Commuting symplectomorphisms
and Dehn twists in divisors

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Two commuting symplectomorphisms of a symplectic manifold give rise to actions on Floer cohomologies of each other. We prove the elliptic relation saying that the supertraces of these two actions are equal. In the case when a symplectomorphism f commutes with a symplectic involution, the elliptic relation provides a lower bound on the dimension of $\mathrm{HF}^*(f)$ in terms of the Lefschetz number of f restricted to the fixed locus of the involution. We apply this bound to prove that Dehn twists around vanishing Lagrangian spheres inside most hypersurfaces in Grassmannians have infinite order in the symplectic mapping class group.

1 Introduction and main results

1.1 Overview

Let X be a symplectic manifold and $\mathrm{Symp}(X)/\mathrm{Ham}(X)$ be the group of all symplectomorphisms of X modulo Hamiltonian isotopy. When X is simply connected, this group is the same as $\pi_0 \mathrm{Symp}(X)$. If one denotes by $\pi_0 \mathrm{Diff}(X)$ the smooth mapping class group, there is an obvious forgetful map

$$\mathrm{Symp}(X)/\mathrm{Ham}(X) \xrightarrow{\text{ forgetful }} \pi_0 \mathrm{Diff}(X).$$

Paul Seidel in his thesis [33] found examples where this map is not injective: if X is any complete intersection of complex dimension 2 other than $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, and $\tau: X \to X$ is a certain symplectomorphism called the Dehn twist, then τ^2 is smoothly isotopic to the identity, but not Hamiltonian isotopic to the identity. Later Seidel proved [35] that the kernel of the forgetful map is infinite for some K3 surfaces, again by considering the group generated by a Dehn twist. Using a new technique, we study Dehn twists in certain divisors (the main examples are divisors in Grassmannians) and extend the range of examples where the forgetful map has infinite kernel.

Suppose X satisfies the so-called W^+ condition, which is slightly stronger than weak monotonicity. We define, for two commuting symplectomorphisms $f, g: X \to X$, their

actions on Floer cohomology $f_{\text{floer}}\colon \operatorname{HF}^*(g) \to \operatorname{HF}^*(g)$, $g_{\text{floer}}\colon \operatorname{HF}^*(f) \to \operatorname{HF}^*(f)$. We then prove a theorem which was proposed by Paul Seidel (cf [39, Remark 4.1]), who suggested it be called the elliptic relation.

Theorem 1.1 (Elliptic relation) *If X is a symplectic manifold satisfying the W^+ condition and $f, g\colon X \to X$ are two commuting symplectomorphisms, then*

$$\operatorname{STr}(f_{\text{floer}}) = \operatorname{STr}(g_{\text{floer}}) \in \Lambda.$$

Here Λ is the Novikov field. In the rest of the introduction, we explain the elliptic relation, state its Lagrangian version, and consider applications to Dehn twists in divisors. We begin by discussing our results regarding Dehn twists.

1.2 Order of Dehn twists in divisors

Let $\operatorname{Gr}(k, n)$ be the Grassmannian of k–planes in $\mathbb{C}^n$. Let $\mathcal{O}(d)$ be the line bundle on $\operatorname{Gr}(k, n)$ which is the pullback of $\mathcal{O}_{\mathbb{P}^N}(d)$ under the Plücker embedding $\operatorname{Gr}(k, n) \subset \mathbb{P}^N$. Consider a smooth divisor $X \subset \operatorname{Gr}(k, n)$ in the linear system $|\mathcal{O}(d)| = \mathbb{P}H^0(\operatorname{Gr}(k, n), \mathcal{O}(d))$. The results below are interesting even for $\operatorname{Gr}(1, n) = \mathbb{P}^{n-1}$, so for simplicity one can take $X \subset \mathbb{P}^{n-1}$ to be a smooth projective hypersurface of degree d throughout this subsection.

For $d \geq 2$, X contains a class of Lagrangian spheres which we call $|\mathcal{O}(d)|$–vanishing Lagrangian spheres, which, briefly, are vanishing cycles for algebraic degenerations of X inside the linear system $|\mathcal{O}(d)|$. To every parametrised Lagrangian sphere $L \subset X$ one associates a symplectomorphism $\tau_L\colon X \to X$ called the Dehn twist around L. (The definitions are given in Section 3.) We prove the following.

Theorem 1.2 *Let $X \subset \operatorname{Gr}(k, n)$ be a smooth divisor in the linear system $|\mathcal{O}(d)|$, and $L \subset X$ be an $|\mathcal{O}(d)|$–vanishing Lagrangian sphere. Suppose*

$$3 \leq d \leq n \quad \text{or} \quad d \geq k(n-k) + n - 2.$$

Then the Hamiltonian isotopy class of τ_L is an element of infinite order in the group $\operatorname{Symp}(X)/\operatorname{Ham}(X)$.

When $d = 2$ and $k = 1$ (X is a projective quadric), τ_L has order 1 or 2 depending on the parity of n; see Smith [42, Lemma 4.2]. While our proof crucially uses $d \geq 3$, further restrictions on d are only needed to make X satisfy the W^+ condition, so that the "classical" definition of Floer cohomology of symplectomorphisms $X \to X$ applies. There are techniques defining Floer cohomology of symplectomorphisms on

arbitrary symplectic manifolds; see Fukaya and Ono [15]. With their help, it should be possible to drop the W^+ condition from Theorem 1.1, and the proof of Theorem 1.2 should work for all $d \geq 3$.

Recall the forgetful map $\mathrm{Symp}(X)/\mathrm{Ham}(X) \to \pi_0 \mathrm{Diff}(X)$. If $\dim_{\mathbb{C}} X$ is odd and $d \geq 3$, the image of τ_L has infinite order in $\pi_0 \mathrm{Diff}(X)$ by the Picard–Lefschetz formula, so Theorem 1.2 becomes trivial. However, when $\dim_{\mathbb{C}} X$ is even, the image of τ_L has finite order in $\pi_0 \mathrm{Diff}(X)$ (see Section 3.4 for details), so Theorem 1.2 is really of symplectic nature in this case. When X is Calabi–Yau ($d = n$), Theorem 1.2 follows from a grading argument of Paul Seidel [35]. Theorem 1.2 is new in all cases when $\dim_{\mathbb{C}} X$ is even and $d \neq n$. For instance, it appears to be new even for the cubic surface $X \subset \mathbb{P}^3$.

Let

$$\Delta \subset \mathbb{P}H^0(\mathrm{Gr}(k,n), \mathcal{O}(d))$$

be the discriminant variety parametrising all singular divisors in $|\mathcal{O}(d)|$. Theorem 1.2 implies a corollary about the fundamental group of the complement to the discriminant. Fix a divisor $X \in |\mathcal{O}(d)|$. For any family $X_t \subset \mathrm{Gr}(k,n)$ of smooth divisors in $|\mathcal{O}(d)|$, $t \in [0,1]$, there is a symplectic parallel transport map, a symplectomorphism $X_0 \to X_1$ which depends up to Hamiltonian isotopy only on the homotopy class of the path X_t relative to its endpoints. Applied to loops, parallel transport gives the symplectic monodromy map

$$\pi_1\big(\mathbb{P}H^0(\mathrm{Gr}(k,n), \mathcal{O}(d)) \setminus \Delta\big) \xrightarrow{\text{monodromy}} \mathrm{Symp}(X)/\mathrm{Ham}(X).$$

The discriminant complement contains a distinguished conjugacy class of loops γ called meridian loops. A meridian loop

$$\gamma \subset \mathbb{P}H^0(\mathrm{Gr}(k,n), \mathcal{O}(d)) \setminus \Delta$$

is the boundary of a 2–disk in $\mathbb{P}H^0(\mathrm{Gr}(n,k), \mathcal{O}(d))$ that intersects Δ transversely once. The image of such a loop under the monodromy map is the Dehn twist τ_L, where $L \subset X$ is an $|\mathcal{O}(d)|$–vanishing Lagrangian sphere. Theorem 1.2 implies the following.

Corollary 1.3 *If $3 \leq d \leq n$ or $d \geq k(n-k)+n-2$, and $\gamma \subset \mathbb{P}H^0(\mathrm{Gr}(k,n), \mathcal{O}(d)) \setminus \Delta$ is a meridian loop, then*

$$[\gamma] \in \pi_1\big(\mathbb{P}H^0(\mathrm{Gr}(k,n), \mathcal{O}(d)) \setminus \Delta\big)$$

is an element of infinite order.

Note that $[\gamma] \in H_1(\mathbb{P}H^0(\mathrm{Gr}(k,n), \mathcal{O}(d)) \setminus \Delta; \mathbb{Z})$ has finite order. For the projective space $\mathrm{Gr}(1,n) = \mathbb{P}^{n-1}$, the fundamental group $\pi_1(\mathbb{P}H^0(\mathbb{P}^{n-1}, \mathcal{O}(d)) \setminus \Delta)$ is computed by Lönne in [22] and implies Corollary 1.3 for $k = 1$. For $k \neq 1$, the corresponding fundamental group seems not to be studied, but Corollary 1.3 should allow a more straightforward proof, suggested to us by Dmitri Panov. Namely, assume $\dim_\mathbb{C} X$ is even (otherwise the corollary follows from the fact that the Dehn twist has infinite order topologically) and consider the $d : 1$ cover of $\mathrm{Gr}(k,n)$ branched along X, which now has odd complex dimension. A nodal degeneration of X provides an A_d–degeneration of the cover, and the monodromy around such a degeneration, which is a composition of Dehn twists around a chain of Lagrangian spheres, has infinite order in the smooth mapping class group (which uses the Picard–Lefschetz formula and the fact that the spheres are now odd-dimensional). This observation is enough to imply Corollary 1.3, bypassing the need to consider the Dehn twist in X itself. However, we decided to keep Corollary 1.3 to add an additional context to the main theorems.

We prove analogues of Theorem 1.2 and Corollary 1.3 for divisors in some very ample line bundles $\mathcal{L} \to Y$, where Y is a Kähler manifold which carries a holomorphic involution with certain properties. The precise statement is postponed to Section 1.7.

1.3 Elliptic relation for commuting symplectomorphisms

To prove Theorem 1.2, we use the elliptic relation (Theorem 1.1), which we now discuss.

Let X be a symplectic manifold satisfying the W^+ condition explained in Section 2; for example, X can be a Kähler manifold which is either Fano or whose canonical class K_X is sufficiently positive. Given a symplectomorphism $f : X \to X$, one defines its Floer cohomology $\mathrm{HF}^*(f)$. It is a $\mathbb{Z}_2$–graded vector space $\mathrm{HF}^*(f) = \mathrm{HF}^0(f) \oplus \mathrm{HF}^1(f)$ over the Novikov field

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i q^{\omega_i} : a_i \in \mathbb{C}, \, \omega_i \in \mathbb{R}, \, \lim_{i \to \infty} \omega_i = +\infty \right\}.$$

For any two commuting symplectomorphisms $f, g : X \to X$ we define invertible automorphisms

$$g_{\mathrm{floer}} : \mathrm{HF}^*(f) \to \mathrm{HF}^*(f) \quad \text{and} \quad f_{\mathrm{floer}} : \mathrm{HF}^*(g) \to \mathrm{HF}^*(g).$$

The construction of $\mathrm{HF}^*(f)$ uses a time-dependent almost complex structure J and a Hamiltonian H to define a vector space $\mathrm{HF}^*(f; J, H)$. This vector space is canonically

isomorphic (on the chain level) to $\mathrm{HF}^*(gfg^{-1}; g^*J, H \circ g)$ by composing all pseudo-holomorphic curves with g. If f, g commute, g_{floer} is the composition of isomorphisms

$$\mathrm{HF}^*(f; J, H) \to \mathrm{HF}^*(gfg^{-1}; g^*J, H \circ g) = \mathrm{HF}^*(f; g^*J, H \circ g) \to \mathrm{HF}^*(f; J, H),$$

where the last arrow is the continuation map associated to a homotopy of data from $(g^*J, H \circ g)$ to (J, H).

The automorphisms $f_{\mathrm{floer}}, g_{\mathrm{floer}}$ have degree zero, and one can define their supertrace

$$\mathrm{STr}(g_{\mathrm{floer}}) := \mathrm{Tr}(g_{\mathrm{floer}}|_{\mathrm{HF}^0(f)}) - \mathrm{Tr}(g_{\mathrm{floer}}|_{\mathrm{HF}^1(f)}) \in \Lambda.$$

Recall that Theorem 1.1 asserts that $\mathrm{STr}(f_{\mathrm{floer}}) = \mathrm{STr}(g_{\mathrm{floer}})$.

Now suppose a symplectomorphism f commutes with a finite-order symplectomorphism ϕ, $\phi^k = \mathrm{Id}$, with fixed locus X^ϕ. Then X^ϕ is a disjoint union of symplectic submanifolds. Using an argument reminiscent of the PSS isomorphism, we show that

$$\mathrm{STr}(f_{\mathrm{floer}}: \mathrm{HF}^*(\phi) \to \mathrm{HF}^*(\phi)) = L(f|_{X^\phi}) \cdot q^0.$$

The right-hand side is the topological Lefschetz number

$$L(f|_{X^\phi}) = \mathrm{Tr}(f^*|_{H^{\mathrm{even}}(X^\phi)}) - \mathrm{Tr}(f^*|_{H^{\mathrm{odd}}(X^\phi)}),$$

where $f^*: H^*(X^\phi) \to H^*(X^\phi)$ is the classical action on the cohomology of X^ϕ. On the other hand, using that ϕ has finite order, we show that

$$\mathrm{STr}(\phi_{\mathrm{floer}}: \mathrm{HF}^*(f) \to \mathrm{HF}^*(f)) = a \cdot q^0,$$

where $|a| \leq \dim_\Lambda \mathrm{HF}^*(f)$. Combining this with the elliptic relation, we obtain the following corollary.

Proposition 1.4 *Let X be a symplectic manifold satisfying the W^+ condition and $f, \phi: X \to X$ two commuting symplectomorphisms with $\phi^k = \mathrm{Id}$. Then*

$$\dim_\Lambda \mathrm{HF}^*(f) \geq |L(f|_{X^\phi})|.$$

Remark 1.5 The fixed locus X^ϕ is allowed to be disconnected, with components of different dimensions.

Remark 1.6 If $f: X \to X$ is a diffeomorphism with smooth fixed locus X^f, such that $\mathrm{Id} - df(x)|_{N_x \Sigma}$ is non-degenerate on the normal space $N_x \Sigma \subset T_x X$ to any connected component $\Sigma \subset X^f$ for every $x \in \Sigma$, then

$$L(f) = \sum_{\Sigma \subset X^f} \mathrm{sign}(\det(\mathrm{Id} - df|_{N_x \Sigma})) \cdot \chi(\Sigma).$$

Consequently, if $\phi, \psi \colon X \to X$ are finite-order symplectomorphisms, we get

$$L(\phi|_{X^\psi}) = L(\psi|_{X^\phi}) = \chi(X^\phi \cap X^\psi),$$

provided the latter intersection is clean. This agrees with the elliptic relation and the topological interpretation of the Floer-homological actions for finite-order maps.

Remark 1.7 It is possible to give a more straightforward proof of Proposition 1.4 which does not appeal to Theorem 1.1, but still requires some analysis in the spirit of Seidel [37, Lemma 14.11]. See Remark 2.23 for more details.

Remark 1.8 Theorem 1.1 holds when f, g commute only up to Hamiltonian isotopy, and more generally when fg^{-1} is isomorphic to Id in the Donaldson category, whose objects are symplectomorphisms of X and $\mathrm{Hom}(f, g) = \mathrm{HF}^*(fg^{-1})$; the proofs require only minor modifications. In Proposition 1.4, f, g can also be allowed to commute up to Hamiltonian isotopy.

1.4 Outline of proof of Theorem 1.1

The complete proof of Theorem 1.1 with all necessary definitions is found in Section 2; here we provide a sketch, illustrated by Figure 1, and indicate the main technical issue we have to solve.
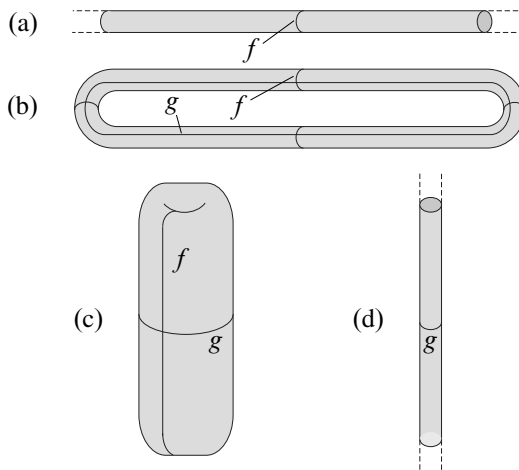

Figure 1: Changing the base of a symplectic fibration in the proof of Theorem 1.1

Let f, g be two commuting symplectomorphisms. By our definition, the supertrace $\mathrm{STr}(g_{\mathrm{floer}})$ is computed by counting certain solutions to Floer's continuation equation, or equivalently by counting holomorphic sections of a certain symplectic fibration

$E_f \to S^1 \times \mathbb{R}$; see Figure 1(a). This fibration has monodromy f along S^1, and the almost complex structure on E_f differs by the action of g over the two ends of the cylinder. We count only those sections whose asymptotics differ by the action of g over the ends of the cylinder. One can therefore glue the fibration, together with the almost complex structure, into a fibration $E_{f,g} \to S^1 \times S^1$. A gluing theorem in symplectic field theory (SFT) gives a bijection between holomorphic sections $S^1 \times \mathbb{R} \to E_f$ (with asymptotics as above) and all holomorphic sections $S^1 \times S^1 \to E_{f,g}$, where $S^1 \times S^1$ is endowed with the complex structure which is very "long" in the direction of the second S^1–factor; see Figure 1(b). We will refer to this bijection by $(*)$ in the next few paragraphs.

On the other hand, the count of holomorphic sections $S^1 \times S^1 \to E_{f,g}$ does not depend on the chosen complex structure on $S^1 \times S^1$. Take another complex structure on $S^1 \times S^1$ which is "long" in the first S^1–factor instead of the second one; see Figure 1(c). The same gluing argument as $(*)$ above implies that the count of holomorphic sections

$$S^1 \times S^1 \to E_{f,g}$$

is equal to the count of holomorphic sections

$$\mathbb{R} \times S^1 \to E_g$$

(with asymptotics differing by the action of f over the ends of the cylinder), where $E_g \to \mathbb{R} \times S^1$ is the fibration obtained by cutting $E_{f,g}$ along the first S^1–factor; see Figure 1(d). Similarly to what we began with, the latter count of holomorphic sections over $\mathbb{R} \times S^1$ gives $\mathrm{STr}(f_{\mathrm{floer}})$.

The key difficulty in upgrading this sketch to a proof is to determine how the bijection $(*)$ behaves with respect to the signs attached to sections over the cylinder (which in general depend on the choice of a "coherent orientation", but are canonical for sections contributing to the supertrace), and signs canonically attached to sections over the torus. The outcome is that $(*)$ multiplies signs by $(-1)^{\deg x}$, where x is a $\pm\infty$ asymptotic periodic orbit of the section over the cylinder. (The $\pm\infty$ asymptotics differ by g and thus have the same degree.) This is (2-27) in Section 2. It explains why Theorem 1.1 is an equality between supertraces and not usual traces. (We have not found (2-27) elsewhere in the literature. Coherent orientations in SFT are discussed in Eliashberg, Givental and Hofer [12] and Bourgeois and Mohnke [8] (see especially [8, Corollary 7]), but these works don't seem to give the result we need).

Remark 1.9 As the proof uses the torus with different complex structures (ie elliptic curves), this justifies the name "elliptic relation". There is some categorical perspective to the elliptic relation, as well: Ben-Zvi and Nadler [4, Theorem 1.2] obtained an

equality between the so-called "secondary traces" in a 2-category, which also comes from cutting the torus into pieces in two different ways (however, not into two different cylinders as we do). To give another informal analogy, the Lagrangian elliptic relation (which is the next thing we discuss) can be compared to an algebro-geometric result of similar flavor proved by Polishchuk [28]; it is tempting to suggest that there is a mirror symmetry connection between these two results.

1.5   Elliptic relation for invariant Lagrangians

Before explaining how the elliptic relation helps to prove Theorem 1.2, let us discuss its Lagrangian version. The coefficient field is still Λ. Definitions and sketch proofs are briefly presented in Section 2.13.

Let X be a connected monotone symplectic manifold (eg a complex Fano variety), and $L_1, L_2 \subset X$ monotone Lagrangians (eg simply connected). Suppose there is a symplectomorphism $\phi: X \to X$ such that $\phi(L_1) = L_1$, $\phi(L_2) = L_2$. Under a condition involving spin structures, formulated later as Hypothesis 2.24, a version of the open-closed string map provides twisted cohomology classes $[L_1]^\phi \in \mathrm{HF}^*(\phi)$, $[L_2]^{\phi^{-1}} \in \mathrm{HF}^*(\phi^{-1})$. Now consider the quantum product $[L_1]^\phi * [L_2]^{\phi^{-1}} \in \mathrm{QH}^*(X)$ and the map $\chi: \mathrm{QH}^*(X) \to \Lambda$ which is the integration over $[X]$ (sending the volume form to 1 and all elements of $H^{<2n}(X)$, seen as elements of $\mathrm{QH}^*(X)$, to 0). Under the assumptions of the next theorem, there is again an action $\phi_{\mathrm{floer}}: \mathrm{HF}^*(L_1, L_2) \to \mathrm{HF}^*(L_1, L_2)$, with Floer cohomology taken over Λ.

Theorem 1.10 (Elliptic relation) *Suppose (X, L_1, L_2) are monotone, $\phi: X \to X$ is a symplectomorphism, $\phi(L_i) = L_i$. If the base field has char $\neq 2$, suppose the L_i are orientable and Hypothesis 2.24 is satisfied (eg the L_i are simply connected). Then*

$$\mathrm{STr}(\phi_{\mathrm{floer}}) = \chi\big([L_1]^\phi * [L_2]^{\phi^{-1}}\big).$$

If $\phi^k = \mathrm{Id}$ and the fixed loci $L_i^\phi \subset X^\phi$ are smooth and orientable, the q^0–term of the right-hand side equals the classical homological intersection $[L_1^\phi] \cdot [L_2^\phi] \in \mathbb{Z}$ inside X^ϕ, where $[L_i^\phi] \in H_{\dim_{\mathbb{R}} X/2}(X; \mathbb{Z})$. On the other hand, eigenvalue decomposition of ϕ_{floer} implies that the left-hand side equals $a \cdot q^0$ with $a \in \mathbb{C}$, $|a| \leq \dim_\Lambda \mathrm{HF}^*(L_1, L_2)$. The elliptic relation yields the following analogue of Proposition 1.4.

Proposition 1.11 *Under the assumptions of Theorem 1.10, if $\phi^k = \mathrm{Id}$ and the fixed loci L_i^ϕ, X^ϕ are smooth and orientable then*

$$\dim_\Lambda \mathrm{HF}^*(L_1, L_2) \geq \big|[L_1^\phi] \cdot [L_2^\phi]\big|.$$

As our Lagrangians are monotone, we can pass from Λ–coefficients to the base field (eg $\mathbb{C}$ or $\mathbb{Z}/2\mathbb{Z}$) without changing the dimensions of Floer cohomology; see eg Wehrheim and Woodward [46, Remark 4.4]. So Proposition 1.11 gives the same bound on $\dim \mathrm{HF}^*(L_1, L_2; \mathbb{C})$ or $\dim \mathrm{HF}^*(L_1, L_2; \mathbb{Z}/2\mathbb{Z})$. However, the proof of Proposition 1.11 crucially uses Theorem 1.10 over Λ, as can be seen from the sketch we presented.

As a simple application of Proposition 1.11 we can recover the following known fact: $\mathbb{RP}^n \subset \mathbb{CP}^n$ is not self-displaceable by a Hamiltonian isotopy, as

$$\dim \mathrm{HF}^*(\mathbb{RP}^n, \mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \geq 1.$$

When n is even, this is true because the Euler characteristic of $\mathbb{RP}^n$ equals 1 over a characteristic–2 field. When n is odd, consider the hyperplane reflection ι on $\mathbb{CP}^n$ so that $(\mathbb{RP}^n)^\iota = \mathbb{RP}^{n-1}$ and apply Proposition 1.11.

In the appendix we provide a more interesting application of Proposition 1.11. Namely, we prove that for $L \subset X$ as in Theorem 1.2, and if X is in addition Fano and even-dimensional, there is an isomorphism of rings $\mathrm{HF}^*(L, L; \mathbb{C}) \cong \mathbb{C}[x]/x^2$. For Lagrangian spheres in the cubic surface, this was proved by Sheridan [41], and after the present paper had appeared, it was observed by Biran and Membrez [6, Section 1.3.2] that for a Lagrangian sphere in a projective hypersurface, which is Fano and of degree at least 3, the isomorphism $\mathrm{HF}^*(L, L; \mathbb{C}) \cong \mathbb{C}[x]/x^2$ follows from the known structure of $\mathrm{QH}^*(X)$, regardless of the complex dimension of X. Our method is completely different: it does not use any knowledge of $\mathrm{QH}^*(X)$, and works for hypersurfaces in Grassmannians as well as in some more abstract cases discussed in the appendix.

Remark 1.12 The action ϕ_{floer} on $\mathrm{HF}^*(L_1, L_2)$ (as well as the actions in the case of two commuting symplectomorphisms) can be defined using functors coming from Lagrangian correspondences; see Wehrheim and Woodward [44; 45]. It is possible that the two versions of the elliptic relation admit a generalisation for Lagrangian correspondences.

1.6 Outline of proof of Theorem 1.2

We have already mentioned that Theorem 1.2 holds for topological reasons when $\dim X$ is odd. Suppose therefore that $\dim_{\mathbb{C}} \mathrm{Gr}(k, n)$ is odd, so that $\dim_{\mathbb{C}} X$ is even. The Grassmannian has an involution ι whose fixed locus contains an even-dimensional connected component $\widetilde{\Sigma} \subset \mathrm{Gr}(k, n)$. For example, when $k = 1$ we can take the involution $(x_1 : x_2 : x_3 : x_4 : \ldots : x_n) \mapsto (-x_1 : -x_2 : -x_3 : x_4 : \ldots : x_n)$ and $\widetilde{\Sigma} = \mathbb{P}^2(x_1 : x_2 : x_3)$.

The key idea of reducing Theorem 1.2 to Proposition 1.4 is the following construction performed in Section 4. We construct a smooth divisor $X \subset \mathrm{Gr}(k,n)$ invariant under ι such that the fixed locus X^ι of the involution $\iota|_X$ contains an odd-dimensional connected component $\Sigma = \widetilde{\Sigma} \cap X$. Next, we construct two ι–invariant $|\mathcal{O}(d)|$–vanishing Lagrangian spheres $L_1, L_2 \subset X$ which intersect each other transversely once. Moreover, the fixed loci $L_i^\iota := L_i \cap \Sigma$, $i = 1, 2$, are Lagrangian spheres in Σ which intersect each other transversely once; see Figure 2. This is where we need $d \geq 3$.
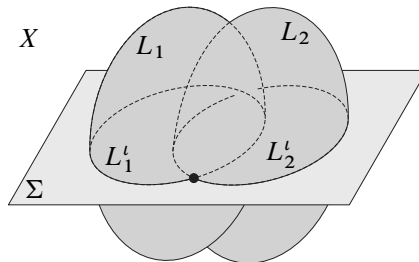

Figure 2: Invariant Lagrangian spheres L_1 and L_2 used in the proof of Theorem 1.2

Theorem 1.2 is proved in Section 5. Consider the product $\tau_{L_1}^{2k} \tau_{L_2}^{2k}$ of iterated Dehn twists. Because L_1, L_2 are ι–invariant, this product can be made ι–equivariant. The Lefschetz number of

$$(\tau_{L_1}^{2k} \tau_{L_2}^{2k})|_\Sigma = \tau_{L_1^\iota}^{2k} \tau_{L_2^\iota}^{2k}$$

is equal to $c - 4k^2$, where c is a constant. This follows from the Picard–Lefschetz formula and crucially uses the fact that $\dim \Sigma$ is odd. If $\dim \Sigma$ were even, the trace would be independent of k. Consequently, by Proposition 1.4, $\dim \mathrm{HF}^*(\tau_{L_1}^{2k} \tau_{L_2}^{2k})$ grows with k.

Finally, we note that L_1, L_2 from our construction can be taken one to another by a symplectomorphism of X. This means τ_{L_1} and τ_{L_2} are conjugate. If $\tau_{L_1}^{2k}$ was Hamiltonian isotopic to Id, then $\tau_{L_2}^{2k}$ and the product $\tau_{L_1}^{2k} \tau_{L_2}^{2k}$ would be as well. This contradicts the growth of Floer cohomology from above, and proves Theorem 1.2 for the specially constructed $|\mathcal{O}(d)|$–vanishing Lagrangian sphere $L_1 \subset X$. If X' is another smooth divisor linearly equivalent to X and $L' \subset X'$ is another $|\mathcal{O}(d)|$–vanishing Lagrangian sphere, Lemma 3.8 says there is a symplectomorphism $X \to X'$ taking L to L'. This implies Theorem 1.2 in general.

1.7 An extension of Theorem 1.2

Theorem 1.2 is a particular case of the more general, but also more technical theorem which we now state. Let $\mathcal{L}$ be a very ample line bundle over a Kähler manifold Y. It gives an embedding $Y \subset \mathbb{P}^N := \mathbb{P}H^0(Y, \mathcal{L})^*$.

Suppose $\iota\colon Y \to Y$ is a holomorphic involution which lifts to an automorphism of $\mathcal{L}$. The map ι induces a linear involution on $H^0(Y,\mathcal{L})^*$, splitting it into the direct sum of the ± 1–eigenspaces $H^0(Y,\mathcal{L})^*_\pm$. Let $\Pi_\pm \subset \mathbb{P}^N$ be the projectivisations of these eigenspaces. The fixed locus $Y^\iota \subset Y$ of the involution ι is

$$Y^\iota = (\Pi_+ \sqcup \Pi_-) \cap Y,$$

where the intersection is taken inside $\mathbb{P}^N$. It is automatically smooth, but can have many connected components because the intersections $\Pi_+ \cap Y$, $\Pi_- \cap Y$ may be disconnected.

Theorem 1.13 *With notation and assumptions as above, fix $d \geq 3$ and let $H^0(Y,\mathcal{L}^{\otimes d})_\pm$ denote the ± 1–eigenspace of the involution on $H^0(Y,\mathcal{L}^{\otimes d})$ induced by ι. Further, suppose one of the following holds:*

(a) *d is even, and Y^ι contains a connected component $\widetilde{\Sigma}$ such that $\dim_\mathbb{C} \widetilde{\Sigma}$ is even.*

(b) *d is odd, there is a smooth divisor in the linear system $\mathbb{P}H^0(Y,\mathcal{L}^{\otimes d})_+$, and $\Pi_+ \cap Y$ contains a connected component $\widetilde{\Sigma}$ such that $\dim_\mathbb{C} \widetilde{\Sigma}$ is even.*

Let $X \subset Y$ be a smooth divisor in the linear system $|\mathcal{L}^{\otimes d}|$ and $L \subset X$ an $|\mathcal{L}^{\otimes d}|$– vanishing Lagrangian sphere. Denote by τ_L the Dehn twist around L, and assume X satisfies the W^+ condition. Then the Hamiltonian isotopy class of τ_L is an element of infinite order in the group $\mathrm{Symp}(X)/\mathrm{Ham}(X)$. The same is true if we replace symbols $+$ with symbols $-$ in case (b).

Like Theorem 1.2, Theorem 1.13 is new when $\dim_\mathbb{C} X$ is even and X is not Calabi–Yau.

In case (a), the existence of a smooth ι–invariant divisor X follows from Bertini's theorem, so it is not included as a condition of the theorem. In case (b), an invariant divisor can sometimes be found using a strong Bertini theorem [10, Corollary 2.4], which gives the following.

Lemma 1.14 *Under the conditions of Theorem 1.13, let d be odd. There is a smooth divisor in the linear system $\mathbb{P}H^0(Y,\mathcal{L}^{\otimes d})_\pm$ if every connected component of $\Pi_\mp \cap Y$ has dimension less than $\frac{1}{2}\dim Y$.*

As in the beginning of the introduction, we have the following corollary.

Corollary 1.15 *Under the conditions of Theorem 1.13, let $\gamma \subset \mathbb{P}H^0(Y,\mathcal{L}^{\otimes d}) \setminus \Delta$ be a meridian loop, defined analogously to one in the paragraph before Corollary 1.3. Then*

$$[\gamma] \in \pi_1\big(\mathbb{P}H^0(Y,\mathcal{L}^{\otimes d}) \setminus \Delta\big)$$

is an element of infinite order.

We prove these statements in Section 5. We have earlier explained the plan of proof of Theorem 1.2; actually we follow this plan to prove the general Theorem 1.13 first, and then derive Theorem 1.2 from it.

1.8 Equivariant transversality approaches

This subsection is not used in the rest of the paper. Computations of Floer cohomology in the presence of a symplectic involution were discussed by Khovanov and Seidel [20] and Seidel and Smith [40]. Both papers imposed restrictive conditions on the involution which allow one to choose a regular equivariant almost complex structure for computing Floer cohomology.

In [40], it is proved that

$$\dim \mathrm{HF}^*(L_1, L_2; \mathbb{Z}/2) \geq \dim \mathrm{HF}^*(L_1^\iota, L_2^\iota; \mathbb{Z}/2)$$

when there exists a stable trivialisation of the normal bundle to X^ι respecting the L_i. In particular, the Chern classes of this normal bundle should vanish. The right-hand side is Floer cohomology inside X^ι, where L_i^ι are the fixed loci of L_i and X^ι is the fixed locus of X. Sometimes the right-hand side is easier to compute than the left-hand side (eg when all intersection points $L_1^\iota \cap L_2^\iota$ have the same sign). However, the condition on the normal bundle makes this estimate inapplicable to divisors in $\mathrm{Gr}(k, n)$.

In a very special case, [20] proves that

$$\dim \mathrm{HF}^*(L_1, L_2; \mathbb{Z}/2) = |L_1^\iota \cap L_2^\iota|,$$

where the right-hand side is the unsigned count of intersection points. The assumption is, roughly, that the fixed locus X^ι has real dimension 2 and $L_1^\iota, L_2^\iota \subset X^\iota$ are curves having minimal intersection in their homotopy class. One could prove a $\mathbb{C}$–version of this equality if the L_i admit ι–equivariant Pin strictures, and apply it to divisors in $\mathbb{P}^{n-1} = \mathrm{Gr}(1, n)$, ie projective hypersurfaces (thus giving an alternative proof of Theorem 1.2 in this case). However, it cannot be applied to divisors in general Grassmannians. When $k > 2$, $\mathrm{Gr}(k, n)$ has no holomorphic involution with a connected component of complex dimension 2; this is easy to check because all holomorphic automorphisms $\mathrm{Gr}(k, n)$ come from linear ones on $\mathbb{C}^n$, with a single exception when $n = 2k$; see Cowen [9, Theorem 1.1 (Chow)].

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2 The elliptic relation

This section proves the elliptic relation for symplectomorphisms (together with its corollary, Proposition 1.4) and sketches a proof of the Lagrangian elliptic relation.

2.1 Floer cohomology and continuation maps

Definition 2.1 (The W^+ condition) A symplectic manifold (X, ω) of dimension $2n$ satisfies the W^+ condition [34] if, for every $A \in \pi_2(X)$,

$$2 - n \le c_1(A) \le -1 \quad \Longrightarrow \quad \omega(A) \le 0.$$

Let (X, ω) be a compact symplectic manifold satisfying the W^+ condition. Fix a symplectomorphism $f: X \to X$. In this subsection we recall the definition of Floer cohomology $\mathrm{HF}^*(f)$; basic references are [24; 11; 34].

Take a family of ω–tame almost complex structures J_s on X, and a family of Hamiltonian functions $H_s: X \to \mathbb{R}$, $s \in \mathbb{R}$. They must be f–periodic:

$$(2\text{-}1) \qquad\qquad H_s = H_{s+1} \circ f, \quad J_s = f^* J_{s+1}.$$

By X_{H_s} we denote the Hamiltonian vector field of H_s, and by $\psi_s: X \to X$ the Hamiltonian flow:

$$(2\text{-}2) \qquad\qquad d\psi_s/ds = X_{H_s} \circ \psi_s, \quad \psi_0 = \mathrm{Id}.$$

The following equation on $u(s,t): \mathbb{R}^2 \to X$ is called Floer's equation:

$$(2\text{-}3) \qquad\qquad \partial u/\partial t + J_s(u)(\partial u/\partial s - X_{H_s}(u)) = 0.$$

This equation comes with the periodicity conditions

$$(2\text{-}4) \qquad\qquad u(s+1, t) = f(u(s,t)).$$

Denote

$$(2\text{-}5) \qquad\qquad f_H := \psi_1^{-1} \circ f \in \mathrm{Symp}(X).$$

(The correct notation would be f_{H_s}, but we stick to f_H for brevity). Suppose the fixed points of f_H are isolated and non-degenerate (that is to say, for every $x \in \mathrm{Fix}\, f_H$, $\ker(\mathrm{Id} - df_H(x)) = 0$). Then finite-energy solutions to Floer's equation have the following convergence property. There exist points x, y such that

$$(2\text{-}6) \qquad \lim_{t \to -\infty} u(s, t) = \psi_s(x), \quad \lim_{t \to +\infty} u(s, t) = \psi_s(y), \quad x, y \in \mathrm{Fix}\, f_H.$$

For $x, y \in \mathrm{Fix}\, f_H$, let $\mathcal{M}(x, y; J_s, H_s)$ be the moduli space of all solutions to Floer's equation (2-3) with limits (2-6). For regular J_s, H_s, the moduli space is a manifold which is a disjoint union of the k–dimensional pieces $\mathcal{M}^k(x, y; J_s, H_s)$. They can be oriented in a way consistent with gluings; such orientations are called coherent [13]. There is an $\mathbb{R}$–action on $\mathcal{M}(x, y; J_s, H_s)$, and once a coherent orientation is fixed, $\mathcal{M}^1(x, y; J_s, H_s)/\mathbb{R}$ is a set of signed points.

The Floer complex associated to $(f; J_s, H_s)$ is the Λ–vector space generated by points in $\mathrm{Fix}\, f_H$:

$$\mathrm{CF}^*(f; J_s, H_s) := \bigoplus_{x \in \mathrm{Fix}\, f_H} \Lambda\langle x \rangle.$$

The differential on $\mathrm{CF}^*(f; J_s, H_s)$ is defined on a generator $x \in \mathrm{Fix}\, f_H$ by

$$(2\text{-}7) \qquad \partial(x) = \sum_{\substack{y \in \mathrm{Fix}\, f_H \\ u \in \mathcal{M}^1(x, y; J_s, H_s)/\mathbb{R}}} \pm q^{\omega(u)} \cdot y.$$

Here the signs are those of the points in $\mathcal{M}^1(x, y; J_s, H_s)/\mathbb{R}$, and

$$(2\text{-}8) \qquad \omega(u) = \int_{s \in [0,1]} \int_{t \in \mathbb{R}} u^* \omega \, ds \, dt.$$

Suppose J_s, H_s and J_s', H_s' are two regular choices of almost complex structures and Hamiltonians that satisfy the f–periodicity condition (2-1). Choose a family of ω–tame complex structures $J_{s,t}$ and Hamiltonians $H_{s,t}$, $s, t \in \mathbb{R}$, such that for each t the condition (2-1) is satisfied and

$$(2\text{-}9) \qquad \begin{aligned} J_{s,t} &\equiv J_s', \ H_{s,t} \equiv H_s' \quad \text{for } t \text{ near } -\infty, \\ J_{s,t} &\equiv J_s, \ H_{s,t} \equiv H_s \quad \text{for } t \text{ near } +\infty. \end{aligned}$$

We call $J_{s,t}, H_{s,t}$ a homotopy from J_s', H_s' to J_s, H_s. Define $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ to be the set of solutions to Floer's continuation equation

$$(2\text{-}10) \qquad \partial u/\partial t + J_{s,t}(u)(\partial u/\partial s - X_{H_{s,t}}(u)) = 0$$

with periodicity condition (2-4) and asymptotic conditions

$$(2\text{-}11) \quad \lim_{t \to -\infty} u(s,t) = \psi_s(x), \quad \lim_{t \to +\infty} u(s,t) = \psi_s(y), \quad x \in \mathrm{Fix}\, f_{H'}, \ y \in \mathrm{Fix}\, f_H.$$

If $J_{s,t}, H_{s,t}$ are regular, $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ is a manifold. Let $\mathcal{M}^0(x, y; J_{s,t}, H_{s,t})$ be its 0–dimensional component, which is a collection of signed points once coherent orientations (consistent with those for J_s, H_s and J'_s, H'_s) are fixed. Define the continuation map $C_{J_{s,t}, H_{s,t}} \colon \mathrm{CF}^*(f; J'_s, H'_s) \to \mathrm{CF}^*(f; J_s, H_s)$ by

$$(2\text{-}12) \qquad\qquad C_{J_{s,t}, H_{s,t}}(x) = \sum_{\substack{y \in \mathrm{Fix}\, f_H \\ u \in \mathcal{M}^0(x, y; J_{s,t}, H_{s,t})}} \pm q^{\omega(u)} \cdot y.$$

Here $x \in \mathrm{Fix}\, f_{H'}$. For regular $J_{s,t}, H_{s,t}$, it is a chain map inducing an isomorphism on cohomology. So one can actually identify the homologies $\mathrm{HF}^*(f; J_s, H_s)$ for all generic J_s, H_s to get a single space $\mathrm{HF}^*(f)$. It is called Floer cohomology of f. It is a $\mathbb{Z}/2$–graded vector space over Λ; we will recall the grading later.

2.2 Commuting symplectomorphisms induce actions on Floer cohomology

As before, let X be a compact symplectic manifold satisfying the W^+ condition. Let $f, g \colon X \to X$ be two commuting symplectomorphisms; we will now define an automorphism $g_{\mathrm{floer}} \colon \mathrm{HF}^*(f) \to \mathrm{HF}^*(f)$.

Pick generic J_s, H_s that satisfy (2-1) to get the complex $\mathrm{CF}^*(f; J_s, H_s)$. Denote

$$(2\text{-}13) \qquad\qquad J'_s := g^* J_s, \quad H'_s := H_s \circ g.$$

This gives us another complex $\mathrm{CF}^*(f; J'_s, H'_s)$. Note that $g \circ \psi_1 = \psi'_1$. Let us check that $f_H = f_{H'} \circ g$:

$$f_{H'} \circ g(x) = (\psi'_1)^{-1} fg(x) = (\psi'_1)^{-1} gf(x) = \psi_1^{-1} f(x) = f_H(x).$$

Consequently, g induces a bijection $\mathrm{Fix}\, f_H \to \mathrm{Fix}\, f_{H'}$. Extend it by Λ–linearity to

$$g_{\mathrm{push}} \colon \mathrm{CF}^*(f; J_s, H_s) \to \mathrm{CF}^*(f; J'_s, H'_s).$$

Similarly, the composition map $u \mapsto g \circ u$ is an isomorphism

$$\mathcal{M}(x, y; J_s, H_s) \xrightarrow{\cong} \mathcal{M}(g(x), g(y); J'_s, H'_s).$$

So g_{push} is tautologically a chain map inducing an isomorphism on cohomology. Now fix a homotopy $J_{s,t}, H_{s,t}$ from J'_s, H'_s to J_s, H_s as in (2-9). Consider the composition

$$\mathrm{CF}^*(f; J_s, H_s) \xrightarrow{g_{\mathrm{push}}} \mathrm{CF}^*(f; J'_s, H'_s) \xrightarrow{C_{J_{s,t}, H_{s,t}}} \mathrm{CF}^*(f; J_s, H_s).$$

Definition 2.2 (Action on Floer cohomology)   We define

$$g_{\text{floer}}\colon \operatorname{HF}^*(f; J_{s,t}, H_{s,t}) \to \operatorname{HF}^*(f; J_{s,t}, H_{s,t})$$

to be the map induced by the composition of chain maps $C_{J_{s,t}, H_{s,t}} \circ g_{\text{push}}$. We will frequently suppress the choice of J_s, H_s and simply write $g_{\text{floer}}\colon \operatorname{HF}^*(f) \to \operatorname{HF}^*(f)$. Also, we will sometimes denote the chain-level map by the same symbol, $g_{\text{floer}} = C_{J_{s,t}, H_{s,t}} \circ g_{\text{push}}$.

As a part of this definition, the signs in formula (2-12) for $C_{J_{s,t}, H_{s,t}}$ must come from a coherent orientation, as explained in Section 2.8 below. In particular, for any $x \in \operatorname{Fix} f_H$, the sign of an element $u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$ is canonical — see Definition 2.16 — and denoted by $\operatorname{sign}(u)$.

Remark 2.3   On the level of cohomology, g_{floer} does not depend on the chosen homotopy $J_{s,t}, H_{s,t}$; this follows from the fact that the continuation map $C_{J_{s,t}, H_{s,t}}$ does not depend on the choice of homotopy; see eg [24, Section 12.1].

Remark 2.4 (An analogue in Morse cohomology)   A similar construction is known in Morse cohomology [31, Section 4.2.2]. Suppose $H\colon X \to \mathbb{R}$ is a Morse–Smale function on a Riemannian manifold (X, g), and $f\colon X \to X$ is a diffeomorphism. Let $C^*(H)$ be the Morse complex of X generated by points in $\operatorname{Crit}(H)$. Pick homotopies H_t from $H \circ f$ to H, and g_t from f^*g to g, and define $f^*\colon C^*(H) \to C^*(H)$ as follows. Take $x, y \in \operatorname{Crit}(H)$ and let the coefficient of $f^*(x)$ on y be the signed count of flowlines of the gradient $\nabla_{g_t} H_t$ going from $f(x)$ to y. The chain map f^* induces an automorphism of $H^*(X)$ known from elementary topology.

In particular, let us note for future use that the Lefschetz number $L(f)$ can be computed as the sum over $x \in \operatorname{Crit}(H)$ of $\nabla_{g_t} H_t$–flowlines going from $f(x)$ to x, counted with signs.

Remark 2.5 (Relation to Seidel elements)   If g is Hamiltonian isotopic to f through symplectomorphisms commuting with f, then one can show $g_{\text{floer}}\colon \operatorname{HF}^*(f) \to \operatorname{HF}^*(f)$ is the identity. If g is just Hamiltonian isotopic to f, g_{floer} need not be the identity, but can be understood as follows. Take a homotopy g_t, $g_0 = g$, $g_1 = f$. The path $\gamma_t := g_t^{-1} f g_t$ is actually a loop in $\operatorname{Symp}(X)$: $\gamma(0) = \gamma(1) = f$ because $g^{-1} f g = f$. To this path one associates its Seidel element $S(\gamma) \in \operatorname{QH}^*(M; \Lambda)$ [34]. Let $*$ be the quantum multiplication $\operatorname{QH}^*(M; \Lambda) \otimes \operatorname{HF}^*(f) \to \operatorname{HF}^*(f)$. One can check that $g_{\text{floer}}(x) = S(\gamma) * x$ for any $x \in \operatorname{HF}^*(f)$. We will not use this observation, so we omit its proof.

2.3 Iterations

If f, g commute then f, g^k also commute for any iteration g^k.

Lemma 2.6 *The following two automorphisms of* $\mathrm{HF}^*(f)$ *are equal:*

$$(g_{\mathrm{floer}})^k = (g^k)_{\mathrm{floer}}.$$

Proof We prove the case $k = 2$; the general case is analogous. Take J_s, H_s as in (2-1), J'_s, H'_s pulled by g as in (2-13) and the homotopy $J_{s,t}, H_{s,t}$ as in (2-9). Denote

$$J''_s = g^* J'_s = (g^2)^* J_s, \quad H''_s = H'_s \circ g = H_s \circ g^2.$$

Compare the two compositions given below. The first one induces $(g_{\mathrm{floer}})^2$ on the homological level:

$$\mathrm{CF}^*(f; J_s, H_s) \xrightarrow{\;g_{\mathrm{push}}\;} \mathrm{CF}^*(f; J'_s, H'_s)$$
$$\xrightarrow{\;C_{J_{s,t}, H_{s,t}}\;} \mathrm{CF}^*(f; J_s, H_s)$$
$$\xrightarrow{\;g_{\mathrm{push}}\;} \mathrm{CF}^*(f; J'_s, H'_s) \xrightarrow{\;C_{J_{s,t}, H_{s,t}}\;} \mathrm{CF}^*(f; J_s, H_s).$$

The second composition gives $(g^2)_{\mathrm{floer}}$, by a gluing theorem for continuation maps:

$$\mathrm{CF}^*(f; J_s, H_s) \xrightarrow{\;g_{\mathrm{push}}\;} \mathrm{CF}^*(f; J'_s, H'_s)$$
$$\xrightarrow{\;g_{\mathrm{push}}\;} \mathrm{CF}^*(f; J''_s, H''_s)$$
$$\xrightarrow{\;C_{J'_{s,t}, H'_{s,t}}\;} \mathrm{CF}^*(f; J'_s, H'_s) \xrightarrow{\;C_{J_{s,t}, H_{s,t}}\;} \mathrm{CF}^*(f; J_s, H_s).$$

By definition of $J'_{s,t}, H'_{s,t}$ (2-13), g maps Floer solutions (2-10) in $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ to those in $\mathcal{M}(g(x), g(y); J'_{s,t}, H'_{s,t})$. This means

$$C_{J'_{s,t}, H'_{s,t}} \circ g_{\mathrm{push}} = g_{\mathrm{push}} \circ C_{J_{s,t}, H_{s,t}}.$$

This proves Lemma 2.6. □

2.4 Supertrace

We continue to use notation from Section 2.1.

Definition 2.7 (Grading on Floer's complex) Let $x \in \mathrm{Fix}\, f_H$. We say $\deg x = 0$ if $\mathrm{sign}\det(\mathrm{Id} - df_H(x)) > 0$ and $\deg x = 1$ otherwise.

This makes $\mathrm{CF}^*(f; J_s, H_s)$ a $\mathbb{Z}_2$–graded vector space over Λ. Floer's differential has degree 1, so the cohomology is also $\mathbb{Z}_2$–graded: $\mathrm{HF}^*(f) = \mathrm{HF}^0(f) \oplus \mathrm{HF}^1(f)$.

Definition 2.8 (Supertrace) Let $V = V^0 \oplus V^1$ be a $\mathbb{Z}_2$–graded vector space and $\phi \colon V \to V$ an automorphism of degree zero, ie $\phi(V^0) \subset V^0$, $\phi(V^1) \subset V^1$. Then $\mathrm{STr}(\phi) := \mathrm{Tr}(\phi|_{V^0}) - \mathrm{Tr}(\phi|_{V^1})$.

The automorphism g_{floer} from Definition 2.2 has degree zero, so it has a well-defined supertrace which is an element of Λ. Supertraces can be computed on the chain level, since all our chain complexes are finite-dimensional. Therefore the following is just a restatement of definitions.

Lemma 2.9 *Let X be a symplectic manifold satisfying the W^+ condition and $f, g \colon X \to X$ be two commuting symplectomorphisms. Take J'_s, H'_s as in (2-13) and a homotopy $J_{s,t}, H_{s,t}$ from J'_s, H'_s to J_s, H_s as in (2-9). Then*

$$\mathrm{STr}\big(g_{\mathrm{floer}} \colon \mathrm{HF}^*(f) \to \mathrm{HF}^*(f)\big) = \sum_{\substack{x \in \mathrm{Fix}\, f_H \\ u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})}} (-1)^{\deg x} \cdot \mathrm{sign}(u) \cdot q^{\omega(u)},$$

where $\mathrm{sign}(u) = \pm 1$ is as in Definition 2.16.

Proof Pick a generator $x \in \mathrm{Fix}\, f_H$ of $\mathrm{CF}^*(f; J_s, H_s)$. Rewriting the definition of g_{floer}, we get

$$g_{\mathrm{floer}}(x) = \sum_{u \in \mathcal{M}^0(g(x), y; J_{s,t}, H_{s,t})} \pm q^{\omega(u)} \cdot y.$$

When we put $x = y$, the sign $\pm$ becomes $\mathrm{sign}(u)$ according to Definition 2.2. $\qquad\square$

2.5 Holomorphic sections

It is useful to reformulate the definition of Floer cohomology using holomorphic sections as in eg [38]. If $f \colon X \to X$ is a symplectomorphism, consider the mapping cylinder

$$(2\text{-}14) \qquad\qquad E_f := \frac{X \times \mathbb{R}^2_{s,t}}{(x, s, t) \sim (f(x), s + 1, t)}.$$

There is a closed 2–form ω_{E_f} on E_f which comes from $\omega \oplus 0$ on $X \times \mathbb{R}^2$, and a natural fibration $p \colon E_f \to S^1 \times \mathbb{R}$ whose fibres are symplectomorphic to X.

The f–periodicity condition (2-4) on $u \colon \mathbb{R}^2 \to X$ means that it can be seen as a section $u \colon S^1 \times \mathbb{R} \to E_f$. Floer's equation itself (2-3) is equivalent to u being a holomorphic

section with respect to the standard complex structure $j_{S^1 \times \mathbb{R}}$ on $S^1 \times \mathbb{R}$ and an almost complex structure $\tilde{J}$ on E_f. In other words, Floer's equation (2-3) becomes

$$(2\text{-}15) \qquad du + \tilde{J} \circ du \circ j_{S^1 \times \mathbb{R}} = 0.$$

The almost complex structure $\tilde{J} := \tilde{J}(J_s, H_s)$ is determined by J_s and H_s; see eg [24, Section 8.1]. Analogously, if $J_{t,s}, H_{t,s}$ is a continuation homotopy (2-9), the moduli space $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ consists of sections $u \colon S^1 \times \mathbb{R} \to E_f$ that are holomorphic with respect to $j_{S^1 \times \mathbb{R}}$ and an almost complex structure $\tilde{J}(J_{s,t}, H_{s,t})$ on E_f.

2.6 Asymptotic linearised Floer's equation

Let E_f be as in (2-14). We denote by

$$T^v E_f = \ker dp$$

the vertical tangent bundle of E_f. The almost complex structures J_s turn $T^v E_f$ into a complex vector bundle. Take a solution $u(s, t)$ to Floer's equation, $u \in \mathcal{M}(x, y; J_s, H_s)$. We regard it as a section $u(s, t) \colon S^1 \times \mathbb{R} \to E_f$ as explained above. The pullback $u^* T^v E_f$ is a complex vector bundle over $S^1 \times \mathbb{R}$. By linearising Floer's equation (2-15), one gets a map

$$(2\text{-}16) \qquad D_u \colon H^{1,p}(u^* T^v E_f) \to L^p(\Omega^{0,1}(u^* T^v E_f)).$$

Here $\Omega^{0,1}(u^* T^v E_f)$ consists of bundle maps $T(S^1 \times \mathbb{R}) \to u^* T^v E_f$ which are complex-antilinear with respect to $\tilde{J}$ and the standard complex structure on $S^1 \times \mathbb{R}$.

We know from (2-6) that u extends to $S^1 \times \{\pm\infty\}$: $u(s, -\infty) = \psi_s(x)$, where ψ_s is the flow (2-2) of X_{H_s}. (The same is true of $t \to +\infty$ and the point y. We will now speak of $t \to -\infty$ only.) Choose a complex trivialisation

$$(2\text{-}17) \qquad \Phi_x \colon u^* T^v E_f|_{S^1 \times \{-\infty\}} \to S^1 \times \mathbb{R}^{2n}.$$

We choose a single trivialisation for each point x; this is possible because $u(s, -\infty) = \psi_s(x)$. The operator D_u is asymptotic, as $t \to -\infty$, to the operator

$$(2\text{-}18) \quad L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s) \colon H^{1,p}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \to L^p(S^1 \times \mathbb{R}, \mathbb{R}^{2n}).$$

Here J_0 is the standard complex structure on $\mathbb{R}^{2n}$, and $A(s) \colon S^1 \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is a map taking values in symmetric matrices. It is known that $A(s)$ is determined by J_s, H_s, the point x and the chosen trivialisation $\Phi(x)$. It does not depend on u as long as the $t \to -\infty$ asymptotic of u stays fixed. A reference for these facts is (among others) the thesis of Schwarz [32, Definition 3.1.6, Theorem 3.1.31]. Although that thesis only considers the case $f = \mathrm{Id}$, the proofs of the results we use are valid for

any f, as these are general results about certain Fredholm operators on bundles over S^1 and $S^1 \times \mathbb{R}$.

Lemma 2.10 [32, proof of Lemma 3.1.33]   *Consider the operator*

$$J_0 \partial/\partial s + A(s) : C^\infty(S^1, \mathbb{R}^{2n}) \to C^\infty(S^1, \mathbb{R}^{2n}).$$

There is a family of linear maps $\Psi(s) : [0, 1] \to \mathrm{Sp}(\mathbb{R}^{2n})$ *such that*

(2-19) $$(J_0 \partial/\partial s + A(s))\Psi(s) = 0, \quad \Psi(0) = \mathrm{Id}$$

and $\Psi(1) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *coincides, under the trivialisation* Φ_x *(2-17), with the differential* $df_H(x)$. □

Remark 2.11   We identify $S^1 = \mathbb{R}/\mathbb{Z}$, so the points $s = 0$ and $s = 1$ of the circle are the same. The statement about $\Psi(1)$ in the lemma above makes sense because $u(0, -\infty) = x$ for some $x \in \mathrm{Fix}\, f_H$; see (2-6) and (2-2). So $df_H(x)$ acts on $T_x X = u^* T^v E_f|_{(0,-\infty)}$. The trivialisation (2-17) identifies this space with $\mathbb{R}^{2n}$.

Remark 2.12   Given $\Psi(s) : [0, 1] \to \mathrm{Sp}(\mathbb{R}^{2n})$, by solving (2-19) we get

(2-20) $$A(s) = -J_0(\partial/\partial s \Psi(s))\Psi(s)^{-1},$$

with symmetric $A(s) : [0, 1] \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. Conversely, we can go from $A(s)$ to $\Psi(s)$ by solving (2-19) as an ODE.

Remark 2.13   For the reader's convenience, we include a correspondence between our notation and that of Schwarz [32], our notation being on the left in each pair: $s \leftrightarrow t$, $t \leftrightarrow s$, $A(s) \leftrightarrow S^\infty(t)$, $\Psi(s) \leftrightarrow \Psi(t)$, and D_u in our notation corresponds to either D_u or DF_h, the latter being the linearisation of Floer's equation at an h which is not necessarily a solution. Equation (2-19) is [32, (3.23)].

2.7   An index problem on the torus

The operator $L_{A(s)}$ (2-18) is Fredholm if and only if $\det(\mathrm{Id} - \Psi(1)) = \det(\mathrm{Id} - df_H(x))$ is non-zero. Now, for later use, consider variables (s, t) belonging to the torus $S^1 \times S^1$ instead of the cylinder $S^1 \times \mathbb{R}$. The same formula (2-18) gives the operator

$$L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s) : C^\infty(S^1 \times S^1, \mathbb{R}^{2n}) \to C^\infty(S^1 \times S^1, \mathbb{R}^{2n}),$$

which is now Fredholm of index zero for any family of symmetric matrices $A(s) : S^1 \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. For the remainder of this subsection, $L_{A(s)}$ denotes the operator on $S^1 \times S^1$ and not on the cylinder.

Lemma 2.14 *Let* $A(s)\colon S^1 \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ *be a family of symmetric matrices. Suppose* $A(s)$ *and* $\Psi(s)$ *satisfy* (2-19). *Then* $\dim \ker L_{A(s)} = \dim \ker(\mathrm{Id} - \Psi(1))$.

Proof Any $\xi(s, t) \in \ker L_{A(s)}$ must be independent of t (for details see [32, proof of Lemma 3.1.33]) so we write $\xi(s, t) \equiv \xi(s)$. The equation on $\xi(s)$ becomes

$$(J_0 \partial/\partial s + A(s))\xi(s) = 0.$$

This is an ODE whose solutions are of the form $\xi(s) = \Psi(s)v$ for some $v \in \mathbb{R}^{2n}$ by (2-19). There are no other solutions by the uniqueness theorem for ODEs, as the $v \in \mathbb{R}^{2n}$ sweep out all initial conditions. In addition, our solutions must close up on the circle, meaning $\xi(1) = \xi(0)$, which forces $\Psi(1)v = v$. $\qquad\square$

Let $A_0(s), A_1(s)\colon S^1 \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ be two families of symmetric matrices with $L_{A_0(s)}, L_{A_1(s)}$ injective. Choose a generic smooth homotopy $A_\tau(s)$ between them, $\tau \in [0, 1]$. Define $\mathrm{sign}(L_{A_0(s)}, L_{A_1(s)}) = (-1)^\epsilon$ where

$$(2\text{-}21) \qquad\qquad \epsilon = \sum_{\tau \in [0,1]} \dim_\mathbb{R} \ker L_{A_\tau(s)}.$$

This sum contains a finite number of non-zero terms as $L_{A_\tau(s)}$ are generically injective, and does not depend modulo 2 on the chosen homotopy.

Lemma 2.15 *For* $A_0(s), A_1(s)$ *as above and* $\Psi_0(s), \Psi_1(s)$ *satisfying* (2-19), *we have*

$$\mathrm{sign}(L_{A_0(s)}, L_{A_1(s)}) = \mathrm{sign}\det(\mathrm{Id} - \Psi_0(1)) \cdot \mathrm{sign}\det(\mathrm{Id} - \Psi_1(1)).$$

Proof For $i = 0, 1$ denote $\widetilde{\Psi}_i(s) = e^{s \log \Psi_i(1)}$, $s \in [0, 1]$, so that $\widetilde{\Psi}_i(0) = \Psi_i(0) = \mathrm{Id}$ and $\widetilde{\Psi}_i(1) = \Psi_i(1)$. Let us compute $\widetilde{A}_i(s)$ from $\widetilde{\Psi}_i(s)$ using (2-20):

$$\widetilde{A}_i(s) = -J_0(\partial/\partial s \, \widetilde{\Psi}_i(s))\widetilde{\Psi}_i(s)^{-1} = -J_0 \log \Psi_i(1).$$

We see that it is a constant s–independent symmetric matrix $\widetilde{A}_i(s) \equiv \widetilde{A}_i$. Our first claim is that

$$(2\text{-}22) \qquad\qquad \mathrm{sign}(L_{A_i(s)}, L_{\widetilde{A}_i}) = +1.$$

Indeed, choose the homotopy

$$(\Psi_i)_\tau(s) = e^{\tau s \log \Psi_i(1)} e^{(1-\tau) \log \Psi_i(s)}$$

from $\Psi_i(s)$ to $\widetilde{\Psi}_i(s)$, where $\tau \in [0, 1]$, and observe that this homotopy has fixed endpoints: we have $(\Psi_i)_\tau(0) = \Psi_i(0)$ for each τ, and also $(\Psi_i)_\tau(1) = \Psi_i(1)$. Passing from $(\Psi_i)_\tau(s)$ to $(A_i)_\tau(s)$ by formula (2-19) we get the linear homotopy

$$(A_i)_\tau(s) = \tau A_i(s) + (1 - \tau)\widetilde{A}_i$$

from $A_i(s)$ to $\widetilde{A}_i$. The corresponding operator $L_{(A_i)_\tau(s)}$ is injective for all τ by Lemma 2.14 because we are given $\ker(\mathrm{Id} - \Psi_i(1)) = 0$. This implies that

$$\mathrm{sign}(L_{A_i(s)}, L_{\widetilde{A}_i}) = +1,$$

as desired.

Let us compute $\mathrm{sign}(L_{\widetilde{A}_0}, L_{\widetilde{A}_1})$ for two constant matrices $\widetilde{A}_i(s) \equiv \widetilde{A}_i$, $i = 0, 1$. By linear algebra, one can find a smooth path of matrices $\widetilde{A}_\tau$ from $\widetilde{A}_0$ to $\widetilde{A}_1$ such that

$$(-1)^{\sum_\tau \dim \ker \widetilde{A}_\tau} = \mathrm{sign} \det \widetilde{A}_0 \cdot \mathrm{sign} \det \widetilde{A}_1.$$

We will now show that $\dim \ker \widetilde{A}_\tau = \dim \ker L_{\widetilde{A}_\tau}$ for each τ, and this will immediately imply that

(2-23) $$\mathrm{sign}(L_{\widetilde{A}_0}, L_{\widetilde{A}_1}) = \mathrm{sign} \det \widetilde{A}_0 \cdot \mathrm{sign} \det \widetilde{A}_1.$$

For the rest of the paragraph, redenote $\widetilde{A}_\tau$ (for some fixed τ) by A: this is an arbitrary symmetric matrix, and if we consider it as an s–independent family and solve (2-20) with respect to Ψ, we get $\Psi(1) = e^{-J_0 A}$. By Lemma 2.14, we have

$$\ker L_A = \ker(\mathrm{Id} - \Psi(1)) = \ker(\mathrm{Id} - e^{-J_0 A}).$$

The latter equals $\ker A$, as seen by bringing A to the Jordan normal form. So (2-23) is now justified. Combining all the equations above, we get

$$\begin{aligned}
\mathrm{sign}(L_{A_0(s)}, L_{A_1(s)}) &= \mathrm{sign}(L_{A_0(s)}, L_{\widetilde{A}_0}) \cdot \mathrm{sign}(L_{\widetilde{A}_0}, L_{\widetilde{A}_1}) \cdot \mathrm{sign}(L_{\widetilde{A}_1}, L_{A_1(s)}) \\
&= \mathrm{sign}(\det \widetilde{A}_0) \cdot \mathrm{sign}(\det \widetilde{A}_1).
\end{aligned}$$

The first equality is true because we can regard the concatenation of three homotopies between the operators appearing in the middle expression as a single homotopy between the eventual endpoints $L_{A_0(s)}$ and $L_{A_1(s)}$; the second equality follows from (2-22) and (2-23). Finally, recall $\widetilde{A}_i = -J_0 \log \Psi_i(1)$ and observe that $\mathrm{sign} \det \log \Psi_i(1) = \mathrm{sign} \det(\mathrm{Id} - \Psi_i(1))$. This completes the proof. $\qquad\square$

2.8 Signs for the action on Floer cohomology

Let f, g be two commuting symplectomorphisms. We will now complete Definition 2.2 of the action $g_{\mathrm{floer}} \colon \mathrm{HF}^*(f) \to \mathrm{HF}^*(f)$ by specifying the signs appearing there.

Pick regular J_s, H_s to define Floer's complex $\mathrm{CF}^*(f; J_s, H_s)$. For each $x \in \mathrm{Fix}\, f_H$, pick a trivialisation Φ_x (2-17). Then for each x we get a unique asymptotic linearised operator $L_{A_x(s)}$ (2-18).

Let J'_s, H'_s be pulled by g (2-13) and $J_{s,t}, H_{s,t}$ be a homotopy (2-9). Let $u \in \mathcal{M}^0(g(x), y; J_{s,t}, H_{s,t})$ be a solution to Floer's continuation equation, where $x, y \in$ Fix f_H so that $g(x) \in$ Fix $f_{H'}$. Consider the linearisation D_u of Floer's continuation equation at u; its properties are similar to those discussed in Section 2.6. As $t \to +\infty$, D_u is asymptotic to $L_{A_y(s)}$ because, for t close to $+\infty$, $J_{s,t}, H_{s,t}$ are equal to J_s, H_s. On the other hand, as $t \to -\infty$, we can write down D_u in the g–induced trivialisation $\Phi_x \circ dg$ of $u^* T E_f |_{u(-\infty,s)}$. We claim that D_u is asymptotic, as $t \to -\infty$, to $L_{A_x(s)}$. Indeed, the asymptotic operator is determined by the following data: the fixed point $g(x)$, the chosen trivialisation $\Phi_x \circ dg$, and $J_{s,t}, H_{s,t}$ which equal $g^* J_s, H_s \circ g$ for t close to $-\infty$. We see that all of this data is pulled by g from the data x, Φ_x, J_s, H_s which defines the asymptotic linearised operator $A_x(s)$. Clearly, pullback by g does not change the linearised operator at all, so D_u is asymptotic to $L_{A_x(s)}$ as $t \to -\infty$.

The outcome is that the set $\{L_{A_x(s)}\}_{x \in \text{Fix } f_H}$ of asymptotic operators to D_u for $u \in \mathcal{M}(x, y; J_s, H_s)$ (these are solutions to Floer's equations for the differential on $CF^*(f)$, without the second symplectomorphism g involved) is identical to the set of asymptotic operators to D_u for $u \in \mathcal{M}(g(x), y; J_{s,t}, H_{s,t})$ (these are solutions to Floer's continuation equation), provided we use the described trivialisations.

Consequently, the usual definition of coherent orientations [13] on $\mathcal{M}(x, y; J_s, H_s)$ can be applied without any change to orient $\mathcal{M}(g(x), y; J_{s,t}, H_{s,t})$, $x, y \in$ Fix f_H. In Definition 2.2, we pick such a coherent orientation on $\mathcal{M}(g(x), y; J_{s,t}, H_{s,t})$. Instead of repeating the complete definition of coherent orientations, we only recall a piece relevant to the signs appearing in Lemma 2.9 regarding the supertrace of g_{floer}.

Coherent orientations are not unique, but the sign any coherent orientation associates to a point $u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$, $x \in$ Fix f_H, is canonical. We explain its definition following [13] and [24, Appendix A]. As we have seen, D_u is asymptotic as $t \to \pm\infty$ to the same operator

$$L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s),$$

where $A(s) = A_x(s)$ in the notation of the previous paragraphs. Choose a generic homotopy L_τ from D_u to $L_{A(s)}$, $\tau \in [0, 1]$, such that the L_τ are Fredholm operators which stay asymptotic to $L_{A(s)}$ as $t \to \pm\infty$.

Definition 2.16 For $u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$, $x \in$ Fix f_H, define $\text{sign}(u) = (-1)^\epsilon$, where

$$\epsilon = \sum_{\tau \in [0,1]} \dim_{\mathbb{R}} \ker L_\tau.$$

Because the operators L_τ have index zero, the sum is well-defined and does not depend modulo 2 on the chosen path. Let us repeat that, as part of Definition 2.2, these signs appear in Lemma 2.9.

2.9 Holomorphic sections over the torus

Section 2.5 explained that solutions to Floer's equation are holomorphic sections of a fibration $E_f \to S^1 \times \mathbb{R}$, whose monodromy around S^1 equals f. Now, let f, g be two commuting symplectomorphisms of X. In this subsection we define a fibration $p\colon E_{f,g}^{1,R} \to T^{1,R}$ over a 2-torus $T^{1,R}$. The monodromies of this fibration equal f and g around the two basis loops of the torus. After that we recall how to count its holomorphic sections; see [24] for details. We start by defining the torus

$$T^{1,R} := \frac{[0,1] \times [-R, R]}{(s, \{-R\}) \sim (s, \{R\}), \ (\{0\}, t) \sim (\{1\}, t)}$$

and equipping $T^{1,R}$ with the complex structure $j^{1,R}$ which comes from the standard one on $[0,1] \times \sqrt{-1}[-R, R] \subset \mathbb{C}$. Define

$$E_{f,g}^{1,R} := \frac{X \times [0,1] \times [-R, R]}{(x, s, \{-R\}) \sim (g(x), s, \{R\}), \ (x, \{0\}, t) \sim (f(x), \{1\}, t)}.$$

Here $x \in X$, $s \in [0,1]$, $t \in [-R, R]$. Because $fg = gf$, there is a fibration

$$p\colon E_{f,g}^{1,R} \to T^{1,R}$$

and a fibrewise symplectic closed 2–form $\omega_{E_{f,g}^{1,R}}$ coming from the one on X. Fix a generic almost complex structure $\widetilde{J}$ on $E_{f,g}^{1,R}$ such that $\widetilde{J}$ is $\omega_{f,g}^{1,R}$–tame on the fibres and the projection p is $(\widetilde{J}, j^{1,R})$–holomorphic. Let

$$\mathcal{M}(j^{1,R}, \widetilde{J})$$

be the space of all $(j^{1,R}, \widetilde{J})$–holomorphic sections $u\colon T^{1,R} \to E_{f,g}^{1,R}$ satisfying

$$(2\text{-}24) \qquad\qquad du + \widetilde{J}(u) \circ du \circ j^{1,R} = 0.$$

For generic $\widetilde{J}$, this moduli space is a smooth manifold that breaks into components of different dimensions. This manifold has a canonical orientation, and in particular its 0–dimensional part $\mathcal{M}^0(j^{1,R}, \widetilde{J})$ consists of signed points. We will now describe how these signs are defined. Let $u \in \mathcal{M}^0(j^{1,R}, \widetilde{J})$. Consider the linearised equation (2-24) at u,

$$D_u\colon C^\infty(u^* T^v E_{f,g}^{1,R}) \to \Omega^{0,1}(u^* T^v E_{f,g}^{1,R}).$$

Here $T^v E_{f,g}^{1,R} = \ker dp$ and $u^* T^v E_{f,g}^{1,R}$ is a complex bundle over the torus $T^{1,R}$. Because u has index zero, this bundle has Chern number zero and hence is trivial; fix

its trivialisation. Together with the holomorphic coordinates (s,t) on $T^{1,R}$, it induces a trivialisation of $\Omega^{0,1}(u^*TE^{1,R}_{f,g}) = \mathbb{R}^{2n}$. In this trivialisation, D_u is a 0–order perturbation of the Cauchy–Riemann operator:

$$(2\text{-}25) \qquad D_u = \partial/\partial t + J_0 \partial/\partial s + A(s,t)\colon C^\infty(T^{1,R}, \mathbb{R}^{2n}) \to C^\infty(T^{1,R}, \mathbb{R}^{2n}),$$

where $A(s,t)\colon T^{1,R} \to \operatorname{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. This is the same operator as considered in Section 2.7, except that now $A(s,t)$ can depend on t as well as on s. The operator D_u is always Fredholm of index zero.

Fix, once and for all, an injective operator of the above form, for example

$$L_{\mathrm{Id}} = \partial/\partial t + J_0 \partial/\partial s + \mathrm{Id}.$$

(This one is injective by Lemma 2.14, because $\ker(\mathrm{Id} - e^{-J_0}) = 0$.) Find a smooth homotopy of operators L_τ, $\tau \in [0,1]$, from D_u to L_{Id}, by deforming the 0–order part $A(s,t)$ to Id.

Definition 2.17 (Compare [24, page 51 and Appendix A]) For $u \in \mathcal{M}^0(j^{1,R}, \tilde{J})$, define $\operatorname{sign}(u) := (-1)^\epsilon$, where

$$\epsilon = \sum_{\tau \in [0,1]} \dim_{\mathbb{R}} \ker L_\tau.$$

For $u \in \mathcal{M}^0(j^{1,R}, \tilde{J})$, denote $\omega(u) := \int_{T^{1,R}} u^* \omega^{1,R}_{E_{f,g}}$. The following is well known.

Proposition 2.18 *The signed count*

$$\sharp\mathcal{M}^0(j^{1,R}, \tilde{J}) := \sum_{u \in \mathcal{M}^0(j^{1,R}, \tilde{J})} \operatorname{sign}(u) \cdot q^{\omega(u)}$$

is independent of the complex structure $j^{1,R}$ on the torus and of generic $\tilde{J}$. □

2.10 Gluing the fibration over the cylinder to the fibration over the torus

Given a symplectomorphism $f\colon X \to X$, we have constructed a fibration $p\colon E_f \to S^1 \times \mathbb{R}$ (2-14); also, given two commuting symplectomorphisms $f,g\colon X \to X$ and a parameter $R \in \mathbb{R}$, we have constructed a fibration $E^{1,R}_{f,g} \to T^{1,R}$. The fibres of both fibrations are symplectomorphic to X. Now, there is a map

$$(2\text{-}26) \qquad\qquad E_f \supset p^{-1}(S^1 \times [-R,R]) \to E^{1,R}_{f,g}.$$

It glues the boundary component $p^{-1}(S^1 \times \{R\})$ to the other boundary component $p^{-1}(S^1 \times \{-R\})$ via the symplectomorphism $g\colon X \to X$ applied fibrewise along S^1.

Fix regular J_s, H_s (2-1). As in (2-13), set

$$J_s' = g^* J_s, \quad H_s' = H_s \circ g.$$

Choose a homotopy $J_{s,t}, H_{s,t}$ (2-9) between J_s', H_s' and J_s, H_s. This homotopy must be t–independent for large and small t; we assume for convenience that

$$J_{s,t} \equiv J_s', \ H_{s,t} \equiv H_s' \quad \text{for } t \leq -R, \quad \text{and} \quad J_{s,t} \equiv J_s, \ H_{s,t} \equiv H_s \quad \text{for } t \geq R.$$

Finally, let $\widetilde{J} := \widetilde{J}(J_{s,t}, H_{s,t})$ be the almost complex structure on E_f from Section 2.5, which has the property that solutions to Floer's continuation equation are exactly $(j_{S^1 \times \mathbb{R}}, \widetilde{J})$–holomorphic sections $S^1 \times \mathbb{R} \to E_f$.

By definition, $\widetilde{J}|_{p^{-1}(S^1 \times \{R\})}$ is the g–pullback of $\widetilde{J}|_{p^{-1}(S^1 \times \{-R\})}$, which agrees with the gluing (2-26). So $\widetilde{J}$ defines a glued almost complex structure $\mathrm{gl}\widetilde{J}$ on $E_{f,g}^{1,R}$. Let us recall our notation one more time. $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ consists of holomorphic sections over $S^1 \times \mathbb{R}$ which are solutions to Floer's continuation equation (2-10), and $\mathcal{M}(j^{1,R}, \mathrm{gl}\widetilde{J}(J_{s,t}, H_{s,t}))$ consists of holomorphic sections over the torus $T^{1,R}$. We come to an important proposition, of which everything but formula (2-27) is well known.

Proposition 2.19 *For each $A > 0$ there is $R > 0$ such that there is a bijection called the gluing map and denoted by* gl:

$$\mathrm{gl}: \bigsqcup_{x \in \mathrm{Fix}\, f_H} \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})^{<A} \xrightarrow{\ 1-1\ } \mathcal{M}^0(j^{1,R}, \mathrm{gl}\widetilde{J}(J_{s,t}, H_{s,t}))^{<A}.$$

Here the superscripts $*^{<A}$ mean we are taking only those solutions whose ω–area is less than A. The gluing map preserves ω–areas,

$$\int_{S^1 \times \mathbb{R}} u^* \omega_{E_f} = \int_{T^{1,R}} \mathrm{gl}(u)^* \omega_{E_{f,g}^{1,R}},$$

and changes the signs from Definitions 2.16 and 2.17 by $(-1)^{\deg x}$:

(2-27) $$\mathrm{sign}(u) = \mathrm{sign}(\mathrm{gl}(u)) \cdot (-1)^{\deg x}.$$

Here $u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})^{<A}$, and $\deg x$ is as in Definition 2.7.

Proof The existence of the bijection gl is well-known. The map gl is constructed for the case $f = g = \mathrm{Id}$ in [32] (see also [5]) and that proof carries over to arbitrary f, g. Alternatively, one can adopt general SFT gluing and compactness theorems [7].

Let $u(s, t) \in \mathcal{M}(g(x), x; J_{s,t}, H_{s,t})$. By a smooth homotopy this section can be made t–independent for t close to $-\infty$ and $+\infty$. We can glue it into a smooth section over $T^{1,R}$ by applying (2-26). The smooth section over $T^{1,R}$ we obtained is smoothly

homotopic to $\mathrm{gl}(u)$ and hence has the same ω–area as $\mathrm{gl}(u)$; so gluing preserves ω–areas.

Let us explain why gl changes the sign by $(-1)^{\deg x}$. We have illustrated our argument by an informal diagram below; its arrows correspond to homotopies between Fredholm operators, and its labels are the signs determined by the mod 2 count of the dimensions of kernels appearing during the homotopies.

$$
\begin{array}{lll}
\underset{S^1 \times \mathbb{R}}{\text{over}} & D_u \xrightarrow{\ \mathrm{sign}(u)\ } \partial/\partial t + J_0 \partial/\partial s + A(s) \\[4pt]
& \qquad\ \ \text{gluing} \big\Downarrow \\[4pt]
\underset{T^{1,R}}{\text{over}} & D_{\mathrm{gl}(u)} \xrightarrow{\ \mathrm{sign}(u)\ } \partial/\partial t + J_0 \partial/\partial s + A(s) \xrightarrow{\ (-1)^{\deg x}\ } \partial/\partial t + J_0 \partial/\partial s + \mathrm{Id}
\end{array}
$$

$$\underbrace{\phantom{D_{\mathrm{gl}(u)} \xrightarrow{\quad} \partial/\partial t + J_0 \partial/\partial s + A(s) \xrightarrow{\quad\quad\quad}}}_{\mathrm{sign}(\mathrm{gl}(u))}$$

Take $u(s,t) \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$ and consider linearised Floer's operators (2-16) and (2-25):

$$D_u \colon H^{1,p}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \to L^p(S^1 \times \mathbb{R}, \mathbb{R}^{2n}),$$
$$D_{\mathrm{gl}(u)} \colon C^\infty(T^{1,R}, \mathbb{R}^{2n}) \to C^\infty(T^{1,R}, \mathbb{R}^{2n}).$$

Take a homotopy L_τ from D_u to the operator (2-18) $L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s)$. By Definition 2.16,

$$(2\text{-}28) \qquad\qquad\qquad \mathrm{sign}(u) = (-1)^{\sum_\tau \dim \ker L_\tau}.$$

Let L_τ^{gl} be a homotopy from $D_{\mathrm{gl}(u)}$ to the analogous operator $L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s)$ over the torus, considered in Section 2.7. It is well known that

$$(2\text{-}29) \qquad\qquad \sum_{\tau \in [0,1]} \dim \ker L_\tau \equiv \sum_{\tau \in [0,1]} \ker L_\tau^{\mathrm{gl}} \quad \text{mod } 2.$$

(This is a special case of the fact that orientations of moduli spaces of pseudo-holomorphic sections before gluing canonically define orientations on moduli spaces after gluing.)

Take a homotopy $L_{A_\tau(s)}$ from $L_{A(s)}$ to $L_{\mathrm{Id}} = \partial/\partial t + J_0 \partial/\partial s + \mathrm{Id}$. To compute the kernels swept by this homotopy, we will use Lemma 2.15. First, let $\Psi(s)$ be the matrix which solves (2-19) with respect to our given $A(s)$; then $\Psi(1) = df_H(X)$ by Lemma 2.10. By Definition 2.7, $\mathrm{sign}\det(\mathrm{Id} - \Psi(1)) = \deg x$. Second, let $\Psi(s)$ instead be the matrix which solves (2-19) with respect to the s–independent matrix

$A(s) \equiv \mathrm{Id}$; the solution is e^{-sJ_0}, and for it we obtain $\operatorname{sign} \det(\mathrm{Id} - \Psi(1)) = +1$. Now by Lemma 2.15,

$$(2\text{-}30) \qquad \sum_{\tau \in [0,1]} \dim \ker L_{A_\tau(s)} \equiv \deg x \quad \mod 2.$$

The concatenation of homotopies $L_{A_\tau(s)}$ and L_τ^{gl} is a homotopy from $D_{\mathrm{gl}(u)}$ to $\partial/\partial t + J_0 \partial/\partial s + \mathrm{Id}$. So by Definition 2.17,

$$(2\text{-}31) \qquad \operatorname{sign}(\mathrm{gl}(u)) = (-1)^{\sum_\tau \dim \ker L_{A_\tau(s)}} \cdot (-1)^{\sum_\tau \dim \ker L_\tau^{\mathrm{gl}}}.$$

Combining (2-28)–(2-31) we get $\operatorname{sign}(u) = \operatorname{sign}(\mathrm{gl}(u)) \cdot (-1)^{\deg x}$, which completes the proof. □

2.11 Proof of the elliptic relation

Proof of Theorem 1.1 We only need to compile the previous statements. It suffices to prove that for each $A > 0$, the supertraces are equal up to order q^A:

$$\mathrm{STr}(f_{\mathrm{floer}})/q^A = \mathrm{STr}(g_{\mathrm{floer}})/q^A.$$

By Lemma 2.9 and Proposition 2.19, for sufficiently large R we have

$$\mathrm{STr}(g_{\mathrm{floer}})/q^A = \sum_{\substack{x \in \mathrm{Fix}\, f_H \\ u \in \mathcal{M}^0(g(x),x;J_{s,t},H_{s,t})^{<A}}} (-1)^{\deg x} \cdot \operatorname{sign}(u) \cdot q^{\omega(u)}$$

$$= \sharp \mathcal{M}^0(j^{1,R}, \tilde{J}(J_{s,t}, H_{s,t}))^{<A}.$$

One can repeat all constructions after swapping f and g to get

$$\mathrm{STr}\big(f_{\mathrm{floer}} \colon \mathrm{HF}^*(g) \to \mathrm{HF}^*(g)\big) = \sharp \mathcal{M}^0(j^{R,1}, \tilde{J}_1)^{<A}.$$

Here $j^{R,1} = j^{1,1/R}$ is another complex structure on the torus (which is "long" in the s–direction, while $j^{1,R}$ is "long" in the t–direction), and $\tilde{J}_1$ is some other almost complex structure on the total space. Now Theorem 1.1 follows from Proposition 2.18. □

2.12 Finite-order symplectomorphisms

We will now prove two lemmas about the action on Floer cohomology when one of the two commuting symplectomorphisms has finite order, and derive Proposition 1.4. The proof of the next lemma is an extension of [16, Lemma 7.1].

Lemma 2.20 *Let X be a symplectic manifold satisfying the W^+ condition. Let $g, \phi\colon X \to X$ be two commuting symplectomorphisms. Suppose $\phi^k = \mathrm{Id}$ and the fixed point set X^ϕ is a smooth manifold (maybe disconnected, with components of different dimensions). Then*

$$\mathrm{STr}\bigl(g_{\mathrm{floer}}\colon \mathrm{HF}^*(\phi) \to \mathrm{HF}^*(\phi)\bigr) = L(g|_{X^\phi}) \cdot q^0 + \sum_i a_i \cdot q^{\omega_i}, \quad \omega_i > 0.$$

In other words, $\mathrm{STr}(g_{\mathrm{floer}}) \in \Lambda$ contains only summands with non-negative powers of q, and the q^0–coefficient is the topological Lefschetz number of $g|_{X^\phi}$. Using the elliptic relation we will later show that the higher-order terms $a_i q^{\omega_i}$ actually vanish; this is however a separate argument and we first prove the lemma as stated.

Proof First we construct a Hamiltonian function on X of special form. Let $U(X^\phi)$ be a ϕ–equivariant tubular neighbourhood of X^ϕ, $p\colon U(X^\phi) \to X^\phi$ the projection and dist a ϕ–invariant function on $U(X^\phi)$ measuring the distance to X^ϕ in some ϕ–invariant metric. Let H_0 be an arbitrary function on X^ϕ. Define

$$H := H_0 \circ p + \mathrm{dist}^2 .$$

This is a function on $U(X^\phi)$. Extend this function to X in any way and then average it with respect to ϕ (this will not change the function on $U(X^\phi)$). We denote the result by H again. Note that $H|_{X^\phi} = H_0$ and $\mathrm{Crit}(H_0) = \mathrm{Crit}(H) \cap X^\phi$. For the rest of the proof, H will be a generic function constructed this way; in particular, $H|_{X^\phi}$ is also generic.

Because ϕ has finite order, we can choose a ϕ–invariant compatible almost complex structure J on X which preserves TX^ϕ, and such that $J|_{X^\phi}$ is arbitrary. Since J, H are ϕ–invariant, they satisfy (2-1), with $f = \phi$. Thus Floer's equation (2-3) makes sense for such s–independent data J, H. Denote $J' \equiv g^*J$, $H' \equiv H \circ g$ as in (2-13).

Choose an s–independent homotopy (2-9) H_t, J_t from H' to H (resp. from J' to J). For every t, H_t, J_t must be ϕ–invariant, and, as earlier,

(2-32) $$H_t = (H_0)_t \circ p + \mathrm{dist}^2$$

on $U(X^\phi)$, where $(H_0)_t = (H_t)|_{X^\phi}$ can be arbitrary. Note that in general it might not be possible to find s–independent J_t, H_t that would make all solutions of Floer's continuation equation (2-10) regular. However, using [16] we will now argue that some solutions of (2-10) (namely the gradient flowlines of H_t) are still generically regular.

Recall that J_t induces the time-dependent metric $\omega(\,\cdot\,, J_t\,\cdot\,)$ on X by definition of a compatible almost complex structure. If H is a function on X, its gradient and

Hamiltonian vector fields are related by the equation $\nabla H = JX_H$. So s–independent solutions $u(s,t) \equiv x(t)$ of Floer's continuation equation (2-10) are exactly $\omega(\,\cdot\,, J_t \,\cdot\,)$–gradient flowlines of H_t:

$$dx(t)/dt - \nabla H_t = 0.$$

The ϕ–periodicity condition (2-4) now reads $\phi(x(t)) = x(t)$, so we are looking only at gradient flowlines inside X^ϕ. Note that every s–independent solution $u(s,t) \equiv x(t)$ of (2-10) has zero area: $\omega(u) = 0$. Recall that solutions of (2-10) are elements of $\mathcal{M}(x, y; J_t, H_t)$ where $x \in \mathrm{Fix}\,\phi_{H'}$ and $y \in \mathrm{Fix}\,\phi_H$. Also note that $\mathrm{Fix}\,\phi_H = \mathrm{Crit}(H|_{X^\phi})$, and similarly $\mathrm{Fix}\,\phi_{H'} = \mathrm{Crit}(H'|_{X^\phi})$.

The following two facts are proved in [16] when H_t, J_t are t–independent and $\phi = \mathrm{Id}$ (that paper is interested in the equations for Floer's differential rather than continuation maps). The proofs are valid in the general case. For example, one can track that the periodicity condition (2-1), which is the only place where ϕ explicitly appears, is not used in the proof of the facts below.

(1) For any J_t, H_t as above, an s–independent solution $u(s,t) \equiv x(t)$ of (2-10) is regular, ie D_u (2-16) is onto, if and only if the $\omega(\,\cdot\,, J_t \,\cdot\,)$–gradient flow of H_t is Morse–Smale near X^ϕ [30, Corollary 4.3, Theorem 7.3]; compare [16, proof of Theorem 6.1].

(2) There is $\epsilon > 0$ such that every solution $u(s,t)$ of (2-10) with $\omega(u) < \epsilon$ is s–independent [16, Lemma 7.1].

We claim that the gradient flow of a generic H_t constructed above is Morse–Smale near X^ϕ. Indeed, we can choose $H_t|_{X^\phi}$ freely, so we can make the flow of $H_t|_{X^\phi}$ Morse–Smale. Because H_t is quadratic in the normal direction to X^ϕ (2-32), the stable manifolds of H_t are, near X^ϕ, normal disk bundles over those of $H_t|_{X^\phi}$, and the unstable manifolds of H_t lie in X^ϕ and coincide with those of $H_t|_{X^\phi}$. Consequently, H_t is Morse–Smale near X^ϕ if and only if $H_t|_{X^\phi}$ is Morse–Smale.

By Remark 2.4 or [31, Section 4.2.2],

$$(2\text{-}33) \qquad \sum_{\substack{x \in \mathrm{Fix}\,\phi_H \\ u \in \mathcal{M}^0(g(x),x;J_s,H_s):\,\omega(u)\leq 0}} (-1)^{\deg x} \cdot \mathrm{sign}(u) \cdot q^{\omega(u)} = L(g|_{X^\phi}) \cdot q^0.$$

Although the left-hand side looks exactly like the expression for $\mathrm{STr}(g_{\mathrm{floer}})$ from Lemma 2.9, J_t, H_t need not be regular for all continuation equation solutions, while g_{floer} must be computed using a regular Hamiltonian and almost complex structure. To cure this, we slightly perturb J, H and J_t, H_t by allowing them to depend on s, to get J_s, H_s and $J_{s,t}, H_{s,t}$. For a generic such perturbation, all solutions

to (2-10) with respect to $J_{s,t}, H_{s,t}$ become regular. Because s–independent solutions in $\mathcal{M}(x, y; J_t, H_t)$ were already regular, they are in 1–1 correspondence (via the continuation map) with some solutions in $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ of zero ω–area. By item (2) above, every $u \in \mathcal{M}(x, y; J_{s,t}, H_{s,t})$ with $\omega(u) < \epsilon$ actually has zero area and corresponds to an s–independent solution in $\mathcal{M}(x, y; J_t, H_t)$. (See [16, proof of Proposition 7.4] for this argument.) In view of (2-33) this means

$$\sum_{\substack{x \in \mathrm{Fix}\,\phi_H \\ u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t}): \omega(u) \leq 0}} (-1)^{\deg x} \cdot \mathrm{sign}(u) \cdot q^{\omega(u)} = L(g|_{X^\phi}) \cdot q^0.$$

Lemma 2.20 follows from this equality and Lemma 2.9. $\qquad\square$

Lemma 2.21 *Let X be a symplectic manifold satisfying the W^+ condition. Let $g, \phi \colon X \to X$ be two commuting symplectomorphisms. Suppose $\phi^k = \mathrm{Id}$. Then*

$$\mathrm{STr}(\phi_{\mathrm{floer}} \colon \mathrm{HF}^*(g) \to \mathrm{HF}^*(g)) = a \cdot q^0, \quad \text{where } a \in \mathbb{C} \text{ and } |a| \leq \dim_\Lambda \mathrm{HF}^*(\phi).$$

Proof By Lemma 2.6 $(\phi_{\mathrm{floer}})^k = \mathrm{Id}$, so all eigenvalues of ϕ_{floer} are among the roots of unity $\sqrt[k]{1} \cdot q^0 \in \Lambda$. The signed sum of these eigenvalues gives $\mathrm{STr}(\phi_{\mathrm{floer}})$, and Lemma 2.21 follows. $\qquad\square$

The elliptic relation (Theorem 1.1) and Lemma 2.21 imply the following corollary.

Corollary 2.22 *The terms $a_i \cdot q^{\omega_i}$, $\omega_i > 0$ from Lemma 2.20 actually vanish.* $\qquad\square$

Proof of Proposition 1.4 The proposition follows from Lemma 2.20, Lemma 2.21 and Theorem 1.1. $\qquad\square$

Remark 2.23 As promised in Remark 1.7, we now sketch an alternative proof of Proposition 1.4 which does not appeal to Theorem 1.1. Suppose for simplicity that a symplectomorphism $f \colon X \to X$ commutes with a symplectic involution ι and f has non-degenerate isolated fixed points. Note that, for general reasons, $d\iota$ acts by $-\mathrm{Id}$ on the normal bundle to its fixed locus X^ι. To compute $\mathrm{HF}^*(f)$, choose the zero Hamiltonian perturbation and an almost complex structure which is ι–invariant at points $x \in \mathrm{Fix}\, f \cap X^\iota$. Then ι_{floer} only counts constant solutions $u(s, t) \equiv x \in \mathrm{Fix}\, f \cap X^\iota$. (Because f has isolated fixed points, the only zero-area solutions are constant, and because $\iota_{\mathrm{floer}}^2 = \mathrm{Id}$, all positive-area solutions cancel.) However, the sign associated to a constant solution u is not always positive. The reason is that we must write the linearised Floer's operator D_u in a trivialisation of $u^* T_x X = S^1 \times \mathbb{R} \times T_x X$ which differs by $d\iota(x)$ over the two ends of the cylinder, according to the definition

in Section 2.8. Consider the splitting $T_x X = T_x X^\iota \oplus N_x X^\iota$ into the $+1$ and -1 eigenspaces of $d\iota(x)$. We can choose the constant trivialisation of $u^* T_x X^\iota$ and get the $\mathbb{R}$–independent operator on this subspace, which by definition carries the positive sign. However, we are not allowed to choose the constant trivialisation of $u^* N_x X^\iota$ (instead, an allowed choice would be, for example, a rotation from Id to $-$ Id with parameter t), so D_u will not be the canonical $\mathbb{R}$–invariant operator on $N_x X$ and can carry a nontrivial sign from Definition 2.16. We claim that this sign equals sign det(Id $-df(x)|_{N_x X^\iota}$). The computation can essentially be reduced to the index problem considered in Section 2.7, since D_u can still be chosen independent of one variable; a related Lagrangian version of this statement is [37, Lemma 14.11]. Once the signs are known, it is easy to see that $\text{STr}(\iota_{\text{floer}}) = L(f|_{\text{Fix}\,\iota}) \cdot q^0$:

$$\text{STr}(\iota_{\text{floer}}) = \sum_{x \in \text{Fix}\,f \cap X^\iota} (-1)^{\deg x} \cdot \text{sign det}(\text{Id} - df(x)|_{N_x X^\iota}) \cdot q^0$$

$$= \sum_{x \in \text{Fix}\,f \cap X^\iota} \text{sign det}(\text{Id} - df(x)|_{T_x X^\iota}) \cdot q^0 = L(f|_{\text{Fix}\,\iota}) \cdot q^0.$$

The bound dim $\text{HF}^*(f) \geq L(f|_{\text{Fix}\,\iota})$ follows as in Lemma 2.21.

2.13 Lagrangian elliptic relation

In this subsection, we briefly explain Theorem 1.10 and Proposition 1.11. Let X be a monotone symplectic manifold, ie $[\omega(X)] = \lambda c_1(X)$ as elements of $H^2(X;\mathbb{R})$, $\lambda > 0$. Let $\phi \colon X \to X$ be a symplectomorphism, and $L_i \subset X$ be two connected monotone Lagrangian submanifolds such that $\phi(L_i) = L_i$.

In order to define the action $\phi_{\text{floer}} \colon \text{HF}^*(L_1, L_2) \to \text{HF}^*(L_1, L_2)$ over a field of characteristic not equal to two, we must fix the following additional data. First, L_i must be oriented, although ϕ need not preserve the orientations. (In the appendix we use the orientation-reversing case.) Second, the hypothesis below must be satisfied.

Hypothesis 2.24 L_i must be equipped with spin structures S_i together with isomorphisms $\phi^* S_i \to S_i$ if $\phi|_{L_i}$ preserves orientation, and $\phi^* S_i \to \overline{S}_i$ if $\phi|_{L_i}$ reverses orientation. Here $\overline{S}_i$ is the following spin structure on $\overline{L}_i$ (that is, on L_i with the opposite orientation). The original spin structure S_i is a trivialisation of TL_i over the 1–skeleton of L_i which extends over the 2–skeleton and agrees with the orientation on L_i. By definition, $\overline{S}_i$ is the composition of the trivialisation S_i with a fixed orientation-reversing isomorphism $\mathbb{R}^n \to \mathbb{R}^n$, for example the one which multiplies the first coordinate by -1. We note the desired isomorphisms $\phi^* S_i \to S_i$ or $\phi^* S_i \to \overline{S}_i$ always exist if the L_i are simply connected.

In [37, Section 14], similar data (defined only for an involution ϕ, with an extra condition on the "squares" of the above isomorphisms, but also allowing non-orientable Lagrangians) was called an equivariant Pin structure.

Pick some J_s, H_s defining Floer cohomology $\mathrm{HF}^*(L_1, L_2; J_s, H_s, S_i)$; see [26; 14] for a definition in the monotone setting. We have included the choice of spin structures in our notation. The action ϕ_{floer} is the composition

$$\mathrm{HF}^*(L_1, L_2; J_s, H_s, S_i) \longrightarrow \mathrm{HF}^*(L_1, L_2; \phi^* J_s, H_s \circ \phi, \phi^* S_i)$$
$$\longrightarrow \mathrm{HF}^*(L_1, L_2; J_s, H_s, S_i).$$

Here the first map is the tautological chain-level map that takes all chain generators and Floer's solutions to their ϕ–images; we are using that $\phi L_i = L_i$. The second one is the continuation map. We skip the proof of the next lemma.

Lemma 2.25 (Cf [37, Sections (14a) and (14e)]) *If $\phi^k = \mathrm{Id}$ then $(\phi_{\mathrm{floer}})^k = \pm \mathrm{Id}$.* □

Note that, unlike Lemma 2.6 and [37, top of page 310], we do not necessarily get $(\phi_{\mathrm{floer}})^k = \mathrm{Id}$, but having $(\phi_{\mathrm{floer}})^k = \pm \mathrm{Id}$ is enough for future applications.

Choose J_s, H_s (2-1) to define Floer's complex $\mathrm{CF}^*(\phi; J_s, H_s)$. Take the fibration $p: E_\phi \to S^1 \times [0, +\infty)$ with monodromy ϕ around the circle as in (2-14), but now over the semi-infinite cylinder $S^1 \times [0, +\infty)$ instead of $S^1 \times \mathbb{R}$. It contains the "boundary condition" manifold $S^1 \times L \subset p^{-1}(S^1 \times \{0\})$. The symplectic form on X defines a fibrewise symplectic form ω_{E_ϕ} on E_ϕ. Choose a tame almost complex structure $\widetilde{J}$ on E_ϕ which, over $S^1 \times [1, +\infty)$, equals $\widetilde{J}(J_s, H_s)$ for some J_s, H_s (see Section 2.5), and in particular is independent of $t \in [1, +\infty)$.

Take $x \in \mathrm{Fix}\,\phi_H$ (2-5), that is, a generator of $\mathrm{CF}^*(\phi; J_s, H_s)$. We define $\mathcal{M}^0(L, x)$ to be the set of all zero-index $\widetilde{J}$–holomorphic sections $u(s, t): S^1 \times [0, +\infty) \to E_\phi$ which are asymptotic, as $t \to +\infty$, to the Hamiltonian trajectory $\psi_s(x)$ (2-2), and satisfy the Lagrangian boundary condition $u(s, 0) \in S^1 \times L$. Then we define

$$[L]^\phi = \sum_{x \in \mathrm{Fix}\,\phi_H} \sum_{u \in \mathcal{M}^0(L, x)} \pm q^{\omega(u)} \cdot [x] \in \mathrm{HF}^*(\phi).$$

Here $[x] \in \mathrm{HF}^*(\phi)$ is the cohomology class of the chain generator x, and

$$\omega(u) = \int_{S^1 \times [0, +\infty)} u^* \omega_{E_\phi}.$$

The signs are defined using the chosen spin structures on the L_i and coherent orientations for ϕ. This is a version of the open-closed string map; cf [29].

Next we review the quantum product $\mathrm{HF}^*(\phi) \otimes \mathrm{HF}^*(\phi^{-1}) \to \mathrm{HF}^*(\mathrm{Id}) \cong \mathrm{QH}^*(X)$. It counts holomorphic sections of a symplectic fibration over S^2 with three punctures and monodromies $\phi, \phi^{-1}, \mathrm{Id}$ around them. The first two punctures serve as inputs for $\mathrm{HF}^*(\phi)$, $\mathrm{HF}^*(\phi^{-1})$, and the third puncture is the output; see [24] for details. If one caps the output puncture by a disk, the count of sections over the resulting twice-punctured sphere (see the lower part of Figure 3(a)), gives the composition $\mathrm{HF}^*(\phi) \otimes \mathrm{HF}^*(\phi^{-1}) \to \mathrm{HF}^*(\mathrm{Id}) \xrightarrow{\chi} \Lambda$ of the product and the integration map χ (once we identify $\mathrm{HF}^*(\mathrm{Id})$ with $\mathrm{QH}^*(X)$).
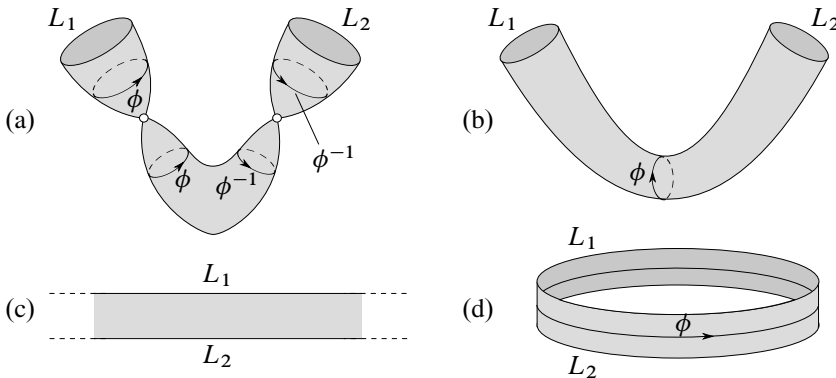

Figure 3: Proving the Lagrangian elliptic relation

Combining the definitions, $\chi([L_1]^\phi * [L_2]^{\phi^{-1}})$ counts holomorphic sections over two cylinders and a twice-punctured sphere which have the same asymptotics over the punctures in two pairs; see Figure 3(a). Here the cylinder $S^1 \times [0, +\infty)$ is seen as a once-punctured disk. This count equals the number of sections of a glued fibration over an annulus, with monodromy ϕ around the core circle, and Lagrangian conditions $S^1 \times L_1$, $S^1 \times L_2$ over the boundary of the annulus. The annulus carries a fixed "long" complex structure; see Figure 3(b).

On the other hand, $\mathrm{STr}(\phi_{\mathrm{floer}})$ counts sections of a trivial fibration over the strip $[0, 1] \times \mathbb{R}$ with Lagrangian boundary conditions $\mathbb{R} \times L_i$ and asymptotics differing by ϕ over $t \to \pm\infty$; see Figure 3(c). We can glue the fibration over the strip, twisting it by ϕ to get a fibration over the annulus which we have already encountered: it carries Lagrangian conditions $S^1 \times L_i$ over the boundary and has monodromy ϕ around the core circle; see Figure 3(d). By gluing, $\mathrm{STr}(\phi_{\mathrm{floer}})$ is equal to the count of holomorphic sections of this fibration, with a fixed ("long", but in the other direction than before) complex structure on the annulus. As the count of sections does not depend on the complex structure on the annulus, we get Theorem 1.10. We omit the discussion of signs, which was carried out in detail for the case of commuting symplectomorphisms.

The signs in the present case can be studied by similar arguments if we superficially deform the Lagrangians so that $T_p L_1 = T_p L_2$ for all intersection points $p \in L_1 \cap L_2$, keeping these points isolated, and then pick non-degenerate Hamiltonians H_1, H_2 to compute $\mathrm{HF}^*(L_1, L_2)$.

Let us now explain Proposition 1.11. The most important step is to prove a Lagrangian analogue of Lemma 2.20: if ϕ is a map of finite order with fixed locus X^ϕ and smooth orientable Lagrangian fixed loci $L_i^\phi \subset X^\phi$ then

$$(2\text{-}34) \qquad \chi([L_1]^\phi * [L_2]^\phi) = ([L_1^\phi] \cdot [L_2^\phi]) \cdot q^0 + \sum_i a_i \cdot q^{\omega_i}, \quad \omega_i > 0.$$

Recall that $[L_1^\phi] \cdot [L_2^\phi] \in \mathbb{Z}$ is the homological intersection of the fixed loci L_1^ϕ, L_2^ϕ inside X^ϕ. (Note that L_i^ϕ are automatically isotropic but not necessarily Lagrangian, although we will only use the case when they are Lagrangian. One can get examples of $(\phi_{\mathrm{floer}})^k = -\mathrm{Id}$ in Lemma 2.25 when the dimensions of L_1^ϕ, L_2^ϕ are different.)

In order to count sections of the configuration on Figure 3(a), we must specify the data $J_{s,t}, H_{s,t}$ over our configuration consisting of two half-cylinders $S^1 \times [0, +\infty)$ and a twice-punctured sphere which we will now see as the cylinder $S^1 \times \mathbb{R}$. Similarly to Lemma 2.20, we choose the data to be of special form, namely independent of the basepoint: $J_{s,t} \equiv J$, $H_{s,t} \equiv H$ (this forces J, H to be ϕ–equivariant). With this data, s–independent ($s \in S^1$) sections become gradient flowlines of the Morse function H inside the fixed locus X^ϕ. Rigid sections over $S^1 \times \mathbb{R}$ are constant, while rigid sections over $S^1 \times [0, +\infty)$ are flowlines from L_i to a critical point of H. This way, the count of s–independent rigid configurations on Figure 3(a) is

$$\sum_{x \in \mathrm{Crit}^n(H|_{X^\phi})} ([L_1^\phi] \cdot [\mathrm{Stab}(x)])([L_2^\phi] \cdot [\mathrm{Stab}(x)]),$$

where $n = \frac{1}{2} \dim_{\mathbb{R}} X^\phi$, Crit^n are index-n critical points, and Stab are stable manifolds in X^ϕ. This sum equals the intersection $[L_1^\phi] \cdot [L_2^\phi]$.

Finally, one must argue that these configurations of flowlines are regular, and are the only zero-area solutions. (There could be other positive-area solutions which are not necessarily regular.) This is a variation on the lemmas cited in the proof of Lemma 2.20. Then one makes the data J, H regular by allowing them to depend on s, t and argues that the count of zero-area solutions (which were already regular) is preserved.

On the other hand, if ϕ is of finite order then $\phi_{\mathrm{floer}} \colon \mathrm{HF}^*(L_1, L_2) \to \mathrm{HF}^*(L_1, L_2)$ is of finite order by Lemma 2.25, and the eigenvalues of ϕ_{floer} are among $\sqrt[2k]{1} \cdot q^0$. Consequently, $\mathrm{STr}(\phi_{\mathrm{floer}}) = a \cdot q^0$, $|a| \leq \dim_\Lambda \mathrm{HF}^*(L_1, L_2)$. Now Theorem 1.10 and formula (2-34) imply Proposition 1.11.

3 Vanishing spheres and Dehn twists

Let Y be a Kähler manifold with a Kähler form ω, and $\mathcal{L} \to Y$ a very ample holomorphic line bundle. Let $X \subset Y$ be a smooth divisor in the linear system $|\mathcal{L}|$. In this section we define $|\mathcal{L}|$–vanishing Lagrangian spheres in the symplectic manifold $(X, \omega|_X)$. They exist if the line bundle $\mathcal{L} \to Y$ has zero defect (see below) and are then unique up to symplectomorphism. Throughout this section, we denote by $D \subset \mathbb{C}$ the unit complex disk.

3.1 Lefschetz fibrations and vanishing cycles

This subsection reviews well known material; see eg [37].

Definition 3.1 (Lefschetz fibration with a unique singularity) Suppose E is a smooth manifold, Ω a closed 2–form on E, and $\pi\colon E \to D$ is a smooth proper map. The triple (E, Ω, π) is called a Lefschetz fibration with a unique singularity if there is a point $p \in E$ (without loss of generality, we assume $\pi(p) = 0 \in D$), and a neighbourhood $U(p)$ such that:

- π is regular outside $U(p)$, and the restriction of Ω on the regular fibres of π is symplectic.

- There exists a complex structure on $U(p)$ with a holomorphic chart $x_1, \ldots, x_n$ such that
$$\pi(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2.$$

- $\Omega|_{U(p)}$ is Kähler with respect to this complex structure.

All smooth fibres $E_t := \pi^{-1}(t)$ contain a Lagrangian sphere, uniquely defined up to Hamiltonian isotopy. Let us sketch its construction, as we will refer to it later in the proof of Lemma 4.8. Because the smooth fibres E_t are symplectomorphic to each other by parallel transport with respect to the Ω–induced connection on E, it suffices to construct a Lagrangian sphere in E_t for a small $t \in \mathbb{R}_+$. Define $L \subset U(p) \subset E$ by the equation
$$x_1^2 + \cdots + x_n^2 = t, \quad x_i \in \mathbb{R}.$$

Clearly L is contained in E_t and is a Lagrangian sphere for $t \in \mathbb{R}_+$ with respect to the standard symplectic structure Ω_{std} on $U(p) \subset \mathbb{C}^n$. However, it is generally not possible to make our form $\Omega|_{U(p)}$ standard by a holomorphic change of coordinates preserving π. Instead, we can follow the argument of [36, Lemma 1.6]: there is a function f on $U(p)$ such that $\Omega|_{U(p)} = \Omega_{\text{std}} + d\,d^c f$. We can deform f to 0 in a

smaller neighbourhood $U'(p) \subset U(p)$ while leaving f unchanged outside of $U(p)$. Let f_r be such a homotopy and define $\Omega_r := \Omega$ outside of $U(p)$, and $\Omega_r|_{U(p)} := \Omega_{\text{std}} + dd^c f_r$. Observe that $\Omega_0 = \Omega$ and $\Omega_1|_{U'(p)} = \Omega_{\text{std}}$. For all r, the smooth fibres $(E_t, \Omega_r|_{E_t})$ are symplectic and the cohomology class of $\Omega_r|_{E_t}$ is constant, so by Moser's lemma the smooth fibres are actually symplectomorphic to each other for any r. In particular, the Lagrangian sphere $L \subset (E_t, \Omega_{\text{std}}|_{E_t})$ constructed above can be mapped by this symplectomorphism to a Lagrangian sphere in $(E_t, \Omega|_{E_t})$.

Definition 3.2 (Vanishing Lagrangian sphere) A Lagrangian sphere in a smooth fibre E_t is called vanishing for the Lefschetz fibration $E \to D$ if it is Hamiltonian isotopic to the one constructed above.

3.2 Defect of a line bundle

Definition 3.3 (Defect of a line bundle) Let Y be a complex manifold and $\mathcal{L} \to Y$ a very ample holomorphic line bundle, giving an embedding $Y \subset (\mathbb{P}^N)^*$ where $\mathbb{P}^N = \mathbb{P}H^0(Y, \mathcal{L})$. The discriminant variety $\Delta \subset \mathbb{P}^N$ is the dual variety to Y, parametrising all hyperplanes in $(\mathbb{P}^N)^*$ which are tangent to $Y \subset \mathbb{P}^N$. Equivalently, it parametrises all singular divisors in the linear system $\mathbb{P}H^0(Y, \mathcal{L})$. The defect of $\mathcal{L}$ is the number

$$\text{def}\,\mathcal{L} = N - 1 - \dim \Delta \geq 0.$$

Line bundles usually have zero defect; for us, it is useful to note the following.

Lemma 3.4 [3, page 532] *Suppose $\mathcal{L} \to Y$ is a very ample line bundle. If $\text{def}\,\mathcal{L} \geq 1$, there exists a smooth rational curve $l \subset Y$ such that $\mathcal{L} \cdot l = 1$.* $\square$

For completeness, let us sketch a proof. Recall that points in Δ^{reg} correspond to generic hyperplanes $H \subset (\mathbb{P}^N)^*$ which are not transverse to Y. If $\text{def}\,\mathcal{L} \geq 1$, for such a hyperplane $H \in \Delta^{\text{reg}}$ the contact locus $(H \cap Y)^{\text{sing}}$ is a linear $\mathbb{P}^{\text{def}\,\mathcal{L}}$ [43, Theorem 1.18]. Take any line $l \cong \mathbb{P}^1$ in H. Obviously it intersects a generic smooth hyperplane section $\widetilde{H} \cap Y$ transversely at a single point, which means $\mathcal{L} \cdot l = 1$.

Corollary 3.5 *Suppose $\mathcal{L} \to Y$ is a very ample line bundle. Then $\text{def}\,\mathcal{L}^{\otimes d} = 0$ for any $d \geq 2$.* $\square$

3.3 $|\mathcal{L}|$–vanishing spheres in divisors

Recall that $D \subset \mathbb{C}$ denotes the unit disk.

Definition 3.6 (Total space of a family of divisors)   Let Y be a Kähler manifold and $\mathcal{L} \to Y$ a very ample line bundle. Take a holomorphic embedding $u: D \to \mathbb{P}H^0(Y, \mathcal{L}) = |\mathcal{L}|$; then each point $t \in D$ defines a divisor $X_{u(t)} \subset Y$. We call $\{X_{u(t)}\}_{t \in D}$ a family of divisors. The total space of the family $\{X_{u(t)}\}_{t \in D}$ is

$$E := \{(x, u(t)) : x \in X_t,\ t \in D\} \subset Y \times \mathbb{P}H^0(Y, \mathcal{L}).$$

The restriction of the product Kähler form from $Y \times \mathbb{P}H^0(Y, \mathcal{L})$ to E makes E a Kähler manifold. There is a canonical projection $\pi: E \to D$ whose fibres are $X_{u(t)}$. In the future we shall write $\{X_t\}_{t \in D}$ instead of $\{X_{u(t)}\}_{t \in D}$.

Definition 3.7 ($|\mathcal{L}|$–vanishing Lagrangian sphere in a divisor)   Let Y be a Kähler manifold and $\mathcal{L} \to Y$ a very ample line bundle with zero defect, and with $\dim \mathbb{P}H^0(Y, \mathcal{L}) \geq 2$. Let $\Delta \subset \mathbb{P}H^0(Y, \mathcal{L})$ be the discriminant variety from Definition 3.3. Let $u: D \to \mathbb{P}H^0(Y, \mathcal{L})$ be a holomorphic embedding such that $u(0) \in \Delta^{\mathrm{reg}}$, $u(t) \notin \Delta$ for $t \neq 0$, and the intersection of $u(D)$ with Δ^{reg} is transverse. Let $\pi: E \to D$ be as in Definition 3.6.

By [23, 1.8], $\pi: E \to D$ is a Lefschetz fibration with a unique singular point over $t = 0$ (in particular, X_0 has a single node). The vanishing sphere $L \subset X_1$ of this fibration is called an $|\mathcal{L}|$–vanishing sphere.

Obviously, every smooth divisor in the linear system $|\mathcal{L}|$ contains an $|\mathcal{L}|$–vanishing sphere, if $\mathcal{L}$ has zero defect. Two different maps $u, u': D \to H^0(Y, \mathcal{L})$ with $u(1) = u'(1)$ can give two $|\mathcal{L}|$–vanishing spheres in X_1 which are not Hamiltonian isotopic and even not homologous to each other, such as in the case of Lemma 4.1. However, $|\mathcal{L}|$–vanishing spheres are unique up to symplectomorphism.

Lemma 3.8   *Let $\mathcal{L} \to Y$ be a very ample line bundle over a Kähler manifold Y, $\mathrm{def}\,\mathcal{L} = 0$. Suppose X, X' are two smooth divisors in the linear system $|\mathcal{L}|$ and $L \subset X$, $L' \subset X'$ are two $|\mathcal{L}|$–vanishing Lagrangian spheres. Then there is a symplectomorphism $\psi: X \to X'$ such that $\psi(L) = L'$.*

This lemma is probably well known, but we don't have a clear reference for it, so we prove it here. An auxiliary lemma is required.

Lemma 3.9   *Let $\pi: X \to D \times [0, 1]$ be a smooth proper map and Ω a closed 2–form on X. Suppose that for every $s \in [0, 1]$, $X_{D;s} := \pi^{-1}(D \times \{s\})$, equipped with the restriction of Ω, is a Lefschetz fibration over D with a unique singularity over $0 \in D$. (In particular, the fibres of π are symplectic.) For $t \in D$, $s \in [0, 1]$ denote by $X_{t;s}$ the fibre $\pi^{-1}(\{t\} \times \{s\})$. Let $L_0 \subset X_{1;0}$ and $L_1 \subset X_{1;1}$ be vanishing spheres of the Lefschetz fibrations $X_{D;0}$ and $X_{D;1}$, respectively. Then there is a symplectomorphism $\psi: X_{1;0} \to X_{1;1}$ such that $\psi(L_0) = L_1$.*

Proof One can choose a smooth family of Lagrangian spheres $L_s \subset X_{1;s}$ such that L_s is vanishing for the fibration on $X_{D,s}$, and L_0, L_1 are the given spheres. This is easily seen from our definition or from [37, proof of Lemma 16.2].

Fix $s \in [0; 1]$ and let $\phi_\epsilon \colon X_{1;s} \to X_{1;s+\epsilon}$ be the parallel transport with respect to Ω [37, Section (15a)] along the s–direction. Look at $\phi_\epsilon(L_s)$ and $L_{s+\epsilon}$: these are two Lagrangian spheres in $X_{1;s+\epsilon}$ which coincide when $\epsilon = 0$, so they remain sufficiently close to each other for ϵ small enough, say $|\epsilon| < \epsilon(s)$. Being sufficiently close, the two spheres are Hamiltonian isotopic inside $X_{1;s+\epsilon}$. By composing ϕ_ϵ with this Hamiltonian isotopy, we get a symplectomorphism $\psi_\epsilon \colon X_{1;s} \to X_{1;s+\epsilon}$ taking L_s to $L_{s+\epsilon}$.

The open cover of $[0, 1]$ consisting of the intervals $\{(s - \epsilon(s), s + \epsilon(s))\}_{s \in [0;1]}$ admits a finite subcover. We know that, for s, s' within a single interval, L_s can be taken to L'_s by a symplectomorphism $X_{1;s} \to X_{1;s'}$; using the finite subcover, we are able to find a finite composition of such maps which is a symplectomorphism $X_{1;0} \to X_{1;1}$ taking L_0 to L_1. $\square$

Proof of Lemma 3.8 Let $u, u' \colon D \to \mathbb{P}H^0(Y, \mathcal{L})$ be two holomorphic maps as in Definition 3.7, and denote $X = X_{u(1)}$, $X' = X_{u'(1)}$. By Definition 3.7, $u(0), u'(0) \in \Delta^{\mathrm{reg}}$. Since Δ^{reg} is connected, one can find a path $\alpha(s) \in \Delta^{\mathrm{reg}}$ from $u(0)$ to $u'(0)$, $s \in [0, 1]$. Next one can find an s–parametric family of holomorphic disks $u_s \colon D \to \mathbb{P}H^0(Y, \mathcal{L})$ such that $u_0 = u$, $u_1 = u'$, $u_s(0) \in \Delta^{\mathrm{reg}}$ and $u_s(D)$ intersects Δ^{reg} transversely. Consider the space

$$E := \{(x, u_s(t)) : t \in D, \ s \in [0, 1], \ x \in X_{u(t)}\} \subset Y \times \mathbb{P}H^0(Y, \mathcal{L}).$$

It carries a closed 2–form which is the restriction of the product Kähler form to Y and $\mathbb{P}H^0(Y, \mathcal{L})$. There is also a canonical projection $E \to D \times [0, 1]$. With these data, E satisfies the conditions of Lemma 3.9. This lemma provides the desired symplectomorphism $\psi \colon X \to X'$ taking a given $|\mathcal{L}|$–vanishing sphere in X to a given one in X'. $\square$

3.4 Dehn twists

We recall the definition of Dehn twists from [37, Section (16c)]. First, one defines the Dehn twist as a compactly supported symplectomorphism of T^*S^n. Fix the standard round metric on S^n, and let $|\xi|$ be the norm function on T^*S^n. It is non-smooth at the 0–section; away from the 0–section, its Hamiltonian flow is the normalised geodesic flow. Take a function $b(r) \colon \mathbb{R} \to \mathbb{R}$ with compact support and such that $b(r) - b(-r) = -r$. The Dehn twist $\tau \colon T^*S^n \to T^*S^n$ is the 2π–flow of the Hamiltonian function $b(|\xi|)$. It extends smoothly to the 0–section by the antipodal map, thanks the special form

of $b(r)$. As result, τ is a compactly supported symplectomorphism of T^*S^n. Its behaviour in T^*S^n is well understood.

Theorem 3.10 (1) *The map τ has infinite order in* $\operatorname{Symp}^c(T^*S^n)/\operatorname{Ham}^c(T^*S^n)$, *the group of compactly supported symplectomorphisms of T^*S^n modulo compactly supported symplectic isotopy.*

(2) *If n is even, τ has finite order in* $\pi_0 \operatorname{Diff}^c(T^*S^n)$, *the group of compactly supported diffeomorphisms of T^*S^n modulo compactly supported isotopy* [21]. □

When $n = 2$ it is further known that τ generates $\pi_0 \operatorname{Symp}^c(T^*S^2) \cong \mathbb{Z}$, and τ^2 is smoothly isotopic to Id in $\operatorname{Diff}^c(T^*S^2)$ [38]; see also [2, Theorem 1.21].

Next, if $L \subset X$ is a Lagrangian sphere in any symplectic manifold, a neighbourhood of L in X is symplectomorphic to a neighbourhood of the 0–section in T^*S^n. So one can pull back τ using this symplectomorphism and then extend it by the identity to get a map $\tau_L \colon X \to X$. It is a symplectomorphism uniquely defined up to Hamiltonian isotopy (once a parametrisation of L is fixed), supported in a neighbourhood of L.

Definition 3.11 (Dehn twist) The symplectomorphism $\tau_L \colon X \to X$ is called the Dehn twist around L.

Lemma 3.12 (Picard–Lefschetz formula [23]) *If* $\dim_{\mathbb{R}} X = 2n$ *and $L \subset X$ is a Lagrangian sphere, then $(\tau_L)_*$ acts by Id on $H_i(X)$, $i \neq n$. For any $[A] \in H_n(X)$,*

$$(\tau_L)_*[A] = [A] - \epsilon \cdot ([L] \cdot [A])[L].$$

Here $\epsilon = (-1)^{\frac{1}{2}n(n-1)}$. Consequently:

(1) *If n is even, then $(\tau_L)_*^2$ acts by Id on $H_*(X)$.*

(2) *If n is odd and $[L] \in H_n(X; \mathbb{R})$ is non-zero, then $(\tau_L)_*$ is an automorphism of infinite order of $H_*(X)$.* □

Summarising Theorem 3.10(2) and Lemma 3.12(2), we arrive at the following well known statement.

Corollary 3.13 *Let X be a compact symplectic manifold, $\dim X_{\mathbb{R}} = 2n$, and $L \subset X$ a Lagrangian sphere non-zero in $H_n(X; \mathbb{R})$.*

(1) *If n is even, τ_L has finite order in $\pi_0 \operatorname{Diff}(X)$.*

(2) *If n is odd, τ_L has infinite order in $\pi_0 \operatorname{Diff}(X)$.* □

The next lemma relates Dehn twists and Lefschetz fibrations; see [37, Section (15b)] for details.

Lemma 3.14 [36; 37] *Let (E, Ω, π) be a Lefschetz fibration with a unique singularity. Let E_1 be its regular fibre and $L \subset E_1$ a vanishing Lagrangian sphere. Then the Dehn twist $\tau_L: E_1 \to E_1$ is Hamiltonian isotopic to the symplectic monodromy map $E_1 \to E_1$ obtained by applying symplectic parallel transport to the fibres E_t along the circle $t \in \partial D$.* □

Remark 3.15 Let X be a symplectic manifold and $L \subset X$ a Lagrangian sphere; assume L is non-zero in $H_n(X)$. There are three main previously known cases when τ_L has infinite order in $\mathrm{Symp}(X)/\mathrm{Ham}(X)$ (if X is non-compact, consider $\mathrm{Symp}^c(X)/\mathrm{Ham}^c(X)$ instead):

(1) $\frac{1}{2} \dim_{\mathbb{R}} X$ is odd, as explained above.

(2) X is exact with contact-type boundary, and L is exact (Seidel, unpublished).

(3) X is Calabi–Yau, and there is another Lagrangian sphere L' intersecting L once transversely [35].

Let $X = \mathrm{Bl}_k \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ in k generic points, $2 \leq k \leq 8$, with the monotone symplectic form, and $L \subset X$ be any Lagrangian sphere. Seidel [38] showed that τ_L has order 2 in $\mathrm{Symp}(X)/\mathrm{Ham}(X)$ when $k = 2, 3, 4$ and has order greater than 2 when $k = 5, 6, 7, 8$, but did not prove it was infinite. Note that $X = \mathrm{Bl}_6 \mathbb{P}^2$ is the cubic surface $X \subset \mathbb{P}^3$, to which Theorem 1.2 applies.

4 Constructing invariant Lagrangian spheres

The aim of this section is to state and prove Proposition 4.2, which will later be used to prove Theorem 1.13. We start by stating an essentially known lemma which can be used to prove the simple case of Theorem 1.2 when $\dim_{\mathbb{C}} X$ is odd.

Lemma 4.1 *Let $\mathcal{L}$ be a very ample line bundle over a Kähler manifold Y. For any $d \geq 3$, every smooth divisor $X \subset Y$ in the linear system $|\mathcal{L}^{\otimes d}|$ contains two $|\mathcal{L}^{\otimes d}|$–vanishing Lagrangian spheres L_1, L_2 that intersect once, transversely.* □

The proposition below should be considered as an equivariant version of Lemma 4.1. It will be used to prove the harder case of Theorem 1.13 when $\dim_{\mathbb{C}} X$ is even.

Proposition 4.2 *Let $\mathcal{L}$ be a very ample line bundle over a Kähler manifold Y, and let $\iota\colon Y \to Y$ be a holomorphic involution which lifts to an automorphism of $\mathcal{L}$. Fix $d \geq 3$ and let $H^0(Y, \mathcal{L}^{\otimes d})_\pm$ denote the ± 1–eigenspaces of the involution on $H^0(Y, \mathcal{L}^{\otimes d})$ induced by ι. Let $\Pi_\pm$ be as in Theorem 1.13. Pick a connected component $\widetilde{\Sigma}$ of $Y^\iota \subset Y$, $\dim \widetilde{\Sigma} \geq 2$. Suppose one of the following holds:*

(a) *d is even.*

(b) *d is odd, $\widetilde{\Sigma} \subset \Pi_+$, and the linear system $\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d})_+$ contains a smooth divisor.*

Then there is a smooth divisor X in the linear system $|\mathcal{L}^{\otimes d}|$ and two $|\mathcal{L}^{\otimes d}|$–vanishing Lagrangian spheres $L_1, L_2 \subset X$ such that:

(1) *$\iota(X) = X$, $\Sigma := X \cap \widetilde{\Sigma}$ is smooth, $\dim \Sigma = \dim \widetilde{\Sigma} - 1$.*

(2) *$\iota(L_1) = L_1$, $\iota(L_2) = L_2$.*

(3) *L_1, L_2 intersect transversely at a single point which belongs to Σ.*

(4) *$L_i^\iota = L_i \cap \Sigma$ are Lagrangian spheres in Σ, $i = 1, 2$.*

(5) *for $i = 1, 2$ one can choose a symplectomorphism τ_{L_i} of X representing the Hamiltonian isotopy class of the Dehn twist around L_i such that τ_{L_i} commutes with ι, and $\tau_{L_i}|_{X^\iota}$ is the Dehn twist around L_i^ι.*

The same is true if we replace symbols $+$ with $-$ in case (b).

4.1 A_2 chains of Lagrangian spheres from A_2 fibrations

Definition 4.3 (A_2 chain of Lagrangian spheres) Let X be a symplectic manifold. A pair (L_1, L_2) of two Lagrangian spheres in X is called an A_2 chain if L_1 and L_2 intersect at a single point, and the intersection is transverse.

In Section 3 we have seen that one can construct Lagrangian spheres as vanishing cycles of Lefschetz fibrations. Similarly, one can get A_2 chains of Lagrangian spheres from fibrations with slightly more complicated singularities.

Definition 4.4 (A_2 fibration) Denote by $D \subset \mathbb{C}$ the open unit disk, and by $B_\epsilon \subset \mathbb{C}$ the open disk of radius ϵ. Both disks are centred at 0.

Suppose E is a smooth manifold, Ω a closed 2–form on E and $\pi\colon E \to D$ is a smooth map. The triple (E, Ω, π) is called an A_2 fibration if there is a point $p \in E$ (without loss of generality, we assume $\pi(p) = 0 \in D$), and a neighbourhood $U(p)$ such that:

- All but a finite number of fibres of π are regular, and the restriction of Ω is symplectic on them.

- There exists a complex structure on $U(p)$ with a holomorphic chart $x_1, \ldots, x_n$, $x_i \in B_\epsilon$ such that

$$\pi(x_1, \ldots, x_n) = x_1^2 + \cdots + x_{n-1}^2 + h(x_n),$$

where $h(x_n)$ is holomorphic.

- $h(x_n)$ has at least three roots within $B_{\epsilon/2}$, counted with multiplicities.

- For any $x_n \in B_{\epsilon/2}$, $\sqrt{h(x_n)} \in B_{\epsilon/2}$.

- $\Omega|_{U(P)}$ is Kähler with respect to the above complex structure.

Remark 4.5 The definition allows π to have singularities outside of $U(p)$. Also, the definition does not require $p\colon E \to D$ to be a proper map, so the smooth fibres E_t need not be symplectomorphic, as we may not be able to integrate the parallel transport vector fields. The generality of this definition is slightly unusual, but it makes no difference to the local construction of A_2 chains of Lagrangian spheres, which is the next thing we discuss.

In order to prove Proposition 4.2, we need to introduce A_2 fibrations with involutions.

Definition 4.6 (Involutive A_2 fibration) Let (E, Ω, π) be an A_2 fibration. It is called an involutive A_2 fibration with involution $\iota\colon E \to E$ if in the holomorphic chart from Definition 4.4 we have in addition

$$\iota(x_1, \ldots, x_l, x_{l+1}, \ldots, x_n) = (-x_1, \ldots, -x_l, x_{l+1}, \ldots, x_n)$$

for some $l < n$. We denote by E^ι the fixed locus of ι.

Remark 4.7 It follows from this definition that $\pi|_{E^\iota}\colon E^\iota \to D$ is also an A_2 fibration. Note that $x \in E^\iota$ is regular for π if and only if it is regular for $\pi|_{E^\iota}$. Indeed, we can decompose $T_x E = T_x E^\iota \oplus N_x$ where N_x is the (-1)–eigenspace of $d\iota(x)$. Since $\pi\iota = \pi$, $N_x \subset \ker d\pi(x)$. So $\operatorname{rk} d\pi(x) = \operatorname{rk} d\pi(x)|_{T_x E^\iota}$. Consequently, for a regular fibre E_t, the fixed locus E_t^ι is smooth.

The following is a slight refinement of [20, Lemma 6.12].

Lemma 4.8 Let $\pi\colon E \to D$ be an A_2 fibration. Then for every sufficiently small $t \in D$ such that the fibre $E_t := \pi^{-1}(t)$ is smooth, E_t contains an A_2 chain of Lagrangian spheres.

We will use the following equivariant analogue of this lemma.

Lemma 4.9 Let $\pi\colon E \to D$ be an involutive A_2 fibration with an involution ι. Then for every sufficiently small $t \in D$ such that the fibre $E_t := \pi^{-1}(t)$ is smooth, E_t contains an A_2 chain of Lagrangian spheres (L_1, L_2) which satisfy properties (2)–(5) from Proposition 4.2 with $X := E_t$, and Σ being the connected component of E_t^ι which is a subset of the connected component of the point p in E^ι.

Remark 4.10 Note that $\dim \Sigma = l - 1$, where l is the number coming from the coordinate chart in Definition 4.6.

Proof of Lemma 4.8 Let $U'(p) \subset U(p)$ be the ball around p given by $|x_i| < \epsilon/2$, $i = 1, \ldots, n$. As in Section 3.1, it suffices to assume that $\Omega|_{U'(p)}$ is the standard symplectic form in the holomorphic chart $(x_1, \ldots, x_n)$ from Definition 4.4.

The condition that E_t is smooth means the equation $h(x_n) = t$ has no multiple roots with $x_n \in B_{\epsilon/2}$. Therefore by Definition 4.4, the equation $h(x_n) = 0$ has at least three roots with $x_n \in B_{\epsilon/2}$. So for sufficiently small t the equation $h(x_n) = t$ also has at least three distinct roots with $x_n \in B_{\epsilon/2}$. Pick three such roots, say $z_1, z_2, z_3 \in B_{\epsilon/2}$: $h(z_i) = t$. Let $\gamma_{12} \subset B_{\epsilon/2}$ be a path from z_1 to z_2 whose interior avoids the roots of $h - t$. Define

$$L_1 := \bigsqcup_{z \in \gamma_{12}} \{(x_1, \ldots, x_n) \in B_{\epsilon/2} \cap \pi^{-1}(t) : |x_i| \in \mathbb{R} \cdot \sqrt{-h(z)}\}.$$

This is a smooth Lagrangian sphere in $\pi^{-1}(t)$ with respect to the restriction of the standard symplectic form on $\mathbb{C}^n$ to $\pi^{-1}(t)$. Similarly, let $\gamma_{23} \subset B_{\epsilon/2} \subset \mathbb{C}$ be a path from z_2 to z_3 and define L_2 by the same formula replacing γ_{12} by γ_{23}. If γ_{12} and γ_{23} are transverse at their common endpoint z_2, then (L_1, L_2) is an A_2 chain of Lagrangian spheres by [20, Lemma 6.12]. Note that L_1, L_2 lie in $U'(p)$ by the fourth condition in Definition 4.4. $\square$

Proof of Lemma 4.9 We use notation from the proof of Lemma 4.8. Arguing ι–invariantly as in that proof, we can again assume Ω is standard on $U'(p)$. The formulas for L_1, L_2 are invariant under the change $x_i \mapsto -x_i$, $i \le l$, so L_1, L_2 are ι–invariant. This proves property (2) from Proposition 4.2. Next, we already know L_1 intersects L_2 transversely at a single point. This point has coordinates $x_1 = 0, \ldots, x_{n-1} = 0$, $x_n = z_2$. (Recall z_2 is a root of $h(x_n) - t$.) This intersection point is ι–invariant, and it obviously belongs to the connected component of the point p in E^ι, so property (3) from Proposition 4.2 holds. Property (4) is true because E^ι locally around π is given by $x_1 = \cdots = x_l = 0$, and so $L_i \cap \Sigma$ are transverse

Lagrangians for the same reason that the L_i are. By their local construction, the L_i do not intersect the connected components of E_t^ι other than Σ.

It remains to explain property (5). Let $S^{n-1} \subset \mathbb{R}^n$ be the standard unit sphere. Let ι_0 be the involution on S^n which changes the sign of the first k coordinates on $\mathbb{R}^n$. It naturally extends to an involution ι_0 on T^*S^n. It is not hard to check that there is an (ι, ι_0)–equivariant diffeomorphism $V(L_1) \to V(S^n)$, where $V(L_1)$ is an ι–invariant tubular neighbourhood of $L_1 \subset X$ and $V(S^n)$ is an ι_0–invariant tubular neighbourhood of the 0–section in T^*S^n. Then there is also an (ι, ι_0)–equivariant symplectomorphism $V(L_1) \to V(S^n)$, by an equivariant analogue of the Weinstein tubular neighbourhood theorem. The Dehn twist in T^*S^n is ι_0–equivariant by definition. Its pullback via the equivariant symplectomorphism $V(L_1) \to V(S^n)$ is the desired ι–equivariant Dehn twist inside E_t. $\square$

4.2 A_2 fibrations of divisors from projective embeddings

One way of constructing an A_2 fibration is to embed all its fibres E_t as divisors $E_t = X_t \subset Y$ in a single Kähler manifold Y. This idea can be used to prove Lemma 4.1, and now we will run such an argument ι–invariantly to prove Proposition 4.2.

Proof of Proposition 4.2 Let us recall the setting. We have a very ample line bundle $\mathcal{L} \to Y$ over a Kähler manifold Y, and a holomorphic involution $\iota\colon Y \to Y$ which lifts to an involution on $\mathcal{L}$. This means ι induces a linear involution on $H^0(Y, \mathcal{L})^*$ splitting it into the direct sum of ± 1 eigenspaces, denoted by $H^0(Y, \mathcal{L})^*_\pm$. The projectivisations of these eigenspaces are denoted by $\Pi_\pm \subset \mathbb{P}H^0(Y, \mathcal{L})^*$. We also denote $\mathbb{P}^N := \mathbb{P}H^0(Y, \mathcal{L})^*$, and the ι–induced involution on $\mathbb{P}^N$ by $\iota_{\mathbb{P}^N}$. The fixed locus of $\iota_{\mathbb{P}^N}$ is $\Pi_+ \sqcup \Pi_- \subset \mathbb{P}^N$.

Because $\mathcal{L}$ is very ample, we have an embedding $Y \subset \mathbb{P}^N$, $\mathcal{L} = \mathcal{O}_Y(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_Y$, Y is invariant under $\iota_{\mathbb{P}^N}$ and $\iota_{\mathbb{P}^N}|_Y = \iota$, and also

$$Y^\iota = (Y \cap \Pi_+) \sqcup (Y \cap \Pi_-).$$

Let $\widetilde{\Sigma}$ be the given connected component of Y^ι (smooth by assumption), and $\dim \widetilde{\Sigma} = l$. Then $\widetilde{\Sigma} \subset \Pi_\epsilon$, where ϵ is one of the two symbols $+$ or $-$. We will also denote by ϵ the correspondingly signed number ± 1.

Choose homogeneous coordinates $(x_0 : \ldots : x_l : x_{l+1} : \ldots : x_N)$ on $\mathbb{P}^N$ with the following properties:

(1) $\iota_{\mathbb{P}^N}(x_0 : \ldots : x_l : x_{l+1} : \ldots : x_N) = (\epsilon x_0 : \ldots : \epsilon x_l : \pm x_{l+1} : \ldots : \pm x_{N+1})$.

(2) $(1 : 0 : \ldots : 0) \in \widetilde{\Sigma}$.

(3) The plane spanned by $(x_0, \ldots, x_l)$ (other coordinates are set to 0) is the tangent plane to $\widetilde{\Sigma}$ at $(1:0:\ldots:0)$.

(4) For some $n \geq l$, the plane spanned by $(x_0, \ldots, x_n)$ (other coordinates are set to 0) is the tangent plane to Y at $(1:0:\ldots:0)$.

The third property implies that $x_0, \ldots, x_l$, seen as sections in $H^0(\mathcal{O}_{\mathbb{P}^N}(1))$, belong to the ϵ–eigenspace of ι. This is in agreement with the first property. So coordinates with these properties exist.
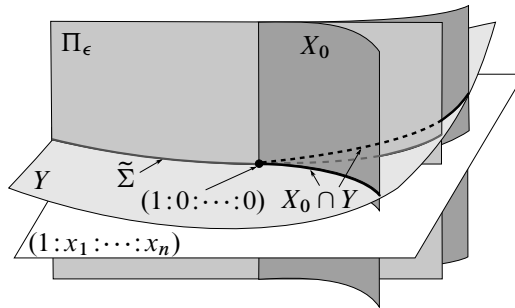

Figure 4: A divisor X_0 from the family X_t constructed in the proof of Proposition 4.2

In the affine chart $x_0 = 1$, the coordinates $(x_1, \ldots, x_n)$ are local coordinates for Y near the origin. In the chart $x_0 = 1$, write (see Figure 4)

$$X_t := x_1^3 + x_2^2 + \cdots + x_n^2 - t.$$

We want X_t to be a section of $\mathcal{O}_{\mathbb{P}^N}(d)$, so in projective coordinates we set

$$X_t := x_0^{d-3} x_1^3 + x_0^{d-2}(x_2^2 + \cdots + x_n^2) - t x_0^d.$$

From property (1) of the coordinates x_i, we see that $X_t \circ \iota = \epsilon^d X_t$ as polynomials. In other words:

(a) If d is even, $X_t \in H^0(\mathcal{O}_{\mathbb{P}^N}(d))_+$.

(b) If d is odd, $X_t \in H^0(\mathcal{O}_{\mathbb{P}^N}(d))_\epsilon$.

For all t, the divisors $\{X_t = 0\}$ and $\{X_t = 0\} \cap Y$ are reducible and hence singular. We want to smooth the family $\{X_t = 0\} \cap Y$ so that a generic divisor in this t–family becomes non-singular.

Suppose d is even. Then the linear system $H^0(\mathcal{O}_{\mathbb{P}^N}(d))_+$ has no base locus as it contains all monomials x_i^d. Then $H^0(\mathcal{O}_Y(d))_+ = H^0(Y, \mathcal{L}^{\otimes d})_+$ also has no base locus. By Bertini's theorem in characteristic 0, there exists $F \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))_+$ such that the divisor $\{F = 0\} \cap Y$ is smooth.

Suppose d is odd. Then the linear systems $H^0(\mathcal{O}_{\mathbb{P}^N}(d))_{\pm}$ have non-empty base loci, namely $\Pi_{\mp}$ (see the proof of Lemma 1.14 below). Therefore it is not a priori clear that these linear systems contain a smooth divisor. This condition is included in the assumptions of Proposition 4.2, case (b). Let $F \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))_{\epsilon}$ be a polynomial such that $\{F = 0\} \cap Y$ is smooth.

The rest of the proof is the same for even and odd d. For all generic $\delta \in \mathbb{C}$, the divisors $\{X_t + \delta F = 0\} \cap Y$ are smooth except for a finite number of t's. Recall that $(x_1, \ldots, x_n)$ is a holomorphic chart for Y around $(1 : 0 : \ldots : 0)$. There is another chart $\tilde{x}_1, \ldots, \tilde{x}_n$ in which the divisors $\{X_t + \delta F = 0\} \cap Y$ are given by

$$h(\tilde{x}_1) + \tilde{x}_2^2 + \cdots + \tilde{x}_n^2 - t + c = 0,$$

where $h(\tilde{x}_1)$ is close to $\tilde{x}_1^3$ (when δ is small) and c is a small constant. Moreover, the change of coordinates from x_i to $\tilde{x}_i$ is ι–equivariant. This follows from an equivariant version of the holomorphic Morse splitting lemma [1].

Consider the family $\{X_t + \delta F = 0\} \cap Y$ of divisors in Y, $t \in D$. They are ι–invariant and belong to the linear system $|\mathcal{L}^{\otimes d}|$. Let $E \to D$ be the total space of this family; see Definition 3.6. It may be singular; if it is, remove its singular locus to get E_0. The involution ι turns $E_0 \to D$ into an involutive fibration in the sense of Definition 4.6. So by Lemma 4.9 a smooth divisor in the family $\{X_t + \delta F = 0\} \cap Y$ has a pair of Lagrangian spheres (L_1, L_2) that satisfy properties (2)–(5) of Proposition 4.2. It is easy to see that Lemma 4.9 constructs L_1, L_2 which are $|\mathcal{L}^{\otimes d}|$–vanishing.

It remains to check that ι satisfies property (1). We have to show that smooth divisors $\{X_t + \delta F = 0\} \cap Y$ intersect $\Sigma = \tilde{\Sigma} \cap Y$ transversely. Suppose $X := \{X_t + \delta F = 0\} \cap Y$ intersects Σ non-transversely at one point p, so $T_p\Sigma \subset T_pX$ (the tangent spaces are taken inside Y). This means T_pX contains $\dim \Sigma$ positive ($+1$) eigenvalues of $d\iota$. Then the same must hold for all intersection points $X \cap \Sigma$, and hence $T_p\Sigma \subset T_pX$ for any $p \in X \cap \Sigma$. But in a neighbourhood of $(1 : 0 : \ldots : 0)$ the intersection $X \cap \Sigma$ is transverse, which is easily verified in the local chart $(x_1, \ldots, x_n)$ from above. So X intersects Σ transversely everywhere. Similarly, every other connected component of Y^ι either intersects X transversely or is contained in X. □

5 Proofs of theorems about Lagrangian spheres in divisors

Proof of Theorem 1.13 Apply Proposition 4.2 to $Y, \mathcal{L}, \tilde{\Sigma}$ given in the hypothesis of Theorem 1.13. Proposition 4.2 returns an $|\mathcal{L}^{\otimes d}|$–divisor $X \subset Y$ and $|\mathcal{L}^{\otimes d}|$–vanishing Lagrangian spheres $L_1, L_2 \subset X$ satisfying the conditions enumerated in the proposition. Because $|\mathcal{L}^{\otimes d}|$–vanishing spheres are unique up to symplectomorphism (Lemma 3.8),

it suffices to show that τ_{L_1} has infinite order in $\mathrm{Symp}(X)/\mathrm{Ham}(X)$. To show this, we compute the Lefschetz number of

$$\tau_{L_1}^{2k}\tau_{L_2}^{2k}|_{X^\iota} = \tau_{L_1^\iota}^{2k}\tau_{L_2^\iota}^{2k}$$

on $H^*(X^\iota)$, where X^ι is the fixed locus of the involution ι on X. Recall that $\Sigma = \widetilde{\Sigma}\cap X$ is a connected component of X^ι. We are given that $\dim\widetilde{\Sigma}$ is even, so $\dim\Sigma = \dim\widetilde{\Sigma}-1$ is odd. Let $X^\iota = \Sigma \sqcup \Sigma_0$ where Σ_0 consists of all other connected components. We identify $H^*(X^\iota)$ with $H_*(X^\iota)$ via Poincaré duality.

Consider the homology classes $[L_1^\iota], [L_2^\iota] \in H_*(\Sigma)$, $[L_1^\iota]\cdot[L_2^\iota] = 1$. Using the Picard–Lefschetz formula (see Section 3.4) and property (5) from Proposition 4.2, we write down the actions of Dehn twists on the 2–dimensional vector space $\mathrm{span}\{[L_1^\iota], [L_2^\iota]\} \subset H_*(X^\iota)$. Let $s = \dim_{\mathbb{C}}\Sigma$ and $\epsilon = (-1)^{\frac{1}{2}s(s-1)}$. These actions are given by

$$(\tau_{L_1^\iota})_*^{2k} : \begin{pmatrix} 1 & k(1+(-1)^{s-1})\epsilon \\ 0 & 1 \end{pmatrix}, \quad (\tau_{L_2^\iota})_*^{2k} : \begin{pmatrix} 1 & 0 \\ k(1+(-1)^{s-1})\epsilon & 1 \end{pmatrix}.$$

Now since $s = \dim_{\mathbb{C}}\Sigma$ is odd, we see that

$$\mathrm{STr}\big((\tau_{L_1^\iota})_*^{2k}(\tau_{L_2^\iota})_*^{2k}|_{\mathrm{span}\{[L_1^\iota],[L_2^\iota]\}}\big) = -4k^2 - 2.$$

(The negative signs appear because we are computing the supertrace). If s were even, we would get the constant 2 instead.

We can extend $[L_1^\iota], [L_2^\iota]$ to a basis of $H_*(X^\iota)$ in which all other elements have zero intersection with $[L_1^\iota], [L_2^\iota]$. By the Picard–Lefschetz formula, $(\tau_{L_i^\iota})_*$ acts by Id on the rest of such a basis. Consequently, the Lefschetz number is

$$L\big((\tau_{L_1^\iota})^{2k}(\tau_{L_2^\iota})^{2k}\big) = -4k^2 + c,$$

where c is a constant independent of k. By Proposition 1.4,

(5-1) $$\dim_\Lambda \mathrm{HF}^*(\tau_{L_1}^{2k}\tau_{L_2}^{2k}) \geq |-4k^2 + c|.$$

Suppose $\tau_{L_1}^{2k}$ is Hamiltonian isotopic to Id for some $k > 0$. Then $\tau_{L_2}^{2k}$ is also Hamiltonian isotopic to Id, because by Lemma 3.8 there is a symplectomorphism of X taking L_1 to L_2. Then the product $\tau_{L_1}^{2k}\tau_{L_2}^{2k}$ is also Hamiltonian isotopic to Id. Since k can be taken arbitrarily large, this contradicts the growth of dimensions in Equation (5-1). Consequently τ_{L_1} has infinite order in the group $\mathrm{Symp}(X)/\mathrm{Ham}(X)$. $\qquad\square$

Next we prove Lemma 1.14. It follows from a strong Bertini theorem which we now quote.

Theorem 5.1 [10, Corollary 2.4] *Let Y be a compact smooth complex manifold and S an effective linear system of divisors on Y. Let B be the base locus of S. If B is reduced and non-singular, and $\dim B < \frac{1}{2} \dim Y$, then a generic divisor in S is smooth.* □

If B is disconnected, the dimensional inequality must hold for every connected component of B.

Proof of Lemma 1.14 We repeat the beginning of the proof of Proposition 4.2. We have $Y \subset \mathbb{P}^N$ and $\mathcal{L}^{\otimes d} = \mathcal{O}_{\mathbb{P}^N}(d)|_Y$. The involution ι acts on sections of $\mathcal{L}$ and so acts on $\mathbb{P}^N$ by a linear involution $\iota_{\mathbb{P}^N}$, and $Y \subset \mathbb{P}^N$ is invariant under it. Find homogeneous coordinates $(x_0 : \ldots : x_N)$ such that

$$\iota_{\mathbb{P}^N}(x_0 : \ldots x_l : x_{l+1} : \ldots : x_N) = (x_0 : \ldots : x_l : -x_{l+1} : -x_N).$$

Recall that d is odd by assumption. Then $H^0(\mathcal{O}_{\mathbb{P}^N}(d))_+$ consists of degree-d polynomials which are sums of monomials of the form

$$x_0^{\text{odd}} \cdots x_l^{\text{odd}} x_{l+1}^{\text{even}} \cdots x_N^{\text{even}}.$$

Here *even* or *odd* denote the parity of a power. The base locus of the linear system $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^N}(d))_+$ is given by

$$x_0 = 0, \quad \ldots, \quad x_l = 0,$$

and so coincides with Π_-. The base locus B of $\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d})_+$ is therefore $\Pi_- \cap Y$. It is smooth because Y^ι is smooth. We are also given that $\dim B < \frac{1}{2} \dim Y$ by hypothesis. Finally, we know that $\iota_{\mathbb{P}^N}|_{\Pi_-} = \text{Id}$, so Y intersects Π_- cleanly (ie transversely in the normal direction to $\Pi_- \cap Y$), and hence $B = \Pi_- \cap Y$ is reduced. Consequently, Lemma 1.14 follows from Theorem 5.1. (The case when the sign symbols $+$ and $-$ are interchanged is analogous.)   □

We now return to divisors in Grassmannians and prove Theorem 1.2. Let $\text{Gr}(k, n) \subset \mathbb{P}^N$ be the Plücker embedding. Then the anti-canonical class of $\text{Gr}(k, n)$ equals $\mathcal{O}_{\mathbb{P}^N}(n)|_{\text{Gr}(k,n)}$ [25, Proposition 1.9]. Consequently, a smooth divisor $X \subset \text{Gr}(k, n)$ in the linear system $\mathcal{O}_{\mathbb{P}^N}(d)|_{\text{Gr}(k,n)}$ satisfies the W^+ condition (see Definition 2.1) if and only if $d \leq n$ or $d \geq k(n-k) + n - 2$, and X is monotone (Fano) if and only if $d < n$.

Proof of Theorem 1.2 We have already mentioned that this theorem is easy and essentially known when $k(n-k)$ is even. (The sphere $L \subset X$ is non-trivial in $H_n(X)$ by Lemmas 4.1 and 3.8. Then apply Corollary 3.13(2).) We will now prove the hard

case when $k(n - k)$ is odd using the general Theorem 1.13. Denote $k = 2p + 1$, $n = 2q$.

Consider a linear involution on $\mathbb{C}^{2q}$ with $q + l$ positive eigenvalues and $q - l$ negative eigenvalues for some l. It induces a non-degenerate involution ι on $\mathrm{Gr}(2p + 1, 2q)$ whose fixed locus is

$$\mathrm{Gr}(2p + 1, 2q)^\iota = \bigsqcup_{t=0}^{2p+1} \mathrm{Gr}(t, q + l) \times \mathrm{Gr}(2p + 1 - t, q - l).$$

This fixed locus consists of $(2p + 1)$–planes that admit a frame in which t vectors lie in the positive eigenspace of the involution on $\mathbb{C}^{2q}$, and the remaining $2p + 1 - t$ vectors lie in the negative eigenspace. We compute

$$(5\text{-}2) \quad \dim \mathrm{Gr}(t, q + l) + \dim \mathrm{Gr}(2p + 1 - t, q - l) - \tfrac{1}{2} \dim \mathrm{Gr}(2p + 1, 2q)$$
$$= -\tfrac{1}{2}(1 + 2p - 2t)(1 + 2p + 2l - 2t).$$

For this paragraph, set $l = 0$. Then the expression (5-2) is less than 0 for any $t \in \mathbb{Z}$. This means $\dim \mathrm{Gr}(2p+1, 2q)^\iota < \tfrac{1}{2} \dim \mathrm{Gr}(2p+1, 2q)$. (The left-hand side is disconnected, and we mean that the inequality holds for each of its connected components.) Therefore we can apply Lemma 1.14 to either of the two linear systems $\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d})_\pm$. In order to apply Theorem 1.13, it remains to check that $\mathrm{Gr}(2p + 1, 2q)^\iota$ contains a connected component of even dimension. A computation shows that a connected component of $\mathrm{Gr}(2p + 1, 2q)^\iota$ has dimension of parity

$$\dim \mathrm{Gr}(t, q) + \dim \mathrm{Gr}(2p + 1 - t, q) \equiv q - 1 \quad \mathrm{mod}\ 2$$

independently of t. We will now consider the case when q is odd, and will discuss the case when q is even in the next paragraph. If d is odd, apply Theorem 1.13(b) taking either of the two sign symbols $+$ or $-$. If d is even, apply Theorem 1.13(a) (this case is easier and does not require the computation of dimensions we have made). This proves Theorem 1.2 for $\mathrm{Gr}(2p + 1, 2q)$ in the case when q is odd.

Now suppose q is even. Set $l = 1$ until the end of the proof. Recall that

$$\mathrm{Gr}(2p + 1, 2q)^\iota = (\Pi_+ \sqcup \Pi_-) \cap \mathrm{Gr}(2p + 1, 2q).$$

The only case when (5-2) fails to be less than zero is when

$$1 + 2p - 2t = -1.$$

This happens for a unique $t \in \mathbb{Z}$. So either

$$\dim \mathrm{Gr}(2p + 1, 2q) \cap \Pi_+ < \tfrac{1}{2} \dim \mathrm{Gr}(2p + 1, 2q),$$

or the same holds with Π_- taken instead. (As above, we mean that the inequality holds for each connected component of the left-hand side.) A computation shows that a connected component of $\mathrm{Gr}(2p + 1, 2q)^t$ has dimension of parity

$$\dim \mathrm{Gr}(t, q + 1) + \dim \mathrm{Gr}(2p + 1 - t, q - 1) \equiv q \bmod 2 \equiv 0 \bmod 2.$$

Therefore we can apply Lemma 1.14 and Theorem 1.13 taking that symbol $+$ or $-$ for which the inequality

$$\dim \mathrm{Gr}(2p + 1, 2q) \cap \Pi_{\mp} < \tfrac{1}{2} \dim \mathrm{Gr}(2p + 1, 2q)$$

holds. Theorem 1.2 is proved in all cases. $\qquad\square$

Proof of Corollaries 1.3 and 1.15. These corollaries follow from Theorems 1.2 and 1.13 and Lemma 3.14. $\qquad\square$

Appendix: Growth of Lagrangian Floer cohomology and ring structures

A.1 Dehn twists around spheres with deformed cohomology

Keating [19] has recently obtained an exact sequence involving iterated Dehn twists in the Fukaya category of a symplectic manifold, extending Seidel's original exact sequence [36]. In this subsection we use it to prove Proposition A.1, which is stated below. Then we apply it to compute Floer cohomology rings of vanishing spheres in some divisors.

Let X be a compact monotone symplectic manifold. Denote by $\mathcal{F}(X)$ its monotone Fukaya category over $\mathbb{C}$, which is a collection of A_∞–categories $\mathcal{F}(X)_\lambda$, $\lambda \in \mathbb{C}$, corresponding to the eigenvalues of multiplication with $c_1(X)$ in $\mathrm{QH}^*(X)$. The basic language of A_∞ and Fukaya categories is explained in [37], and the monotone version of the Fukaya category is discussed in [41]. Our aim is to prove the following.

Proposition A.1 *Let X be a monotone symplectic manifold, $\dim_\mathbb{R} X = 4k$, $L_1 \subset X$ be a Lagrangian sphere and $L_2 \subset X$ another monotone Lagrangian which intersects L_1 once, transversely. Assume L_1, L_2 are included into the same summand $\mathcal{F}(X)_\lambda$. Suppose that $\dim \mathrm{HF}^*(\tau_{L_1}^k L_2, L_2) > 2$ for some $k \in \mathbb{N}$. Then there is an isomorphism of rings $\mathrm{HF}^*(L_1, L_1) \cong \mathbb{C}[x]/x^2$.*

All A_∞–algebras and modules in this section are assumed to be minimal. We need to introduce some notation.

Definition A.2 Let A be a strictly unital $\mathbb{Z}/2$–graded A_∞–algebra with unit $1 \in A$, M a right A_∞–module over A and N a left A_∞–module over A. Fix an augmentation, ie a vector space splitting $A = (1) \oplus \bar{A}$. The k–truncated bar complex is the vector space

$$(M \otimes_A N)_k := \bigoplus_{j=0}^{k-1} M \otimes \bar{A}^{\otimes j} \otimes N$$

with the differential that on the jth summand equals

$$(A\text{-}1) \qquad \sum_{\substack{j+2=p+q+r \\ p,r \geq 0, \, q \geq 2}} (-1)^{\maltese}(-1)^r (\mathrm{Id}^{\otimes p} \otimes \mu^q \otimes \mathrm{Id}^{\otimes r}).$$

Here $\maltese \in \{0,1\}$ depends on the gradings of the arguments: if the input is $m \otimes x_1 \otimes \cdots \otimes x_{k-1} \otimes n$, where $m \in M$, $x_i \in A$, $n \in N$, then $\maltese$ is the sum of gradings of the last r elements of the input. If we put $p=0$ in (A-1), we get the summand $\mu^q \otimes \mathrm{Id}^{\otimes r}$ which involves the module structure map $\mu^q \colon M \otimes A^{\otimes(q-1)} \to M$. Similarly, when we put $r=0$ in (A-1), μ^q will be the module structure map $\mu^q \colon A^{\otimes(q-1)} \otimes N \to N$. When $p, r > 0$, μ^q denotes the algebra structure map $A^{\otimes q} \to A$ composed with the augmentation $A \to \bar{A}$.

Theorem A.3 (Keating [19, Lemma 7.2 and Remark 6.6]) *Suppose $L_1, L, L_2 \subset X$ are three Lagrangian submanifolds which are objects of $\mathcal{F}(X)_\lambda$, and L is a sphere. Then there is an exact sequence of vector spaces:*

$$\mathrm{HF}^*(L_1, L_2) \longrightarrow \mathrm{HF}^*(\tau_L^k L_1, L_2)$$

$$H\big(\mathrm{Hom}(L, L_1) \otimes_{\mathrm{Hom}(L,L)} \mathrm{Hom}(L_2, L)\big)_k \qquad\qquad \square$$

Note that [19] states this theorem for exact X and over $\mathbb{Z}/2$; in particular, it does not mention the signs in (A-1). The proof uses a theorem of Seidel [37, Corollary 17.17] which says that $\tau_L L_1$ is quasi-isomorphic to the cone of a certain evaluation map, as an object of the (category of twisted complexes over the) Fukaya category. This allows one to write $\tau_L^k L_1$ as an iterated cone, which automatically provides some exact sequence of the type above. Keating proves Theorem A.3 by simplifying the iterated cone in a purely algebraic way: by identifying and killing some acyclic sub-complexes in it. We know that the initial Seidel's theorem holds for the monotone Fukaya category and over $\mathbb{C}$ (see eg Oh [27] for the homological version), and the proof of Theorem A.3 works in the monotone case and over $\mathbb{C}$ by virtue of being purely algebraic. The signs

in (A-1) will be enforced for algebraic reasons, and it is a matter of book-keeping to check that they are the ones that we expect to see in a bar complex. In addition to Theorem A.3, we will need some auxiliary lemmas.

Lemma A.4 (Formality) *Every A_∞–algebra with cohomology ring $\mathbb{C}[x]/(x^2 - 1)$ is quasi-isomorphic to the A_∞–algebra $\mathbb{C}[x]/(x^2 - 1)$ with vanishing higher multiplications $\mu^j = 0$, $j > 2$.*

Proof This follows from [18, Corollary 4] and [17, Proposition 2.2]. □

Lemma A.5 *Take the A_∞–algebra $\mathbb{C}[x]/(x^2 - 1)$ with vanishing μ^j, $j > 2$. Every strictly unital A_∞–module M over this algebra with vanishing μ^1 necessarily has vanishing μ^j, $j > 2$.*

Proof Take the minimal j such that $\mu^j(m, x^{\otimes(j-1)}) \neq 0$ for some $m \in M$. If $j > 1$, the A_∞–relation for the tuple $(m, x^{\otimes(j-1)}, 1)$ gives $\mu^j(m, x^{\otimes(j-1)}) = 0$, a contradiction. □

Lemma A.6 [19, Lemma 3.1] *Let (M, A, N) be a c-unital A_∞–category consisting of an A_∞–algebra A, a left A_∞–module M and a right A_∞–module N. Let A' be a strictly unital A_∞–algebra quasi-isomorphic to A. Then there are strictly unital A_∞–modules M', N' over A' such that the category (M, A, N) is quasi-isomorphic to (M', A', N'). The underlying Hom–spaces of (M, A, N) and (M', A', N') are the same.* □

Lemma A.7 (Cf [19, Lemma 7.3]) *Let (M, A, N) and (M', A', N') be two strictly unital A_∞–categories consisting of an algebra, a left module and a right module. If they are quasi-isomorphic, the associated bar complexes $(M \otimes_A N)_k$ and $(M' \otimes_{A'} N')_k$ are quasi-isomorphic.* □

Remark A.8 Let $\dim_\mathbb{R} X = 2n$. Suppose $L \subset X$ is a Lagrangian sphere. The $\mathbb{Z}/2$–graded Floer chain complex $\mathrm{CF}^*(L, L)$ can be realised as a 2–dimensional vector space $\mathbb{C} \oplus \mathbb{C}$ with two generators: the unit 1, $\deg 1 = 0$ and the second generator x, $\deg x \equiv n \mod 2$. The differential has degree 1.

If n is even, Floer's differential must vanish and $\mathrm{HF}^*(L, L)$ is a unital 2–dimensional commutative algebra. Up to isomorphism, this leaves only two possibilities: $\mathbb{C}[x]/x^2$ or $\mathbb{C}[x]/(x^2 - 1)$.

If n is odd, $\mathrm{HF}^*(L, L)$ is zero or 2–dimensional. In the latter case, we necessarily have $x^2 = 0$ because $\mathrm{HF}^*(L, L)$ is graded commutative, so $\mathrm{HF}^*(L, L) \cong \mathbb{C}[x]/x^2$.

The minimal Chern number of X is the maximal integer N such that $c_1(X)$ is divisible by N in integral cohomology $H^2(X; \mathbb{Z})$. Floer cohomology of a Lagrangian sphere can be made $\mathbb{Z}/2N$ graded, and our generators have gradings $\deg 1 = 0$ and $\deg x \equiv n \bmod 2N$. If n is even and $n \neq 0 \bmod N$, for grading reasons we get $x^2 = 0$ and $\mathrm{HF}^*(L, L) \cong \mathbb{C}[x]/x^2$.

Proof of Proposition A.1 We want to prove that $\mathrm{HF}^*(L_1, L_1) \cong \mathbb{C}[x]/x^2$. Suppose this is not the case. Then by Remark A.8, $\mathrm{HF}^*(L_1, L_1; \mathbb{C}) \cong \mathbb{C}[x]/(x^2 - 1)$. Recall that n is even.

Inside $\mathcal{F}(X)_\lambda$, take the subcategory consisting of the A_∞–algebra $\mathrm{Hom}(L_1, L_1)$, its left module $\mathrm{Hom}(L_1, L_2)$ and its right module $\mathrm{Hom}(L_2, L_1)$. Because $|L_1 \cap L_2| = 1$, $\mathrm{Hom}(L_1, L_2)$ and $\mathrm{Hom}(L_2, L_1)$ are 1–dimensional as vector spaces. Denote their generators by

$$\mathrm{Hom}(L_1, L_2) = \langle m \rangle, \quad \mathrm{Hom}(L_2, L_1) = \langle n \rangle.$$

By Lemma A.4, we see that $\mathrm{Hom}(L_1, L_1)$ is quasi-isomorphic to the associative algebra $\mathbb{C}[x]/(x^2 - 1)$ with trivial higher multiplications. By Lemma A.6 and Lemma A.5, modules $\mathrm{Hom}(L_1, L_2)$ and $\mathrm{Hom}(L_2, L_1)$ are quasi-isomorphic to those with trivial higher multiplications over $\mathbb{C}[x]/(x^2 - 1)$. The module μ^2–operations, however, must be non-trivial because $x^2 = 1$:

$$\mu^2(m, x) = \epsilon_m m, \quad \mu^2(x, n) = \epsilon_n n, \quad \text{where } \epsilon_m, \epsilon_n = \pm 1.$$

Lemma A.7 allows to compute the homology of the bar complex

$$B_k := \big(\mathrm{Hom}(L_1, L_2) \otimes_{\mathrm{Hom}(L_1, L_1)} \mathrm{Hom}(L_2, L_1)\big)_k$$

using the simple associative model we obtained. In this model, the bar complex B_k is based on the k–dimensional vector space

$$\bigoplus_{j=1}^{k-1} m \otimes x^{\otimes j} \otimes n.$$

The differential comes only from $\mu^2(m, x)$ and $\mu^2(x, n)$:

$$\partial(m \otimes x^{\otimes j} \otimes n) = \big((-1)^j \epsilon_n + \epsilon_m\big) m \otimes x^{\otimes(j-1)} \otimes n.$$

Note that the sign $(-1)^{\maltese}$ from Equation (A-1) equals $+1$ because we are given $\deg x = 0$ and may assume $\deg n = 0$. We see that $\dim H(B_k) = 0$ or 1, depending on the parity of k. By the exact sequence of Theorem A.3, we get $\dim \mathrm{HF}^*(\tau_{L_1}^k L_2, L_2; \mathbb{C}) \leq 2$, which contradicts the hypothesis. $\qquad\square$

Remark A.9 If $HF^*(L_1, L_1; \mathbb{C}) \cong \mathbb{C}[x]/x^2$, it might still happen that $\mathrm{Hom}(L_1, L_1)$ is formal, for example when X is exact. Running the above proof, from $x^2 = 0$ we conclude that $\mu^2(m, x) = \mu^2(x, n) = 0$. So the differential on the k–dimensional model for B_k written above vanishes, and $\dim H(B_k) = k$. This agrees with the growth of $\dim HF^*(\tau_{L_1}^k L_2, L_2)$.

A.2 Floer cohomology rings of Lagrangian spheres in divisors

We now combine Proposition A.1 with previous results (Propositions 1.11 and 4.2) to compute the ring $HF^*(L, L; \mathbb{C})$ for vanishing Lagrangian spheres L in certain divisors. The statement uses notation from Section 1.7. We also provide a corollary which specialises to divisors in Grassmannians.

Proposition A.10 *In addition to the conditions of Theorem 1.13 (a) or (b), suppose that X is Fano and $\dim_{\mathbb{C}} X$ is even. Then there is a ring isomorphism $HF^*(L, L; \mathbb{C}) \cong \mathbb{C}[x]/x^2$.*

Corollary A.11 *Let $X \subset \mathrm{Gr}(k, n)$ be a smooth divisor of degree $3 \le d < n$, $\dim_{\mathbb{C}} X$ even. Let $L \subset X$ be an $|\mathcal{O}(d)|$–vanishing Lagrangian sphere. Then there is a ring isomorphism $HF^*(L, L; \mathbb{C}) \cong \mathbb{C}[x]/x^2$.*

The possibility ruled out by these two statements is that the deformed ring $HF^*(L, L)$ is isomorphic to $\mathbb{C}[x]/(x^2-1)$. An example of a sphere with $HF^*(L, L) \cong \mathbb{C}[x]/(x^2-1)$ is the antidiagonal $L \subset \mathbb{P}^1 \times \mathbb{P}^1$. Note that for this sphere τ_L has order 2 in $\pi_0 \mathrm{Symp}(\mathbb{P}^1 \times \mathbb{P}^1)$ [38]. It seems natural to ask whether there is a general relation between the isomorphism $HF^*(L, L) \cong \mathbb{C}[x]/(x^2 - 1)$ and τ_L being of finite order (both cases are rare). Observe that for many, but not all, pairs (k, n), Corollary A.11 follows from the grading consideration in Remark A.8.

Proof of Proposition A.10 As in the beginning of the proof of Theorem 1.13, take X, L_1, L_2 as constructed in Proposition 4.2. By Lemma 3.8, it suffices to prove that $HF^*(L_1, L_1) \cong \mathbb{C}[x]/x^2$.

From the Picard–Lefschetz formula (Lemma 3.12), given that $|L_1 \cap L_2| = 1$ and $\dim L_i^\iota$ is odd, we get the equality

$$[\tau_{L_1^\iota}^k L_2^\iota] = [L_2^\iota] - \epsilon k [L_1^\iota]$$

in the homology of the fixed locus $H_*(X^\iota)$. Consequently,

$$[\tau_{L_1^\iota}^k L_2^\iota] \cdot [L_2^\iota] = -\epsilon k.$$

By Proposition 1.11, dim $\mathrm{HF}^*(\tau_{L_1}^k L_2, L_2) \geq k$. By Proposition A.1, $\mathrm{HF}^*(L_1, L_1) \cong \mathbb{C}[x]/x^2$. □

Proof of Corollary A.11 Repeat the proof of Theorem 1.2 but refer to Proposition A.10 instead of Theorem 1.13. Recall the condition $d < n$ means that X is Fano. □

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