msp

Geometry & Topology 19 (2015) 3603–3643

Bounded cohomology via partial differential equations, I

TOBIAS HARTNICK
ANDREAS OTT

We present a new technique that employs partial differential equations in order to explicitly construct primitives in the continuous bounded cohomology of Lie groups. As an application, we prove a vanishing theorem for the continuous bounded cohomology of $\mathrm{SL}(2,\mathbb{R})$ in degree 4, establishing a special case of a conjecture of Monod.

20J06; 22E41, 35F35

1 Introduction

Ever since Gromov's seminal paper [24], bounded cohomology of discrete groups has proved a useful tool in geometry, topology and group theory. In recent years the scope of bounded cohomology has widened considerably. An important step was taken by Burger and Monod [13] and Monod [37], who extended the theory to the category of locally compact second countable groups under the name of continuous bounded cohomology. Not only did this lead to a breakthrough in the understanding of bounded cohomology of lattices in Lie groups [13], but also triggered a series of discoveries in rigidity theory (eg Burger, Monod and Iozzi [14], Burger and Iozzi [8; 9], Bucher, Burger and Iozzi [6], Monod and Shalom [41; 42] Chatterji, Fernós and Iozzi [17] and Hamenstädt [25; 26]), higher Teichmüller theory (eg Burger, Iozzi and Wienhard [10; 11; 12] and Ben Simon, Burger, Hartnick, Iozzi and Wienhard [1]) and symplectic geometry (eg Polterovich [44] and Entov and Polterovich [19]). At the same time, our understanding of the second bounded cohomology has improved. In particular, the approach originally developed for surface groups by Brooks [4], free groups by Grigorchuk [23] and hyperbolic groups by Epstein and Fujiwara [20] has been extended to larger classes of groups including mapping class groups by Bestvina and Fujiwara [2] and acylindrically hyperbolic groups by Hull and Osin [29] and Fujiwara, Bestvina and Bromberg [21]. Moreover, there has been some progress in constructing bounded cohomology classes in higher degree due to Mineyev [36], Hartnick and Ott [27], Bucher and Monod [7] and Goncharov [22].

Published: 6 January 2016 DOI: 10.2140/gt.2015.19.3603

On the other hand, our knowledge on vanishing results for bounded cohomology in higher degree is still very poor. It was already known to Johnson [31] that the bounded cohomology of an amenable group vanishes in all positive degrees. Here the primitive of a given cocycle is obtained by applying an invariant mean. In contrast, for non-amenable groups no general technique for constructing primitives is available so far. In particular, there is not a single non-amenable group whose bounded cohomology is known in all degrees. Actually, the situation is even worse. One may define the *bounded cohomological dimension* of a group Γ to be

$$\mathrm{bcd}(\Gamma) := \sup\{n \mid H^n_{\mathrm{b}}(\Gamma;\mathbb{R}) \neq 0\},$$

where $H^n_{\mathrm{b}}(\Gamma;\mathbb{R})$ is the n^{th} bounded cohomology of Γ with coefficients in the trivial module $\mathbb{R}$. At present we do not even know whether there exists any group Γ with $\mathrm{bcd}(\Gamma) \notin \{0,\infty\}$.

The few vanishing results we have for the bounded cohomology of non-amenable groups are all based on the vanishing of all cocycles in the respective degree in some resolution. The most far-reaching results in this direction were achieved by Monod [39] by choosing efficient resolutions. However, no such resolutions are known for dealing with the continuous bounded cohomology $H^n_{\mathrm{cb}}(H;\mathbb{R})$ of non-amenable connected Lie groups H. For such groups the most efficient resolution that is presently available is the boundary resolution of Ivanov [30] and Burger and Monod [13]. In this particular resolution cocycles vanish only in degree at most 3; in degree greater than 3 there will inevitably be nonzero cocycles and one faces the problem of finding primitives. This explains why the few existing vanishing results such as in [13; 15] do not go beyond degree 3.

Our goal in this article is to develop a new approach to the construction of primitives in continuous bounded cohomology for real semisimple Lie groups. To demonstrate its effectiveness we settle the following special case of a conjecture due to Monod [38, Problem A]:

Theorem 1.1 *Let G be a connected real Lie group that is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Then*

$$H^4_{\mathrm{cb}}(G;\mathbb{R}) = 0.$$

Actually, for such G, Monod conjectured that $\mathrm{bcd}(G) = 2$, ie that $H^n_{\mathrm{cb}}(G;\mathbb{R}) = 0$ for all $n > 2$. In degree $n = 3$ there are no nonzero cocycles in the boundary resolution at all — see Burger and Monod [15] — but this is no longer true for $n > 3$. In this sense, Theorem 1.1 is the prototype of a vanishing theorem that requires the construction of primitives. We believe that our method of proof generalizes to arbitrary $n > 3$ nd possibly to other Lie groups. This will be addressed in future work.

Monod's conjecture about the bounded cohomology of $\mathrm{SL}_2(\mathbb{R})$ is a special case of a more general conjecture, which would allow one to compute the continuous bounded cohomology of arbitrary connected Lie groups. In fact, since continuous bounded cohomology is invariant under division by the amenable radical [13; 37], it is sufficient to compute the continuous bounded cohomology of semisimple Lie groups H without compact factors and with finite center. For such groups it is conjectured — see Dupont [18] and Monod [38] — that the natural comparison map between the continuous bounded cohomology and the continuous cohomology is an isomorphism. This would imply that $\mathrm{bcd}(H)$ coincides with the dimension of the associated symmetric space, thereby providing examples of groups of arbitrary bounded cohomological dimension. Plenty is known by now about surjectivity of the comparison map — see Dupont [18], Gromov [24], Bucher [5], Lafont and Schmidt [34], Goncharov [22] and Hartnick and Ott [27] — while injectivity still remains mysterious in higher degrees. Indeed, injectivity has so far been established only in degree 2 for arbitrary H by Burger and Monod [13] and for some rank 1 groups in degree 3 by Burger and Monod [15], Bloch [3] and Pieters [43]. Theorem 1.1 is the first result in degree greater than 3. Incidentally, it has an application to the existence of solutions to perturbations of the Spence–Abel functional equation for Rogers' dilogarithm, along the lines suggested in [15]. This will be discussed in our forthcoming [28].

For the proof of Theorem 1.1 we shall reformulate the problem of constructing bounded primitives in terms of a fixed point problem for the action of G on a certain function space. The main idea is then to describe the fixed points as solutions of a certain system of linear first-order partial differential equations. In this way, we obtain primitives by solving the corresponding Cauchy problem. We show that, for carefully chosen initial conditions, particular solutions have additional discrete symmetries, which we finally use to deduce boundedness. We will give a more detailed outline of our strategy of proof in Section 2.7 after introducing some notation.

Acknowledgements We are most indebted to B Tugemann for generously conducting numerous computer experiments at an early stage of this project and for many helpful discussions. We further thank U Bader, M Björklund, M Bucher, M Burger, A Derighetti, J L Dupont, H Hedenmalm, A Iozzi, A Karlsson, H Knörrer, A Laptev, N Monod, A Nevo, E Sayag, I Smith, J Swoboda, and A Wienhard for discussions and suggestions. We are very grateful to the Institut Mittag-Leffler and the organizers of the program "Geometric and Analytic Aspects of Group Theory", the Max Planck Institute for Mathematics and the organizers of the program "Analysis on Lie Groups", and École

Polytechnique Fédérale de Lausanne for their hospitality and for providing us with excellent working conditions.

We thank the referee for corrections and suggestions.

Hartnick received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013), ERC Grant agreement no 306706.

Ott wishes to thank the Institut des Hautes Études Scientifiques, the Isaac Newton Institute for Mathematical Sciences, the Max Planck Institute for Mathematics, the Centre de Recerca Matemàtica Barcelona, the Hausdorff Research Institute for Mathematics and the Department of Pure Mathematics and Mathematical Statistics at Cambridge University for their hospitality and excellent working conditions. He was supported by a grant from the Klaus Tschira Foundation, by grant EP/F005431/1 from the Engineering and Physical Sciences Research Council, and partially supported by grant ERC-2007-StG-205349 from the European Research Council.

2 Preliminaries on continuous bounded cohomology

2.1 The boundary action of $\mathbf{PSL_2(\mathbb{R})}$

The goal of this section is to describe a model for the continuous bounded cohomology of $\mathrm{SL}_2(\mathbb{R})$. Since continuous bounded cohomology of connected Lie groups is invariant under local isomorphisms by Corollary 7.5.10 of Monod [37], we have

$$H^\bullet_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{R});\mathbb{R}) \cong H^\bullet_{\mathrm{cb}}(\mathrm{PSL}_2(\mathbb{R});\mathbb{R}) \cong H^\bullet_{\mathrm{cb}}(\mathrm{PU}(1,1);\mathbb{R}),$$

where the latter isomorphism is induced by the Cayley transform. We prefer to carry out our computations in the group $G := \mathrm{PU}(1,1)$. The group G acts by fractional linear transformations on the Poincaré disc $\mathbb{D}$. We can identify G with the group of orientation-preserving isometries of $\mathbb{D}$, which is an index-2 subgroup in the full isometry group $\widehat{G}$. The actions of G and $\widehat{G}$ extend continuously to the boundary S^1 of $\mathbb{D}$ and the corresponding actions on S^1 will be referred to as the *boundary action* of G and $\widehat{G}$, respectively. The action of $\widehat{G}$ on S^1 (but not on $\mathbb{D}$) may be identified with the action of $\mathrm{PGU}(1,1)$ by fractional linear transformations. It is well known that this action is strictly 3–transitive; see Kerby [32, Theorem 11.1].

Explicitly, elements of $\mathrm{PGU}(1,1)$ can be represented by matrices of the form

$$g_{a,b} := \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 \in \{\pm 1\}$. We denote by $[g_{a,b}]$ the equivalence class of the matrix $g_{a,b}$ in $\mathrm{PGU}(1,1)$. Given $\xi, s, t \in \mathbb{R}$, we abbreviate

$$k_\xi := [g_{e^{i\xi/2},0}], \quad a_s := [g_{\cosh(-s/2),\sinh(-s/2)}], \quad n_t := [g_{1+\frac{1}{2}it,-\frac{1}{2}it}].$$

We also set $\epsilon := [g_{0,1}]$. Then $\widehat{G} \cong G \rtimes \langle \epsilon \rangle$, where G is given by equivalence classes of matrices of determinant 1 and ϵ acts on G via

$$\epsilon [g_{a,b}] \epsilon^{-1} = [g_{\overline{a},\overline{b}}].$$

Next we observe that $K := \{k_\xi \mid \xi \in \mathbb{R}\}$, $A := \{a_s \mid s \in \mathbb{R}\}$ and $N := \{n_t \mid t \in \mathbb{R}\}$ are subgroups of G. Moreover, K is a maximal compact subgroup of G and A normalizes N. In particular, $P = AN$ is a subgroup of G, which is in fact a parabolic subgroup. The group K can also be described as the stabilizer of 0 under the G–action on $\mathbb{D}$ and, similarly, the group P is the stabilizer of 1 for the boundary action of G. Since N is the unipotent radical of P we have $\mathrm{Fix}(N) = \{1\}$, whereas $\mathrm{Fix}(A) = \{\pm 1\}$. Moreover, we obtain an Iwasawa decomposition $G = KAN$ and every elliptic (respectively hyperbolic, parabolic) element in G is conjugate to an element in K (respectively A, N).

Our parametrization of elements of K, A and N is chosen in such a way that the maps $\xi \mapsto k_\xi$, $s \mapsto a_s$ and $t \mapsto n_t$ define 1–parameter subgroups of G, ie smooth homomorphisms $\mathbb{R} \to G$. (Both the homomorphisms and their images are commonly referred to as 1–parameter subgroups. We reserve this term for the homomorphisms.) These parametrizations are not quite standard, but turn out to be convenient for certain computations in local coordinates; see Lemma 3.2 below.

2.2 Cocycles and strict cocycles

We keep the notation introduced in the last section and denote by μ_K the unique K–invariant probability measure on S^1. Given $n \geq 0$, we shall write $\mathcal{M}((S^1)^{n+1})$ for the space of $\mu_K^{\otimes(n+1)}$–measurable real-valued functions on $(S^1)^n$ and $\mathcal{L}^\infty((S^1)^n)$ for the subspace of bounded functions. The quotients of these spaces obtained by identifying $\mu_K^{\otimes(n+1)}$–almost everywhere coinciding functions will be denoted by $M((S^1)^{n+1})$ and $L^\infty((S^1)^{n+1})$, respectively.

We define the homogeneous differential $d\colon \mathcal{M}((S^1)^{n+1}) \to \mathcal{M}((S^1)^{n+2})$ by

$$df(z_0, \dots, z_{n+1}) := \sum_{j=0}^{n+1} (-1)^j \, f(z_0, \dots, \widehat{z}_j, \dots, z_{n+1}).$$

This induces differentials on $\mathcal{L}^\infty((S^1)^n)$, $M((S^1)^{n+1})$ and $L^\infty((S^1)^{n+1})$. Elements in the kernels of these four differentials are referred to as *strict n–cocycles*, *strict bounded n–cocycles*, *n–cocycles* and *bounded n–cocycles*, respectively.

The group $\widehat{G}$ acts diagonally on $(S^1)^{n+1}$ and this action commutes with the action of the symmetric group $\mathfrak{S}_{n+1}$ by permutation of the variables. We thus obtain a $\widehat{G} \times \mathfrak{S}_{n+1}$–action on each of the spaces $\mathcal{M}((S^1)^{n+1})$, $\mathcal{L}^\infty((S^1)^{n+1})$, $M((S^1)^{n+1})$ and $L^\infty((S^1)^{n+1})$. It is immediate from the explicit formula that all homogeneous differentials are $\widehat{G}$–equivariant; in particular, the $\widehat{G}$–action maps cocycles to cocycles. Given a subgroup $H < \widehat{G}$, an H–invariant (strict, bounded) cocycle is simply called a (strict, bounded) *H–cocycle* and it is called a (strict, bounded) *H–coboundary* if it is contained in the image of the H–invariants under d.

A major technical inconvenience is caused by the failure of surjectivity of the map $\mathcal{M}((S^1)^n)^G \to M((S^1)^n)^G$, which means that a G–cocycle c may not admit an *invariant representative*, ie a strict G–cocycle which coincides with c almost everywhere. Fortunately, for bounded function classes, existence of invariant representatives follows from Monod [40, Theorem A].

Lemma 2.1 (Invariant representatives) *The maps $\mathcal{L}^\infty((S^1)^{n+1})^G \to L^\infty((S^1)^{n+1})^G$, $n \geq 0$, are surjective. In fact, they admit a family of sections compatible with the homogeneous differentials.*

2.3 Orbitwise smooth functions

We say that a function $f \in \mathcal{M}((S^1)^{n+1})$ is *orbitwise smooth* if for each $(z_0, \dots, z_n)$ the map $G \to \mathbb{R}$ given by

$$g \mapsto f(g.z_0, \dots, g.z_n)$$

is smooth. We record the following basic properties of such functions for later reference:

Lemma 2.2 (i) *Every smooth function is orbitwise smooth. If $n \leq 2$, then every orbitwise smooth function in $\mathcal{M}((S^1)^{n+1})$ restricts to a smooth function on $(S^1)^{(n+1)}$.*

(ii) *Orbitwise smooth functions in $\mathcal{M}((S^1)^{n+1})$ form an $\mathbb{R}$–algebra.*

(iii) *If $f \in \mathcal{L}^\infty((S^1)^{n+1})$ is orbitwise smooth, then so is the function $I(f)$ in $\mathcal{L}^\infty((S^1)^n)$ given by*

$$I(f)(z_1, \dots, z_n) = \int_{S^1} f(z, z_1, \dots, z_n)\, d\mu_K(z).$$

(iv) *Every class in $L^\infty((S^1)^{n+1})^G$ can be represented by an orbitwise smooth function.*

Proof The first statement of (i) follows from smoothness of the orbit map of the diagonal G–action and the second statement follows from the fact that $(S^1)^{(n+1)}$ is a union of open G–orbits if $n \leq 2$. (ii) is obvious. Concerning (iii), we observe that

$$\begin{aligned} I(f)(g.z_0, \dots, g.z_n) &= \int_{S^1} f(z, g.z_1, g.z_2, g.z_3)\, d\mu_K(z) \\ &= \int_{S^1} \frac{d(g.\mu_K)}{d\mu_K}(z) f(z, z_1, z_2, z_3)\, d\mu_K(z). \end{aligned}$$

Orbitwise smoothness of this function now follows from smoothness of the Radon–Nikodym derivative (see eg Knapp [33, Proposition 8.43]). Finally, (iv) follows from Lemma 2.1 and the observation that every constant function is smooth. □

We emphasize that, in general, the analog of (iv) fails for classes in $M((S^1)^{n+1})^G$, since boundedness is essential for the construction of invariant representatives. We will often apply Lemma 2.2 in the following form, which combines (i)–(iv) of the lemma:

Corollary 2.3 *If $f \in L^\infty((S^1)^{n+1})^G$ and $h \in C^\infty((S^1)^k)$, then the class g in $L^\infty((S^1)^{n-k+1})$ given by*

$$\begin{aligned} &g(z_0, \dots, z_{n-k}) \\ &\qquad := \int_{(S^1)^k} h(w_1, \dots, w_k) f(w_1, \dots, w_k, z_0, \dots, z_{n-k})\, d\mu_K^{\otimes k}(w_1, \dots, w_k) \end{aligned}$$

admits an orbitwise smooth representative.

Convention 2.4 Whenever we are given a function class f in $M((S^1)^{n+1})$ or $L^\infty((S^1)^{n+1})$ that admits an orbitwise smooth representative, we will use the same letter f to denote an orbitwise smooth representative.

2.4 The boundary model for continuous bounded cohomology

A function class $f \in L^\infty((S^1)^{n+1})$ is called *alternating* provided $\sigma.f = (-1)^\sigma \cdot f$ for all $\sigma \in \mathfrak{S}_{n+1}$; we denote by $L^\infty_{\mathrm{alt}}((S^1)^{n+1}) < L^\infty((S^1)^{n+1})$ the subspace of alternating function classes. Since the actions of $\widehat{G}$ and $\mathfrak{S}_{n+1}$ commute, this subspace is $\widehat{G}$–invariant. Moreover, the homogeneous differential maps alternating function classes to alternating functions classes, whence $(L^\infty_{\mathrm{alt}}((S^1)^{\bullet+1}), d)$ is a subcomplex of $(L^\infty((S^1)^{\bullet+1}), d)$.

Proposition 2.5 (Boundary model [37, Theorem 7.5.3]) *Given a closed subgroup $H < \widehat{G}$, the continuous bounded cohomology of H is given by the cohomology of the complex $(L^\infty_{\mathrm{alt}}((S^1)^{\bullet+1})^H, d)$, ie*

$$H^n_{\mathrm{cb}}(H; \mathbb{R}) \cong H^n(L^\infty_{\mathrm{alt}}((S^1)^{\bullet+1})^H, d) \quad \textit{for all } n \geq 0.$$

2.5 Even and odd cocycles

Given $f \in L^\infty_{\mathrm{alt}}((S^1)^{n+1})$, we denote by $f^\pm$ the projections of f onto the ± 1–eigenspaces of ϵ. Explicitly, we have $f^\pm = \frac{1}{2}(f \pm \epsilon.f)$ and $f = f^+ + f^-$. We say that f is *even* if $f = f^+$ and *odd* if $f = f^-$. Since ϵ normalizes G, the projections $f \mapsto f^\pm$ preserve the subspace of G–invariants. They also commute with homogeneous differentials, since ϵ does. In particular, every G–invariant alternating (strict) cocycle decomposes uniquely into a sum of a G–invariant alternating (strict) even cocycle and a G–invariant alternating (strict) odd cocycle, and similarly for coboundaries. On the level of cohomology this yields a decomposition

$$H^\bullet_{\mathrm{cb}}(G;\mathbb{R}) \cong H^\bullet_{\mathrm{cb}}(G;\mathbb{R})_{\mathrm{ev}} \oplus H^\bullet_{\mathrm{cb}}(G;\mathbb{R})_{\mathrm{odd}}.$$

The first summand $H^n_{\mathrm{cb}}(G;\mathbb{R})_{\mathrm{ev}}$ can be identified with the continuous bounded cohomology $H^n_{\mathrm{cb}}(\widehat{G};\mathbb{R})$ of the extended group $\widehat{G}$. Similarly, if we denote by $\mathbb{R}_\epsilon$ the unique non-trivial 1–dimensional $\widehat{G}$–module, then $H^\bullet_{\mathrm{cb}}(G;\mathbb{R})_{\mathrm{odd}} \cong H^\bullet_{\mathrm{cb}}(\widehat{G};\mathbb{R}_\epsilon)$, whence the above decomposition can also be written as

$$H^\bullet_{\mathrm{cb}}(G;\mathbb{R}) \cong H^\bullet_{\mathrm{cb}}(\widehat{G};\mathbb{R}) \oplus H^\bullet_{\mathrm{cb}}(\widehat{G};\mathbb{R}_\epsilon).$$

In particular, the vanishing of $H^\bullet_{\mathrm{cb}}(G;\mathbb{R})$ is equivalent to the vanishing of both $H^\bullet_{\mathrm{cb}}(\widehat{G};\mathbb{R})$ and $H^\bullet_{\mathrm{cb}}(\widehat{G};\mathbb{R}_\epsilon)$. It turns out that the vanishing of $H^\bullet_{\mathrm{cb}}(\widehat{G};\mathbb{R}_\epsilon)$ can be deduced using only combinatorial arguments; see Proposition 2.6 below. The vanishing of the first summand is considerably harder and its proof will occupy the rest of this article.

Proposition 2.6 *Every alternating G–invariant bounded 4–cocycle is even. In particular,*

$$H^4_{\mathrm{cb}}(G;\mathbb{R})_{\mathrm{odd}} \cong H^4_{\mathrm{cb}}(\widehat{G};\mathbb{R}_\epsilon) = 0.$$

The proof of Proposition 2.6 will be given in Appendix A. We remind the reader that non-zero even alternating G–invariant bounded 4–cocycles *do* exist; see Burger and Monod [15].

2.6 Primitives in the boundary model

Given a bounded G–cocycle $c \in L^\infty((S^1)^5)^G$ and any closed subgroup $H \subset G$, we denote by

$$\mathcal{P}(c)^H := \{p \in M((S^1)^4)^H \mid dp = c\}$$

the space of H–invariant primitives of c and by

$$\mathcal{P}^\infty(c)^H := \{p \in L^\infty((S^1)^4)^H \mid dp = c\}$$

the subspace of bounded H–invariant primitives. In view of Proposition 2.5, the proof of Theorem 1.1 amounts to showing that $\mathcal{P}^\infty(c)^G \neq \varnothing$ for any bounded alternating G–cocycle c and, by Proposition 2.6, we may furthermore assume that c is even. Under this assumption we will explicitly construct elements in $\mathcal{P}^\infty(c)^G$. For this purpose, we first define an operator

$$I\colon \mathcal{L}^\infty((S^1)^{n+1}) \to \mathcal{L}^\infty((S^1)^n)$$

by

$$I(c)(z_1,\dots,z_n) := \int_{S^1} c(z,z_1\dots,z_n)\, d\mu_K(z). \tag{2-1}$$

It induces an operator $I\colon L^\infty((S^1)^{n+1}) \to L^\infty((S^1)^n)$, which by abuse of notation we denote by the same symbol. By integrating the cocycle equation $dc = 0$, we see that $d(I(c)) = c$ for every cocycle c.

From now on we fix a bounded G–cocycle $c \in L^\infty((S^1)^5)^G$. For the moment we do not need to assume that c is either alternating or even. By K–invariance of the measure μ_K, we see from formula (2-1) that the function $I(c)$ is K–invariant, hence a K–invariant primitive of c. It will, however, in general not be G–invariant. In order to obtain G–invariant primitives we amend the operator I in the following way.

We denote by $(S^1)^{(n)} \subset (S^1)^n$ the subset of n–tuples of pairwise distinct points in S^1. Note in particular that the G–action on $(S^1)^{(3)}$ is free and has two open orbits, given by positively and negatively oriented triples. We write $C^\infty((S^1)^{(3)})^K$ for the space of K–invariant real-valued smooth functions on $(S^1)^{(3)}$ and consider it as a subspace of $M((S^1)^3)$. We then define an operator

$$P_c\colon C^\infty((S^1)^{(3)})^K \to M((S^1)^4), \quad f \mapsto I(c) + df. \tag{2-2}$$

A key observation is that all G–invariant bounded primitives of c necessarily lie in the image of the operator P_c. This will allow us to express primitives in terms of smooth (rather than measurable) solutions to differential equations.

Proposition 2.7 *The image of the operator P_c satisfies*

$$\mathcal{P}^\infty(c)^G \subset P_c(C^\infty((S^1)^{(3)})^K) \subset \mathcal{P}(c)^K.$$

Proof We have already seen that $I(c) \in \mathcal{P}^\infty(c)^K$. We conclude that $P_c(f) \in \mathcal{P}(c)^K$ for all $f \in C^\infty((S^1)^{(3)})^K$. Concerning the other inclusion, we observe that, if $p \in \mathcal{P}^\infty(c)^G$ is any bounded primitive of c, then $dp = c = dI(c)$, whence $p - I(c)$ is a K–invariant cocycle. In particular, if we define $f := I(p - I(c))$, then $df = p - I(c)$ and thus

$$p = I(c) + df.$$

By Lemma 2.2 the function classes p and c, and hence also f, can be represented by an orbitwise smooth function. Since G–orbits are open in $(S^1)^{(3)}$, we infer that this representative is actually smooth on $(S^1)^{(3)}$ and thus $p = P_c(f) \in P_c(C^\infty((S^1)^{(3)})^K)$. □

2.7 Strategy of proof

We briefly outline the strategy for the proof of Theorem 1.1. We shall proceed in three steps:

(1) In Section 3, we show that for any function $f \in C^\infty((S^1)^{(3)})^K$ the primitive $P_c(f)$ is G–invariant if and only if f satisfies a certain system of linear first-order partial differential equations.

(2) In Section 4, we explicitly construct solutions f of this system of differential equations, showing that $\mathcal{P}(c)^G \neq \varnothing$.

(3) In Section 5, we prove that there exist particular solutions f with certain additional discrete symmetries. For such functions f we then show boundedness of $P_c(f)$, establishing that $\mathcal{P}^\infty(c)^G \neq \varnothing$.

While the constructions in (1) and (2) work for arbitrary bounded G–cocycles c, the construction in (3) relies on c being alternating and even.

3 Partial differential equations

3.1 The boundary action in local coordinates

In order to describe the boundary action of $G = \mathrm{PU}(1,1)$ explicitly, we introduce coordinates as follows. We consider S^1 as a subset of $\mathbb{C}$ and write $z \in S^1$ for a complex number z of modulus 1. In addition, it will often be convenient to work with the angular coordinate $\theta \in [0, 2\pi)$ defined by $z = e^{i\theta}$. Correspondingly, on $(S^1)^n$ we will use the two sets of coordinates $(z_0, \dots, z_{n-1})$ and $(\theta_0, \dots, \theta_{n-1})$. Note that, in angular coordinates on S^1, the measure μ_K is given by

$$\int_{S^1} f(z)\, d\mu_K(z) = \fint f(\theta)\, d\theta := \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta.$$

Convention 3.1 Throughout, all operations on angular coordinates will implicitly be understood modulo 2π. For example, $\theta_2 - \theta_1$ denotes the unique point in the interval $[0, 2\pi)$ that is congruent to $\theta_2 - \theta_1$ modulo 2π.

The G–action on S^1 induces a G–action on the interval $[0,2\pi)$ by the relation $g.e^{i\theta}=e^{ig.\theta}$. Note that in particular $k_\xi.\theta=\theta+\xi$.

Lemma 3.2 (Infinitesimal actions in angular coordinates) *For every $\eta\in[0,2\pi)$ we have*

$$\frac{d}{ds}(a_s.\eta)=\sin(a_s.\eta)\quad\text{and}\quad\frac{d}{dt}(n_t.\eta)=1-\cos(n_t.\eta).$$

Proof To prove the first formula, we compute

$$\frac{d}{ds}\bigg|_{s=0}(a_s.\phi)=\frac{1}{ie^{i\phi}}\frac{d}{ds}\bigg|_{s=0}e^{ia_s.\phi}=\frac{1}{ie^{i\phi}}\frac{d}{ds}\left(\frac{\cosh\left(-\frac{1}{2}s\right)e^{i\phi}+\sinh\left(-\frac{1}{2}s\right)}{\sinh\left(-\frac{1}{2}s\right)e^{i\phi}+\cosh\left(-\frac{1}{2}s\right)}\right)\bigg|_{s=0}$$
$$=\sin(\phi).$$

Since the map $s\mapsto a_s$ is a 1–parameter group we further infer that

$$\frac{d}{ds}(a_s.\eta)=\frac{d}{d\sigma}\bigg|_{\sigma=0}(a_{s+\sigma}.\eta)=\frac{d}{d\sigma}\bigg|_{\sigma=0}(a_\sigma(a_s.\eta))=\sin(a_s.\eta).$$

Likewise, for the second formula we compute

$$\frac{d}{dt}\bigg|_{t=0}(n_t.\phi)=\frac{1}{ie^{i\phi}}\frac{d}{dt}\bigg|_{t=0}e^{in_t.\phi}=\frac{1}{ie^{i\phi}}\frac{d}{dt}\left(\frac{\left(1+\frac{1}{2}it\right)e^{i\phi}-\frac{1}{2}it}{\frac{1}{2}ite^{i\phi}+1-\frac{1}{2}it}\right)\bigg|_{t=0}=1-\cos\phi$$

and conclude as above. □

3.2 Fundamental vector fields

We denote by $L_K^{(n)}$, $L_A^{(n)}$ and $L_N^{(n)}$ the differential operators that appear as fundamental vector fields for the infinitesimal action of the 1–parameter groups $\xi\mapsto k_\xi$, $s\mapsto a_s$ and $t\mapsto n_t$ on $(S^1)^{(n)}$, respectively. By Lemma 3.2, they are given in angular coordinates by

$$L_K^{(n)}=\sum_{j=0}^{n-1}\frac{\partial}{\partial\theta_j},\quad L_A^{(n)}=\sum_{j=0}^{n-1}\sin\theta_j\frac{\partial}{\partial\theta_j}\quad\text{and}\quad L_N^{(n)}=\sum_{j=0}^{n-1}(1-\cos\theta_j)\frac{\partial}{\partial\theta_j}.$$

Note that $L_K^{(n)}f$ is well-defined for any orbitwise smooth function $f\colon(S^1)^n\to\mathbb{R}$, and similarly for $L_A^{(n)}$ and $L_N^{(n)}$. The next lemma is crucial for applications of the operators $L_K^{(n)}$, $L_A^{(n)}$ and $L_N^{(n)}$ in cohomology.

Lemma 3.3 *Let $L^{(n)}$ denote one of the operators $L_K^{(n)}$, $L_A^{(n)}$ and $L_N^{(n)}$. Then $L^{(n)}$ commutes with the homogeneous differential in the sense that*

$$d^n\circ L^{(n)}=L^{(n+1)}\circ d^n\quad\textit{for every } n>0.$$

Proof Let $\lambda \in C^\infty([0,2\pi))$ and consider the differential operators

$$L_\lambda^{(n)} := \sum_{j=0}^{n-1} \lambda(\theta_j)\frac{\partial}{\partial\theta_j}$$

for every $n > 0$. For any orbitwise smooth $(n-1)$–cochain q we compute

$$\begin{aligned}(L_\lambda^{(n+1)}(d^n q))(\theta_0,\dots,\theta_n) &= \sum_{j=0}^{n} \lambda(\theta_j)\frac{\partial}{\partial\theta_j}\Big(\sum_{\ell=0}^{n}(-1)^\ell q(\theta_0,\dots,\widehat{\theta}_\ell,\dots,\theta_n)\Big)\\ &= \sum_{\ell=0}^{n}(-1)^\ell \sum_{j\neq\ell}\lambda(\theta_j)\frac{\partial q}{\partial\theta_j}(\theta_0,\dots,\widehat{\theta}_\ell,\dots,\theta_n)\\ &= \sum_{\ell=0}^{n}(-1)^\ell (L_\lambda^{(n)} q)(\theta_0,\dots,\widehat{\theta}_\ell,\dots,\theta_n)\\ &= (d^n(L_\lambda^{(n)} q))(\theta_0,\dots,\theta_n).\end{aligned}$$

□

3.3 Infinitesimal invariance of primitives

We now return to the setting of Section 2.6. In particular, $c \in L^\infty((S^1)^5)^G$ is a cocycle and the operator

$$P_c\colon C^\infty((S^1)^{(3)})^K \to \mathcal{P}(c)^K, \quad f \mapsto I(c)+df,$$

is defined as in (2-2). We will characterize G–invariance of primitives $P_c(f)$ in terms of differential equations for the function f. First, let us define function classes $c^\sharp, c^\flat \in L^\infty((S^1)^3)$ by

$$c^\sharp(\theta_0,\theta_1,\theta_2) := ⨍⨍ \cos(\varphi)\, c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi \tag{3-1}$$

and

$$c^\flat(\theta_0,\theta_1,\theta_2) := ⨍⨍ \sin(\varphi)\, c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi. \tag{3-2}$$

By Corollary 2.3 the function classes $c^\sharp$ and $c^\flat$ admit orbitwise smooth representatives. By Lemma 2.2(i) they can thus be represented by smooth functions on $(S^1)^{(3)}$. Following Convention 2.4, we denote these smooth representatives by the same letters $c^\sharp$ and $c^\flat$.

The next proposition achieves the first step in the agenda outlined in Section 2.7.

Proposition 3.4 *Let $f \in C^\infty((S^1)^{(3)})^K$ and $c^\sharp, c^\flat \in C^\infty((S^1)^{(3)})$ as above. Then the primitive $P_c(f) \in \mathcal{P}(c)^K$ is G–invariant if and only if there exist functions $v^\sharp$, $v^\flat \in C^\infty((S^1)^{(2)})$ such that the triple $(f, v^\sharp, v^\flat)$ satisfies the system of partial differential equations*

$$\begin{cases} L_A^{(3)} f = c^\sharp + dv^\sharp, \\ L_N^{(3)} f = c^\flat + dv^\flat. \end{cases} \tag{3-3}$$

Note that all functions appearing in (3-3) are smooth, so all derivatives can be understood classically. The proof of Proposition 3.4 relies on the following lemma:

Lemma 3.5 *The function class $I(c) \in L^\infty((S^1)^4)$ admits an orbitwise smooth representative that satisfies*

$$\begin{cases} L_A^{(4)} I(c) = -dc^\sharp, \\ L_N^{(4)} I(c) = -dc^\flat, \end{cases}$$

where $c^\sharp$ and $c^\flat$ are orbitwise smooth representatives of the function classes defined in (3-1) and (3-2).

Proof By Corollary 2.3 the function class $I(c)$ admits an orbitwise smooth representative. Fix an invariant representative $\tilde{c}$ of c by Lemma 2.1. Using the A–invariance of $\tilde{c}$ and Lemma 3.2, we compute

$$\begin{aligned} L_A^{(4)}(I(c))(\theta_0, \dots, \theta_3) &= \frac{d}{ds}\bigg|_{s=0} \fint \tilde{c}(\varphi, a_s.\theta_0, \dots, a_s.\theta_3)\, d\varphi \\ &= \frac{d}{ds}\bigg|_{s=0} \fint \frac{d(a_s.\varphi)}{d\varphi} \tilde{c}(\varphi, \theta_0, \dots, \theta_3)\, d\varphi \\ &= \fint \frac{d}{d\varphi} \frac{d(a_s.\varphi)}{ds}\bigg|_{s=0} \tilde{c}(\varphi, \theta_0, \dots, \theta_3)\, d\varphi \\ &= \fint \frac{d}{d\varphi} \sin(a_s.\varphi)\bigg|_{s=0} \tilde{c}(\varphi, \theta_0, \dots, \theta_3)\, d\varphi \\ &= \fint \cos(\varphi) c(\varphi, \theta_0, \dots, \theta_3)\, d\varphi. \end{aligned}$$

On the other hand, the cocycle identity

$$\begin{aligned} 0 &= dc(\eta, \varphi, \theta_0, \dots, \theta_3) \\ &= c(\varphi, \theta_0, \dots, \theta_3) - c(\eta, \theta_0, \dots, \theta_3) + \sum_{j=0}^{3} (-1)^j\, c(\eta, \varphi, \theta_0, \dots, \widehat{\theta}_j, \dots, \theta_3) \end{aligned}$$

integrates against $\cos\varphi$ to

$$\begin{aligned}0 &= \iint \cos(\varphi)c(\varphi,\theta_0,\dots,\theta_3)\,d\eta\,d\varphi - 0 + \sum_{j=0}^{3}(-1)^j c^\sharp(\theta_0,\dots,\widehat{\theta}_j,\dots,\theta_3)\\ &= \int \cos(\varphi)\,c(\varphi,\theta_0,\dots,\theta_3)\,d\varphi + dc^\sharp(\theta_0,\dots,\theta_3).\end{aligned}$$

This establishes the first identity. Likewise, to prove the second identity we compute

$$\begin{aligned}L_N^{(4)}(I(c))(\theta_0,\dots,\theta_3) &= \frac{d}{dt}\bigg|_{t=0}\int \tilde{c}(\varphi,n_t.\theta_0,\dots,n_t.\theta_3)\,d\varphi\\ &= \int \frac{d}{d\varphi}\frac{d(n_t.\varphi)}{dt}\bigg|_{t=0}\tilde{c}(\varphi,\theta_0,\dots,\theta_3)\,d\varphi\\ &= \int \frac{d}{d\varphi}(1-\cos(n_t.\varphi))\bigg|_{t=0}\tilde{c}(\varphi,\theta_0,\dots,\theta_3)\,d\varphi\\ &= \int \sin(\varphi)\tilde{c}(\varphi,\theta_0,\dots,\theta_3)\,d\varphi\end{aligned}$$

and then integrate the above cocycle identity against $\sin\varphi$. □

Proof of Proposition 3.4 By Corollary 2.3, the function class

$$P_c(f) = I(c) + df$$

admits an orbitwise smooth representative for every $f \in C^\infty((S^1)^{(3)})^K$. This representative $P_c(f)$ is then smooth on $(S^1)^{(3)}$ by Lemma 2.2(i), hence G–invariant if and only if it is infinitesimally G–invariant. Since the subgroups K, A and N generate the group G, this in turn is equivalent to $P_c(f)$ satisfying the system of partial differential equations

$$\begin{cases} L_K^{(4)}P_c(f) = 0,\\ L_A^{(4)}P_c(f) = 0,\\ L_N^{(4)}P_c(f) = 0.\end{cases}$$

The first equation is automatically satisfied, since $P_c(f)$ is K–invariant by Proposition 2.7. Writing out the definition of $P_c(f)$ and applying Lemmas 3.3 and 3.5, the remaining two equations are seen to be equivalent to the system

$$\begin{cases} d(L_A^{(3)}f) = dc^\sharp,\\ d(L_N^{(3)}f) = dc^\flat.\end{cases} \tag{3-4}$$

If (3-3) admits a solution $(f, v^\sharp, v^\flat) \in C^\infty((S^1)^{(3)}) \times C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)})$, then we see by applying the operator d to both sides of (3-3) that f also satisfies

(3-4) and thus $P_c(f)$ is G–invariant. Conversely, if $P_c(f)$ is G–invariant, then (3-4) holds and, if we define measurable function classes $v^\sharp := I(L_A^{(3)} f - c^\sharp)$ and $v^\flat := I(L_N^{(3)} f - c^\flat)$, then the triple $(f, v^\sharp, v^\flat)$ satisfies (3-3). Moreover, $v^\sharp, v^\flat \in L^\infty((S^1)^2)$ admit smooth representatives on $(S^1)^{(2)}$, by Corollary 2.3 and Lemma 2.2(i). Thus, $P_c(f)$ is G–invariant if and only if (3-3) admits a solution

$$(f, v^\sharp, v^\flat) \in C^\infty((S^1)^{(3)}) \times C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)}). \qquad \square$$

3.4 The Frobenius integrability condition

We now turn to the problem of finding solutions of system (3-3). As we shall see in Proposition 3.7 below, it follows from the classical Frobenius theorem that this system admits a smooth solution $(f, v^\sharp, v^\flat)$ if and only if the functions $v^\sharp$ and $v^\flat$ satisfy a certain integrability condition. In order to state the result we define a function class $\check{c} \in L^\infty((S^1)^2)$ by

$$\check{c}(\phi_1, \phi_2) := \fint\fint\fint \sin(\eta - \varphi)\, c(\eta, \varphi, \psi, \phi_1, \phi_2)\, d\eta\, d\varphi\, d\psi. \tag{3-5}$$

Lemma 3.6 *The function class $\check{c} \in L^\infty((S^1)^2)$ can be represented by a smooth K–invariant function on $(S^1)^{(2)}$.*

Proof Smoothness is immediate from Corollary 2.3 and Lemma 2.2(i). By K–invariance of c and of the measure, for every $\xi \in [0, 2\pi]$ we have

$$\begin{aligned}
\check{c}(k_\xi.\phi_1, k_\xi.\phi_2) &= \fint\fint\fint \sin(\eta - \varphi)\, c(\eta, \varphi, \psi, \phi_1 + \xi, \phi_2 + \xi)\, d\eta\, d\varphi\, d\psi \\
&= \fint\fint\fint \sin(\eta - \varphi) c(\eta - \xi, \varphi - \xi, \psi - \xi, \phi_1, \phi_2)\, d\eta\, d\varphi\, d\psi \\
&= \fint\fint\fint \sin((\eta + \xi) - (\varphi + \xi)) c(\eta, \varphi, \psi, \phi_1, \phi_2)\, d\eta\, d\varphi\, d\psi \\
&= \check{c}(\phi_1, \phi_2). \qquad \square
\end{aligned}$$

Proposition 3.7 (Integrability condition) *There exists a solution $(f, v^\sharp, v^\flat)$ to (3-3) in $C^\infty((S^1)^{(3)}) \times C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)})$ if and only if the pair $(v^\sharp, v^\flat)$ in $C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)})$ satisfies the system of partial differential equations*

$$\begin{cases} d(L_K^{(2)} v^\sharp + v^\flat) = 0, \\ d(L_K^{(2)} v^\flat - v^\sharp) = 0, \\ d(L_K^{(2)} v^\sharp - L_N^{(2)} v^\sharp + L_A^{(2)} v^\flat - \check{c}) = 0. \end{cases} \tag{3-6}$$

The proof of the proposition relies on the Frobenius theorem and will be deferred to Appendix B. In the sequel, we shall refer to system (3-6) as the *Frobenius system*. It will be enough for our purposes to find a function f satisfying system (3-3) for some pair $(v^\sharp, v^\flat)$. We will hence not attempt to find all solutions of system (3-6). Rather, we will construct a single special solution $(v^\sharp, v^\flat)$.

Proposition 3.8 *Let $r \in C^\infty((0,2\pi),\mathbb{C})$ be a smooth complex-valued solution of the ordinary differential equation*

$$(1-e^{-i\phi})\cdot\frac{dr}{d\phi} = i\,r(\phi) - \check{c}(0,\phi). \tag{3-7}$$

By Convention 3.1, we may define a function $v \in C^\infty((S^1)^{(2)},\mathbb{C})$ by

$$v(\theta_1,\theta_2) := e^{i\theta_1}\,r(\theta_2-\theta_1). \tag{3-8}$$

Then the pair $(v^\sharp, v^\flat) := (\mathrm{Re}(v),\mathrm{Im}(v)) \in C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)})$ is a solution of the Frobenius system (3-6).

Proof We remind the reader that throughout the proof we adhere to Convention 3.1. First observe that, if v is a solution of the system

$$\begin{cases} L_K^{(2)} v = i\,v, \\ e^{-i\theta_1}\dfrac{\partial v}{\partial\theta_1} + e^{-i\theta_2}\dfrac{\partial v}{\partial\theta_2} = \check{c}, \end{cases} \tag{3-9}$$

then $(v^\sharp, v^\flat) := (\mathrm{Re}(v),\mathrm{Im}(v))$ is a solution of (3-6). Indeed, taking real and imaginary parts of the first equation in (3-9) we obtain

$$L_K^{(2)} v^\sharp = -v^\flat, \quad L_K^{(2)} v^\flat = v^\sharp,$$

while taking the real part of the second equation yields

$$L_K^{(2)} v^\sharp - L_N^{(2)} v^\sharp + L_A^{(2)} v^\flat = \check{c}.$$

Next consider the transformation

$$u(\theta_1,\theta_2) := e^{-i\theta_1}\,v(\theta_1,\theta_2).$$

Then the first equation in (3-9) is equivalent to

$$L_K^{(2)} u = 0 \tag{3-10}$$

and the second equation is equivalent to

$$\text{(3-11)}\qquad \begin{aligned}\check{c}(\theta_1,\theta_2) &= e^{-i\theta_1}\frac{\partial}{\partial\theta_1}(e^{i\theta_1}u(\theta_1,\theta_2)) + e^{-i\theta_2}\frac{\partial}{\partial\theta_2}(e^{i\theta_1}u(\theta_1,\theta_2))\\ &= iu(\theta_1,\theta_2) + \frac{\partial u}{\partial\theta_1} + e^{i(\theta_1-\theta_2)}\frac{\partial u}{\partial\theta_2}\\ &= iu(0,\theta_2-\theta_1) + (1-e^{i(\theta_1-\theta_2)})\frac{\partial u}{\partial\theta_1}.\end{aligned}$$

Here we used that $\partial_{\theta_1}u = -\partial_{\theta_2}u$, by (3-10). Now let r be a solution of (3-7) and set $v(\theta_1,\theta_2) := e^{i\theta_1}r(\theta_2-\theta_1)$. Then $u(\theta_1,\theta_2) = r(\theta_2-\theta_1)$ obviously satisfies (3-10). By the K–invariance of $\check{c}$ from Lemma 3.6, we have $\check{c}(\theta_1,\theta_2) = \check{c}(0,\theta_2-\theta_1)$. It follows that u solves (3-11). □

4 Construction of primitives

4.1 Solving the Frobenius system

Throughout this section we fix a bounded G–cocycle $c \in L^\infty((S^1)^5)^G$ and denote by

$$P_c\colon C^\infty\big((S^1)^{(3)}\big)^K \to \mathcal{P}(c)^K, \quad f \mapsto I(c) + df,$$

the operator from (2-2). Our goal is to solve the system (3-3) in order to construct a function $f \in C^\infty((S^1)^{(3)})^K$ such that $\mathcal{P}_c(f) \in \mathcal{P}(c)^G$ is a G–invariant primitive for c. The first step in the construction of this primitive is to solve the differential equation (3-7) for the function r. As we have seen in Propositions 3.8 and 3.7, the function r then gives rise to a special solution $(v^\sharp, v^\flat)$ of the Frobenius system (3-6), which in turn determines the inhomogeneities in (3-3) in such a way that this system admits a solution f.

The complex ordinary differential equation (3-7) can be solved by applying the method of variation of constants. Its general solution $r \in C^\infty((0,2\pi),\mathbb{C})$ is given by

$$r(\phi) = (1-e^{i\phi})\cdot\left(C_0 - \frac{1}{2}\int_\pi^\phi \frac{\check{c}(0,\zeta)}{1-\cos\zeta}\,d\zeta\right),$$

where C_0 is an arbitrary complex constant. Note that different choices of C_0 lead to cohomologous cochains v. We may therefore assume $C_0 = 0$, obtaining

$$\text{(4-1)}\qquad r(\phi) = -\tfrac{1}{2}\big(1-e^{i\phi}\big)\cdot\int_\pi^\phi \frac{\check{c}(0,\zeta)}{1-\cos\zeta}\,d\zeta.$$

We shall henceforth be working with the function r defined by this formula. A crucial observation is the following lemma:

Lemma 4.1 (Boundedness of the inhomogeneity) *The function $r \in C^\infty((0,2\pi),\mathbb{C})$ given by (4-1) is bounded. In particular, if $(v^\sharp, v^\flat) \in C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)})$ are defined as in Proposition 3.8, then the inhomogeneities in the system (3-3) are bounded.*

Proof For all $\phi \in (0,2\pi)$ we have

$$|r(\phi)| \le \tfrac{1}{2}|1-e^{i\phi}| \cdot \left| \int_\pi^\phi \frac{\check{c}(0,\zeta)}{1-\cos\zeta}\, d\zeta \right|. \tag{4-2}$$

To estimate this further, observe that, on the one hand,

$$|1-e^{i\phi}| = \sqrt{2}\cdot\sqrt{1-\cos\phi} \tag{4-3}$$

and, on the other hand, we have

$$\left| \int_\pi^\phi \frac{\check{c}(0,\zeta)}{1-\cos\zeta}\, d\zeta \right| \le \|\check{c}\|_\infty \cdot \left| \int_\pi^\phi \frac{1}{1-\cos\zeta}\, d\zeta \right| \le \|c\|_\infty \cdot \frac{\sqrt{1+\cos\phi}}{\sqrt{1-\cos\phi}}. \tag{4-4}$$

Here we used that $\|\check{c}\|_\infty \le \|c\|_\infty$, as well as

$$\int_\pi^\phi \frac{1}{1-\cos\zeta}\, d\zeta = -\cot(\tfrac{1}{2}\phi)$$

and

$$\left|\cot\left(\tfrac{1}{2}\phi\right)\right| = \frac{\sqrt{1+\cos\phi}}{\sqrt{1-\cos\phi}}.$$

Note that $\sqrt{1+\cos\phi} \le \sqrt{2}$. Hence, plugging (4-3) and (4-4) into (4-2) we arrive at

$$|r(\phi)| \le \tfrac{1}{2}\sqrt{2}\cdot\sqrt{1+\cos\phi}\cdot\|c\|_\infty \le \|c\|_\infty$$

for all $\phi \in (0,2\pi)$, which proves the lemma. □

4.2 Cauchy initial value problem

Recall that solutions to a first-order linear partial differential equation may be constructed explicitly by integration along its characteristic curves, with initial values prescribed on some non-characteristic hypersurface; see Carathéodory [16, Chapter 3]. We shall now apply this principle in order to explicitly construct solutions of the system (3-3).

Let $r \in C^\infty((0,2\pi),\mathbb{C})$ be given by (4-1). By Proposition 3.8 and Lemma 4.1, this determines a bounded solution $(v^\sharp, v^\flat) := (\mathrm{Re}(v), \mathrm{Im}(v)) \in C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)})$ of the Frobenius system (3-6) by

$$v(\theta_1,\theta_2) := e^{i\theta_1}\, r(\theta_2-\theta_1). \tag{4-5}$$

We shall henceforth keep the function v defined that way. Then, by Proposition 3.7, there exists a function $f \in C^\infty((S^1)^{(3)})$ satisfying the system of equations

$$L_K^{(3)} f = 0, \tag{4-6a}$$

$$L_A^{(3)} f = c^\sharp + dv^\sharp, \tag{4-6b}$$

$$L_N^{(3)} f = c^\flat + dv^\flat. \tag{4-6c}$$

We know from Section 3.2 that the characteristic curves for these equations are precisely the orbits for the actions of the subgroups K, A and N of G on $(S^1)^{(3)}$, respectively. In order to construct the function f we may therefore proceed as follows:

(1) We use (4-6a) in order to construct f with initial values $f_0 := f|_{H_0}$ prescribed on the hypersurface

$$H_0 := \{(z_0, z_1, z_2) \in (S^1)^{(3)} \mid z_0 = 1\}.$$

Of course, (4-6a) just says that f is constant along the K–orbits in $(S^1)^{(3)}$. The hypersurface H_0 is non-characteristic for (4-6a), since it intersects transversally with the K–orbits in $(S^1)^{(3)}$. Note that, in order for f to be compatible with the remaining equations (4-6b) and (4-6c), the hypersurface H_0 has to be invariant under the actions of A and N. This, however, is indeed the case since the point 1 remains fixed under these two actions.

(2) We use (4-6c) in order to construct f_0 with initial values $f_1 := f_0|_{H_1}$ prescribed on the union of A–orbits

$$H_1 := \{a_s.(1, i, -i) \mid -\infty < s < \infty\} \cup \{a_s.(1, -i, i) \mid -\infty < s < \infty\} \subset H_0.$$

Note that H_1 is non-characteristic for (4-6c) since it intersects transversally with the N–orbits in H_0. Moreover, in order for f_0 to be compatible with the remaining equation (4-6b), the curve H_1 has to be invariant under the action of A, which is obviously the case.

(3) We use (4-6b) in order to construct f_1 with initial values $f_2 := f_1|_{H_2}$ prescribed on the set of *base points*

$$H_2 := \{(1, e^{2\pi i/3}, e^{4\pi i/3}),\ (1, e^{4\pi i/3}, e^{2\pi i/3})\} \subset H_1.$$

Note that (4-6b), when restricted to the curve H_1, becomes an ordinary differential equation, which can be solved directly.

In particular, we see that solutions of the system (3-3) are uniquely determined by the initial values of f_2 on the set of base points H_2 and hence form a 2–parameter family. We will work out the details of (1) in Section 4.3, while the details of (2) and (3) will be worked out in Section 4.4.

4.3 Reduction of variables

In angular coordinates, the hypersurface H_0 introduced in the previous section is given by

$$H_0 = \{(\theta_0, \theta_1, \theta_2) \in [0, 2\pi)^3 \mid \theta_0 = 0,\ \theta_1 \neq \theta_2,\ \theta_1 \neq 0 \neq \theta_2\}.$$

The canonical projection $(\theta_0, \theta_1, \theta_2) \mapsto (\theta_1, \theta_2)$ identifies H_0 with the open subset $\Omega := (0, 2\pi)^2 \setminus \Delta$ of the square, where $\Delta \subset (0, 2\pi)^2$ denotes the diagonal in the open square. The coordinates in Ω will be denoted by (ϕ_1, ϕ_2). Moreover, we write $\Omega_\pm := \{(\phi_1, \phi_2) \mid \phi_1 \lessgtr \phi_2\}$ for the open subsets corresponding to the two G–orbits in $(S^1)^{(3)}$ consisting of positively and negatively oriented triples. The restriction $f_0 := f|_{H_0}$ is then given by $f_0(\phi_1, \phi_2) = f(0, \phi_1, \phi_2)$. Note that by K–invariance the function f is recovered from f_0 by

$$f(\theta_0, \theta_1, \theta_2) = f_0(\theta_1 - \theta_0, \theta_2 - \theta_0). \tag{4-7}$$

Since the hypersurface H_0 is invariant under the actions of A and N, the system (3-3) restricts to the system

$$\begin{cases} L_A^{(2)} f_0 = c_0^\sharp + (dv^\sharp)_0, \\ L_N^{(2)} f_0 = c_0^\flat + (dv^\flat)_0. \end{cases} \tag{4-8}$$

Here we denote by $c_0^\sharp$, $c_0^\flat$, $(dv^\sharp)_0$ and $(dv^\flat)_0$ the respective restrictions of the functions $c^\sharp$, $c^\flat$, $dv^\sharp$ and $dv^\flat$, ie

$$c_0^\sharp(\phi_1, \phi_2) := c^\sharp(0, \phi_1, \phi_2), \quad c_0^\flat(\phi_1, \phi_2) := c^\flat(0, \phi_1, \phi_2) \tag{4-9}$$

and

$$(dv^\sharp)_0(\phi_1, \phi_2) := dv^\sharp(0, \phi_1, \phi_2), \quad (dv^\flat)_0(\phi_1, \phi_2) := dv^\flat(0, \phi_1, \phi_2). \tag{4-10}$$

Note that these functions are smooth on Ω.

4.4 Method of characteristics

We now construct the function f_0 by integrating equations (4-6b) and (4-6c) along their characteristic curves. Recall that the characteristics for these equations are precisely the orbits for the actions of the subgroups A and N on Ω (see Figure 1).

It will be convenient to abbreviate the inhomogeneities appearing on the right-hand side of the system (4-8) by

$$F_c^\sharp := c_0^\sharp + (dv^\sharp)_0 \quad \text{and} \quad F_c^\flat := c_0^\flat + (dv^\flat)_0. \tag{4-11}$$

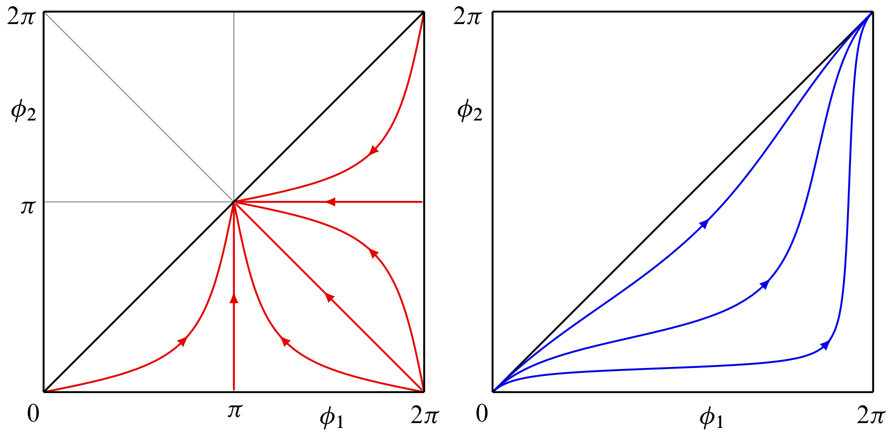

Figure 1: A–orbits (left) and N–orbits (right) in the domain Ω_-

If we prescribe the value of f_0 on a single point in each of the orbits Ω_+ and Ω_-, the function f_0 will be uniquely determined by the relations

$$f_0(a_S.\phi_1, a_S.\phi_2) - f_0(\phi_1, \phi_2) = \int_0^S F_c^\sharp(a_s.\phi_1, a_s.\phi_2)\, ds$$

and

$$f_0(n_T.\phi_1, n_T.\phi_2) - f_0(\phi_1, \phi_2) = \int_0^T F_c^\flat(n_t.\phi_1, n_t.\phi_2)\, dt.$$

More precisely, let us denote by

$$\Delta^{\mathrm{op}} := \{(\phi, 2\pi - \phi) \mid \phi \in (0, 2\pi) \setminus \{\pi\}\} \subset \Omega$$

the antidiagonal in Ω, which corresponds to the hypersurface H_1 introduced in Section 4.2. Note that it has two connected components. In order to compute the function f_0 we first introduce new coordinates on Ω that are adapted to the N–orbits. For every point $(\phi_1, \phi_2) \in \Omega$ we define $\Phi(\phi_1, \phi_2) \in (0, \pi) \cup (\pi, 2\pi)$ in such a way that $(\Phi(\phi_1, \phi_2), 2\pi - \Phi(\phi_1, \phi_2))$ is the point of intersection of the antidiagonal with the unique N–orbit passing through the point (ϕ_1, ϕ_2). We then define $T(\phi_1, \phi_2) \in (-\infty, \infty)$ by the relation

$$(\phi_1, \phi_2) = n_{T(\phi_1, \phi_2)}.(\Phi(\phi_1, \phi_2), 2\pi - \Phi(\phi_1, \phi_2)).$$

For later reference we note that

$$T(\phi_1, \phi_2) = -\tfrac{1}{2}\bigl(\cot\bigl(\tfrac{1}{2}\phi_1\bigr) + \cot\bigl(\tfrac{1}{2}\phi_2\bigr)\bigr), \tag{4-12}$$

which follows from the formula $n_t.\phi = 2\,\mathrm{arccot}(-t + \cot(\tfrac{1}{2}\phi))$. Integrating the second equation in (4-8) along the N–orbits in Ω, with initial values f_1 prescribed on the

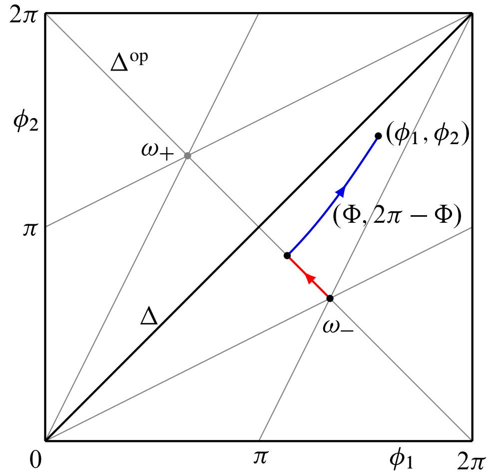

Figure 2: A path traveling along A– and N–orbits from the basepoint ω_- to some point (ϕ_1, ϕ_2) in Ω_-

antidiagonal Δ^{op}, we then obtain

$$\text{(4-13)}\quad f_0(\phi_1,\phi_2) = f_1(\Phi(\phi_1,\phi_2), 2\pi - \Phi(\phi_1,\phi_2)) + \int_0^{T(\phi_1,\phi_2)} F_c^\flat\big(n_t.\Phi(\phi_1,\phi_2), n_t.(2\pi - \Phi(\phi_1,\phi_2))\big)\,dt$$

for every $(\phi_1, \phi_2) \in \Omega$. It remains to compute the function f_1 along the antidiagonal. Let

$$\omega_+ := \left(\tfrac{2}{3}\pi, \tfrac{4}{3}\pi\right) \quad \text{and} \quad \omega_- := \left(\tfrac{4}{3}\pi, \tfrac{2}{3}\pi\right)$$

be the points in Ω corresponding to the base points in H_2 introduced in Section 4.2. Note that ω_+ and ω_- coincide with the barycenters of the triangles enclosing the domains Ω_+ and Ω_- (see Figure 2). Define a new coordinate $S(\phi) \in (-\infty, \infty)$ on each component of the antidiagonal Δ^{op} by the relation

$$(\phi, 2\pi - \phi) = a_{S(\phi)}.\omega_\pm,$$

depending on whether the point $(\phi, 2\pi - \phi)$ lies in Ω_+ or Ω_-.

Integrating the first equation in (4-8) along the A–orbits in Δ^{op}, with initial values f_2 prescribed on the base points $\{\omega_+, \omega_-\}$, we get

$$\text{(4-14)}\quad f_1(\phi, 2\pi - \phi) = f_2(\omega_\pm) + \int_0^{S(\phi)} F_c^\sharp(a_s.\omega_\pm)\,ds$$

for every $\phi \in (0, \pi) \cup (\pi, 2\pi)$.

4.5 Explicit primitives

Combining the results from the previous sections, we are now in a position to give the following explicit characterization of primitives:

Proposition 4.2 (Explicit primitives) *Let $(v^\sharp, v^\flat) \in C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)})$ be the real and imaginary parts of the function $v \in C^\infty((S^1), \mathbb{C})$ defined by (4-5), where the function $r \in C^\infty((0, 2\pi), \mathbb{C})$ is as in (4-1). Then the following hold:*

(i) *Let $f \in C^\infty((S^1)^{(3)})^K$. The primitive $P_c(f) \in \mathcal{P}(c)^K$ is G–invariant if and only if the function f solves the system (3-3).*

(ii) *There is a one-to-one correspondence between solutions $f \in C^\infty((S^1)^{(3)})^K$ of (3-3) and solutions $f_0 \in C^\infty(\Omega)$ of (4-8) via the relation*

$$f(\theta_0, \theta_1, \theta_2) = f_0(\theta_1 - \theta_0, \theta_2 - \theta_0).$$

(iii) *Every pair $(f_0(\omega_+), f_0(\omega_-)) \in \mathbb{R}^2$ of initial values uniquely determines a smooth solution $f_0 \in C^\infty(\Omega)$ of (4-8) by the formula*

$$f_0(\phi_1, \phi_2) = f_0(\omega_\pm) + \int_0^{S(\Phi(\phi_1,\phi_2))} F_c^\sharp(a_s.\omega_\pm)\, ds + \int_0^{T(\phi_1,\phi_2)} F_c^\flat\big(n_t.\Phi(\phi_1,\phi_2), n_t.(2\pi - \Phi(\phi_1,\phi_2))\big)\, dt, \tag{4-15}$$

where the functions $F_c^\sharp$ and $F_c^\flat$ are as in (4-11). Conversely, any smooth solution of (4-8) arises in this way.

Proof Let $f \in C^\infty((S^1)^{(3)})^K$. Assertion (i) holds by Proposition 3.4, while (ii) was proved in Section 4.3. Finally, our considerations in Section 4.4 show that the function f_0 satisfies (4-8) if and only if it is given by formulas (4-13) and (4-14), in terms of integration along the unique path in Ω starting at $\omega_\pm$ and traveling to (ϕ_1, ϕ_2) along A– and N–orbits via the point $(\Phi(\phi_1,\phi_2), 2\pi - \Phi(\phi_1,\phi_2))$ on the antidiagonal (see Figure 2), with initial values prescribed at $\omega_\pm$. This proves (iii). □

The proposition achieves the second step in the agenda outlined in Section 2.7. In particular, it shows that solutions f_0 of (4-8) form a 2–parameter family.

5 Boundedness of primitives

5.1 Symmetries

Our construction of primitives in the previous section was valid for arbitrary G–cocycles $c \in L^\infty((S^1)^5)^G$. However, for the proof of boundedness of primitives, which we shall discuss in this section, it turns out to be essential that c be alternating and even. As we have seen in Section 2.6, this is not a loss of generality. The following proposition discusses symmetries of the various functions appearing in the construction of primitives resulting from these additional assumptions.

Proposition 5.1 (Symmetries) *Assume that the cocycle $c \in L^\infty((S^1)^5)^G$ is alternating and even. Let $r \in C^\infty((0,2\pi),\mathbb{C})$ be given by (4-1) and let $(v^\sharp, v^\flat)$ in $C^\infty((S^1)^{(2)}) \times C^\infty((S^1)^{(2)})$ be defined as in Proposition 3.8. Then the inhomogeneities on the right-hand sides of systems (3-3) and (4-8) have the following properties:*

(i) *The functions $c^\sharp + dv^\sharp$ and $c^\flat + dv^\flat$ are alternating.*

(ii) *The function $F_c^\sharp = c_0^\sharp + (dv^\sharp)_0$ is antisymmetric about the antidiagonal Δ^{op} in Ω, ie*

$$F_c^\sharp(\phi_1,\phi_2) = -F_c^\sharp(-\phi_2,-\phi_1)$$

for all $(\phi_1,\phi_2) \in \Omega$. In particular, it vanishes along the antidiagonal.

(iii) *The function $F_c^\flat = c_0^\flat + (dv^\flat)_0$ is symmetric about the antidiagonal Δ^{op} in Ω, ie*

$$F_c^\flat(\phi_1,\phi_2) = F_c^\flat(-\phi_2,-\phi_1)$$

for all $(\phi_1,\phi_2) \in \Omega$.

Proof We begin with the following observation. The function $\check{c}$ defined in (3-5) is alternating since c is assumed to be alternating. By Lemma 3.6, the function $\check{c}$ is K–invariant. Hence

$$\check{c}(0,\zeta) = \check{c}(-\zeta,0) = -\check{c}(0,-\zeta).$$

By Convention 3.1, replacing ζ by $2\pi-\zeta$ we infer from this that

$$\int_\pi^\phi \frac{\check{c}(0,\zeta)}{1-\cos\zeta}\,d\zeta = -\int_\pi^{2\pi-\phi} \frac{\check{c}(0,-\zeta)}{1-\cos(-\zeta)}\,d\zeta = \int_\pi^{-\phi} \frac{\check{c}(0,\zeta)}{1-\cos\zeta}\,d\zeta \tag{5-1}$$

for every $\phi \in (0,2\pi)$. Recall moreover from Section 4.2 that $(v^\sharp, v^\flat) := (\mathrm{Re}(v), \mathrm{Im}(v))$, where

$$v(\theta_1,\theta_2) = e^{i\theta_1} r(\theta_2-\theta_1)$$

and r is as in (4-1). Let us prove (i). Since c is alternating, it is immediate from (3-1) and (3-2) that $c^\sharp$ and $c^\flat$ are alternating. By (4-1) and (5-1) we have

$$v(\theta_1,\theta_2) = e^{i\theta_1} r(\theta_2-\theta_1) \tag{5-2}$$
$$= -\tfrac{1}{2}(e^{i\theta_1}-e^{i\theta_2})\cdot\int_\pi^{\theta_2-\theta_1}\frac{\check{c}(0,\zeta)}{1-\cos\zeta}\,d\zeta$$
$$= \tfrac{1}{2}(e^{i\theta_2}-e^{i\theta_1})\cdot\int_\pi^{\theta_1-\theta_2}\frac{\check{c}(0,\zeta)}{1-\cos\zeta}\,d\zeta = -v(\theta_2,\theta_1).$$

It follows that dv, and hence $dv^\sharp$ and $dv^\flat$, are alternating. This proves (i). For the proof of (ii) and (iii) we have to show that

$$F_c^\sharp(\phi_1,\phi_2) = -F_c^\sharp(-\phi_2,-\phi_1) \quad\text{and}\quad F_c^\flat(\phi_1,\phi_2) = F_c^\flat(-\phi_2,-\phi_1). \tag{5-3}$$

To this end, we first note that, by (4-9) and since c is even, we have

$$c_0^\sharp(-\phi_2,-\phi_1) = \iint \cos(\varphi)c(\eta,\varphi,0,-\phi_2,-\phi_1)\,d\eta\,d\varphi \tag{5-4}$$
$$= \iint \cos(\varphi)c(-\eta,-\varphi,0,\phi_2,\phi_1)\,d\eta\,d\varphi$$
$$= \iint \cos(-\varphi)c(\eta,\varphi,0,\phi_2,\phi_1)\,d\eta\,d\varphi$$
$$= -\iint \cos(\varphi)c(\eta,\varphi,0,\phi_1,\phi_2)\,d\eta\,d\varphi$$
$$= -c_0^\sharp(\phi_1,\phi_2)$$

and, similarly,

$$c_0^\flat(-\phi_2,-\phi_1) = \iint \sin(\varphi)c(\eta,\varphi,0,-\phi_2,-\phi_1)\,d\eta\,d\varphi \tag{5-5}$$
$$= \iint \sin(-\varphi)c(\eta,\varphi,0,\phi_2,\phi_1)\,d\eta\,d\varphi$$
$$= \iint \sin(\varphi)c(\eta,\varphi,0,\phi_1,\phi_2)\,d\eta\,d\varphi = c_0^\flat(\phi_1,\phi_2).$$

Applying (5-1) as in the proof of (i) above, we obtain

$$v(-\theta_1,-\theta_2) = -\tfrac{1}{2}(e^{-i\theta_1}-e^{-i\theta_2})\cdot\int_\pi^{-\theta_2+\theta_1}\frac{\check{c}(0,\zeta)}{1-\cos\zeta}\,d\zeta$$
$$= -\tfrac{1}{2}(e^{-i\theta_1}-e^{-i\theta_2})\cdot\int_\pi^{\theta_2-\theta_1}\frac{\check{c}(0,\zeta)}{1-\cos\zeta}\,d\zeta = \overline{v(\theta_1,\theta_2)}.$$

Combining this with (5-2) we arrive at

$$v(\theta_1,\theta_2) = -\overline{v(-\theta_2,-\theta_1)}.$$

Consider the function $(dv)_0(\phi_1,\phi_2) := dv(0,\phi_1,\phi_2)$. The previous two identities imply that

$$\begin{aligned}(dv)_0(\phi_1,\phi_2) &= v(\phi_1,\phi_2) - v(0,\phi_2) + v(0,\phi_1)\\ &= -\overline{(v(-\phi_2,-\phi_1) - v(0,-\phi_1) + v(0,-\phi_2))}\\ &= -\overline{(dv)_0(-\phi_2,-\phi_1)}.\end{aligned}$$

Recall from (4-10) that $(dv^\sharp)_0$ and $(dv^\flat)_0$ are the real and imaginary parts of $(dv)_0$. Hence we conclude that

$$\begin{aligned}(dv^\sharp)_0(\phi_1,\phi_2) &= -(dv^\sharp)_0(-\phi_2,-\phi_1),\\ (dv^\flat)_0(\phi_1,\phi_2) &= (dv^\flat)_0(-\phi_2,-\phi_1).\end{aligned} \tag{5-6}$$

The identities (5-3) now follow from (5-4), (5-5) and (5-6), which proves (ii)–(iii). □

Next we consider symmetries of the solutions of (4-8). We introduce some notation first. The $\mathfrak{S}_3$–action on $(S^1)^{(3)}$ commutes with the K–action, whence it descends to an action on Ω. To describe this action explicitly, we denote by s_1 and s_2 the Coxeter generators of $\mathfrak{S}_3$ that act on $(S^1)^{(3)}$ by swapping coordinates in the pairs (θ_0,θ_1) and (θ_1,θ_2), respectively. Then, with respect to the coordinates (ϕ_1,ϕ_2) on Ω, the actions of s_1 and s_2 are given by

$$s_1.(\phi_1,\phi_2) = (-\phi_1,\phi_2-\phi_1) \quad \text{and} \quad s_2.(\phi_1,\phi_2) = (\phi_2,\phi_1).$$

A function $h_0 \in C^\infty(\Omega)$ will be called *alternating under the action of* $\mathfrak{S}_3$ if $s.h_0 = (-1)^s h_0$ for all $s \in \mathfrak{S}_3$. Thus a function $h_0 \in C^\infty(\Omega)$ is alternating under the action of $\mathfrak{S}_3$ if and only if the function $h \in C^\infty((S^1)^{(3)})^K$ defined by

$$h(\theta_0,\theta_1,\theta_2) = h_0(\theta_1-\theta_0,\theta_2-\theta_0)$$

is alternating in the usual sense.

Proposition 5.2 (Alternating solutions) *Assume that the cocycle $c \in L^\infty((S^1)^5)^G$ is alternating and even. A solution $f_0 \in C^\infty(\Omega)$ of (4-8) is alternating under the action of $\mathfrak{S}_3$ if and only if*

$$f_0(\omega_+) = -f_0(\omega_-). \tag{5-7}$$

In this case the primitive $P_c(f) \in \mathcal{P}(c)^G$, where $f \in C^\infty((S^1)^{(3)})^K$ is defined by (4-7), is alternating.

Proof First of all, we observe that $\mathfrak{S}_3$ acts on the base points $\{\omega_+, \omega_-\}$ by

$$s_1.\omega_\pm = \omega_\mp, \quad s_2.\omega_\pm = \omega_\mp. \tag{5-8}$$

Hence, if f_0 is alternating under the action of $\mathfrak{S}_3$ it follows that

$$f_0(\omega_+) = (-1)^{s_1} f_0(s_1.\omega_+) = -f_0(\omega_-).$$

Conversely, let f_0 be a solution of system (4-8) that satisfies (5-7). We will prove that f_0 coincides with its antisymmetrization under the action of $\mathfrak{S}_3$. By Proposition 4.2(ii), the function f_0 corresponds to a solution $f \in C^\infty((S^1)^{(3)})^K$ of (3-3) via

$$f(\theta_0, \theta_1, \theta_2) = f_0(\theta_1 - \theta_0, \theta_2 - \theta_0).$$

Now let

$$\widehat{f} := \frac{1}{6} \cdot \sum_{s \in \mathfrak{S}_3} (-1)^s \, s.f$$

be the antisymmetrization of f. Then $\widehat{f} \in C^\infty((S^1)^{(3)})^K$, and we further claim that $\widehat{f}$ solves (3-3) as well. To see this, observe that in (3-3) the operators $L_A^{(3)}$ and $L_N^{(3)}$ are symmetric, while by Proposition 5.1(i) the inhomogeneities $c^\sharp + dv^\sharp$ and $c^\flat + dv^\flat$ are alternating. Now, by K–invariance, the function $\widehat{f}$ gives rise to a function $\widehat{f_0} \in C^\infty(\Omega)$ via

$$\widehat{f}(\theta_0, \theta_1, \theta_2) = \widehat{f_0}(\theta_1 - \theta_0, \theta_2 - \theta_0).$$

Then Proposition 4.2(ii) implies that $\widehat{f_0}$ solves the system (4-8). Moreover, we have

$$\widehat{f_0} = \frac{1}{6} \cdot \sum_{s \in \mathfrak{S}_3} (-1)^s \, s.f_0,$$

whence $\widehat{f_0}$ is alternating under the action of $\mathfrak{S}_3$. It follows from (5-7) and (5-8) that $\widehat{f_0}(\omega_\pm) = f_0(\omega_\pm)$. The uniqueness statement in Proposition 4.2(iii) implies that $\widehat{f_0}$ coincides with f_0. □

The proposition shows that solutions f_0 of (4-8) that are alternating under the action of $\mathfrak{S}_3$ form a 1–parameter family.

5.2 Boundedness

In order to complete the proof of Theorem 1.1 it remains to show that, among the G–invariant primitives we constructed in Section 4, there actually exist bounded ones. This is the content of the next proposition, which crucially relies on the symmetries unveiled in the previous section.

Proposition 5.3 (Boundedness) *Assume that the cocycle $c \in L^\infty((S^1)^5)^G$ is alternating and even. Let $f_0 \in C^\infty(\Omega)$ be a solution of (4-8) that is alternating under the action of $\mathfrak{S}_3$. Define $f \in C^\infty((S^1)^{(3)})^K$ by (4-7). Then the corresponding primitive $P_c(f) \in \mathcal{P}(c)^G$ is bounded.*

The proof of the proposition relies on the following three basic observations:

Lemma 5.4 *Let the function $f_0 \in C^\infty(\Omega)$ be defined by (4-15). If f_0 is bounded along the line segments*

$$\big(0, \tfrac{2}{3}\pi\big) \cup \big(\tfrac{2}{3}\pi, 2\pi\big) \ni \xi \mapsto \big(\tfrac{2}{3}\pi, \xi\big)$$

and

$$\big(0, \tfrac{4}{3}\pi\big) \cup \big(\tfrac{4}{3}\pi, 2\pi\big) \ni \xi \mapsto \big(\tfrac{4}{3}\pi, \xi\big),$$

and $f \in C^\infty((S^1)^{(3)})^K$ is given by (4-7), then the corresponding primitive $P_c(f)$ in $\mathcal{P}(c)^G$ is bounded.

Proof By Proposition 4.2, $P_c(f) = I(c) + df$ is G–invariant. By 3–transitivity of the G–action on S^1 and since $I(c)$ is bounded, we therefore deduce that $P_c(f)$ is bounded if and only if the function

$$z \mapsto df(1, e^{2\pi i/3}, e^{4\pi i/3}, z)$$

is bounded. Writing $z = e^{i\xi}$, we may express this function as

$$\begin{aligned}\xi \mapsto f\big(\tfrac{2}{3}\pi, \tfrac{4}{3}\pi, \xi\big) - f\big(0, \tfrac{4}{3}\pi, \xi\big) + f\big(0, \tfrac{2}{3}\pi, \xi\big) - f\big(0, \tfrac{2}{3}\pi, \tfrac{4}{3}\pi\big)\\ = f_0\big(\tfrac{2}{3}\pi, \xi - \tfrac{2}{3}\pi\big) - f_0\big(\tfrac{4}{3}\pi, \xi\big) + f_0\big(\tfrac{2}{3}\pi, \xi\big) - f_0\big(\tfrac{2}{3}\pi, \tfrac{4}{3}\pi\big).\end{aligned}$$

The lemma follows. □

Lemma 5.5 *Let C be a compact subset of the open square $(0, 2\pi)^2$. If the function $f_0 \in C^\infty(\Omega)$ defined by (4-15) is bounded along the antidiagonal Δ^{op} in Ω, then it is bounded on the subset $C \cap \Omega$ of Ω.*

Proof By Lemma 4.1 the function $F_c^\flat = c_0^\flat + (dv^\flat)_0$ is bounded. Moreover, by assumption, the function f_0 is bounded along the antidiagonal Δ^{op} in Ω, ie there exists a number $M > 0$ such that $|f_0|_{\Delta^{\mathrm{op}}}| \leq M$. Hence we obtain from (4-15) the estimate

$$|f_0(\phi_1, \phi_2)| \leq M + \|F_c^\flat\|_\infty \cdot |\phi_1, \phi_2)|$$

for all $(\phi_1, \phi_2) \in \Omega$. It remains to show that the function T is bounded on $C \cap \Omega$. By compactness of C it will be enough to prove that the function $T\colon \Omega \to \mathbb{R}$ extends to a continuous function on the open square $(0, 2\pi)^2$. This, however, is immediate from (4-12). □

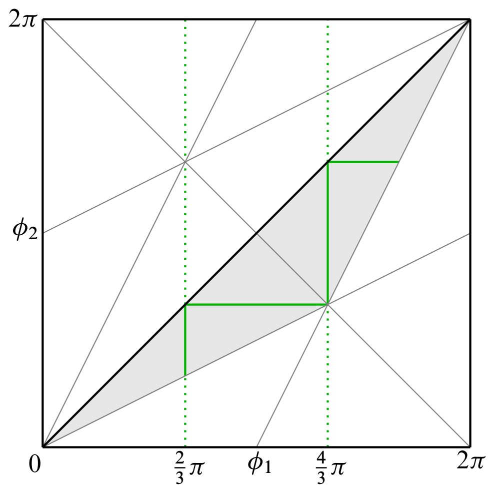

Figure 3: The images of the line segments $\xi \mapsto (\frac{2}{3}\pi, \xi)$ and $\xi \mapsto (\frac{4}{3}\pi, \xi)$ under the $\mathfrak{S}_3$–action on Ω that are contained in a fundamental domain (shaded) for this action

Lemma 5.6 *Assume that the cocycle $c \in L^\infty((S^1)^5)^G$ is alternating. Then the function $f_0 \in C^\infty(\Omega)$ defined by (4-15) is locally constant along the antidiagonal Δ^{op} in Ω.*

Proof Since c is alternating, the inhomogeneity $F_c^\sharp$ vanishes along the antidiagonal Δ^{op} by Proposition 5.1(ii). The lemma now follows from (4-15). □

Example 5.7 Assume that the cocycle c is alternating. Consider the special solution f_0 determined by the initial values $f_0(\omega_\pm) = 0$. It is alternating under the action of $\mathfrak{S}_3$ by Proposition 5.2. Moreover, by Lemma 5.6 it vanishes along the antidiagonal Δ^{op}. Since under the action of $\mathfrak{S}_3$ the components of Δ^{op} get identified with the medians of the triangles enclosing the domains Ω_+ and Ω_-, we further infer that f_0 also vanishes along these medians. Moreover, by Proposition 5.1(iii) the function $F_c^\flat$ is symmetric about the antidiagonal. Thus we see from (4-15) that the special solution f_0 is antisymmetric with respect to the antidiagonal and, hence, antisymmetric with respect to all medians.

We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3 By Proposition 4.2(iii) the function f_0 is given by (4-15). Hence, by Lemma 5.4 it suffices to show that f_0 is bounded along the line segments $\xi \mapsto (\frac{2}{3}\pi, \xi)$ and $\xi \mapsto (\frac{4}{3}\pi, \xi)$. Since f_0 is alternating under the action of $\mathfrak{S}_3$, it suffices to prove that f_0 is bounded along the images of these line segments in any fundamental domain for the $\mathfrak{S}_3$–action on Ω. In fact, we may choose the fundamental domain in such a way that the image line segments lie inside a compact subset of $(0, 2\pi)^2$ (see Figure 3). By Lemmas 5.5 and 5.6, the function f_0 is then bounded on these line segments. □

Theorem 1.1 follows by combining Propositions 4.2, 5.2 and 5.3.

Appendix A: Vanishing of odd cocycles

The goal of this appendix is to prove Proposition 2.6, which states that every bounded alternating G–invariant 4–cocycle is necessarily even. Our strategy here is inspired by Burger and Monod [15] in that we consider Fourier transforms of cocycles and study the conditions imposed on them by G–invariance. Given $n \in \mathbb{N}_0$, we denote by $c(\mathbb{Z}^{n+1})$ the space of complex-valued sequences indexed by $\mathbb{Z}^{n+1}$ and by $\ell^2(\mathbb{Z}^{n+1})$ the subspace of square-summable sequences. We denote by $e_0, \dots, e_n$ the standard basis of $\mathbb{Z}^{n+1}$ and use the multi-index notation

$$\underline{k} := (k_0, \dots, k_n) := \sum_{j=0}^{n} k_j e_j.$$

We then write $c_{\mathrm{alt}}(\mathbb{Z}^{n+1})$ and $\ell^2_{\mathrm{alt}}(\mathbb{Z}^{n+1})$, respectively, for the corresponding subspaces of alternating sequences and define two linear operators $A_{\pm}^{(n)} : \ell^2_{\mathrm{alt}}(\mathbb{Z}^{n+1}) \to c_{\mathrm{alt}}(\mathbb{Z}^{n+1})$ by

$$(A_{+}^{(n)} F)(\underline{k}) := \sum_{j=0}^{n} (k_j + 1) \cdot F(\underline{k} + e_j),$$

$$(A_{-}^{(n)} F)(\underline{k}) := \sum_{j=0}^{n} (k_j - 1) \cdot F(\underline{k} - e_j).$$

Definition A.1 An element $C \in \ell^2_{\mathrm{alt}}(\mathbb{Z}^{n+1})$ is called a *combinatorial n–cocycle* if the following hold:

(i) $C(k_0, \dots, k_n) = 0$ unless $k_j = 0$ for precisely one $j \in \{0, \dots, n\}$.

(ii) $C(k_0, \dots, k_n) = 0$ unless $k_0 + \dots + k_n = 0$.

(iii) $C \in \ker(A_{+}^{(n)}) \cap \ker(A_{-}^{(n)})$.

According to [15, Section 3.1], the Fourier transform $\hat{c}$ of a G–invariant alternating bounded n–cocycle c is a combinatorial n–cocycle. Observe that c is even if and only if $\hat{c}$ is *even* in the sense that $\hat{c}(-\underline{k}) = \hat{c}(\underline{k})$. Proposition 2.6 will therefore be a consequence of the following combinatorial result:

Proposition A.2 *Every combinatorial* 4*–cocycle is even.*

For the proof of Proposition A.2 we need two preparatory lemmas. Given a combinatorial n–cocycle C we let $\operatorname{supp}(C) := \{\underline{k} \in \mathbb{Z}^{n+1} \mid C(\underline{k}) \neq 0\}$ be the *support* of C.

Lemma A.3 *Let* C *be a combinatorial* 4*–cocycle and* $(k_0, k_1, k_2, k_3, k_4) \in \operatorname{supp}(C)$*. Then there exists* $\sigma \in \mathfrak{S}_5$ *such that*

$$k_{\sigma(0)} < k_{\sigma(1)} < k_{\sigma(2)} = 0 < k_{\sigma(3)} < k_{\sigma(4)}.$$

Proof Since C is alternating and vanishes unless precisely one of its entries is 0, it suffices to show that C vanishes on those $\underline{k} \in \mathbb{Z}^5$ which satisfy either

$$k_0 > k_1 = 0 > k_2 > k_3 > k_4 \tag{A-1}$$

or $k_0 < k_1 = 0 < k_2 < k_3 < k_4$. We are going to show $C(k_0, 0, k_2, k_3, k_4) = 0$ whenever $\underline{k}$ satisfies (A-1) and leave the second, analogous case to the reader. Our proof will be by induction on k_0.

If $k_0 \leq 5$ then the condition $k_2 + k_3 + k_4 = -k_0$ cannot be satisfied for $\underline{k}$ satisfying (A-1), hence $C(\underline{k}) = 0$. Otherwise, we use $C \in \ker(A_+^{(4)})$ to deduce that

$$\begin{aligned} 0 &= (A_+^{(4)} C)(k_0 - 1, 0, k_2, k_3, k_4) \\ &= k_0 \cdot C(k_0, 0, k_2, k_3, k_4) + 0 + (k_2 + 1) \cdot C(k_0 - 1, 0, k_2 + 1, k_3, k_4) \\ &\qquad + (k_3+1) \cdot C(k_0 - 1, 0, k_2, k_3 + 1, k_4) + (k_2 + 1) \cdot C(k_0 - 1, 0, k_2, k_3, k_4 + 1). \end{aligned}$$

The third summand on the right-hand side vanishes by antisymmetry if $k_2 + 1 = 0$, and by the induction hypothesis otherwise. Similarly, the last two summands vanish. Now the assumption $k_0 \neq 0$ implies $C(k_0, 0, k_2, k_3, k_4) = 0$. □

Lemma A.4 *Let* C *and* D *be combinatorial* 4*–cocycles such that*

$$C(-n-1, -n, 0, n, n+1) = D(-n-1, -n, 0, n, n+1)$$

holds for all $n > 0$*. Then* $C = D$*.*

Proof Since the space of combinatorial cocycles is linear, we may assume $D = 0$ and hence

$$C(-n-1, -n, 0, n, n+1) = 0 \quad \text{for all } n > 0.$$

We have to show that $C=0$. Using that C is alternating and Lemma A.3, it suffices to prove that $C(k_0,k_1,0,k_3,k_4)=0$ whenever $k_0<k_1<0<k_3<k_4$. Since $C\in\ker(A_+^{(4)})$ we have

$$\begin{aligned}0&=(A_+^{(4)}C)(k_0,k_1,0,k_3,k_4-1)\\&=(k_0+1)\cdot C(k_0+1,k_1,0,k_3,k_4-1)+(k_1+1)\cdot C(k_0,k_1+1,0,k_3,k_4-1)+0\\&\qquad+(k_3+1)\cdot C(k_0,k_1,0,k_3+1,k_4-1)+k_4\cdot C(k_0,k_1,0,k_3,k_4).\end{aligned}$$

We may rewrite this as

$$\text{(A-2)}\qquad \begin{aligned}C(k_0,k_1,0,k_3,k_4)=&-\frac{k_0+1}{k_4}\cdot C(k_0+1,k_1,0,k_3,k_4-1)\\&-\frac{k_1+1}{k_4}\cdot C(k_0,k_1+1,0,k_3,k_4-1)\\&-\frac{k_3+1}{k_4}\cdot C(k_0,k_1,0,k_3+1,k_4-1).\end{aligned}$$

Now we iterate this recursion. In each step we get a sum of terms of the form $C(k_0,k_1,0,k_3,k_4)$, where the distance between k_3 and k_4 is smaller than in the previous step. We can thus run the iteration until we arrive at terms of the form $C(k_0,k_1,0,n,n+1)$ with $0<n$. We may furthermore assume that $k_0<k_1<0$, since C is alternating. It then remains to show that

$$\text{(A-3)}\qquad C(k_0,k_1,0,n,n+1)=0\quad\text{for all } k_0<k_1<0<n.$$

We prove this by a double induction on n and $|k_0|$.

If $n=1$ then the condition $k_0+k_1=-2n-1$ forces $(k_0,k_1)=(-2,-1)$ and we are done by hypothesis. Now assume that $n>1$ is arbitrary. The condition $k_0+k_1=-2n-1$ forces $|k_0|\geq n+1$. If $|k_0|=n+1$ then $(k_0,k_1)=(-n-1,-n)$ and we are again done by hypothesis. It thus remains to show (A-3) for $n>1$ and $|k_0|>n+1$, where we assume

$$C(k_0',k_1,0,n',n'+1)=0$$

if either $n'<n$, or $n=n'$ and $|k_0'|<|k_0|$.

Now since $C\in\ker(A_-^{(4)})$ we have

$$\begin{aligned}0&=(A_-^{(4)}C)(k_0+1,k_1,0,n,n+1)\\&=k_0\cdot C(k_0,k_1,0,n,n+1)+(k_1-1)\cdot C(k_0+1,k_1-1,0,n,n+1)+0\\&\qquad+(n-1)\cdot C(k_0+1,k_1,0,n-1,n+1)+n\cdot C(k_0+1,k_1,0,n,n).\end{aligned}$$

The second of the five summands on the right-hand side vanishes by induction hypothesis of the inner induction on k_0, while the last summand vanishes by antisymmetry. Since $k_0 < -n-1 < -2$ we have $k_0 \neq 0$ and thus

$$C(k_0, k_1, 0, n, n+1) = -\frac{n-1}{k_0} \cdot C(k_0+1, k_1, 0, n-1, n+1).$$

To deal with the expression on the right-hand side we again apply (A-2) with $k_3 = n-1$ and $k_4 = n+1$. We thereby find

$$\begin{aligned} C(k_0, k_1, 0, n, n+1) = & \frac{(n-1)\cdot(k_0+1)}{k_0 \cdot (n+1)} C(k_0+1, k_1, 0, n-1, n) \\ & + \frac{(n-1)\cdot(k_1+1)}{k_0 \cdot (n+1)} C(k_0, k_1+1, 0, n-1, n) \\ & + \frac{(n-1)\cdot n}{k_0 \cdot (n+1)} C(k_0, k_1, 0, n, n). \end{aligned}$$

Here, the first two summands vanish by the induction hypothesis of the outer induction on n and the last summand vanishes by antisymmetry. This shows that $C(k_0, k_1, 0, n, n+1) = 0$ and finishes the proof of the lemma. □

Proof of Proposition A.2 Given a combinatorial 4–cocycle C we define a function $D : \mathbb{Z}^5 \to \mathbb{C}$ by $D(k_0, \dots, k_4) := C(-k_0, \dots, -k_4)$. We claim that D is a combinatorial cocycle. Since C is alternating and in ℓ^2, D is alternating and in ℓ^2 as well. Conditions (i) and (ii) are obvious. Since $C \in \ker(A_-^{(4)})$ we have

$$\begin{aligned} (A_+^{(4)} D)(\underline{k}) &= \sum_{j=0}^{4} (k_j+1) \cdot D(\underline{k}+e_j) = -\sum_{j=0}^{4} (-k_j - 1) \cdot C(-(\underline{k}+e_j)) \\ &= -\sum_{j=0}^{4} (-k_j-1) \cdot C(-\underline{k} - e_j) = -(A_-^{(4)} C)(-k_0, \dots, -k_4) = 0, \end{aligned}$$

which shows that $D \in \ker(A_+^{(4)})$. Dually, $C \in \ker(A_+^{(4)})$ implies $D \in \ker(A_-^{(4)})$, which finishes the proof that D is a combinatorial cocycle.

On the other hand, antisymmetry of C yields

$$D(-n-1, -n, 0, n, n+1) = C(n+1, n, 0, -n, -n-1) = C(-n-1, -n, 0, n, n+1).$$

Now Lemma A.4 implies that $C(\underline{k}) = D(\underline{k}) = C(-\underline{k})$, which means that C is even. □

This finishes the proof of Proposition 2.6.

Appendix B: The Frobenius integrability condition

The goal of this appendix is to prove Proposition 3.7. We have to show that the system

$$\begin{cases} L_K^{(3)} f = 0, \\ L_A^{(3)} f = c^\sharp + dv^\sharp, \\ L_N^{(3)} f = c^\flat + dv^\flat, \end{cases} \tag{B-1}$$

admits a solution $(f, v^\sharp, v^\flat)$ if and only if the pair $(v^\sharp, v^\flat)$ satisfies the Frobenius system (3-6). Here we consider f and $v^\sharp$, $v^\flat$ as smooth functions on the domains $D := [0, 2\pi)^{(3)}$ and $[0, 2\pi)^{(2)}$, respectively. It will be convenient to replace (B-1) by the equivalent system

$$\begin{cases} L_K^{(3)} f = 0, \\ L_A^{(3)} f = c^\sharp + dv^\sharp, \\ L_N^{(3)} f - L_K^{(3)} f = c^\flat + dv^\flat. \end{cases} \tag{B-2}$$

Consider the product $D \times \mathbb{R}$. We denote the coordinates on D by $(\theta_0, \theta_1, \theta_2)$ and the coordinate on $\mathbb{R}$ by θ_3. The graph $\Gamma_f := \{((\theta_0, \theta_1, \theta_2), f(\theta_0, \theta_1, \theta_2))\}$ of the function f is a 3–dimensional submanifold of $D \times \mathbb{R}$. Define vector fields on $D \times \mathbb{R}$ by

$$\begin{aligned} X &:= L_K^{(3)}, \\ Y &:= L_A^{(3)} + \left(c^\sharp + dv^\sharp\right) \partial_{\theta_3}, \\ Z &:= L_N^{(3)} - L_K^{(3)} + \left(c^\flat + dv^\flat\right) \partial_{\theta_3}. \end{aligned}$$

Since G acts strictly 3–transitively on D, it follows that these vector fields span a distribution E of constant rank 3 on $D \times \mathbb{R}$. Then a triple $(f, v^\sharp, v^\flat)$ is a solution of (B-2) if and only if the graph Γ_f is an integral manifold for E. Hence the Frobenius theorem (see eg Lee [35, Chapter 11]) implies that (B-2) admits a solution $(f, v^\sharp, v^\flat)$ if and only if the distribution E is integrable, ie the vector fields X, Y, Z form an involutive system. Note that

$$[L_K^{(3)}, L_A^{(3)}] = L_K^{(3)} - L_N^{(3)}, \quad [L_K^{(3)}, L_N^{(3)} - L_K^{(3)}] = L_A^{(3)}, \quad [L_A^{(3)}, L_N^{(3)} - L_K^{(3)}] = L_K^{(3)}.$$

Hence the vector fields X, Y, Z form an involutive system if and only if

$$[X, Y] = -Z, \quad [X, Z] = Y, \quad [Y, Z] = X. \tag{B-3}$$

We shall now make these conditions explicit. We start with two preliminary lemmas.

Lemma B.1 *The functions $c^\sharp$ and $c^\flat$ defined in (3-1) and (3-2) satisfy*

$$L_K^{(3)} c^\sharp = -c^\flat, \quad L_K^{(3)} c^\flat = c^\sharp.$$

Proof More generally, we prove that, for any function $\lambda \in C^\infty((0,2\pi))$, the function

$$c_\lambda(\theta_0,\theta_1,\theta_2) := \fint\!\!\fint \lambda(\varphi)\, c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi$$

satisfies $L^{(3)}_K c_\lambda = c_{\lambda'}$. Indeed, by K–invariance of the cocycle c and the measure, we have

$$\begin{aligned}
L^{(3)}_K c_\lambda(\theta_0,\theta_1,\theta_2) &= \frac{d}{d\xi}\bigg|_{\xi=0} \fint\!\!\fint \lambda(\varphi) c(\eta,\varphi,\theta_0+\xi,\theta_1+\xi,\theta_2+\xi)\, d\eta\, d\varphi \\
&= \frac{d}{d\xi}\bigg|_{\xi=0} \fint\!\!\fint \lambda(\varphi) c(\eta-\xi,\varphi-\xi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi \\
&= \fint\!\!\fint \frac{d}{d\xi}\bigg|_{\xi=0} \lambda(\varphi+\xi) c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi \\
&= \fint\!\!\fint \lambda'(\varphi) c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi. \qquad \square
\end{aligned}$$

Lemma B.2 *The function $\check{c}$ defined in (3-5) satisfies*

$$L^{(3)}_K c^\sharp - L^{(3)}_N c^\sharp + L^{(3)}_A c^\flat = -d\check{c}. \tag{B-4}$$

Proof Let us consider the left-hand side of (B-4). In a first step, using G–invariance of c and K–invariance of the measure, we compute

$$\begin{aligned}
L^{(3)}_K c^\sharp(\theta_0,\theta_1,\theta_2) &= \frac{d}{d\xi}\bigg|_{\xi=0} \fint\!\!\fint \cos(\varphi) c(\eta,\varphi,\theta_0+\xi,\theta_1+\xi,\theta_2+\xi)\, d\eta\, d\varphi \\
&= \fint\!\!\fint \frac{d}{d\xi} \cos(\varphi+\xi)\bigg|_{\xi=0} c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi
\end{aligned}$$

and

$$\begin{aligned}
&L^{(3)}_N c^\sharp(\theta_0,\theta_1,\theta_2) \\
&\quad = \frac{d}{dt}\bigg|_{t=0} \fint\!\!\fint \cos(\varphi) c(\eta,\varphi,n_t.\theta_0,n_t.\theta_1,n_t.\theta_2)\, d\eta\, d\varphi \\
&\quad = \fint\!\!\fint \frac{d}{dt}\left(\cos(n_t.\varphi)\frac{d(n_t.\varphi)}{d\varphi}\frac{d(n_t.\eta)}{d\eta}\right)\bigg|_{t=0} c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi
\end{aligned}$$

and, similarly,

$$
\begin{aligned}
&L_A^{(3)} c^\flat(\theta_0,\theta_1,\theta_2)\\
&\qquad = \frac{d}{ds}\bigg|_{s=0} \iint \sin(\varphi) c(\eta,\varphi,a_s.\theta_0,a_s.\theta_1,a_s.\theta_2)\,d\eta\,d\varphi\\
&\qquad = \iint \frac{d}{ds}\left(\sin(a_s.\varphi)\frac{d(a_s.\varphi)}{d\varphi}\frac{d(a_s.\eta)}{d\eta}\right)\bigg|_{s=0} c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi.
\end{aligned}
$$

Second, using Lemma 3.2 we compute the derivatives appearing in the above formulas. Firstly, we have $d\cos(\varphi+\xi)/d\xi\big|_{\xi=0} = -\sin(\varphi)$. Moreover,

$$
\begin{aligned}
&\frac{d}{dt}\left(\cos(n_t.\varphi)\frac{d(n_t.\varphi)}{d\varphi}\frac{d(n_t.\eta)}{d\eta}\right)\bigg|_{t=0}\\
&\quad = -\sin(n_t.\varphi)\frac{d(n_t.\varphi)}{dt}\frac{d(n_t.\varphi)}{d\varphi}\frac{d(n_t.\eta)}{d\eta}\bigg|_{t=0} + \cos(n_t.\varphi)\frac{d}{d\varphi}\frac{d(n_t.\varphi)}{dt}\frac{d(n_t.\eta)}{d\eta}\bigg|_{t=0}\\
&\qquad\qquad + \cos(n_t.\varphi)\frac{d(n_t.\varphi)}{d\varphi}\frac{d}{d\eta}\frac{d(n_t.\eta)}{dt}\bigg|_{t=0}\\
&\quad = -\sin(n_t.\varphi)(1-\cos(n_t.\varphi))\frac{d(n_t.\varphi)}{d\varphi}\frac{d(n_t.\eta)}{d\eta}\bigg|_{t=0}\\
&\qquad\qquad + \cos(n_t.\varphi)\frac{d}{d\varphi}(1-\cos(n_t.\varphi))\frac{d(n_t.\eta)}{d\eta}\bigg|_{t=0}\\
&\qquad\qquad + \cos(n_t.\varphi)\frac{d(n_t.\varphi)}{d\varphi}\frac{d}{d\eta}(1-\cos(n_t.\eta))\bigg|_{t=0}\\
&\quad = -\sin\varphi(1-\cos\varphi) + \cos\varphi\sin\varphi + \cos\varphi\sin\eta\\
&\quad = 2\sin\varphi\cos\varphi + \cos\varphi\sin\eta - \sin\varphi.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\frac{d}{ds}\left(\sin(a_s.\varphi)\frac{d(a_s.\varphi)}{d\varphi}\frac{d(a_s.\eta)}{d\eta}\right)\bigg|_{s=0}\\
&\quad = \cos(a_s.\varphi)\frac{d(a_s.\varphi)}{ds}\frac{d(a_s.\varphi)}{d\varphi}\frac{d(a_s.\eta)}{d\eta}\bigg|_{s=0} + \sin(a_s.\varphi)\frac{d}{d\varphi}\frac{d(a_s.\varphi)}{ds}\frac{d(a_s.\eta)}{d\eta}\bigg|_{s=0}\\
&\qquad\qquad + \sin(a_s.\varphi)\frac{d(a_s.\varphi)}{d\varphi}\frac{d}{d\eta}\frac{d(a_s.\eta)}{ds}\bigg|_{s=0}\\
&\quad = \cos(a_s.\varphi)\sin(a_s.\varphi)\frac{d(a_s.\varphi)}{d\varphi}\frac{d(a_s.\eta)}{d\eta}\bigg|_{s=0} + \sin(a_s.\varphi)\frac{d}{d\varphi}\sin(a_s.\varphi)\frac{d(a_s.\eta)}{d\eta}\bigg|_{s=0}\\
&\qquad\qquad + \sin(a_s.\varphi)\frac{d(a_s.\varphi)}{d\varphi}\frac{d}{d\eta}\sin(a_s.\eta)\bigg|_{s=0}
\end{aligned}
$$

$$= 2\sin\varphi\cos\varphi + \sin\varphi\cos\eta.$$

Summing up, we obtain

$$\begin{aligned}\text{(B-5)}\quad L_K^{(3)}c^\sharp(\theta_0,\theta_1,\theta_2) - L_N^{(3)}c^\sharp(\theta_0,\theta_1,\theta_2) + L_A^{(3)}c^\flat(\theta_0,\theta_1,\theta_2) \\ = \fint\!\!\fint(\sin\varphi\cos\eta - \cos\varphi\sin\eta)c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi \\ = -\fint\!\!\fint\sin(\eta-\varphi)c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi.\end{aligned}$$

Now we turn to the computation of the right-hand side of (B-4). The cocycle identity for c yields

$$\begin{aligned}0 = dc(\eta,\varphi,\psi,\theta_0,\theta_1,\theta_2) = c(\varphi,\psi,\theta_0,\theta_1,\theta_2) - c(\eta,\psi,\theta_0,\theta_1,\theta_2) + c(\eta,\varphi,\theta_0,\theta_1,\theta_2) \\ - \sum_{j=0}^{2}(-1)^j c(\eta,\varphi,\psi,\theta_0,\dots,\widehat{\theta}_j,\dots,\theta_2).\end{aligned}$$

We multiply this identity by $\sin(\eta-\varphi)$ and integrate over the variables η, φ and ψ. Integrating the first term, we get

$$\begin{aligned}\fint\!\!\fint\!\!\fint \sin(\eta-\varphi)c(\varphi,\psi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi\,d\psi \\ = \fint\!\!\fint\Bigl(\fint\sin(\eta-\varphi)\,d\eta\Bigr)c(\varphi,\psi,\theta_0,\theta_1,\theta_2)\,d\varphi\,d\psi = 0.\end{aligned}$$

Likewise, the integral of the second term vanishes. We are thus left with

$$\begin{aligned}&\fint\!\!\fint\sin(\eta-\varphi)c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi \\ &\quad= \fint\!\!\fint\!\!\fint\sin(\eta-\varphi)\Bigl(\sum_{j=0}^{2}(-1)^j c(\eta,\varphi,\psi,\theta_0,\dots,\widehat{\theta}_j,\dots,\theta_2)\Bigr)d\eta\,d\varphi\,d\psi \\ &\quad= \sum_{j=0}^{2}(-1)^j\fint\!\!\fint\!\!\fint\sin(\eta-\varphi)c(\eta,\varphi,\psi,\theta_0,\dots,\widehat{\theta}_j,\dots,\theta_2)\,d\eta\,d\varphi\,d\psi \\ &\quad= d\check{c}(\theta_0,\theta_1,\theta_2).\end{aligned}$$

Comparing this with (B-5) above, formula (B-4) follows. □

We are now in a position to finish the proof of Proposition 3.7 by spelling out the integrability conditions (B-3). Consider the first identity in (B-3). We have

$$[X,Y] = [L_K^{(3)}, L_A^{(3)}] + (L_K^{(3)}(c^\sharp + dv^\sharp))\,\partial_{\theta_3} = L_K^{(3)} - L_N^{(3)} + (L_K^{(3)}(c^\sharp + dv^\sharp))\,\partial_{\theta_3}.$$

Recall from Lemma B.1 and Lemma 3.3 that

$$L_K^{(3)} c^\sharp = -c^\flat, \quad L_K^{(3)} dv^\sharp = dL_K^{(2)} v^\sharp.$$

Thus,

$$[X, Y] = L_K^{(3)} - L_N^{(3)} + (-c^\flat + dL_K^{(2)} v^\sharp)\,\partial_{\theta_3}.$$

Comparing this to $-Z$ we find

$$[X, Y] = -Z \iff d(L_K^{(2)} v^\sharp + v^\flat) = 0. \tag{B-6}$$

Likewise, for the second identity in (B-3) we have

$$[X, Z] = Y \iff d(L_K^{(2)} v^\flat - v^\sharp) = 0. \tag{B-7}$$

Finally, observe that

$$\begin{aligned}[Y, Z] &= [L_A^{(3)}, L_N^{(3)} - L_K^{(3)}] + [L_A^{(3)}, (c^\flat + dv^\flat)\,\partial_{\theta_3}] - [L_N^{(3)} - L_K^{(3)}, (c^\sharp + dv^\sharp)\,\partial_{\theta_3}] \\ &= L_K^{(3)} + ((L_K^{(2)} - L_N^{(2)})\, dv^\sharp + L_A^{(3)}\, dv^\flat + (L_K^{(3)} - L_N^{(3)})c^\sharp + L_A^{(3)} c^\flat)\,\partial_{\theta_3}.\end{aligned}$$

By Lemma B.2 and Lemma 3.3, this becomes

$$[Y, Z] = L_K^{(3)} + (d(L_K^{(2)} - L_N^{(2)})v^\sharp + dL_A^{(2)} v^\flat - d\check{c})\,\partial_{\theta_3}.$$

We deduce that

$$[Y, Z] = X \iff d(L_K^{(2)} v^\sharp - L_N^{(2)} v^\sharp + L_A^{(2)} v^\flat - \check{c}) = 0. \tag{B-8}$$

Combining (B-6), (B-7) and (B-8), Proposition 3.7 now follows from (B-3).

References

[1] **G Ben Simon**, **M Burger**, **T Hartnick**, **A Iozzi**, **A Wienhard**, *On weakly maximal representations of surface groups*, preprint (2013) arXiv:1305.2620 to be published in J. Diff. Geom.

[2] **M Bestvina**, **K Fujiwara**, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. 6 (2002) 69–89 MR1914565

[3] **S J Bloch**, *Higher regulators, algebraic K–theory, and zeta functions of elliptic curves*, CRM Monograph Series 11, Amer. Math. Soc. (2000) MR1760901

[4] **R Brooks**, *Some remarks on bounded cohomology*, from: "Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook conference", (I Kra, B Maskit, editors), Ann. of Math. Stud. 97, Princeton Univ. Press (1981) 53–63 MR624804

[5] **M Bucher-Karlsson**, *Finiteness properties of characteristic classes of flat bundles*, Enseign. Math. 53 (2007) 33–66 MR2343344

[6] **M Bucher**, **M Burger**, **A Iozzi**, *A dual interpretation of the Gromov–Thurston proof of Mostow rigidity and volume rigidity for representations of hyperbolic lattices*, from: "Trends in harmonic analysis", (M A Picardello, editor), Springer INdAM Ser. 3, Springer, Milan (2013) 47–76 MR3026348

[7] **M Bucher**, **N Monod**, *The norm of the Euler class*, Math. Ann. 353 (2012) 523–544 MR2915547

[8] **M Burger**, **A Iozzi**, *Bounded Kähler class rigidity of actions on Hermitian symmetric spaces*, Ann. Sci. École Norm. Sup. 37 (2004) 77–103 MR2050206

[9] **M Burger**, **A Iozzi**, *Bounded cohomology and totally real subspaces in complex hyperbolic geometry*, Ergodic Theory Dynam. Systems 32 (2012) 467–478 MR2901355

[10] **M Burger**, **A Iozzi**, **A Wienhard**, *Tight homomorphisms and Hermitian symmetric spaces*, Geom. Funct. Anal. 19 (2009) 678–721 MR2563767

[11] **M Burger**, **A Iozzi**, **A Wienhard**, *Surface group representations with maximal Toledo invariant*, Ann. of Math. 172 (2010) 517–566 MR2680425

[12] **M Burger**, **A Iozzi**, **A Wienhard**, *Higher Teichmüller spaces: from* $\mathrm{SL}(2,\mathbb{R}$ *to other Lie groups*, from: "Handbook of Teichmüller theory, IV", (A Papadopoulos, editor), IRMA Lect. Math. Theor. Phys. 19, Eur. Math. Soc., Zürich (2014) 539–618 MR3289711

[13] **M Burger**, **N Monod**, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. (JEMS) 1 (1999) 199–235 MR1694584

[14] **M Burger**, **N Monod**, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. 12 (2002) 219–280 MR1911660 With an appendix by M Burger and A Iozzi

[15] **M Burger**, **N Monod**, *On and around the bounded cohomology of* SL_2, from: "Rigidity in dynamics and geometry", (M Burger, A Iozzi, editors), Springer, Berlin (2002) 19–37 MR1919394

[16] **C Carathéodory**, *Calculus of variations and partial differential equations of the first order, I: Partial differential equations of the first order*, Holden-Day (1965) MR0192372

[17] **I Chatterji**, **T Fernós**, **A Iozzi**, *The median class and superrigidity of actions on CAT(0) cube complexes*, preprint (2012) `arXiv:1212.1585`

[18] **J L Dupont**, *Bounds for characteristic numbers of flat bundles*, from: "Algebraic topology", (J L Dupont, I H Madsen, editors), Lecture Notes in Math. 763, Springer, Berlin (1979) 109–119 MR561216

[19] **M Entov**, **L Polterovich**, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. 2003 (2003) 1635–1676 MR1979584

[20] **D B A Epstein**, **K Fujiwara**, *The second bounded cohomology of word-hyperbolic groups*, Topology 36 (1997) 1275–1289 MR1452851

[21] **K Fujiwara**, **M Bestvina**, **K Bromberg**, *Bounded cohomology with coefficients in uniformly convex Banach spaces*, preprint (2013) arXiv:1306.1542

[22] **A B Goncharov**, *Geometry of configurations, polylogarithms, and motivic cohomology*, Adv. Math. 114 (1995) 197–318 MR1348706

[23] **R I Grigorchuk**, *Some results on bounded cohomology*, from: "Combinatorial and geometric group theory", (A J Duncan, N D Gilbert, J Howie, editors), London Math. Soc. Lecture Note Ser. 204, Cambridge Univ. Press (1995) 111–163 MR1320279

[24] **M Gromov**, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. (1982) 5–99 MR686042

[25] **U Hamenstädt**, *Isometry groups of proper hyperbolic spaces*, Geom. Funct. Anal. 19 (2009) 170–205 MR2507222

[26] **U Hamenstädt**, *Isometry groups of proper* CAT(0)*–spaces of rank one*, Groups Geom. Dyn. 6 (2012) 579–618 MR2961285

[27] **T Hartnick**, **A Ott**, *Surjectivity of the comparison map in bounded cohomology for Hermitian Lie groups*, Int. Math. Res. Not. 2012 (2012) 2068–2093 MR2920824

[28] **T Hartnick**, **A Ott**, *Perturbations of the Spence–Abel equation and deformations of the dilogarithm function*, in preparation (2015)

[29] **M Hull**, **D Osin**, *Induced quasi-cocycles on groups with hyperbolically embedded subgroups*, preprint (2012) arXiv:1203.5436

[30] **N V Ivanov**, *Foundations of the theory of bounded cohomology*, Zap. Nauchn. Sem. LOMI 143 (1995) 69–109 In Russian; translated in J. Soviet Math. 37 (1987) 1090–1115

[31] **B E Johnson**, *Cohomology in Banach algebras*, Memoirs of the American Mathematical Society 127, Amer. Math. Soc. (1972) MR0374934

[32] **W Kerby**, *On infinite sharply multiply transitive groups*, Hamburger Mathematische Einzelschriften 6, Vandenhoeck & Ruprecht, Göttingen (1974) MR0384938

[33] **A W Knapp**, *Lie groups beyond an introduction*, 2nd edition, Progress in Mathematics 140, Birkhäuser, Boston (2002) MR1920389

[34] **J-F Lafont**, **B Schmidt**, *Simplicial volume of closed locally symmetric spaces of non-compact type*, Acta Math. 197 (2006) 129–143 MR2285319

[35] **J M Lee**, *Manifolds and differential geometry*, Graduate Studies in Mathematics 107, Amer. Math. Soc. (2009) MR2572292

[36] **I Mineyev**, *Bounded cohomology characterizes hyperbolic groups*, Q. J. Math. 53 (2002) 59–73 MR1887670

[37] **N Monod**, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics 1758, Springer, Berlin (2001) MR1840942

[38] **N Monod**, *An invitation to bounded cohomology*, from: "International Congress of Mathematicians, II", (M Sanz-Solé, J Soria, J L Varona, J Verdera, editors), Eur. Math. Soc., Zürich (2006) 1183–1211 MR2275641

[39] **N Monod**, *On the bounded cohomology of semi-simple groups, S–arithmetic groups and products*, J. Reine Angew. Math. 640 (2010) 167–202 MR2629693

[40] **N Monod**, *Equivariant measurable liftings*, Fund. Math. 230 (2015) 149–165 MR3337223

[41] **N Monod**, **Y Shalom**, *Cocycle superrigidity and bounded cohomology for negatively curved spaces*, J. Differential Geom. 67 (2004) 395–455 MR2153026

[42] **N Monod**, **Y Shalom**, *Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. 164 (2006) 825–878 MR2259246

[43] **H Pieters**, *Continuous cohomology of the isometry group of hyperbolic space realizable on the boundary*, preprint (2015) arXiv:1507.04915

[44] **L Polterovich**, *Growth of maps, distortion in groups and symplectic geometry*, Invent. Math. 150 (2002) 655–686 MR1946555

Mathematics Department, Technion
32000 Haifa, Israel

Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany

tobias.hartnick@gmail.com, aott@mathi.uni-heidelberg.de

Proposed: Danny Calegari Received: 11 October 2014
Seconded: Leonid Polterovich, Jesper Grodal Accepted: 20 April 2015