF}(R(D_{\mathrm{O}}))$.

Proposition 4.4 *Let D_{I} and D'_{I} be two braid graphs related as in the second line of Figure 3, and let $C_p(\mathrm{I})$ be the matrix factorization*

$$\mathbb{Q}[X_1, X_2]\{1,-2\} \underset{p'_{12}}{\overset{0}{\rightleftarrows}} \mathbb{Q}[X_1, X_2]\{-1,0\},$$

where $p'_{12} = (p'(X_1) - p'(X_2))/(X_1 - X_2)$. Then $C_p(D_{\mathrm{I}}) \cong C_p(D'_{\mathrm{I}}) \otimes_{\mathbb{Q}[X_1]} C_p(\mathrm{I})$ in $\mathrm{GMF}(R(D_{\mathrm{I}}))$.

Proposition 4.5 *Let D_{II} and D'_{II} be two braid graphs formed by taking the union of a fixed graph D with the diagrams in the last line of Figure 3. Then*

$$C_p(D_{\mathrm{II}}) \sim C_p(D'_{\mathrm{II}})\{-1,0\} \oplus C_p(D'_{\mathrm{II}})\{1,0\}$$

in $\mathrm{GMF}(R(D))$.

Proposition 4.6 *Let D_{IIIa}, D_{IIIb}, D'_{IIIa}, and D'_{IIIb} be braid graphs formed by taking the union of a fixed graph D with the diagrams in Figure 4. Then*

$$C_p(D_{\mathrm{IIIa}}) \oplus C_p(D_{\mathrm{IIIb}}) \sim C_p(D'_{\mathrm{IIIa}}) \oplus C_p(D'_{\mathrm{IIIb}})$$

in the category $\mathrm{GMF}(R(D))$.

As an immediate consequence, we have relations

$$\begin{aligned}
&(\mathrm{O}) \quad \widetilde{H}(D_{\mathrm{O}}) \cong (\widetilde{H}(D'_{\mathrm{O}})\{0,-2\} \oplus \widetilde{H}(D'_{\mathrm{O}})) \otimes_{\mathbb{Q}} \mathbb{Q}[x],\\
&(\mathrm{I}) \quad \widetilde{H}(D_{\mathrm{I}}) \cong (\widetilde{H}(D'_{\mathrm{I}})\{1,-2\} \oplus \widetilde{H}(D'_{\mathrm{I}})\{-1,0\}) \otimes_{\mathbb{Q}} \mathbb{Q}[x],\\
&(\mathrm{II}) \quad \widetilde{H}(D_{\mathrm{II}}) \cong \widetilde{H}(D'_{\mathrm{II}})\{-1,0\} \oplus \widetilde{H}(D'_{\mathrm{II}})\{1,0\},\\
&(\mathrm{III}) \quad \widetilde{H}(D_{\mathrm{IIIa}}) \oplus \widetilde{H}(D_{\mathrm{IIIb}}) \cong \widetilde{H}(D'_{\mathrm{IIIa}}) \oplus \widetilde{H}(D'_{\mathrm{IIIb}}),
\end{aligned}$$

which closely parallel the MOY relations for the HOMFLY polynomial. In fact, these relations are proved by Khovanov and Rozansky in [13], where they are used to show that $\widetilde{H}$ is invariant under braidlike Reidemeister moves.

Like the HOMFLY polynomial, the HOMFLY homology of a braid graph is determined by the MOY relations. In fact, the two carry precisely the same information. More specifically, let

$$\widetilde{\mathcal{P}}(D) = \sum_{i,j} (-1)^{j/2} a^j q^i \dim \widetilde{H}^{i,j}(D)$$

be the signed Poincaré polynomial of $\widetilde{H}(D)$. Then we have:

Proposition 4.7 *If D is a closed braid graph on b strands, $\widetilde{\mathcal{P}}(D) = (-aq)^{-b}\, \widetilde{P}(D)$.*

Proof We induct on the complexity of D. The base case is the empty diagram, which has complexity 0, HOMFLY polynomial 1, and KR–homology $\mathbb{Q}$ supported in bigrading $(0,0)$. For the induction step, we apply Lemma 4.1 to see that D is related to diagrams of lesser complexity by an MOY move. To complete the proof, we need only check that the MOY relations for $\widetilde{P}$ are consistent with the corresponding MOY decompositions for $\widetilde{H}$.

For example, consider MOY move O. By the induction hypothesis, we know that $\widetilde{\mathcal{P}}(D'_{\mathrm{O}}) = (-aq)^{1-b}\widetilde{P}(D'_{\mathrm{O}})$. On the other hand, relation O above shows that

$$\begin{aligned}\widetilde{\mathcal{P}}(D_{\mathrm{O}}) &= (1-a^{-2})\Big(\sum_{i=0}^{\infty} q^i\Big)\widetilde{\mathcal{P}}(D'_{\mathrm{O}}) = (-aq)^{-1}\Big(\frac{a-a^{-1}}{q-q^{-1}}\Big)\widetilde{\mathcal{P}}(D'_{\mathrm{O}})\\ &= (-aq)^{-b}\Big(\frac{a-a^{-1}}{q-q^{-1}}\Big)\widetilde{P}(D'_{\mathrm{O}}) = (-aq)^{-b}\widetilde{P}(D_{\mathrm{O}}),\end{aligned}$$

so the claim holds for D_{O} as well.

We leave it to the reader to check the remaining MOY moves. The argument for move I is very similar to the one for move O, and moves II and III are even easier, since all diagrams involved have the same number of strands. □

Using the MOY relations it is not difficult to see that if D is a braid graph on b strands, the denominator of $\widetilde{P}(D)$ is $(q-q^{-1})^b$. This fact is nicely reflected in the module structure of $\widetilde{H}(D)$. To see this, write D as the closure of an open braid graph D_o, and label the outgoing edges of D_o by $1,2,\dots,b$. (In D, these are identified with the incoming edges of D_o.) The ring $R_b = \mathbb{Q}[X_1, X_2, \dots, X_b]$ is a subring of $R(D)$.

Proposition 4.8 *If D is a closed braid graph on b strands, $\widetilde{H}(D)$ is a free module of finite rank over R_b.*

Proof Again, we induct on the complexity of D. The base case is when D is the empty diagram, and $\widetilde{H}(D) \cong \mathbb{Q}$ is free of rank 1 over $\mathbb{Q}$. For the induction step, we use Lemma 4.1 to see that D can be simplified either by a MOY O or I move, or by a MOY II or III move which takes place in the open braid D_o. We consider each of these four possibilities separately.

For move O, it follows from Proposition 4.3 that $\widetilde{H}(D_{\mathrm{O}})$ is a direct sum of two copies of $\widetilde{H}(D'_{\mathrm{O}})$ tensored over $\mathbb{Q}$ with $\mathbb{Q}[X_k]$, where the strand to be eliminated has label k. By the induction hypothesis, $\widetilde{H}(D'_{\mathrm{O}})$ is a free module of finite rank over $\mathbb{Q}[X_1,\dots,X_{k-1},X_{k+1},\dots,X_b]$, so $C_p(D_{\mathrm{O}})$ will be free of finite rank over R_b. The argument for move I is similar.

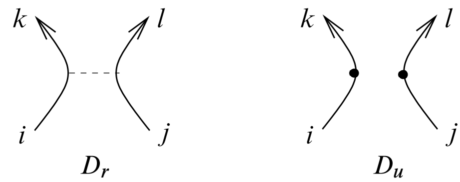


Figure 5: Left: the diagram D_r, which represents a four-valent vertex. Right: the diagram D_u, which represents a pair of two-valent vertices.

For moves of type II and III, the fact that the move takes place in D_o implies that R_b is contained in the ring $R(D)$ over which the relations of Propositions 4.5 and 4.6 hold. Thus these decompositions also hold over R_b. The result for move II follows easily from this, since $\widetilde{H}(D_{\mathrm{II}})$ is a direct sum of two copies of $\widetilde{H}(D'_{\mathrm{II}})$, which is free of finite rank by the induction hypothesis.

For move III, the induction hypothesis implies that $\widetilde{H}(D'_{\mathrm{IIIa}})$ and $\widetilde{H}(D'_{\mathrm{IIIb}})$ are free. It follows that $\widetilde{H}(D_{\mathrm{IIIa}}) \oplus \widetilde{H}(D_{\mathrm{IIIb}})$ is free as well, so $\widetilde{H}(D_{\mathrm{IIIa}})$ is a projective module over the polynomial ring R_b. By the theorem of Quillen and Suslin (see eg [14]), any such module is free. Finally, $\widetilde{H}(D'_{\mathrm{IIIa}})$ and $\widetilde{H}(D'_{\mathrm{IIIb}})$ are of finite rank, so the same must be true for $\widetilde{H}(D_{\mathrm{IIIa}})$. □

4.4 States and the KR–complex

Now that we understand the relation between the MOY state model and $\widetilde{C}_p(D)$ for braid graphs, we consider what happens when D is an arbitrary braid.

Lemma 4.9 *Suppose D is a closed braid diagram. Then*

$$\widetilde{H}^+(D) \cong \bigoplus_{\sigma} \widetilde{H}(D_\sigma)\{\mu(\sigma), 0, -2\mu(\sigma)\},$$

where the sum runs over MOY states of D.

Proof We temporarily enlarge our notion of a tangle diagram to include a fourth sort of crossing D_r, represented by the diagram of Figure 5. The local factor associated to such a crossing is

$$C_p(D_r) = \quad R\{0,-2,0\} \underset{p_i}{\overset{(X_k - X_i)}{\rightleftarrows}} R\{0,0,0\}.$$

The definition of the KR–complex is otherwise unchanged.

Referring to the diagrams in Section 2.5, we see that if we ignore the vertical differential, there are decompositions

$$C_p(D_+) = C_p(D_s)\{1,0,-2\} \oplus C_p(D_r),$$
$$C_p(D_-) = C_p(D_r) \oplus C_p(D_s)\{-1,0,2\}.$$

$C_p(D)$ is a tensor product of factors, one for each crossing of D. If we ignore d_v, then $\widetilde{C}_p(D)$ will split into a direct sum of 2^n summands, where n is the number of ordinary crossings of D. By assigning the summand $C_p(D_r)$ to the oriented resolution of a crossing and $C_p(D_s)$ to its singular resolution, we get a bijection between summands and MOY states of D. Comparing the grading shifts with the weights in Figure 2, we see that

$$\widetilde{C}_p(D) \cong \bigoplus_{\sigma} \widetilde{C}_p(D(\sigma))\{\mu(\sigma), 0, -2\mu(\sigma)\},$$

where the diagram $D(\sigma)$ is obtained by replacing each ordinary crossing of D with either D_s or D_r, depending on σ. Note that this is not quite the same the diagram D_σ, which is obtained by replacing each ordinary crossing with either D_s or the oriented resolution D_u. To remedy this discrepancy we use the following lemma, whose proof is given in the next section.

Lemma 4.10 *Suppose D is a closed tangle diagram containing a crossing of type D_r, and let D' be the diagram obtained by replacing this crossing by a pair of marks, as illustrated by the diagram D_u in Figure 5. Then $\widetilde{C}_p(D)$ is quasi-isomorphic to $\widetilde{C}_p(D')$ over $R(D')$.*

Applying this lemma repeatedly, we see that $\widetilde{C}_p(D(\sigma))$ is quasi-isomorphic to $\widetilde{C}_p(D_\sigma)$. Thus $\widetilde{H}^+(D(\sigma)) \cong \widetilde{H}^+(D_\sigma)$, and the claim is proved. □

The similarity between the HOMFLY homology and the original Khovanov homology [9] is now evident. Like the chain complex used to define the Khovanov homology, the summands of $\widetilde{H}_p^+(D)$ naturally lie at the vertices of the "cube of resolutions" of D, each of whose vertices corresponds to a MOY state. The components of the induced differential d_v^* correspond to edges of the cube. This analogy can be used to give an alternate proof of Theorem 2.11, which is easily seen to be equivalent to the statement below.

Proposition 4.11 $\widetilde{P}(L) = \sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \widetilde{H}^{i,j,k}(L).$

The argument is similar to the proof that the Euler characteristic of the Khovanov homology is the Jones polynomial, but with the MOY state model in place of the Kauffman state model.

Proof Recall that if D is a closed braid diagram representing L,

$$\widetilde{H}(L) = H(\widetilde{H}^+(D), d_v^*)\{-w+b, w+b, w-b\}.$$

Since d_v^* preserves both q and gr_h, the graded Euler characteristic

$$\chi(\widetilde{H}(L)) = \sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \widetilde{H}^{i,j,k}(L)$$

can be computed from $\widetilde{H}^+(D)$. We find

$$\begin{aligned}\chi(\widetilde{H}(L)) &= (-1)^{-b} a^{w+b} q^{-w+b} \sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \widetilde{H}_p^{+\ i,j,k}(D) \\ &= (aq^{-1})^w \sum_{\sigma} (-q)^{\mu(\sigma)} (-aq)^b \sum_{i,j} (-1)^{j/2} a^j q^i \dim \widetilde{H}^{i,j}(D_\sigma) \\ &= (aq^{-1})^w \sum_{\sigma} (-q)^{\mu(\sigma)} \widetilde{P}(D_\sigma) = \widetilde{P}(L).\end{aligned}$$

□

4.5 MOY decompositions

We now prove the various technical results used throughout the section. We begin with the proof of Lemma 4.10, which asserted that a "crossing" of type D_r was equivalent to its oriented resolution.

Proof of Lemma 4.10 Either D and D' have the same number of connected components, or D' has one more component than D. Suppose we are in the first case. Then X_i and X_k are independent linear elements of the polynomial ring $R(D)$, and we can use Lemma 3.8 to exclude the linear factor $X_k - X_i$ appearing in $\widetilde{C}_p(D_r)$. We obtain a quasi-isomorphic complex C' defined over the ring $R(D)/(X_k - X_i)$. The ideals generated by $(X_k - X_i, X_l - X_j)$ and $(X_k - X_i, X_k + X_l - X_i - X_j)$ are clearly equal, so $R(D)/(X_k - X_i) \cong R(D')$. Then C' and $\widetilde{C}_p(D')$ are Koszul factorizations over $R(D')$ with the same Koszul matrices, so $C' \cong \widetilde{C}_p(D)$.

Now suppose that replacing D_r with D_u increases the number of components in D. In this case, $X_i = X_k$ and $X_j = X_l$ in $R(D)$, so $R(D) \cong R(D')$. We compute

$$\begin{aligned} p_i &= \frac{p(X_k) + p(X_l) - p(X_i) - p(X_j)}{X_k - X_i} \\ &= \frac{p(X_k) - p(X_i)}{X_k - X_i} + \frac{p(X_j + X_i - X_k) - p(X_j)}{X_k - X_i} \\ &= p'(X_i) - p'(X_j), \end{aligned}$$

so $C_p(D_r)$ is given by the short factorization $\{p'(X_i) - p'(X_j), 0\}$.

Recall that $\widetilde{C}_p(D)$ is obtained by tensoring $C_p(D)$ with short factorizations of the form $\{p'(X_n), 0\}$, where we pick one edge n for each component of D. By Proposition 3.14, we may assume that the component containing D_r has marked edge i. Thus $\widetilde{C}_p(D)$ has short factors $\{p'(X_i), 0\}$ and $\{p'(X_i) - p'(X_j), 0\}$. Applying a Koszul row operation, we see that this is isomorphic to a factorization with short factors $\{p'(X_i), 0\}$ and $\{p'(X_j), 0\}$. If we choose j as the marked edge on the new component, this is the factorization for $\widetilde{C}_p(D')$. □

Next, we take up the task of proving the MOY decompositions stated in Propositions 4.3–4.6. In each case, the argument follows the proofs of the corresponding results in [12; 13], although some additional care is required for the MOY III move. The proof for the MOY O move is easiest.

Proof of Proposition 4.3 The diagram D'_{O} is obtained from D_{O} by deleting a small loop consisting of a single edge, labeled 1, attached at both ends to a single mark. The relation $\rho(v)$ associated to this mark is 0, so $R(D_{\mathrm{O}}) \cong R(D'_{\mathrm{O}}) \otimes_{\mathbb{Q}} \mathbb{Q}[X_1]$. Both diagrams have the same set of crossings, so the only difference between $\widetilde{C}_p(D_{\mathrm{O}})$ and $\widetilde{C}_p(D'_{\mathrm{O}})$ comes from the factor associated to the deleted component. This is precisely the factorization $C_p(\mathrm{O})$ from the statement of the proposition. □

The argument for the MOY I move is not much harder.

Proof of Proposition 4.4 We start by considering the case when D_{I} is the open diagram shown in Figure 3. Then

$$R(D_{\mathrm{I}}) \cong \mathbb{Q}[X_1, X_2, X_3]/(X_3 + X_2 - X_2 - X_1) \cong \mathbb{Q}[X_1, X_2],$$

while $R(D'_{\mathrm{I}}) = \mathbb{Q}[X_1]$, so $R(D_{\mathrm{I}}) \cong R(D'_{\mathrm{I}}) \otimes \mathbb{Q}[X_2]$. The diagram D'_{I} has no crossings, so $C_p(D'_{\mathrm{I}}) = R(D'_{\mathrm{I}})$, while $C_p(D_{\mathrm{I}})$ is the short factorization

$$R\{1, -2, 0\} \underset{p_{12}}{\overset{(X_3 - X_1)(X_3 - X_2)}{\rightleftarrows}} R\{-1, 0, 0\},$$

where $R = R(D_{\mathrm{I}})$. Since $X_1 = X_3$ in $R(D_1)$, the entry on the upper arrow is 0. To compute p_{12}, we first take the quotient

$$\frac{p(X_3) + p(X_4) - p(X_2) - p(X_1)}{(X_3 - X_1)(X_3 - X_2)}$$

in the ring $\mathbb{Q}[X_1, X_2, X_3, X_4]/(X_3 + X_4 - X_1 - X_2)$ and then set $X_2 = X_4$. In other words,

$$p_{12} = \frac{1}{X_1 - X_4}\left[\frac{p(X_1 + X_2 - X_4) - p(X_1)}{X_2 - X_4} + \frac{p(X_4) - p(X_2)}{X_2 - X_4}\right]\Bigg|_{X_2 = X_4},$$

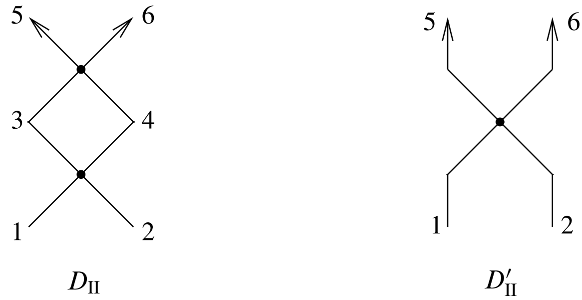


Figure 6: Diagrams for the MOY II move

which reduces to $(p'(X_1) - p'(X_2))/(X_1 - X_2)$. Thus $C_p(D_{\mathrm{I}})$ is exactly the factorization $C_p(\mathrm{I})$ described in the statement of the proposition, and

$$C_p(D_{\mathrm{I}}) \cong C_p(D'_{\mathrm{I}}) \otimes_{\mathbb{Q}[X_1]} C_p(\mathrm{I}).$$

More generally, suppose that $\overline{D}_{\mathrm{I}}$ and $\overline{D}'_{\mathrm{I}}$ are formed by gluing a fixed graph D to D_{I} and D'_{I}. Then Lemma 2.8 tells us that $\widetilde{C}_p(\overline{D}'_{\mathrm{I}}) \cong \widetilde{C}_p(D) \otimes_{\mathbb{Q}[X_1]} C_p(D'_{\mathrm{I}})$, so

$$\begin{aligned}\widetilde{C}_p(\overline{D}_{\mathrm{I}}) &\cong \widetilde{C}_p(D) \otimes_{\mathbb{Q}[X_1]} C_p(D_{\mathrm{I}})\\ &\cong \widetilde{C}_p(D) \otimes_{\mathbb{Q}[X_1]} C_p(D'_{\mathrm{I}}) \otimes_{\mathbb{Q}[X_1]} C_p(\mathrm{I})\\ &\cong \widetilde{C}_p(\overline{D}'_{\mathrm{I}}) \otimes C_p(\mathrm{I}),\end{aligned}$$

and the general case follows from the local one. □

The proof of the MOY II relation follows its counterpart in [13] almost verbatim.

Proof of Proposition 4.5 As before, we start by assuming that D_{II} and D'_{II} are the open graphs shown in Figure 6. We label their edges as shown in the figure. $C_p(D_{\mathrm{II}})$ is an order-two Koszul factorization over the ring

$$R = \mathbb{Q}[X_1, \dots, X_6]/(X_5 + X_6 - X_3 - X_4, X_3 + X_4 - X_1 - X_2) \cong R_0[X_3],$$

where $R_0 = \mathbb{Q}[X_1, X_2, X_5, X_6]/(X_5 + X_6 - X_1 - X_2) \cong \mathbb{Q}[X_1, X_2, X_5]$ is isomorphic to both $R_e(D_{\mathrm{II}})$ and $R_e(D'_{\mathrm{II}})$. It is given by a Koszul matrix of the form

$$\begin{pmatrix} * & -(X_3 - X_1)(X_3 - X_2) \\ * & (X_3 - X_5)(X_3 - X_6) \end{pmatrix}.$$

We use the entry $-(X_3 - X_1)(X_3 - X_2)$ in the first row to exclude the internal variable X_3. The result is a new factorization C_1 which is quasi-isomorphic to $C_p(D_{\mathrm{II}})$ over the ring R_0. C_1 is an order-one Koszul factorization defined over the ring

$$R_1 = R/(X_3^2 - (X_1 + X_2)X_3 + X_1 X_2).$$

We can write $C_1 = \{P, Q\}$, where Q is obtained by substituting

$$X_3^2 = (X_1 + X_2)X_3 - X_1X_2$$

into the lower right entry of the factorization above. We have

$$\begin{aligned} Q &= (X_1 + X_2)X_3 - X_1X_2 - (X_5 + X_6)X_3 + X_5X_6 \\ &= -X_1X_2 + X_5X_6 = -(X_5 - X_2)(X_5 - X_1). \end{aligned}$$

Thus, although Q is *a priori* an element of R_1, we find that actually $Q \in R_0$. Since the product $PQ = w_p(D_{\mathrm{II}})$ is also contained in R_0, $P \in R_0$ as well. Thus $C_1 \cong C_2 \otimes_{R_0} R_1$, where C_2 is the short factorization over R_0 defined by the pair $\{P, Q\}$. In other words, $C_2 = C_p(D'_{\mathrm{II}})$. Viewed as a module over R_0, we have $R_1 \cong R_0 \oplus X_3 R_0$, so, over R_0,

$$C_1 \cong C_2 \oplus X_3C_2 = C_p(D'_{\mathrm{II}}) \oplus X_3C_p(D'_{\mathrm{II}}).$$

Next, we check the grading shifts of the two summands. $C_p(D_{\mathrm{II}})$ is a direct sum of four copies of R, with grading shifts $\{-2,0\}, \{0,-2\}, \{0,-2\}$ and $\{2,-4\}$. When we exclude X_3 to get C_1, we are left with two copies of R_1, with grading shifts $\{-2,0\}$ and $\{0,-2\}$. Since

$$R_1 \cong R_0 \oplus X_3 R_0 = R_0 \oplus R_0\{2,0\},$$

C_1 is a direct sum of four copies of R_0, with grading shifts $\{-2,0\}, \{0,0\}, \{0,-2\}$ and $\{2,-2\}$. On the other hand, $C_p(D'_{\mathrm{II}})$ is a direct sum of two copies of R_0 with grading shifts $\{-1,0\}$ and $\{1,-2\}$. Thus C_1 must decompose as $C_p(D'_{\mathrm{II}})\{-1,0\} \oplus C_p(D'_{\mathrm{II}})\{1,0\}$.

Finally, we consider the general situation, in which D_{II} and D'_{II} are formed by attaching the diagrams shown in the figure to an arbitrary graph D. In this case the result follows from the special case considered above, the local nature of the KR–complex (Lemma 2.8), and the fact that if $A \sim B$ over R, then $A \otimes_R C \sim B \otimes_R C$. □

Lastly, we turn to the MOY III move. As usual, it suffices to prove the statement of Proposition 4.6 for the graphs shown in Figure 4 and then appeal to the local nature of the KR–complex to show that it holds in general. We number the edges of the diagram D_{IIIa} as shown in Figure 7, and label the external edges of D_{IIIb}, D'_{IIIa}, and D'_{IIIb} to match. All four diagrams share the same potential

$$W = p(X_4) + p(X_5) + p(X_6) - p(X_1) - p(X_2) - p(X_3)$$

and the same exterior ring

$$R_0 = \mathbb{Q}[X_1, \dots X_6]/(X_4 + X_5 + X_6 - X_1 - X_2 - X_3).$$

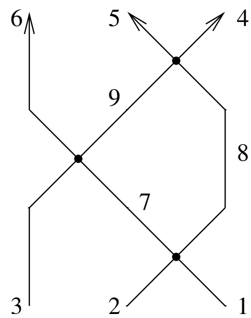


Figure 7: The diagram D_{IIIa}

In [12; 13], Khovanov and Rozansky introduce an additional factorization Υ defined over R_0 and with potential W. Υ is a order-two Koszul factorization given by the Koszul matrix

$$\begin{pmatrix} P_2 & s_2(X_1,X_2,X_3)-s_2(X_4,X_5,X_6) \\ P_3 & X_1X_2X_3-X_4X_5X_6 \end{pmatrix},$$

where $s_2(x,y,z)=xy+xz+yz$ is the degree-two symmetric polynomial. Since W is symmetric in X_1,X_2,X_3 and X_4,X_5,X_6, it is not difficult to see that it is in the ideal generated by the symmetric differences $X_1+X_2+X_3-X_4-X_5-X_6$, $s_2(X_1,X_2,X_3)-s_2(X_4,X_5,X_6)$, and $X_1X_2X_3-X_4X_5X_6$. The first symmetric difference vanishes in R_0, so we can find $P_2, P_3 \in R_0$ so that

$$(s_2(X_1,X_2,X_3)-s_2(X_4,X_5,X_6))P_2+(X_1X_2X_3-X_4X_5X_6)P_3=W$$

in R_0. The choice of P_2 and P_3 is not unique, but since

$$s_2(X_1,X_2,X_3)-s_2(X_4,X_5,X_6) \quad \text{and} \quad X_1X_2X_3-X_4X_5X_6$$

are relatively prime, Lemma 3.10 implies that any two choices are related by a twist. For definiteness, we fix some values of P_2 and P_3 which are symmetric in X_1,X_2,X_3 and (separately) in X_4,X_5,X_6.

In [13], Khovanov and Rozansky exhibit homotopy equivalences

$$f\colon C_p(D_{\mathrm{IIIa}})_+ \to \Upsilon_+ \oplus C_p(D'_{\mathrm{IIIb}})_+,$$
$$g\colon C_p(D_{\mathrm{IIIb}})_+ \to \Upsilon_+ \oplus C_p(D'_{\mathrm{IIIa}})_+.$$

Our goal is to show that f and g are quasi-isomorphisms. We will prove the following:

Proposition 4.12 *$C_p(D_{\mathrm{IIIa}}) \sim \Upsilon \oplus C_p(D'_{\mathrm{IIIb}})$ over R_0.*

For the moment, let us assume that this proposition is true. Observe that D'_{IIIa} is essentially the same graph as D_{IIIa}, but with the labels on edges 1 and 3 and 4 and 6 reversed. Since Υ is symmetric in both X_1, X_2, X_3 and X_4, X_5, X_6, $C_p(D'_{\mathrm{IIIa}})$ will be quasi-isomorphic to $\Upsilon \oplus C_p(D_{\mathrm{IIIb}})$. It follows that $C_p(D_{\mathrm{IIIa}}) \oplus C_p(D_{\mathrm{IIIb}})$ and $C_p(D'_{\mathrm{IIIa}}) \oplus C_p(D'_{\mathrm{IIIb}})$ are both quasi-isomorphic to $\Upsilon \oplus C_p(D_{\mathrm{IIIb}}) \oplus C_p(D'_{\mathrm{IIIb}})$. Thus Proposition 4.6 is implied by Proposition 4.12. To prove the latter, we follow the argument given in [13] step by step.

The factorization $C_p(D_{\mathrm{IIIa}})$ is defined over the ring

$$R = R(D_{\mathrm{IIIa}}) = \mathbb{Q}[X_1, \dots, X_9]/I,$$

where

$$I = (X_7 + X_8 - X_1 - X_2,\ X_6 + X_9 - X_3 - X_7,\ X_4 + X_5 - X_8 - X_9).$$

We use these relations to eliminate X_1, X_7 and X_8, thus expressing R as a polynomial ring in variables $X_2, X_3, X_4, X_5, X_6, X_9$. In this ring, $C_p(D_{\mathrm{IIIa}})$ is an order-three Koszul factorization, with Koszul matrix

$$\begin{pmatrix} * & (X_2 - X_8)(X_2 - X_7) \\ * & (X_3 - X_9)(X_3 - X_6) \\ * & (X_9 - X_4)(X_9 - X_5) \end{pmatrix}.$$

Eliminating X_7 and X_8, this becomes

$$\begin{pmatrix} * & (X_2 + X_9 - X_4 - X_5)(X_2 + X_3 - X_6 - X_9) \\ * & (X_3 - X_9)(X_3 - X_6) \\ * & (X_9 - X_4)(X_9 - X_5) \end{pmatrix}.$$

We use the right-hand entry of the last row to exclude the internal variable X_9. The result is an order-two Koszul factorization C_1 over the ring $R_1 = R/(X_9 - X_4)(X_9 - X_5)$, with Koszul matrix

$$\begin{pmatrix} * & X_9(X_3 - X_6) + X_4X_5 + (X_2 - X_4 - X_5)(X_2 + X_3 - X_6) \\ * & (X_3 - X_9)(X_3 - X_6) \end{pmatrix}.$$

After a row operation in which we add the bottom entry in the right-hand row to the top one, we get

$$\begin{pmatrix} * & (X_2 - X_4)(X_2 - X_5) + (X_2 + X_3 - X_4 - X_5)(X_3 - X_6) \\ * & (X_3 - X_9)(X_3 - X_6) \end{pmatrix}.$$

More explicitly, this factorization is represented by the diagram

$$\begin{array}{ccc} R_1 & \overset{b_1}{\underset{a_1}{\rightleftarrows}} & R_1 \\ b_2 \Big\uparrow\Big\downarrow a_2 & & b_2 \Big\uparrow\Big\downarrow a_2 \\ R_1 & \overset{-b_1}{\underset{-a_1}{\rightleftarrows}} & R_1 \end{array}$$

where a_1 and a_2 are unknown, and

$$\begin{aligned} b_1 &= (X_2 - X_4)(X_2 - X_5) + (X_2 + X_3 - X_4 - X_5)(X_3 - X_6), \\ b_2 &= (X_3 - X_9)(X_3 - X_6). \end{aligned}$$

We now think of C_1 as an object of $\mathrm{GMF}(R_0)$, and R_1 as a free module of rank 2 over R_0. Following [13], we choose an explicit basis $\{1, X_9 + X_3 - X_4 - X_5\}$ for the two copies of R_1 in the lower row of the diagram, and the basis $\{1, X_9 - X_3\}$ for the two copies of R_1 in the upper row. With respect to these bases, C_1 takes the form

$$\begin{array}{ccc} R_0 \oplus R_0 & \overset{B_1'}{\underset{A_1'}{\rightleftarrows}} & R_0 \oplus R_0 \\ B_2 \Big\uparrow\Big\downarrow A_2 & & B_2 \Big\uparrow\Big\downarrow A_2 \\ R_0 \oplus R_0 & \overset{-B_1}{\underset{-A_1}{\rightleftarrows}} & R_0 \oplus R_0 \end{array}$$

where $A_1, A_1', B_1, B_1', A_2, B_2$ are 2×2 matrices over R_0 representing multiplication by a_1, b_1, a_2 and b_2. The pairs A_1 and A_1' and B_1 and B_1' represent the same linear maps with respect to different bases, so they are conjugate. For B_1, this is irrelevant — X_9 does not appear in b_1, so B_1 is a multiple of the identity map:

$$B_1 = B_1' = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

where $x = (X_2 - X_4)(X_2 - X_5) + (X_2 + X_3 - X_4 - X_5)(X_3 - X_6)$. Direct computation shows that

$$B_2 = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix},$$

where $y = (X_3 - X_4)(X_3 - X_5)(X_3 - X_6)$ and $z = X_6 - X_3$.

Lemma 4.13 *A_1 and A_1' may be expressed in the form*

$$A_1 = \begin{pmatrix} a & yc/z \\ c & a-qc \end{pmatrix}, \quad A_1' = \begin{pmatrix} a-qc & yc/z \\ c & a \end{pmatrix},$$

where $q = X_4 + X_5 - 2X_3$.

Proof Suppose that $A_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Changing basis, we see that

$$A_1' = \begin{pmatrix} a-qc & b+q(a-d)-q^2c \\ c & d+qc \end{pmatrix},$$

where $q = X_4 + X_5 - 2X_3$. The component of $d_+d_- + d_-d_+$ which maps from the bottom right corner of the square to the upper left must vanish, so $A_1'B_2 = B_2A_1$, or, more explicitly

$$\begin{pmatrix} a-qc & b+q(a-d)-q^2c \\ c & d+qc \end{pmatrix} \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Multiplying out and equating terms, we find that we must have $d = a - qc$ and $yc = zb$. □

Lemma 4.14 $A_2 = \begin{pmatrix} -xc' & \beta \\ \gamma & -xc' \end{pmatrix}$*, where $zc' = c$.*

Proof Suppose that $A_2 = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. Inspecting the component of $d_-d_+ + d_+d_-$ which maps the lower left-hand corner of the diagram to itself, we see that $A_2B_2 + B_1A_1 = W \cdot \mathrm{Id}$, or

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} + \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} a & yc/z \\ c & a-qc \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}.$$

Inspecting the off-diagonal elements, we find that $z\delta + xc = y\alpha + xyc/z = 0$. Since x and z are relatively prime, we must have $c = zc'$, $\delta = -xc'$ for some $c' \in R_0$. Substituting into the second equation, we see that $\alpha = \delta$. □

Thus we can write A and A' in the form

$$A_1 = \begin{pmatrix} a & yc' \\ zc' & a-qzc' \end{pmatrix}, \quad A_1' = \begin{pmatrix} a-qzc' & yc' \\ zc' & a \end{pmatrix}.$$

Consider the map H from the upper right-hand copy of $R_0 \oplus R_0$ to the lower left given by the matrix $H = \left(\begin{smallmatrix} -c' & 0 \\ 0 & -c' \end{smallmatrix}\right)$. The twisted factorization $C_2(H)$ has the same positive differentials as C_2, but the negative differentials are given by matrices

$$A_1(H) = \begin{pmatrix} a & 0 \\ 0 & a-qzc' \end{pmatrix}, \quad A_1'(H) = \begin{pmatrix} a-qzc' & 0 \\ 0 & a \end{pmatrix}, \quad A_2(H) = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.$$

It is now clear that $C_2(H)$ decomposes as a direct sum: one summand consists of the first copies of R_0 in the top row and the second copies in the bottom, and the other of the second copies in the top and the first in the bottom. Both summands are order-two Koszul factorizations over R_0, with Koszul matrices

$$\begin{pmatrix} a-qzc' & x \\ \gamma & z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & x \\ \beta & y \end{pmatrix}.$$

Recalling that

$$y=(X_3-X_4)(X_3-X_5)(X_3-X_6), \quad z=X_3-X_6,$$
$$x=z(X_4+X_5-X_2-X_3)+(X_2-X_4)(X_2-X_5),$$

and using row operations to simplify the right-hand columns, we see that these are equivalent to Koszul matrices of the form

$$\begin{pmatrix} * & (X_2-X_4)(X_2-X_5) \\ * & X_3-X_6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & s_2(X_1,X_2,X_3)-s_2(X_4,X_5,X_6) \\ * & X_1X_2X_3-X_4X_5X_6 \end{pmatrix}.$$

By Lemma 3.10, the first factorization is a twisted version of

$$\begin{pmatrix} p_{1245} & (X_2-X_4)(X_2-X_5) \\ p_{36} & X_3-X_6 \end{pmatrix},$$

where

$$p_{1245}=\frac{p(X_4)+p(X_5)-p(X_1)-p(X_2)}{X_4+X_5-X_1-X_2} \quad \text{and} \quad p_{36}=\frac{p(X_6)-p(X_3)}{X_6-X_3}.$$

Arguing as in the proof of Lemma 4.10, we see that this factorization is quasi-isomorphic to $C_p(D'_{\mathrm{IIIb}})$. Likewise, Lemma 3.10 shows that the second factorization is a twisted version of Υ.

To recap, we have shown that $C_p(D_{\mathrm{IIIa}}) \sim C_1$. By Lemma 3.6, $C_1 \sim C_1(H)$, which decomposes into a direct sum of two order-two Koszul complexes. Finally, a further application of Lemmas 3.6 and 3.10 shows that these are quasi-isomorphic to $C_p(D'_{\mathrm{IIIb}})$ and Υ. We leave it as an exercise for the reader to check that the two summands have the correct bigrading. This concludes the proof of Proposition 4.6. □

5 Relation between $\overline{H}$ and $\overline{H}_N$

We are now in a position to address the relation between the HOMFLY and $sl(N)$ homologies. Here is our main result.

Theorem 5.1 *Suppose $L \subset S^3$ is a link, and let i be a marked component of L. For each $p \in \mathbb{Q}[x]$, there is a spectral sequence $E_k(p)$ with $E_1(p) \cong \overline{H}(L)$ and $E_\infty(p) \cong \overline{H}_p(L,i)$. For all $k > 0$, the isomorphism type of $E_k(p)$ is an invariant of the pair (L,i).*

Corollary 5.2 *The isomorphism type of $\overline{H}_p(L,i)$ is an invariant of (L,i).*

The relation of these sequences with the various gradings may be summarized as follows. Let $d_k(p)\colon E_k(p) \to E_k(p)$ be the k^{th} differential in the sequence. If $p(x) = x^{N+1}$, then $d_k(p)$ is homogenous of degree $(2Nk, -2k, 2-2k)$ with respect to the triple grading on $\overline{H}(K)$. In particular, each $d_k(p)$ preserves the grading $\mathrm{gr}'_N = q + 2N\,\mathrm{gr}_h$. The grading on $E_\infty(p)$ induced by gr'_N is equal to the polynomial grading $\mathrm{gr}_N = q + (N-1)\,\mathrm{gr}_h$ on $\overline{H}_N(L,i)$.

For general values of p, $d_k(p)$ is no longer homogenous with respect to the q–grading, but it is still the case that $d_k(p)$ shifts gr_h by $-k$ and gr_v by $1-k$. Thus the $d_k(p)$ are all homogenous of degree 1 with respect to the grading gr_-. The grading induced by gr_- on the E_∞ term is equal to the homological grading on $\overline{H}_p(k)$.

A few other remarks on the theorem are in order. First, it is possible to prove an analogous result for the unreduced homology. The argument is very similar to the one in the reduced case, except that we don't need to worry about keeping track of a marked edge. Second, in terms of invariance, the spectral sequence suffers from the same drawback as the HOMFLY homology — we can show that any two diagrams representing the same link give rise to isomorphic spectral sequences, but not that the isomorphism is canonical. Finally, we expect that $\overline{H}_p(L,i)$ should be determined by the order of vanishing of $p'(x)$ at $x=0$, and that $\widetilde{H}_p(L)$ should be determined by the multiplicities of the roots of $p'(x)$. (This idea has its source in the work of Gornik [4]. Although we will not pursue it here, some supporting evidence has been provided by Mackaay and Vaz [17].) In particular, it seems unlikely that the set of all homologies $\overline{H}_p(L,i)$, $p \in \mathbb{Q}[x]$ contains more information than is present in the $sl(N)$ homologies.

5.1 Definition and basic properties

We now construct the spectral sequence $E_k(p)$. Given a link L with a marked component, we fix a braid diagram D representing L and an edge i belonging to the marked component. The complex $\overline{C}_p(D,i)$ is endowed with differentials d_+, d_- and d_v. Since D is a closed diagram, all three differentials anticommute. It follows that $\overline{H}^+_p(D,i)$ inherits a pair of anticommuting differentials d^*_- and d^*_v. The differential d_-^* lowers gr_h by 1 and preserves gr_v, while d_v^* raises gr_v by 1 and preserves gr_h. Thus the triple $(\overline{H}_p^+(D,i), d_v^*, d_-^*)$ defines a double complex with total differential $d_{v-} = d_v^* + d_-^*$ and total grading $\mathrm{gr}_- = \mathrm{gr}_v - \mathrm{gr}_h$. Like any double complex, $(\overline{H}_p^+(D,i), d_{v-})$ comes with two natural filtrations: a *horizontal filtration* induced by gr_h, and a *vertical filtration* induced by gr_v.

Definition 5.3 $E_k(p)$ is the spectral sequence induced by the horizontal filtration on the complex $(\overline{H}_p^+(D,i), d_{v-})$.

As we did in the definition of $\overline{H}$, we shift the triple grading on $E_k(p)$ by a factor of $\{-w+b-1, w+b-1, w-b+1\}$, where w and b are the writhe and number of strands in the diagram D. With this normalization, the first claim of Theorem 5.1 is easily verified. $E_0(p) = \overline{H}_p^+(D,i)\{-w+b-1, w+b-1, w-b+1\}$, and the differential $d_0(p)\colon E_0(p) \to E_0(p)$ is the part of $d_v^* + d_-^*$ which preserves gr_h. In other words, $d_0(p) = d_v^*$, so

$$E_1(p) = H(\overline{H}_p^+(D,i), d_v^*)\{-w+b-1, w+b-1, w-b+1\} \cong \overline{H}(L,i).$$

To complete the proof of the theorem, we must show that the total homology satisfies

$$H(\overline{H}_p^+(D,i), d_{v-}) \cong \overline{H}_p(D,i),$$

and that the sequence is an invariant of the pair (L,i). Before we doing this, we pause to discuss some elementary properties of $E_k(p)$. First, note that when p is a linear polynomial, the differential d_- is identically zero, and the spectral sequence converges trivially to $\overline{H}_p(D,i) \cong \overline{H}(D)$. Thus the sequence is only interesting when $\deg p > 1$. For the rest of the section, we will assume that this is the case.

Next, we address the issue of gradings.

Lemma 5.4 *The differential $d_k(p)$ is homogenous of degree $-k$ with respect to gr_h and degree $1-k$ with respect to gr_v. In addition, if $p(x) = x^{N+1}$, then $d_k(p)$ is homogenous of degree $2Nk$ with respect to the q–grading.*

Proof When $p(x) = x^{N+1}$, this follows immediately from the fact that d_- and d_v are homogenous of degrees $(2N,-2,0)$ and $(0,0,2)$ with respect to the triple grading on $\overline{H}_p^+(D,i)$. For general values of p, d_- is no longer homogenous with respect to the q–grading, but its behavior with respect to the homological gradings remains unchanged. □

When $p(x) = x^{N+1}$, the differentials $d_k(p)$ all preserve the quantity $\mathrm{gr}'_N = q + 2N\,\mathrm{gr}_h$, so the graded Euler characteristic of $H(\overline{H}_p^+(D,i), d_{v-})$ with respect to gr'_N will be the same as the graded Euler characteristic of $E_1(p)$. Using Theorem 2.11, we compute that

$$\begin{aligned}\chi(E_1(p)) &= \sum (-1)^{\mathrm{gr}_-} q^{\mathrm{gr}'_N} \dim \overline{H}(L,i) \\ &= \sum_{i,j,k} (-1)^{k-j} q^{i+Nj} \dim \overline{H}^{i,j,k}(L,i) \\ &= \sum_{i,j,k} (-1)^{k-j} a^j q^i \dim \overline{H}^{i,j,k}(L,i)\big|_{a=q^N} \\ &= P_L(q^N, q).\end{aligned}$$

Since the graded Euler characteristic of $H(\overline{H}_p^+(D,i), d_{v-})$ is given by the $sl(N)$ polynomial, it's at least plausible that the homology should agree with $\overline{H}_N$.

Next, we consider the relation between $E_k(p)$ for different values of p. In the original complex $\overline{C}_p(D,i)$, the underlying group and the differentials d_+ and d_v are independent of p. Thus $E_0(p) = \overline{H}_p^+(D,i)$ is independent of p, as is $d_0 = d_v^*$. It follows that we can view $E_1(p) = \overline{H}(L,i)$ as being equipped with an infinite-dimensional family of differentials $d_1(p)$; one for each $p \in \mathbb{Q}[x]$.

Lemma 5.5 $$d_1(ap+bq) = ad_1(p) + bd_1(q)$$

Proof Denote the differential d_- on $\overline{C}_p(D)$ by $d_-(p)$. We claim that

$$d_-(ap+bq) = ad_-(p) + bd_-(q).$$

Since $d_1(p)$ is the map induced by $d_-(p)$, the claim implies the statement of the lemma. To prove it, observe that, for an elementary tangle D, the potential

$$W_p = p(X_3) + p(X_4) - p(X_1) - p(X_2)$$

satisfies $W_{ap+bq} = aW_p + bW_q$. The coefficients of $d_-(p)$ are quotients of W_p by fixed polynomials, so they are also linear in p. Finally, it is easy to see that the linearity property is preserved under tensor product, so the claim holds. □

Corollary 5.6 *For all $p, q \in \mathbb{Q}[x]$, $d_1(p)$ and $d_1(q)$ anticommute.*

Proof We have $d_1(p)d_1(q) + d_1(q)d_1(p) = (d_1(p)+d_1(q))^2 = d_1(p+q)^2 = 0$. □

5.2 The total homology

Our goal in this section is to calculate the homology group $H(\overline{H}_p^+(D,i), d_{v-})$ to which $E_k(p)$ converges. (Throughout, we continue to assume that $\deg p > 1$.) To do this, we use two more spectral sequences — one which converges to $H(\overline{H}_p^+(D,i), d_{v-})$, and another which can be used to calculate $\overline{H}_p(D,i)$. The key point is to show that both of these sequences converge at the E_2 term, and that the E_2 terms agree.

We start off with some notation. Suppose that C^* is a graded matrix factorization with potential 0, so that d_+ anticommutes with d_-. Then we define

$$H^{\pm}(C^*) = H(H(C^*, d_+), d_-^*).$$

When $C^* = C_p(D,i)$, we abbreviate this to $H^{\pm}(D,i)$. Both d_+ and d_- are homogenous with respect to the homological grading on C (albeit with different degrees), so this grading descends to a well-defined grading on $H^{\pm}(C^*)$.

Lemma 5.7 *If C^* and D^* are quasi-isomorphic factorizations with potential 0, then $H^{\pm}(C^*) \cong H^{\pm}(D^*)$.*

Proof The definition of quasi-isomorphism implies that the complex $(H(C^*, d_+), d_-^*)$ is isomorphic to $(H(D^*, d_+), d_-^*)$. □

Lemma 5.8 *If D is a closed braid graph on b strands, then $\overline{H}_p^{\pm}(D,i)$ is supported in horizontal grading $\mathrm{gr}_h = 1-b$.*

Proof As in Section 4.3, we use the MOY relations to induct on the complexity of D. In the base case, D is a single circle, and $\overline{H}_p^{\pm}(D,i) \cong \overline{C}_p(D,i) \cong \mathbb{Q}$ is supported in grading $\mathrm{gr}_h = 0$.

For the induction step, we use Lemma 4.1 to see that D can be related to diagrams of lesser complexity using the MOY moves. In fact, we claim that D can be simplified by a MOY move which has the marked edge i as an external edge. To see this, write D as the closure of an open braid in such a way that i is one of the edges which appear in the closure. Then i will be an external edge for any MOY II or III move provided by the lemma. If i lies on a loop which could be eliminated by a MOY O move, we ignore it and simplify using some other MOY move. Finally, if i is a small loop which is about to be eliminated by a MOY I move, i must be the rightmost strand in D. In this case, we consider the mirror image $\overline{D}$ of D. It's easy to see that $\overline{C}_p(D,i) \cong \overline{C}_p(\overline{D},i)$, and the marked edge in $\overline{D}$ is on the leftmost strand. We now use Lemma 4.1 to simplify $\overline{D}$ as before.

Thus, in order to prove the lemma, it is enough to check that if the statement holds for the less complex diagram(s) in each of the four MOY moves, it also holds for the more complex one. For example, suppose D_{O} and D'_{O} are related by a MOY O move, so that

$$C_p(D_{\mathrm{O}}) \cong C_p(D'_{\mathrm{O}}) \otimes_{\mathbb{Q}} C_p(\mathrm{O}).$$

Then

$$\overline{C}_p(D_{\mathrm{O}}, i) \cong \overline{C}_p(D'_{\mathrm{O}}, i) \otimes_{\mathbb{Q}} C_p(\mathrm{O}),$$

and $d_+ = 0$ on $C_p(\mathrm{O})$, so

$$\overline{H}^+_p(D_{\mathrm{O}}, i) \cong \overline{H}^+_p(D'_{\mathrm{O}}, i) \otimes_{\mathbb{Q}} C_p(\mathrm{O}).$$

By the Kunneth formula,

$$\overline{H}^\pm_p(D_{\mathrm{O}}, i) \cong \overline{H}^\pm_p(D'_{\mathrm{O}}, i) \otimes_{\mathbb{Q}} H(C_p(\mathrm{O}), d_-).$$

D'_{O} has one fewer strand than $H_p(D)$, so $H_p(D'_{\mathrm{O}}, i)$ is supported in $\mathrm{gr}_h = 2-b$ by the induction hypothesis. When $\deg(p) > 1$, $H(C_p(\mathrm{O}), d_-)$ is supported in $\mathrm{gr}_h = -1$. Thus $\overline{H}^\pm_p(D_{\mathrm{O}}, i)$ is supported in $\mathrm{gr}_h = 1-b$ as claimed.

Similarly, if D_{I} and D'_{I} are related by a MOY I move,

$$C_p(D_{\mathrm{I}}, i) \cong C_p(D'_{\mathrm{I}}, i) \otimes_{\mathbb{Q}[X_1]} C_p(\mathrm{I}) \quad \text{and} \quad \overline{H}^+_p(D_{\mathrm{I}}, i) \cong \overline{H}^+_p(D'_{\mathrm{I}}, i) \otimes_{\mathbb{Q}[X_1]} C_p(\mathrm{I}).$$

The complex $(C_p(\mathrm{I}), d_-)$ has the form

$$\mathbb{Q}[X_1, X_2]\{0, -2\} \xleftarrow{\;p'_{12}\;} \mathbb{Q}[X_1, X_2]\{0, -0\},$$

where $p'_{12} = (p'(X_1) - p'(X_2))/(X_1 - X_2)$. When $\deg p > 1$, its homology is a free module over $\mathbb{Q}[X_1]$, supported in grading $\mathrm{gr}_h = -1$. As in the previous case, we apply the Kunneth formula to conclude that $\overline{H}^\pm_p(D_{\mathrm{I}}, i)$ is supported in $\mathrm{gr}_h = 1-b$.

Next, suppose that D_{II} and D'_{II} are related by a MOY II move which takes place away from the marked edge i. Proposition 4.5 tells us that

$$C_p(D_{\mathrm{II}}) \sim C_p(D'_{\mathrm{II}})\{-1, 0\} \oplus C_p(D'_{\mathrm{II}})\{1, 0\}$$

over a ring R which contains $\mathbb{Q}[X_i]$ as a subring. By Corollary 3.9, it follows that

$$C_p(D_{\mathrm{II}}, i) \sim C_p(D'_{\mathrm{II}}, i)\{-1, 0\} \oplus C_p(D'_{\mathrm{II}}, i)\{1, 0\}.$$

Applying Lemma 5.7, we see that

$$\overline{H}^\pm_p(D_{\mathrm{II}}, i) \sim \overline{H}^\pm_p(D'_{\mathrm{II}}, i)\{-1, 0\} \oplus \overline{H}^\pm_p(D'_{\mathrm{II}}, i)\{1, 0\}.$$

Since both diagrams have the same number of strands, the result follows from the induction hypothesis. The argument for the MOY III move is very similar, and is left to the reader. □

Corollary 5.9 *If D is a closed braid on b strands, $\overline{H}_p^{\pm}(D,i)$ is supported in horizontal grading* $\mathrm{gr}_h = 1-b$.

Proof Since we are only taking homology with respect to d_+ and d_-, $\overline{H}_p^{\pm}(D,i)$ decomposes as a direct sum over MOY states of D (cf Lemma 4.9):

$$\overline{H}_p^{\pm}(D,i) \cong \bigoplus_{\sigma} \overline{H}_p^{\pm}(D_\sigma,i)\{\mu(\sigma),0,-\mu(\sigma)\}.$$

If D is a braid on b strands, each diagram D_σ will be a braid graph on b strands. There are no shifts in gr_h, so each summand $\overline{H}_p^{\pm}(D_\sigma,i)$ is supported in $\mathrm{gr}_h = 1-b$. □

Proposition 5.10 *If D is a closed braid, then*

$$H(\overline{H}_p^{+}(D,i),d_{v-}) \cong H(\overline{H}_p^{\pm}(D,i),d_v^*).$$

Proof We compute the total homology using the spectral sequence induced by the vertical filtration on $(\overline{H}_p^{+}(D,i),d_v^*,d_-^*)$. In this sequence, $d_0 = d_-^*$, so the E_1 term is

$$H(\overline{H}_p^{+}(D,i),d_-^*) = \overline{H}_p^{\pm}(D,i).$$

By Corollary 5.9, this group is supported in a single horizontal grading.

The differential d_1 is the induced map $d_v^*\colon \overline{H}_p^{\pm}(D,i) \to \overline{H}_p^{\pm}(D,i)$. Thus the E_2 term of the sequence is the group $H(\overline{H}_p^{\pm}(D,i),d_v^*)$. For $n>1$, the differential d_n raises gr_v by n and gr_h by $n-1$. Since the E_1 term (and thus the E_2 term) is supported in a single horizontal grading, $d_n \equiv 0$ for all $n>1$, and the sequence converges at the E_2 term. *A priori*, this implies that the graded group $H(\overline{H}_p^{\pm}(D,i),d_v^*)$ is isomorphic to the associated graded group of $H(\overline{H}_p^{+}(D,i),d_{v-})$. In fact, for each value of the homological grading gr_-, the former group is supported in a unique value of the filtration grading gr_v. Thus the two groups are canonically isomorphic. □

Our next task is to relate the group $H(\overline{H}_p^{\pm}(D,i),d_v^*)$ to $\overline{H}_p(D,i)$. To do so, we use a slightly different spectral sequence. Recall that if $(C^*,d_\pm)$ is a matrix factorization with potential 0, we can form the total differential $d_{\mathrm{tot}} = d_+ + d_-$.

Lemma 5.11 *If C^* is a matrix factorization with potential 0, there is a spectral sequence with E_2 term $H^{\pm}(C^*)$ which converges to $H(C^*,d_{\mathrm{tot}})$.*

Proof We define an increasing filtration of (C^*, d_{tot}) by $F^n = \bigoplus_{i<n} C^i \oplus \ker d_+^n$, where d_+^n denotes the component of d_+ which maps C^n to C^{n+1}. The E_0 term of the spectral sequence induced by this filtration is the associated graded complex

$$E_0^n = \frac{F^n}{F^{n-1}} \cong \frac{C^{n-1}}{\ker d_+^{n-1}} \oplus \ker d_+^n.$$

The differential $d_0\colon E_0^n \to E_0^n$ is given by $d_0(x,y) = (d_- y, d_+ x)$. If $d_0(x,y) = 0$, then $d_+ x = 0$, so $x = 0$ as an element of $C^{n-1}/\ker d_+^{n-1}$. Conversely, $d_+ d_- y = -d_- d_+ y = 0$ for any $y \in \ker d_+^n$, so $d_- y = 0$ as an element of $C^{n-1}/\ker d_+^{n-1}$. Thus

$$\ker d_0 = \{(0,y) \mid y \in \ker d_+^n\} \cong \ker d_+^n.$$

Similarly, $\operatorname{im} d_0 \cong \operatorname{im} d_+^{n-1}$, so $E_1^n = H(E_0^n, d_0) \cong H^n(C^*, d^+)$.

Next, we consider the differential $d_1\colon E_1^n \to E_1^{n-1}$. An element of E_1^n can be represented by $x \in \ker d_+^n$, and $d_1 x$ is the image of $d_{\mathrm{tot}} x = d_- x$ in E_1^{n-1}. In other words, d_1 is given by $d_-^*\colon H^n(C^*, d^+) \to H^{n-1}(C^*, d^+)$, and the E_2 term is $H(H(C^*, d_+), d_-^*) = H^{\pm}(C^*)$. □

Lemma 5.12 *If D is a closed braid graph, then $H(\overline{C}_p(D,i), d_{\mathrm{tot}}) \cong \overline{H}_p^{\pm}(D,i)$.*

Proof We apply the sequence of the preceding lemma to the complex $(\overline{C}_p(D,i), d_{\mathrm{tot}})$. By Lemma 5.8, the E_2 term is supported in a single homological grading, and thus in a single filtration grading as well. It follows that the sequence has converged at the E_2 term. As in the proof of Proposition 5.10, the fact that the E_∞ term is supported in a single filtration grading implies that it is canonically isomorphic to the total homology. □

Proposition 5.13 *If D is a closed braid, then $\overline{H}_p(D,i) \cong H(\overline{H}_p^{\pm}(D,i), d_v^*)$.*

Proof By definition, $\overline{H}_p(D,i)$ is the homology of the complex $(H(\overline{C}_p(D,i), d_{\mathrm{tot}}), d_v^*)$. To prove the proposition, it suffices to show that this complex is isomorphic to $(\overline{H}_p^{\pm}(D,i), d_v^*)$. The complex $(\overline{C}_p(D,i), d_{\mathrm{tot}})$ splits as a direct sum over MOY states of D, so, by the previous lemma, the underlying group $H(\overline{C}_p(D,i), d_{\mathrm{tot}})$ is isomorphic to $\overline{H}_p^{\pm}(D,i)$.

To see that the differentials are identified under this isomorphism, we must check that the following diagram commutes:

$$\begin{array}{ccc} H(\overline{C}_p(D,i), d_{\mathrm{tot}}) & \xrightarrow{d_v^*} & H(\overline{C}_p(D,i), d_{\mathrm{tot}}) \\ \| & & \| \\ \overline{H}_p^{\pm}(D,i) & \xrightarrow{\quad d_v^* \quad} & \overline{H}_p^{\pm}(D,i) \end{array}$$

To show this, we filter $C_{\mathrm{tot}} = (\overline{C}_p(D,i), d_{\mathrm{tot}})$ as in the proof of Lemma 5.11. Since d_v commutes with d_+, the map $d_v\colon C_{\mathrm{tot}} \to C_{\mathrm{tot}}$ preserves the filtration. It thus induces a morphism of spectral sequences $(d_v)_k\colon E_k \to E_k$ which converges to $d_v^*\colon H_{\mathrm{tot}} \to H_{\mathrm{tot}}$. In particular, the map $(d_v)_2$ is the induced map $d_v^*\colon \overline{H}_p^{\pm}(D,i) \to \overline{H}_p^{\pm}(D,i)$. Since both sequences converge at the E_2 term, $(d_v)_2$ is also equal to the associated graded map of $d_v^*\colon H_{\mathrm{tot}} \to H_{\mathrm{tot}}$. Since H_{tot} is supported in a single filtration grading, the two maps are actually equal. □

To sum up, we have:

Proposition 5.14 *The spectral sequence $E_k(p)$ converges to $\overline{H}_p(D,i)$. The grading gr_- on $E_k(p)$ corresponds to the homological grading on $\overline{H}_p(D,i)$, and if $p(x) = x^{N+1}$, the grading gr'_N on $E_k(p)$ corresponds to the polynomial grading gr_N on $\overline{H}_p(D,i)$.*

Proof The first statement is immediate from Propositions 5.10 and 5.13, so we just need to check that the gradings agree. The triple grading on $E_1(p)$ is the grading on $\overline{C}_p(D,i)$, shifted by $\{-w+b-1, w+b-1, w-b+1\}$, where w and b are the writhe and number of strands in D. We write $\mathrm{gr}_-(E)$ and $\mathrm{gr}_-(C)$ for the shifted and unshifted gradings, so that

$$\mathrm{gr}_-(E) = \mathrm{gr}_v(E) - \mathrm{gr}_h(E) = \mathrm{gr}_v(C) - \mathrm{gr}_h(C) - b + 1$$

and

$$\begin{aligned}\mathrm{gr}'_N(E) &= q(E) + 2N\,\mathrm{gr}_h(E)\\ &= q(C) + 2N\,\mathrm{gr}_h(C) + (N+1)(b-1) + (N-1)w.\end{aligned}$$

On $E_\infty(p)$, $\mathrm{gr}_-(C)$ and $\mathrm{gr}'_N(C)$ agree with the gradings gr_- and gr'_N on the total homology $H(\overline{H}_p^+(D,i), d_{v-})$. The gradings on the latter group can be computed using the spectral sequence of Proposition 5.10, whose E_∞ term is supported in $\mathrm{gr}_h = 1-b$. Substituting, we find that

$$\mathrm{gr}_-(E) = \mathrm{gr}_v(C) - (1-b) - b + 1 = \mathrm{gr}_v(C),$$

which is the homological grading on $\overline{H}_p(D,i)$, and

$$\begin{aligned}\mathrm{gr}'_N(E) &= q(C) - (N-1)(b-1) + (N-1)w\\ &= q(C) + (N-1)\,\mathrm{gr}_h(C) + (N-1)w\\ &= \mathrm{gr}_N(C) + (N-1)w,\end{aligned}$$

which is the polynomial grading on $\overline{H}_N(D,i)$. □

5.3 Change of marked edge

We now turn to the last part of Theorem 5.1; the invariance of $E_k(p)$. The first step is to show that, for a fixed braid diagram D representing L, $E_k(p)$ depends only on the component of L containing the marked edge i, and not on i itself.

Suppose for the moment that D is an arbitrary tangle diagram. If i is an internal edge of D, multiplication by X_i defines an endomorphism $X_i\colon C_p(D) \to C_p(D)$. Viewing $C_p(D)$ as an object of $\mathrm{Kom}(\mathrm{GMF}_w(R(D)))$, we can form the mapping cone $\mathrm{Cone}(X_i)$, which is also an object of $\mathrm{Kom}(\mathrm{GMF}_w(R(D)))$. We would like to view $\mathrm{Cone}(X_i)$ as a factorization over the ring $R_i(D) = R(D)/(X_i)$. To do so, we observe that, since X_i is a generator of the polynomial ring $R(D)$, there is an inclusion $R_i(D) \subset R(D)$ with the property that $R_i(D)[X_i] \cong R(D)$.

Lemma 5.15 $\mathrm{Cone}(X_i)$ *is homotopy equivalent to* $\overline{C}_p(D,i)$ *in* $\mathrm{Kom}(\mathrm{GMF}_w(R_i(D)))$.

Proof As a matrix factorization, $\mathrm{Cone}(X_i) = C_p(D) \oplus C_p(D)$. The vertical differential is given by

$$d_v(x,y) = (d_v x, d_v y + (-1)^{\mathrm{gr}_v x} X_i \cdot x).$$

Let $\pi_0\colon C_p(D) \to C_p(D,i)$ be the projection, and define $\pi\colon \mathrm{Cone}(X_i) \to \overline{C}_p(D,i)$ by $\pi(x,y) = \pi_0(y)$. Since $C_p(D)$ is a free module over $R(D) = R_i(D)[X_i]$, we have an injection $\iota_0\colon \overline{C}_p(D,i) \to C_p(D)$ modeled on the inclusion $R_i(D) \subset R(D)$. We extend this to an inclusion $\iota\colon \overline{C}_p(D,i) \to \mathrm{Cone}(X_i)$ given by $\iota(z) = (0, \iota_0(z))$. The composition $\pi\iota$ is the identity map, and $\iota\pi$ is homotopic to the identity via a homotopy $H\colon \mathrm{Cone}(X_i) \to \mathrm{Cone}(X_i)$ given by $H(x,y) = ((y - \iota\pi y)/X_i, 0)$. □

Lemma 5.16 *Suppose D is a tangle diagram, and let j and k be two edges of D which belong to the same component of the underlying tangle and are separated by a single ordinary crossing. (For example, the edges labeled j and k in Figure 1.) Then X_j and X_k are homotopic morphisms from $C_p(D)$ to itself.*

Proof Suppose that D is the elementary diagram D_+. The complex $C_p(D_+)$ has the form

$$\begin{array}{ccc}
R\{0,-2,0\} & \underset{p_1}{\overset{(X_k - X_i)}{\rightleftarrows}} & R\{0,0,0\} \\
\uparrow {\scriptstyle (X_j - X_k)} & & \uparrow {\scriptstyle 1} \\
R\{2,-2,-2\} & \underset{p_{12}}{\overset{-(X_k - X_i)(X_k - X_j)}{\rightleftarrows}} & R\{0,0,-2\}
\end{array}$$

where $R = R(D_+)$. The homotopy H is given by the vertical arrows in the diagram

$$\begin{array}{ccc} R\{0,-2,0\} & \underset{p_1}{\overset{(X_k-X_i)}{\rightleftarrows}} & R\{0,0,0\} \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle (X_j-X_k)} \\ R\{2,-2,-2\} & \underset{p_{12}}{\overset{-(X_k-X_i)(X_k-X_j)}{\rightleftarrows}} & R\{0,0,-2\} \end{array}$$

It's easy to see that H commutes with d_+ and d_- and that $d_v H + H d_v = X_j - X_k$. Since $X_j - X_k = X_l - X_i$ in $R(D_+)$, multiplication by X_i and X_l are also homotopic. The reader can easily check that there is a similar homotopy when D is the elementary diagram D_-.

For a general diagram D, the local nature of the KR–complex implies that we can write $C_p(D) \cong C_p(D_\pm) \otimes_{R(D_\pm)} C_p(D')$, where $D_\pm$ is the crossing separating j from k, and D' is the rest of the diagram. In terms of this decomposition, $X_j\colon C_p(D) \to C_p(D)$ can be written as $X_j \otimes 1$ and similarly for X_k. Clearly $f \sim g$ implies $f \otimes 1 \sim g \otimes 1$, so we are done. □

Next, we need a general result from homological algebra.

Lemma 5.17 *Suppose $\mathcal{A}$ is an additive category. If $f\colon A \to B$ and $g\colon A \to B$ are homotopic in* $\mathrm{Kom}(\mathcal{A})$*, then* $\mathrm{Cone}(f) \cong \mathrm{Cone}(g)$.

Proof If $H\colon A \to B$ is the homotopy from f to g, then the map $H\colon \mathrm{Cone}(f) \to \mathrm{Cone}(g)$ defined by $h(a,b) = (a, b - Ha)$ is an isomorphism with inverse $h^{-1}(a,b) = (a, b + Ha)$. □

By repeatedly applying the lemmas, we see that if i and j belong to the same component of L then $\overline{C}_p(D,i)$ and $\overline{C}_p(D,j)$ are homotopy equivalent as objects of $\mathrm{Kom}(\mathrm{GMF}_0(\mathbb{Q}))$.

Proposition 5.18 *Suppose D is a diagram representing a link L and that i and j are edges of D which belong to the same component of L. If we denote by $E_k(p,D,i)$ the spectral sequence associated to the pair (D,i), then $E_k(p,D,i) \cong E_k(p,D,j)$ for all $k > 0$.*

Proof Let $f\colon \overline{C}_p(D,i) \to \overline{C}_p(D,j)$ be a homotopy equivalence. Recall the functor $\mathcal{H}^+\colon \mathrm{GMF}_0(\mathbb{Q}) \to \mathrm{Kom}(\mathbb{Q})$, which takes a factorization C^* to the complex $(H^+(C^*), d_-^*)$. We apply $\mathcal{H}^+$ to f and get a homotopy equivalence

$$f^+\colon \overline{H}_p^+(D,i) \to \overline{H}_p^+(D,j)$$

in the category Kom(Kom($\mathbb{Q}$)) (ie double complexes over $\mathbb{Q}$). Since f^+ respects the horizontal filtration on $\overline{H}^+_p$, it induces a map of spectral sequences $f^+_k\colon E_k(p,i) \to E_k(p,j)$. The map $f^+_1\colon E_1(p,i) \to E_1(p,j)$ is the induced map

$$(f^+)^*\colon H(\overline{H}^+_p(D,i), d^*_v) \to H(\overline{H}^+_p(D,i), d^*_v).$$

Since f^+ is a homotopy equivalence with respect to d^*_v, this map is an isomorphism. This proves the claim when $k=1$. Finally, it is a well-known property of spectral sequences that if f^+_r is an isomorphism then f^+_k is an isomorphism for all $k > r$ as well. (See eg [18, Theorem 3.4].) □

5.4 Invariance under Reidemeister moves

The final step in the proof of Theorem 5.1 is to show that $E_k(p)$ remains invariant when we vary the diagram D. Following [13], we make some preliminary simplifications of the problem. By assumption, the diagram representing L is a braid diagram. Any two braid diagrams representing the same link L can be joined by a sequence of the five moves shown in Figure 8, so it is enough to prove that $E_k(p)$ is invariant under these moves. Also, using Proposition 5.18, we can assume that the marked edge i does not participate in the move.

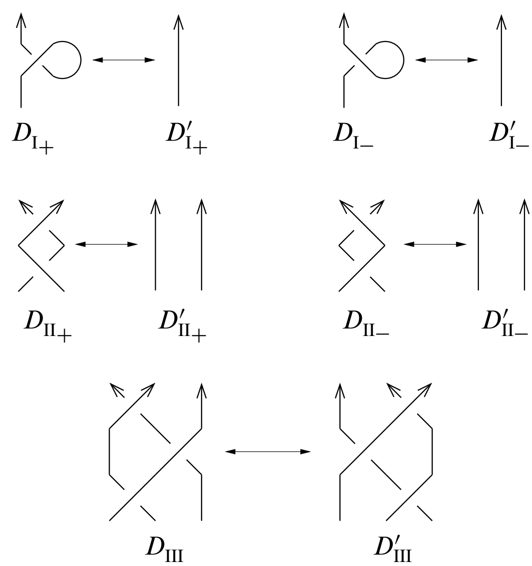


Figure 8: Braidlike Reidemeister moves

In what follows, it will be important to keep track of the category we are working in. To help with this, we introduce the following notation, writing

$$\mathcal{C}_j = \mathrm{Kom}(\mathrm{GMF}_{w(D_j)}(R_e(D_j))), \qquad \mathcal{C}_j^+ = \mathrm{Kom}(\mathrm{Kom}(R_e(D_j))),$$
$$\mathcal{K}_j = \mathrm{Kom}(\mathrm{hmf}(R_e(D_j))), \qquad \mathcal{K}_j^+ = \mathrm{Kom}(\mathrm{Mod}(R_e(D_j))),$$

where $j = \mathrm{I}, \mathrm{II}, \mathrm{III}$. The same symbols without subscripts indicate the corresponding category for a closed diagram (for example $\mathcal{C} = \mathrm{Kom}(\mathrm{GMF}_0(\mathbb{Q}))$). There is a commutative square of functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{H}^+} & \mathcal{C}^+ \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathcal{K} & \xrightarrow{\mathcal{H}^+} & \mathcal{K}^+ \end{array}$$

where $\mathcal{F}$ is the forgetful functor which corresponds to ignoring the differential d_- (or d_-^*) and $\mathcal{H}^+$ is the functor which corresponds to taking homology with respect to d_+.

Suppose that $\overline{D}_j$ and $\overline{D}'_j$ are closed diagrams related by the j^{th} Reidemeister move. Below, we will show that there are morphisms

$$\sigma_j \colon \overline{H}_p^+(\overline{D}_j, i) \to \overline{H}_p^+(\overline{D}'_j, i) \quad (j = \mathrm{I}_\pm, \mathrm{II}_\pm, \mathrm{III})$$

in the category $\mathcal{C}^+$ with the property that $\mathcal{F}(\sigma_j)$ is a homotopy equivalence in $\mathcal{K}^+$. This is sufficient to prove the theorem. Indeed, arguing as in proof of Proposition 5.18, we see that σ_j induces a morphism of spectral sequences $(\sigma_j)_k \colon E_k(P, \overline{D}_j, i) \to E_k(P, \overline{D}'_j, i)$ which is an isomorphism for $k > 0$.

Most of the work involved in constructing the σ_j and showing they are homotopy equivalences has already been done by Khovanov and Rozansky. In [13], they prove invariance of the HOMFLY homology by exhibiting homotopy equivalences

$$\rho_j \colon \mathcal{F}(C_p(D_j)) \to \mathcal{F}(C_p(D'_j))$$

in the category $\mathcal{K}_j$. From the local nature of the KR–complex, it follows that there are homotopy equivalences

$$\rho_j \otimes 1 \colon \mathcal{F}(\overline{C}_p(\overline{D}_j, i)) \to \mathcal{F}(\overline{C}_p(\overline{D}'_j, i))$$

in $\mathcal{K}$. The morphism σ_j will be derived from ρ_j, in the sense that $\mathcal{F}(\sigma_j) = \mathcal{H}^+(\rho_j \otimes 1)$.

Reidemeister I move In this case, we can work directly in the category $\mathcal{C}_{\mathrm{I}}$.

Lemma 5.19 *There are morphisms $\rho_{\mathrm{I}_\pm} \colon C_p(D_{\mathrm{I}_\pm}) \to C_p(D'_{\mathrm{I}_\pm})$ in $\mathcal{C}_{\mathrm{I}}$ with the property that $\mathcal{F}(\rho_{\mathrm{I}_\pm})$ is a homotopy equivalence.*

Proof The ring $R_e(R_{\mathrm{I}_\pm})$ is isomorphic to $\mathbb{Q}[X_1]$. Since R'_{I} has no crossings, $C_p(R'_{\mathrm{I}}) = \mathbb{Q}[X_1]$ as well. Arguing as in the proof of Proposition 4.4, we see that the complex $C_p(\mathrm{I}_+)$ has the form

$$\begin{array}{ccc}
\mathbb{Q}[X_1, X_2]\{0,-2,0\} & \underset{p'(X_1)-p'(X_2)}{\overset{0}{\rightleftarrows}} & \mathbb{Q}[X_1, X_2]\{0,0,0\} \\
{\scriptstyle X_2-X_1}\big\uparrow & & \big\uparrow{\scriptstyle 1} \\
\mathbb{Q}[X_1, X_2]\{2,-2,-2\} & \underset{p'_{12}}{\overset{0}{\rightleftarrows}} & \mathbb{Q}[X_1, X_2]\{0,0,-2\}
\end{array}$$

The morphism ρ_{I_+} takes the copy of $\mathbb{Q}[X_1, X_2]$ in the top left to $C_p(R'_{\mathrm{I}}) = \mathbb{Q}[X_1]$ by substituting $X_2 = X_1$, and is zero elsewhere. Similarly, $C_p(\mathrm{I}_-)$ is the complex

$$\begin{array}{ccc}
\mathbb{Q}[X_1, X_2]\{0,-2,2\} & \underset{p'_{12}}{\overset{0}{\rightleftarrows}} & \mathbb{Q}[X_1, X_2]\{-2,0,2\} \\
{\scriptstyle 1}\big\uparrow & & \big\uparrow{\scriptstyle X_2-X_1} \\
\mathbb{Q}[X_1, X_2]\{0,-2,0\} & \underset{p'(X_1)-p'(X_2)}{\overset{0}{\rightleftarrows}} & \mathbb{Q}[X_1, X_2]\{0,0,0\}
\end{array}$$

and the morphism ρ_{I_-} takes the copy of $\mathbb{Q}[X_1, X_2]$ in the top right to $C_p(R'_{\mathrm{I}}) = \mathbb{Q}[X_1]$ by substituting $X_2 = X_1$, and is zero elsewhere. The reader can easily verify that both ϕ_{I_+} and ϕ_{I_-} are morphisms in $\mathcal{C}_{\mathrm{I}}$ and that their restrictions to $\mathcal{K}_{\mathrm{I}}$ are homotopy equivalences. (In fact, they are the morphisms $\rho_{\mathrm{I}_\pm}$ defined in [13].) □

By the local nature of the KR–complex, there are morphisms

$$\rho_{\mathrm{I}_\pm} \otimes 1\colon \overline{C}_p(\overline{D}, i) \to \overline{C}_p(\overline{D}, i)$$

in $\mathcal{C}$ which restrict to homotopy equivalences in $\mathcal{K}$. Finally, the morphism

$$\sigma_{\mathrm{I}_\pm}\colon \overline{H}^+_p(\overline{D}_{\mathrm{I}_\pm}, i) \to \overline{H}^+_p(\overline{D}'_{\mathrm{I}_\pm}, i)$$

is defined to be $\mathcal{H}^+(\rho_{\mathrm{I}_\pm} \otimes 1)$. The fact that $\mathcal{F}(\sigma_{\mathrm{I}_\pm})$ is a homotopy equivalence follows from the relation $\mathcal{F}\mathcal{H}^+ = \mathcal{H}^+\mathcal{F}$. This concludes the proof of Reidemeister I invariance. □

Reidemeister II move As shown in Figure 8, there are two different versions of the oriented Reidemeister II move. We discuss only the first one; the proof for the other is virtually identical. We can represent $\overline{C}_p(\overline{D}_{\mathrm{II}_+}, i)$ by the diagram on the left-hand

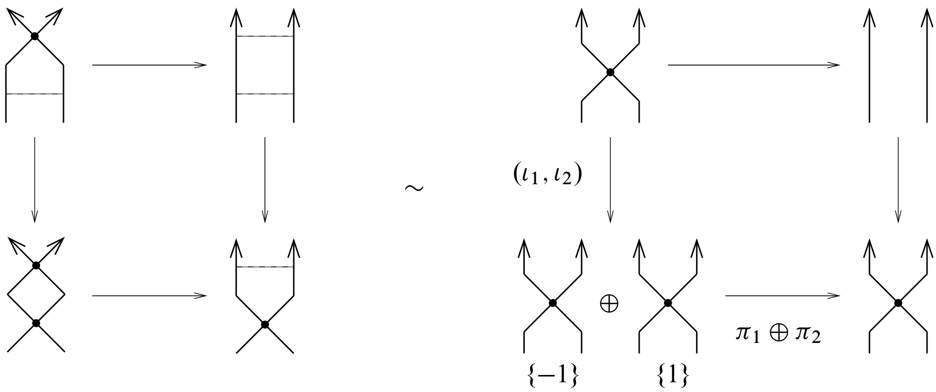


Figure 9: The complex $\overline{H}^+_p(D_{\mathrm{II}_+}, i)$

side of Figure 9. Each corner of the square is an object of $\mathcal{C}$, and the edges represent additional components of d_v going between them. We apply the functor $\mathcal{H}^+$ to get $\overline{H}^+_p(\overline{D}_{\mathrm{II}_+}, i)$. Using the MOY II decomposition on each factorization in the complex at the bottom left of the square (and Lemma 4.10 on the other corners) we see that $\overline{H}^+_p(\overline{D}_{\mathrm{II}_+}, i)$ has the form shown on the right-hand side of the figure.

Consider the maps $\iota_1\colon \overline{H}^+_p(\times, i) \to \overline{H}^+_p(\times, i)$ (from the upper left-hand corner to the first summand on the bottom left) and $\pi_2\colon \overline{H}^+_p(\times, i) \to \overline{H}^+_p(\times, i)$ (from the second summand to the bottom right). We claim that both ι_1 and π_2 are isomorphisms in $\mathcal{C}^+$. To see this, we return to $\overline{C}_p(D_{\mathrm{II}}, i)$, and apply the functor $\mathcal{F}$. In $\mathcal{K}$, we can use the MOY II decomposition directly, without applying $\mathcal{H}^+$ first. We get a diagram like that on the right-hand side of Figure 9, with corresponding morphisms $\widetilde{\iota}_1$ and $\widetilde{\pi}_2$. The main ingredient in the proof of invariance under the Reidemeister II move in [13] is to show that $\widetilde{\iota}_1$ and $\widetilde{\pi}_2$ are isomorphisms. From this, it follows that $\mathcal{H}^+(\widetilde{\iota}_1) = \mathcal{F}(\iota_1)$ and $\mathcal{H}^+(\widetilde{\pi}_2) = \mathcal{F}(\pi_2)$ are isomorphisms. But if f is a morphism in $\mathcal{C}^+$ with the property that $\mathcal{F}(f)$ is an isomorphism in $\mathcal{K}^+$, then f is an isomorphism in $\mathcal{C}^+$. (In plainer language, a chain map that is an isomorphism at the level of modules is an isomorphism.) This proves the claim.

At this point, a standard cancellation argument like that used in the proof of invariance under the Reidemeister II move in [9] or [12] shows that $\overline{H}^+_p(\overline{D}_{\mathrm{II}}, i)$ is homotopy equivalent to $\overline{H}^+_p(\overline{D}'_{\mathrm{II}}, i)$ in $\mathcal{C}^+$. This concludes the proof of Reidemeister II invariance □

Reidemeister III move The argument in this case is similar to the one for the Reidemeister II move. We start out with the complex $\overline{C}_p(\overline{D}_{\mathrm{III}}, i)$, which has the form

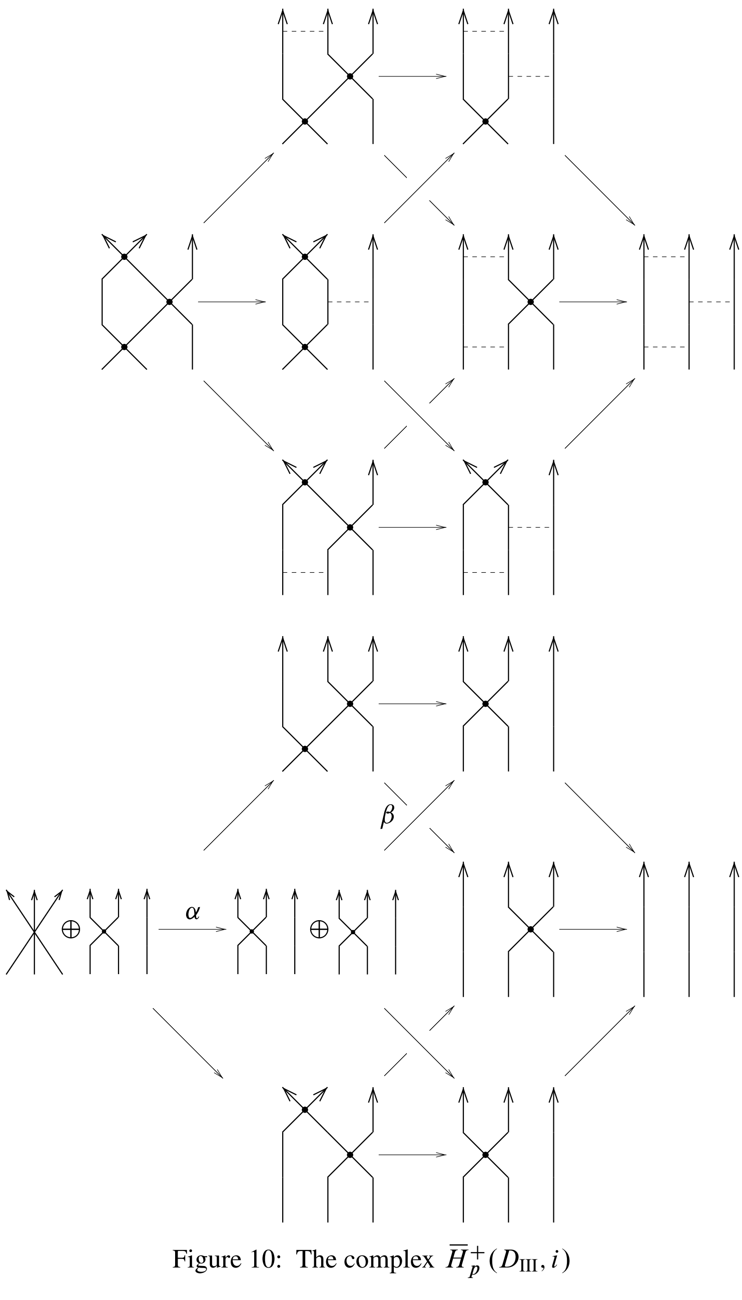


Figure 10: The complex $\overline{H}_p^+(D_{\mathrm{III}}, i)$

illustrated in the top half of Figure 10. After applying the functor $\mathcal{H}^+$ and using the MOY II and III decompositions, we get the diagram for $\overline{H}_p^+(\overline{D}_{\mathrm{III}})$ shown in the bottom half of the figure. Our first claim is that the maps labeled α and β are isomorphisms. The proof is the same as it was for the Reidemeister II move — we consider the

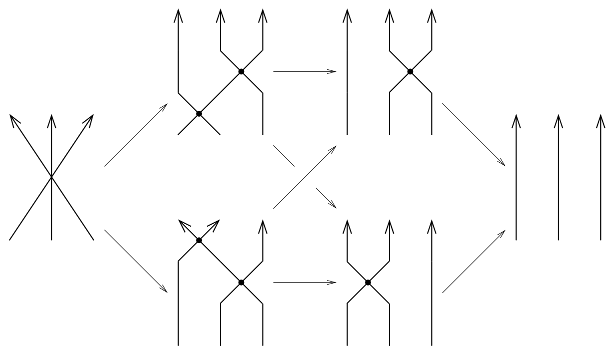


Figure 11: Simplified version of $\overline{H}_p^+(D_{\mathrm{III}}, i)$

analogous decomposition of $\overline{C}_p(\overline{D}_{\mathrm{III}}, i)$ in the category $\mathcal{K}$, where Khovanov and Rozansky proved that the corresponding maps $\widetilde{\alpha}$ and $\widetilde{\beta}$ are isomorphisms. (This is the first part of the proof of [13, Proposition 8].) Canceling the summands connected by α and β, we see that $\overline{H}_p^+(\overline{D}_{\mathrm{III}}, i)$ is homotopy equivalent (in $\mathcal{C}^+$) to a complex C of the form shown in Figure 11.

An analogous simplification of $\overline{H}_p^+(D'_{\mathrm{III}})$ shows that it is homotopy equivalent to a complex C' which also has the form shown in Figure 11. To be precise, it has the same six subquotients as C. *A priori*, however, the morphisms between them may be different. We claim that in reality this is not the case; the maps in C and C' corresponding to a fixed arrow in the diagram are the same up to multiplication by a nonzero element of $\mathbb{Q}$. It follows that $C \cong C'$ in $\mathcal{C}^+$, which gives us the desired homotopy equivalence between $\overline{H}_p^+(D_{\mathrm{III}}, i)$ and $\overline{H}_p^+(D'_{\mathrm{III}}, i)$.

To prove the claim, we go back to the proof of the Reidemeister III move in [13]. There, Khovanov and Rozansky consider the complexes of the open diagrams $C_p(D_{\mathrm{III}})$ and $C_p(D'_{\mathrm{III}})$ in the category $\mathcal{K}_{\mathrm{III}}$. As we have described above, they show that they are homotopy equivalent to complexes C_o and C'_o of the form illustrated in Figure 11. Moreover, they show that the morphisms in C_o and C'_o corresponding to a fixed edge in the figure are nonzero multiples of each other. Going from this statement to our claim is just a matter of applying functoriality. More precisely, suppose f and f' are morphisms in C and C' associated to some edge of the diagram, and that f_o and f'_o are the corresponding morphisms in C_o and C'_o. Then

$$\mathcal{F}(f) = \mathcal{H}^+(f_o \otimes 1) \quad \text{and} \quad \mathcal{F}(f') = \mathcal{H}^+(f'_o \otimes 1)$$

It follows that $\mathcal{F}(f)$ is a nonzero multiple of $\mathcal{F}(f')$. Since a morphism of complexes is determined by its action on modules, f is a nonzero multiple of f'. This concludes the proof of Reidemeister III invariance. □

6 An additional sequence

We now turn our attention to the spectral sequence described in Theorem 3, which is a special case of the following result:

Theorem 6.1 *Suppose $L \subset S^3$ is an ℓ–component link, and let U^ℓ be the ℓ–component unlink. There's a spectral sequence $E_k(-1)$ which has E_2 term $\overline{H}(L)$ and converges to $\overline{H}(U^\ell)$. The differentials in this sequence raise the cohomological grading gr_+ by 1 and preserve the polynomial grading $\mathrm{gr}'_{-1} = q - 2\,\mathrm{gr}_h$.*

More precisely, the differential d_k is homogenous of degree $(2-2k, 2-2k, 2k)$ with respect to the triple grading on $\overline{H}(L)$.

Compared with the sequences of the preceding section, this construction of $E_k(-1)$ is quite simple. Indeed, the fact that such a sequence should exist is well-known to experts in the field. It is more surprising, however, that the behavior of this sequence with respect to the triple grading should so closely match the behavior predicted in [3] for the "canceling differential" d_{-1}.

Proof We represent L by a braidlike diagram D, and consider the globally reduced complex $C_r(D)$ defined in Section 2.8. The triple $(C_r(D), d_+, d_v)$ is a double complex with respect to the bigrading $(\mathrm{gr}_h, \mathrm{gr}_v)$. The total differential on this complex is $d_+ + d_v$, and the total grading is $\mathrm{gr}_h + \mathrm{gr}_v = \mathrm{gr}_+$. In addition, since d_+ and d_v have degrees $(2,2,0)$ and $(0,0,2)$ with respect to the triple grading on $C_r(D)$, $d_+ + d_v$ preserves the polynomial grading $\mathrm{gr}'_{-1} = q - 2\,\mathrm{gr}_h$.

The spectral sequence of the theorem is induced by the horizontal filtration on the complex $(C_r(D), d_+ + d_v)$. This sequence has E_2 term $H(H(C_r(D), d_+), d_v^*) = \overline{H}(L)$, and converges to the total homology $H(C_r(D), d_v + d_+)$.

To compute the latter group, we consider the spectral sequence induced by the vertical filtration on $C_r(D)$. The E_1 term of this sequence is $H(C_r(D), d_v)$. Recall that $C_r(D)$ is a tensor product of factors, one for each crossing of D:

$$C_r(D) = \bigotimes_c C_r(D_c).$$

If we ignore the differential d_+, the factor associated to a crossing with sign ± 1 has the form

$$C_r(D_c) = M_c\{\pm 1, \mp 1, \mp 1\} \oplus N_c\{0, \pm 1, \mp 1\},$$

where

$$M_c = R_r(D)\{1,-1,-1\} \xrightarrow{(X_k - X_j)} R_r(D)\{-1,-1,1\},$$

$$N_c = R_r(D)\{0,-1,-1\} \xrightarrow{\quad 1 \quad} R_r(D)\{0,-1,1\}.$$

The complex N_c is contractible, so $C_r(D)$ is the direct sum of

$$M = \bigotimes_c M_c\{\pm 1, \mp 1, \mp 1\}$$

and a contractible complex. It follows that $H(C_r(D), d_v) \cong H(M, d_v)$. Each factor M_c is supported in a single value of gr_h, so M and $H(M, d_v)$ are supported in a single value of gr_h as well. Thus the spectral sequence has converged at the E_1 term, and

$$H(C_r(D), d_v + d_+) \cong H(M, d_v).$$

To evaluate $H(M, d_v)$, we observe that M_c has the same form as the complex $C_p(D_r)$ introduced in the proof of Lemma 4.9, but with d_v in place of d_h. Thus $H(M, d_v) \cong \overline{H}^+(D')$, where D' is the abstract graph obtained by replacing each crossing of D with a "crossing" of type D_r. Since the differential in M_c is multiplication by $X_k - X_j$, the ends labeled j and k lie on a solid segment of D_r, as do the ends labeled i and l. By Lemma 4.10, $\overline{H}^+(D') \cong \overline{H}^+(D'')$, where D'' is the abstract graph obtained by erasing the dashed lines in each copy of D_r. In other words, D'' is obtained by thinking of L as a topological space and entirely forgetting its embedding in $\mathbb{R}^3$. Thus D'' is a disjoint union of ℓ circles and $H(M, d_v) \cong \overline{H}(U^\ell)$. □

When $\ell = 1$, $H(M)$ is one-dimensional, and is supported in the top homological grading of M. If we compute the gradings gr_+ and gr'_{-1} for this generator, we find that they are given by $\mathrm{gr}_+ = -2w$ and $\mathrm{gr}'_{-1} = 2w$, where w is the writhe of D. Together with the overall shift of $\{-w+b-1, w+b-1, w-b+1\}$ in the triple grading, this means that the total homology $H(C_r(D), d_v + d_+)$ is supported in gradings $\mathrm{gr}_+ = \mathrm{gr}'_{-1} = 0$. More generally, the total homology will have Poincaré polynomial

$$\sum_{\substack{i=\mathrm{gr}_+ \\ j=\mathrm{gr}'_{-1}}} t^i q^j \dim H^{i,j}(C_r(D), d_v + d_+) = \left(\frac{1+t^{-1}}{1-q}\right)^{\ell-1}.$$

We can also prove an analog of Corollary 5.6.

Lemma 6.2 *The differential $d_k(-1)$ anticommutes with $d_1(p)$ for any value of p.*

Proof In this situation, it's more convenient to work with $\overline{C}_p(D,i)$ than $C_r(D)$. The isomorphism between the two described in Lemma 2.12 clearly respects their structure as double complexes, so we can think of $E_k(-1)$ as being the spectral sequence induced by the horizontal filtration on $(\overline{C}_p(D,i), d_+, d_v)$. Since d_- anticommutes with both d_+ and d_v, it defines a morphism $d_-\colon \overline{C}_p(D,i) \to \overline{C}_p(D,i)$ in the category $\mathrm{Kom}(\mathrm{Kom}(R(D)))$. There is an induced morphism $(d_-)_k\colon E_k(-1) \to E_k(-1)$ which anticommutes with $d_k(-1)$. The map $(d_-)_k$ is induced by the action of d_- on $E_k(-1)$. In particular, $(d_-)_2$ is the map induced by d_- on $E_2(-1) \cong \overline{H}(L)$ — in other words, $(d_-)_2 = d_1(p)$. This proves the claim. □

7 Examples

In this section, we compute the KR–homology of some simple knots. We begin by giving a quick proof of Theorem 1. Next, we discuss the notion of a KR–thin knot and show that two-bridge knots are KR–thin. We then derive a skein exact sequence which is useful for making calculations. Combining this with some computations of Webster [24], we are able to determine the KR–homology of all knots with 9 crossings or fewer.

7.1 Homology of knots

The reduced homology of a knot has the following important property:

Proposition 7.1 *If K is a knot, then $\overline{H}(K)$ is a finite-dimensional vector space over $\mathbb{Q}$.*

Proof Suppose D is a diagram representing a link L. The complex $\overline{C}_p(D,i)$ is a finitely generated module over the ring $R(D)$, so $\overline{H}(D,i)$ is finitely generated over the ring $\mathbb{Q}[X_j]$, where j runs over the edges of D. According to Lemma 5.16, multiplication by X_j and X_k are homotopic as morphisms of $\overline{C}_p(D,i)$ whenever j and k belong to the same component of L. In particular, if $L = K$ is a knot, multiplication by any X_j is homotopic to multiplication by X_i, which is the zero map on $\overline{C}_p(D,i)$. It follows that all X_j act trivially on $\overline{H}(K,i) \cong \overline{H}(K)$, so this group is finitely generated over $\mathbb{Q}$. □

Proof of Theorem 1 Since $\overline{H}(K)$ is finite-dimensional, it is supported in finitely many q–gradings. Consider the spectral sequence $E_k(N)$ which relates $\overline{H}(K)$ to $\overline{H}_N(K)$. The k^{th} differential in this sequence raises the q–grading by $2kN$. Thus, when N is sufficiently large, all the differentials beyond d_0 must vanish, and the sequence has converged at the E_1 term. □

7.2 KR–thin knots

In both Khovanov homology and knot Floer homology, the simplest knots exhibit a very similar pattern of behavior, in which there is a linear relation between the two gradings and the signature of the knot. Such knots are said to be *thin*. An analogous definition of thinness in the context of KR–homology was proposed in [3]. In terms of our current normalizations, it is:

Definition 7.2 A knot $K \subset S^3$ is *KR–thin* if $\overline{H}^{i,j,k}(K) = 0$ whenever $i + j + k \neq \sigma(K)$.

Our sign convention for σ is that positive knots have positive signature. The quantity $\delta = i + j + k$ which appears in the definition occurs frequently, and it is often convenient to think of the grading on $\overline{H}$ as being determined by the triple (i, j, δ), rather than i, j, k. From this point of view, it is clear that the HOMFLY homology of a KR–thin knot is completely determined by its signature and HOMFLY polynomial. The same statement holds for the $sl(N)$ homology as well:

Proposition 7.3 *If K is KR–thin, then the isomorphism of Theorem 1 holds for all $N > 1$.*

Proof Consider the spectral sequence $E_k(N) := E_k(x^{N+1})$ which relates $\overline{H}(K)$ to $\overline{H}_N(K)$. The differential $d_k(N)$ is triply graded of degree $(2kN, -2k, 2-2k)$, so it raises $\delta = i + j + k$ by $2 + 2k(N-2)$. This quantity is positive whenever $N > 1$. Since $E_1(N) \cong \overline{H}(K)$ is supported in a single value of δ, it follows that $d_k \equiv 0$ for all $k > 0$, and the sequence converges at the E_1 term. □

In [21], knots for which $\overline{H}_N$ took this form were called N–thin. In this language, the proposition says that if a knot is KR–thin, then it is N–thin for all $N > 1$. Conversely, we have the following result, which is an immediate consequence of Theorem 1.

Proposition 7.4 *If K is N–thin for all sufficiently large N, then K is KR–thin.*

The main result of [21] says that two-bridge knots are N–thin for all $N > 4$, so they are KR–thin as well. We thus arrive at the statement of Corollary 1 from the introduction.

Curiously, it seems difficult to prove this result without appealing to the $sl(N)$ homology. The issue is that $\overline{H}_N(K)$ can be computed using any planar diagram of K, whereas the definition of $\overline{H}(K)$ requires that we use a braid diagram. Any two-bridge knot admits a simple plat diagram of the form shown in Figure 12, which can be used to

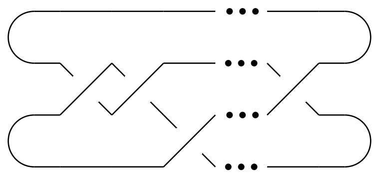


Figure 12: Plat diagram of a two-bridge knot

compute $\overline{H}_N(K)$. In contrast, the minimal braid diagram of such a knot can be quite complicated, and there does not seem to be an easy way to compute $\overline{H}(K)$ from it.

By a well-known theorem of Lee [15], the Khovanov homology of any alternating knot is thin. In [20], Ozsváth and Szabó proved a similar result for the knot Floer homology. As observed in [3], however, it cannot be the case that all alternating knots are KR–thin. Indeed, the HOMFLY polynomial of a KR–thin knot must be *alternating*, in the sense that

$$P_K(a,q) = (-1)^{\sigma(K)} \sum_{i,j} c_{ij} a^{2j} (-q^2)^i,$$

with $c_{ij} \geq 0$. On the other hand, it is not difficult to find alternating knots whose HOMFLY polynomials are not alternating.

We conclude our discussion of KR–thin knots by considering their behavior with respect to the spectral sequences $E_k(1)$ and $E_k(-1)$. We have already seen that the sequences $E_k(N)$ are essentially trivial for $N > 1$. This cannot be true for $E_k(\pm 1)$, however, since they converge to $\mathbb{Q}$. Instead, we have:

Lemma 7.5 *If K is KR–thin, then the spectral sequences $E_k(1)$ and $E_k(-1)$ converge after the first differential on $\overline{H}(K)$. (That is, at $E_2(1)$ and $E_3(-1)$.)*

Proof As we saw in the proof of Proposition 7.3, the differential $d_k(1)$ shifts δ by $2-2k$. Since $E_1(1) \cong \overline{H}(K)$ is supported in a single δ–grading, $d_k(1)$ is trivial for all $k > 1$. Similarly, the differential $d_k(-1)$ is triply graded of degree $(2-2k, 2-2k, 2k)$, so it shifts δ by $4-2k$. Thus it vanishes for all $k > 2$. □

It follows that the spectral sequence of a KR–thin knot behaves as conjectured in [3].

7.3 A skein exact sequence

Suppose D is a planar diagram representing a two-component link L, and that i and j are edges of D belonging to the two components of L. Let $\overline{C}_N(D,i) = (H(\overline{C}_p(D,i), d_{\mathrm{tot}}), d_v^*)$ be the $sl(N)$ chain complex, and form the mapping cone

$$\overline{C}_N(D,i,j) = \overline{C}_N(D,i)\{1,0,-1\} \xrightarrow{X_j} \overline{C}_N(D,i)\{-1,0,1\},$$

where X_j denotes the map induced by multiplication by X_j. The grading shifts are chosen so that X_j —like d_v^* —is homogenous of degree $(0,0,2)$ with respect to the triple grading. We call the homology of $\overline{\overline{C}}_N(D,i,j)$ the *totally reduced homology* of L and denote it by $\overline{\overline{H}}_N(L)$. Using Lemma 5.16, it is not difficult to see that $\overline{\overline{H}}_N(L)$ is independent of the choice of i and j, although we will not use this fact here.

The group $\overline{H}_N(L,i)$ can naturally be viewed as a $\mathbb{Q}[X]$ module, where X acts as multiplication by X_j. If we understand the module structure of $\overline{H}_N(L,i)$, we can easily determine the totally reduced homology from the long exact sequence

$$\cdots \longrightarrow \overline{\overline{H}}_N(L) \longrightarrow \overline{H}_N(L,i) \xrightarrow{X_j} \overline{H}_N(L,i) \longrightarrow \overline{\overline{H}}_N(L) \longrightarrow \cdots .$$

One such case is when L is a two-bridge link. In [21], it was shown that $\overline{H}_N(L,i)$ is composed of a number of summands on which X acts trivially, together with a single summand isomorphic to $\mathbb{Q}[X]/X^{N-1}$. The generators of each summand have δ–grading congruent to $\sigma(L) \mod (2N-4)$. (On $\overline{H}_N$, we can't tell the difference between a^2 and q^{2N}, so the δ–grading is only defined modulo $(2N-4)$.) From this, it is not difficult to see that $\overline{\overline{H}}_N(L)$ is also thin, in the sense that it is supported in δ–gradings congruent to $\sigma(L)$. In analogy with the case of knots, we say that L is KR–thin if $\overline{\overline{H}}_N(L)$ is thin for all $N \gg 0$.

For the moment, our interest in the group $\overline{\overline{H}}(L)$ arises from the following skein exact sequence, which generalizes the oriented skein relation for the $sl(N)$ polynomial.

Proposition 7.6 *Suppose L_+ and L_- are two knots related by a crossing change, and L_0 is the two-component link obtained by resolving the crossing. Then there is a long exact sequence*

$$\cdots \xrightarrow{(0,0)} \overline{H}_N(L_-) \xrightarrow{(-N,1)} \overline{\overline{H}}_N(L_0) \xrightarrow{(N,1)} \overline{H}_N(L_+) \xrightarrow{(0,0)} \overline{H}_N(L_-) \xrightarrow{(-N,1)} \cdots .$$

The numbers over each arrow indicate the degree of the corresponding map with respect to the (q,δ) bigrading on $\overline{H}_N$. For example, the map $\overline{\overline{H}}_N(L_0) \to \overline{H}_N(L_+)$ raises the q–grading by N and δ by 1.

Proof The complex $\overline{C}_N(L_-)$ is the mapping cone of the map $\chi_0\colon \overline{C}_N(L_0) \to \overline{C}_N(L_s)$, where L_s is the diagram obtained by replacing the crossing in question with a singular point. Similarly, $\overline{C}_N(L_+)$ is the mapping cone of $\chi_1\colon \overline{C}_N(L_s) \to \overline{C}_N(L_0)$, from which it follows that $\overline{C}_N(L_s)$ is homotopy equivalent to the mapping cone of the inclusion $i\colon \overline{C}_N(L_0) \to \overline{C}_N(L_+)$. An explicit homotopy equivalence is given by the map

$$\iota\colon \overline{C}_N(L_s) \to \overline{C}_N(L_0) \oplus \overline{C}_N(L_s) \oplus \overline{C}_N(L_0)$$

which sends $a \in \overline{C}_N(L_s)$ to $(-\chi_1(a), a, 0)$. It is easy to see that ι is the inclusion in a strong deformation retract, in the sense of Bar-Natan [2]. By [2, Lemma 4.5], it follows that $\overline{C}_N(L_-)$ is homotopy equivalent to the mapping cone of $\iota\chi_0$: $\overline{C}_N(L_0) \to \mathrm{Cone}(i)$. This complex has a three-step filtration, as illustrated in the diagram

$$\overline{C}_N(L_0) \xrightarrow{-\chi_1\chi_0} \overline{C}_N(L_0) \longrightarrow \overline{C}_N(L_+).$$

It follows that there is a short exact sequence

$$0 \longrightarrow \overline{C}_N(L_+) \longrightarrow \mathrm{Cone}(\iota\chi_0) \longrightarrow \mathrm{Cone}(\chi_1\chi_0) \longrightarrow 0.$$

Considering the associated long exact sequence, we see that, to prove the lemma, it suffices to show that $\mathrm{Cone}(\chi_1\chi_0) \cong \overline{C}_N(D, i, j)$. To show this, recall that the composition $\chi_1\chi_0$ is multiplication by $X_j - X_i$, where i and j are the edges of L_0 adjacent to the resolution. Taking these two edges to be the edges i and j which appear in the definition of $\overline{\overline{H}}(L)$ gives the desired isomorphism. Finally, the bigrading of each map in the sequence can easily be determined from [21, Lemma 3.3]. This is left as an exercise to the reader. □

As an application, we have the following criterion for showing that a knot is KR–thin. It is a slight generalization of [21, Criterion 5.4].

Corollary 7.7 *Suppose that L_-, L_0 and L_+ are as above, that L_- and L_0 are both KR–thin, and that* $\det L_- + 2\det L_0 = \det L_+$. *Then L_+ is KR–thin as well.*

Proof Suppose N is very large. Then all three terms in the exact sequence stabilize — L_0 by hypothesis, and L_- and L_+ by Theorem 1. We have

$$\mathrm{rank}\, \overline{H}_N(L_+) \geq \det L_+ = \mathrm{rank}\, \overline{H}_N(L_-) + \mathrm{rank}\, \overline{\overline{H}}_N(L_0)$$

since both L_- and L_0 are thin. For this to happen, the map $\overline{H}_N(L_-) \to \overline{\overline{H}}_N(L_0)$ in the exact sequence must vanish. To show that L_+ is N–thin, it is enough to check that $\sigma(L_+) = \sigma(L_-) = \sigma(L_0) + 1$. This follows from the usual skein-theoretic constraint on the signature. (See the proof of [21, Criterion 5.4] for details.) Finally, since L_+ is N–thin for all large N, it is KR–thin as well. □

The analogous statement with the roles of L_- and L_+ reversed also holds.

7.4 Connected sums

In applying the skein exact sequence of the previous section, one often encounters non-prime knots and links. For this reason, it is convenient to understand the behavior of the KR–homology under connected sum.

Suppose L_1 and L_2 are oriented links with marked components i_1 and i_2. Up to isotopy, there is a unique way to form their orientation-preserving connected sum along i_1 and i_2. We denote the resulting link by $L_1 \#_{i_1=i_2} L_2$.

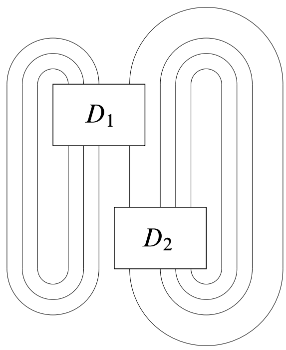


Figure 13: The connected sum of braids D_1 and D_2

Lemma 7.8 *There are isomorphisms*

$$\overline{H}(L_1 \#_{i_1=i_2} L_2) \cong \overline{H}(L_1) \otimes \overline{H}(L_2),$$
$$\overline{H}_N(L_1 \#_{i_1=i_2} L_2, i_1) \cong \overline{H}_N(L_1, i_1) \otimes \overline{H}_N(L_2, i_2).$$

Proof Suppose L_i is represented by a braid diagram D_i on b_i strands. Without loss of generality, we may arrange the diagrams D_i so i_1 is the rightmost strand of D_1 and i_2 is the leftmost strand of D_2. Then the connected sum $L_1 \#_{i_1=i_2} L_2$ can be represented by a braid diagram $D_1 \# D_2$ on $b_1 + b_2 - 1$ strands, as illustrated in Figure 13. Let D_1^o be the open diagram obtained by removing a neighborhood of the connected sum point in D_1, and let i_1^+ and i_1^- be its free ends. Then $X_{i_1^+} = X_{i_1^-}$ in $R(D_1^o)$, and

$$C_p(D_1) \cong C_p(D_1^o)|_{X_{i_1^+}=X_{i_1^-}} \cong C_p(D_1^o).$$

From the local nature of the KR–complex, we see that

$$\begin{aligned} C_p(D_1 \# D_2) &\cong C_p(D_1^o) \otimes C_p(D_2^o)|_{X_{i_1^+}=X_{i_2^-}, X_{i_1^-}=X_{i_2^+}} \\ &\cong C_p(D_1) \otimes C_p(D_2)|_{X_{i_1}=X_{i_2}}. \end{aligned}$$

It follows that the reduced KR–complex has the form

$$C_p(D_1 \# D_2, i_1) \cong C_p(D_1, i_1) \otimes C_p(D_2, i_2).$$

Applying the Kunneth formula twice gives the statement of the lemma. □

Corollary 7.9 *The connected sum of two KR–thin knots is KR–thin.*

Similarly, it is not difficult to see that the connected sum of a KR–thin knot and a KR–thin link is also KR–thin.

7.5 Small knots

We conclude by describing the KR–homology of knots with 9 or fewer crossings. Previous computations of $\overline{H}$ have been made by Khovanov [11], who showed that the $(2,n)$ torus knots are KR–thin, and by Webster [24], who wrote a computer program for this purpose. Using it, he computed $\overline{H}$ for knots up through 7 crossings, all of which are KR–thin. For larger knots, the program is very effective at computing the homology of knots which can be represented as closures of three-strand braids, but less useful in other cases. Fortunately, many of the small knots with large braid index are two-bridge, and thus covered by Corollary 1. The remainder can be analyzed using the skein exact sequence of Proposition 7.6. Combining the information from these various sources, we have the following result:

Proposition 7.10 *The only knots with* 9 *crossings or fewer which are not KR–thin are* 8_{19}, 9_{42}, 9_{43}, *and* 9_{47}. *(Numbering as in Rolfsen* [22]*.)*

Remarks The homology of 8_{19} (the $(3,4)$ torus knot) was computed by Webster [24]. The homology of the remaining three knots is illustrated in Figure 14. In all four cases, the homologies are symmetric; the sequences $E_k(-1), E_k(1)$ and $E_k(2)$ converge after the first differential on $\overline{H}(K)$; and $\overline{H}_N(K) \cong \overline{H}(K)$ for $N > 2$. In addition, the calculated values of $\overline{H}(8_{19})$ and $\overline{H}(9_{42})$ agree with the predictions made in [3].

Proof The only knots with 8 or fewer crossings which are not two-bridge are $8_5, 8_{10}$ and 8_{15}–8_{21}. Among these, all but 8_{15} have braid index 3 and were computed by Webster [24]. The only one which is not thin is the $(3,4)$ torus knot 8_{19}. In [21], it was shown that 8_{15} is N–thin for all $N > 4$. Thus it is also KR–thin.

For the 9–crossing knots, we have to work a little harder. The knots 9_{16}, 9_{22}, 9_{24}, 9_{25}, 9_{28}, 9_{29}, 9_{30} and 9_{32}–9_{49} are not two-bridge. Only one — 9_{16} — is the closure of a three-strand braid, and Webster's program shows that it is KR–thin. Of the rest,

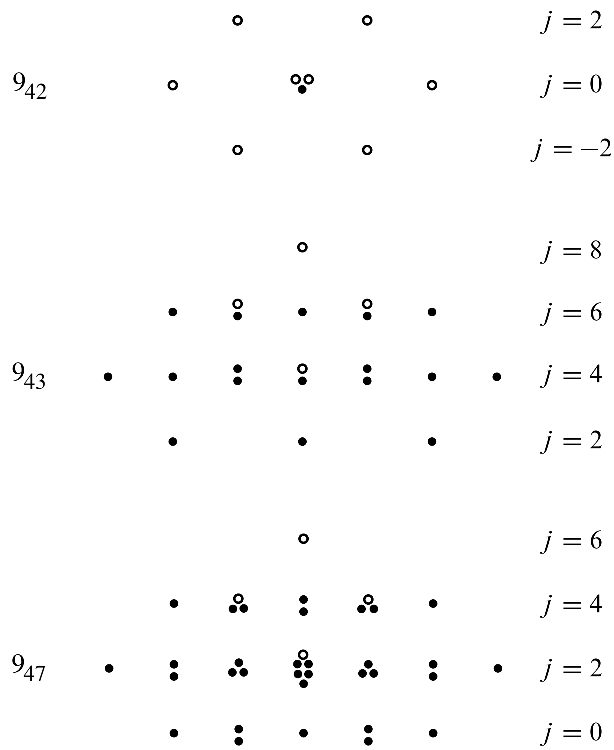
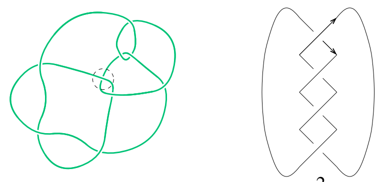


Figure 14: HOMFLY homology of the knots 9_{42}, 9_{43} and 9_{47}, represented by "dot diagrams". Each dot represents a generator of the homology. The i and j gradings are indicated by the positions in the horizontal and vertical directions, respectively ($i = 0$ corresponds to the axis of symmetry). The solid and hollow dots have different δ–gradings. For 9_{42}, hollow dots have $\delta = -2$, and the solid dot has $\delta = 0$. For 9_{43}, the values are $\delta = 2$ and $\delta = 4$, respectively, and for 9_{47}, they are $\delta = 0$ and $\delta = 2$.

all but five — 9_{29}, 9_{42}, 9_{43}, 9_{46} and 9_{47} — can be shown to be KR–thin using the criterion of Corollary 7.7. These knots, and the crossing to which the criterion can be applied, are shown in Figures 17 and 18 at the end of the paper.

The knot 9_{29} can be seen to be KR–thin by a similar but slightly more elaborate argument. If we change the marked crossing on the right side of the knot in Figure 17, we get the two-bridge knot 7_6. Resolving the crossing gives the link 7^2_4. We claim that this link is KR–thin. To see this, we consider the second marked crossing in the figure.



9_{42} $\qquad$ 4^2_1

Figure 15: We consider the skein exact sequence associated to the circled crossing in diagram of 9_{42} shown above. L_0 is the link 4^2_1, oriented as shown.

Changing this crossing gives the connected sum of the Hopf link and the trefoil knot, which is KR–thin and has determinant 6. Resolving the crossing gives the knot 5_1, which has determinant 5. Since the determinant of 7^2_4 is $16 = 6 + 2 \cdot 5$, it's not difficult to see that 7^2_4 is KR–thin. Then 9_{29} must be KR–thin as well.

To analyze the four remaining knots, we resort to a more detailed study of the skein exact sequence. We illustrate this process in the case of the knot 9_{42}, which is shown in Figure 15. If we change the circled crossing in the figure from positive to negative, the result is the connected sum of the negative trefoil and the figure-eight knot. Resolving the crossing, we get the two-bridge link 4^2_1 shown on the right-hand side of the figure. Thus we have a long exact sequence

$$\cdots \longrightarrow \overline{H}_N(3_1 \# 4_1) \longrightarrow \overline{\overline{H}}_N(4^2_1) \longrightarrow \overline{H}_N(9_{42}) \longrightarrow \overline{H}_N(3_1 \# 4_1) \longrightarrow \cdots .$$

When N is large, all three terms in this sequence stabilize. Both $3_1 \# 4_1$ and 4^2_1 are KR–thin, so their homologies are determined by their HOMFLY polynomials. In Figure 16, we have superimposed diagrams representing the homology of $3_1 \# 4_1$ (hollow dots) and 4^2_1 (solid dots). The j–gradings are shifted so that they correspond to the power of a in the HOMFLY polynomial of 9_{42}. Under the assumption that N is large, nontrivial components of the map

$$\overline{H}_N(3_1 \# 4_1) \to \overline{\overline{H}}_N(4^2_1)$$

must preserve the position of the generators. In other words, a generator corresponding to a hollow dot at any of the lettered positions can map nontrivially to a solid dot at the same position, but not to anything else.

From the figure, we can deduce some constraints on the group $\overline{H}_N(9_{42})$. For example, the group at position c must have rank either 2 or 0, depending on whether the map from the hollow to the solid generator at that position is trivial or nontrivial. Since $\overline{H}_N(9_{42}) \cong \overline{H}(9_{42})$ when N is large, the same is true for $\overline{H}$ as well.

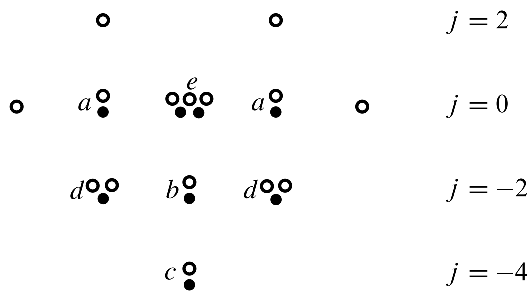


Figure 16: Possible generators of $\overline{H}(9_{42})$

We can now use Theorems 2 and 3 to deduce the exact value of the homology. For example, suppose the two generators labeled a on the right-hand side of the figure survive in $\overline{H}(9_{42})$. They must die in the spectral sequence $E_k(1)$, but there is nothing to kill them. We conclude that these generators could not have survived. A similar argument using $E_k(-1)$ shows that the two left-hand generators labeled a do not survive either. It is now easy to see that generators labeled b must kill each other too. □

To eliminate the generators labeled c, we consider the spectral sequence $E_k(2)$, which converges to the usual Khovanov homology. Clearly, if these generators survive in $\overline{H}(K)$, they will also appear in $\overline{H}_2(K)$, where they will have q–grading -8. On the other hand, it is well known that $\overline{H}_2(9_{42})$ has Poincaré polynomial

$$\mathcal{P}_2(9_{42}) = q^{-6}t^{-4} + q^{-4}t^{-3} + q^2t^{-2} + 2t^{-1} + 1 + q^2 + q^4t + q^6t^2.$$

There is no term with q^{-8}, so the generators in position c must die. Next, we consider the positions labeled d, where we have a map from a two-dimensional space generated by the hollow dots to a one-dimensional space generated by the solid dot. Now that we know that there is nothing in position c, considering $E_k(\pm 1)$ shows that both maps must be surjective. Finally, in position e, we have a map from a space of dimension 3 to a space of dimension 2. Examining the sequence $E_k(2)$ shows that this map must have rank 1. Thus the homology is as shown in Figure 14.

Similar considerations may be applied to compute the homology of the knots $9_{43}, 9_{46}$ and 9_{47}. Rather than go into details, we simply indicate an appropriate crossing for each knot in Figure 18, and leave it to the interested reader to check the rest.

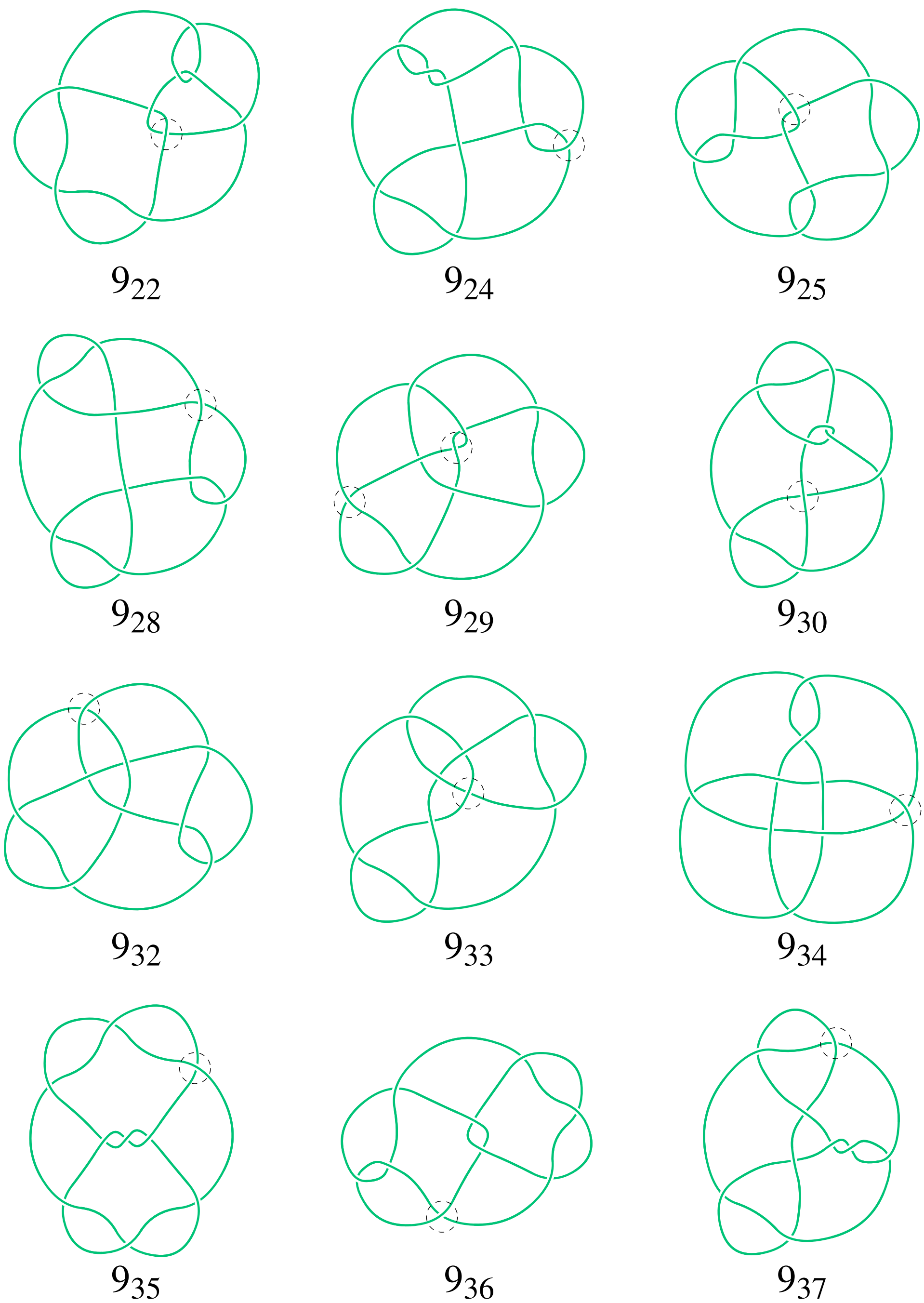


Figure 17: 9-crossing knots (I). Figures drawn by *Knotscape* [6].

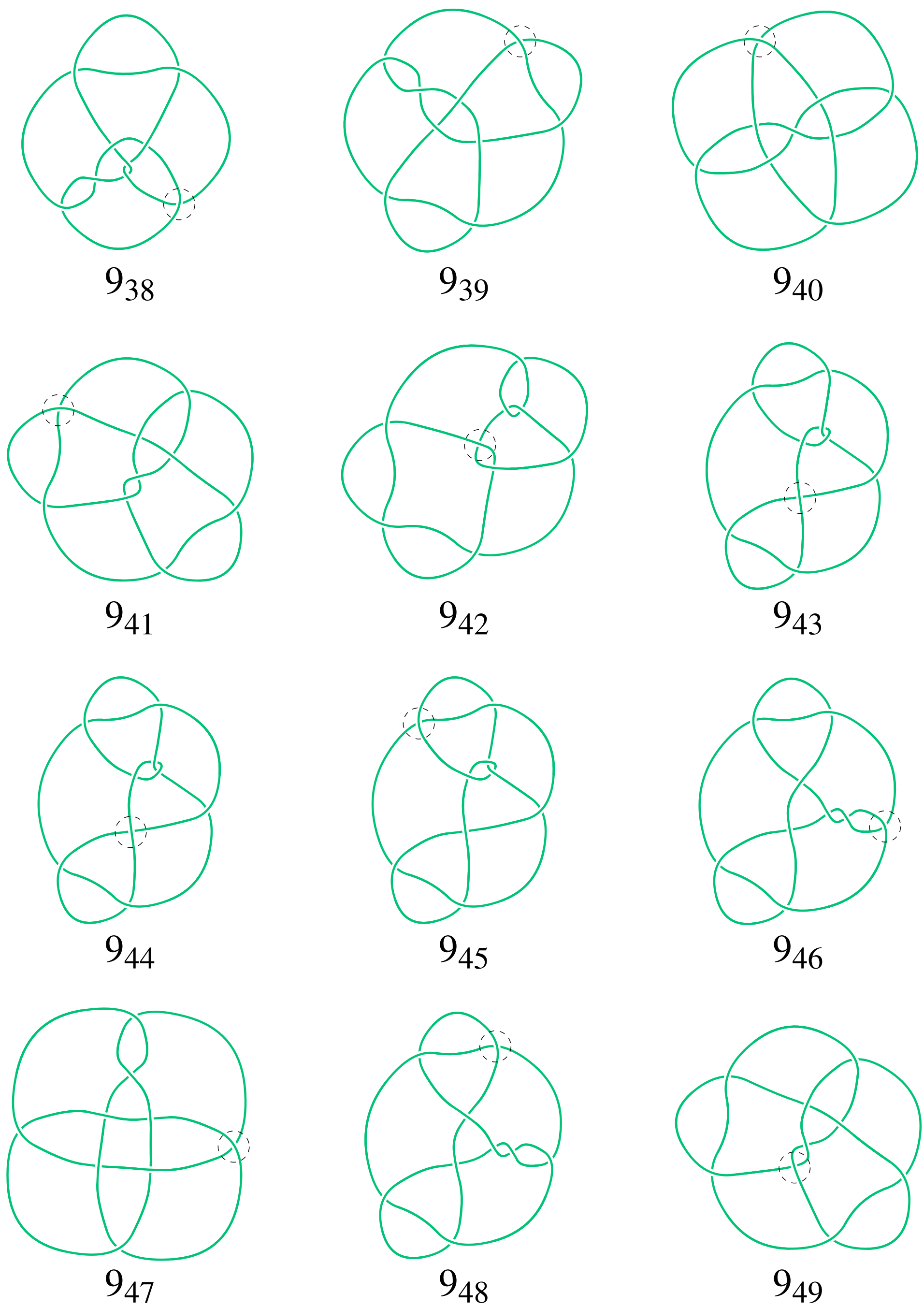


Figure 18: 9-crossing knots (II). Figures drawn by *Knotscape* [6].

References

[1] **D Bar-Natan**, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. 2 (2002) 337–370 MR1917056

[2] **D Bar-Natan**, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. 9 (2005) 1443–1499 MR2174270

[3] **N M Dunfield**, **S Gukov**, **J Rasmussen**, *The superpolynomial for knot homologies*, Experiment. Math. 15 (2006) 129–159 MR2253002

[4] **B Gornik**, *Note on Khovanov link cohomology* `arXiv:math/0402266`

[5] **S Gukov**, **A Schwarz**, **C Vafa**, *Khovanov–Rozansky homology and topological strings*, Lett. Math. Phys. 74 (2005) 53–74 MR2193547

[6] **J Hoste**, **M Thistlethwaite**, *Knotscape* (1999) computer program. Available at `http://www.math.utk.edu/~morwen/knotscape.html`

[7] **V F R Jones**, *On knot invariants related to some statistical mechanical models*, Pacific J. Math. 137 (1989) 311–334 MR990215

[8] **L H Kauffman**, *State models and the Jones polynomial*, Topology 26 (1987) 395–407 MR899057

[9] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426 MR1740682

[10] **M Khovanov**, *Patterns in knot cohomology, I*, Experiment. Math. 12 (2003) 365–374 MR2034399

[11] **M Khovanov**, *Triply-graded link homology and Hochschild homology of Soergel bimodules*, Internat. J. Math. 18 (2007) 869–885 MR2339573

[12] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology*, Fund. Math. 199 (2008) 1–91 MR2391017

[13] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology, II*, Geom. Topol. 12 (2008) 1387–1425 MR2421131

[14] **S Lang**, *Algebra*, 3rd edition, Graduate Texts in Mathematics 211, Springer, New York (2002) MR1878556

[15] **E S Lee**, *The support of the Khovanov's invariants for alternating knots* `arXiv:math/0201105`

[16] **E S Lee**, *An endomorphism of the Khovanov invariant*, Adv. Math. 197 (2005) 554–586 MR2173845

[17] **M Mackaay**, **P Vaz**, *The universal* sl_3*–link homology*, Algebr. Geom. Topol. 7 (2007) 1135–1169 MR2336253

[18] **J McCleary**, *User's guide to spectral sequences*, Mathematics Lecture Series 12, Publish or Perish, Wilmington, DE (1985) MR820463

[19] **H Murakami**, **T Ohtsuki**, **S Yamada**, *Homfly polynomial via an invariant of colored plane graphs*, Enseign. Math. 44 (1998) 325–360 MR1659228

[20] **P Ozsváth**, **Z Szabó**, *Heegaard Floer homology and alternating knots*, Geom. Topol. 7 (2003) 225–254 MR1988285

[21] **J Rasmussen**, *Khovanov–Rozansky homology of two-bridge knots and links*, Duke Math. J. 136 (2007) 551–583 MR2309174

[22] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish, Houston, TX (1990) MR1277811

[23] **V Turaev**, *Torsion invariants of* Spin^c*–structures on* 3*–manifolds*, Math. Res. Lett. 4 (1997) 679–695 MR1484699

[24] **B Webster**, *KR.m2* (2005) Macaulay2 program. Available at `http://tinyurl.com/nmcglnk`

[25] **H Wu**, *Braids, transversal links and the Khovanov–Rozansky theory*, Trans. Amer. Math. Soc. 360 (2008) 3365–3389 MR2386230

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge
Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK

j.rasmussen@dpmms.cam.ac.uk

Proposed: Peter S Ozsváth Received: 13 September 2006
Seconded: Ciprian Manolescu, Haynes Miller Accepted: 21 January 2015