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The homotopy theory of cyclotomic spectra

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We describe spectral model category structures on the categories of cyclotomic spectra and p–cyclotomic spectra (in orthogonal spectra) with triangulated homotopy categories. We show that the functors TR and TC are corepresentable in these categories. Specifically, the derived mapping spectrum out of the sphere spectrum in the category of cyclotomic spectra corepresents the finite completion of TC and the derived mapping spectrum out of the sphere spectrum in the category of p–cyclotomic spectra corepresents the p–completion of $TC(-;p)$.

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1 Introduction

Topological cyclic homology (TC) has proved to be an enormously successful tool for studying algebraic K–theory and K–theoretic phenomena. After finite completion, relative K–theory for certain pairs is equivalent to relative TC via the cyclotomic trace map and TC can be computed using the methods of equivariant stable homotopy theory.

The construction of TC begins with a cyclotomic spectrum; this is a $\mathbb{T}$–equivariant spectrum equipped with additional structure that mimics the structure seen on the free suspension spectrum of the free loop space, $\Sigma^\infty_+ \Lambda X$. Here $\mathbb{T}$ denotes the circle group of unit complex numbers and ΛX the $\mathbb{T}$–space of maps from $\mathbb{T}$ to X. The n^{th} root map induces an isomorphism $\rho_n\colon \mathbb{T} \cong \mathbb{T}/C_n$, which induces an isomorphism of $\mathbb{T}$–spaces $\rho_n^*(\Lambda X)^{C_n} \cong \Lambda X$, where C_n is the cyclic subgroup of order n and ρ_n^* is the change of group functor along ρ_n. This isomorphism then gives rise to an equivalence of $\mathbb{T}$–spectra $\rho_n^* \Phi^{C_n} \Sigma^\infty_+ \Lambda X \simeq \Sigma^\infty_+ \Lambda X$, where Φ^{C_n} denotes the derived geometric fixed point functor and ρ_n^* denotes both change of groups and change of universe.

Although TC has been around for over twenty years, there has been relatively little investigation of the nature of cyclotomic spectra or the homotopy theory associated to the category of cyclotomic spectra. Recently, in the course of proving the degeneration of the noncommutative "Hodge to de Rham" spectral sequence, Kaledin has described the close connection between cyclotomic spectra and Dieudonné modules and the

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relationship between TC and syntomic cohomology. This work led him to make conjectures [7, Section 7] regarding the structure of the category of cyclotomic spectra and its relationship to TC. The purpose of this paper is to prove these conjectures.

After a review of background, we begin in Section 4 by setting up a point-set category of cyclotomic spectra as a category of orthogonal $\mathbb{T}$–spectra with extra structure; see Definition 4.10. Topological Hochschild homology (THH) provides the primary source of examples of cyclotomic spectra. We also set up a (significantly simpler) point-set category of p–cyclotomic spectra: a p–cyclotomic spectrum is an orthogonal $\mathbb{T}$–spectrum X together with a map of $\mathbb{T}$–spectra

$$t\colon \rho_p^*\Phi^{C_p}T \longrightarrow T$$

from the (point-set) geometric fixed points of T back to T such that the composite in the homotopy category from the derived geometric fixed points is an $\mathcal{F}_p$–equivalence, i.e., induces an isomorphism on homotopy groups $\pi_*^{C_{p^n}}$ for all $n \geq 0$ (Definition 4.5). Since in most examples one works with the p–cyclotomic structure, in the remainder of this introduction we focus on this case for expositional simplicity.

Since the definition of cyclotomic spectra includes a homotopy-theoretic constraint, it is unreasonable to expect any category of cyclotomic spectra to be closed under general (or even finite) limits or colimits. Thus, we cannot expect a model category of cyclotomic spectra; nevertheless, we show that our category of cyclotomic spectra admits a *model structure* in the sense of Definition 1.1.3 of Hovey [6]: It has subcategories of cofibrations, fibrations and weak equivalences that satisfy Quillen's closed model category axioms. Moreover, the category of cyclotomic spectra admits finite coproducts and products, pushouts over cofibrations, and pullbacks over fibrations. These limits and colimits suffice to construct the entirety of the homotopy theory set up in Chapter I of Quillen [12] and much of the abstract homotopy theory developed since; e.g., see Radulescu-Banu [13], where this is worked out in even greater generality. As an example, since the category of cyclotomic spectra additionally admits filtered colimits along cofibrations, we can deduce that it has all homotopy colimits.

1.1 Definition A *model* category* is a category that has a model structure [6, Definition 1.1.3] and admits finite coproducts and products, pushouts over cofibrations, and pullbacks over fibrations.

One big advantage of model* categories over model categories is that any subcategory of a model* category that is closed under weak equivalences, finite products and coproducts, pushouts over cofibrations, and pullbacks over fibrations is a model* category with the inherited model structure. We take advantage of this in the proof of the following theorem, which is our main theorem on the homotopy theory of p–cyclotomic spectra.

1.2 Theorem *The category of p–cyclotomic spectra is a model* category with weak equivalences the weak equivalences of the underlying non-equivariant orthogonal spectra and fibrations the $\mathcal{F}_p$–fibrations* (see [8, Theorem IV.6.5] or Theorem 3.5 below) *of the underlying orthogonal $\mathbb{T}$–spectra, where $\mathcal{F}_p = \{C_{p^n}\}$.*

The model* category of p–cyclotomic spectra has additional structure. Clearly, it inherits an enrichment over spaces from the category of orthogonal $\mathbb{T}$–spectra. We show in Section 4 that it in fact inherits an enrichment over non-equivariant orthogonal spectra. This enrichment is compatible with the model structure in the sense that the analogue of Quillen's SM7 axiom holds; see Theorem 5.9. In particular, the homotopy category of p–cyclotomic spectra becomes triangulated with the usual definition of distinguished triangles and we have a good construction of intrinsic mapping spectra.

1.3 Theorem *The model structure on p–cyclotomic spectra has an enrichment over orthogonal spectra. The homotopy category of p–cyclotomic spectra is triangulated with the shift functor given by suspension and the distinguished triangles determined by the cofiber sequences specified by the model structure* (see [6, Definition 6.2.6]).

In fact, the homotopy type of the mapping spectra turns out to have a relatively straightforward description in terms of the underlying orthogonal spectra and the structure map t. See Theorem 5.12 for a precise statement.

Finally, $TC(-;p)$ has an intrinsic interpretation in the context of the homotopy category of p–cyclotomic spectra (after p–completion). The sphere spectrum S has a canonical cyclotomic structure using the canonical identification of the geometric fixed points of S as S (see Example 4.11). We can identify the p–completion of the right derived functor of $TC(-;p)$ as the derived mapping spectrum out of S.

1.4 Theorem *Let T be a p–cyclotomic spectrum. Then the derived mapping spectrum from the sphere spectrum S to T in the homotopy category of p–cyclotomic spectra becomes naturally isomorphic to the right derived functor of $TC(T;p)$ after p–completion. Moreover, the natural isomorphism of p–completed right derived functors is canonical.*

This confirms the conjecture of Kaledin [7, Remark 7.9]. Also, this theorem gives a motivic interpretation of $TC(-;p)$, viewing the triangulated homotopy category of

p–cyclotomic spectra as a category of "p–cyclotomic motives" associated to non-commutative schemes (viewed as spectral categories, with *THH* as the realization functor).

See Section 6 for additional corepresentability results.

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2 Review of orthogonal $\mathbb{T}$–spectra

In this section, we give a brief review of the definition of orthogonal $\mathbb{T}$–spectra and the geometric fixed point functors. Along the way we provide some new technical results that are needed in later sections. We begin with some preliminaries about the categories of $\mathbb{T}$–spaces we work with.

We work throughout with the category $\mathcal{U}$ of compactly generated weak Hausdorff spaces, the objects of which we call *spaces*. (As we never use more general topological spaces, this will cause no confusion.) We use $\mathcal{T}$ to denote the category of *based spaces*, which is the undercategory in $\mathcal{U}$ of the one-point space $* = \{\{\}\}$. The category $\mathcal{U}$ is complete, cocomplete and cartesian closed. The category $\mathcal{T}$ is complete, cocomplete and closed symmetric monoidal under the smash product; we also regard $\mathcal{T}$ as enriched over $\mathcal{U}$ by the forgetful functor, which is lax symmetric monoidal. The categories $\mathbb{T}\mathcal{U}$ and $\mathbb{T}\mathcal{T}$ of $\mathbb{T}$*–spaces* and *based* $\mathbb{T}$*–spaces* are by definition the category of $\mathbb{T}$–objects in $\mathcal{U}$ and $\mathcal{T}$, respectively, where $\mathbb{T}$ denotes the circle group, the Lie group of unit complex numbers. These categories are complete and cocomplete with the limits and colimits constructed in $\mathcal{U}$ and $\mathcal{T}$, respectively. The categories $\mathbb{T}\mathcal{U}$ and $\mathbb{T}\mathcal{T}$ are closed symmetric monoidal, with product given by the cartesian product and smash product, respectively, and with function objects the function spaces from $\mathcal{U}$ and $\mathcal{T}$ endowed with the conjugation $\mathbb{T}$–action. The $\mathbb{T}$–fixed point functors $\mathbb{T}\mathcal{U} \to \mathcal{U}$ and $\mathbb{T}\mathcal{T} \to \mathcal{T}$ are symmetric monoidal and give $\mathcal{U}$ and $\mathcal{T}$ enrichments, respectively.

There are several equivalent formulations of the category of orthogonal $\mathbb{T}$–spectra. The simplest definition of orthogonal $\mathbb{T}$–spectra in Mandell and May [8, Section II.2] turns out to be less technically convenient for our purposes than the reformulation in [8, Section II.4] in terms of diagram spaces. Recall that, for a skeletally small category $\mathcal{D}$ enriched in $\mathbb{T}\mathcal{T}$, a *$\mathcal{D}$–space* is a $\mathbb{T}\mathcal{T}$–enriched functor from $\mathcal{D}$ to $\mathbb{T}\mathcal{T}$ and a morphism of $\mathcal{D}$–spaces is an enriched natural transformation of enriched functors. The category of $\mathcal{D}$–spaces then has an enrichment in $\mathbb{T}\mathcal{T}$ given by the usual limit formula.

For brevity, in what follows we write *orthogonal $\mathbb{T}$–representation* to mean finite-dimensional real inner product space with $\mathbb{T}$–action by isometries (and not the isomorphism class of such an object). As in [8, Definition II.4.1], for orthogonal $\mathbb{T}$–representations V and W, let $\mathscr{I}(V,W)$ denote the $\mathbb{T}$–space of (non-equivariant) linear isometries from V to W. Let $E(V,W)$ denote the subbundle of the product $\mathbb{T}$–bundle $\mathscr{I}(V,W)\times W$ consisting of the points (f,x) where x is in the orthogonal complement of $f(V)$. Let $\mathscr{J}_{\mathbb{T}}(V,W)$ denote the Thom $\mathbb{T}$–space of $E(V,W)$. Composition of isometries and addition in the codomain vector space induces composition maps

$$\mathscr{J}_{\mathbb{T}}(W,Z)\wedge\mathscr{J}_{\mathbb{T}}(V,W)\longrightarrow\mathscr{J}_{\mathbb{T}}(V,Z),$$

which together with the obvious identity elements make $\mathscr{J}_{\mathbb{T}}$ a category enriched in based $\mathbb{T}$–spaces (with objects the orthogonal $\mathbb{T}$–representations).

2.1 Definition [8, Theorem II.4.3] The category of *orthogonal $\mathbb{T}$–spectra* is the category of $\mathscr{J}_{\mathbb{T}}$–spaces.

As discussed above, the category of orthogonal $\mathbb{T}$–spectra inherits an enrichment in $\mathbb{T}\mathcal{T}$. In addition, the category of orthogonal $\mathbb{T}$–spectra is a closed symmetric monoidal category under the smash product constructed in [8, Section II.3] and in particular has internal function objects: for X, Y orthogonal $\mathbb{T}$–spectra, we let $F_{\mathbb{T}}(X,Y)$ denote the orthogonal $\mathbb{T}$–spectrum of maps (analogous to the $\mathbb{T}$–space of maps between $\mathbb{T}$–spaces). We write its $\mathbb{T}$–fixed point non-equivariant orthogonal spectrum as $F^{\mathbb{T}}(X,Y)$, which we regard as the spectrum of $\mathbb{T}$–equivariant maps from X to Y.

We now turn to the discussion of the fixed point functors. The advantage of the diagram space definition (Definition 2.1) over the spacewise definition of orthogonal $\mathbb{T}$–spectra is that the diagram space definition makes it easier to define the (point-set) fixed point and geometric fixed point functors. For $C\le\mathbb{T}$ a closed subgroup, consider $\mathscr{J}_{\mathbb{T}}^{C}(V,W)$, the $\mathbb{T}/C$–space of C–fixed points of $\mathscr{J}_{\mathbb{T}}(V,W)$. Identity elements and composition in $\mathscr{J}_{\mathbb{T}}$ restrict appropriately, making $\mathscr{J}_{\mathbb{T}}^{C}$ a category enriched in based $\mathbb{T}/C$–spaces. Moreover, we have an evident $\mathbb{T}/C$–enriched functor $\tilde{q}_C\colon\mathscr{J}_{\mathbb{T}/C}\to\mathscr{J}_{\mathbb{T}}^{C}$ induced by regarding an orthogonal $\mathbb{T}/C$–representation V as an orthogonal $\mathbb{T}$–representation q_C^*V via the quotient map $q_C\colon\mathbb{T}\to\mathbb{T}/C$. Given an orthogonal $\mathbb{T}$–spectrum X, the fixed points $(X(V))^C$ form a $\mathscr{J}_{\mathbb{T}}^{C}$–space, i.e., a based $\mathbb{T}/C$–space enriched functor from $\mathscr{J}_{\mathbb{T}}^{C}$ to based $\mathbb{T}/C$–spaces. We can then compose with the enriched functor $\tilde{q}_C$ to obtain an orthogonal $\mathbb{T}/C$–spectrum.

2.2 Definition [8, Definition V.3.1] Let X be an orthogonal $\mathbb{T}$–spectrum. For $C \leq \mathbb{T}$ a closed subgroup, let X^C be the orthogonal $\mathbb{T}/C$–spectrum defined by $X^C(V) = (X(q_C^* V))^C$, with $\mathscr{I}_{\mathbb{T}/C}$–space structure induced via the enriched functor $\tilde{q}_C$ as above. We call this functor $(-)^C$ from orthogonal $\mathbb{T}$–spectra to orthogonal $\mathbb{T}/C$–spectra the (point-set) *categorical fixed point functor*.

We also have a based $\mathbb{T}/C$–space enriched functor $\phi\colon \mathscr{I}_{\mathbb{T}}^C \to \mathscr{I}_{\mathbb{T}/C}$ which sends the orthogonal $\mathbb{T}$–representation V to the orthogonal $\mathbb{T}/C$–representation V^C. Enriched left Kan extension along ϕ constructs a functor from $\mathscr{I}_{\mathbb{T}}^C$–spaces to $\mathscr{I}_{\mathbb{T}/C}$–spaces. Applying this to the fixed point $\mathscr{I}_{\mathbb{T}}^C$–space obtained from a orthogonal $\mathbb{T}$–spectrum, we get the (point-set) geometric fixed point functor.

2.3 Definition [8, Definition V.4.3] Let X be an orthogonal $\mathbb{T}$–spectrum. For $C \leq \mathbb{T}$ a closed subgroup, let $\mathrm{Fix}^C X$ be the $\mathscr{I}_{\mathbb{T}}^C$–space defined by $\mathrm{Fix}^C X(V) = (X(V))^C$ and let $\Phi^C X$ be the orthogonal $\mathbb{T}/C$–spectrum obtained from $\mathrm{Fix}^C X$ by enriched left Kan extension along the enriched functor $\phi\colon \mathscr{I}_{\mathbb{T}}^C \to \mathscr{I}_{\mathbb{T}/C}$. We call the functor Φ^C from orthogonal $\mathbb{T}$–spectra to orthogonal $\mathbb{T}/C$–spectra the (point-set) *geometric fixed point functor*.

The categorical fixed point functor is a right adjoint [8, Proposition V.3.4] and so preserves all limits. The geometric fixed point functor is not a left adjoint but has the feel of a left adjoint, preserving all the colimits preserved by the fixed point functor on based $\mathbb{T}$–spaces; this includes pushouts over levelwise closed inclusions and sequential colimits of levelwise closed inclusions. In particular, the geometric fixed point functor preserves homotopy colimits. The two functors also have right-hand and left-hand relationships (respectively) to the fixed point functor of spaces. The zeroth space (or n^{th} space) of the categorical fixed point functor is the fixed point space of the zeroth space (or n^{th} space) of the orthogonal $\mathbb{T}$–spectrum:

$$X^C(\mathbb{R}^n) = (X(\mathbb{R}^n))^C.$$

Likewise, the geometric fixed point functor of a suspension $\mathbb{T}$–spectrum (or n–shift desuspension $\mathbb{T}$–spectrum) is the suspension $\mathbb{T}$–spectrum (or n–shift desuspension $\mathbb{T}$–spectrum) of the fixed point space

$$\Phi^C(F_{\mathbb{R}^n} A) \cong F_{\mathbb{R}^n}(A^C)$$

(using the notation of [8, Definition II.4.6]).

It is evident from the definitions that both the categorical fixed point functor and the geometric fixed point functor have a canonical enrichment in based spaces. For the construction of function spectra for cyclotomic spectra in Section 5, we need an enrichment in orthogonal spectra of the geometric fixed point functors for finite subgroups. To obtain this, we now describe a natural transformation

$$F^{\mathbb{T}}(X,Y) \longrightarrow F^{\mathbb{T}/C}(\Phi^C X, \Phi^C Y) \cong F^{\mathbb{T}}(\rho^*\Phi^C X, \rho^*\Phi^C Y). \tag{2.4}$$

Here the isomorphism $F^{\mathbb{T}/C}(\Phi^C X, \Phi^C Y) \cong F^{\mathbb{T}}(\rho^*\Phi^C X, \rho^*\Phi^C Y)$ on the right is induced from the evident space-level isomorphism induced by the $(\#C)^{\text{th}}$ root isomorphism $\rho\colon \mathbb{T} \to \mathbb{T}/C$ (the unique orientation-preserving isomorphism of compact connected 1–dimensional Lie groups). The left-hand map, on the other hand, arises as a direct consequence of the following theorem (as we explain below).

2.5 Theorem *For X an orthogonal $\mathbb{T}$–spectrum, A a cofibrant non-equivariant orthogonal spectrum and C a closed subgroup of $\mathbb{T}$, the canonical natural map*

$$\Phi^C(X) \wedge A \longrightarrow \Phi^C(X \wedge A)$$

is an isomorphism.

As the previous theorem is a special case of a new foundational observation about equivariant orthogonal spectra that holds for any compact Lie group G, we state and prove it in the more general context in the appendix.

To deduce (2.4), we recall that $F^{\mathbb{T}}(X,Y)$ is the orthogonal $\mathbb{T}$–spectrum with n^{th} space $F^{\mathbb{T}}(X,Y)(\mathbb{R}^n)$ the space of $\mathbb{T}$–equivariant maps from $X \wedge F_{\mathbb{R}^n} S^0$ to Y. As the geometric fixed point functor is enriched in based spaces, we get an induced map from $F^{\mathbb{T}}(X,Y)(\mathbb{R}^n)$ to the space of $\mathbb{T}/C$–equivariant maps from $\Phi^C(X \wedge F_{\mathbb{R}^n} S^0)$ to $\Phi^C Y$, which Theorem 2.5 then identifies as the n^{th} space of $F^{\mathbb{T}/C}(\Phi^C X, \Phi^C Y)$. This then assembles to the map of orthogonal spectra in (2.4).

Finally, for later use we need a new observation on iterated geometric fixed point functors. Using the r^{th} root isomorphism $\rho_r\colon \mathbb{T} \to \mathbb{T}/C_r$ and the concomitant functor ρ_r^* from $\mathbb{T}/C_r$–spaces to $\mathbb{T}$–spaces, for a $\mathbb{T}$–space A we have a canonical natural identification of $\mathbb{T}$–spaces

$$\rho_{mn}^* A^{C_{mn}} = \rho_m^*(\rho_n^* A^{C_n})^{C_m}$$

for m, $n \in \mathbb{N}$. Using the analogous functor from orthogonal $\mathbb{T}/C_r$–spectra to $\mathbb{T}$–spectra, we have the following orthogonal spectrum version of this natural transformation for the geometric fixed point functors.

2.6 Proposition *For every m, $n \in \mathbb{N}$, there is a canonical natural map of orthogonal $\mathbb{T}$–spectra*

$$c_{m,n}\colon \rho_{mn}^* \Phi^{C_{mn}} X \longrightarrow \rho_m^* \Phi^{C_m}(\rho_n^* \Phi^{C_n} X)$$

making the following diagram commute:

$$\begin{array}{ccc}
\rho^*_{mnp}\Phi^{C_{mnp}}X & \xrightarrow{\;c_{mn,p}\;} & \rho^*_{mn}\Phi^{C_{mn}}(\rho^*_p\Phi^{C_p}X) \\
{\scriptstyle c_{m,np}}\downarrow & & \downarrow{\scriptstyle c_{m,n}} \\
\rho^*_m\Phi^{C_m}(\rho^*_{np}\Phi^{C_{np}}X) & \xrightarrow[\rho^*_m\Phi^m c_{n,p}]{} & \rho^*_m\Phi^{C_m}(\rho^*_n\Phi^{C_n}(\rho^*_p\Phi^{C_p}X))
\end{array}$$

The map $c_{m,n}$ is an isomorphism when $X = F_V A$ or, more generally, when X is cofibrant (see Theorem 3.3).

Proof In order to construct $c_{m,n}$, it suffices to construct a natural transformation of $\rho^*_{mn}\mathscr{J}_{\mathbb{T}}^{C_{mn}}$–spaces from $\rho^*_{mn}\operatorname{Fix}^{C_{mn}}X$ to $\rho^*_m\Phi^{C_m}(\rho^*_n\Phi^{C_n}X)$. For this, it suffices to construct a natural transformation of $\rho^*_{mn}\mathscr{J}_{\mathbb{T}}^{C_{mn}}$–spaces from $\rho^*_{mn}\operatorname{Fix}^{C_{mn}}X$ to $\rho^*_m\operatorname{Fix}^{C_m}(\rho^*_n\Phi^{C_n}X)$ and, for this, it suffices to construct a natural transformation of $\rho^*_{mn}\mathscr{J}_{\mathbb{T}}^{C_{mn}}$–spaces from $\rho^*_{mn}\operatorname{Fix}^{C_{mn}}X$ to $\rho^*_m(\operatorname{Fix}^{C_m}\rho^*_n(\operatorname{Fix}^{C_n}X))$. This is induced by the space-level identity

$$\rho^*_{mn}(X(V))^{C_{mn}} = \rho^*_m(\rho^*_n(X(V))^{C_n})^{C_m}.$$

For the diagram, we observe that both composites are ultimately induced by the same space-level canonical natural isomorphism. For $X = F_V A$, both $\rho^*_{mn}\Phi^{C_{mn}}X$ and $\rho^*_m\Phi^{C_m}(\rho^*_n\Phi^n X)$ are isomorphic to $F_{V^{C_{mn}}}A^{C_{mn}}$ and it follows that $c_{m,n}$ is an isomorphism from the universal property of F_V. The isomorphism for a general cofibrant X then follows from the fact that the functors ρ^*_r and Φ^{C_r} preserve pushouts over cofibrations and sequential colimits of cofibrations. □

3 Review of the homotopy theory of orthogonal $\mathbb{T}$–spectra

In the previous section we reviewed the definition of orthogonal $\mathbb{T}$–spectra and the construction of the point-set fixed point functors. In this section, we give a brief review of relevant aspects of the homotopy theory of orthogonal $\mathbb{T}$–spectra, including the homotopy groups, the model structures, and the derived functors of the fixed point functors.

We begin with the level model structure on orthogonal $\mathbb{T}$–spectra, which is mainly a tool for construction of the stable model structure, but in the non-equivariant setting plays a role in later sections allowing us to simplify hypotheses in certain statements (see Theorems 5.12 and 5.17 and the arguments in Section 6). In the level model structure, the weak equivalences are the *level equivalences*, which are the maps $X \to Y$ that are equivariant weak equivalences $X(V) \to Y(V)$ for all orthogonal $\mathbb{T}$–representations V, or in other words, the maps that are non-equivariant weak equivalences $X(V)^C \to Y(V)^C$ for all closed subgroups $C \leq \mathbb{T}$. The level fibrations are the maps that are equivariant Serre fibrations $X(V) \to Y(V)$ for all V, or in other words, the maps that are non-equivariant Serre fibrations $X(V)^C \to Y(V)^C$ for all closed subgroups $C \leq \mathbb{T}$. In particular, every object is fibrant. The cofibrations are defined by the left lifting property and are precisely the retracts of relative cell complexes (defined using sequential colimits; see e.g., Definition 5.4 of Mandell, May, Schwede and Shipley [9]) built out of the V–desuspension $\mathbb{T}$–spectra of standard $\mathbb{T}$–space n–cells

$$F_V(\mathbb{T}/C \times S^{n-1})_+ \longrightarrow F_V(\mathbb{T}/C \times D^n)_+$$

for any orthogonal $\mathbb{T}$–representation V and any $n \geq 0$, where D^n denotes the unit disk in $\mathbb{R}^n$, S^{n-1} its boundary and $C \leq \mathbb{T}$ a closed subgroup. These cofibrations are also the cofibrations in the stable model structure described next.

In the stable model structure, we define the weak equivalences in terms of homotopy groups. In fact, we describe two different versions of homotopy groups, both of which define the same weak equivalences. The homotopy groups of an orthogonal $\mathbb{T}$–spectrum are the homotopy groups of the underlying $\mathbb{T}$–(pre-)spectrum, which can be defined concretely as follows. Because of our emphasis on cyclotomic spectra and TC, we will work in terms of the specific complete "$\mathbb{T}$–universe" (countable-dimensional equivariant real inner product space) usually used in this context. (The specifics do not play a significant role here; rather, we follow the notation and exposition of [1, Section 4] as closely as possible.) Let

$$U = \bigoplus_{n=0}^{\infty} \bigoplus_{r=1}^{\infty} \mathbb{C}(n),$$

where $\mathbb{C}(0)$ denotes the complex numbers with trivial $\mathbb{T}$–action, $\mathbb{C}(1)$ denotes the complex numbers with the standard $\mathbb{T}$–action and $\mathbb{C}(n)$ denotes the complex numbers with $\mathbb{T}$ acting through the n^{th} power map, all regarded as real vector spaces. Here the inner product is induced by the standard ($\mathbb{T}$–invariant) hermitian product on $\mathbb{C}(n)$ and orthogonal direct sum. Every orthogonal $\mathbb{T}$–representation is then isometric to a finite-dimensional $\mathbb{T}$–stable subspace of U. Notationally, we write $V < U$ to denote a finite-dimensional $\mathbb{T}$–stable subspace of U and for $V < W < U$, we denote by $W - V$ the orthogonal complement of V in W.

3.1 Definition For an orthogonal $\mathbb{T}$–spectrum X, we define the *homotopy groups* by

$$\pi_q^C X = \begin{cases} \operatorname*{colim}_{V<U} \pi_q((\Omega^V X(V))^C) & q \geq 0, \\ \operatorname*{colim}_{\mathbb{C}(0)^{-q}<V<U} \pi_{-q}((\Omega^{V-\mathbb{C}(0)^{-q}} X(V))^C) & q < 0, \end{cases}$$

for $q \in \mathbb{Z}$ and C a closed subgroup of $\mathbb{T}$ [8, Definition III.3.2].

The expression above for the homotopy groups π_*^C provides an intrinsic construction of the homotopy groups of the underlying non-equivariant spectrum of the right derived categorical fixed point functor. Analogously, we can construct the homotopy groups of the underlying non-equivariant spectrum of the left derived geometric fixed point functor. We call these the "geometric homotopy groups" and use the notation $\pi_*^{\Phi C}$ to emphasize the analogy with π_*^C. The geometric homotopy groups were denoted as ρ_q^H in [8, Section V.4].

3.2 Definition For an orthogonal $\mathbb{T}$–spectrum X, we define the *geometric homotopy groups* by

$$\pi_q^{\Phi C} X = \begin{cases} \operatorname*{colim}_{V<U} \pi_q(\Omega^{V^C}(X(V)^C)) & q \geq 0, \\ \operatorname*{colim}_{\mathbb{C}(0)^{-q}<V<U} \pi_{-q}(\Omega^{V^C-\mathbb{C}(0)^{-q}}(X(V)^C)) & q < 0, \end{cases}$$

for $q \in \mathbb{Z}$ and C a closed subgroup of $\mathbb{T}$ [8, Definition V.4.8(iii) and Proposition V.4.12].

It is clear from the formula that $\pi_q^{\Phi C}(-)$ sends level equivalences of orthogonal $\mathbb{T}$–spectra to isomorphisms of abelian groups. Since, by [8, Proposition V.4.12], $\pi_*^{\Phi C} X \cong \pi_*(\Phi^C X)$ when X is level cofibrant, it follows that, for any X, $\pi_*^{\Phi C} X$ calculates the homotopy groups of the underlying non-equivariant spectrum of the left derived geometric fixed point functor applied to X,

$$\pi_*^{\Phi C} X \cong \pi_*(\mathbb{L}\Phi^C X).$$

Both the homotopy groups and the geometric homotopy groups detect the weak equivalences on the stable model structure on orthogonal $\mathbb{T}$–spectra: we define a *weak equivalence* of orthogonal $\mathbb{T}$–spectra (or *stable equivalence* when necessary to distinguish from other notions of weak equivalence) to be a map that induces an isomorphism on all homotopy groups or, equivalently (by [10, Section XVI.6.4] and [8, Equation VI.5.1 or Proposition V.4.17]), a map that induces an isomorphism on all geometric homotopy groups. The cofibrations in the stable model structure are the same as the cofibrations in the level model structure and we define the fibrations by the right lifting property. By [8, Proposition III.4.8], a map $X \to Y$ is a fibration in the stable model structure exactly when it is a level fibration such that the diagram

$$\begin{array}{ccc} X(V) & \longrightarrow & \Omega^W X(V \oplus W) \\ \downarrow & & \downarrow \\ Y(V) & \longrightarrow & \Omega^W Y(V \oplus W) \end{array}$$

is a homotopy pullback for all orthogonal $\mathbb{T}$–representations V and W. In particular, an object is cofibrant if and only if it is the retract of a cell complex, and an object is fibrant if and only if it is an equivariant Ω–spectrum.

3.3 Theorem (Stable model structure [8, Theorem III.4.2]) *The category of orthogonal $\mathbb{T}$–spectra is a cofibrantly generated closed model category in which a map $X \to Y$ is*

- *a weak equivalence if the induced map on homotopy groups π_q^C is an isomorphism for all $q \in \mathbb{Z}$ and all closed subgroups $C \leq \mathbb{T}$ or, equivalently, if the induced map on geometric homotopy groups $\pi_q^{\Phi C}$ is an isomorphism for all $q \in \mathbb{Z}$ and all closed subgroups $C \leq \mathbb{T}$;*
- *a cofibration if it is a retract of a relative cell complex; and*
- *a fibration if it satisfies the right lifting property with respect to the acyclic cofibrations.*

Moreover, the model structure is compatible with the enrichment of orthogonal $\mathbb{T}$–spectra over orthogonal spectra, meaning that the analogue of Quillen's axiom SM7 *is satisfied.*

For our purposes, we need model structures for some localized homotopy categories. We start with the homotopy categories local to a family.

3.4 Definition [8, Definition IV.6.1] Let $\mathcal{F}$ be a family of subgroups of $\mathbb{T}$, i.e., a collection of closed subgroups of $\mathbb{T}$ closed under taking closed subgroups (and conjugation). Let X and Y be orthogonal $\mathbb{T}$–spectra. An *$\mathcal{F}$–local equivalence* (or *$\mathcal{F}$–equivalence*) is a map $X \to Y$ that induces an isomorphism on homotopy groups π_*^C for all C in $\mathcal{F}$ or, equivalently, on all geometric homotopy groups $\pi_*^{\Phi C}$ for all C in $\mathcal{F}$.

An *$\mathcal{F}$–cofibration* is a map built as a retract of a relative cell complex using cells

$$F_V(\mathbb{T}/C \times S^{n-1})_+ \longrightarrow F_V(\mathbb{T}/C \times D^n)_+,$$

where we require $C \in \mathcal{F}$. The $\mathcal{F}$–fibrations are then defined by the right lifting property. Explicitly, a map $X \to Y$ of orthogonal $\mathbb{T}$–spectra is an $\mathcal{F}$–fibration exactly when it is levelwise an $\mathcal{F}$–fibration of spaces (the maps $(X(V))^C \to (Y(V))^C$ are non-equivariant Serre fibrations for each orthogonal $\mathbb{T}$–representation V and each $C \in \mathcal{F}$) such that the diagram

$$\begin{array}{ccc} (X(V))^C & \longrightarrow & (\Omega^W X(V \oplus W))^C \\ \downarrow & & \downarrow \\ (Y(V))^C & \longrightarrow & (\Omega^W Y(V \oplus W))^C \end{array}$$

is a homotopy pullback for all orthogonal $\mathbb{T}$–representations V and W and all $C \in \mathcal{F}$.

3.5 Theorem ($\mathcal{F}$–local model structure [8, Theorem IV.6.5]) *The category of orthogonal $\mathbb{T}$–spectra is a cofibrantly generated closed model category in which the weak equivalences, cofibrations and fibrations are the $\mathcal{F}$–equivalences, $\mathcal{F}$–cofibrations and $\mathcal{F}$–fibrations, respectively. The model structure is compatible with the enrichment of orthogonal $\mathbb{T}$–spectra over orthogonal spectra, meaning that the analogue of Quillen's axiom* SM7 *is satisfied.*

For our corepresentability results, we use model structures based on the finite complete or p–complete homotopy categories. Letting M^1_p denote the mod-p Moore space in dimension 1, we define a p–equivalence to be a map that becomes a weak equivalence after smashing with M^1_p. Likewise, we define a p–$\mathcal{F}$–equivalence to be a map that becomes an $\mathcal{F}$–equivalence after smashing with M^1_p. (Note that since M^1_p is a CW complex, smash product with it preserves $\mathcal{F}$–equivalences for any family $\mathcal{F}$.) We also have the more general notions of finite equivalence and finite $\mathcal{F}$–equivalence, which are the maps that are p–equivalences and p–$\mathcal{F}$–equivalences, respectively, for all p. The argument for [8, Theorem IV.6.3], applied directly starting with the $\mathcal{F}$–local model structure rather than the standard stable model structure, proves the following theorem:

3.6 Theorem ($\mathcal{F}$–local finite complete model structure) *The category of orthogonal $\mathbb{T}$–spectra is a cofibrantly generated closed model category in which the weak equivalences are the finite $\mathcal{F}$–equivalences (resp., finite p–$\mathcal{F}$–equivalences) and the cofibrations are the $\mathcal{F}$–cofibrations. The model structure is compatible with the enrichment of orthogonal $\mathbb{T}$–spectra over orthogonal spectra, meaning that the analogue of Quillen's axiom* SM7 *is satisfied with respect to the fibrations in the finite complete (resp., p–complete) model category of orthogonal spectra.*

3.7 Remark The model structures reviewed in this section all admit explicit descriptions of the generating cofibrations and acyclic cofibrations; for details, see the cited references where they are constructed.

4 The categories of cyclotomic and pre-cyclotomic spectra

The work of the previous two sections provides the background we need for the work in this section to define the categories of p–cyclotomic and cyclotomic spectra and for the work in the next section to construct model structures on these categories. Here we start with the easier category of p–cyclotomic spectra and then turn to the more complicated category of cyclotomic spectra.

The definition of a p–cyclotomic spectrum requires both extra structure on an orthogonal $\mathbb{T}$–spectrum and a homotopical condition, which we break into separate pieces.

4.1 Definition (Pre-*p*–cyclotomic spectra) A *pre-p–cyclotomic spectrum* X is a pair (X,t) consisting of an orthogonal $\mathbb{T}$–spectrum X together with a map of orthogonal $\mathbb{T}$–spectra

$$t\colon \rho_p^*\Phi^{C_p}X \longrightarrow X,$$

where ρ_p is the p^{th} root isomorphism $\mathbb{T}\to\mathbb{T}/C_p$. A morphism of pre-*p*–cyclotomic spectra $(X,t_X)\to(Y,t_Y)$ consists of a map of orthogonal $\mathbb{T}$–spectra $X\to Y$ such that the diagram

$$\begin{array}{ccc} \rho_p^*\Phi^{C_p}X & \xrightarrow{t_X} & X \\ \downarrow & & \downarrow \\ \rho_p^*\Phi^{C_p}Y & \xrightarrow{t_Y} & Y \end{array}$$

commutes.

To avoid unnecessary verbosity we will say simply "cyclotomic maps" rather than "pre-*p*–cyclotomic maps" when the context is clear.

Clearly the category of pre-*p*–cyclotomic spectra inherits an enrichment over spaces, with the set of cyclotomic maps topologized using the subspace topology from the space of maps of orthogonal $\mathbb{T}$–spectra. In fact, the category of pre-*p*–cyclotomic spectra inherits an enrichment over spectra.

4.2 Proposition *The category of pre-p–cyclotomic spectra inherits an enrichment over orthogonal spectra from the enrichment on orthogonal $\mathbb{T}$–spectra and the enrichment of the functor $\rho^*\Phi^{C_p}$.*

Proof Using the canonical orthogonal spectrum enrichment on ρ_p^* and the orthogonal spectrum enrichment on Φ^{C_p} from (2.4), for orthogonal $\mathbb{T}$–spectra X and Y we get the spectrum of cyclotomic maps

$$F_{\mathrm{Cyc}}(X,Y)=\mathrm{Eq}[F^{\mathbb{T}}(X,Y)\rightrightarrows F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}X,Y)],$$

formed as the equalizer of the map

$$F^{\mathbb{T}}(X,Y) \longrightarrow F^{\mathbb{T}}(\rho_p^* \Phi^{C_p} X, Y),$$

induced by the structure map for X, and the composite

$$F^{\mathbb{T}}(X,Y) \longrightarrow F^{\mathbb{T}}(\rho_p^* \Phi^{C_p} X, \rho_p^* \Phi^{C_p} Y) \longrightarrow F^{\mathbb{T}}(\rho_p^* \Phi^{C_p} X, Y),$$

induced by (2.4) and the structure map for Y. Because equalizers are formed spacewise, the zeroth space of this mapping spectrum is the space of cyclotomic maps from X to Y. Composition in $F^{\mathbb{T}}$ induces composition on F_{Cyc}, which is compatible with the composition of cyclotomic maps. □

The category of pre-p–cyclotomic spectra is complete (has all limits) but only has certain colimits: the natural map goes from the colimit of the geometric fixed points to the geometric fixed points of the colimit, so we can typically only construct those colimits where this map is an isomorphism.

4.3 Proposition *The category of pre-p–cyclotomic spectra has all limits. It has all coproducts, pushouts along maps that are levelwise closed inclusions, and sequential colimits of maps that are levelwise closed inclusions.*

Proof Limits are created in the category of orthogonal $\mathbb{T}$–spectra, using the natural map from the geometric fixed points of a limit to the limit of the geometric fixed points as the structure map. The natural map from the colimit of the geometric fixed points to the geometric fixed points of the colimit is an isomorphism for all of the colimits in the statement since the space-level fixed point functor commutes with these colimits. □

For indexed limits and colimits, similar observations apply.

4.4 Proposition *For a cofibrant non-equivariant orthogonal spectrum A, $(-)\wedge A$ extends to an endofunctor on pre-p–cyclotomic spectra that provides the tensor with A in the orthogonal spectrum enrichment of pre-p–cyclotomic spectra. For an arbitrary non-equivariant orthogonal spectrum A, $F(A,-)$ extends to an endofunctor on pre-p–cyclotomic spectra that provides the cotensor with A in the orthogonal spectrum enrichment of pre-p–cyclotomic spectra.*

Proof For $X \wedge A$, the structure map is induced by the structure map on X and the map

$$\rho_p^* \Phi^{C_p}(X \wedge A) \longrightarrow \rho_p^* \Phi^{C_p} X \wedge A$$

in Theorem 2.5. For $F(A, X)$, the structure map is induced by the structure map on X and the map adjoint to the map

$$\rho_p^* \Phi^{C_p}(F(A, X)) \wedge A \longrightarrow \rho_p^* \Phi^{C_p}(F(A, X) \wedge A) \longrightarrow \rho_p^* \Phi^{C_p} X.$$

An easy comparison of equalizers shows that $X \wedge A$ is the tensor and $F(A, X)$ is the cotensor of X with A. □

The previous proposition in particular shows that the category of pre-p–cyclotomic spectra has cotensors by all spaces and tensors by cofibrant spaces. In fact, the category of pre-p–cyclotomic has tensors by all spaces since the smash product with spaces commutes with geometric fixed points.

We define a p–cyclotomic spectrum to be a pre-p–cyclotomic spectrum that satisfies the homotopical condition that the structure map induces an $\mathcal{F}_p$–equivalence in the equivariant stable category

$$\rho_p^* \mathbb{L}\Phi^{C_p} X \longrightarrow X,$$

where $\mathbb{L}\Phi^{C_p}$ denotes the left derived functor of Φ^{C_p} (see [8, Proposition V.4.5]) and $\mathcal{F}_p$ denotes the family of p–groups, $\mathcal{F}_p = \{C_{p^n}\}$. Since the geometric homotopy groups of $\rho_p^* \mathbb{L}\Phi^{C_p} X$ are canonically isomorphic to the geometric homotopy groups of X [8, Proposition V.4.12],

$$\pi_*^{\Phi C_m}(\rho_p^* \mathbb{L}\Phi^{C_p} X) \cong \pi_*^{\Phi C_{mp}}(X),$$

we can write this condition concisely as follows:

4.5 Definition (p–cyclotomic spectra) The category of *p–cyclotomic spectra* is the full subcategory of the category of pre-p–cyclotomic spectra consisting of those objects X for which the map $\pi_q^{\Phi C_{p^{n+1}}}(X) \to \pi_q^{\Phi C_{p^n}}(X)$ induced by

$$t(V)\colon \rho_p^*(X(V))^{C_p} \longrightarrow X(V)$$

is an isomorphism for all $n \geq 0$, $q \in \mathbb{Z}$.

As a full subcategory, the category of p–cyclotomic spectra inherits mapping spectra and has those limits and colimits whose objects remain in the category. More specifically:

4.6 Proposition *The category of p–cyclotomic spectra has finite products and pullbacks over fibrations. It has all coproducts, pushouts over maps that are levelwise closed inclusions, and sequential colimits over levelwise closed inclusions.*

Proof The assertion for colimits follows from the fact that the fixed point functor on spaces preserves the colimits in the statement. We can deduce the existence of finite products and pullbacks over fibrations from the fact that these homotopy limits are naturally weakly equivalent to homotopy colimits. □

Because the derived geometric fixed point functor commutes with derived smash product with non-equivariant spectra [8, Proposition V.4.7], p–cyclotomic spectra are closed under tensor (smash product) with cofibrant non-equivariant orthogonal spectra. Likewise, p–cyclotomic spectra are closed under cotensor (function spectrum construction) with cofibrant non-equivariant orthogonal spectra whose underlying object in the stable category is finite.

We now turn to pre-cyclotomic and cyclotomic spectra. We require structure maps for all primes p, with some compatibility relations.

4.7 Definition (Pre-cyclotomic spectra) A *pre-cyclotomic spectrum* X consists of an orthogonal $\mathbb{T}$–spectrum X together with structure maps

$$t_n\colon \rho_n^* \Phi^{C_n} X \longrightarrow X$$

for all $n \geq 1$ such that the following diagram commutes for all $m, n \in \mathbb{N}$:

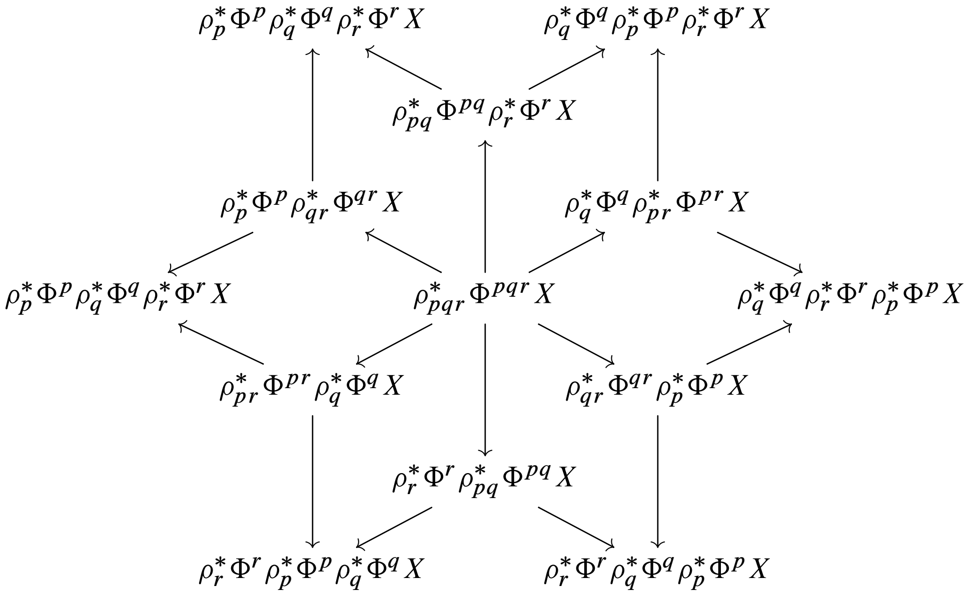
$$\begin{array}{ccc}
\rho_{mn}^* \Phi^{C_{mn}} X & \xrightarrow{\quad t_{mn} \quad} & X \\
{\scriptstyle c_{m,n}} \big\downarrow & & \big\uparrow {\scriptstyle t_m} \\
\rho_m^* \Phi^{C_m} (\rho_n^* \Phi^{C_n} X) & \xrightarrow[\rho_m^* \Phi^{C_m} t_n]{} & \rho_m^* \Phi^{C_m} X
\end{array}$$

A map of pre-cyclotomic spectra is a map of orthogonal $\mathbb{T}$–spectra that commutes with the structure maps.

4.8 Remark Clearly the structure of a pre-cyclotomic spectrum is determined by the maps t_p for p prime. Vigleik Angeltveit has verified that the relation

$$t_p \circ (\rho_p^* t_q) \circ c_{p,q} = t_q \circ (\rho_q^* t_p) \circ c_{q,p} \tag{4.9}$$

for all primes p and q implies the relation in the previous definition for all m and n. Specifically, for any orthogonal $\mathbb{T}$–spectrum X there is a commutative diagram (for primes p, q and r):



If X is equipped with pre-cyclotomic structure maps, then this diagram yields six different maps $\rho_{pqr}^*\Phi^{pqr}X \to X$. The relation in (4.9) now implies (after a little diagram-chasing) that these six maps are equal.

4.10 Definition (Cyclotomic spectra) The category of *cyclotomic spectra* is the full subcategory of the category of pre-cyclotomic spectra consisting of those objects for which the map $\pi_q^{\Phi C_{mn}}(X) \to \pi_q^{\Phi C_m}(X)$ induced by

$$t_n(V)\colon \rho_n^*(X(V))^{C_n} \longrightarrow X(V)$$

is an isomorphism for all $m, n \geq 1$ and $q \in \mathbb{Z}$.

Once again, we get a spectrum of cyclotomic maps of pre-cyclotomic (or cyclotomic) spectra as the equalizer

$$F_{\mathrm{Cyc}}(X,Y) = \mathrm{Eq}\Big[F^{\mathbb{T}}(X,Y) \rightrightarrows \prod_{n\geq 1} F^{\mathbb{T}}(\rho_n^*\Phi^{C_n}X, Y)\Big]$$

of the maps determined by the maps

$$F^{\mathbb{T}}(X,Y) \longrightarrow F^{\mathbb{T}}(\rho_n^*\Phi^{C_n}X, Y),$$

induced by the structure map for X, and the composites

$$F^{\mathbb{T}}(X,Y) \longrightarrow F^{\mathbb{T}}(\rho_n^*\Phi^{C_n}X, \rho_n^*\Phi^{C_n}Y) \longrightarrow F^{\mathbb{T}}(\rho_n^*\Phi^{C_n}X, Y),$$

induced by (2.4) and the structure map for Y. Propositions analogous to the ones above hold for the categories of pre-cyclotomic spectra and cyclotomic spectra.

4.11 Example The S^1–equivariant sphere spectrum has a canonical structure as a cyclotomic spectrum induced by the canonical isomorphisms $\rho_n^* \Phi^{C_n} S \cong S$.

We close by comparing the definition of cyclotomic spectra here to definitions in previous work. As far as we know the only definition of a point-set category of cyclotomic spectra entirely in the context of orthogonal $\mathbb{T}$–spectra is our [1, Definition 4.2] (compare Hesselholt and Madsen [4, Section 1.2]), where the definition and construction of TC is compared with older definitions in the context of Lewis–May spectra, e.g., Hesselholt and Madsen [3]. In [1], the authors lacked Proposition 2.6 and so wrote a spacewise definition. An easy check of universal properties reveals that the definition here coincides with the definition there.

5 Model structures on cyclotomic spectra

We now move on to the homotopy theory of p–cyclotomic spectra and cyclotomic spectra, which we express in terms of model structures. The model structures are inherited from the ambient categories of pre-p–cyclotomic and pre-cyclotomic spectra, where they are significantly easier to set up, using standard arguments for categories of algebras over monads.

5.1 Construction For an orthogonal $\mathbb{T}$–spectrum X, let

$$\begin{aligned} \mathbb{C}_p X &= X \vee \rho_p^* \Phi^{C_p} X \vee \rho_p^* \Phi^{C_p} (\rho_p^* \Phi^{C_p} X) \vee \cdots, \\ \mathbb{C} X &= \bigvee_{n \geq 1} \rho_n^* \Phi^{C_n} X. \end{aligned}$$

The functor $\mathbb{C}_p$ is a monad on the category of orthogonal $\mathbb{T}$–spectra, the free monad generated by the endofunctor $\rho_p^* \Phi^{C_p}$. Clearly, the category of pre-p–cyclotomic spectra is precisely the category of $\mathbb{C}_p$–algebras in orthogonal $\mathbb{T}$–spectra. Because of the apparent failure of the canonical map

$$\rho_{mn}^* \Phi^{C_{mn}} X \longrightarrow \rho_n^* \Phi^{C_n} (\rho_m^* \Phi^{C_m} X)$$

of Proposition 2.6 to be an isomorphism, $\mathbb{C}$ does not appear to be a monad on the category of orthogonal $\mathbb{T}$–spectra; however, it is a monad on the full subcategory of cofibrant orthogonal $\mathbb{T}$–spectra, on which the map is an isomorphism. The unit is the inclusion of X as $\rho^*_{C_1}\Phi^{C_1}X$. The multiplication is induced by the inverse isomorphisms

$$\rho^*_n\Phi^{C_n}(\rho^*_m\Phi^{C_m}X) \longrightarrow \rho^*_{mn}\Phi^{C_{mn}}X.$$

Pre-cyclotomic spectra with cofibrant underlying orthogonal $\mathbb{T}$–spectra are precisely the $\mathbb{C}$–algebras in the category of cofibrant orthogonal $\mathbb{T}$–spectra. More generally, every pre-cyclotomic spectrum comes with a canonical natural map $\xi\colon \mathbb{C}X \to X$ and a cyclotomic map is precisely a map of orthogonal $\mathbb{T}$–spectra $f\colon X \to Y$ that makes the diagram

$$\begin{array}{ccc} \mathbb{C}X & \xrightarrow{\mathbb{C}f} & \mathbb{C}Y \\ \xi\downarrow & & \downarrow\xi \\ X & \xrightarrow[f]{} & Y \end{array}$$

commute. We cannot say much more, except for the following proposition, which allows us to treat $\mathbb{C}X$ like a free pre-cyclotomic spectrum functor.

5.2 Proposition *Let A be a cofibrant orthogonal $\mathbb{T}$–spectrum. Then $\mathbb{C}_pA$ is a pre-p–cyclotomic spectrum and $\mathbb{C}A$ is a pre-cyclotomic spectrum with structure maps induced by the monad multiplication. If X is a pre-p–cyclotomic or pre-cyclotomic spectrum, then maps of orthogonal $\mathbb{T}$–spectra from A to X are in one-to-one correspondence with cyclotomic maps $\mathbb{C}_pA \to X$ or $\mathbb{C}A \to X$, respectively.*

The usual theory of model structures on algebra categories tells us to define cells of pre-p–cyclotomic spectra and pre-cyclotomic spectra using the free functor applied to cells in the model structure on the underlying category, in this case the $\mathcal{F}_p$–local or $\mathcal{F}_{\text{fin}}$–local model structure on orthogonal $\mathbb{T}$–spectra, respectively (where $\mathcal{F}_p$ is the family of p–subgroups and $\mathcal{F}_{\text{fin}}$ is the family of finite subgroups of $\mathbb{T}$). Specifically, for pre-p–cyclotomic spectra, the cells are

$$\mathbb{C}_pF_V(\mathbb{T}/C \times S^{n-1})_+ \longrightarrow \mathbb{C}_pF_V(\mathbb{T}/C \times D^n)_+$$

for V an orthogonal $\mathbb{T}$–representation, $n \geq 0$ and $C < \mathbb{T}$ a p–subgroup; and, for pre-cyclotomic spectra, the cells are

$$\mathbb{C}F_V(\mathbb{T}/C \times S^{n-1})_+ \longrightarrow \mathbb{C}F_V(\mathbb{T}/C \times D^n)_+$$

for V an orthogonal $\mathbb{T}$–representation, $n \geq 0$ and $C < \mathbb{T}$ a finite subgroup. The first model structure theorem is then the following:

5.3 Theorem *The category of pre-p–cyclotomic spectra has a cofibrantly generated model structure with*

- *weak equivalences the $\mathcal{F}_p$–equivalences of the underlying orthogonal $\mathbb{T}$–spectra,*
- *cofibrations the retracts of relative cell complexes built out of the cells above, and*
- *fibrations the $\mathcal{F}_p$–fibrations of the underlying orthogonal $\mathbb{T}$–spectra.*

The category of pre-cyclotomic spectra has a cofibrantly generated model structure with

- *weak equivalences the $\mathcal{F}_{\text{fin}}$–equivalences of the underlying orthogonal $\mathbb{T}$–spectra,*
- *cofibrations the retracts of relative cell complexes built out of the cells above, and*
- *fibrations the $\mathcal{F}_{\text{fin}}$–fibrations of the underlying orthogonal $\mathbb{T}$–spectra.*

Thus, pre-p–cyclotomic spectra and pre-cyclotomic spectra form model categories with the above model structures.*

Proof As in [9, Proposition 5.13] (and [8, Section III.8]), the model structure statements follow from a "Cofibration Hypothesis" [9, 5.3] about pushouts and sequential colimits. Specifically, recall that a map $X \to Y$ of orthogonal $\mathbb{T}$–spectra is an h–cofibration if it satisfies the homotopy extension property. The cofibration hypothesis is satisfied for a collection of maps $\mathcal{I}$ when the following two conditions hold:

(i) Let $i\colon A \to B$ be a coproduct of maps in $\mathcal{I}$. In any pushout

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ B & \longrightarrow & Y \end{array}$$

of pre-p–cyclotomic (or p–cyclotomic) spectra, the cobase change j is an h–cofibration of orthogonal $\mathbb{T}$–spectra.

(ii) The sequential colimit of a sequence of maps f_i in pre-p–cyclotomic (or p–cyclotomic) spectra that are h–cofibrations of orthogonal $\mathbb{T}$–spectra is computed as the sequential colimit in the category of orthogonal $\mathbb{T}$–spectra.

In order to construct the model structures, it suffices to show that the cofibration hypothesis holds for the candidate generating cofibrations and acyclic cofibrations produced by applying $\mathbb{C}_p$ and $\mathbb{C}$ to the generating cofibrations and acyclic cofibrations in the $\mathcal{F}_p$–local and $\mathcal{F}_{\text{fin}}$–local model structures on orthogonal $\mathbb{T}$–spectra. But this is clear from the fact that the geometric fixed point functor preserves the colimits in question.

The last statement then follows from Proposition 4.3 and the corresponding proposition for pre-cyclotomic spectra. □

Similarly, starting with the $\mathcal{F}$–local p– and finite complete model structures on orthogonal $\mathbb{T}$–spectra (Theorem 3.6), we obtain the following "complete" model structures on pre-p–cyclotomic spectra and pre-cyclotomic spectra.

5.4 Theorem *The category of pre-p–cyclotomic spectra has a cofibrantly generated model structure with*

- *weak equivalences the p–$\mathcal{F}_p$–equivalences of the underlying orthogonal $\mathbb{T}$–spectra,*
- *cofibrations the retracts of relative cell complexes built out of the cells above, and*
- *fibrations the fibrations of the underlying orthogonal $\mathbb{T}$–spectra in the $\mathcal{F}_p$–local p–complete model structure.*

The category of pre-cyclotomic spectra has a cofibrantly generated model structure with

- *weak equivalences the finite complete $\mathcal{F}_{\text{fin}}$–equivalences of the underlying orthogonal $\mathbb{T}$–spectra,*
- *cofibrations the retracts of relative cell complexes built out of the cells above, and*
- *fibrations the fibrations of the underlying orthogonal $\mathbb{T}$–spectra in the $\mathcal{F}_{\text{fin}}$–local finite complete model structure.*

Thus, pre-p–cyclotomic spectra and pre-cyclotomic spectra form model categories with the above model structures.*

Turning to p–cyclotomic and cyclotomic spectra, because $\mathcal{F}$–equivalences are defined in terms of the geometric homotopy groups (Definition 3.4) we have the following simpler description of weak equivalences in this context.

5.5 Proposition *A cyclotomic map of p–cyclotomic spectra is an $\mathcal{F}_p$–equivalence of the underlying orthogonal $\mathbb{T}$–spectra if and only if it is a weak equivalence of the underlying non-equivariant orthogonal spectra. A cyclotomic map of cyclotomic spectra is an $\mathcal{F}_{\text{fin}}$–equivalence of the underlying orthogonal $\mathbb{T}$–spectra if and only if it is a weak equivalence of the underlying non-equivariant orthogonal spectra.*

Proof We give the argument for cyclotomic spectra; the proof for p–cyclotomic spectra is analogous. Clearly, a map of cyclotomic spectra which is an $\mathcal{F}_{\text{fin}}$–equivalence of the underlying orthogonal $\mathbb{T}$–spectra is a weak equivalence of underlying non-equivariant orthogonal spectra. Conversely, suppose we are given a map $X \to Y$

of p–cyclotomic spectra which is a weak equivalence of underlying non-equivariant orthogonal spectra. In the diagram

$$\begin{array}{ccccc} \rho_n^* \mathbb{L}\Phi^{C_n} X & \longrightarrow & \rho_n^* \Phi^{C_n} X & \xrightarrow{t} & X \\ \downarrow & & \downarrow & & \downarrow \\ \rho_n^* \mathbb{L}\Phi^{C_n} Y & \longrightarrow & \rho_n^* \Phi^{C_n} Y & \xrightarrow{t} & Y, \end{array}$$

the composite horizontal maps and the right-hand vertical map are weak equivalences of underlying non-equivariant spectra, so we conclude that so is the left-hand vertical map. Therefore, $\pi_*^{\Phi C_n} X \to \pi_*^{\Phi C_n} Y$ is an isomorphism. Inductively, we conclude that $X \to Y$ is an $\mathcal{F}_{\text{fin}}$–equivalence of $\mathbb{T}$–spectra. □

Theorem 1.2, the main theorem on the homotopy theory of p–cyclotomic spectra, is now an immediate consequence.

Proof of Theorem 1.2 Any subcategory of a model* category that is closed under weak equivalences, finite products and coproducts, pushouts over cofibrations, and pullbacks over fibrations is itself a model* category with the inherited model structure. Clearly, the p–cyclotomic spectra regarded as a subcategory of the model* category of pre-p–cyclotomic spectra (with the $\mathcal{F}_p$–equivalences) satisfies these conditions. Proposition 5.5 now yields the characterization of the weak equivalences given in the statement of the theorem. □

Similarly, Proposition 5.5 also implies the corresponding theorem for cyclotomic spectra.

5.6 Theorem *The category of cyclotomic spectra is a model* category with weak equivalences the weak equivalences of the underlying non-equivariant orthogonal spectra and fibrations the $\mathcal{F}_{\text{fin}}$–fibrations of the underlying orthogonal $\mathbb{T}$–spectra.*

For the "complete" model structures, we should look at the closure of the categories of p–cyclotomic spectra and cyclotomic spectra under the weak equivalences in those model structures. We refer to these as "weak" p–cyclotomic and cyclotomic spectra.

5.7 Definition The category of *weak p–cyclotomic spectra* is the full subcategory of the category of pre-p–cyclotomic spectra consisting of those objects X for which the map $\pi_q^{\Phi C_{p^{n+1}}}(X \wedge M_p^1) \to \pi_q^{\Phi C_{p^n}}(X \wedge M_p^1)$ induced by

$$t(V)\colon \rho_p^*(X(V) \wedge M_p^1)^{C_p} \longrightarrow X(V) \wedge M_p^1$$

is an isomorphism for all $n \geq 0$ and $q \in \mathbb{Z}$, where M_p^1 denotes the mod-p Moore space in dimension 1.

The category of *weak cyclotomic spectra* is the full subcategory of the category of pre-cyclotomic spectra consisting of those objects for which the maps

$$\pi_q^{\Phi C_{mn}}(X \wedge M_p^1) \longrightarrow \pi_q^{\Phi C_m}(X \wedge M_p^1)$$

are isomorphisms for all m, $n \geq 1$, $q \in \mathbb{Z}$ and p prime.

Once again, the categories of weak p–cyclotomic spectra and weak cyclotomic spectra are suitable subcategories of model* categories. Along with Proposition 5.5, this implies the following result:

5.8 Theorem *The category of weak p–cyclotomic spectra is a model* category with weak equivalences the p–equivalences of the underlying non-equivariant orthogonal spectra and fibrations the $\mathcal{F}_p$–local p–complete fibrations of the underlying orthogonal $\mathbb{T}$–spectra.*

The category of weak cyclotomic spectra is a model category with weak equivalences the finite equivalences of the underlying non-equivariant orthogonal spectra and fibrations the $\mathcal{F}_{\text{fin}}$–local finite complete fibrations of the underlying orthogonal $\mathbb{T}$–spectra.*

Quillen's SM7 axiomatizes the compatibility between the model structure and the enrichment. In our context of an enrichment over orthogonal spectra, the statement is the following theorem:

5.9 Theorem *Let $i\colon W \to X$ be a cofibration of pre-p–cyclotomic (resp., pre-cyclotomic) spectra and let $f\colon Y \to Z$ be a fibration of pre-p–cyclotomic (resp., pre-cyclotomic) spectra in either of the model structures above. Then the map*

$$F_{\text{Cyc}}(X,Y) \longrightarrow F_{\text{Cyc}}(X,Z) \times_{F_{\text{Cyc}}(W,Z)} F_{\text{Cyc}}(W,Y)$$

is a fibration of orthogonal spectra, and is a weak equivalence if either i or f is.

Proof By the usual adjunction argument (using the adjunction of Proposition 4.4), it is equivalent to show that for every cofibration $j\colon A \to B$ of orthogonal spectra, the map

$$F(A,Y) \longrightarrow F(B,Z) \times_{F(A,Z)} F(B,Y)$$

is a fibration of pre-p–cyclotomic (resp., pre-cyclotomic) spectra and a weak equivalence whenever j or f is. But since fibrations of pre-p–cyclotomic (resp., pre-cyclotomic) spectra are just $\mathcal{F}_p$–fibrations (resp., $\mathcal{F}_{\text{fin}}$–fibrations) of the underlying orthogonal $\mathbb{T}$–spectra, this is clear from the corresponding fact in the category of orthogonal $\mathbb{T}$–spectra. Note that in the p–complete (resp., finite complete) model structure, we actually obtain a fibration in the p–complete (resp., finite complete) model structure on orthogonal spectra. □

We proved SM7 using one of the usually equivalent adjoint formulations. The other adjoint formulation, called the pushout–product axiom, is not equivalent in this context because not all the relevant pushouts and tensors exist; however, the pushout–product axiom does follow for those pushouts and tensors that exist in the category. Specifically, given a cofibration of pre-p–cyclotomic (resp., pre-cyclotomic) spectra $j\colon X \to Y$ and a cofibration of cofibrant orthogonal spectra $i\colon A \to B$, the map

$$(Y \wedge A) \cup_{(X \wedge A)} (X \wedge B) \longrightarrow Y \wedge B$$

is a cofibration and a weak equivalence if either i or j is cofibration. An immediate consequence of this formula is the fact that the model structures on pre-p–cyclotomic and pre-cyclotomic spectra are stable in the sense that suspension (smash with S^1) is an equivalence on the homotopy category with inverse equivalence smash with $F_{\mathbb{R}} S^0$. As in [6, Section 7], this implies that the associated homotopy categories become triangulated with the Quillen Puppe cofibration sequences defining the distinguished triangles and the Quillen suspension defining the shift. In fact, Theorem 5.9 directly gives the triangulated structure: mapping out of a cofibration sequence of cofibrant objects into a fibrant object, we get a fibration sequence on the mapping spectra, and mapping into a fibration sequence of fibrant objects from a cofibrant object, we get a fibration sequence of mapping spectra. Summarizing, we have the following proposition:

5.10 Proposition *The homotopy categories of pre-p–cyclotomic spectra and pre-cyclotomic spectra are triangulated, with the distinguished triangles determined by the cofiber sequences specified by the model structure* (see [6, Definition 6.2.6]) *and suspension inducing the shift. The homotopy categories of p–cyclotomic spectra and cyclotomic spectra are full triangulated subcategories of the homotopy categories of pre-p–cyclotomic spectra and pre-cyclotomic spectra, respectively.*

To compute the derived mapping spectrum using the enrichment, we take the orthogonal spectrum of maps from a cofibrant cyclotomic spectrum X to a fibrant cyclotomic spectrum Y. As cofibrant cyclotomic spectra do not typically arise in nature, we offer the following construction, which will allow us to construct the derived mapping spectra using the more general class of cyclotomic spectra whose underlying orthogonal $\mathbb{T}$–spectra are cofibrant.

5.11 Construction For pre-p–cyclotomic spectra X and Y, let $F^h_{\mathrm{Cyc}}(X,Y)$ be the homotopy equalizer of the maps x_p, $y_p\colon F(X,Y)\to F(\rho_p^*\Phi^{C_p}X,Y)$, where x_p is induced by the structure map for X and y_p is the composite

$$F^{\mathbb{T}}(X,Y)\longrightarrow F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}X,\rho_p^*\Phi^{C_p}Y)\longrightarrow F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}X,Y)$$

induced by (2.4) and the structure map for Y. Specifically, we construct $F^h_{\mathrm{Cyc}}(X,Y)$ as the pullback

$$\begin{array}{ccc} F^h_{\mathrm{Cyc}}(X,Y) & \longrightarrow & F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}X,Y)^I \\ \big\downarrow & & \big\downarrow \\ F^{\mathbb{T}}(X,Y) & \xrightarrow[(x_p,y_p)]{} & F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}X,Y)\times F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}X,Y) \end{array}$$

where $(-)^I=F(I_+,-)$ is the orthogonal spectrum of unbased maps out of the unit interval and the vertical map is induced by the restriction to $\{0,1\}\subset I$.

5.12 Theorem *Let X and Y be pre-p–cyclotomic spectra. If X is cofibrant, then the canonical map from the equalizer to the homotopy equalizer $F_{\mathrm{Cyc}}(X,Y)\to F^h_{\mathrm{Cyc}}(X,Y)$ is a level equivalence.*

Proof First consider the case when $X=\mathbb{C}_pA$ for A an $\mathcal{F}_p$–cofibrant orthogonal $\mathbb{T}$–spectrum. In this case we are looking at the canonical map from the pullback with $J=*$ to the pullback with $J=I$ (induced by $I\to *$) in the following diagram:

$$\begin{array}{ccc} & & \prod_{n\ge 1} F^{\mathbb{T}}(\rho_{p_n}^*\Phi^{C_{p^n}}A,Y)^J \\ & & \big\downarrow \\ \prod_{n\ge 0} F^{\mathbb{T}}(\rho_{p_n}^*\Phi^{C_{p^n}}A,Y) & \longrightarrow & \prod_{n\ge 1} F^{\mathbb{T}}(\rho_{p_n}^*\Phi^{C_{p^n}}A,Y)\times\prod_{n\ge 1} F^{\mathbb{T}}(\rho_{p_n}^*\Phi^{C_{p^n}}A,Y) \end{array}$$

This is the map

$$F^{\mathbb{T}}(A,Y)\longrightarrow F^{\mathbb{T}}(A,Y)\times_{\prod_{n\ge 1}F^{\mathbb{T}}(\rho_{p_n}^*\Phi^{C_{p^n}}A,Y)}\Big(\prod_{n\ge 1}F^{\mathbb{T}}(\rho_{p_n}^*\Phi^{C_{p^n}}A,Y)\Big)^{I_+}$$

from $F^{\mathbb{T}}(A,Y)$ to the mapping path object, which is always a level equivalence of orthogonal spectra. Looking at pushouts over cofibrations and sequential colimits over cofibrations, it follows from [8, Theorem III.2.7] that the map is a level equivalence for cell pre-p–cyclotomic spectra. Finally, since level equivalences are preserved by retracts, it follows that the map is a level equivalence for cofibrant pre-p–cyclotomic spectra. □

5.13 Corollary *Let X be a pre-p–cyclotomic spectrum whose underlying orthogonal $\mathbb{T}$–spectrum is $\mathcal{F}_p$–cofibrant and let $\bar{X} \to X$ be a cofibrant replacement in the model* category of pre-p–cyclotomic spectra. Then, for any fibrant pre-p–cyclotomic spectrum Y, the maps*

$$F^h_{\mathrm{Cyc}}(X, Y) \longrightarrow F^h_{\mathrm{Cyc}}(\bar{X}, Y) \longleftarrow F_{\mathrm{Cyc}}(\bar{X}, Y)$$

are weak equivalences; thus, when Y is fibrant, $F^h_{\mathrm{Cyc}}(X, Y)$ represents the derived mapping spectrum.

Note that, if X is just cofibrant as an orthogonal $\mathbb{T}$–spectrum, $X \wedge E\mathcal{F}_{p+}$ is $\mathcal{F}_p$–cofibrant; in practice, one expects to apply the previous corollary to $X \wedge E\mathcal{F}_{p+}$ (which inherits a canonical pre-p–cyclotomic structure from a pre-p–cyclotomic structure on X).

We have an analogous construction in the context of pre-cyclotomic spectra. Because the definition of pre-cyclotomic spectra involves compatibility relations, we need a homotopy limit over a more complicated category. As above, but for each $n > 1$, we have maps

$$x_n, y_n \colon F(X, Y) \longrightarrow F(\rho_n^* \Phi^{C_n} X, Y)$$

induced by the structure map on X and the structure map on Y, respectively. More generally, we write $x_{n;m}$ and $y_{n;m}$ for the maps

$$F(\rho_m^* \Phi^{C_m} X, Y) \longrightarrow F(\rho_{mn}^* \Phi^{C_{mn}} X, Y)$$

induced by the map $\rho_m^* \Phi^{C_m} t_n$ on X,

$$\rho_{mn}^* \Phi^{C_{mn}} X \longrightarrow \rho_m^* \Phi^{C_m} (\rho_n^* \Phi^{C_n} X) \xrightarrow{\rho_m^* \Phi^{C_m} t_n} X,$$

and

$$F^{\mathbb{T}}(\rho_m^* \Phi^{C_m} X, Y) \longrightarrow F^{\mathbb{T}}(\rho_n^* \Phi^{C_n}(\rho_m^* \Phi^{C_m} X), \rho_n^* \Phi^{C_n} Y) \longrightarrow F^{\mathbb{T}}(\rho_{mn}^* \Phi^{C_{mn}} X, Y),$$

induced by the map t_n on Y. These maps satisfy the following relations:

$$(5.14)\quad x_{n;\ell m} \circ x_{m;\ell} = x_{mn;\ell}, \quad x_{n;\ell m} \circ y_{m;\ell} = y_{m;\ell n} \circ x_{n;\ell}, \quad y_{n;\ell m} \circ y_{m;\ell} = y_{mn;\ell}.$$

The first and last equations follow from the compatibility requirement relating t_{mn} and $t_n \circ t_m$ (for X and Y, respectively). The middle equation follows from the functoriality of $\rho_m^* \Phi^{C_m}$.

5.15 Construction Let Θ be the category whose objects are the positive integers and maps freely generated by maps $x_{n;m}$, $y_{n;m}: m \to mn$ for all $m, n \in \mathbb{N}$, subject to the relations (5.14). Construct $F^h_{\mathrm{Cyc}}(X, Y)$ as the homotopy limit of the functor $F^{\mathbb{T}}(\rho^*_m \Phi^{C_m} X, Y)$ from Θ to orthogonal spectra.

The category Θ is (non-canonically) isomorphic to the opposite of the category $\mathbb{I}$ used in the construction of TC (see [3, Section 3.1] or Definition 6.2 below).

To explain the relationship of this construction to Construction 5.11, let Θ_p be the full subcategory of Θ consisting of the objects p^s for $s \geq 0$. The inclusion of the full subcategory consisting of 1 and p induces a map to the homotopy equalizer of $x_{p;1}$ and $y_{p;1}$ from the homotopy limit over Θ_p.

5.16 Proposition *Let F be any functor from Θ_p to orthogonal spectra. The canonical map to the homotopy equalizer of $x_{p;1}$ and $y_{p;1}$ from the homotopy limit over Θ_p is a level equivalence.*

Proof By [2, Section XI.9.1], it suffices to see that the inclusion of $\{1, p\}$ into Θ_p is left cofinal, that is, for every s the category of maps in Θ_p from $\{1, p\}$ to p^s has weakly contractible nerve. The maps from 1 to p^s are in one-to-one correspondence with monomials of the form $x^i y^{s-i}$ for $0 \leq i \leq s$ and maps from p to p^s are in one-to-one correspondence with monomials of the form $x^i y^{s-i-1}$ for $0 \leq i \leq s-1$. For x^s and y^s, there is exactly one non-identity map in the category, namely $x^s \to x^{s-1}$ and $y^s \to y^{s-1}$, respectively. For every other monomial $x^i y^{s-i}$, there are exactly two maps, $x_{p;1}$ and $y_{p;1}$, to $x^{i-1} y^{s-i}$ and $x^i y^{s-i-1}$, respectively. The nerve of this category is therefore a generalized interval with $s+1$ 1–simplices. □

Observe that the limit of the functor $F^{\mathbb{T}}(\rho^*_m \Phi^{C_m} X, Y)$ from Θ to orthogonal spectra is precisely $F_{\mathrm{Cyc}}(X, Y)$. As in Theorem 5.12, the canonical map from the limit to the homotopy limit is a level equivalence when X is cofibrant and therefore $F^h_{\mathrm{Cyc}}(X, Y)$ provides an explicit model of the derived mapping space in the category of pre-cyclotomic spectra when Y is fibrant.

5.17 Theorem *Let X and Y be pre-cyclotomic spectra. If X is cofibrant, then the canonical map from the limit to the homotopy limit $F_{\mathrm{Cyc}}(X, Y) \to F^h_{\mathrm{Cyc}}(X, Y)$ is a level equivalence.*

Proof By the usual argument, this reduces to the case of the domain and codomain of a cell in pre-cyclotomic spectra, $X = \mathbb{C} F_V(\mathbb{T}/C_n \times B)_+$, where $B = S^{q-1}$ or $B = D^q$. Let $P = \{p_1, \dots, p_r\}$ be the distinct prime factors of n and let $\Theta_{\hat{P}}$ be the full subcategory of Θ consisting of the integers not divisible by the primes in P. Then we have that

$$\Theta = \Theta_{p_1} \times \cdots \times \Theta_{p_r} \times \Theta_{\widehat{P}}$$

and the Fubini theorem for homotopy limits gives a level equivalence

$$F^h_{\mathrm{Cyc}}(X,Y) \simeq \mathrm{holim}_{\Theta_{p_1}} \cdots \mathrm{holim}_{\Theta_{p_r}} \mathrm{holim}_{\Theta_{\widehat{P}}} F^{\mathbb{T}}(\rho_m^* \Phi^{C_m} X, Y),$$

compatibly with the map from the limit. Since $\Phi^{C_m} X = *$ for all m in $\Theta_{\widehat{P}}$ except $m = 1$, the equivalence above reduces to a level equivalence

$$F^h_{\mathrm{Cyc}}(X,Y) \simeq \mathrm{holim}_{\Theta_{p_1}} \cdots \mathrm{holim}_{\Theta_{p_r}} F^{\mathbb{T}}(\rho_m^* \Phi^{C_m} X, Y).$$

Using the previous proposition, we can identify this homotopy limit as an iterated homotopy equalizer. As in the proof of Theorem 5.12, we can then identify the map in question as the map from $F_V(\mathbb{T}/C_n \times B)_+$ to an iterated mapping path object. □

The analogue of Corollary 5.13 also holds in this context:

5.18 Corollary *Let X be a pre-cyclotomic spectrum whose underlying orthogonal $\mathbb{T}$–spectrum is $\mathcal{F}_{\mathrm{fin}}$–cofibrant and let $\overline{X} \to X$ be a cofibrant replacement in the model* category of pre-cyclotomic spectra. Then, for any fibrant pre-cyclotomic spectrum Y, the maps*

$$F^h_{\mathrm{Cyc}}(X,Y) \longrightarrow F^h_{\mathrm{Cyc}}(\overline{X},Y) \longleftarrow F_{\mathrm{Cyc}}(\overline{X},Y)$$

are weak equivalences; thus, $F^h_{\mathrm{Cyc}}(X,Y)$ represents the derived mapping spectrum when Y is fibrant.

6 Corepresentability of *TR* and *TC*

We apply the work of the previous section to deduce corepresentability results for *TC* and related functors. We begin with a brief review of the constructions, starting with the "restriction" and "Frobenius" maps.

Following [8], we omit notation for the forgetful functor from orthogonal $\mathbb{T}$–spectra to non-equivariant orthogonal spectra.

6.1 Definition Let X be cyclotomic spectrum. The *restriction map* R_n is the map of non-equivariant orthogonal spectra

$$R_n\colon X^{C_{mn}} \longrightarrow (\rho_n^* \Phi^{C_n})^{C_m} \longrightarrow X^{C_m}$$

induced by t_n. The *Frobenius map* F_n is the map of non-equivariant orthogonal spectra

$$F_n\colon X^{C_{mn}} \longrightarrow X^{C_m}$$

induced by the inclusion of fixed points.

By convention, $R_1 = F_1 = \mathrm{id}\colon X \to X$. It is easy to see that R_n and F_n satisfy the following formulas:

$$R_m R_n = R_{mn}, \quad F_m R_n = R_n F_m, \quad F_m F_n = F_{mn}.$$

We let $\mathbb{I}$ be the category with objects the positive integers and maps R_n, $F_n\colon nm \to m$ subject to the relations above. We write R for the subcategory consisting of all the objects and the restriction maps, and F for the subcategory consisting of all the objects and the Frobenius maps.

A cyclotomic spectrum X determines a functor $m \mapsto X^{C_m}$ from $\mathbb{I}$ to non-equivariant orthogonal spectra.

6.2 Definition For a fibrant cyclotomic spectrum X, let

$$\begin{aligned} TR(X) &= \mathrm{holim}_R\, X^{C_m}, \\ TF(X) &= \mathrm{holim}_F\, X^{C_m}, \\ TC(X) &= \mathrm{holim}_{\mathbb{I}}\, X^{C_m}. \end{aligned}$$

We have defined *TR*, *TF* and *TC* as point-set functors on the subcategory of fibrant objects. By taking fibrant approximations, we obtain right derived functors *TR*, *TF* and *TC* from the homotopy category of cyclotomic spectra to the stable category.

For a p–cyclotomic spectrum, we have the maps R_p and F_p, and the corresponding full subcategories R_p, F_p and $\mathbb{I}_p$ of R, F and $\mathbb{I}$, respectively, consisting of the subset of objects $\{1, p, p^2, \dots\}$.

6.3 Definition For a fibrant p–cyclotomic spectrum X, let

$$\begin{aligned} TR(X;p) &= \mathrm{holim}_{R_p}\, X^{C_{p^m}}, \\ TF(X;p) &= \mathrm{holim}_{F_p}\, X^{C_{p^m}}, \\ TC(X;p) &= \mathrm{holim}_{\mathbb{I}_p}\, X^{C_{p^m}}. \end{aligned}$$

We now explain how to realize *TR* and *TC* as suitable mapping objects in cyclotomic spectra.

6.4 Construction Let $S_{TR;p}$ be the p–cyclotomic spectrum with underlying orthogonal $\mathbb{T}$–spectrum

$$S_{TR;p} = \bigvee_{s \geq 0} F_0(\mathbb{T}/C_{p^s})_+$$

and structure map from

$$\rho_p^* \Phi^{C_p} S_{TR;p} \cong \bigvee_{s\geq 0} \rho_p^* \Phi^{C_p}(F_0(\mathbb{T}/C_{p^s})_+) \cong \bigvee_{s \geq 1} F_0(\mathbb{T}/C_{p^{s-1}})_+ = \bigvee_{s\geq 0} F_0(\mathbb{T}/C_{p^s})_+$$

to $S_{TR;p}$ induced by the canonical isomorphism.

For any orthogonal $\mathbb{T}$–spectrum X, we have a canonical natural isomorphism

$$F^{\mathbb{T}}(S_{TR;p}, X) \cong \prod_{s \geq 0} X^{C_{p^s}}.$$

For X a p–cyclotomic spectrum, $F^h_{\mathrm{Cyc}}(S_{TR;p}, X)$ is then the homotopy equalizer

$$F^h(S_{TR;p}, X) \cong \mathrm{hoEq}\Big[\prod X^{C_{p^s}} \rightrightarrows \prod X^{C_{p^s}}\Big],$$

where one map is the identity (induced by the p–cyclotomic structure map on $S_{TR;p}$) and the other is the product of the maps $R_p\colon X^{C_{p^s}} \to X^{C_{p^{s-1}}}$. This construction is the "mapping microscope" of the maps R_p, which is a model for the homotopy limit over R_p [11, Definition 2.2.8]. Since the underlying orthogonal $\mathbb{T}$–spectrum of $S_{TR;p}$ is $\mathcal{F}_p$–cofibrant, we obtain the following theorem as a corollary of Theorem 5.12:

6.5 Theorem *The right derived functor of $TR(-;p)$ is corepresentable in the homotopy category of p–cyclotomic spectra, with corepresenting object $S_{TR;p}$.*

For $TC(-;p)$, we begin with the following pre-p–cyclotomic spectrum:

6.6 Construction Let $S_{TC^s;p}$ (for $s > 0$) be the pre-p–cyclotomic spectrum whose underlying orthogonal $\mathbb{T}$–spectrum is $F_0(\mathbb{T}/C_{p^s})_+$ and whose structure map is

$$\rho_p^* \Phi^{C_p} S_{TC^s;p} \cong F_0(\mathbb{T}/C_{p^{s-1}})_+ \longrightarrow F_0(\mathbb{T}/C_{p^s})_+ = S_{TC^s;p},$$

induced by the quotient map $\mathbb{T}/C_{p^{s-1}} \to \mathbb{T}/C_{p^s}$. The map $S_{TC^s;p} \to S_{TC^{s+1};p}$ induced by the quotient is then a map of pre-p–cyclotomic spectra; let $S_{TC;p}$ be the telescope.

We can identify $TC(-;p)$ in terms of maps of pre-p–cyclotomic spectra out of $S_{TC;p}$.

6.7 Theorem *For pre-p–cyclotomic spectra X, there is a natural level equivalence*

$$\operatorname{holim}_{\mathbb{I}_p} X^{C_{p^m}} \xrightarrow{\sim} F^h_{\mathrm{Cyc}}(S_{TC;p}, X).$$

Thus, if X is a fibrant p–cyclotomic spectrum, there is a natural level equivalence

$$TC(X;p) \xrightarrow{\sim} F^h_{\mathrm{Cyc}}(S_{TC;p}, X).$$

Proof Since $S_{TC;p}$ is the telescope of $S_{TC^s;p}$, commuting homotopy limits, we see that $F^h_{\mathrm{Cyc}}(S_{TC;p}, X)$ is the mapping microscope of the orthogonal spectra $F^h_{\mathrm{Cyc}}(S_{TC^s;p}, X)$. Since $F^{\mathbb{T}}(S_{TC^s;p}, X)$ and $F^{\mathbb{T}}(\rho_p^* \Phi^{C_p} S_{TC^s;p}, X)$ are canonically isomorphic to $X^{C_{p^s}}$ and $X^{C_{p^{s-1}}}$, respectively, $F^h_{\mathrm{Cyc}}(S_{TC^s;p}, X)$ is canonically isomorphic to a homotopy equalizer of the form

$$F^h_{\mathrm{Cyc}}(S_{TC^s;p}, X) = \operatorname{hoEq}[X^{C_{p^s}} \rightrightarrows X^{C_{p^{s-1}}}],$$

with the maps induced by the structure map on $S_{TC^s;p}$ and the structure map on X. The structure map on X induces the map $R_p\colon X^{C_{p^s}} \to X^{C_{p^{s-1}}}$ and the structure map on $S_{TC^s;p}$ induces the map $F_p\colon X^{C_{p^s}} \to X^{C_{p^{s-1}}}$. Therefore, we have a natural isomorphism

$$F^h_{\mathrm{Cyc}}(S_{TC;p}, X) \cong \operatorname{Mic}_s \operatorname{hoEq}[X^{C_{p^s}} \rightrightarrows X^{C_{p^{s-1}}}],$$

where the maps on the homotopy equalizers are induced by the inclusion of fixed points, i.e., by the maps F_p. It is a standard fact in TC theory that the above homotopy limit is level equivalent to the homotopy limit $\operatorname{holim}_{\mathbb{I}_p} X^{C_{p^m}}$; we now review the argument. Let $\mathbb{I}_p^{\leq s}$ denote the full subcategory of $\mathbb{I}_p$ consisting of the objects m for $m \leq s$ and let $\mathbb{I}_p^{\{s-1,s\}}$ denote the full subcategory of $\mathbb{I}_p$ consisting of the objects s and $s-1$. The argument of Proposition 5.16 then shows that the inclusion of $\mathbb{I}_p^{\{s-1,s\}}$ in $\mathbb{I}_p^{\leq s}$ induces a level equivalence

$$\operatorname{holim}_{\mathbb{I}_p^{\leq s}} X^{C_{p^m}} \longrightarrow \operatorname{holim}_{\mathbb{I}_p^{\{s-1,s\}}} X^{C_{p^m}} = \operatorname{hoEq}[X^{C_{p^s}} \rightrightarrows X^{C_{p^{s-1}}}].$$

For fixed $s \geq 0$, the diagram

$$\begin{array}{ccc} \operatorname{holim}_{\mathbb{I}_p^{\leq s+1}} X^{C_{p^m}} & \longrightarrow & \operatorname{holim}_{\mathbb{I}_p^{\{s,s+1\}}} X^{C_{p^m}} \\ \downarrow & & \downarrow \\ \operatorname{holim}_{\mathbb{I}_p^{\leq s}} X^{C_{p^m}} & \longrightarrow & \operatorname{holim}_{\mathbb{I}_p^{\{s-1,s\}}} X^{C_{p^m}} \end{array}$$

commutes up to a canonical natural homotopy, where the vertical map on the left is induced by the inclusion of $\mathbb{I}_p^{\leq s}$ in $\mathbb{I}_p^{\leq s+1}$ and the vertical map on the right is the map in the homotopy limit system for $F_{\mathrm{Cyc}}(S_{TC;p}, X)$. To see this, note that the down-then-right map is induced by the natural inclusion i of $\mathbb{I}_p^{\{s-1,s\}}$ in $\mathbb{I}_p^{\leq s+1}$ and the right-then-down map is induced by the functor $j\colon \mathbb{I}_p^{\{s-1,s\}} \to \mathbb{I}_p^{\leq s+1}$ that sends $s-1$ to s and s to $s+1$ (with the morphisms F_p and R_p from s to $s-1$ going to the corresponding morphisms from $s+1$ to s) together with F_p viewed as a natural transformation to the functor $X^{C_{p^m}}$ on $\mathbb{I}_p^{\{s-1,s\}}$ from j^* of the functor $X^{C_{p^m}}$ on $\mathbb{I}_p^{\leq s+1}$. The maps F_p then become a natural transformation from i to j and induce a homotopy making the diagram commute, naturally in X. Incorporating these canonical natural homotopies, we get a canonical map

$$\mathrm{Mic}_s \operatorname{holim}_{\mathbb{I}_p^{\leq s}} X^{C_{p^m}} \longrightarrow \mathrm{Mic}_s \operatorname{holim}_{\mathbb{I}_p^{\{s-1,s\}}} X^{C_{p^m}} \cong F_{\mathrm{Cyc}}(S_{TC;p}, X)$$

that is a level equivalence and natural in X. The composite

$$\operatorname{holim}_{\mathbb{I}_p} X^{C_{p^m}} \cong \lim_s \operatorname{holim}_{\mathbb{I}_p^{\leq s}} X^{C_{p^m}} \xrightarrow{\sim} \mathrm{Mic}_s \operatorname{holim}_{\mathbb{I}_p^{\leq s}} X^{C_{p^m}} \xrightarrow{\sim} F_{\mathrm{Cyc}}(S_{TC}, X)$$

is then the natural level equivalence in the statement. □

The preceding theorem now leads to the following characterization:

6.8 Theorem *For X a fibrant weak p–cyclotomic spectrum, $TC(X;p)$ is naturally weakly equivalent to $F^h_{\mathrm{Cyc}}(S \wedge E\mathcal{F}_{p+}, X)$, which represents the derived mapping spectrum $\mathbb{R}F_{\mathrm{Cyc}}(S, X)$.*

Proof We have compatible canonical maps from $S_{TC^s;p}$ to the sphere spectrum S, induced by the collapse map $\mathbb{T}/C_{p^s} \to *$. Using the canonical cyclotomic structure on S of Example 4.11, we obtain a canonical map of pre-p–cyclotomic spectra from $S_{TC;p}$ to S. Looking at mod-p homology, we see that the unique map from the telescope of $\mathbb{T}/C_{p^s}$ to a point is a p–equivalence and it follows that the map of pre-p–cyclotomic spectra $S_{TC;p} \to S$ is a p–$\mathcal{F}_p$–equivalence of the underlying orthogonal $\mathbb{T}$–spectra. We then have a zigzag of p–$\mathcal{F}_p$–equivalences

$$S_{TC;p} \longleftarrow S_{TC;p} \wedge E\mathcal{F}_{p+} \longrightarrow S \wedge E\mathcal{F}_{p+},$$

where all of the above weak p–cyclotomic spectra have underlying orthogonal $\mathbb{T}$–spectra that are cofibrant in both the $\mathcal{F}_p$–local model structure and the $\mathcal{F}_p$–local p–complete model structure. Since we have assumed that X is fibrant in the $\mathcal{F}_p$–local p–complete model structure, the functors $F^{\mathbb{T}}(-, X)$ and $F^{\mathbb{T}}(\rho_p^* \Phi^{C_p}(-), X)$ convert the p–$\mathcal{F}_p$–equivalences of weak p–cyclotomic spectra above to weak equivalences of orthogonal spectra. Thus, we obtain a zigzag of weak equivalences

$$F^h_{\mathrm{Cyc}}(S_{TC;p}, X) \longrightarrow F^h_{\mathrm{Cyc}}(S_{TC;p} \wedge E\mathcal{F}_{p+}, X) \longleftarrow F^h_{\mathrm{Cyc}}(S \wedge E\mathcal{F}_{p+}, X).$$

Theorem 6.7 gives a weak equivalence with $TC(X;p)$ and Corollary 5.13 shows that $F^h_{\mathrm{Cyc}}(S \wedge E\mathcal{F}_{p+}, X)$ represents the derived mapping spectrum $\mathbb{R}F_{\mathrm{Cyc}}(S, X)$. □

We also have the following standard relationship between p–completion and mapping spectra:

6.9 Lemma *Let X be a fibrant p–cyclotomic spectrum and $X^\wedge_p$ a fibrant replacement for X in the category of weak p–cyclotomic spectra. For any pre-p–cyclotomic spectrum Y that is $\mathcal{F}_p$–cofibrant as an orthogonal $\mathbb{T}$–spectrum, $F^h_{\mathrm{Cyc}}(Y,X) \to F^h(Y,X^\wedge_p)$ is a p–equivalence.*

Proof Because the orthogonal $\mathbb{T}$–spectra Y and $\rho_p^*\Phi^{C_p}Y$ are $\mathcal{F}_p$–cofibrant and the orthogonal $\mathbb{T}$–spectra X and $X^\wedge_p$ are $\mathcal{F}_p$–fibrant, the non-equivariant orthogonal spectra $F^{\mathbb{T}}(Y,X)$, $F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}Y,X)$, $F^{\mathbb{T}}(Y,X^\wedge_p)$ and $F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}Y,X^\wedge_p)$ are all fibrant. For fibrant orthogonal spectra, a map is a p–equivalence if and only if the induced map on $F(M^1_p,-)$ is a weak equivalence. It follows that the maps

$$F^{\mathbb{T}}(Y,X) \longrightarrow F^{\mathbb{T}}(Y,X^\wedge_p) \quad \text{and} \quad F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}Y,X) \longrightarrow F^{\mathbb{T}}(\rho_p^*\Phi^{C_p}Y,X^\wedge_p)$$

are p–equivalences and that the induced map on homotopy equalizers

$$F^h_{\mathrm{Cyc}}(Y,X) \longrightarrow F^h(Y,X^\wedge_p)$$

is a p–equivalence. □

Finally, to prove Theorem 1.4, we need the following standard fact about the relationship between $TC(-)^\wedge_p$ and $TC(-;p)$, rewritten in the terminology of Section 5.

6.10 Theorem *Let X be a fibrant p–cyclotomic spectrum and $X^\wedge_p$ a fibrant replacement for X in the category of weak p–cyclotomic spectra. Then the canonical map $TC(X;p) \to TC(X^\wedge_p;p)$ is a p–equivalence and $TC(X^\wedge_p;p)$ is fibrant in the p–complete model structure on orthogonal spectra.*

Proof The p–equivalence follows from the previous lemma and Theorem 6.7. To see that $TC(X^\wedge_p;p)$ is fibrant in the p–complete model structure on orthogonal spectra, we note that each $(X^\wedge_p)^{C_{p^s}} = F^{\mathbb{T}}(F_0(\mathbb{T}/C_{p^s})_+, X^\wedge_p)$ is fibrant in the p–complete model structure of orthogonal spectra and homotopy limits of fibrant objects are fibrant [5, Theorem 18.5.2]. □

Theorem 1.4 follows directly from the statements above, but to illustrate how the constructions fit together, we write out the argument as a whole.

Proof of Theorem 1.4 Given a p–cyclotomic spectrum X, let $\widetilde{X}$ be a fibrant replacement in the model* category of p–cyclotomic spectra and let $X^\wedge_p$ be a fibrant replacement of $\widetilde{X}$ in the model* category of weak p–cyclotomic spectra; then $X^\wedge_p$ is also a fibrant replacement for X in the model* category of weak p–cyclotomic spectra. We then have a commutative diagram

$$\begin{array}{ccc}
F^h_{\mathrm{Cyc}}(S \wedge E\mathcal{F}_{p+}, \widetilde{X}) & \xrightarrow{\simeq_p} & F^h_{\mathrm{Cyc}}(S \wedge E\mathcal{F}_{p+}, X^\wedge_p) \\
\downarrow & & \downarrow\wr \\
F^h_{\mathrm{Cyc}}(S_{TC;p} \wedge E\mathcal{F}_{p+}, \widetilde{X}) & \xrightarrow{\simeq_p} & F^h_{\mathrm{Cyc}}(S_{TC;p} \wedge E\mathcal{F}_{p+}, X^\wedge_p) \\
\uparrow & & \uparrow\wr \\
F^h_{\mathrm{Cyc}}(S_{TC;p}, \widetilde{X}) & \xrightarrow{\simeq_p} & F^h_{\mathrm{Cyc}}(S_{TC;p}, X^\wedge_p) \\
\uparrow & & \uparrow\wr \\
TC(\widetilde{X}; p) & \xrightarrow{\simeq_p} & TC(X^\wedge_p; p)
\end{array}$$

which (varying X) extends to a diagram of natural transformations from the homotopy category of p–cyclotomic spectra to the stable category. The maps marked "$\simeq$" are isomorphisms in the stable category (see Theorem 6.7 and the proof of Theorem 6.8) and the maps marked "$\simeq_p$" become isomorphisms after p–completion (see Lemma 6.9 and Theorem 6.10). The top-left entry $F^h_{\mathrm{Cyc}}(S \wedge E\mathcal{F}_{p+}, \widetilde{X})$ represents the derived mapping spectrum $\mathbb{R}F_{\mathrm{Cyc}}(S, X)$ and the bottom-left entry represents the derived functor $TC(X; p)$. □

We have corresponding corepresentability results for cyclotomic spectra.

6.11 Construction Let S_{TR} be the cyclotomic spectrum with underlying orthogonal $\mathbb{T}$–spectrum

$$S_{TR} = \bigvee_{m \geq 0} F_0(\mathbb{T}/C_m)_+$$

and structure map t_n from

$$\rho_n^* \Phi^{C_n} S_{TR} \cong \bigvee_{m \geq 0} \rho_n^* \Phi^{C_n}(F_0(\mathbb{T}/C_m)_+) \cong \bigvee_{\substack{n|m \\ m \geq 0}} \rho_n^* F_0(\mathbb{T}/C_m)_+ \cong \bigvee_{m \geq 0} F_0(\mathbb{T}/C_m)_+$$

to S_{TR} induced by the canonical isomorphism.

6.12 Theorem *The right derived functor of $TR(-)$ is corepresentable in the homotopy category of cyclotomic spectra, with corepresenting object S_{TR}.*

Proof For any pre-cyclotomic spectrum X, we have

$$F^h(S_{TR}, X) \cong \operatorname*{holim}_{n \in \Theta} F^{\mathbb{T}}(\rho_*^n \Phi^{C_n} S_{TR}, X) \cong \operatorname*{holim}_{n \in \Theta} \prod_{m \geq 1} X^{C_m}, \tag{6.13}$$

where the x maps are the identity and the y maps are induced by the cyclotomic structure maps on X. We show that this is level equivalent to $\operatorname{holim}_R X^{C_m}$, which is $TR(X)$ when X is a fibrant cyclotomic spectrum. As in the proof of Theorem 5.17, we use the Fubini theorem for homotopy limits combined with Proposition 5.16 to compare the homotopy limit over Θ_p with the homotopy equalizer of x_p and y_p.

Let $P_r = \{p_1, \dots, p_r\}$ denote the first r prime numbers and let Θ_{P_r} denote the full subcategory of Θ consisting of the objects n that only have elements of P_r in their prime factorization. Then Θ_{P_r} is canonically isomorphic to $\Theta_{p_1} \times \cdots \times \Theta_{p_r}$. Using the Bousfield–Kan model for the homotopy limit, we have a canonical isomorphism

$$\lim_r \operatorname{holim}_{\Theta_{p_r}} \cdots \operatorname{holim}_{\Theta_{p_1}} F \cong \lim_r \operatorname{holim}_{\Theta_{P_r}} F \cong \operatorname{holim}_{\Theta} F$$

for any functor F from Θ to orthogonal spectra. Likewise, letting R_{P_r} be the corresponding subcategory of R maps, we have a canonical isomorphism

$$\lim_r \operatorname{holim}_{R_{p_r}} \cdots \operatorname{holim}_{R_{p_1}} F \cong \lim_r \operatorname{holim}_{R_{P_r}} F \cong \operatorname{holim}_R F.$$

We note that all of the sequential limits above are limits of towers of levelwise fibrations (since on the cosimplicial objects at each cosimplicial level the map is a product projection). Let

$$F(p^{s_1}, \dots, p^{s_r}) = F^{\mathbb{T}}(\rho_*^n \Phi^{C_n} S_{TR}, X), \quad n = p_1^{s_1} \cdots p_r^{s_r},$$

be the restriction of $F^{\mathbb{T}}(\rho_*^n \Phi^{C_n} S_{TR}, X)$ to Θ_{P_r}. Applying Proposition 5.16, we have a level equivalence

$$\operatorname{holim}_{\Theta_{p_r}} \cdots \operatorname{holim}_{\Theta_{p_1}} F \longrightarrow \operatorname{hoEq}_{x_{p_r}, y_{p_r}}(\cdots(\operatorname{hoEq}_{x_{p_1}, y_{p_1}} F)\cdots).$$

Applying (6.13), the homotopy equalizer over x_p and y_p is the pullback of the diagram

$$\begin{array}{ccc} & & \prod_m (X^{C_m})^I \\ & & \downarrow \\ \prod_m X^{C_m} & \xrightarrow[(\mathrm{id}, R_p)]{} & \prod_m X^{C_m} \times \prod_m X^{C_m}, \end{array}$$

which we can identify as the microscope of

$$s \mapsto \prod_{p \nmid m} X^{C_{mp^s}}$$

over the maps R_p. By induction, we get compatible maps

$$\operatorname{holim}_{\Theta_{p_r}} \cdots \operatorname{holim}_{\Theta_{p_1}} F(s_1, \dots, s_r) \longrightarrow \prod \operatorname{Mic}_{p^{s_r}} \cdots \operatorname{Mic}_{p^{s_1}} X^{C_{mn}},$$

where $n = p_1^{s_1} \cdots p_r^{s_r}$ and the product is over m not divisible by any element of P_r. As we vary r, the tower on the right is again a tower of level fibrations and the limit is then isomorphic (via projection onto the $m = 1$ factor) to the limit

$$\lim_r \mathrm{Mic}_{s_r} \cdots \mathrm{Mic}_{s_1} X^{C_n}.$$

A functor out of R_p is just a tower, so we have the canonical level equivalence

$$\operatorname{holim}_R X^{C_n} \cong \lim_r \operatorname{holim}_{R_{p_r}} \cdots \operatorname{holim}_{R_{p_1}} X^{C^n} \longrightarrow \lim_r \mathrm{Mic}_{s_r} \cdots \mathrm{Mic}_{s_1} X^{C_n}. \qquad \square$$

We next construct the representing pre-cyclotomic spectrum for TC. Now, instead of being a telescope, it will arise as the homotopy colimit over the partially ordered set of positive integers under divisibility, which we denote by $\mathbb{J}$.

6.14 Construction Let S_{TC^m} (for $m \geq 1$) be the pre-cyclotomic spectrum whose underlying orthogonal $\mathbb{T}$–spectrum is $F_0(\mathbb{T}/C_m)_+$ and whose structure map

$$t_n \colon \rho_*^n \Phi^{C_n} S_{TC^m} = F_0 \rho_*^n (\mathbb{T}/C_m)_+^{C_n} \longrightarrow F_0(\mathbb{T}/C_m)_+ = S_{TC^m}$$

is either the trivial map $* \to S_{TC^m}$ if $n \nmid m$ or induced by the quotient map

$$\rho_*^n (\mathbb{T}/C_m)^{C_n} = \mathbb{T}/C_{m/n} \longrightarrow \mathbb{T}/C_m$$

if $n | m$. For any $n \geq 1$, we have a map $S_{TC^m} \to S_{TC^{mn}}$ induced by the quotient $\mathbb{T}/C_m \to \mathbb{T}/C_{mn}$ that is a map of pre-cyclotomic spectra; let S_{TC} be the homotopy colimit of S_{TC^m} over $\mathbb{J}$.

We can identify $TC(-)$ in terms of maps of pre-cyclotomic spectra out of S_{TC}.

6.15 Theorem *For pre-cyclotomic spectra X, there is a natural level equivalence*

$$\operatorname{holim}_{\mathbb{I}} X^{C_m} \xrightarrow{\sim} F^h_{\mathrm{Cyc}}(S_{TC}, X).$$

Thus, if X is a fibrant cyclotomic spectrum, there is a natural level equivalence

$$TC(X) \xrightarrow{\sim} F^h_{\mathrm{Cyc}}(S_{TC}, X).$$

Proof Write $\mathbb{I}(m)$ for the full subcategory of $\mathbb{I}$ consisting of m and its divisors. Looking at $F^h(S_{TC^m}, X)$ and using the fact that $\Phi^{C_n} S_{TC^m} = *$ for $n \nmid m$, we can identify $F^h_{\mathrm{Cyc}}(S_{TC^m}, X)$ as the homotopy limit

$$F^h_{\mathrm{Cyc}}(S_{TC^m}, X) = \operatorname{holim}_{\mathbb{I}(m)} X^{C_n},$$

with the usual interpretation of F and R. Since $S_{TC} = \operatorname{hocolim}_{\mathbb{J}} S_{TC^m}$, we have

$$F^h_{\mathrm{Cyc}}(S_{TC}, X) \cong \operatorname{holim}_{m \in \mathbb{J}} \operatorname{holim}_{n \in \mathbb{I}(m)} X^{C_n}.$$

The natural map in the statement is then the map

$$\operatorname{holim}_{\mathbb{I}} X^{C_n} \cong \lim_{m \in \mathbb{J}} \operatorname{holim}_{n \in \mathbb{I}(m)} X^{C_n} \longrightarrow F^h_{\mathrm{Cyc}}(S_{TC}, X).$$

Taking a cofinal sequence of m in $\mathbb{J}$ and applying [2, Section XI.9.1], we see that the map is a level equivalence. □

The collapse maps $\mathbb{T}/C_m \to *$ induce a map of pre-cyclotomic spectra from S_{TC} to the cyclotomic spectrum S, which is clearly a finite complete $\mathcal{F}_{\mathrm{fin}}$–equivalence of the underlying orthogonal $\mathbb{T}$–spectra. The argument of Theorem 6.8 now generalizes to prove the following theorem:

6.16 Theorem *For X a fibrant weak cyclotomic spectrum, $TC(X)$ is naturally weakly equivalent to $F^h_{\mathrm{Cyc}}(S \wedge E\mathcal{F}_{\mathrm{fin}+}, X)$, which represents the derived mapping spectrum $\mathbb{R}F_{\mathrm{Cyc}}(S, X)$.*

Appendix: A technical result on the geometric fixed point functor for orthogonal G–spectra

Let G be a compact Lie group and U a G–universe, and consider the category of orthogonal G–spectra modeled on U [8, Chapter II]. Fix a closed normal subgroup $N \lhd G$. For cofibrant orthogonal G–spectra X and Y, the canonical natural map (of orthogonal G/N–spectra)

$$\Phi^N X \wedge \Phi^N Y \longrightarrow \Phi^N (X \wedge Y)$$

is an isomorphism [8, Proposition V.4.7]. In this appendix, we generalize this statement to the case when X is not cofibrant.

A.1 Lemma *Let X and Y be orthogonal G–spectra and assume that Y is cofibrant. The canonical natural map*

$$\Phi^N X \wedge \Phi^N Y \longrightarrow \Phi^N (X \wedge Y)$$

is an isomorphism.

The proof occupies the entirety of the appendix.

Since both sides commute with pushouts over Hurewicz cofibrations, the statement reduces to the case when $Y = F_A B_+$ for B a G–CW complex (space) and $A < U$. Since smashing with an unbased G–space commutes appropriately with geometric fixed points [8, Corollary V.4.6 and Proposition V.4.7],

$$\Phi^N T \wedge (B_+)^N \cong \Phi^N (T \wedge B_+),$$

the statement reduces to the case when $B = *$ and $Y = F_A S^0$.

Following the notation of [8, Definition V.4.1], we write E for the extension

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\epsilon} J \longrightarrow 1,$$

where $J = G/N$, and we write $\mathscr{I}_E$ for $(\mathscr{I}_G)^N$. Recall that the geometric fixed point functor $\Phi^N X$ is formed from the $\mathscr{I}_E$–space $\mathrm{Fix}^N X$

$$\mathrm{Fix}^N X(V) = (X(V))^N$$

by enriched left Kan extension to $\mathscr{I}_J$ along the functor $\phi\colon \mathscr{I}_E \to \mathscr{I}_J$ sending V to V^N. The evident functor

$$\oplus\colon \mathscr{I}_E \wedge \mathscr{I}_E \longrightarrow \mathscr{I}_E$$

sending (V, W) to $V \oplus W$ induces a smash product of $\mathscr{I}_E$–spaces. Since the diagram

$$\begin{array}{ccc} \mathscr{I}_E \wedge \mathscr{I}_E & \xrightarrow{\oplus} & \mathscr{I}_E \\ {\scriptstyle \phi\wedge\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathscr{I}_J \wedge \mathscr{I}_J & \xrightarrow[\oplus]{} & \mathscr{I}_J \end{array}$$

commutes, enriched left Kan extension along ϕ takes the smash product of $\mathscr{I}_E$–spaces to the smash product of orthogonal G/N–spectra. Thus, we are reduced to showing that the canonical natural map

$$\mathrm{Fix}^N X \wedge \mathrm{Fix}^N (F_A S^0) \longrightarrow \mathrm{Fix}^N (X \wedge F_A S^0)$$

is an isomorphism.

We now write formulas for $(\mathrm{Fix}^N X \wedge \mathrm{Fix}^N (F_A S^0))(V)$ and $\mathrm{Fix}^N (X \wedge F_A S^0)(V)$. By definition, $X \wedge F_A S^0$ is the enriched left Kan extension of the functor $\mathscr{I}_G \wedge \mathscr{I}_G \to \mathscr{I}_G$, the V^{th} space of which we can write as the coequalizer:

$$\begin{array}{c} \displaystyle\bigvee_{W,W',Z,Z'<U} \mathscr{I}_G(W' \oplus Z', V) \wedge (\mathscr{I}(W, W') \wedge \mathscr{I}_G(Z, Z')) \wedge X(W) \wedge F_A S^0(Z) \\ \downarrow\downarrow \\ \displaystyle\bigvee_{W,Z<U} \mathscr{I}_G(W \oplus Z, V) \wedge X(W) \wedge F_A S^0(Z) \end{array}$$

From the universal property of $F_A S^0$, we have that $F_A S^0(Z) = \mathscr{J}_G(A, Z)$ and so the coequalizer above simplifies to the coequalizer:

$$\begin{array}{c}\bigvee_{W,W'<U} \mathscr{J}_G(W' \oplus A, V) \wedge \mathscr{J}(W, W') \wedge X(W) \\ \downarrow\downarrow \\ \bigvee_{W<U} \mathscr{J}_G(W \oplus A, V) \wedge X(W)\end{array}$$

Let $a = \dim |A|$. We have that $\mathscr{J}_G(W \oplus A, V) = *$ when $\dim V < a$, so the coequalizer is just the basepoint unless $\dim V \geq a$. Let $n = \dim V - a$. Since in $\mathscr{J}_G$ every W is isomorphic to $\mathbb{R}^m$ for some m, every map in $\mathscr{J}_G$ from $W \oplus A$ to V factors through a map from $\mathbb{R}^n \oplus A$ to V. The coequalizer above then reduces to the coequalizer:

$$\begin{array}{c}\bigvee_{W<U} \mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge \mathscr{J}(W, \mathbb{R}^n) \wedge X(W) \\ \downarrow\downarrow \\ \mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge X(\mathbb{R}^n)\end{array}$$

Using the same observation on $\mathbb{R}^n$, we can identify this with the orbit space

$$\mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n).$$

An analogous argument for $(\mathrm{Fix}^N X \wedge \mathrm{Fix}^N(F_A S^0))(V)$ yields an identification as the orbit space

$$\mathscr{J}_E(W \oplus A, V) \wedge_{O(W)^N} X(W)^N,$$

where V is isomorphic to $W \oplus A$ as an orthogonal N–representation. (In the case when no such decomposition exists, $(\mathrm{Fix}^N X \wedge \mathrm{Fix}^N(F_A S^0))(V)$ is just the basepoint.) Thus, we must show that the map

$$\theta\colon \mathscr{J}_E(W \oplus A, V) \wedge_{O(W)^N} X(W)^N \longrightarrow (\mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n))^N \tag{A.2}$$

is a homeomorphism. We prove this by constructing an explicit inverse homeomorphism below. First, it is useful to describe θ concretely in terms of elements. For this, choose and fix an orthonormal basis of W, which we can regard as a (non-equivariant) isometric isomorphism $h\colon \mathbb{R}^n \to W$. For a representative element (f, x) on the left, with $f \in \mathscr{J}_E(W \oplus A, V)$ and $x \in X(W)^N$,

$$\theta(f, x) = (f \circ (h \oplus \mathrm{id}_A), (h^{-1})_* x),$$

where $(h^{-1})_*$ denotes the (non-equivariant) map $X(W) \to X(\mathbb{R}^n)$ associated to $h^{-1}\colon W \to \mathbb{R}^n$. In the formula,

$$f \circ (h \oplus \mathrm{id}_A) \in \mathscr{J}_G(\mathbb{R}^n \oplus A, V), \quad (h^{-1})_* x \in X(\mathbb{R}^n),$$

and by the abstract definition of the map we must have that

$$(f \circ (h \oplus \mathrm{id}_A), (h^{-1})_* x) \in \mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n)$$

is

(i) independent of the choice of h,

(ii) independent of the choice of representative (f, x), and

(iii) in the N–fixed point subspace $(\mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n))^N$;

however, it is trivial to check each of these facts explicitly in terms of the elementwise formula for θ, and doing this check will (we hope) help give some insight into how the formula works. For (i), if we chose a different $h'\colon \mathbb{R}^n \to W$, then we would have $h' = h \circ j$ for some j in $O(n)$, and

$$(f \circ (h' \oplus \mathrm{id}_A), (h'^{-1})_* x) = \bigl(f \circ (h \oplus \mathrm{id}_A) \circ (j \oplus \mathrm{id}_A), j_*^{-1}((h^{-1})_* x)\bigr)$$

represents the same element of $\mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n)$ as $(f \circ (h \oplus \mathrm{id}_A), (h^{-1})_* x)$ since the right $O(n)$ action on $\mathscr{J}_G(\mathbb{R}^n \oplus A, V)$ is by right multiplication by $j \oplus \mathrm{id}_A$. For (ii), any other representative of (f, x) is $(f \circ (\eta \oplus \mathrm{id}_A), \eta_*^{-1} x)$ for some $\eta \in O(W)^N$, and

$$\theta(f \circ (\eta \oplus \mathrm{id}_A), \eta_*^{-1} x) = \bigl(f \circ ((\eta \circ h) \oplus \mathrm{id}_A), ((\eta \circ h)^{-1})_* x\bigr).$$

Since $h' = \eta \circ h$ is a (non-equivariant) isometric isomorphism $\mathbb{R}^n \to W$, it follows by (i) that this represents the same element of $\mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n)$ as $\theta(f, x) = (f \circ (h \oplus \mathrm{id}_A), (h^{-1})_* x)$. Finally, for (iii), first note that θ is G–equivariant; this follows from the abstract definition of the map, but again, we check it explicitly: The action of G on $\mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n)$ is diagonal, so

$$\begin{aligned} g \cdot \theta(f, x) &= \bigl(g \cdot (f \circ (h \oplus \mathrm{id}_A)), g \cdot ((h^{-1})_* x)\bigr) \\ &= \bigl((g \cdot f) \circ ((g \cdot h) \oplus \mathrm{id}_A), ((g \cdot h)^{-1})_* (g \cdot x)\bigr) \\ &= ((g \cdot f) \circ (h \oplus \mathrm{id}_A), (h^{-1})_* (g \cdot x)) && \text{(by (i), taking } h' = g \cdot h) \\ &= \theta(g \cdot f, g \cdot x). \end{aligned}$$

In the second equality, the G–action distributes over composition with the isometric isomorphism $h \oplus \mathrm{id}_A$ on $\mathscr{J}_E$ by inspection and over the action by the isometric isomorphism h^{-1} on X by the definition of orthogonal spectrum. When $g \in N$, we have $g \cdot f = f$ since $f \in \mathscr{J}_E(W \oplus A, V) = (\mathscr{J}_G(W \oplus A, V))^N$, and $g \cdot x = x$ since $x \in X(W)^N$.

We construct an inverse to the map θ of (A.2) as follows. We note that $\mathscr{J}_G(\mathbb{R}^n \oplus A, V)$ is the G–equivariant space of (non-equivariant) isometric isomorphisms from $\mathbb{R}^n \oplus A$

to V plus a disjoint basepoint. Let Ψ denote the subset of $\mathscr{I}_G(\mathbb{R}^n \oplus A, V) \wedge X(\mathbb{R}^n)$ which the quotient map sends into the N fixed points $(\mathscr{I}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n))^N$. Writing (f, x) for a typical element of $\mathscr{I}_G(\mathbb{R}^n \oplus A, V) \wedge X(\mathbb{R}^n)$, consider an element $(f, x) \in \Psi$, where x is not the basepoint. Then, for every ν in N,

$$(\nu \cdot f, \nu \cdot x) = (f \circ (j_\nu^{-1} \oplus \mathrm{id}_A), j_{\nu *} x)$$

for some j_ν in $O(n)$; note that j_ν is determined uniquely since $O(n)$ acts freely on $\mathscr{I}_G(\mathbb{R}^n \oplus A, V)$. In particular, the restriction of f to A must be N–equivariant and the image $f(\mathbb{R}^n)$ of the restriction to $\mathbb{R}^n$ must be stable under the action of N. Thus, f induces an N–equivariant isometric isomorphism from $f(\mathbb{R}^n) \oplus A$ to V. In particular, in the case when V does not contain a N–representation isomorphic to A, $(\mathscr{I}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n))^N$ consists of just the basepoint and the map θ of (A.2) is a homeomorphism.

Otherwise, we fix the N–equivariant isometric isomorphism $V \cong W \oplus A$ and, for each non-basepoint element (f, x) in Ψ, we choose an N–equivariant isometric isomorphism

$$g_{(f,x)} \colon f(\mathbb{R}^n) \longrightarrow W.$$

Then $g_{(f,x)}^{-1}$ together with the restriction of f to A specify an element $\gamma_{(f,x)}$ in

$$(\mathscr{I}_G(W \oplus A, V))^N = \mathscr{I}_E(W \oplus A, V)$$

and $g_{(f,x)} \circ f$ is an element of $\mathscr{I}_G(\mathbb{R}^n, W)$. The hypothesis that (f, x) is in Ψ then implies that $(g_{(f,x)} \circ f)_* x \in X(W)$ is an N fixed point:

$$\begin{aligned} \nu \cdot ((g_{(f,x)} \circ f)_* x) &= (\nu \cdot (g_{(f,x)} \circ f))_* (\nu \cdot x) = (\nu \cdot g_{(f,x)} \circ \nu \cdot f)_* (\nu \cdot x) \\ &= (g_{(f,x)} \circ f \circ j_\nu^{-1})_* (j_{\nu *} x) = (g_{(f,x)} \circ f)_* x. \end{aligned}$$

We then get a basepoint-preserving function ψ on Ψ that sends (f, x) to

$$(\gamma_{(f,x)}, (g_{(f,x)} \circ f)_* x) \in \mathscr{I}_E(W \oplus A, V) \wedge_{O(W)^N} X(W)^N.$$

Because any two possible choices of $g_{(f,x)}$ are related by an element of $O(W)^N$, it follows that ψ is independent of the choice of $g_{(f,x)}$. It is easy to see that ψ is continuous at the basepoint and, since, for any non-basepoint x, we can choose $g_{(f,x)}$ locally to be a continuous function of (f, x), it follows that ψ is continuous. Finally, for $j \in O(n)$,

$$\begin{aligned} \psi(f \circ (j^{-1} \oplus \mathrm{id}_A), j_* x) &= (\gamma_{(f,x)}, (g_{(f,x)} \circ f \circ j^{-1})_* (j_* x)) \\ &= (\gamma_{(f,x)}, (g_{(f,x)} \circ f)_* x) = \psi(f, x), \end{aligned}$$

so ψ descends to a continuous map

$$(\mathscr{J}_G(\mathbb{R}^n \oplus A, V) \wedge_{O(n)} X(\mathbb{R}^n))^N \longrightarrow \mathscr{J}_E(W \oplus A, V) \wedge_{O(W)^N} X(W)^N.$$

Using the formula above, both composites of θ and ψ are easily seen to be the appropriate identity maps.

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