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# Derived functors of the divided power functors

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We study the derived functors of the components $\Gamma^d_{\mathbb{Z}}(A)$ of the divided power algebra $\Gamma_{\mathbb{Z}}(A)$ associated to an abelian group $A$, with special emphasis on the $d=4$ case. While our results have applications both to representation theory and to algebraic topology, we illustrate them here by providing a new functorial description of certain integral homology groups of the Eilenberg–Mac Lane spaces $K(A,n)$ for $A$ a free abelian group. In particular, we give a complete functorial description of the groups $H_*(K(A,3);\mathbb{Z})$ for such $A$.

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## 1 Introduction

We will consider here functors from the category of free $R$–modules to that of $R$–modules (for $R$ a commutative ring) and particularly the functor from such a module $M$ to the module of $d$–fold invariant tensors $(M^{\otimes d})^{\Sigma_d}$. This module of invariant tensors is canonically isomorphic to the degree $d$ components $\Gamma^d_R(M)$ of the divided power algebra $\Gamma_R(M)$. Though these are non-additive functors of $M$ whenever $d>1$, they may be derived by the Dold–Puppe theory [11]. This associates to $M$ a family of $i^{\text{th}}$ derived functors $L_i\Gamma^d_R(M,n)$, which depends on an additional positive integer $n$. In a wider perspective, $L_i\Gamma^d_R(M,n)$ is the $i^{\text{th}}$ homotopy group of Quillen's left-derived object $L\Gamma^d_R(M[n])$, where $M[n]$ is the module $M$, viewed as a chain complex concentrated in degree $n$; see Quillen [24]. When $R$ is a field, the functors $L_i\Gamma^d_R(M,n)$ are known. Our aim is to compute certain of its values when $R=\mathbb{Z}$. Such computations are motivated by applications in both algebraic topology and representation theory.

In algebraic topology, the functors $L_i\Gamma^d_{\mathbb{Z}}(A,n)$, where $A$ is an arbitrary abelian group, are closely related to the integral homology of the Eilenberg–Mac Lane spaces $K(A,n+2)$, ie those spaces whose only nontrivial homotopy group is $A$ in degree $n+2$. Indeed, even though a truly functorial description of the homology groups

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$H_*(K(A,n+2);\mathbb{Z})$ is complicated, they are endowed with a natural filtration, inducing for all $n \geq 0$ functorial isomorphisms

$$\bigoplus_{d\geq 0} L_{*-2d}\Gamma^d_{\mathbb{Z}}(A,n) \simeq \mathrm{gr}(H_*(K(A,n+2);\mathbb{Z})) \tag{1-1}$$

on the associated graded components. This filtration splits functorially when we restrict ourselves to the subcategory of free abelian groups, so that there are then functorial isomorphisms

$$\bigoplus_{d\geq 0} L_{*-2d}\Gamma^d_{\mathbb{Z}}(A,n) \simeq H_*(K(A,n+2);\mathbb{Z}) \tag{1-2}$$

for all $n \geq 0$. When $A$ is non-free, such an isomorphism still exists, but is no longer functorial in $A$ (we refer to Appendix B for a further discussion of this issue). The abelian groups $L_i\Gamma^d_{\mathbb{Z}}(A,n)$ are thus quite fundamental for topology and the description of their dependence on the group $A$ carries much useful information, which remains hidden when such a (finitely generated) group is decomposed into a direct sum of cyclic groups.

In representation theory, the representations of integral Schur algebras can be described in terms of the strict polynomial functors of Friedlander and Suslin [14]. Such strict polynomial functors can be thought of as functors from finitely generated free abelian groups to abelian groups, equipped with an additional strict polynomial structure. The derived category of weight $d$ homogeneous strict polynomial functors $\boldsymbol{D}(\mathcal{P}_{d,\mathbb{Z}})$ is equipped with a Ringel duality operator $\Theta$, which is a self-equivalence of $\boldsymbol{D}(\mathcal{P}_{d,\mathbb{Z}})$. For $A$ free and finitely generated there is an isomorphism

$$H^*(\Theta^n\Gamma^d_{\mathbb{Z}}(A)) \simeq L_{nd-*}\Gamma^d_{\mathbb{Z}}(A,n).$$

The case $n=1$ can be written more explicitly as follows, where $\underline{\mathrm{Ext}}^i_{\mathcal{P}_{d,\mathbb{Z}}}$ refers to the $i^{\text{th}}$ derived functor of the internal Hom in $\mathcal{P}_{d,\mathbb{Z}}$:

$$\underline{\mathrm{Ext}}^i_{\mathcal{P}_{d,\mathbb{Z}}}(\Lambda^d_{\mathbb{Z}},\Gamma^d_{\mathbb{Z}})(A) = L_{d-i}\Gamma^d_{\mathbb{Z}}(A,n).$$

Such an isomorphism holds when $\Gamma^d_{\mathbb{Z}}$ is replaced by an arbitrary strict polynomial functor, but the case of divided powers is fundamental, since these functors constitute a family of projective generators of the category. In this context, the description of the $L_*\Gamma^d_{\mathbb{Z}}(A,n)$ as functors of $A$ is essential, since the functoriality is necessary in order to determine the action of the Schur algebra, and hence to obtain the expressions $H^*(\Theta^n\Gamma^d_{\mathbb{Z}}(A))$ as representations and not simply as abelian groups. Actually we need more than the functoriality in order to understand the action of the Schur algebra; we really need to describe the strict polynomial structure of these functors.

Let us briefly review what was previously known regarding the derived functors $L_*\Gamma^d_{\mathbb{Z}}(A,n)$. The integral homology of Eilenberg–Mac Lane spaces was computed by H Cartan: a non-functorial description of these homology groups is given in [9, Exposé 11, théorème 1], and a functorial one in [9, Exposé 11, théorème 5]. Work in this direction was pursued by two students of Mac Lane, R M Hamsher [15] and G J Decker [10], who studied the cases $n=1$ and $n\geq 1$, respectively. By the isomorphism (1-1), it is possible to retrieve from these results a description of the derived functors $L_*\Gamma^d_{\mathbb{Z}}(A,n)$. The answers, however, are very complicated. For example, the homology groups $H_i(K(A,n);\mathbb{Z})$ for a finitely generated abelian group $A$ are described functorially by an infinite list of generators and relations, even though they are of finite type. Some additional progress in computing the functors $L_*\Gamma^d_{\mathbb{Z}}(A,n)$ was made by a direct study of its properties by A K Bousfield [5; 6], and a complete description of these functors for any $A$ was obtained for $d=2$ by H-J Baues and T Pirashvili [2] and by two of us [8] for $d=3$. On the other hand, no description of the strict polynomial structure of the functors $L_*\Gamma^d_{\mathbb{Z}}(A,n)$ has been given so far.

We now list the results which we obtain in this paper. We give full description (as strict polynomial functors) of:

(1) $L_*\Gamma^d_{\mathbb{Z}}(A,1)$ for all $d$ and $A$ free. By (1-2), this determines a new and functorial description of the groups $H_*(K(A,3);\mathbb{Z})$ for $A$ a free abelian group.

(2) $L_*\Gamma^d_{\mathbb{Z}}(A,n)$ for $d\leq 4$ and $A$ free.

Our treatment is elementary and self-contained and does not use computations from the literature. The strict polynomial structures come in as a help in the computations. Our computations have a number of interesting byproducts, in particular:

(3) Short proofs of some computations first obtained by one of us [29], including that of $L_*\Gamma^d_{\Bbbk}(V,n)$, where $\Bbbk$ is a field of characteristic 2.

(4) A new family of exact complexes "of Koszul type" involving divided powers over a field of characteristic 2.

The kernels of these new complexes yield new families of functors, related to the 2–primary component of $L_*\Gamma^d_{\mathbb{Z}}(A,1)$. We describe these functors in a variety of ways. The simplest one is the 2–primary component of $L_3\Gamma^4_{\mathbb{Z}}(A,1)$, which we denote by $\Phi^4(A)$. This can also be described as the cokernel of the natural transformation $\Lambda^4_{\mathbb{F}_2}(A/2)\to\Gamma^4_{\mathbb{F}_2}(A/2)$ determined by the algebra structure of $\Gamma^*_{\mathbb{F}_2}(A/2)$ (such a map only exists in characteristic 2).

(5) We give a description of the groups $H_{n+i}(K(A,n);\mathbb{Z})$ for $A$ free in the range $0\leq i\leq 10$ which is both more natural and more precise than Cartan's [9].

Finally, in Section 11, we return to the derived functors $L_*\Gamma^d_{\mathbb{Z}}(A,n)$ for all $n$, $d$ and all abelian groups $A$. We present a conjectural description of these functors, up to a filtration, in terms of simpler derived functors of divided and exterior powers. We then make use of our results here and of our previous ones from [8] in order to provide some evidence in support of this conjecture.

We will now discuss the content of this article in more detail. In Sections 2 and 3 we collect the main properties of divided power algebras, strict polynomial functors and derived functors which will be required. In Section 4, we introduce the useful notion of a quasi-trivial filtration, which will allow us to prove that certain spectral sequences degenerate. For $V$ a vector space over a characteristic $p$ field $\mathbb{k}$, the divided power algebra $\Gamma^*_{\mathbb{k}}(V)$ possesses such a filtration, which we call the principal filtration. We use this filtration in Section 5 to compute those values of the derived functors $L_i\Gamma^d_{\mathbb{k}}(V,n)$ which will be needed in the sequel.

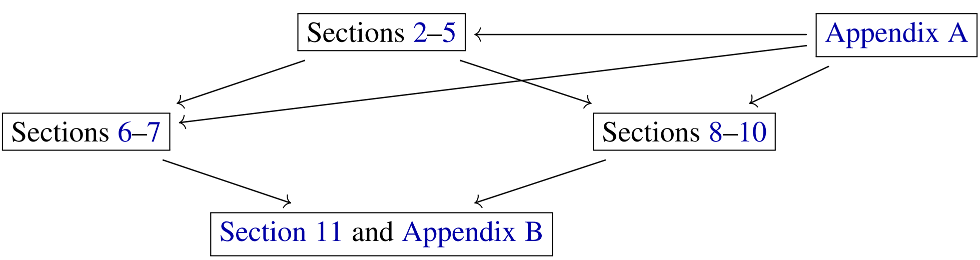
In Sections 6 and 7 we compute the strict polynomial functors $L_i\Gamma^d_{\mathbb{Z}}(A,1)$ for any free abelian group $A$, with the help of the Bockstein maps. These may be viewed as a family of differentials on the mod $p$ graded groups $L_*\Gamma^d_{\mathbb{F}_p}(A/p,1)$ and can be explicitly described. While in odd characteristic these mod $p$ derived functors together the Bockstein differentials simply provide us with a standard dual Koszul complex, the result is slightly different in characteristic $2$. In that case the Bockstein differentials determine on $L_*\Gamma^d_{\mathbb{F}_2}(A/2,1)$ a sort of dual Koszul complex structure, in which the differentials are twisted by a Frobenius map. We call this characteristic $2$ complex the skew Koszul complex. In both odd and even characteristic the integral homology groups $L_*\Gamma^d_{\mathbb{Z}}(A/p,1)$ are simply the groups of cycles in these Koszul and twisted Koszul complexes, and we are able to analyze these more precisely in a number of situations.

In Sections 8–10, we present another method for computing the derived functors of $\Gamma^d_{\mathbb{Z}}(A)$. Here we use the adic filtration associated to the augmentation ideal of the algebra $\Gamma^*_{\mathbb{Z}}(A)$. We call this filtration the maximal filtration on $\Gamma^*_{\mathbb{Z}}(A)$. It was the filtration chosen in [8] in order to study the derived functors of $\Gamma^3_{\mathbb{Z}}(A)$. With the help of the associated spectral sequence, we determine the values of $L_*\Gamma^4_{\mathbb{Z}}(A,n)$. The easiest cases $n=1$, $2$ are studied in Section 9, where we pay particular attention to the first new functor occurring among the derived functors of $\Gamma^4_{\mathbb{Z}}(A)$. This is the functor $\Phi^4(A)$ mentioned above, the 2–primary component of $L_3\Gamma^4_{\mathbb{Z}}(A,1)$, which we describe in three distinct ways. As $n$ increases, the situation becomes more involved, and a detailed study of the differentials in the maximal filtration spectral sequence for $\Gamma^4_{\mathbb{Z}}(A,n)$ for a general $n$ is carried out in Section 10. The computation of certain differentials is delicate and involves functorial considerations and careful dimension counts. This yields an inductive formula for $L_*\Gamma^4_{\mathbb{Z}}(A,n)$ (Theorem 10.2), from which the complete description of the derived functors of $\Gamma^4_{\mathbb{Z}}(A,n)$ for all $n$ follows directly (Theorem 10.1).

In Section 11, we return to the derived functors $L_*\Gamma^d_{\mathbb{Z}}(A,n)$ for all $n$, $d$ and all abelian groups $A$. We propose a conjectural description, up to a filtration, of the functors $L_r\Gamma^d_{\mathbb{Z}}(A,n)$. To state this requires that we first discuss the stable homology groups $H_r(K(A,n);\mathbb{Z})$, ie those for which $n \le r < 2n$. These additive groups are discussed in a number of sources, and we first review here for the reader's convenience certain aspects of the admissible words formalism of Cartan. We then reformulate in Theorem 11.3 his admissible words in terms of a simpler labelling, which only involves those words that do not involve the transpotence operation. We refer in Proposition 11.4 to Cartan's computation of these stable values, and this is the only place in the text where we make use (for simplicity) of his results. We then state our Conjecture 11.5 and we verify that it is compatible with Theorems 6.3 and 10.1 as well as with our results in [8].

We view the appendices as an integral part of our text. In Appendix A, we review some classical methods of computations of some Hom and $\mathrm{Ext}^1$ in functor categories which are used many times in our arguments. Appendix B begins with a discussion of the relation between the derived functors of the functor $\Gamma^d_{\mathbb{Z}}(A)$ and the integral homology of $K(A,n)$. We then display a complete table of the functorial values of the groups $H_{n+i}(K(A,n);\mathbb{Z})$ for $A$ free in the range $0 \le i \le 10$. While the constraints in choosing this range of values of $i$ were to some extent typographical, the fact that we only know the complete set of derived functors of $\Gamma^d_{\mathbb{Z}}(A)$ for $d \le 4$ would have precluded the display of a complete table for much higher values of $i$. This table already features most of the unexpected phenomena which we encounter in our computations, as will be seen from the discussion which precedes the table. Finally, Appendix C provides for convenience a complete list of nontrivial derived functors $L_*\Gamma^4_{\mathbb{Z}}(A,n)$ for $n \le 4$.

The following diagram summarizes the relations between the various parts of our text:




**Notation** Throughout the text the notation $A/p$, where $A$ is an abelian group and $p$ prime number, stands for $A \otimes_{\mathbb{Z}} \mathbb{Z}/p$. For any abelian group $A$, the notation $\Gamma(A)$ will stand for $\Gamma_{\mathbb{Z}}(A)$, the divided power algebra associated to the $\mathbb{Z}$–module $A$, unless otherwise specified. On the other hand, for any $\mathbb{F}_p$–vector space $V$, the notation $\Gamma(V)$ stands for the divided power algebra $\Gamma_{\mathbb{F}_p}(V)$ in the category of $\mathbb{F}_p$–vector spaces. Finally, we will often denote functors by their values, in order to avoid cumbersome notations. For example, we will write $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ rather than $\Gamma^2_{\mathbb{F}_2} \circ I^{(1)} \circ I/2$, where $I$ is the identity functor.

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# 2 Classical functorial algebras

## 2.1 The divided power algebra

Let $M$ be an $R$–module, where $R$ is a commutative ring with unit. The symmetric algebra $S_R(M)$ and the exterior algebra $\Lambda_R(M)$ are well known. The divided power algebra $\Gamma_R(M)$ deserves equal attention. We recall here its basic properties, referring to Roby [26] for the proofs. The algebra $\Gamma_R(M)$ is defined as the commutative $R$–algebra generated, for all $x \in M$ and all nonnegative integers $i$, by elements $\gamma_i(x)$ which satisfy the following relations for all $x$, $y \in M$ and $\lambda \in R$:

(2-1) $$\gamma_0(x) = 1, \quad x \neq 0,$$

(2-2) $$\gamma_s(x)\gamma_t(x) = \binom{s+t}{s}\gamma_{s+t}(x),$$

(2-3) $$\gamma_n(x+y) = \sum_{s+t=n} \gamma_s(x)\gamma_t(y), \quad n \geq 1,$$

(2-4) $$\gamma_n(\lambda x) = \lambda^n \gamma_n(x), \quad n \geq 1.$$

Setting $s = t = 1$ in (2-2), one finds by induction on $n$ that $\gamma_1(x)^n = n!\gamma_n(x)$. The definition $\Gamma_R(M)$ is functorial with respect to $M$: an $R$–linear map $f\colon M \to N$ induces a morphism of $R$–algebras $\Gamma_R(M) \to \Gamma_R(N)$ which sends $\gamma_i(x)$ to $\gamma_i(f(x))$.

The relations (2-1)–(2-4) are homogeneous, so that $\Gamma_R(M)$ can be given a graded algebra structure by setting $\deg \gamma_i(x) = i$. We denote by $\Gamma^d_R(M)$ the homogeneous component of degree $d$, so that there is a functorial decomposition

$$\Gamma_R(M) = \bigoplus_{d\geq 0} \Gamma_R^d(M).$$

There are in addition functorial $R$–linear isomorphisms $\Gamma_R^0(M) \simeq R$ and $\Gamma_R^1(M) \simeq M$. The latter identifies $\gamma_1(x)$ with $x$, so that

$$\gamma_n(x) = \frac{x^n}{n!}$$

whenever $n!$ is invertible in the ring $R$. It is for this reason that $\Gamma_R(M)$ is called the divided power algebra.

Consider the graded commutative algebra $TM := \bigoplus_{d\geq 0} V^{\otimes d}$, equipped with the shuffle product, and let us denote by $TS(M)$ the graded subalgebra of invariant tensors $TS(M) := \bigoplus_{d\geq 0} (V^{\otimes d})^{\Sigma_d}$. There is a well-defined functorial homomorphism of graded algebras

$$\Gamma_R(M) \to TS(M) \tag{2-5}$$

which sends $\gamma_n(x)$ to $x^{\otimes n}$. The morphism (2-5) is an isomorphism if $M$ is projective [26, Proposition IV.5]. In particular, for a projective $R$–module $M$ we have functorial isomorphisms

$$\Gamma_R^d(M) \simeq (M^{\otimes d})^{\Sigma_d}.$$

This description by invariant tensors of $\Gamma_R^d(M)$ is no longer valid for arbitrary $R$–modules $M$. For example, it follows by inspection from the relations (2-1)–(2-4) that

$$\Gamma_{\mathbb{Z}}^2(\mathbb{Z}/2) \simeq \mathbb{Z}/4, \tag{2-6}$$

so that this group cannot live in $\mathbb{Z}/2 \otimes \mathbb{Z}/2$. This example also shows that divided power algebras behave differently from symmetric and exterior algebras with respect to torsion. Indeed, $S_{\mathbb{Z}}(M)$ and $S_{\mathbb{F}_p}(M)$ coincide for any $\mathbb{F}_p$–module $M$, as do $\Lambda_{\mathbb{Z}}(M)$ and $\Lambda_{\mathbb{F}_p}(M)$, but the similar assertion is not true for $\Gamma_{\mathbb{Z}}(M)$ and $\Gamma_{\mathbb{F}_p}(M)$. For any $\mathbb{F}_p$–module $M$, we will always carefully distinguish between the functors $\Gamma_{\mathbb{Z}}(M)$ and $\Gamma_{\mathbb{F}_p}(M)$, with the convention mentioned above that $\Gamma(V)$ will always stand for $\Gamma_{\mathbb{F}_p}(V)$.

## 2.2 Exponential functors

A functor $F$ from $R$–modules to $R$–algebras is said to be an exponential functor if, for all pairs of $R$–modules $M$ and $N$, the composite

$$F(M)\otimes F(N) \xrightarrow{F(i_1)\otimes F(i_2)} F(M\oplus N)\otimes F(M\oplus N) \xrightarrow{\text{mult}} F(M\oplus N) \tag{2-7}$$

induced by the canonical inclusions $i_1$ and $i_2$ of $M$ and $N$ into $M \oplus N$ is an isomorphism. We refer to the map (2-7) as the *exponential isomorphism* for $F$. For example, the algebras $S_R(M)$, $\Lambda_R(M)$, and $\Gamma_R(M)$ determine exponential functors (see [26, théorème III.4]).

Exponential functors are endowed with a canonical bialgebra structure. Indeed, there is a coproduct induced by the diagonal inclusion $\Delta$ of $M$ into $M \oplus M$:

$$F(M) \xrightarrow{F(\Delta)} F(M \oplus M) \simeq F(M) \otimes F(M).$$

It follows from (2-3) that the coalgebra structure obtained in this way on $\Gamma_R(M)$ is the one determined by the comultiplication map $\gamma_i(x) \mapsto \sum_{i=j+k} \gamma_j(x) \otimes \gamma_k(x)$.

## 2.3 Duality

Given an $R$–module $M$, we let $M^\vee := \mathrm{Hom}_R(M, R)$. The (restricted) dual $\Gamma_R(M)^\sharp$ of the divided power algebra is the $R$–module

$$\Gamma_R(M)^\sharp := \bigoplus_{d \geq 0} (\Gamma^d_R(M^\vee))^\vee.$$

The bialgebra structure on $\Gamma_R(M^\vee)$ defines a bialgebra structure on $\Gamma_R(M)^\sharp$. The dual of symmetric and exterior algebras are defined similarly. An explicit computation shows that for all projective $R$–modules $M$, there are natural isomorphisms of $R$–bialgebras

$$\Gamma_R(M)^\sharp \simeq S_R(M), \quad S_R(M)^\sharp \simeq \Gamma_R(M), \quad \Lambda_R(M)^\sharp \simeq \Lambda_R(M).$$

For a generalization of such constructions in the context of strict polynomial functors, see (3-4) .

## 2.4 Base change

For any $R$–module $M$ and any commutative $R$–algebra $A$, there is a base change isomorphism of $A$–algebras [26, théorème III.3], which sends $\gamma_n(x) \otimes 1$ to $\gamma_n(x \otimes 1)$:

$$\Gamma_R(M) \otimes_R A \simeq \Gamma_A(M \otimes_R A) \tag{2-8}$$

There are similar base change isomorphisms for symmetric and exterior algebras:

$$S_R(M) \otimes_R A \simeq S_A(M \otimes_R A) \quad \text{and} \quad \Lambda_R(M) \otimes_R A \simeq \Lambda_A(M \otimes_R A).$$

# 3 Derived functors and strict polynomial functors

## 3.1 Derived functors

The Dold–Kan correspondence states that the normalized chain complex functor $\mathcal{N}$ is an equivalence of categories preserving homotopy equivalences, with inverse $K$ (also preserving homotopy equivalences):

$$\mathcal{N}\colon \mathrm{simpl}(R\text{-Mod}) \rightleftarrows \mathrm{Ch}_{\geq 0}(R\text{-Mod}) : K.$$

If $F$ is a functor from $R$–modules to $R'$–modules, Dold and Puppe [11] defined its derived functors $L_i F(M,n)$ by the formula

$$L_i F(M,n) = \pi_i FK(P^M[n]), \tag{3-1}$$

where $P^M$ is a projective resolution of the $R$–module $M$ and $[n]$ is the degree-$n$ shift of complexes (ie $C[n]_i = C_{i-n}$). More generally, if $C$ is a complex of $R$–modules, we denote by $LF(C)$ the simplicial $R$–module $F(K(P))$, where $P$ is a complex of projective $R$–modules quasi-isomorphic to $C$, and by $L_i F(C)$ its homotopy groups.

**Remark 3.1** The definition of the derived functors of $F$ only depends on the restriction of $F$ to the category of free $R$–modules. Furthermore, if $F$ commutes with directed colimits of free $R$–modules (as the divided power functors do), then $F$ is completely determined by its restriction to the category of free finitely generated $R$–modules. For example, for a free $R$–module $M$, we have an isomorphism $L_*\Gamma^d(M,n) = \lim_i L_*\Gamma^d(M_i,n)$, where the limit is taken over the directed system of free finitely generated submodules $M_i$ of $M$.

## 3.2 Strict polynomial functors

We are mainly interested in the divided powers functors and functors related to these. All these functors actually belong to a class of very rigid functors called "strict polynomial functors", introduced by Friedlander and Suslin [14] in the context of the cohomology of affine algebraic group schemes and related to Bousfield's "homogeneous functors" [5]. We recommend Krause [19] for a presentation of strict polynomial functors. We will only recall here the basic facts required for our computations.

**3.2.1 The category of strict polynomial functors** Let $R$ be a commutative ring. Strict polynomial functors can be thought of as functors from the category of finitely generated projective $R$–modules to the category of $R$–modules, equipped with an additional "scheme-theoretic" structure. This additional structure determines a notion of weight for strict polynomial functors (this weight is called "degree" in [14]; in the present article, we prefer to reserve the term degree for the homological degrees).

For all $d \geq 0$, we denote by $\mathcal{P}_{d,R}$ the abelian category of homogeneous strict polynomial functors of weight $d$, as in [19]. Let $\mathcal{F}_R$ denote the category of functors from finitely generated projective $R$–modules to $R$–modules. There is an exact faithful forgetful functor

$$\mathcal{P}_R \to \mathcal{F}_R. \tag{3-2}$$

The functors lying in the image of this forgetful functor are the homogeneous functors of Bousfield [5]. Typical examples of functors lying in the image of the forgetful functor are the symmetric powers $S^d_R$, the exterior powers $\Lambda^d_R$ and the divided powers $\Gamma^d_R$. For each of these, there is (up to an isomorphism of strict polynomial functors) a unique way to see it as a strict polynomial functor, so we use the same notation to denote the corresponding strict polynomial functor.

We define the category of strict polynomial functors as the product category

$$\mathcal{P}_R = \prod_{d \geq 0} \mathcal{P}_{d,R}.$$

Thus, a typical strict polynomial functor $F$ is a family of homogeneous functors $F^d$ of weight $d$, which we denote usually as a direct sum $F = \bigoplus_{d \geq 0} F^d$. Gathering all the forgetful functors (3-2), we obtain an exact faithful forgetful functor preserving colimits:

$$\mathcal{P}_R \to \mathcal{F}_R. \tag{3-3}$$

Let us mention some useful structures which equip the category $\mathcal{P}_R$. First, the usual tensor product of functors (defined objectwise, ie $(F \otimes G)(M) = F(M) \otimes_R G(M)$) lifts to a tensor product in the category of strict polynomial functors. Thus, the forgetful functor (3-3) preserves tensor products. Similarly the composition of strict polynomial functors is well defined, and the forgetful functor (3-3) preserves composition. Finally, we have a (restricted) duality functor

$$\sharp\colon \mathcal{P}_R^{\oplus} \to \mathcal{P}_R, \tag{3-4}$$

which sends a strict polynomial functor $F = \bigoplus_{d \geq 0} F^d$ to the functor $F^\sharp$ such that $F^\sharp(M) = \bigoplus_{d \geq 0} F^d(M^\vee)^\vee$, where $^\vee$ denotes $R$–linear duality, ie $M^\vee = \mathrm{Hom}_R(M, R)$. When $R$ is a field, this duality functor is exact.

Observe that the category of strict polynomial functors discussed by Friedlander and Suslin [14] is the direct sum of the categories $\mathcal{P}_{d,R}$. Taking the product instead of the direct sum is a rather cosmetic change, which we make here in order to be able to consider the infinite sum $\Gamma_R = \bigoplus_{d\geq 0} \Gamma_R^d$ of all the divided powers as a strict polynomial functor, and the multiplication of the divided power algebra as a morphism of strict polynomial functors $\Gamma_R \otimes \Gamma_R \to \Gamma_R$.

The advantage of working with strict polynomial functors rather than with ordinary functors from finitely generated projective $R$–modules to $R$–modules is twofold:

(1) The category of strict polynomial functors is graded by the weight. This weight actually plays an important role in organizing the computations of derived functors of the divided power algebra. Now, in all computations involving strict polynomial functors this information is automatically and transparently carried. So we do not have to pay special attention to the bookkeeping by the weights, since the strict polynomial setting does it for us. For example, the functors $S_R^d$, $\Lambda_R^d$ and $\Gamma_R^d$ are homogenous of weight $d$. If $F$ and $G$ are homogeneous of respective weights $d$ and $e$ then $F\otimes G$ and $F\circ G$ are homogeneous of weights $d+e$ and $de$, respectively. Moreover all morphisms of strict polynomial functors preserve the weight. However, when working with strict polynomial functors, one has to take care that a given ordinary functor may sometimes be given several non-isomorphic scheme-theoretic structures, as the example of the Frobenius twists in Section 3.2.2 shows.

(2) It is easier to compute extensions in $\mathcal{P}_R$ than in $\mathcal{F}_R$, and there are often fewer possible ones in $\mathcal{P}_R$. This fact will be of great help in solving certain extension problems arising from spectral sequences. We have gathered in Appendix A some standard methods and results regarding the computation of extensions in these categories. An illustration of the difference between extensions in $\mathcal{P}_R$ and in $\mathcal{F}_R$ is provided by comparing the results from Lemma A.11 with those of (a)–(g) in Section 11.5.

**3.2.2 Frobenius twists functors** We now take a field $\Bbbk$ of positive characteristic $p$ as our ground ring $R$. We denote by $I^{(r)}$ the kernel of the morphism

$$S_{\Bbbk}^{p^r} \to \bigoplus_{k=1}^{p^r-1} S_{\Bbbk}^k \otimes S_{\Bbbk}^{p^r-k}$$

induced by the comultiplication of the symmetric power bialgebra. Thus, for all finite-dimensional $\Bbbk$–vector spaces $V$, the vector space $I^{(r)}(V)$, also denoted by $V^{(r)}$, can be identified with the subspace of $S_{\Bbbk}^{p^r}(V)$ generated by the $(p^r)^{\text{th}}$ powers of the elements of $V$. The strict polynomial functor $I^{(r)}$ is called the $r^{th}$ *Frobenius twist functor*. It has an important role in the theory of representations of affine algebraic group schemes in positive characteristic [14] and it will also appear in our computations. It enjoys the following basic properties:

(1) $I^{(r)}$ is a homogeneous strict polynomial functor of weight $p^r$.

(2) $I^{(r)}$ is additive.

(3) The dimension of $I^{(r)}(V) = V^{(r)}$ is equal to the dimension of $V$.

(4) The functors can be composed according to the rule $I^{(r)} \circ I^{(s)} = I^{(r+s)}$ and $I^{(0)}$ is the identity functor.

(5) The Frobenius twist functors are self dual: $(I^{(r)})^\sharp \simeq I^{(r)}$, so that $I^{(r)}$ is also the cokernel of the map $\bigoplus_{k=1}^{p^r-1} \Gamma^k_{\Bbbk} \otimes \Gamma^{p^r-k}_{\Bbbk} \to \Gamma^{p^r}_{\Bbbk}$ induced by the multiplication of the algebra of divided powers.

The inclusion $I^{(r)} \hookrightarrow S^{p^r}_{\Bbbk}$ is called the Frobenius morphism and the dual epimorphism $\Gamma^{p^r}_{\Bbbk} \twoheadrightarrow I^{(r)}$ therefore deserves to be called the Verschiebung morphism, following the usage in arithmetic. As explained in Appendix A, these two morphisms provide bases of the vector spaces $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(I^{(r)}, S^{p^r}_{\Bbbk})$ and $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\Gamma^{p^r}_{\Bbbk}, I^{(r)})$, respectively.

Observe that if $\Bbbk = \mathbb{F}_q$ with $q$ dividing $p^r$, the strict polynomial functors $I^{(nr)}$, $n \geq 0$, are not isomorphic to each other since they do not have the same weight. However, if we forget their scheme-theoretic structure and view them as ordinary functors, that is as objects of $\mathcal{F}_{\mathbb{F}_q}$, they all become isomorphic to the identity functor.

## 3.3 Derived functors and differential graded $\mathcal{P}_R$–algebras

We are interested in computing the derived functors of the divided power functors. We have seen that these divided power functors are not mere functors: they are strict polynomial ones and, moreover, their direct sum form an algebra. We will say that they form a strict polynomial algebra (a $\mathcal{P}_R$–algebra, for short). In this section we explain how such a structure is inherited by its derived functors.

**Definition 3.2** A *differential graded strict polynomial algebra over* $R$ (dg–$\mathcal{P}_R$–algebra, for short) is a graded strict polynomial functor

$$A = \bigoplus_{i \geq 0} A_i \in \mathcal{P}_R,$$

equipped with a multiplication $A_i \otimes A_j \to A_{i+j}$ and a differential $\partial\colon A_i \to A_{i-1}$, satisfying the usual axioms of a differential graded algebra. A *morphism of dg–$\mathcal{P}_R$–algebras* is a morphism of strict polynomial functors $A \to B$ commuting with multiplications and differentials.

**Remark 3.3** Strict polynomial functors are always graded by their weight and morphisms of strict polynomial functors preserve the weights. Thus, dg–$\mathcal{P}_R$–algebras are actually implicitly bigraded. To be more specific, let us denote by $A_i^d$ the homogeneous summand of weight $d$ of the strict polynomial functor $A_i$. Then $A=\bigoplus_{d\geq 0, i\geq 0} A_i^d$ and the multiplication and the differential restrict to morphisms $A_i^d\otimes A_j^e\to A_{i+j}^{d+e}$ and $A_i^d\to A_{i-1}^d$, respectively.

Definition 3.2 admits obvious variants (eg dg–$\mathcal{P}_R$–coalgebras), whose formulation is left to the reader. A dg–$\mathcal{P}_R$–algebra with zero differential is called a graded $\mathcal{P}_R$–algebra, and a graded $\mathcal{P}_R$–algebra concentrated in degree zero is simply a $\mathcal{P}_R$–algebra. Here are some basic examples of graded $\mathcal{P}_R$–algebras:

- We denote by $\Gamma_R(M[i])$ the divided power algebra generated by a finitely generated projective $R$–module $M$ placed in degree $i$. This yields a graded $\mathcal{P}_R$–algebra whose homogeneous component of degree $di$ and weight $d$ is the functor $\Gamma_R^d$.
- If $R=\Bbbk$ is a field of positive characteristic $p$ and $V$ is a finite-dimensional $\Bbbk$–vector space, we similarly denote by $\Gamma_{\Bbbk}(V^{(r)}[i])$ the divided power algebra generated by a copy of $V^{(r)}$ placed in degree $i$. This yields a graded $\mathcal{P}_R$–algebra with $\Gamma_{\Bbbk}^d\circ I^{(r)}$ the homogeneous part of degree $di$ and weight $dp^r$.
- There are of course similar examples based on exterior and symmetric algebras.

Now let us have a look at the effect of derivation of functors on dg–$\mathcal{P}_R$–algebras. For all finitely generated projective $R$–modules $M$ and all homogeneous strict polynomial functors $F$ of weight $d$, we denote by $\mathcal{N}F(M,n)$ the normalized chains of the simplicial object $F(K(M[n]))=F(K(R[n])\otimes M)$. This complex is a complex of homogeneous strict polynomial functors of weight $d$ in the variable $M$ [28, Observation 2.5]. In particular, derivation induces (weight-preserving) functors:

$$\begin{aligned}\mathcal{P}_R&\to\mathcal{P}_R,\\ F&\mapsto[M\mapsto L_iF(M,n)],\end{aligned}$$

consistently with (3-1). In the case of a $\mathcal{P}_R$–algebra $A$, the composition of the shuffle map [30] in the Eilenberg–Zilber theorem with the map induced by the product on $A$

$$\mathcal{N}A(M,n)\otimes\mathcal{N}A(M,n)\to\mathcal{N}(A\otimes A)(M,n)\to\mathcal{N}A(M,n)$$

defines a dg–$\mathcal{P}_R$–algebra structure on $\mathcal{N}A(M,n)$. Thus we obtain functors

$$\begin{aligned}\{\mathcal{P}_R\text{–algebras}\}&\to\{\text{dg–}\mathcal{P}_R\text{–algebras}\},\\ A&\mapsto[M\mapsto\mathcal{N}A(M,n)],\end{aligned}\tag{3-5}$$

$$\begin{aligned}\{\mathcal{P}_R\text{–algebras}\}&\to\{\text{graded }\mathcal{P}_R\text{–algebras}\},\\ A&\mapsto[M\mapsto L_*A(M,n)].\end{aligned}\tag{3-6}$$

More generally, if $A$ is a dg–$\mathcal{P}_R$–algebra, we define $\mathcal{N}A(M,n)$ by placing the degree-$j$ object of the complex $\mathcal{N}A_i(M)$ in degree $i+j$, with product defined up to a Koszul sign by the shuffle map and the product of $A$, and with differential defined as the sum of the differential of $A$ and the differential arising from the simplicial structure. We thus obtain functors

$$\begin{aligned}\{\text{dg–}\mathcal{P}_R\text{–algebras}\} &\to \{\text{dg–}\mathcal{P}_R\text{–algebras}\},\\ A &\mapsto [M \mapsto \mathcal{N}A(M,n)],\end{aligned} \tag{3-7}$$

$$\begin{aligned}\{\text{dg–}\mathcal{P}_R\text{–algebras}\} &\to \{\text{graded } \mathcal{P}_R\text{–algebras}\},\\ A &\mapsto [M \mapsto L_*A(M,n)].\end{aligned} \tag{3-8}$$

**Remark 3.4** If the dg–$\mathcal{P}_R$–algebra $A$ is graded commutative and is an exponential functor, then $\mathcal{N}A(M,n)$ coincides (up to homotopy equivalence) with the $n$–fold bar construction of $A(M)$ [20, Chapter X]. For arbitrary dg–$\mathcal{P}_R$–algebras, these two constructions are different.

## 3.4 Some basic facts regarding derived functors of the divided power algebra

Before starting our computations, we recall in this section some basic facts regarding the derived functors of the symmetric algebras, the exterior algebras and the divided powers algebras, which provide a clearer picture of the situation. The following formula is due to Bousfield [5] and Quillen [25] (see also Illusie [16, Chapitre I, paragraphe 4.3.2]).

**Proposition 3.5** *Let $R$ be a commutative ring and let $M$ be a finitely generated projective $R$–module. There are isomorphisms of graded $\mathcal{P}_R$–algebras*

$$\bigoplus_{i,d\geq 0} L_i\Gamma^d_R(M,n) \simeq \bigoplus_{i,d\geq 0} L_{i+d}\Lambda^d_R(M,n+1) \simeq \bigoplus_{i,d\geq 0} L_{i+2d}S^d_R(M,n+2). \tag{3-9}$$

In the sequel, following Illusie [16; 17], we will refer to either of these isomorphisms as a décalage isomorphism and to their composite as the double décalage. If $R$ is a $\mathbb{Q}$–algebra, there is an isomorphism of graded $\mathcal{P}_R$–algebras $S_R(M[i]) \simeq \Gamma_R(M[i])$. Thus the décalage formula implies the following statement:

**Corollary 3.6** *If $R$ is a $\mathbb{Q}$–algebra and $M$ is a finitely generated $R$–module, then the graded $\mathcal{P}_R$–algebra $L_*\Gamma_R(M,n)$ is isomorphic to $\Gamma_R(M[n])$ if $n$ is even and to $\Lambda_R(M[n])$ if $n$ is odd.*

Thus computing derived functors of derived power algebras over $\mathbb{Q}$–algebras $R$ is not an issue. Here are some elementary properties of divided powers when $R$ is noetherian:

**Proposition 3.7** *Let $R$ be noetherian and let $M$ be a finitely generated projective $R$–module. Then the $R$–module $L_j\Gamma^d_R(M,n)$ is:*

- *Zero if $j<n$ or $j>nd$.*
- *A finitely generated $R$–module if $n\le j\le nd$. If $R=\mathbb{Z}$ then, for $n\le j<nd$, $L_j\Gamma^d_R(M,n)$ is a finite abelian group.*

*Finally, the graded $\mathcal{P}_R$–subalgebra*

$$\bigoplus_{d\ge 0} L_{nd}\Gamma^d_R(M,n)\subset \bigoplus_{d,j\ge 0} L_j\Gamma^d_R(M,n)=L_*\Gamma_R(M,n)$$

*is equal to $\Lambda_R(M)$ if $n$ is odd and $2\neq 0$ in $R$, and to $\Gamma_R(M)$ otherwise.*

**Proof** The first two assertions follow from Dold and Puppe [11, Satz 4.22] and the final statement is easily verified by making use of the décalage isomorphisms (3-9). □

# 4 A quasi-trivial filtration of the divided power algebra

In this section, we fix a field $\Bbbk$ of positive characteristic $p>0$. All the functors considered are strict polynomial functors defined over $\Bbbk$. We write $\Gamma^d$, $\Lambda^d$ and $S^d$ for $\Gamma^d_{\Bbbk}$, $S^d_{\Bbbk}$ and $\Lambda^d_{\Bbbk}$. A generic finite-dimensional $\Bbbk$–vector space will be denoted by the letter $V$. The goal of this section is to introduce some particularly nice filtrations of dg–$\mathcal{P}_{\Bbbk}$–algebras, which we call "quasi-trivial", and to exhibit a quasi-trivial filtration of the divided power algebra.

## 4.1 Quasi-trivial filtrations

A nonnegative decreasing filtration on a dg–$\mathcal{P}_{\Bbbk}$–algebra $A$ is a family of graded subfunctors of $A$

$$\bigcap_{i\ge 0} F^iA=0\subset\cdots\subset F^{i+1}A\subset F^iA\subset\cdots\subset F^0A=A$$

such that the differential and the multiplication in $A$ restrict to morphisms $F^iA\to F^iA$ and $F^iA\otimes F^jA\to F^{i+j}A$. Since the $F^iA$ are graded subfunctors of $A$ and there are no morphisms between homogeneous functors of different degrees, the filtration is actually a direct sum over the indices $d$ and $j$ of the filtrations

$$\bigcap_{i\geq 0} F^i A^d_j = 0 \subset \cdots \subset F^{i+1} A^d_j \subset F^i A^d_j \subset \cdots \subset F^0 A^d_j = A^d_j,$$

where $A^d_j$ is the homogeneous component of weight $d$ and degree $j$ of the dg–$\mathcal{P}_{\Bbbk}$–algebra $A$ and $F^i A^d_j := (F^i A) \cap A^d_j$. The graded object associated to this filtration is then a dg–$\mathcal{P}_{\Bbbk}$–algebra, which we denote by $\operatorname{gr} A$. It follows that

$$\operatorname{gr} A = \bigoplus_{i,j,d\geq 0} \operatorname{gr}^i A^d_j = \bigoplus_{i,j,d\geq 0} \frac{F^i A^d_j}{F^{i+1} A^d_j}$$

and we denote by $\operatorname{gr} A^d_j$ the homogeneous component of degree $j$ and weight $d$ of $\operatorname{gr} A$, so that

$$\operatorname{gr} A^d_j = \bigoplus_{i\geq 0} \operatorname{gr}^i A^d_j.$$

**Definition 4.1** Let $A$ be a dg–$\mathcal{P}_{\Bbbk}$–algebra equipped with a nonnegative decreasing filtration $(F^i A)_{i\geq 0}$. We will say that this filtration is quasi-trivial if:

(1) The graded $\mathcal{P}_{\Bbbk}$–algebra $\operatorname{gr} A$ is an exponential functor and, for every finite-dimensional vector space $V$, the vector spaces $\operatorname{gr} A^d_j(V)$ are finite-dimensional for all $j$ and $d$.

(2) There is a weight-preserving isomorphism of differential graded $\Bbbk$–algebras $\phi\colon A(\Bbbk) \xrightarrow{\simeq} \operatorname{gr} A(\Bbbk)$.

The next lemma gives some straightforward consequences of Definition 4.1. Lemma 4.2(c) says that the graded $\mathcal{P}_{\Bbbk}$–algebras $A$ and $\operatorname{gr} A$ are "as close as possible": the filtration modifies the functoriality but not the algebra structure of $A(V)$.

**Lemma 4.2** *Let $A$ be a dg–$\mathcal{P}_{\Bbbk}$–algebra equipped with a quasi-trivial filtration. Then:*

(a) *The filtration of each summand $A^d_j$ is bounded.*

(b) *The dg–$\mathcal{P}_{\Bbbk}$–algebra $A$ is an exponential functor.*

(c) *The choice of a basis of $V$ determines a non-functorial, weight-preserving isomorphism of differential graded $\Bbbk$–algebras $A(V) \simeq \operatorname{gr} A(V)$.*

**Proof** (a) Since each $A^d_j$ is a strict polynomial functor with finite-dimensional values, it is a finite functor and, in particular, all filtrations are bounded (see Touzé [29, Lemma 14.1]).

(b) The map $\psi\colon A(V)\otimes A(W)\to A(V\oplus W)$ induced by the multiplication preserves filtrations, and the associated graded map $\mathrm{gr}\,\psi\colon \mathrm{gr}\,A(V)\otimes \mathrm{gr}\,A(W)\xrightarrow{\simeq}\mathrm{gr}\,A(V\oplus W)$ is an isomorphism by the first condition of Definition 4.1. If we restrict ourselves to homogeneous components of a given weight $d$, the filtrations on $A(V)\otimes A(W)$ and $A(V\oplus W)$ are finite, so that $\psi$ is an isomorphism.

(c) A basis of $V$ determines an isomorphism $V\simeq \mathbb{k}^s$. We obtain the required isomorphism of differential graded algebras as the composite

$$A(\mathbb{k}^s)\simeq A(\mathbb{k})^{\otimes s}\xrightarrow[\phi^{\otimes s}]{\simeq}(\mathrm{gr}\,A(\mathbb{k}))^{\otimes s}\simeq \mathrm{gr}\,A(\mathbb{k}^s),$$

where the first and third isomorphisms are induced by the multiplications (and are isomorphisms since $A(V)$ and $\mathrm{gr}\,A(V)$ are exponential). □

A key property of quasi-trivial filtrations is that they commute with derivation.

**Proposition 4.3** *Let $A$ be a graded $\mathcal{P}_{\mathbb{k}}$–algebra, endowed with a quasi-trivial filtration. For all $n\geq 0$, there exists a filtration of the graded $\mathcal{P}_{\mathbb{k}}$–algebra $L_*A(V,n)$ and a functorial isomorphism of graded $\mathcal{P}_{\mathbb{k}}$–algebras*

$$\mathrm{gr}(L_*A(V,n))\simeq L_*(\mathrm{gr}\,A)(V,n).$$

**Proof** The filtration of $A$ induces a filtration of $\mathcal{N}A(V,n)$. The associated spectral sequence of graded $\mathcal{P}_{\mathbb{k}}$–algebras has the form

$$E^1_{i,j}=H_{i+j}(\mathrm{gr}^{-i}\,\mathcal{N}A(V,n))\Longrightarrow H_{i+j}(\mathcal{N}A(V,n)). \tag{4-1}$$

While this is a second quadrant spectral sequence, there is no problem with convergence since it splits as a direct sum of spectral sequences of homogeneous strict polynomial functors of given weight $d$, and Lemma 4.2(a) ensures that each summand boundedly converges (see Weibel [30, Theorem 5.5.1]). The first page of the spectral sequence may be rewritten as $E^1_{i,j}=L_{i+j}(\mathrm{gr}^{-i}\,A)(V,n)$. To prove Proposition 4.3 it therefore suffices to prove that the spectral sequence (4-1) degenerates at $E_1$. This will be the case if we are able to prove that, for all $i$ and $d$, the homogeneous components of degree $i$ and weight $d$ of the graded $\mathcal{P}_{\mathbb{k}}$–algebras $L_*A(V,n)$ and $L_*(\mathrm{gr}\,A)(V,n)$ have the same dimension (note that we already know that they both are finite-dimensional by Proposition 3.7).

This equality of dimensions follows directly from the observation that $A(K(V,n))$ and $\mathrm{gr}\,A(K(V,n))$ coincide as semi-simplicial $\mathbb{k}$–vector spaces. Indeed the graded $\mathbb{k}$–algebras with weights $A(\mathbb{k})$ and $\mathrm{gr}\,A(\mathbb{k})$ are isomorphic and we claim that, for an exponential graded $\mathcal{P}_{\mathbb{k}}$–algebra $E$, the graded $\mathbb{k}$–algebra with weights $E(\mathbb{k})$ determines completely the semi-simplicial $\mathbb{k}$–vector space $E(K(V,n))$.

The latter claim follows from the explicit construction of the Dold–Kan functor $K$ [30, Section 8.4]. The simplicial $\Bbbk$–vector space $K(V[n])$ is degreewise finite-dimensional, say of some dimension $d_k$ in degree $k$. If we choose a basis of $V$, each $K(V[n])_k$ has a canonical basis determined by the basis of $V$ and, if we use coordinates relative to these bases, then for each $i$ the face operator $\partial_i\colon K(V[n])_k \to K(V[n])_{k-1}$ is given by a formula $\partial_i(x_1,\dots,x_{d_k}) = (y_1,\dots,y_{d_{k-1}})$, where $y_j = \sum_{i\in I_j} x_i$ and $I_1,\dots,I_{d_{k-1}}$ is some partition of the set $\{1,\dots,d_k\}$. We therefore have commutative diagrams

$$\begin{array}{ccc}
E(\Bbbk)^{\otimes d_k} & \xrightarrow[\mu]{\simeq} & E(K(V[n])_k) \\
\downarrow{\scriptstyle \bar\partial_i} & & \downarrow{\scriptstyle \partial_i} \\
E(\Bbbk)^{\otimes d_{k-1}} & \xrightarrow[\mu]{\simeq} & E(K(V[n])_{k-1})
\end{array}$$

where the horizontal isomorphisms $\mu$ are induced by the multiplication of $E(V)$ and $\bar\partial_i$ sends $x_1\otimes\cdots\otimes x_{d_k}$ to $y_1\otimes\cdots\otimes y_{d_{k-1}}$, with $y_j = \prod_{i\in I_j} x_i$. This proves our claim and finishes the proof of Proposition 4.3. □

Another useful property of quasi-trivial filtrations is their compatibility with kernels. To be more specific, let $A$ be a filtered dg–$\mathcal{P}_\Bbbk$–algebra and denote by $Z$ the subalgebra of cycles of $A$. The filtration of $A$ induces a filtration on $Z$ by setting $F^iZ = F^iA\cap Z$. Let $Z'$ be the subalgebra of cycles of $\operatorname{gr} A$. Then we have a canonical injective morphism of algebras

$$\operatorname{gr} Z \hookrightarrow Z'. \tag{4-2}$$

In general, this morphism is not surjective, but this nevertheless turns out to be the case if the filtration of $A$ is quasi-trivial.

**Proposition 4.4** *Let $A$ be a dg–$\mathcal{P}_\Bbbk$–algebra, endowed with a quasi-trivial filtration. Let us denote by $Z$ the cycles of $A$ and by $Z'$ the cycles of $\operatorname{gr} A$. The canonical morphism (4-2) is an isomorphism of graded $\mathcal{P}_\Bbbk$–algebras.*

**Proof** It suffices to prove that the homogeneous components $(\operatorname{gr} Z)^d_i$ and $Z^d_i$ have the same dimension for all degrees $i$ and all weights $d$ or, equivalently, that the maps $\partial\colon A^d_i \to A^d_{i-1}$ and $\operatorname{gr}\partial\colon (\operatorname{gr} A)^d_i \to (\operatorname{gr} A)^d_{i-1}$ have the same rank. This follows from Lemma 4.2(c). □

## 4.2 Truncated polynomial algebras

The truncated polynomial algebra $Q(V)$ is the $\mathcal{P}_\Bbbk$–algebra obtained as the quotient of $S(V)$ by the ideal generated by $V^{(1)}$. For all $i\geq 0$ we denote by $Q(V[i])$ the truncated polynomial algebra on a generator $V$ placed in degree $i$. This is defined in a similar way, as the quotient of $S(V[i])$ by the ideal generated by $V^{(1)}[i]$. Truncated polynomial algebras enjoy the following properties:

(1) If $p=2$, $Q(V)=\Lambda(V)$ (but this is no longer true in odd characteristics).

(2) The $\mathcal{P}_{\Bbbk}$–algebra $Q(V)$ is an exponential functor (in particular a $\mathcal{P}_{\Bbbk}$–bialgebra).

(3) Let us denote by $\phi\colon S(V)\to\Gamma(V)$ the unique morphism of $\mathcal{P}_{\Bbbk}$–algebras whose restriction $V=S^1(V)\to\Gamma^1(V)=V$ to the summand of weight 1 is equal to the identity. It follows that $Q(V)$ is equal to the image of $\phi$.

(4) $Q(V)$ is self-dual, ie there is an isomorphism of $\mathcal{P}_{\Bbbk}$–bialgebras $Q(V)\simeq Q^\sharp(V)$.

All these properties are well known and easy to check; we just indicate how to retrieve (4) from (3). Since $\phi$ is a morphism between exponential $\mathcal{P}_R$–algebras, it is actually a morphism of $\mathcal{P}_R$–bialgebras. Hence its dual yields a morphism of $\mathcal{P}_R$–bialgebras $\phi^\sharp\colon S(V)\simeq\Gamma^\sharp(V)\to S^\sharp(V)\simeq\Gamma(V)$. Since $\phi$ and $\phi^\sharp$ coincide when restricted to the homogeneous summand of weight 1, they must be equal. So we obtain $Q(V)=\operatorname{Im}\phi\simeq\operatorname{Im}\phi^\sharp=Q^\sharp(V)$.

We finish this paragraph with a slightly less well-known result on truncated polynomials, namely the construction of the functorial resolution of $Q(V)$. We equip the graded $\mathcal{P}_{\Bbbk}$–algebra $S(V)\otimes\Lambda(V^{(1)}[1])$ with a differential $\partial$ defined as the composite

$$S^d(V)\otimes\Lambda^e(V^{(1)})\to S^d(V)\otimes V^{(1)}\otimes\Lambda^{e-1}(V^{(1)})\to S^{d+p}(V)\otimes\Lambda^{e-1}(V^{(1)}),$$

where the first map is induced by the comultiplication in $\Lambda(V^{(1)})$ and the second by composition of the inclusion $V^{(1)}\hookrightarrow S^p(V)$ and the multiplication in $S(V)$. The composite morphism of graded $\mathcal{P}_{\Bbbk}$–algebras $S(V)\otimes\Lambda(V^{(1)}[1])\twoheadrightarrow S(V)\twoheadrightarrow Q(V)$ induces a morphism of differential graded algebras

$$f\colon (S(V)\otimes\Lambda(V^{(1)}[1]),\partial)\to(Q(V),0). \tag{4-3}$$

**Proposition 4.5** *The morphism $f$ is a quasi-isomorphism.*

**Proof** The graded $\mathcal{P}_{\Bbbk}$–algebras $S(V)\otimes\Lambda(V^{(1)}[1])$ and $Q(V)$ are exponential functors. Hence for $V=\Bbbk^d$ there is a commutative diagram of differential graded $\Bbbk$–algebras whose vertical isomorphisms are induced by the multiplication:

$$\begin{array}{ccc}
\big(S(\Bbbk)\otimes\Lambda(\Bbbk^{(1)}[1])\big)^{\otimes d} & \xrightarrow{f^{\otimes d}} & Q(\Bbbk)^{\otimes d}\\
\downarrow\simeq & & \downarrow\simeq\\
S(\Bbbk^d)\otimes\Lambda(\Bbbk^{d(1)}[1]) & \xrightarrow{f} & Q(\Bbbk^d)
\end{array}$$

Thus, by the Künneth formula, the proof reduces to the easy case $V=\Bbbk$. □

**Example 4.6** In characteristic 2, the weight-4 component of the morphism of differential graded algebras (4-3) determines the following resolution of $\Lambda^4(V)$, where $\partial_1(x \wedge y) = x^2 \otimes y - y^2 \otimes x$ and $\partial_0(xy \otimes z) = xyz^2$:

$$(4\text{-}4)\quad 0 \longrightarrow \Lambda^2(V^{(1)}) \xrightarrow{\partial_1} S^2(V) \otimes V^{(1)} \xrightarrow{\partial_0} S^4(V) \xrightarrow{f_4} \Lambda^4(V) \longrightarrow 0.$$

## 4.3 The principal filtration on the divided power algebra

We denote by $\mathcal{I}(V)$ the ideal of $\Gamma(V)$ generated by $V = \Gamma^1(V)$. We call this ideal the principal ideal of $\Gamma(V)$ (although it is not strictly speaking a principal ideal). The adic filtration relative to $\mathcal{I}(V)$ will be called the principal filtration of $\Gamma(V)$. The associated graded object is the $\mathcal{P}_{\Bbbk}$–algebra

$$\operatorname{gr}\Gamma(V) := \bigoplus_{n\geq 0} \operatorname{gr}^n(\Gamma(V)) = \bigoplus_{n\geq 0} \mathcal{I}(V)^n/\mathcal{I}(V)^{n+1}.$$

In this section, we will compute in Proposition 4.9 the graded object associated to this principal filtration. This result deserves to be compared to the following well-known assertion [9, Exposé 9, page 9-07]:

**Proposition 4.7** *The choice of a basis of the finite-dimensional vector space $V$ determines a (non-natural) weight-preserving algebra isomorphism*

$$\Gamma(V) \xrightarrow{\sim} Q(V) \otimes \Gamma(V^{(1)}).$$

**Proof** By the exponential properties of $\Gamma(V)$ and $Q(V) \otimes \Gamma(V^{(1)})$ (as in the proof of Lemma 4.2), the proof reduces to the case $V = \Bbbk$, which is a straightforward computation. □

To describe the $\mathcal{P}_{\Bbbk}$–algebra $\operatorname{gr}\Gamma(V)$, we first need to interpret $\mathcal{I}(V)$ as a kernel. By the universal property of the symmetric algebra, the inclusion $V^{(1)} \hookrightarrow S^{p^r}_{\Bbbk}(V)$ induces an injective morphism of $\mathcal{P}_{\Bbbk}$–algebras $S_{\Bbbk}(V^{(1)}) \hookrightarrow S_{\Bbbk}(V)$. Since $S_{\Bbbk}(V)$ is an exponential functor and Frobenius twists are additive functors, $S_{\Bbbk}(V^{(1)})$ is also an exponential functor. Thus the natural inclusion above is also a morphism of $\mathcal{P}_{\Bbbk}$–bialgebras and it induces by duality an epimorphism of $\mathcal{P}_{\Bbbk}$–bialgebras $\Gamma(V) \twoheadrightarrow \Gamma(V^{(1)})$.

**Lemma 4.8** *The principal ideal $\mathcal{I}(V)$ is the kernel of the morphism $\Gamma(V) \twoheadrightarrow \Gamma(V^{(1)})$. In other words, the multiplication in $\Gamma(V)$ yields an exact sequence*

$$(4\text{-}5)\qquad V \otimes \Gamma(V) \xrightarrow{\text{mult}} \Gamma(V) \twoheadrightarrow \Gamma(V^{(1)}) \to 0.$$

**Proof** If $V = V_1 \oplus V_2$, by using the exponential properties of $\Gamma(V)$ and $\Gamma(V^{(1)})$ we obtain that (4-5) is isomorphic to the exact sequence

$$V_1 \otimes \Gamma(V_1) \otimes \Gamma(V_2) \oplus \Gamma(V_1) \otimes V_2 \otimes \Gamma(V_2) \to \Gamma(V_1) \otimes \Gamma(V_2) \twoheadrightarrow \Gamma(V_1^{(1)}) \otimes \Gamma(V_2^{(1)}) \to 0.$$

Hence it suffices to check exactness for $V = \Bbbk$, which is easy. □

**Proposition 4.9** *There is an isomorphism of $\mathcal{P}_{\Bbbk}$–algebras, which maps $\operatorname{gr}^n \Gamma(V)$ isomorphically onto $Q^n(V) \otimes \Gamma(V^{(1)})$:*

$$\operatorname{gr} \Gamma(V) \simeq Q(V) \otimes \Gamma(V^{(1)}).$$

**Proof** Let $\mathcal{I}(V)^n_d$ be the direct summand of $\mathcal{I}(V)^n$ contained in $\Gamma^d(V)$, so that

$$\operatorname{gr}^n(\Gamma^d(V)) = \mathcal{I}(V)^n_d / \mathcal{I}(V)^{n+1}_d.$$

Then $\mathcal{I}(V)^n_d = 0$ if $d < n$ and, for $d \geq n$, the multiplication of $\Gamma(V)$ induces an epimorphism

$$V^{\otimes n} \otimes \Gamma^{d-n}(V) \twoheadrightarrow \mathcal{I}(V)^n_d. \tag{4-6}$$

Since the multiplication $V^{\otimes n} \to \Gamma^n(V)$ factors through the canonical inclusion of $Q^n(V)$ in $\Gamma^n(V)$, the maps (4-6) induce commutative diagrams:

$$\begin{array}{ccc} Q^n(V) \otimes V \otimes \Gamma^{d-n-1}(V) & \twoheadrightarrow & \mathcal{I}(V)^{n+1}_d \\ \downarrow{\scriptstyle Q^n(V)\otimes \mathrm{mult}} & & \downarrow \\ Q^n(V) \otimes \Gamma^{d-n}(V) & \twoheadrightarrow & \mathcal{I}(V)^n_d \end{array}$$

By Lemma 4.8, the cokernel of the multiplication $V \otimes \Gamma^{d-n-1}(V) \to \Gamma^{d-n}(V)$ is equal to $\Gamma^{(d-n)/p}(V^{(1)})$ if $p$ divides $d-n$ and to zero otherwise. Hence, if $p$ divides $d-n$, the map $Q^n(V) \otimes \Gamma^{d-n}(V) \twoheadrightarrow \mathcal{I}(V)^n_d$ induces an epimorphism

$$Q^n(V) \otimes \Gamma^{(d-n)/p}(V^{(1)}) \twoheadrightarrow \mathcal{I}(V)^n_d / \mathcal{I}(V)^{n+1}_d$$

and the quotient $\mathcal{I}(V)^n_d / \mathcal{I}(V)^{n+1}_d$ equals zero if $p$ does not divide $d-n$. We thus have a surjective morphism of $\mathcal{P}_{\Bbbk}$–algebras

$$Q(V) \otimes \Gamma(V^{(1)}) \twoheadrightarrow \bigoplus_{n \geq 0} \operatorname{gr}^n \Gamma(V), \tag{4-7}$$

which sends $Q^n(V) \otimes \Gamma(V^{(1)})$ onto $\operatorname{gr}^n \Gamma(V)$. To finish the proof, we observe that the epimorphism (4-7) is actually an isomorphism for dimension reasons: it follows from Proposition 4.7 that the direct summands of a given weight $d$ of the source and the target of the epimorphism (4-7) have the same finite dimension. □

Propositions 4.7 and 4.9 have the following consequence:

**Corollary 4.10** *The principal filtration on $\Gamma(V)$ is quasi-trivial and determines an isomorphism of $\mathcal{P}_{\Bbbk}$–algebras* $\operatorname{gr}\Gamma(V) \simeq Q(V) \otimes \Gamma(V^{(1)})$.

In the previous statements, we considered the divided power algebra $\Gamma(V)$ as a non-graded algebra (or equivalently as a graded algebra concentrated in degree zero). But we can define an extra degree on the divided power algebra by placing the generator $V$ in degree $i$. In that case the statements of Propositions 4.7 and 4.9 and Corollary 4.10 remain valid with $V$ replaced by $V[i]$ and $V^{(1)}$ replaced by $V^{(1)}[pi]$, since all the morphisms in those propositions preserve the weights, and the extra degree is equal to $i$ times the weight. By iterating Corollary 4.10 we then obtain the following result:

**Corollary 4.11** *For any nonnegative integer $i$, there exists a quasi-trivial filtration on $\Gamma(V[i])$ and an isomorphism of graded $\mathcal{P}_{\Bbbk}$–algebras*

$$\operatorname{gr}\Gamma(V[i]) \simeq \bigotimes_{r\geq 0} Q(V^{(r)}[ip^r]).$$

# 5 The derived functors of $\Gamma^d_{\Bbbk}(V)$ in positive characteristic

In this section, $\Bbbk$ is a field of positive characteristic $p$. All the functors considered are strict polynomial functors defined over $\Bbbk$. In particular, we write $\Gamma^d$, $\Lambda^d$ and $S^d$ for $\Gamma^d_{\Bbbk}$, $S^d_{\Bbbk}$ and $\Lambda^d_{\Bbbk}$. A generic finite-dimensional $\Bbbk$–vector space will be denoted by the letter $V$.

The main results of nhis section are Theorems 5.1 and 5.6, which describe the derived functors of $\Gamma(V)$. These results were already proved by one of us in [29], where the proof was rather technical and relied heavily on the computations of Cartan [9]. The proofs which we will give here are more elementary and independent of [29] and [9]. We will require Theorems 5.1 and 5.6 as an input for the computations of Sections 6, 9 and 10.

## 5.1 The description of $L_*\Gamma^d(V,n)$ in characteristic 2

**Theorem 5.1** *Let $\Bbbk$ be a field of characteristic 2, and let $V$ be a finite-dimensional $\Bbbk$–vector space. For all $n \geq 1$, there is an isomorphism of graded $\mathcal{P}_{\Bbbk}$–algebras*

$$L_*\Gamma(V,n) \simeq \bigotimes_{r_1,\dots,r_n\geq 0} \Gamma(V^{(r_1+\cdots+r_n)}[2^{r_2+\cdots+r_n}+2^{r_3+\cdots+r_n}+\cdots+2^{r_n}+1]). \tag{5-1}$$

For example, there is an isomorphism $L_*\Gamma(V,1) \simeq \bigotimes_{r\geq 0}\Gamma(V^{(r)}[1])$. The homogeneous component of weight $d$ of the graded $\mathcal{P}_{\Bbbk}$–algebra $L_*\Gamma(V,n)$ provides us the derived functors of $\Gamma^d(V)$. Let us spell this out in the $d=4$ case.

**Example 5.2** For $n\geq 1$, the derived functors $L_*\Gamma^4(V,n)$ are given by the following formula (where $F(V)[k]$ means a copy of the strict polynomial functor $F(V)$ placed in degree $k$):

$$\begin{aligned} L_*\Gamma^4(V,n) &\\ \simeq \Gamma^4(V)[4n] &\oplus \bigoplus_{i=1,\dots,n} \Gamma^2(V)\otimes V^{(1)}[3n+i-1] \\ &\oplus \bigoplus_{1\leq i<j\leq n} V^{(1)}\otimes V^{(1)}[2n+i+j-2] \oplus \bigoplus_{i=1,\dots,n} \Gamma^2(V^{(1)})[2n+2i-2] \\ &\oplus \bigoplus_{1\leq i\leq j\leq n} V^{(2)}[n+2i+j-3]. \end{aligned}$$

**Explanation of Example 5.2** In order to unpack the compact formula (5-1), we list those generators of the graded $\mathcal{P}_{\Bbbk}$–algebra $L_*\Gamma(V,n)$ which can contribute (after applying a divided power functor or after taking tensor products) to a summand of weight 4 of $L_*\Gamma(V,n)$. These generators are of the following four distinct types:

(i) One generator $V[n]$, corresponding to the $n$–tuple $(0,\dots,0)$.

(ii) $n$ generators of the form $V^{(1)}[n+i-1]$, corresponding to the $n$–tuples $(r_1,\dots,r_n)$ with $r_i=1$ and $r_k=0$ if $k\neq i$.

(iii) $n$ generators of the form $V^{(2)}[n+3i-3]$, corresponding to the $n$–tuples $(r_1,\dots,r_n)$ with $r_i=2$ and $r_k=0$ if $k\neq i$.

(iv) $\frac{1}{2}n(n-1)$ generators of the form $V^{(2)}[n+j+2i-3]$, corresponding to the $n$–tuples $(r_1,\dots,r_n)$ with $r_i=r_j=1$ for a given pair $\{i,j\}$, $i\neq j$, and $r_k=0$ if $k\neq i,\ j$.

Then we determine all possible manners in which these generators can contribute to a direct summand of weight 4 of $L_*\Gamma(V,n)$:

- The generator (i) can contribute to a summand of weight 4 in two ways, namely (a) via a summand $\Gamma^4(V)$, and (b) via a summand $\Gamma^2(V)\otimes V^{(1)}$, where $V^{(1)}$ is a generator of type (ii).
- The generators of type (ii) can contribute to a summand of weight 4 in three ways. First of all by the method (b) listed before, secondly via a summand $V^{(1)}\otimes V^{(1)}$ where two generators of type (ii) are involved, or thirdly via a summand $\Gamma^2(V^{(1)})$.

- The generators of type (iii) and (iv) are already of weight 4, hence they can only contribute to the part of weight 4 as summands of the form $V^{(2)}$.

Finally, we compute the degree of each of these summands of weight 4 and thereby obtain the sought-after expression for $L_*\Gamma^4(V,n)$. □

More generally, we may extract from Theorem 5.1 the homogeneous component of an arbitrary given weight $d$. This yields the following result:

**Corollary 5.3** *Let $\Bbbk$ be a field of characteristic 2, $d$ be a positive integer and $V$ be a finite-dimensional $\Bbbk$–vector space. There exists an isomorphism of strict polynomial functors*

$$L_i\Gamma^d(V,n) \simeq \bigoplus_{\delta} \bigotimes_{r_1,\dots,r_n\geq 0} \Gamma^{\delta(r_1,\dots,r_n)}(V^{(r_1+\dots+r_n)}[2^{r_2+\dots+r_n}+2^{r_3+\dots+r_n}+\dots+2^{r_n}+1]),$$

*where the sum is taken over all the maps $\delta\colon \mathbb{N}^n\to\mathbb{N}$ satisfying the following two summability conditions:*

$$(1)\quad \sum_{r_1,\dots,r_n\geq 0}\delta(r_1,\dots,r_n)2^{r_1+\dots+r_n}=d.$$

$$(2)\quad \sum_{r_1,\dots,r_n\geq 0}\delta(r_1,\dots,r_n)(2^{r_2+\dots+r_n}+2^{r_3+\dots+r_n}+\dots+2^{r_n}+1)=i.$$

## 5.2 Proof of Theorem 5.1

In this proof, we will constantly use the graded $\mathcal{P}_{\Bbbk}$–algebras $\Gamma(V^{(r)}[i])$ and $\Lambda(V^{(r)}[i])$. To keep formulas in a compact form and to handle the degrees and the twists in a confortable way, we denote these graded $\mathcal{P}_{\Bbbk}$–algebras by $\Gamma^{(r,i]}(V)$ and $\Lambda^{(r,i]}(V)$, respectively. For example, the homogeneous summand of weight $dp^r$ and degree $di$ of $\Gamma^{(r,i]}(V)$ is $\Gamma^d(V^{(r)})$, and the homogeneous summand of weight $dp^r$ and degree $di+j$ of $L_*\Gamma^{(r,i]}(V,n)$ is equal to $L_{j+di}\Gamma^d(V^{(r)},n)$. With these notations, Theorem 5.1 appears as the special case $r=i=0$ of the following theorem, which is the statement that we actually prove.

**Theorem 5.4** *Let $\Bbbk$ be a field of characteristic 2 and let $V$ be a finite-dimensional $\Bbbk$–vector space. For all $n\geq 1$ and $r$, $i\geq 0$, the graded $\mathcal{P}_{\Bbbk}$–algebra $L_*\Gamma^{(r,i]}(V,n)$ is isomorphic to the tensor product*

$$\bigotimes_{r_1,\dots,r_n\geq 0}\Gamma(V^{(r+r_1+\dots+r_n)}[i\,2^{r_1+r_2+\dots+r_n}+2^{r_2+r_3+\dots+r_n}+\dots+2^{r_n}+1]).$$

**Proof** We break the proof into three steps.

**Step 1** (The quasi-trivial filtration of the divided power algebra) Corollary 4.11 yields a quasi-trivial filtration of $\Gamma^{(r,i]}(V)$ with associated graded $\mathcal{P}_{\Bbbk}$–algebra

$$\operatorname{gr}\Gamma^{(r,i]}(V)\simeq\bigotimes_{s\geq 0}\Lambda^{(r+s,i2^s]}(V). \tag{5-2}$$

By Proposition 4.3, derivation commutes with quasi-trivial filtrations. Moreover, by the Eilenberg–Zilber theorem and the Künneth formula, derivation also commutes with tensor products. We therefore have isomorphisms of graded $\mathcal{P}_{\Bbbk}$–algebras

$$\operatorname{gr}(L_*\Gamma^{(r,i]}(V,n))\simeq L_*(\operatorname{gr}\Gamma^{(r,i]})(V,n)\simeq\bigotimes_{s\geq 0}L_*\Lambda^{(r+s,i2^s]}(V,n). \tag{5-3}$$

**Step 2** (Décalage) Now we use the décalage formula of Proposition 3.5,

$$L_*\Lambda^{(r+s,i2^s]}(V,n)\simeq L_*\Gamma^{(r+s,i2^s+1]}(V,n-1),$$

to rewrite the right-hand side of (5-3). In this way we obtain, for all $n\geq 1$, an isomorphism

$$\operatorname{gr}(L_*\Gamma^{(r,i]}(V,n))\simeq\bigotimes_{s\geq 0}L_*\Gamma^{(r+s,i2^s+1]}(V,n-1). \tag{5-4}$$

**Step 3** (Induction) We now prove Theorem 5.4 by induction on $n$. For $n=1$, $L_*\Gamma^{(r+s,i2^s+1]}(V,0)$ is isomorphic to $\Gamma^{(r+s,i2^s+1]}(V)$, so that Theorem 5.4 holds. Let us assume that we have computed $L_*\Gamma^{(r+s,i2^s+1]}(V,n-1)$. By inserting the formula giving $L_*\Gamma^{(r+s,i2^s+1]}(V,n-1)$ in the right-hand side of (5-4), we obtain that the isomorphism of Theorem 5.4 holds, up to a filtration. To prove Theorem 5.4, it remains to verify that the filtration on the left-hand side of (5-4) is trivial. This is a direct consequence of the following proposition. □

**Proposition 5.5** [29, Proposition 14.5] *Let $\Bbbk$ be a field of characteristic 2 and let $A(V)$ be a filtered graded commutative $\mathcal{P}_{\Bbbk}$–algebra whose summands $A_i^d(V)$ are finite-dimensional. Assume that $\operatorname{gr}A(V)$ is isomorphic to a tensor product of graded $\mathcal{P}_{\Bbbk}$–algebras of the form $\Gamma(V^{(r)}[i])$. Then there exists an isomorphism of graded $\mathcal{P}_{\Bbbk}$–algebras $A(V)\simeq\operatorname{gr}A(V)$.*

**Proof** The proof is given in Touzé [29]; we sketch it here for the sake of completeness. The starting point is the vanishing of Proposition A.6, which yields an isomorphism of graded strict polynomial functors $f\colon A(V)\xrightarrow{\sim}\operatorname{gr}A(V)$. The isomorphism $f$ is not an isomorphism of algebras, but we can use it to build one in the following way. Let us first recall that $\operatorname{gr}A(V)$ is exponential, hence $A(V)$ also is by the same reasoning as in Lemma 4.2. In particular, both algebras also have a coalgebra structure determined by the multiplication, as explained in Section 2. One can check that the primitives of an exponential functor form an additive functor. So the isomorphism $f$ shows that the primitives of $A(V)$ form a direct summand of the primitives of $\operatorname{gr} A(V)$. Let us denote by $F(V)$ the primitive part of $\operatorname{gr} A(V)$, that is, the direct sum of all the generators $V^{(r)}[i]$. Then $f$ induces a monomorphism $f\colon A(V) \hookrightarrow F(V)$. But $\operatorname{gr} A(V)$ is the universal cofree coalgebra on $F(V)$, hence $f$ extends uniquely to a morphism of graded $\mathcal{P}_{\Bbbk}$–coalgebras $\bar{f}\colon A(V) \to \operatorname{gr} A(V)$. Since $\bar{f}$ induces an injection between the primitives of $A(V)$ and those of $\operatorname{gr} A(V)$, it is injective, and it is an isomorphism for dimension reasons. Finally, the coalgebra structure of an exponential functor uniquely determines its algebra structure and vice versa, so the isomorphism $\bar{f}$ is also a morphism of algebras. □

## 5.3 The computation of $L_*\Gamma^d(V,1)$ over a field of odd characteristic

**Theorem 5.6** *Let $\Bbbk$ be a field of odd characteristic $p$ and let $V$ be a finite-dimensional $\Bbbk$–vector space. There is an isomorphism of graded $\mathcal{P}_{\Bbbk}$–algebras*

$$L_*\Gamma(V,1) \simeq \bigotimes_{r\geq 1} \Gamma(V^{(r)}[2]) \otimes \bigotimes_{r\geq 0} \Lambda(V^{(r)}[1]). \tag{5-5}$$

**Remark 5.7** The reader might find it surprising that derived functors of $\Gamma$ are so different in characteristic 2 from what they are in odd characteristic. However, Proposition 4.7 shows that Theorem 5.6 is valid in characteristic 2 in a nonnatural way. Also, Proposition 4.9 shows that Theorem 5.6 remains valid in characteristic 2, up to a filtration.

**Proof** The proof is similar to the characteristic 2 case, ie to the proof of Theorem 5.4.

**Step 1** (The quasi-trivial filtration of the divided power algebra) Corollary 4.11 yields a quasi-trivial filtration of $\Gamma(V)$ with associated graded $\mathcal{P}_{\Bbbk}$–algebra $\bigotimes_{s\geq 0} Q(V^{(s)})$. Thus, by Proposition 4.3 and the Eilenberg–Zilber theorem, the graded $\mathcal{P}_{\Bbbk}$–algebra $L_*\Gamma(V,1)$ is filtered and we have isomorphisms

$$\operatorname{gr}(L_*\Gamma(V,1)) \simeq L_*(\operatorname{gr}\Gamma)(V^{(s)},1) \simeq \bigotimes_{s\geq 0} L_*Q(V^{(s)},1). \tag{5-6}$$

**Step 2** (Derived functors of truncated polynomials) In characteristic 2, the derived functors of truncated polynomials (ie of exterior powers) can be computed by the décalage formula of (3-9). This is no longer the case in odd characteristic. The aim of step 2 is to prove an analogue of the décalage formula (for $n = 1$), namely an isomorphism of graded $\mathcal{P}_{\Bbbk}$–algebras

$$L_* Q(V^{(s)}, 1) \simeq \Lambda(V^{(s)}[1]) \otimes \Gamma(V^{(s+1)}[2]). \tag{5-7}$$

Since Frobenius twist functors are additive, derivation commutes with precomposition by the Frobenius twist. So it suffices to prove (5-7) for $s = 0$. Let us denote by $A(V)$ the differential graded $\mathcal{P}_{\Bbbk}$–algebra $(S(V) \otimes \Lambda(V^{(1)}[1]), \partial)$ introduced in (4-3). There is a quasi-isomorphism $f\colon A(V) \to Q(V)$, where the target has zero differential. By deriving this quasi-isomorphism, we obtain a morphism of differential graded $\mathcal{P}_{\Bbbk}$–algebras

$$\mathcal{N} A(V, 1) \to \mathcal{N} Q(V, 1). \tag{5-8}$$

By definition, $\mathcal{N} A(V, 1)$ is the total object of a bigraded $\mathcal{P}_{\Bbbk}$–algebra equipped with two differentials: the differential $\partial$ of the dg–$\mathcal{P}_{\Bbbk}$–algebra $A(V)$ and the differential $d$ coming from the simplicial structure. Thus we have two spectral sequences of graded $\mathcal{P}_{\Bbbk}$–algebras converging to the homology of $\mathcal{N} A(V, 1)$. The first one is obtained by computing first the homology along the differential $\partial$ and secondly the homology along the differential $d$. Thus, its second page is given by $E^2_{0,t} = L_t Q(V, 1)$ and $E^2_{s,t} = 0$ if $s \neq 0$, which proves that (5-8) is a quasi-isomorphism. The second spectral sequence is given by computing first the homology along the simplicial differential $d$. By the décalage formula of (3-9), its first page is given by

$$`E^1_{s,t} = \Lambda^{t-s}(V) \otimes \Gamma^s(V^{(1)}),$$

with the convention that $\Lambda^{t-s}(V)$ is zero if $t - s < 0$. We observe that $`E^1_{s,t}$ is a strict polynomial functor of weight $t + (p-1)s$. Since the differentials of the spectral sequence are weight-preserving maps $d_i\colon `E^i_{s,t} \to `E^i_{s+i,t-i+1}$, they must be zero for lacunary reasons. Hence $`E^1_{s,t} = `E^\infty_{s,t}$. Thus we can conclude that there exists a filtration on the graded $\mathcal{P}_{\Bbbk}$–algebra $L_* Q(V, 1)$ such that the quasi-isomorphism (5-8) induces an isomorphism of graded $\mathcal{P}_{\Bbbk}$–algebras

$$\mathrm{gr}(L_* Q(V^{(s)}, 1)) \simeq \Lambda(V^{(s)}[1]) \otimes \Gamma(V^{(s+1)}[2]).$$

This is almost the formula (5-7) that we want to prove. To finish the proof, we have to get rid of the filtration on the left-hand side. This follows from Proposition 5.8 below.

**Step 3** (Conclusion) The isomorphisms (5-6) and (5-7) together yield the formula of Theorem 5.6 up to a filtration. To finish the proof of Theorem 5.6, we prove that this filtration splits by applying Proposition 5.8 once more. □

In the course of the proof of Theorem 5.6, we have made use of the following statement, whose proof is similar to that of Proposition 5.5.

**Proposition 5.8** [29] *Let $\Bbbk$ be a field of odd characteristic and let $A(V)$ be a filtered graded commutative $\mathcal{P}_{\Bbbk}$–algebra whose summands $A_i^d(V)$ are finite-dimensional. If $\operatorname{gr} A(V)$ is isomorphic to a tensor product of graded $\mathcal{P}_{\Bbbk}$–algebras of the form $\Gamma(V^{(r)}[2i])$ or $\Lambda(V^{(r)}[2i+1])$, then there exists an isomorphism of graded $\mathcal{P}_{\Bbbk}$–algebras $A(V) \simeq \operatorname{gr} A(V)$.*

# 6 The first derived functors of $\Gamma$ over the integers

In this section, we work over the ground ring $\mathbb{Z}$. In particular, we write $\Gamma^d$, $\Lambda^d$ and $S^d$ for $\Gamma^d_{\mathbb{Z}}$, $S^d_{\mathbb{Z}}$ and $\Lambda^d_{\mathbb{Z}}$. A generic free finitely generated abelian group will be denoted by the letter $A$, and we will denote by $A/p$ the quotient $A/pA$. Strict polynomial functors defined over prime fields will enter the picture in the following form: if $F \in \mathcal{P}_{d,\mathbb{F}_p}$, the functor $A \mapsto F(A/p)$ lives in $\mathcal{P}_{d,\mathbb{Z}}$. In particular, Frobenius twist functors yield strict polynomial functors $(A/p)^{(r)}$. We most often drop the parentheses and simply denote those functors by $A/p^{(r)}$. We denote by $\Gamma^d_{\mathbb{F}_p}$, $\Lambda^d_{\mathbb{F}_p}$ and $S^d_{\mathbb{F}_p}$ the symmetric, exterior and divided powers functors considered as objects of $\mathcal{P}_{d,\mathbb{F}_p}$.

The goal of this section is to compute the derived functors $L_*\Gamma(A,1)$. The main result in this section is Theorem 6.3, which gives a first description of these derived functors. In Section 6.1 we present and illustrate this result. The proof will be given in Section 6.4, while Sections 6.2 and 6.3 contain preliminary results which will be needed for this proof. We will further elaborate on this theorem in Section 7 to obtain other forms of the result, in the hope that one or another of these descriptions will be of direct use to the reader.

## 6.1 The description of $L_*\Gamma(A,1)$

Recall from Proposition 3.7 that the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $L_*\Gamma(A,1)$ decomposes as

$$L_*\Gamma(A,1) = \underbrace{\bigoplus_{d\geq 0} L_d\Gamma^d(A,1)}_{D(A)} \oplus \underbrace{\bigoplus_{0<i<d} L_i\Gamma^d(A,1)}_{I(A)},$$

where the diagonal subalgebra $D(A)$ is isomorphic to $\Lambda(A[1])$ and the ideal $I(A)$ has values in torsion abelian groups. Thus, if ${}_{(p)}L_*\Gamma(A,1)$ denotes the $p$–primary part of the abelian group $L_*\Gamma(A,1)$, there is an equality

$$I(A) = \bigoplus_{p \text{ prime}} {}_{(p)}L_*\Gamma(A,1).$$

To describe the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $L_*\Gamma(A,1)$, it therefore suffices to compute the $p$–primary summands ${}_{(p)}L_*\Gamma(A,1)$ as graded $\mathcal{P}_{\mathbb{Z}}$–algebras (without unit) and to describe their $D(A)$–module structure $D(A)\otimes {}_{(p)}L_*\Gamma(A,1)\to {}_{(p)}L_*\Gamma(A,1)$.

We shall describe the $p$–primary summands ${}_{(p)}L_*\Gamma(A,1)$ by the means of "Koszul kernel algebras" (for odd $p$) and "skew Koszul kernel algebras" (for $p=2$), which we now introduce.

**Definition 6.1** Let $p$ be a prime.

(1) We denote by $\partial_{\mathrm{Kos}}$ the unique differential of graded $\mathcal{P}_{\mathbb{Z}}$–algebras on the connected algebra

$$\mathcal{L}_{\mathbb{F}_p}(A/p) := \Lambda_{\mathbb{F}_p}(A/p[1])\otimes \bigotimes_{r\geq 1}\big(\Gamma_{\mathbb{F}_p}(A/p^{(r)}[2])\otimes \Lambda_{\mathbb{F}_p}(A/p^{(r)}[1])\big)$$

sending the generators $A/p^{(r)}[2]=\Gamma^1(A/p^{(r)}[2])$ identically to the corresponding generators $A/p^{(r)}[1]=\Lambda^1(A/p^{(r)}[1])$.

(2) We define the Koszul kernel algebra $K_{\mathbb{F}_p}(A/p)$ as the graded $\mathcal{P}_{\mathbb{Z}}$–subalgebra of $\mathcal{L}_{\mathbb{F}_p}(A/p)$, consisting of the cycles relative to the differential $\partial_{\mathrm{Kos}}$.

As a consequence of Corollary 6.11, the dg–$\mathcal{P}_{\mathbb{Z}}$–algebra $(\mathcal{L}_{\mathbb{F}_p}(A/p),\partial_{\mathrm{Kos}})$ is the tensor product of the algebras $\Gamma_{\mathbb{F}_p}(A/p^{(r)}[2])\otimes\Lambda_{\mathbb{F}_p}(A/p^{(r)}[1])$ with a Koszul differential and of the algebra $\Lambda_{\mathbb{F}_p}(A/p[1])$ with the zero differential. This justifies the name "Koszul kernel algebra" and the notation $\partial_{\mathrm{Kos}}$. We showed in Theorem 5.6 that $\mathcal{L}_{\mathbb{F}_p}(A/p)$ is isomorphic to the algebra $L_*\Gamma_{\mathbb{F}_p}(A/p,1)$ when $p$ is odd. When $p=2$, the algebra $L_*\Gamma_{\mathbb{F}_p}(A/p,1)$ has a different description, which leads us to introduce the following skew Koszul kernel algebra $SK_{\mathbb{F}_2}(A/2)$. We will prove in Corollary 6.20 that the latter is closely related to the Koszul kernel algebra $K_{\mathbb{F}_2}(A/2)$.

**Definition 6.2** (1) We let $\partial_{\mathrm{SKos}}$ be the unique differential of graded $\mathcal{P}_{\mathbb{Z}}$–algebras on

$$L_*\Gamma_{\mathbb{F}_2}(A/2,1)=\bigotimes_{r\geq 0}\Gamma_{\mathbb{F}_2}(A/2^{(r)}[1])$$

whose restriction to each of the summands $\Gamma^2_{\mathbb{F}_2}(A/2^{(r)}[1])$, $r\geq 0$ is equal to the Verschiebung map

$$\pi\colon \Gamma^2_{\mathbb{F}_2}(A/2^{(r)}[1])\to A/2^{(r+1)}[1].$$

(2) We define the skew Koszul kernel algebra $SK_{\mathbb{F}_2}(A/2)$ as the graded $\mathcal{P}_{\mathbb{Z}}$–subalgebra of $L_*\Gamma_{\mathbb{F}_2}(A/2,1)$, consisting of the cycles relative to the differential $\partial_{\mathrm{SKos}}$.

The next theorem provides our first description of the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $L_*\Gamma(A,1)$. It will be proved in Section 6.4.

**Theorem 6.3** *Let $A$ be a finitely generated free abelian group.*

(i) *The diagonal algebra $D(A)$ is isomorphic to $\Lambda(A[1])$.*

(ii) *For any prime number $p$, the $p$–primary component ${}_{(p)}L_*\Gamma(A,1)$ is entirely $p$–torsion. In particular, there is an isomorphism of graded $\mathcal{P}_{\mathbb{Z}}$–algebras*

$$L_*\Gamma(A,1)\otimes\mathbb{F}_p \simeq D(A)\otimes\mathbb{F}_p \oplus {}_{(p)}L_*\Gamma(A,1).$$

(iii) *There are isomorphisms of graded $\mathcal{P}_{\mathbb{Z}}$–algebras*

$$L_*\Gamma(A,1)\otimes\mathbb{F}_p \simeq K_{\mathbb{F}_p}(A/p) \qquad \text{if } p \text{ is an odd prime,} \tag{6-1}$$

$$L_*\Gamma(A,1)\otimes\mathbb{F}_2 \simeq SK_{\mathbb{F}_2}(A/2) \qquad \text{if } p=2. \tag{6-2}$$

**Remark 6.4** The description of the $D(A)$–module structure on ${}_{(p)}L_*\Gamma(A,1)$ is contained in Theorem 6.3(iii). Indeed, part (ii) yields an isomorphism

$$D(A)\otimes {}_{(p)}L_*\Gamma(A,1)\simeq (D(A)\otimes\mathbb{F}_p)\otimes {}_{(p)}L_*\Gamma(A,1),$$

so that the $D(A)$–module structure is obtained by restriction of the multiplication of $L_*\Gamma(A,1)\otimes\mathbb{F}_p$ to $(D(A)\otimes\mathbb{F}_p)\otimes {}_{(p)}L_*\Gamma(A,1)$.

The differentials $\partial_{\mathrm{Kos}}$ and $\partial_{\mathrm{SKos}}$ will be very precisely described in Sections 6.2 and 6.3, so that one can easily write down explicitly the homogeneous component of weight $d$ of each of the differential graded algebras $(L_*\Gamma_{\mathbb{F}_p}(A/p,1),\partial_{\mathrm{Kos}})$ and $(L_*\Gamma_{\mathbb{F}_2}(A/2,1),\partial_{\mathrm{SKos}})$, and thereby compute explicitly $L_*\Gamma^d(A,1)$ for a given $d$. More details regarding the systematic description of $L_*\Gamma^d(A,1)$ will be given in Section 7.

For the moment, we simply provide the flavour of the explicit description of $L_*\Gamma^d(A,1)$ by writing down in detail, for low $d$, the homogeneous summands of weight $d$ of $(L_*\Gamma_{\mathbb{F}_2}(A/2,1),\partial_{\mathrm{SKos}})$. The family of complexes of functors obtained here does not seem to have appeared elsewhere in the literature. In weight $1$, the complex consists simply of a copy of $A/2$, placed in degree $1$. The complexes corresponding to the homogeneous summands of weight $d$ ranging from $2$ to $6$ are the following ones, where each differential can be characterized as the unique nonzero morphism having the specified strict polynomial functors as source and target:

$$\underbrace{\Gamma^2_{\mathbb{F}_2}(A/2)}_{\text{deg } 2} \xrightarrow{f_2} \underbrace{A/2^{(1)}}_{\text{deg } 1}, \tag{6-3}$$

$$\underbrace{\Gamma^3_{\mathbb{F}_2}(A/2)}_{\text{deg } 3} \xrightarrow{f_3} \underbrace{A/2 \otimes A/2^{(1)}}_{\text{deg } 2}, \tag{6-4}$$

$$\underbrace{\Gamma^4_{\mathbb{F}_2}(A/2)}_{\text{deg } 4} \xrightarrow{f_4} \underbrace{\Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}}_{\text{deg } 3} \xrightarrow{g_4} \underbrace{\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})}_{\text{deg } 2} \xrightarrow{h_4} \underbrace{A/2^{(2)}}_{\text{deg } 1}, \tag{6-5}$$

$$\underbrace{\Gamma^5_{\mathbb{F}_2}(A/2)}_{\text{deg } 5} \xrightarrow{f_5} \underbrace{\Gamma^3_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}}_{\text{deg } 4} \xrightarrow{g_5} \underbrace{A/2 \otimes \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})}_{\text{deg } 3} \xrightarrow{h_5} \underbrace{A/2 \otimes A/2^{(2)}}_{\text{deg } 2}, \tag{6-6}$$

$$\underbrace{\Gamma^6_{\mathbb{F}_2}(A/2)}_{\text{deg } 6} \xrightarrow{f_6} \underbrace{\Gamma^4_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}}_{\text{deg } 5} \xrightarrow{g_6} \underbrace{\Gamma^2_{\mathbb{F}_2}(A/2) \otimes \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})}_{\text{deg } 4} \tag{6-7}$$

$$\xrightarrow{\left[\begin{smallmatrix} h_6 \\ h_6' \end{smallmatrix}\right]} \underbrace{\Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(2)} \oplus \Gamma^3_{\mathbb{F}_2}(A/2^{(1)})}_{\text{degree } 3} \xrightarrow{k_6 + f_3^{(1)}} \underbrace{A/2^{(1)} \otimes A/2^{(2)}}_{\text{degree } 2}.$$

More explicitly, the morphisms $f_n$ above are defined as the composites

$$\Gamma^n_{\mathbb{F}_2}(A/2) \to \Gamma^{n-2}_{\mathbb{F}_2}(A/2) \otimes \Gamma^2_{\mathbb{F}_2}(A/2) \to \Gamma^{n-2}_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)},$$

where the first map is induced by the comultiplication of $\Gamma_{\mathbb{F}_2}(A/2)$ and the second one by the Verschiebung morphism. The morphisms $g_n$ are defined as the composites

$$\Gamma^{n-2}_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} \to \Gamma^{n-4}_{\mathbb{F}_2}(A/2) \otimes \Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} \to \Gamma^{n-4}_{\mathbb{F}_2}(A/2) \otimes \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}),$$

where the first map is induced by the comultiplication of $\Gamma_{\mathbb{F}_2}(A/2)$ and the second one by the Verschiebung morphism and the multiplication $A/2^{(1)} \otimes A/2^{(1)} \to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$. The maps $h_n$ and $k_n$ are induced by the Verschiebung morphism and the map $h_6'$ is induced by the Verschiebung morphism and the multiplication of the algebra $\Gamma_{\mathbb{F}_2}(A/2)$.

**Example 6.5** The 2–primary component of $L_*\Gamma^4(A,1)$ is given by

$${}_{(2)}L_*\Gamma^4_{\mathbb{Z}}(A,1) = A/2^{(2)}[1] \oplus \Lambda^2_{\mathbb{F}_2}(A/2^{(1)})[2] \oplus \Phi^4(A)[3],$$

where $\Phi^4(A)$ is the kernel of the morphism $g_4\colon \Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} \to \Lambda^2_{\mathbb{F}_2}(A/2^{(1)})$, and a term $F(A)[i]$ means a copy of the functor $F(A)$ placed in degree $i$.

## 6.2 The Koszul kernel algebra

The purpose of this section is to justify Definition 6.1, that is, to define the differential $\partial_{\mathrm{Kos}}$ and to study some of its properties.

**6.2.1 Koszul algebras** Let $M$ be a projective finitely generated module over a commutative ring $R$. We equip the graded $\mathcal{P}_R$–algebra $\Gamma_R(M[2]) \otimes \Lambda_R(M[1])$ with the differential $d_{\mathrm{Kos}}$, defined as the composite

$$\Gamma_R^d(M) \otimes \Lambda_R^e(M) \to \Gamma_R^{d-1}(M) \otimes M \otimes \Lambda_R^e(M) \to \Gamma_R^{d-1}(M) \otimes \Lambda_R^{e+1}(M),$$

where the first map is induced by the comultiplication in $\Gamma_R(M)$ and the second one by the multiplication in $\Lambda_R(M)$ (if $d = 0$, the differential with source $\Lambda_R^e(M)$ is zero). The resulting commutative dg–$\mathcal{P}_R$–algebra $(\Gamma_R(M[2]) \otimes \Lambda_R(M[1]), d_{\mathrm{Kos}})$ is called the Koszul algebra (on $M$).

**Proposition 6.6** *The homology of the Koszul algebra is equal to $R$ in degree zero and is zero in all other degrees.*

**Proof** Using the fact that $\Gamma_R(M[2]) \otimes \Lambda_R(M[1])$ is exponential (proceed as in the proof of Proposition 4.5), the proof reduces to the elementary case $M = R$. □

**Remark 6.7** The Koszul algebra is a particular case of more general constructions (Illusie [16; 17] and Franjou, Friedlander, Scorichenko and Suslin [13]). Its name is illustrated by the fact that its summand of weight $d$ is the dual (via the duality $^\sharp$) of the more familiar Koszul complex

$$\Lambda_R^d(M) \to S_R^1(M) \otimes \Lambda_R^{d-1}(M) \to \cdots \to S_R^{d-1}(M) \otimes \Lambda_R^1(M) \to S_R^d(M).$$

**6.2.2 The Koszul differential on $L_*\Gamma_{\mathbb{F}_p}(A/p, 1)$** Let $p$ be a prime integer. To be concise, we denote by $\mathcal{L}_{\mathbb{F}_p}(A/p)$ the graded commutative $\mathcal{P}_{\mathbb{Z}}$–algebra

$$\mathcal{L}_{\mathbb{F}_p}(A/p) = \Lambda_{\mathbb{F}_p}(A/p[1]) \otimes \bigotimes_{r \geq 1} \big(\Gamma_{\mathbb{F}_p}(A/p^{(r)}[2]) \otimes \Lambda_{\mathbb{F}_p}(A/p^{(r)}[1])\big). \tag{6-8}$$

If $p$ is odd, $\mathcal{L}_{\mathbb{F}_p}(A/p)$ is isomorphic to the derived functors $L_*\Gamma_{\mathbb{F}_p}(A/p, 1)$ by Theorem 5.6, but this is not the case for $p = 2$. However, the algebra $\mathcal{L}_{\mathbb{F}_2}(A/2)$ will be considered later on. We can endow $\mathcal{L}_{\mathbb{F}_p}(A/p)$ with the structure of a commutative dg–$\mathcal{P}_{\mathbb{Z}}$–algebra in the following way. Let us consider the factor $\Lambda_{\mathbb{F}_p}(A/p[1])$ as a differential graded algebra with zero differential and the other factors of (6-8) as Koszul algebras on the vector spaces $A/p^{(r)}$. We define $(\mathcal{L}_{\mathbb{F}_p}(A/p), \partial_{\mathrm{Kos}})$ to be the tensor product of these differential graded $\mathcal{P}_{\mathbb{Z}}$–algebras. By the Künneth formula and Proposition 6.6 we have:

**Proposition 6.8** *The homology of $(\mathcal{L}_*\Gamma_{\mathbb{F}_p}(A/p,1), \partial_{\mathrm{Kos}})$ is isomorphic to the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $\Lambda_{\mathbb{F}_p}(A/p[1])$.*

We will now justify Definition 6.1, that is, we will characterize the differential $\partial_{\mathrm{Kos}}$ by its values on the generators $\Gamma^1(A/p^{(r)}[2])$ of $\mathcal{L}_{\mathbb{F}_p}(A/p)$. For this, we use the following tool:

**Lemma 6.9** (Uniqueness principle) *Let $\Bbbk$ be a field of positive characteristic $p$ and let $V$ be a finite-dimensional $\Bbbk$–vector space. Let $A(V)$ be a graded commutative $\mathcal{P}_{\Bbbk}$–algebra of the form*

$$A(V) = \Big(\bigotimes_k \Gamma_{\Bbbk}(V^{(r_k)}[i_k])\Big) \otimes \Big(\bigotimes_\ell \Lambda_{\Bbbk}(V^{(r_\ell)}[j_\ell])\Big).$$

*We denote by $G(V)$ the graded functor*

$$G(V) = \Big(\bigoplus_k \Gamma^1_{\Bbbk}(V^{(r_k)}[i_k])\Big) \oplus \Big(\bigoplus_\ell \Lambda^1_{\Bbbk}(V^{(r_\ell)}[j_\ell])\Big)$$

*and by $\pi\colon A(V) \twoheadrightarrow G(V)$ the surjection induced by the projections of $\Gamma_{\Bbbk}(V^{(r_k)}[i_k])$ and $\Lambda_{\Bbbk}(V^{(r_\ell)}[j_\ell])$ onto $V^{(r_k)}[i_k]$ and $V^{(r_\ell)}[j_\ell]$, respectively. Then the map $\partial \mapsto \pi\circ\partial$ induces an injection between the set of differentials on the graded $\mathcal{P}_{\Bbbk}$–algebra $A(V)$ and the set of degree $-1$ morphisms of graded functors $A(V)\to G(V)$:*

$$\{\text{differentials on } A(V)\} \hookrightarrow \mathrm{Hom}_{-1}(A(V), G(V)),$$
$$\partial \mapsto \pi\circ\partial.$$

**Proof of Lemma 6.9** Since $A(V)$ is exponential and graded commutative, the multiplication in $A(V)$ induces an isomorphism of bialgebras $\phi\colon A(V)\otimes A(W) \simeq A(V\oplus W)$. A morphism $\partial\colon A(V)\to A(V)$ of degree $-1$ is a derivation if and only if $\partial$ commutes with $\phi$, which holds if and only if $\partial^\sharp$ commutes with $\phi^\sharp$, which holds if and only if $\partial^\sharp\colon A(V)^\sharp \to A(V)^\sharp$ is a derivation. By duality, Lemma 6.9 is therefore equivalent to the statement that differentials on $A^\sharp(V)$ are completely determined by their restriction $G^\sharp(V)\to A^\sharp(V)$. The latter statement is true since $A^\sharp(V)$ is the free graded commutative algebra on $G^\sharp(V)$. □

**Proposition 6.10** *Let $\partial$ be a differential on the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $\mathcal{L}_{\mathbb{F}_p}(A/p)$.*

(i) *$\partial$ is determined by its restriction to the summands $\Gamma^1_{\mathbb{F}_p}(A/p^{(r)}[2])$ for all $r\geq 1$.*

(ii) *The restriction of $\partial$ sends the summand $\Gamma^1_{\mathbb{F}_p}(A/p^{(r)}[2])$ into $\Lambda^1_{\mathbb{F}_p}(A/p^{(1)}[1])$.*

**Proof** To prove (ii), we observe that, by definition, $\partial$ must send the summand $\Gamma^1(A/p^{(r)}[2])$ into the homogeneous summand of degree 1 and weight $p^r$, which is equal to $\Lambda^1(A/p^{(1)}[1])$. We now prove (i). By Lemma 6.9, $\partial$ is uniquely determined by the morphism

$$\pi \circ \partial_{\mathrm{Kos}}\colon \mathcal{L}_{\mathbb{F}_p}(A/p) \to A/p[1] \oplus \bigoplus_{r\geq 0}(A/p^{(r)}[2] \oplus A/p^{(r)}[1]). \tag{6-9}$$

If we denote by $\pi_r$, $\pi'_r$ and $\pi'_0$ the canonical projections of the right-hand side of (6-9) onto the summands $A/p^{(r)}[2]$, $A/p^{(r)}[1]$ and $A/p[1]$, respectively, then

$$\pi \circ \partial = \pi'_0 \circ (\pi \circ \partial) + \sum \pi'_r \circ (\pi \circ \partial) + \sum \pi_r \circ (\pi \circ \partial).$$

The source of the morphism $\pi'_0 \circ (\pi \circ \partial)$ is the direct summand of weight 1 and degree 2 of $\mathcal{L}_{\mathbb{F}_p}(A/p)$, which is equal to zero, so that $\pi'_0 \circ (\pi \circ \partial) = 0$.

The source of $\pi'_r \circ (\pi \circ \partial)$ is the summand of weight $p^r$ and degree 2 of $\mathcal{L}_{\mathbb{F}_p}(A/p)$, which equals $\Gamma^1_{\mathbb{F}_p}(A/p^{(r)}[2])$ if $p$ is odd and $\Gamma^1_{\mathbb{F}_2}(A/2^{(r)}[2]) \oplus \Lambda^2_{\mathbb{F}_2}(A/2^{(r-1)}[1])$ if $p=2$. Now an easy computation shows that $\mathrm{Hom}_{\mathcal{P}_{\mathbb{F}_2}}(\Lambda^2(V^{(r-1)}), V^{(r)})$ is zero so that, for any prime $p$, $\pi'_r \circ (\pi \circ \partial)$ is determined by the restriction of $\partial$ to the direct summand $\Gamma^1_{\mathbb{F}_p}(A/p^{(r)}[2])$.

Similarly, the source of $\pi_r \circ (\pi \circ \partial)$ is the summand of weight $p^r$ and degree 3 of $\mathcal{L}_{\mathbb{F}_p}(A/p)$. The latter is equal to 0 if $p \geq 5$, to $\Lambda^3_{\mathbb{F}_3}(A/3A^{(r-1)}[1])$ if $p=3$ and to

$$\Gamma^1_{\mathbb{F}_2}(A/2^{(r-1)}[2]) \otimes \Lambda^1_{\mathbb{F}_2}(A/2^{(r-1)}[1]) \oplus \Lambda^2_{\mathbb{F}_2}(A/2^{(r-2)}[1]) \otimes \Lambda^1_{\mathbb{F}_2}(A/2^{(r-1)}[1])$$

if $p=2$. An easy computation shows that there are no nonzero morphisms from such functors to the functor $A/p^{(r)}$ such that $\pi_r \circ (\pi \circ \partial) = 0$. It follows that $\pi \circ \partial$ (hence $\partial$) is completely determined by the restriction of $\partial$ to the summands $\Gamma^1_{\mathbb{F}_p}(A/p^{(r)}[2])$. □

**Corollary 6.11** *The morphism $\partial_{\mathrm{Kos}}$ is the unique differential on the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $\mathcal{L}_{\mathbb{F}_p}(A/p)$ that sends the generators $\Gamma^1_{\mathbb{F}_p}(A/p^{(r)}[2])$ identically to $\Lambda^1_{\mathbb{F}_p}(A/p[1])$.*

The following variant of Corollary 6.11 will be useful in the proof of Theorem 6.3.

**Corollary 6.12** *Let $\partial$ be a differential of the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $\mathcal{L}_{\mathbb{F}_p}(A/p)$. Assume that all the summands $\Lambda^1(A/p^{(r)}[1])$, $r \geq 1$, lie in the image of $\partial$. Then there exists an isomorphism of dg–$\mathcal{P}_{\mathbb{Z}}$–algebras*

$$(\mathcal{L}_{\mathbb{F}_p}(A/p), \partial) \simeq (\mathcal{L}_{\mathbb{F}_p}(A/p), \partial_{\mathrm{Kos}}).$$

**Proof** The only morphisms of strict polynomial functors $f\colon A/p^{(r)} \to A/p^{(r)}$ are the scalar multiples of the identity. By Proposition 6.10, $\delta$ is completely determined by its restrictions to $\Gamma^1(A/p^{(r)}[2])$. These restrictions must be nonzero in order to ensure that the expressions $\Lambda^1(A/p^{(r)}[1])$ lie in the image of $\delta$, so they are of the form $\lambda_r \operatorname{Id}$ with $\lambda_r \in \mathbb{F}_p^*$. Now the required isomorphism is induced by the automorphism of the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $\mathcal{L}_{\mathbb{F}_p}(A/p)$ which sends the generators $A/p[1]$ and $A/p^{(r)}[1]$, $r \geq 1$, identically to themselves and whose restrictions to the generators $A/p^{(r)}[2]$, $r \geq 1$, are equal to $\lambda_r \operatorname{Id}$. □

## 6.3 The skew Koszul kernel algebra

The purpose of this section is to justify Definition 6.2, that is, to define the differential $\partial_{\mathrm{SKos}}$ and to study some of its properties. We also prove that the skew Koszul kernel algebra is, up to a filtration, isomorphic to the Koszul kernel algebra introduced in Definition 6.1.

**6.3.1 The skew Koszul algebras in characteristic 2** Let $\Bbbk$ be a field of characteristic 2 and let $V$ be a finite-dimensional $\Bbbk$–vector space. Consider the graded $\mathcal{P}_{\Bbbk}$–algebra $\Gamma_{\Bbbk}(V[1]) \otimes \Gamma_{\Bbbk}(V^{(1)}[1])$, equipped with the differential $d_{\mathrm{SKos}}$ defined as a composite

$$\Gamma^d_{\Bbbk}(V) \otimes \Gamma^e_{\Bbbk}(V^{(1)}) \to \Gamma^{d-2}_{\Bbbk}(V) \otimes \Gamma^2_{\Bbbk}(V) \otimes \Gamma^e_{\Bbbk}(V^{(1)}) \to \Gamma^{d-2}_{\Bbbk}(V) \otimes \Gamma^{e+1}_{\Bbbk}(V^{(1)}),$$

where the first map is induced by the comultiplication of $\Gamma_{\Bbbk}(V)$ and the second one is induced by the Verschiebung map $\Gamma^2_{\Bbbk}(V) \twoheadrightarrow V^{(1)}$ and the multiplication of $\Gamma_{\Bbbk}(V^{(1)})$ (if $d \leq 1$, the differential with source $\Gamma^0_{\Bbbk}(V) \otimes \Gamma^e_{\Bbbk}(V^{(1)})$ is zero). The resulting commutative differential graded $\mathcal{P}_{\Bbbk}$–algebra $(\Gamma_{\Bbbk}(V[1]) \otimes \Gamma_{\Bbbk}(V^{(1)}[1]), d_{\mathrm{SKos}})$ will be called the *skew Koszul algebra* (on $V$). This name is justified by the following result, which is a differential graded algebra version of Corollary 4.10.

**Proposition 6.13** *Let $V$ be a finite-dimensional vector space over a field $\Bbbk$ of characteristic 2. The tensor product of the principal filtrations of $\Gamma_{\Bbbk}(V[1])$ and of $\Gamma_{\Bbbk}(V^{(1)}[1])$ yields a quasi-trivial filtration of the skew Koszul algebra and the associated graded object is isomorphic to the dg–$\mathcal{P}_{\Bbbk}$–algebra*

$$\Lambda_{\Bbbk}(V[1]) \otimes \big(\Gamma_{\Bbbk}(V^{(1)}[2]) \otimes \Lambda_{\Bbbk}(V^{(1)}[1])\big) \otimes \Gamma_{\Bbbk}(V^{(2)}[2])$$

*via* $\operatorname{Id}_{\Lambda_{\Bbbk}(V[1])} \otimes d_{\mathrm{Kos}} \otimes \operatorname{Id}_{\Gamma_{\Bbbk}(V^{(2)}[2])}$.

**Proof** Let us denote by $(A(V), d_{\mathrm{SKos}})$ the skew Koszul algebra and by $(B(V), d)$ the tensor product of $(\Lambda_{\Bbbk}(V[1]), 0)$, of the Koszul algebra on a generator $V^{(1)}[1]$ and of $(\Gamma_{\Bbbk}(V^{(2)}[2], 0)$.

The tensor product of the principal filtration of $\Gamma_{\Bbbk}(V[1])$ and $\Gamma_{\Bbbk}(V^{(1)}[1])$ coincides with the adic filtration of $A(V)$ relative to the ideal $J(V)$ that is generated by $\Gamma^1_{\Bbbk}(V[1]) \oplus \Gamma^1_{\Bbbk}(V^{(1)}[1])$. By definition, the image of the differential $d_{\mathrm{SKos}}$ is contained in the image of the multiplication $A(V) \otimes V^{(1)}[2] \to A(V)$. In particular, $d_{\mathrm{SKos}}$ sends $J(V)$ to $J(V)$, so the $J(V)$–adic filtration yields a filtration on the dg–$\mathcal{P}_{\Bbbk}$–algebra $(A(V), d_{\mathrm{SKos}})$.

By Proposition 4.9, we have an isomorphism of graded $\mathcal{P}_{\Bbbk}$–algebras $\operatorname{gr} A(V) \simeq B(V)$. We have to prove that $\operatorname{gr}(d_{\mathrm{SKos}}) = d$. By Proposition 6.10 it suffices to show that the restriction of $\operatorname{gr}(d_{\mathrm{SKos}})$ to the direct summand $\Gamma^1_{\Bbbk}(V^{(1)}[2]) = V^{(1)}[2]$ sends this generator identically to $V^{(1)}[1] = \Lambda^1_{\Bbbk}(V^{(1)}[1])$. To prove this, we write down explicitly the homogeneous weight-2 component of the $J(V)$–adic filtration. We have

$$A(V)_2 = \Gamma^2_{\Bbbk}(V[1]) \oplus \Gamma^1_{\Bbbk}(V^{(1)}[1]), \quad J(V)_2 = \Lambda^2_{\Bbbk}(V[1]),$$

and the component of weight 2 of the power $J(V)^n$ is zero for all $n \geq 2$. The restriction of $d_{\mathrm{SKos}}$ to $\Gamma^2_{\Bbbk}(V[1])$ is the Verschiebung map, so that the arrow

$$\operatorname{gr}(d_{\mathrm{SKos}})\colon \Lambda^2_{\Bbbk}(V[1]) \oplus V^{(1)}[2] \to V^{(1)}[1]$$

is zero on the summand $\Lambda^2_{\Bbbk}(V[1])$ and maps $V^{(1)}[2]$ identically to $V^{(1)}[1]$. The fact that $(A(\Bbbk), d_{\mathrm{SKos}})$ is isomorphic to $(B(\Bbbk), d)$ is easily proved by direct inspection. □

**6.3.2 The skew Koszul differential on $L_*\Gamma_{\mathbb{F}_2}(A/2,1)$** We are going to define a differential $\partial_{\mathrm{SKos}}$ on the graded $\mathcal{P}_{\mathbb{Z}}$–algebra

$$L_*\Gamma_{\mathbb{F}_2}(A/2,1) \simeq \bigotimes_{r\geq 0} \Gamma_{\mathbb{F}_2}(A/2^{(r)}[1]).$$

To do this, we consider for all $r \geq 0$ the factor $\Gamma_{\mathbb{F}_2}(A/2^{(r)}[1]) \otimes \Gamma_{\mathbb{F}_2}(A/2^{(r+1)}[1])$ as the skew Koszul algebra (on the vector space $A/2^{(r)}$). Tensoring by identities on the left and the right, this defines a differential $\partial_r$ on $L_*\Gamma_{\mathbb{F}_2}(A/2,1)$. In other words, each element in $L_*\Gamma_{\mathbb{F}_2}(A/2,1)$ can be written as a finite tensor product of elements $x_i \in \Gamma_{\mathbb{F}_2}(A/2^{(i)}[1])$ and $\partial_r$ is given by

$$\partial_r(x_0 \otimes \cdots \otimes x_r \otimes x_{r+1} \otimes \cdots \otimes x_n) = x_0 \otimes \cdots \otimes d_{\mathrm{SKos}}(x_r \otimes x_{r+1}) \otimes \cdots \otimes x_n.$$

**Lemma 6.14** *The differentials $\partial_r$ commute with each other.*

**Proof** We have to check that $\partial_i \circ \partial_j = \partial_j \circ \partial_i$. Since the $\partial_i$ are differentials of algebras, we can use the exponential property to reduce the proof to the trivial case in which $A/2$ is one-dimensional. □

We define the skew Koszul differential $\partial_{\mathrm{SKos}}$ as the sum $\partial_{\mathrm{SKos}} = \sum_{r\geq 0}\partial_r$ (this infinite sum reduces to a finite one on each summand of given degree and weight). We will now justify Definition 6.2, that is, characterize $\partial_{\mathrm{SKos}}$ by its value on the summands $\Gamma^2_{\mathbb{F}_2}(A/2^{(r)})$. The following proposition is proved in the same way as Proposition 6.10.

**Proposition 6.15** *Let $\partial$ be a differential of the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $L_*\Gamma_{\mathbb{F}_2}(A/2,1)$.*

(i) *$\partial$ is determined by its restriction to the summands $\Gamma^2_{\mathbb{F}_2}(A/2^{(r)}[1])$ for $r\geq 0$.*

(ii) *The restriction of $\partial$ sends the summand $\Gamma^2_{\mathbb{F}_2}(A/2^{(r)}[1])$ into $\Gamma^1_{\mathbb{F}_2}(A/2^{(r+1)}[1])$.*

**Corollary 6.16** *The morphism $\partial_{\mathrm{SKos}}$ is the unique differential on the graded $\mathcal{P}_{\mathbb{Z}}$–algebra*

$$L_*\Gamma_{\mathbb{F}_2}(A/2,1) = \bigotimes_{r\geq 0}\Gamma_{\mathbb{F}_2}(A/2^{(r)}[1])$$

*whose restriction to the summands $\Gamma^2_{\mathbb{F}_2}(A/2^{(r)}[1])$, $r\geq 0$, equals the Verschiebung map $\Gamma^2_{\mathbb{F}_2}(A/2^{(r)}[1])\to A/2^{(r+1)}[1]$.*

We also have the analogue of Corollary 6.12. Since the Verschiebung is the only nonzero morphism $\Gamma^2_{\mathbb{F}_2}(A/2^{(r)})\to A/2^{(r+1)}$, this characteristic 2 analogue yields a slightly stronger statement:

**Corollary 6.17** *The differential $\partial_{\mathrm{SKos}}$ is the unique differential on the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $L_*\Gamma_{\mathbb{F}_2}(A/2,1)$ whose image contains all the generators $\Gamma^1_{\mathbb{F}_2}(A/2^{(r)}[1])$ for $r\geq 1$.*

**6.3.3 Koszul versus skew Koszul kernels** The definition of the Koszul differential on $\mathcal{L}_{\mathbb{F}_2}(A/2)$ and of the Skew Koszul differential on $L_*\Gamma_{\mathbb{F}_2}(A/2,1)$ are completely parallel. We now compare explicitly these two constructions. The following proposition follows directly from Proposition 6.13.

**Proposition 6.18** *The tensor product of the principal filtrations on the graded $\mathcal{P}_{\mathbb{Z}}$–algebras $\Gamma_{\mathbb{F}_2}(A/2^{(r)}[1])$, $r\geq 1$, yields a quasi-trivial filtration on $(L_*\Gamma_{\mathbb{F}_2}(A/2,1),\partial_{\mathrm{SKos}})$ and its associated graded object is the dg–$\mathcal{P}_{\mathbb{Z}}$–algebra $(\mathcal{L}_{\mathbb{F}_2}(A/2,1),\partial_{\mathrm{Kos}})$.*

**Corollary 6.19** *The homology of $(L_*\Gamma_{\mathbb{F}_2}(A/2,1),\partial_{\mathrm{SKos}})$ is isomorphic to the graded $\mathcal{P}_{\mathbb{Z}}$–algebra $\Lambda_{\mathbb{F}_2}(A/2[1])$.*

**Proof of Corollary 6.19** The morphism of algebras $\Lambda_{\mathbb{F}_2}(A/2) \hookrightarrow \Gamma_{\mathbb{F}_2}(A/2)$ induces a morphism of algebras $\Lambda_{\mathbb{F}_2}(A/2[1]) \hookrightarrow L_*\Gamma_{\mathbb{F}_2}(A/2,1)$. The image of this morphism consists of cycles and, since $(L_*\Gamma_{\mathbb{F}_2}(A/2,1))_i^d$ is zero for $i > d$, there is an injective morphism

$$\Lambda_{\mathbb{F}_2}(A/2[1]) \hookrightarrow H_*(L_*\Gamma_{\mathbb{F}_2}(A/2,1), \partial_{\mathrm{SKos}}). \tag{6-10}$$

We want to prove that the morphism (6-10) is an isomorphism. For this purpose, it suffices to check that its source and its target have the same dimensions in each summand of a given weight and degree. But Proposition 6.18 and Lemma 4.2(c) yield a *non-functorial* isomorphism of differential graded algebras which preserves the weights:

$$(L_*\Gamma_{\mathbb{F}_2}(A/2,1), \partial_{\mathrm{SKos}}) \simeq (\mathcal{L}_{\mathbb{F}_2}(A/2), \partial_{\mathrm{Kos}})$$

By Proposition 7.1, the homology of $(\mathcal{L}_{\mathbb{F}_2}(A/2), \partial_{\mathrm{Kos}})$ is isomorphic to $\Lambda_{\mathbb{F}_2}(A/2[1])$. Hence the dimensions of the source and the target of morphism (6-10) agree. □

We also mention another consequence of Proposition 6.18, which shows that the skew Koszul kernel algebra is very close to the Koszul kernel algebra. It follows directly from the properties of quasi-trivial filtrations (Lemma 4.2 and Proposition 4.4).

**Corollary 6.20** *Let $A$ be a finitely generated free abelian group. The choice of a basis of $A$ determines a non-functorial isomorphism of algebras which preserves the weights:*

$$SK_{\mathbb{F}_2}(A/2) \simeq K_{\mathbb{F}_2}(A/2).$$

*Moreover, there is a filtration of the graded $\mathcal{P}_{\mathbb{Z}}$ algebra $SK_{\mathbb{F}_2}(A/2)$ and a functorial isomorphism of graded $\mathcal{P}_{\mathbb{Z}}$–algebras*

$$\mathrm{gr}\, SK_{\mathbb{F}_2}(A/2) \simeq K_{\mathbb{F}_2}(A/2).$$

## 6.4 Proof of Theorem 6.3

Theorem 6.3(i) is already known by Proposition 3.7. In this section, we will prove Theorem 6.3(ii)–(iii) in Corollary 6.23. The proof is carried out by examining the Bockstein spectral sequence. We first need the following result:

**Lemma 6.21** *Let $r$ be a positive integer and let $p$ be a prime integer. Then the $p$–primary part of the functor $L_1\Gamma^{p^r}(A,1)$ is equal to $A/p^{(r)}$. Moreover, the morphism induced by* mod $p$ *reduction yields an isomorphism*

$$_{(p)}L_1\Gamma^d(A,1) = L_1\Gamma^d(A,1) \otimes \mathbb{F}_p \xrightarrow{\simeq} L_1\Gamma^d_{\mathbb{F}_p}(A/p,1).$$

**Proof** The complex $\mathcal{N}\Gamma^{p^r}(A,1)\otimes\mathbb{F}_p$ is isomorphic to the complex $\mathcal{N}\Gamma^{p^r}_{\mathbb{F}_p}(A/p,1)$ and $\mathcal{N}\Gamma^{p^r}(A,1)$ is zero in degree zero. Thus, from the long exact sequence associated to the short exact sequence of complexes

$$0\to\mathcal{N}\Gamma^{p^r}(A,1)\xrightarrow{\times p}\mathcal{N}\Gamma^{p^r}(A,1)\to\mathcal{N}\Gamma^{p^r}_{\mathbb{F}_p}(A/p,1)\to 0,$$

we obtain an isomorphism $L_1\Gamma^{p^r}(A,1)\otimes\mathbb{F}_p\simeq L_1\Gamma^{p^r}_{\mathbb{F}_p}(A/p,1)$. But we know by [11, Korollar 10.2] that the $p$–primary part of $L_1\Gamma^d(A,1)$ only contains $p$–torsion. The result follows. □

Let $p$ be a prime number. The Bockstein spectral sequence, as in Weibel [30, Section 5.9.9], is a device which enables one to recover the $p$–primary part of homology of a complex $C$ of free abelian groups from the homology of the complex $C\otimes\mathbb{F}_p$ (provided one is able to compute the differentials in the spectral sequence). We consider the Bockstein spectral sequence for the complex $C=\mathcal{N}\Gamma(A,1)$, which computes the derived functors of $\Gamma(A)$. Since $\mathcal{N}\Gamma(A,1)\otimes\mathbb{F}_p$ is isomorphic to $\mathcal{N}\Gamma_{\mathbb{F}_p}(A/p,1)$, the Bockstein spectral sequence is a spectral sequence of graded $\mathcal{P}_\mathbb{Z}$–algebras, starting at page $0$,

$$E^0(A)_i=L_i\Gamma_{\mathbb{F}_p}(A/p,1)\Longrightarrow(L_i\Gamma(A,1)/\,\mathrm{Torsion})\otimes\mathbb{F}_p=\Lambda_{\mathbb{F}_p}(A/p[1]),$$

with differentials $d^r\colon E^r(A)_i\to E^r(A)_{i-1}$. The differentials in this spectral sequence are related to the $p$–primary torsion of $L_*\Gamma(A,1)$ in the following way:

(i) Given an integer $r\geq 0$, the $p$–torsion of $L_*\Gamma(A,1)$ is killed by multiplication by $p^r$ if and only if $E^r(A)=E^\infty(A)$.

(ii) The injective map induced by the mod $p$ reduction of the complex $L\Gamma(A,1)$,

$$L_*\Gamma(A,1)\otimes\mathbb{F}_p\to L_*\Gamma_{\mathbb{F}_p}(A/p,1)=E^0(A)_*,\tag{6-11}$$

has an image contained in the permanent cycles of the spectral sequence, and the image of the $p$–torsion part of $L_*\Gamma(A,1)$ lies in the image of $d^0$.

We will now completely compute this Bockstein spectral sequence.

**Proposition 6.22** *The Bockstein spectral sequence degenerates at the first page: $E^1(A)=E^\infty(A)$, and the kernel of $d^0$ equals $K_{\mathbb{F}_p}(A/p)$ if $p$ is odd and $SK_{\mathbb{F}_2}(A/2)$ if $p=2$.*

**Proof** By condition (ii) and Lemma 6.21, we know that the image of $d^0$ must contain the generators $A/p^{(1)}[1]$. Thus, by Corollary 6.12 in odd characteristic $p$ or by Corollary 6.17 in characteristic $p=2$, we know that the dg–$\mathcal{P}_\mathbb{Z}$–algebra is isomorphic to $(L_*\Gamma_{\mathbb{F}_p}(A/p,1),\partial_{\mathrm{Kos}})$ if $p$ is odd and to $(L_*\Gamma_{\mathbb{F}_2}(A/2,1),\partial_{\mathrm{SKos}})$ if $p=2$. This proves the statement about the kernel of $d^0$. Both dg–$\mathcal{P}_{\mathbb{Z}}$–algebras have homology equal to $\Lambda_{\mathbb{F}_p}(A/p[1])$, whence the degeneration of the spectral sequence at the $E^1$ page. □

**Corollary 6.23** (Theorem 6.3(ii)–(iii)) *For all primes $p$, the $p$–primary component of $L_*\Gamma(A,1)$ consists only of $p$–torsion and there are isomorphisms of graded $\mathcal{P}_{\mathbb{Z}}$–algebras*

$$L_*\Gamma(A,1)\otimes\mathbb{F}_p \simeq K_{\mathbb{F}_p}(A/p) \quad \text{if } p \text{ is an odd prime,} \tag{6-12}$$

$$L_*\Gamma(A,1)\otimes\mathbb{F}_2 \simeq SK_{\mathbb{F}_2}(A/2) \quad \text{if } p=2. \tag{6-13}$$

**Proof** Since the Bockstein spectral sequence degenerates at the page $E^1$, the $p$–primary component of $L_*\Gamma(A,1)$ is entirely $p$–torsion. In particular, if $0<i<d$ the $p$–primary component ${}_{(p)}L_i\Gamma^d(A,1)$ is isomorphic to $L_i\Gamma^d(A,1)\otimes\mathbb{F}_p$. Hence the map (6-11) sends ${}_{(p)}L_i\Gamma^d(A,1)$ to the weight $d$ and degree $i$ boundaries of $d^0$. The homogeneous part of $E^1(A)$ of degree $i$ and weight $d$ is zero (since this is the case at the $E^\infty(A)$ level). It follows that these boundaries are equal to the weight $d$ and degree $i$ cycles of $d^0$. □

# 7 Further descriptions of the derived functors of $\Gamma$ over the integers

We keep the notations of Section 6, in particular the notation $\mathcal{L}_{\mathbb{F}_p}(A/p)$ from (6-8). The purpose of this section is to give a more detailed description of the derived functors $L_i\Gamma^d(A,1)$ than the one given in Theorem 6.3. By Theorem 6.3, we know that for all $d>0$

$$L_i\Gamma^d(A,1)\simeq\begin{cases}0 & \text{if } i>d \text{ or } i=0,\\ \Lambda^d(A) & \text{if } i=d,\\ SK_{\mathbb{F}_2}(A/2)^d_i\oplus\bigoplus_{p \text{ odd prime}} K_{\mathbb{F}_p}(A/p)^d_i & \text{if } 0<i<d,\end{cases}$$

where the functors $K_{\mathbb{F}_p}(A/p)^d_i$ and $SK_{\mathbb{F}_2}(A/2)^d_i$ denote the homogeneous components of weight $d$ and degree $i$ of the Koszul and skew Koszul kernel algebras. Thus, to give a more detailed description of $L_i\Gamma^d(A,1)$, we need to describe these functors more precisely.

The following description of $K_{\mathbb{F}_p}(A/p)^d_i$ follows directly from the definition of the Koszul kernel algebra and the computation of its homology in Proposition 6.8.

**Proposition 7.1** *Let $p$ be a prime. Then:*

(0) $K_{\mathbb{F}_p}(A/p)^d_d = \Lambda^d_{\mathbb{F}_p}(A/p)$.

(1) $K_{\mathbb{F}_p}(A/p)^d_i = 0$ *if* $i < d < p$ or $d-p+1 < i < d$.

(2) *The nontrivial component* $K_{\mathbb{F}_p}(A/p)^d_i$ *of highest degree* $i < d$ *is given by*

$$K_{\mathbb{F}_p}(A/p)^d_{d-p+1} = \begin{cases} \Lambda^{d-p}_{\mathbb{F}_p}(A/p) \otimes A/p^{(1)} & \text{if } p \neq 2, \\ \bigoplus_{1 \le k \le d/p} \Lambda^{d-kp}_{\mathbb{F}_2}(A/2) \otimes \Gamma^k_{\mathbb{F}_2}(A/2^{(1)}) & \text{if } p = 2. \end{cases}$$

(3) *For any positive integer* $i < d-p+1$, $K_{\mathbb{F}_p}(A/p)^d_i$ *can equally be described as*

(a) *the kernel of the map* $\partial_{\mathrm{Kos}} \colon (\mathcal{L}_{\mathbb{F}_p}(A/p))^d_i \to (\mathcal{L}_{\mathbb{F}_p}(A/p))^d_{i-1}$,

(b) *the image of the map* $\partial_{\mathrm{Kos}} \colon (\mathcal{L}_{\mathbb{F}_p}(A/p))^d_{i+1} \to (\mathcal{L}_{\mathbb{F}_p}(A/p))^d_i$,

(c) *the cokernel of the map* $\partial_{\mathrm{Kos}} \colon (\mathcal{L}_{\mathbb{F}_p}(A/p))^d_{i+2} \to (\mathcal{L}_{\mathbb{F}_p}(A/p))^d_{i+1}$.

**Proof** Let us describe explicitly the terms of highest degrees in the homogeneous component of weight $d$ of $\mathcal{L}_{\mathbb{F}_p}(A/p)$ (an expression such as $F(A)[k]$ means a copy of the functor $F(A)$ placed in degree $k$ and the exterior powers with a negative weight are by convention equal to zero, so that the following direct sums are actually finite):

$$\begin{aligned} \mathcal{L}_{\mathbb{F}_p}(A/p) &= \Lambda^d_{\mathbb{F}_p}(A/p)[d] \oplus \bigoplus_{k \ge 1} \Lambda^{d-kp}_{\mathbb{F}_p}(A/p) \otimes \Gamma^k_{\mathbb{F}_p}(A/p^{(1)})[d-kp+2k] \\ &\quad \oplus \bigoplus_{k \ge 1} \Lambda^{d-kp}_{\mathbb{F}_p}(A/p) \otimes \Gamma^{k-1}_{\mathbb{F}_p}(A/p^{(1)}) \otimes \Lambda^1_{\mathbb{F}_p}(A/p^{(1)})[d-kp+2k-1] \\ &\quad \oplus \text{terms of degree less than } d-kp+2k-1. \end{aligned}$$

The differential $\partial_{\mathrm{Kos}}$ vanishes on the term $\Lambda^d_{\mathbb{F}_p}(A/p)[d]$ and maps the rest of the first line injectively into the terms of the second line. This proves (0) and (1). Moreover, by Proposition 6.8 the homogeneous component of weight $d$ of the complex $(\mathcal{L}_{\mathbb{F}_p}(A/p), \partial_{\mathrm{Kos}})$ is exact in degrees less than $d$. Thus, the degree $d-p+1$ component of the kernel of $\partial_{\mathrm{Kos}}$ is given by some terms from the first line. If $p = 2$ all the terms of the first line have degree $d$, so all of them but $\Lambda^d_{\mathbb{F}_p}(A/p)[d]$ contribute to the degree $d-1$ of the kernel of $\partial_{\mathrm{Kos}}$, whereas if $p \ge 3$ the only contribution to the component of degree $d-1$ of $\partial_{\mathrm{Kos}}$ is that of $\Lambda^{d-p}_{\mathbb{F}_p}(A/p) \otimes A/p^{(1)}$. Finally, (3) follows from the decomposition of the homogeneous component of weight $d$ of the complex $(\mathcal{L}_{\mathbb{F}_p}(A/p), \partial_{\mathrm{Kos}})$ into short exact sequences. □

The following description of the expressions $SK_{\mathbb{F}_p}(A/2)^d_i$ is proved exactly in the same fashion as Proposition 7.1.

**Proposition 7.2** *The following holds:*

(0) $SK_{\mathbb{F}_2}(A/2)^d_d = \Lambda^d_{\mathbb{F}_2}(A/2)$.

(1) $SK_{\mathbb{F}_2}(A/2)^d_{d-1} = \Phi^d(A)$, *where* $\Phi^1(A) = 0$, $\Phi^2(A) = A/2^{(1)}$, $\Phi^3(A) = A/2 \otimes A/2^{(1)}$ *and, for* $d \geq 4$, $\Phi^d(A)$ *can equivalently be described as:*

   (a) *The kernel of the unique nonzero morphism (induced by the comultiplication of* $\Gamma_{\mathbb{F}_2}(A/2)$, *the Verschiebung morphism* $\Gamma^2_{\mathbb{F}_2}(A/2) \twoheadrightarrow A/2^{(1)}$ *and the multiplication* $(A/2^{(1)})^{\otimes 2} \to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$*)*

$$\Gamma^{d-2}_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} \to \Gamma^{d-4}_{\mathbb{F}_2}(A/2) \otimes \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}).$$

   (b) *The image of the unique nonzero morphism (induced by the comultiplication of* $\Gamma_{\mathbb{F}_2}(A/2)$ *and the Verschiebung morphism* $\Gamma^2_{\mathbb{F}_2}(A/2) \twoheadrightarrow A/2^{(1)}$*)*

$$\Gamma^d_{\mathbb{F}_2}(A/2) \to \Gamma^{d-2}_{\mathbb{F}_2}(A/2) \otimes (A/2)^{(1)}.$$

   (c) *The cokernel of the canonical inclusion*

$$\Lambda^d_{\mathbb{F}_2}(A/2) \hookrightarrow \Gamma^d_{\mathbb{F}_2}(A/2).$$

(2) *More generally, for* $i < d-1$, $SK_{\mathbb{F}_2}(A/2)^d_i$ *can be equivalently described as*

   (a) *the kernel of* $\partial_{\mathrm{SKos}}\colon (L_*\Gamma_{\mathbb{F}_2}(A/2,1))^d_i \to (L_*\Gamma_{\mathbb{F}_2}(A/2,1))^d_{i-1}$,

   (b) *the image of* $\partial_{\mathrm{SKos}}\colon (L_*\Gamma_{\mathbb{F}_2}(A/2,1))^d_{i+1} \to (L_*\Gamma_{\mathbb{F}_2}(A/2,1))^d_i$,

   (c) *the cokernel of* $\partial_{\mathrm{SKos}}\colon (L_*\Gamma_{\mathbb{F}_2}(A/2,1))^d_{i+2} \to (L_*\Gamma_{\mathbb{F}_2}(A/2,1))^d_{i+1}$.

Propositions 7.1 and 7.2 yield descriptions of the $L_i\Gamma^d(A,1)$ as kernels, images or cokernels of some very explicit complexes. However, most of these kernels, cokernels or images yield "new" functors which are not direct sums of some familiar functors. For example, by Corollary 6.20, the functors $\Phi^d(A)$ are filtered, with associated graded object equal to the functor $K_{\mathbb{F}_2}(A/2)^d_{d-1}$ described in Proposition 7.1. In particular, for $d = 4$ there is a short exact sequence, which is non-split (as we prove in Proposition 9.2):

$$0 \to \Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} \to \Phi^4(A) \to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \to 0. \tag{7-1}$$

To describe the derived functors $L_i\Gamma^d(A,1)$ in more familiar terms (in terms of classical Weyl functors), we will describe them up to a filtration in Theorem 7.5 below.

By Corollary 6.20, the description of the derived functors $L_i\Gamma^d(A,1)$ up to a filtration reduces to describing the Koszul kernel algebras $K_{\mathbb{F}_p}(A/p)$ up to a filtration for all primes $p$. These $\mathcal{P}_{\mathbb{Z}}$–graded algebras are the cycles of the tensor product of the algebra $\Lambda_{\mathbb{F}_p}(A/p[1])$ with trivial differential and of the Koszul algebras

$(\Lambda_{\mathbb{F}_p}(A/p^{(r)}[1]) \otimes \Gamma_{\mathbb{F}_p}(A/p^{(r)}[2]), d_{\mathrm{Kos}})$ for $r \geq 1$. To obtain a description of the cycles of such a tensor product of complexes, we will use the following result from elementary algebra:

**Lemma 7.3** *Let $C_\bullet(A)$ and $D_\bullet(A)$ be finite exact complexes of functors with values in finite-dimensional $\mathbb{F}_p$–vector spaces and denote by $\delta$ the differential on the tensor product $C_\bullet(A) \otimes D_\bullet(A)$. There is a natural filtration of $\ker\delta$, whose associated graded object is given by*

$$\mathrm{gr}((\ker\delta)_k) \simeq \bigoplus_{i+j=k-1} \mathrm{Ker}\, d_i^C \otimes \mathrm{Ker}\, d_j^D \oplus \bigoplus_{i+j=k} \mathrm{Ker}\, d_i^C \otimes \mathrm{Ker}\, d_j^D$$

**Proof** The proof is very much in the spirit of the quasi-trivial filtrations of Section 4. Let us denote by $X_k(A)$ the kernel of the differential $d_k^C \colon C_k(A) \to C_{k-1}(A)$. We define a two-step decreasing filtration of each functor $C_k(A)$ by

$$F^{k+1}C_k(A) = 0, \quad F^k C_k(A) = X_k(A), \quad F^{k-1}C_k(A) = C_k(A).$$

Then $C_\bullet(A)$ becomes a filtered complex and the associated graded complex is isomorphic to the split exact complex

$$\cdots \to \underbrace{X_k(A) \oplus X_{k-1}(A)}_{\text{degree } k} \xrightarrow{(0,\mathrm{Id})} \underbrace{X_{k-1}(A) \oplus X_{k-2}(A)}_{\text{degree } k-1} \to \cdots. \tag{7-2}$$

There is a non-functorial isomorphism between $C_\bullet(A)$ and the split complex (7-2). The complex $D_\bullet(A)$ is filtered similarly. The tensor product of these two filtrations yields a filtration of the complex $C_\bullet(A) \otimes D_\bullet(A)$, whose associated graded complex $(\mathrm{gr}(C_\bullet(A) \otimes D_\bullet(A)), \mathrm{gr}\,\delta)$ is the tensor product of two split complexes. Moreover, the filtration of the complex $C_\bullet(A) \otimes D_\bullet(A)$ induces a filtration of its cycles $\mathrm{Ker}\,\delta$, defined by the rule $F^i(\mathrm{Ker}\,\delta) := (\mathrm{Ker}\,\delta) \cap F^i(C_\bullet(A) \otimes D_\bullet(A))$, and there is a canonical injection

$$\mathrm{gr}(\mathrm{Ker}\,\delta) \hookrightarrow \mathrm{Ker}(\mathrm{gr}\,\delta). \tag{7-3}$$

Now, the complex $(\mathrm{gr}(C_\bullet(A) \otimes D_\bullet(A)), \mathrm{gr}\,\delta)$ is non-functorially isomorphic to the complex $(C_\bullet(A) \otimes D_\bullet(A), \delta)$, hence the ranks of the differentials $\delta$ and $\mathrm{gr}\,\delta$ are equal. Thus the source and the target of the canonical morphism (7-3) have the same dimension and the morphism (7-3) is an isomorphism. It follows that, up to a filtration, $\mathrm{Ker}\,\delta$ is the kernel of the differential of the tensor product of the split complex (7-2) and the similar complex for $D_\bullet(A)$. The formula of Lemma 7.3 follows. □

The following result follows from Lemma 7.3 by induction on $n$:

**Lemma 7.4** *Let $C^i_\bullet(A)$, $1 \le i \le n$, be a family of finite exact complexes of functors with values in finite-dimensional $\mathbb{F}_p$–vector spaces and let us denote by $\delta$ the induced differential on the $n$–fold tensor product $\bigotimes_{i=1}^n C^i_\bullet(A)$. There is a natural filtration of $\ker\delta$, whose associated graded object is given by*

$$\mathrm{gr}((\ker\delta)_k) \simeq \bigoplus_{j=0}^{n-1} \bigoplus_{i_1+\cdots+i_n=k-j} (\mathrm{Ker}\, d^{C^1}_{i_1} \otimes \cdots \otimes \mathrm{Ker}\, d^{C^n}_{i_n})^{\oplus\binom{n-1}{j}}.$$

To state our description of the derived functors $L_i\Gamma^d(A,1)$, we need to introduce the following combinatorial device. For a fixed prime number $p$, we can consider all possible decompositions of a positive integer $k$ as a sum $k = \sum_i k_i p^{r_i}$, where the $k_i$ are positive integers and the $r_i$ are distinct positive integers. Since the $r_i$ are positive, the existence of such a decomposition implies that $p$ divides $k$. There might however be many such decompositions. Each of these may be identified with a finite sequence of pairs of integers $((r_1,k_1),\dots,(r_i,k_i),\dots)$ satisfying the following conditions:

(i) The integers $r_i$ and $k_i$ are positive.

(ii) For all $i$, $r_i < r_{i+1}$.

(iii) $\sum k_i p^{r_i} = k$.

We denote by $\mathrm{Decomp}(p,k)$ the set of all such finite sequences.

**Theorem 7.5** *Let $V$ be a finite-dimensional $\mathbb{F}_p$–vector space and let $W^d_{k,\mathbb{F}_p}(V)$ in $\mathcal{P}_{d,\mathbb{F}_p}$ denote the kernel of the Koszul differential*

$$d_{\mathrm{Kos}}\colon \Gamma^k_{\mathbb{F}_p}(V)\otimes\Lambda^{d-k}_{\mathbb{F}_p}(V) \to \Gamma^{k-1}_{\mathbb{F}_p}(V)\otimes\Lambda^{d-k+1}_{\mathbb{F}_p}(V).$$

*By convention, $W^d_{k,\mathbb{F}_p}(V)$ is zero if $k>d$ or if $k<0$. For $0<i<d$, the $p$–primary part of $L_i\Gamma^d(A,1)$ is, up to a filtration, isomorphic to the direct sum*

$$\bigoplus_{k=0}^{d} \bigoplus_{\substack{((r_i,k_i))_{i=1}^n \\ \text{in } \mathrm{Decomp}(p,k)}} \bigoplus_{j=0}^{n-1} \bigoplus_{\substack{i_1+\cdots+i_n \\ =i+k-d-j}} \Big(\Lambda^{d-k}_{\mathbb{F}_p}(A/p)\otimes\bigotimes_{\ell=1}^{n} W^{k_\ell}_{i_\ell-k_\ell,\mathbb{F}_p}(A/p^{(r_\ell)})\Big)^{\oplus\binom{n-1}{j}}.$$

**Proof** Up to a filtration, the $p$–primary part of $L_i\Gamma^d(A,1)$ is given by the functor $K_{\mathbb{F}_p}(A/p)^d_i$. Let us denote by $\kappa^d_\bullet(V)$ the complex given by the homogeneous component of weight $d$ of the Koszul algebra $(\Lambda_{\mathbb{F}_p}(V[1])\otimes\Gamma_{\mathbb{F}_p}(V[2]), d_{\mathrm{Kos}})$. By definition, $K_{\mathbb{F}_p}(A/p)^d_i$ is equal to the cycles of degree $i$ in the complex

$$\bigoplus_{k=0}^{d}\Lambda^{d-k}_{\mathbb{F}_p}(A/p[1])\otimes\Big(\bigoplus_{(r_i,k_i)\in\mathrm{Decomp}(p,k)} \kappa^{k_1}_\bullet(A/p^{(r_1)})\otimes\cdots\otimes\kappa^{k_n}_\bullet(A/p^{(r_n)})\Big),$$

with the convention that direct sums over empty sets are equal to zero. The left-hand factors $\Lambda^{d-k}_{\mathbb{F}_p}(A/p[1])$ are concentrated in degree $d-k$. We therefore need to compute the cycles of degree $i+k-d$ of the right-hand factors. The homogeneous summand of degree $i_\ell$ of the cycles of the complex $\kappa^{k_\ell}_\bullet(A/p^{(r_\ell)})$ is $W^{k_\ell}_{i_\ell-k_\ell,\mathbb{F}_p}(A/p^{(r_\ell)})$. Given a sequence $((r_1,k_1),\dots,(r_n,k_n))$, we use Lemma 7.4 to verify that the cycles of degree $i+k-d$ in the tensor product $\kappa^{k_1}_\bullet(A/p^{(r_1)})\otimes\cdots\otimes\kappa^{k_n}_\bullet(A/p^{(r_n)})$ are (up to a filtration) isomorphic to

$$\bigoplus_{j=0}^{n-1}\ \bigoplus_{i_1+\cdots+i_n=(i+k-d)-j} W^{k_1}_{i_1-k_1,\mathbb{F}_p}(A/p^{(r_1)})\otimes\cdots\otimes W^{k_n}_{i_n-k_n,\mathbb{F}_p}(A/p^{(r_n)})^{\oplus\binom{n-1}{j}}.$$

This proves the formula of Theorem 7.5. □

We choose the notation $W^d_{k,\mathbb{F}_p}(V)$ to remind the reader that these kernels of the Koszul complex are well known to representation theorists as Weyl functors [1; 4]. To be more specific, recall that, given a partition $\lambda=(\lambda_1,\dots,\lambda_n)$ of $d$ and a commutative ring $R$, the Weyl functor $W_\lambda(M)\in\mathcal{P}_{d,R}$ is the dual of the Schur functor associated to the partition $\lambda$. The functor $W_\lambda(M)$ may be defined as the image of a certain composite map

$$\Gamma^{\lambda_1}(M)\otimes\cdots\otimes\Gamma^{\lambda_n}(M)\hookrightarrow M^{\otimes d}\xrightarrow{\sigma_{\lambda'}} M^{\otimes d}\twoheadrightarrow\Lambda^{\lambda'_1}(M)\otimes\cdots\otimes\Lambda^{\lambda'_k}(M),$$

where $\sigma_{\lambda'}$ is a specific combinatorial isomorphism and $\lambda'=(\lambda'_1,\dots,\lambda'_k)$ is the partition dual to $\lambda$. The Weyl functor $W_\lambda(M)$ is denoted by $K_\lambda(M)$ in [1; 4]. For example $K_{(d)}(M)=\Gamma^d_R(M)$ and $K_{(1^d)}(M)=\Lambda^d_R(M)$. In particular, our functors $W^d_{k,\mathbb{F}_p}(V)$ are the Weyl functors associated to hook partitions, ie there is an isomorphism $W^d_{k,\mathbb{F}_p}(V)\simeq W_{(k+1,1^{d-k-1})}(V)$ (see eg [4, Chapter III.1] for more details).

# 8 The maximal filtration on $\Gamma(A)$

In this section, we work over the ground ring $\mathbb{Z}$. We use the same notations as in Section 6. In particular, we write $\Gamma^d$, $\Lambda^d$ and $S^d$ for $\Gamma^d_{\mathbb{Z}}$, $S^d_{\mathbb{Z}}$ and $\Lambda^d_{\mathbb{Z}}$. A generic free finitely generated abelian group will be denoted by the letter $A$. We will denote by $A/p$ the quotient $A/pA$ and we will abuse notations and write $A/p^{(r)}$ instead of $(A/pA)^{(r)}$ for simplicity. The purpose in this section is to introduce the maximal filtration of $\Gamma(A)$ and the associated spectral sequence, which will be our main tool for the computation of the derived functors $L_*\Gamma^d(A,n)$ for low $d$ and all $n$. As a warm-up, we finish the section by running through the spectral sequence in the baby cases $d=2$ and $d=3$.

## 8.1 The maximal filtration

We denote by $\mathcal{J}(A)$ the augmentation ideal of the divided power algebra $\Gamma(A)$:

$$\mathcal{J}(A) := \Gamma^{>0}(A) = \ker(\Gamma(A) \to \Gamma^0(A) = \mathbb{Z}).$$

The adic filtration relative to $\mathcal{J}(A)$ will be called the maximal filtration on $\Gamma(A)$ (even though $\mathcal{J}(A)$ is not strictly speaking a maximal ideal in this algebra). The associated graded object is the $\mathcal{P}_{\mathbb{Z}}$–algebra

$$\operatorname{gr}\Gamma(A) := \bigoplus_{i\geq 0} \operatorname{gr}_{-i}(\Gamma(A)) = \bigoplus_{i\geq 0} \mathcal{J}(A)^i/\mathcal{J}(A)^{i+1}.$$

**Remark 8.1** The maximal filtration is different from the principal filtration on $\Gamma(A)$ defined in Section 4.3 (compare the definition of $\mathcal{J}(A)$ with Lemma 4.8).

Restricting ourselves to the weight $d$ component of $\Gamma(A)$, the principal filtration yields a filtration of $\Gamma^d(A)$, with

$$F_{-i}\Gamma^d(A) := \mathcal{J}(A)^i \cap \Gamma^d(A)$$

and associated graded components

$$\operatorname{gr}_{-i}\Gamma^d(A) := F_{-i}\Gamma^d(A)/F_{-i-1}\Gamma^d(A).$$

By definition, the terms of the filtration can be concretely described as

$$F_{-i}\Gamma^d(A) = \operatorname{Im}\Big(\bigoplus \Gamma^{k_1}(A)\otimes\cdots\otimes\Gamma^{k_i}(A) \longrightarrow \Gamma^m(A)\Big), \tag{8-1}$$

where the sum is taken over all $i$–tuples of positive integers $(k_1,\dots,k_i)$ whose sum equals $d$. In particular $F_{-i}\Gamma^d(A) = 0$ for $i > d$, so that the filtration is bounded and

$$\operatorname{gr}_{-d}\Gamma^d(A) = F_{-d}\Gamma^d(A) = \operatorname{Im}(A^{\otimes d} \to \Gamma^d(A)) = S^d(A), \tag{8-2}$$

where the inclusion of $S^d(A)$ into $\Gamma^d(A)$ is determined by the commutative algebra structure on $\Gamma(A)$. It is also easy to identify the graded component $\operatorname{gr}_{-1}\Gamma^d(A)$.

**Lemma 8.2** *For any free abelian group $A$, $\operatorname{gr}_{-1}\Gamma^d(A) = 0$ if $d$ is not a power of a prime $p$ and $\operatorname{gr}_{-1}\Gamma^d(A) = A/p^{(r)}$ if $d = p^r$.*

**Proof** The composite $\Gamma^d(A) \xrightarrow{\text{comult}} \Gamma^k(A)\otimes\Gamma^\ell(A) \xrightarrow{\text{mult}} \Gamma^d(A)$ equals the multiplication by $\binom{d}{k}$, so by (8-1) the integral torsion of $\operatorname{gr}_{-1}\Gamma^d(A)$ is bounded by the gcd of the binomials $\binom{d}{k}$, $0<k<d$. This gcd equals $p$ if $d$ is a power of a prime $p$ and 1 otherwise. Hence $\operatorname{gr}_{-1}\Gamma^d(A) = 0$ if $d$ is not a power of a prime and is an $\mathbb{F}_p$–vector

space if $d = p^r$. In the latter case, by base change, $\mathrm{gr}_{-1}\,\Gamma^d(A)$ identifies with the cokernel of the map

$$\bigoplus \Gamma^k_{\mathbb{F}_p}(A/pA)\otimes\Gamma^\ell_{\mathbb{F}_p}(A/pA)\xrightarrow{\text{mult}}\Gamma^d_{\mathbb{F}_p}(A/pA),$$

where the sum is taken over all pairs $(k,\ell)$ of positive integers with $k+\ell=d$. □

For $1<i<d$, the graded components $\mathrm{gr}_{-i}\,\Gamma^d(A)$ are more complicated and their description involves new classes of functors. We will use the strict polynomial functors $\sigma_{(1,n)}(V)$, defined for $n\geq 2$ on the category of $\mathbb{F}_2$–vector spaces by

(8-3) $$\sigma_{(1,n)}(V)=\mathrm{Coker}\big(\Lambda^2_{\mathbb{F}_2}(V^{(1)})\otimes S^{n-2}_{\mathbb{F}_2}(V)\xrightarrow{u}V^{(1)}\otimes S^n_{\mathbb{F}_2}(V)\big),$$

where $u\colon (x\wedge y)\otimes z\mapsto x\otimes(y^2z)-y\otimes(x^2z)$. The map $u$ appears in the resolution of truncated polynomials introduced in Section 4.2. In particular, Proposition 4.5 implies that $\sigma_{(1,n)}(V)$ lives in a characteristic 2 exact sequence

(8-4) $$0\to\sigma_{(1,n)}(V)\to S^{n+2}_{\mathbb{F}_2}(V)\to\Lambda^{n+2}_{\mathbb{F}_2}(V)\to 0.$$

**Remark 8.3** The functors $\sigma_{(1,n)}(V)$ belong to a family of functors ${}^p\sigma^e_{(\alpha,\beta)}(V)$ introduced by F Jean in [18, Appendix A].

**Lemma 8.4** *Let $d\geq 4$. Then $\mathrm{gr}_{-d+1}\,\Gamma^d(A)\simeq\sigma_{(1,d-2)}(A/2)$ for any free abelian group $A$.*

**Proof** By definition of the maximal filtration, there is a commutative diagram with exact rows (the exactness of the upper row follows from Lemma 8.2 for $d=2$):

$$\begin{array}{ccccccc}
S^{d-2}(A)\otimes S^2(A) & \xrightarrow{\text{mult}} & S^{d-2}(A)\otimes\Gamma^2(A) & \longrightarrow & S^{d-2}(A)\otimes A/2^{(1)} & \longrightarrow & 0\\
\downarrow\!\!\downarrow{\scriptstyle\text{mult}} & & \downarrow\!\!\downarrow{\scriptstyle\text{mult}} & & & & \\
F_{-d}\Gamma^d(A) & \longrightarrow & F_{-d+1}\Gamma^d(A) & \longrightarrow & \mathrm{gr}_{-d+1}\,\Gamma^d(A) & \longrightarrow & 0
\end{array}$$

Hence the surjective morphism $S^{d-2}(A)\otimes\Gamma^2(A)\twoheadrightarrow\mathrm{gr}_{-d+1}\,\Gamma^d(A)$ factors through a surjective morphism

(8-5) $$S^{d-2}(A)\otimes A/2^{(1)}\twoheadrightarrow\mathrm{gr}_{-d+1}\,\Gamma^d(A).$$

Now we check that the composite of the morphism $u$ occurring in (8-3) and the morphism (8-5) is zero. For this, let us take $x\in S^{d-4}(A)$, $y$, $z\in A$, and let us denote by $\bar{y}$ and $\bar{z}$ the images of $\gamma_2(y)$ and $\gamma_2(z)$ in $A/2^{(1)}$. Then it suffices to check that the surjection (8-5) sends $xy^2\otimes\bar{z}-xz^2\otimes\bar{y}$ to zero. This follows from the fact that the elements $xy^2 \otimes \gamma_2(z)$ and $xz^2 \otimes \gamma_2(y)$ are both sent to the same element $2x\gamma_2(y)\gamma_2(z)$ in $F_{-d+1}\Gamma^d(A)$. Hence the morphism (8-5) factors through a surjective morphism

$$\sigma_{(1,d-2)}(A/2) \twoheadrightarrow \mathrm{gr}_{-d+1}\,\Gamma^d(A). \tag{8-6}$$

Finally we check that (8-6) is an isomorphism by dimension-counting (the source and the target are $\mathbb{F}_2$–vector spaces). Let $(a_1,\dots,a_n)$ be a basis of the free abelian group $A$. Then the products $a_{i_1}\cdots a_{i_d}$ with $i_1 \le \cdots \le i_d$ form a basis of $F_{-d}\Gamma^d(A)$. A basis of $F_{-d+1}\Gamma^d(A)$ is given by the products $a_{i_1}\cdots a_{i_d}$ for $i_1 < \cdots < i_d$ and the products $a_{i_1}\cdots a_{i_{d-2}}\gamma_2(a_{i_{d-1}})$ for $i_1 \le \cdots \le i_{d-2}$. The latter elements can be written as the elements $\frac{1}{2}a_{i_1}\cdots a_{i_d}$ for $i_1 \le \cdots \le i_d$ where at least two of the $i_k$ are equal. Thus the dimension of $\mathrm{gr}_{-d+1}\,\Gamma^d(A)$ is equal to the number of $d$–tuples $(i_1,\dots,i_d)$ with $i_1 \le \cdots \le i_d$ and at least two $i_k$ equal. By the short exact sequence (8-4), this is exactly the dimension of $\sigma_{(1,n)}(A/2)$. □

We can identify $\mathrm{gr}_{-2}\,\Gamma^d(A)$ in a similar (and somewhat simpler) fashion.

**Lemma 8.5** *Let $d \ge 3$. For any free abelian group $A$, $\mathrm{gr}_{-2}\,\Gamma^d(A)$ is a torsion abelian group whose $p$–primary part ${}_{(p)}\,\mathrm{gr}_{-2}\,\Gamma^d(A)$ is given by*

$${}_{(p)}\,\mathrm{gr}_{-2}\,\Gamma^d(A) = \begin{cases} 0 & \text{if } d \text{ is not a sum of two powers of } p,\\ A/p^{(k)} \otimes A/p^{(\ell)} & \text{if } d = p^k + p^\ell \text{ with } k \neq \ell,\\ S^2_{\mathbb{F}_p}(A/p^{(k)}) & \text{if } d = 2p^k. \end{cases}$$

**Proof** Let $T^2(A) = \bigoplus_{k+\ell=d} \Gamma^k(A) \otimes \Gamma^\ell(A)$ and let $T^3(A)$ denote the direct sum

$$T^3(A) := \bigoplus_{k+\ell=d}\Bigg(\bigoplus_{k_1+k_2=k} \Gamma^{k_1}(A)\otimes\Gamma^{k_2}(A)\otimes\Gamma^\ell(A) \oplus \bigoplus_{\ell_1+\ell_2=\ell}\Gamma^k(A)\otimes\Gamma^{\ell_1}(A)\otimes\Gamma^{\ell_2}(A)\Bigg).$$

The multiplication in the divided power algebra defines a map $T^3(A) \to T^2(A)$. Let us denote by $C(A)$ its cokernel. By Lemma 8.2, $C(A)$ is a torsion abelian group and its $p$–primary part equals $A/p^{(k)} \otimes A/p^{(\ell)}$ if $d = p^k + p^\ell$ for a prime $p$ and zero otherwise. By definition of the maximal filtration, there is a commutative diagram

$$\begin{array}{ccc} T^3(A) & \longrightarrow & T^2(A) \\ \big\downarrow{\scriptstyle \text{mult}} & & \big\downarrow{\scriptstyle \text{mult}} \\ F_{-3}\Gamma^d(A) & \longrightarrow & F_{-2}(A) \end{array}$$

so the canonical surjection $T^2(A) \twoheadrightarrow \mathrm{gr}_{-2}\,\Gamma^d(A)$ factors through a surjective morphism $C(A) \twoheadrightarrow \mathrm{gr}_{-2}\,\Gamma^d(A)$. In particular, $\mathrm{gr}_{-2}\,\Gamma^d(A)$ is a torsion abelian group and, for a given prime $p$, we have ${}_{(p)}\mathrm{gr}_{-2}\,\Gamma^d(A) = 0$ if $d$ is not of the form $p^k + p^\ell$ and we have a surjective morphism

$$A/p^{(k)} \otimes A/p^{(\ell)} \twoheadrightarrow {}_{(p)}\mathrm{gr}_{-2}\,\Gamma^d(A). \tag{8-7}$$

If $k = \ell$, the commutativity of the product in $\Gamma(A)$ implies that the morphism (8-7) factors further through a surjective morphism

$$S^2_{\mathbb{F}_p}(A/p^{(k)}) \twoheadrightarrow {}_{(p)}\mathrm{gr}_{-2}\,\Gamma^d(A). \tag{8-8}$$

To finish the proof, it suffices to check that the morphisms (8-7) and (8-8) are isomorphisms for $d = p^k + p^\ell$, $k \neq \ell$, and $d = 2p^k$, respectively. This follows from a dimension-counting argument similar to that in the proof of Lemma 8.4. □

We collect the descriptions of the graded components $\mathrm{gr}_{-i}\,\Gamma^d(A)$ for low $d$ in the following example. The cases $d \le 3$ already appear in [8].

**Example 8.6** For any free abelian group $A$, $\mathrm{gr}_{-1}\,\Gamma^1_{\mathbb{Z}}(A) = 0$ and the only nontrivial values of $\mathrm{gr}_{-i}\,\Gamma^j_{\mathbb{Z}}(A)$ for $2 \le j \le 4$ are:

$$\mathrm{gr}_{-i}\,\Gamma^2(A) = \begin{cases} A/2^{(1)} & i = 1, \\ S^2(A) & i = 2, \end{cases} \tag{8-9}$$

$$\mathrm{gr}_{-i}\,\Gamma^3(A) = \begin{cases} A/3^{(1)} & i = 1, \\ A \otimes A/2^{(1)} & i = 2, \\ S^3(A) & i = 3, \end{cases} \tag{8-10}$$

$$\mathrm{gr}_{-i}\,\Gamma^4(A) = \begin{cases} A/2^{(2)} & i = 1, \\ (A \otimes A/3^{(1)}) \oplus S^2_{\mathbb{F}_2}(A/2^{(1)}) & i = 2, \\ \sigma_{(1,2)}(A/2) & i = 3, \\ S^4(A) & i = 4. \end{cases} \tag{8-11}$$

## 8.2 The spectral sequence associated to the maximal filtration

In order to study derived functors of $\Gamma^d(A)$ for $d > 2$, we consider the spectral sequence associated to the maximal filtration of $\Gamma^d(A)$. This is obtained by filtering the simplicial abelian group $\Gamma^d K(A[n])$ componentwise according to the maximal filtration. This yields the homological spectral sequence

$$E^1_{p,q} = L_{p+q}(\mathrm{gr}_p\,\Gamma^d)(A,n) \Longrightarrow L_{p+q}\Gamma^d(A,n) \tag{8-12}$$

with $d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ as usual. The maximal filtration on $\Gamma^d(A)$ is bounded for any fixed integer $d$, so that the associated spectral sequence (8-12) converges to $L_*\Gamma^d(A,n)$ in the strong sense.

We now use this spectral sequence in order to compute the most elementary cases, that is, the derived functors $L_i\Gamma^d(A,n)$ for $A$ free and $d=2$ or $3$.

## 8.3 The derived functors of $\Gamma^2(A)$ for $A$ free

For $d=2$, there are only two nontrivial terms in the graded object associated to the maximal filtration of $\Gamma^2(A)$ by (8-9), so that the spectral sequence (8-12) has only two nonzero columns. For any additive functor $F$,

$$L_iF(A,n) = \begin{cases} F(A) & \text{if } i=n, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore the Frobenius twist functors are additive, so that

$$E^1_{-1,q} = \begin{cases} A/2^{(1)} & \text{if } q=n+1, \\ 0 & \text{otherwise,} \end{cases} \tag{8-13}$$

$$E^2_{-2,q} = L_{q-2}S^2(A,n). \tag{8-14}$$

If $n=1$, then $L_iS^2(A,1)=\Lambda^2(A)$ if $i=2$ and is zero otherwise. The spectral sequence degenerates at $E^1$ for lacunary reasons and we obtain

$$L_*\Gamma^2(A,1) = A/2^{(1)}[1] \oplus \Lambda^2(A)[2]. \tag{8-15}$$

If $n\geq 2$, $L_iS^2(A,n) = L_{i-4}\Gamma^2(A,n-2)$ by the double décalage isomorphism (3-9). So the spectral sequence gives a relation between $L_*\Gamma^2(A,n)$ and $L_*\Gamma^2(A,n-2)$. This gives us the following result:

**Proposition 8.7** *For any $n\geq 0$ and free abelian group $A$, the only nontrivial values of $L_i\Gamma^2(A,n)$ are*

$$L_i\Gamma^2(A,n) = \begin{cases} A/2^{(1)} & \text{if } i=n,n+2,n+4,\dots,2n-2, \ n \text{ even}, \\ A/2^{(1)} & \text{if } i=n,n+2,n+4,\dots,2n-1, \ n \text{ odd}, \\ \Gamma^2(A) & \text{if } i=2n, \ n \text{ even}, \\ \Lambda^2(A) & \text{if } i=2n, \ n \text{ odd}. \end{cases} \tag{8-16}$$

**Proof** The assertion is trivial for $n=0$ and is known for $n=1$ by (8-15). We prove by induction that if it is true for $n-2$, then it is also valid for $n$. Indeed, the spectral sequence degenerates at $E^1$ for lacunary reasons. Thus the result for $n$ is valid up to a filtration. And since there is at most one nonzero term in each total degree in the spectral sequence, the filtration on the abutment is trivial. □

## 8.4 The derived functors of $\Gamma^3(A)$ for $A$ free

By (8-10), there are only three nontrivial terms in the graded object associated to the maximal filtration of $\Gamma^3(A)$. So, for $d = 3$, the spectral sequence (8-12) has only three nonzero columns, namely

(8-17) $$E^1_{-1,q} = A/3^{(1)} \qquad \text{if } q = n+1 \text{ and zero if } q \neq n+1,$$

(8-18) $$E^1_{-2,q} = A \otimes (A/2)^{(1)} \qquad \text{if } q = 4 \text{ and zero if } q \neq 4,$$

(8-19) $$E^1_{-3,q} = L_{q-3}S^3(A,n).$$

If $n = 1$, then by décalage $L_*S^3(A,1) = \Lambda^3(A)[3]$, so the spectral sequence degenerates at $E^1$ for lacunary reasons and we obtain

(8-20) $$L_*\Gamma^3(A,1) = A/3^{(1)}[1] \oplus A \otimes A/2^{(1)}[2] \oplus \Lambda^3(A)[3].$$

If $n \geq 2$, then $L_iS^3(A,n) = L_{i-6}\Gamma^3(A,n-2)$ by the double décalage of Proposition 3.5, so the spectral sequence essentially gives a relation between $L_*\Gamma^3(A,n)$ and $L_*\Gamma^3(A,n-2)$. The next result is proved in exactly the same way as Proposition 8.7.

**Proposition 8.8** *Let $A$ be a free abelian group.*

(i) *For any odd positive integer $n$, the only nontrivial values of $L_i\Gamma^3(A,n)$ are*

(8-21) $$L_i\Gamma^3(A,n) = \begin{cases} A/3^{(1)} & \text{if } i = n, n+4, n+8, \dots, 3n-2, \\ A \otimes A/2^{(1)} & \text{if } i = 2n, 2n+2, 2n+4, \dots, 3n-1, \\ \Lambda^3(A) & \text{if } i = 3n. \end{cases}$$

(ii) *For any even nonnegative integer $n$, the only nontrivial values of $L_i\Gamma^3(A,n)$ are*

(8-22) $$L_i\Gamma^3(A,n) = \begin{cases} A/3^{(1)} & \text{if } i = n, n+4, n+8, \dots, 3n-4, \\ A \otimes A/2^{(1)} & \text{if } i = 2n, 2n+2, 2n+4, \dots, 3n-2, \\ \Gamma^3(A) & \text{if } i = 3n. \end{cases}$$

**Remark 8.9** For a general abelian group $A$, the values of the derived functors $L_*\Gamma^2(A,n)$ and $L_*\Gamma^3(A,n)$ are more complicated and we refer to [8, Proposition 4.1 and Theorem 5.2] for a complete discussion. The present discussion of the derived functors $L_*\Gamma^3(A,n)$ is related to that in [8], since the functor $W_3(A)$ emphasized there is the quotient group $F_{-1}\Gamma^3(A)/F_{-3}\Gamma^3(A)$ for the maximal filtration.

# 9 The derived functors $L_*\Gamma^4(A,1)$ and $L_*\Gamma^4(A,2)$ for $A$ free

In this section, we keep the notations of Section 8, in particular $A$ denotes a generic free finitely generated abelian group and $\Gamma(A)$ stands for $\Gamma_{\mathbb{Z}}(A)$. We compute the derived functors $L_*\Gamma^4(A,1)$ and $L_*\Gamma^4(A,2)$, by the same methods as in Propositions 8.7 and 8.8. The situation is slightly more complicated here. Indeed, although the spectral sequence (8-12) degenerates at $E^1$ for lacunary reasons as in the computations of the derived functors of $\Gamma^2(A)$ and $\Gamma^3(A)$, we will have to solve nontrivial extension problems both to compute the $E^1$ page of the spectral sequence and to recover the derived functors of $\Gamma^4(A)$ from the $E^\infty$–term of the spectral sequence.

The derived functors $L_*\Gamma^4(A,1)$ were already computed in Sections 6 and 7. The proof given here is more elementary and independent from the techniques developed there.

## 9.1 The derived functor $L_*\Gamma^4(A,1)$ for $A$ free

The description of $\mathrm{gr}\,\Gamma^4(A)$ is given in Example 8.6. Most of the graded terms are elementary, so that the corresponding initial terms of the spectral sequence (8-12) for $d=4$ and $n=1$ are easy to compute:

$$E^1_{-1,q}=\begin{cases}A/2^{(2)} & \text{if } q=2,\\ 0 & \text{if } q\neq 2,\end{cases}$$
$$E^1_{-2,q}=\begin{cases}A\otimes A/3^{(1)}\oplus\Lambda^2_{\mathbb{F}_2}(A/2^{(1)}) & \text{if } q=4,\\ 0 & \text{if } q\neq 4,\end{cases}$$
$$E^1_{-4,q}=\begin{cases}\Lambda^4(A) & \text{if } q=8,\\ 0 & \text{if } q\neq 8.\end{cases}$$

To complete the description of the first page, we have to describe the column $E^1_{-3,*}$, that is, the derived functors of $\mathrm{gr}_{-3}\,\Gamma^4(A)=\sigma_{(1,2)}(A/2)$. The presentation (8-3) of $\sigma_{(1,2)}(A/2)$ determines for each $n$ a short exact sequence of simplicial $\mathbb{F}_2$–vector spaces (since, for $\sigma_{(1,2)}(A/2)$, the map $u$ is injective):

$$0\to\Lambda^2_{\mathbb{F}_2}K(A/2^{(1)}[n])\to S^2_{\mathbb{F}_2}K(A/2[n])\otimes K(A/2^{(1)}[n])\to\sigma_{(1,2)}K(A/2[n])\to 0. \tag{9-1}$$

For $n=1$ the associated long exact sequence of homotopy groups induces, by the décalage isomorphisms (3-9), a short exact sequence

$$0\to\Lambda^2_{\mathbb{F}_2}(A/2)\otimes(A/2)^{(1)}\to\pi_3(\sigma_{1,2}K(A/2[1]))\to\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})\to 0 \tag{9-2}$$

in the category of $\mathbb{F}_2$–vector spaces. By (8-11), this describes $E^1_{-3,6}$ as the middle term in (9-2). It also follows from (9-1) that the terms $E^1_{-3,q}$ are trivial for $q \neq 6$. The spectral sequence degenerates for lacunary reasons and, since it has only one nontrivial term in each total degree, there is no filtration issue on the abutment. So we immediately obtain the following result:

**Proposition 9.1** *We have*

$$L_i\Gamma^4(A,1) = \begin{cases} A/2^{(2)} & i=1, \\ (A \otimes A/3^{(1)}) \oplus \Lambda^2_{\mathbb{F}_2}(A/2^{(1)}) & i=2, \\ \Lambda^4(A) & i=4, \end{cases} \tag{9-3}$$

*and there is a short exact sequence*

$$0 \to \Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} \to L_3\Gamma^4(A,1) \to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \to 0 \tag{9-4}$$

*whose middle term was denoted $\Phi^4(A)$ in (7-1) (see also Lemma A.12).*

To solve the extension issue involved in this description of $L_3\Gamma^4(A,1)$, we prove the following statement:

**Proposition 9.2** *The extension (9-4) cannot be functorially split. In fact,*

$$\mathrm{Ext}^1(\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}), \Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}) \simeq \mathbb{Z}/2 \tag{9-5}$$

*in either the category of strict polynomial functors or the category of ordinary functors and $L_3\Gamma^4(A,1)$ can be characterized as the unique nontrivial extension of $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ by $\Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}$.*

**Proof** Suppose that the exact sequence (9-4) is functorially split, so that the 2–primary component ${}_{(2)}L_3\Gamma^4(A,2)$ is isomorphic to $\Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} \oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$. In that case the universal coefficient theorem yields a short exact sequence

$$0 \to L_3\Gamma^4_{\mathbb{Z}}(A,1) \otimes \mathbb{Z}/2 \to L_3\Gamma^4_{\mathbb{F}_2}(A/2,1) \to \mathrm{Tor}(L_2\Gamma^4_{\mathbb{Z}}(A,1), \mathbb{Z}/2) \to 0.$$

The term in the middle is the value for $V := A/2$ of $L_3\Gamma^4_{\mathbb{F}_2}(V,1)$. By Example 5.2, this is equal to $\Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}$. We would therefore have an injective map

$$\Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} \oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \hookrightarrow \Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}.$$

By the methods of Section A.2, we see that there is no nonzero morphism from $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ to $\Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}$, hence the exact sequence (9-4) cannot be functorially split. We refer to Lemma A.12 for the proof of formula (9-5). □

| $q$ \ $p$ | $-4$ | $-3$ | $-2$ | $-1$ |
|---|---|---|---|---|
| 12 | $\Gamma^4(A)$ | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 |
| 9 | 0 | $\Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)}$ | 0 | 0 |
| 8 | 0 | $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ | 0 | 0 |
| 7 | 0 | $A/2^{(2)}$ | 0 | 0 |
| 6 | 0 | 0 | $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})\oplus(A\otimes A/3^{(1)})$ | 0 |
| 5 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | $A/2^{(2)}$ |

Table 1: The initial terms $E^1_{p,q}$ of the maximal filtration spectral sequence for $L\Gamma^4(A,2)$.

## 9.2 The derived functor $L_*\Gamma^4(A,2)$ for $A$ free

We compute the derived functors of $\mathrm{gr}_{-3}\,\Gamma^4(A)=\sigma_{(1,2)}(A/2)$ as in Section 9.1, that is, by the exact sequence (9-1). If $n\geq 2$, there is no extension problem arising from the induced long exact sequence and we obtain

(9-6)
$$L_i\sigma_{(1,2)}(A/2,n)=\begin{cases}\Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)} & \text{if } i=3n,\\ A/2^{(1)}\otimes A/2^{(1)} & \text{if } 2n+2\leq i\leq 3n-1,\\ \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) & \text{if } i=2n+1,\\ A/2^{(2)} & \text{if } n+2\leq i\leq 2n.\end{cases}$$

So we can compute the $E^1$ page of the spectral sequence (8-12) for $\Gamma^4$ for $n=2$. The result is displayed in Table 1.

In particular, the spectral sequence degenerates at $E^1$ for lacunary reasons. So we obtain the following result:

**Proposition 9.3** *For $A$ free, the only nontrivial values of $L_i\Gamma^4(A,2)$ are*

(9-7)
$$L_i\Gamma^4(A,2)=\begin{cases}\Gamma^4(A) & \text{if } i=8,\\ \Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)} & \text{if } i=6,\\ \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) & \text{if } i=5,\\ A/2^{(2)} & \text{if } i=2,\end{cases}$$

*together with $L_4\Gamma^4(A,2)$, which fits into a short exact sequence*

(9-8)
$$0\to A/2^{(2)}\to L_4\Gamma^4(A,2)\to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})\oplus(A\otimes A/3^{(1)})\to 0.$$

To complete the description of $L_*\Gamma^4(A,2)$, we have to solve the extension issue involved in this description of the functor $L_4\Gamma^4(A,2)$. This is the purpose of the following result:

**Proposition 9.4** *The extension (9-8) is not split. More precisely,*

$$\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2^{(1)} \oplus A \otimes A/3^{(1)}, A/2^{(2)}) = \mathbb{Z}/2, \tag{9-9}$$

*so that $L_4\Gamma^4(A,2)$ is the unique nontrivial extension of $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \oplus A \otimes A/3^{(1)}$ by $A/2^{(2)}$. In particular, we obtain an isomorphism of strict polynomial functors:*

$$L_4\Gamma^4(A,2) \simeq \Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \oplus A \otimes A/3^{(1)}.$$

**Proof** The universal coefficient theorem yields a short exact sequence

$$0 \to L_5\Gamma^4_{\mathbb{Z}}(A,2) \otimes \mathbb{Z}/2 \to L_5\Gamma^4_{\mathbb{F}_2}(A/2,2) \to \mathrm{Tor}(L_4\Gamma^4_{\mathbb{Z}}(A,2), \mathbb{Z}/2) \to 0. \tag{9-10}$$

The term in the middle was already computed in Example 5.2. If the extension (9-8) was functorially split, then ${}_{(2)}L_4\Gamma^4(A,2)$ would be isomorphic to $A/2^{(2)} \oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$, so that the short exact sequence (9-10) could be restated as

$$0 \longrightarrow \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \xrightarrow{i} (A/2^{(1)} \otimes A/2^{(1)}) \oplus A/2^{(2)} \xrightarrow{j} A/2^{(2)} \oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \longrightarrow 0.$$

A dimension count in the category of $\mathbb{F}_2$–vector spaces makes it clear that such a short exact sequence cannot exist. It follows that (9-8) is not split. The formula (9-9) and the identification of $L_4\Gamma^4(A,2)$ follow from Lemma A.11. □

# 10 The derived functors of $\Gamma^4(A)$ for $A$ free

In this section we assume that $A$ is a free abelian group. We will give a complete description of the derived functors $L_i\Gamma^4(A,n)$ for all $i \geq 0$ and $n \geq 1$.

## 10.1 The description of $L_*\Gamma^4(A, n)$ for $A$ free

The main result of Section 10 is the following computation:

**Theorem 10.1** *Let $n$ be a positive integer. If $n = 2m+1$ then we have an isomorphism of graded strict polynomial functors (where $F(A)[k]$ denotes a copy of $F(A)$ placed* *in degree $k$ and sums over empty sets mean zero):*

$$\begin{aligned}
&L_*\Gamma^4(A,n)\\
&\simeq \Lambda^4(A)[4n]\oplus\Phi^4(A)[4n-1]\oplus\bigoplus_{i=0}^{m-1}\Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)}[3n+2i]\\
&\qquad\oplus\bigoplus_{i=0}^{m}\Lambda^2_{\mathbb{F}_2}(A/2^{(1)})[2n+4i]\oplus\bigoplus_{i=0}^{m-1}\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})[2n+4i+1]\\
&\qquad\oplus\bigoplus_{i=0}^{m-1}\bigoplus_{j=2i}^{n-3}A/2^{(1)}\otimes A/2^{(1)}[2n+2i+j+2]\oplus\bigoplus_{i=0}^{m}A\otimes A/3^{(1)}[2n+4i]\\
&\qquad\oplus\bigoplus_{i=0}^{m}A/2^{(2)}[n+6i]\oplus\bigoplus_{i=0}^{m-1}\bigoplus_{j=2i}^{n-2}A/2^{(2)}[n+4i+j+2].
\end{aligned}$$

*Here $\Phi^4(A):=L_3\Gamma^4(A,1)$ is, as shown in Propositions 9.1 and 9.2, the unique nontrivial extension of $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ by $\Lambda^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)}$. Similarly, if $n=2m$ there is an isomorphism of graded strict polynomial functors*

$$\begin{aligned}
&L_*\Gamma^4(A,n)\\
&\simeq\Gamma^4(A)[4n]\oplus\bigoplus_{i=0}^{m-1}\Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)}[3n+2i]\oplus\bigoplus_{i=0}^{m-1}\Gamma^2_{\mathbb{Z}}(A/2^{(1)})[2n+4i]\\
&\quad\oplus\bigoplus_{i=0}^{m-1}\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})[2n+4i+1]\oplus\bigoplus_{i=0}^{m-2}\bigoplus_{j=2i}^{n-3}A/2^{(1)}\otimes A/2^{(1)}[2n+2i+j+2]\\
&\quad\oplus\bigoplus_{i=0}^{m-1}A\otimes A/3^{(1)}[2n+4i]\oplus\bigoplus_{i=0}^{m-1}A/2^{(2)}[n+6i]\oplus\bigoplus_{i=0}^{m-2}\bigoplus_{j=2i}^{n-3}A/2^{(2)}[n+4i+j+2].
\end{aligned}$$

Although the formulas describing the derived functors $L_*\Gamma^4(A,n)$ look very similar for $n$ even and odd, there are at least two major differences for the 2–primary part. First of all, if $n$ is even there is some 4–torsion in $L_*\Gamma^4(A,n)$, provided by the summands $\Gamma^2_{\mathbb{Z}}(A/2^{(1)})$. On the contrary, when $n$ is odd, $L_*\Gamma^4(A,n)$ has only 2–torsion. Secondly, the functor $\Phi^4(A)$ appears as a direct summand (with multiplicity one) in $L_*\Gamma^4(A,n)$ when $n$ is odd and does not appear in the formula when $n$ is even. We observe that the formula for $L_*\Gamma^4_{\mathbb{F}_2}(V,n)$ in Example 5.2 did not depend on the parity of $n$.

For $n=1$, Theorem 10.1 is equivalent to Proposition 9.1 and, for $n=2$, it is equivalent to Proposition 9.3. Now Theorem 10.1 easily follows, by induction on $n$, from the following statement:

**Theorem 10.2** *Let $n \geq 3$. If $n$ is odd, there is an isomorphism of graded strict polynomial functors*

$$\begin{aligned}(10\text{-}1)\quad & L_*\Gamma^4(A,n)\\ &= L_*\Gamma^4(A,n-2)[8] \oplus A/2^{(2)}[n] \oplus \bigoplus_{i=n+2}^{2n} A/2^{(2)}[i]\\ &\quad \oplus \bigoplus_{i=2n+2}^{3n-1} A/2^{(1)} \otimes A/2^{(1)}[i] \oplus \Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}[3n]\\ &\quad \oplus \Lambda^2_{\mathbb{F}_2}(A/2^{(1)})[2n] \oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})[2n+1] \oplus A \otimes A/3[2n].\end{aligned}$$

*Similarly, if $n$ is even, there is an isomorphism*

$$\begin{aligned}(10\text{-}2)\quad & L_*\Gamma^4(A,n)\\ &= L_*\Gamma^4(A,n-2)[8] \oplus A/2^{(2)}[n] \oplus \bigoplus_{i=n+2}^{2n-1} A/2^{(2)}[i]\\ &\oplus \bigoplus_{i=2n+2}^{3n-1} A/2^{(1)} \otimes A/2^{(1)}[i] \oplus \Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}[3n] \oplus \Gamma^2_{\mathbb{Z}}(A/2^{(1)})[2n]\\ &\oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})[2n+1] \oplus A \otimes A/3[2n].\end{aligned}$$

The remainder of Section 10 is devoted to the proof of Theorem 10.2. The proof goes along the same lines as the computation of $L_*\Gamma^4(A,1)$ and $L_*\Gamma^4(A,2)$ in Section 9, that is, the relation between $L_*\Gamma^4(A,n)$ and $L_*\Gamma^4(A,n-2)$ is provided by the analysis of the spectral sequence (8-12) induced by the maximal filtration of $\Gamma^4(A)$. However, for $n \geq 3$, the analysis of the spectral sequence is more delicate than in Section 9 as there now are nontrivial differentials in the spectral sequence. Also, we will have to solve extension problems in order to recover $L_*\Gamma^4(A,n)$ from the $E^\infty$ page of the spectral sequence.

## 10.2 Proof of Theorem 10.2

From now on, $n \geq 3$ and we assume that Theorem 10.1 has been proved for $n-2$. We are going to prove that Theorem 10.2 holds for $n$. The first page of the maximal filtration spectral sequence (8-12) for $d=4$ and $n \geq 3$ can be computed by the methods introduced in Sections 9.1 and 9.2 for $n=1, 2$. In particular, the terms in the $p=-3$

| $q$ \ $p$ | $-4$ | $-3$ | $-2$ | $-1$ |
|---|---|---|---|---|
| $4n+4$ | $L_{4n-8}\Gamma^4(A,n-2)$ | $0$ | $0$ | $0$ |
| $4n+3$ | $L_{4n-9}\Gamma^4(A,n-2)$ | $0$ | $0$ | $0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $3n+4$ | $L_{3n-8}\Gamma^4(A,n-2)$ | $0$ | $0$ | $0$ |
| $3n+3$ | $L_{3n-9}\Gamma^4(A,n-2)$ | $\Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)}$ | $0$ | $0$ |
| $3n+2$ | $L_{3n-10}\Gamma^4(A,n-2)$ | $A/2^{(1)}\otimes A/2^{(1)}$ | $0$ | $0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $2n+5$ | $L_{2n-7}\Gamma^4(A,n-2)$ | $A/2^{(1)}\otimes A/2^{(1)}$ | $0$ | $0$ |
| $2n+4$ | $\boldsymbol{L_{2n-8}\Gamma^4(A,n-2)}$ | $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ | $0$ | $0$ |
| $2n+3$ | $L_{2n-9}\Gamma^4(A,n-2)$ | $\boldsymbol{A/2^{(2)}}$ | $0$ | $0$ |
| $2n+2$ | $L_{2n-10}\Gamma^4(A,n-2)$ | $A/2^{(2)}$ | $\boldsymbol{\Gamma^2_{F_2}(A/2^{(1)})\oplus A\otimes A/3^{(1)}}$ | $0$ |
| $2n+1$ | $L_{2n-11}\Gamma^4(A,n-2)$ | $A/2^{(2)}$ | $A/2^{(2)}$ | $\mathbf{0}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n+5$ | $L_{n-7}\Gamma^4(A,n-2)=0$ | $A/2^{(2)}$ | $A/2^{(2)}$ | $0$ |
| $n+4$ | $0$ | $0$ | $A/2^{(2)}$ | $0$ |
| $n+3$ | $0$ | $0$ | $0$ | $0$ |
| $n+2$ | $0$ | $0$ | $0$ | $0$ |
| $n+1$ | $0$ | $0$ | $0$ | $A/2^{(2)}$ |

Table 2: The initial terms $E^1_{p,q}$ of the maximal filtration spectral sequence for $\Gamma^4(A,n)$ with $n>2$.

column follow from (9-6) and those in the $p=-4$ column from the double décalage formula (3-9). The $E^1$ page therefore has the form depicted in Table 2.

The boldface expressions in Table 2 are the terms of total degree $2n$. They will play a special role in the proof. We will indeed prove that all the differentials of the spectral sequence are zero, except some of the differentials with terms of total degree $2n$ as source or target. In order to obtain some information regarding the differentials of the spectral sequence, we are going to use mod 2 reduction, in the spirit of the proof of Propositions 9.2 and 9.4. The universal coefficient theorem yields short exact sequences of strict polynomial functors (where ${}_2G$ denotes the 2–torsion subgroup of an abelian group $G$):

$$0\to L_i\Gamma^4(A,n)\otimes\mathbb{F}_2\to L_i\Gamma^4_{\mathbb{F}_2}(A/2,n)\to {}_2L_{i-1}\Gamma^4(A,n)\to 0, \tag{10-3}$$

$$0\to L_i\Gamma^4(A,n-2)\otimes\mathbb{F}_2\to L_i\Gamma^4_{\mathbb{F}_2}(A/2,n-2)\to {}_2L_{i-1}\Gamma^4(A,n-2)\to 0. \tag{10-4}$$

Moreover, we have already computed $L_i\Gamma^4_{\mathbb{F}_2}(A/2,n)$ in Example 5.2. The following mod 2 analogue of Theorem 10.1 is a straightforward consequence of Example 5.2:

**Lemma 10.3** *There are isomorphisms of strict polynomial functors*

$$L_i\Gamma^4_{\mathbb{F}_2}(A/2,n) \simeq L_{i-8}\Gamma^4_{\mathbb{F}_2}(A/2,n-2) \oplus C_i(A,n),$$

*where*

$$C_i(A,n) \simeq \begin{cases} 0 & \text{if } i > 3n+1,\\ \Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)} & \text{if } i = 3n+1,\\ \Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)} \oplus A/2^{(1)}\otimes A/2^{(1)} & \text{if } i = 3n,\\ A/2^{(1)}\otimes A/2^{(1)} \oplus A/2^{(1)}\otimes A/2^{(1)} & \text{if } 2n+2 < i < 3n,\\ A/2^{(1)}\otimes A/2^{(1)} \oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) & \text{if } i = 2n+2,\\ A/2^{(1)}\otimes A/2^{(1)} \oplus A/2^{(2)} & \text{if } i = 2n+1,\\ \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \oplus A/2^{(2)} & \text{if } i = 2n,\\ A/2^{(2)} \oplus A/2^{(2)} & \text{if } n+2 < i < 2n,\\ A/2^{(2)} & \text{if } n \le i \le n+2. \end{cases}$$

Let $E^r_i$ denote the part of total degree $i$ of the $r^{\text{th}}$ page of the spectral sequence, $E^r_i := \bigoplus_{j+k=i} E^r_{j,k}$, and let $d^r_i\colon E^r_i \to E^r_{i-1}$ denote the total differential. We distinguish three steps in the analysis of the spectral sequence:

- We first analyze the spectral sequence in total degrees $i < 2n$. Formulas (10-5), (10-6) and (10-7) show that Theorem 10.2 holds in degrees $i < 2n$.
- Then we analyze the spectral sequence in total degrees $i > 2n$. Formulas (10-9) and (10-10) show that Theorem 10.2 holds in degrees $i > 2n$.
- Finally, we analyze the spectral sequence in total degree $2n$. Formulas (10-16) and (10-17) show that Theorem 10.2 holds in degree $2n$.

**10.2.1 The spectral sequence in total degree $i < 2n$** For $i = n$ or $n+1$, we have $E^1_i = E^\infty_i$ for lacunary reasons. Since there is only one nontrivial term in total degree $i$ there is no extension issue to recover the abutment. Hence we obtain

$$L_n\Gamma^4(A,n) = A/2^{(2)} = A/2^{(2)} \oplus L_{n-8}\Gamma^4(A,n-2), \tag{10-5}$$

$$L_{n+1}\Gamma^4(A,n) = 0 = L_{n+1-8}\Gamma^4(A,n-2). \tag{10-6}$$

The case $i \ge n+2$ is slightly more involved.

**Proposition 10.4** *For $n+2 \le i \le 2n-1$ we have*

$$L_i\Gamma^4(A,n) \simeq L_i\Gamma^4(A,n-2) \oplus A/2^{(2)}. \tag{10-7}$$

*Moreover, if $n$ is even, the differentials $d^1_{2n}\colon E^1_{2n} \to E^1_{2n-1}$ and $d^2_{2n}\colon E^2_{2n} \to E^2_{2n-1}$ are zero. If $n$ is odd, one of the two differentials $d^1_{2n}$ and $d^2_{2n}$ has $A/2^{(2)}$ as its image and the other one is the zero map.*

**Proof** First of all, by Theorem 10.1 (which we assume to be proved for $n-2$) we know the terms in the column $E^1_{-4,*}$. In particular, all the expressions $E^1_i$ for $i < 2n$ are direct sums of terms $A/2^{(2)}$. A subquotient of a direct sum of copies of $A/2^{(2)}$ is once again a direct sum of copies of $A/2^{(2)}$, so that $E^\infty_i$ is also a direct sum of copies of $A/2^{(2)}$ if $i < 2n$. Finally, the strict polynomial functor $A/2^{(2)}$ has no self-extensions of degree 1, so that

$$E^\infty_i \simeq L_i\Gamma^4(A,n) \quad \text{for } i < 2n.$$

Since only functors of the form $A/2^{(2)}$ appear in total degrees $i < 2n$ in the spectral sequence, there are no functorial issues involved in these degrees and analyzing this part of the spectral sequence amounts to analyzing a spectral sequence of $\mathbb{F}_2$–vector spaces. To be more specific, if $d_i(n)$ denotes the dimension of the $\mathbb{F}_2$–vector space $L_i\Gamma^4(\mathbb{Z},n)$ for $i < 2n$, then the formula (10-7) is equivalent to the following equality for $n+2 \le i \le 2n-1$:

$$d_i(n) = d_i(n-2) + 1. \tag{10-8}$$

We now prove (10-8) by induction on $i$. We have $d_{n+1}(n) = d_{n+1}(n-2) = 0$ by (10-6) and, if we denote by $\delta_i(n)$ the dimension of the $\mathbb{F}_2$–vector space $L_i\Gamma^4_{\mathbb{F}_2}(\mathbb{Z}/2,n)$, the exact sequences (10-3) and (10-4) and Lemma 10.3 yield equalities

$$\begin{aligned} d_{n+2}(n) &= \delta_{n+2}(n),\\ d_{n+2}(n-2) &= \delta_{n+2}(n-2),\\ \delta_{n+2}(n) &= \delta_{n+2}(n-2)+1. \end{aligned}$$

This shows that (10-8) holds for $i = n+2$. Now assume that $n+2 < i < 2n$. Then the exact sequences (10-3) and (10-4) and Lemma 10.3 yield equalities

$$\begin{aligned} d_i(n) &= \delta_i(n) - d_{i-1}(n),\\ d_i(n-2) &= \delta_i(n-2) - d_{i-1}(n-2),\\ \delta_{n+2}(n) &= \delta_{n+2}(n-2)+2. \end{aligned}$$

Thus, assuming that (10-8) holds for $i-1$, we obtain that (10-8) holds for $i$.

It remains to prove the assertion on the differentials of the spectral sequence. By (10-7), for $n+2 \le i < 2n$ $E^1_i$ has one more copy of $A/2^{(2)}$ than $E^\infty_i$. Hence there are only two possibilities for each $i$:

($a_i$) The maps $d_i^1$ and $d_i^2$ are both zero, one of the maps $d_{i+1}^1$ or $d_{i+1}^2$ is zero and the other one has image $A/2^{(2)}$.

($b_i$) One of the maps $d_i^1$ or $d_i^2$ is zero and the other one has image $A/2^{(2)}$, and the maps $d_{i+1}^1$ and $d_{i+1}^2$ are both zero.

We observe that $(a_i) \Longrightarrow (b_{i+1}) \Longrightarrow (a_{i+2})$. Since $E_{n+1}^1 = 0$, $(a_{n+2})$ holds. We can therefore deduce the result by induction on $i$. □

**10.2.2 The spectral sequence in total degree $i > 2n$** If $i > 3n$ then $E_i^1 = E_i^\infty$ for lacunary reasons and, since there is only one nontrivial term in total degree $i$, we have

$$L_i\Gamma(A,n) \simeq L_{i-8}\Gamma(A,n-2). \tag{10-9}$$

To analyze the spectral sequence in total degree $i$ with $2n < i \leq 3n$, we will use mod 2 reduction. So we first recall basic facts regarding mod $p$ reduction. First of all, for any finite abelian group $G$, the $p$–torsion subgroup ${}_pG$ has the same dimension (as an $\mathbb{F}_p$–vector space) as the mod $p$ reduction $G \otimes \mathbb{F}_p$. Using this basic fact, one easily proves the following very rough estimate, which will be useful in order to compare the $E^1$ and the $E^\infty$ pages of the spectral sequence modulo $p$.

**Lemma 10.5** *Let $(G_*, \partial_*)$ be a degreewise finite differential graded abelian group with $\partial_i\colon G_i \to G_{i-1}$. Given a prime $p$, we denote by $({}_pG_*, {}_{(p)}\partial)$ the subcomplex of $p$–torsion elements of $G_*$. Then we have*

$$\dim_{\mathbb{F}_p}(H_i(G)\otimes \mathbb{F}_p) \leq \dim_{\mathbb{F}_p}(G_i \otimes \mathbb{F}_p) - \mathrm{rk}({}_p\partial_i).$$

The next elementary lemma is useful for a modulo $p$ comparison of $L_*\Gamma^4(A,n)$ and the $E^\infty$ page of the spectral sequence.

**Lemma 10.6** *Let $G$ be a filtered finite abelian group. For all primes $p$ we have*

$$\dim_{\mathbb{F}_p}(G\otimes \mathbb{F}_p) \leq \dim_{\mathbb{F}_p}(\mathrm{gr}(G)\otimes \mathbb{F}_p).$$

*Moreover, the equality holds if and only if there exists an isomorphism of groups between the $p$–primary parts, ${}_pG \simeq {}_{(p)}\mathrm{gr}(G)$.*

**Proposition 10.7** *Let $2n < i \leq 3n$. There is an isomorphism*

$$L_i\Gamma^4(A,n) \simeq L_{i-8}\Gamma^4(A,n-2) \oplus \begin{cases} \Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)} & i = 3n, \\ A/2^{(1)}\otimes A/2^{(1)} & 2n+1 < i < 3n, \\ \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) & i = 2n+1. \end{cases} \tag{10-10}$$

*Moreover, the differential $d^1_{-3,2n+4}\colon \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \to L_{2n-8}\Gamma^4(A,n-2)$ is zero.*

**Proof** Lemmas 10.6 and 10.5 yield the inequalities

$$\dim(E^1_i \otimes \mathbb{F}_2) - \mathrm{rk}(d^1_i) \geq \dim(E^\infty_i \otimes \mathbb{F}_2) \geq \dim(L_i\Gamma^4(A,n) \otimes \mathbb{F}_2), \tag{10-11}$$

$$\dim({}_2E^1_{i-1}) - \mathrm{rk}(d^1_{i-1}) \geq \dim({}_2E^\infty_{i-1}) \geq \dim({}_2L_{i-1}\Gamma^4(A,n)). \tag{10-12}$$

We will now verify that the expressions (10-11)–(10-12) are actually equalities for $2n+1 < i \leq 3n$, thereby proving that the total differential $d^1_i$ is zero in degrees $2n < i \leq 3n$. The universal coefficient exact sequence (10-3) yields an equality

$$\dim(L_i\Gamma^4(A,n) \otimes \mathbb{F}_2) + \dim({}_2L_{i-1}\Gamma^4(A,n)) = \dim L_i\Gamma^4_{\mathbb{F}_2}(A/2,n). \tag{10-13}$$

The short exact sequence (10-4) and Lemma 10.3 imply that

$$\dim L_i\Gamma^4_{\mathbb{F}_2}(A/2,n) = \dim(E^1_{-4,i+4} \otimes \mathbb{F}_2) + \dim({}_2E^1_{-4,i+3}) + \dim C_i(A/2,n).$$

We observe that $E^1_{-3,i+3} \oplus E^1_{-3,i+2} \simeq C_i(A/2,n)$ for $2n+1 < i \leq 3n$. It follows that

$$\dim L_i\Gamma^4_{\mathbb{F}_2}(A/2,n) = \dim(E^1_i \otimes \mathbb{F}_2) + \dim({}_2E^1_{i-1}). \tag{10-14}$$

By comparing the sum of the inequalities (10-11) and (10-12) with the equality provided by (10-13) and (10-14), we can now conclude that the expression (10-11) and (10-12) are actually equalities for $2n+1 < i \leq 3n$.

Since the total differential $d^1_i$ is zero in degrees $2n < i \leq 3n$ (and $d^1_{3n+1}$ is zero by lacunarity), we have $d^1_{-3,2n+4} = 0$ and $E^1_i = E^\infty_i$ for $2n < i \leq 3n$. Furthermore, since (10-11) and (10-12) are equalities, Lemma 10.6 yields a non-functorial isomorphism

$$E^\infty_{i-4,4} \oplus E^\infty_{i-3,3} = E^\infty_i \simeq L_i\Gamma^4(A,n). \tag{10-15}$$

To finish the proof, we have to prove a functorial isomorphism $E^\infty_{i-4,4} \oplus E^\infty_{i-4,3} \simeq L_i\Gamma^4(A,n)$. But the 2–primary part of $E^\infty_{i-4,4}$ is a direct sum of functors of the types

$$A/2^{(2)},\quad \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}),\quad A/2^{(1)} \otimes A/2^{(1)},\quad \Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \quad\text{and}\quad \Lambda^2_{\mathbb{F}_2}(A/2^{(1)})$$

(as no term of the form $\Gamma^4(A)$, $\Lambda^4(A)$, $\Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}$ or $\Phi^4(A)$ occurs in the degrees which we are considering here). In addition, $E^\infty_{i-4,3}$ is one of the functors

$$\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}),\quad A/2^{(1)} \otimes A/2^{(1)} \quad\text{or}\quad \Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}.$$

It follows from the $\mathrm{Ext}^1$ computations of Appendix A that there can be no non-split extension of $E^\infty_{i-4,3}$ by $E^\infty_{i-4,4}$, except in the case $E^\infty_{i-4,3} = \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$. In the latter case, the only possible nontrivial extension is an extension of $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ by a functor of the form $(A/2^{(2)})^{\oplus k}$. The middle term of such a nontrivial extension is a functor $\Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \oplus (A/2^{(2)})^{\oplus k-1}$, which has 4–torsion. Such a nontrivial extension is therefore excluded, as this would contradict the isomorphism (10-15). Since all possible extensions are split, we obtain an isomorphism of functors $E_i^\infty \simeq L_i\Gamma^4(A,n)$. This finishes the proof of Proposition 10.7. □

**10.2.3 The spectral sequence in total degree $i = 2n$** The study of the spectral sequence in total degrees $i > 2n$ and $i < 2n$ has already provided us with some partial information regarding the situation for $i = 2n$. Let us sum up what we know so far:

- The differential $d^1_{2n+1}$ is zero, hence $E^\infty_{2n}$ is a subfunctor of $E^1_{2n}$.
- If $n$ is even, then $d^1_{2n}$ and $d^2_{2n}$ are zero, hence $E^\infty_{2n} = E^1_{2n}$.
- If $n$ is odd, then one of the differentials $d^1_{2n}$ and $d^2_{2n}$ is zero and the other has image equal to $A/2^{(2)}$. Hence there are only two possibilities:
  (a) $E^\infty_{2n} = L_{2n-8}\Gamma^4(A,n-2) \oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \oplus A \otimes A/3^{(1)}$.
  (b) $E^\infty_{2n} = L_{2n-8}\Gamma^4(A,n-2) \oplus A/2^{(1)} \oplus \Lambda^2_{\mathbb{F}_2}(A/2^{(1)}) \oplus A \otimes A/3^{(1)}$.

**Proposition 10.8** *If $n$ is even, then*

$$L_{2n}\Gamma^4(A,n) = L_{2n-8}\Gamma^4(A,n-2) \oplus \Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \oplus A \otimes A/3^{(1)}. \tag{10-16}$$

**Proof** We already know that $E^1_{2n} = E^\infty_{2n}$, so we just have to retrieve $L_{2n}\Gamma^4(A,n)$ from $E^\infty_{2n}$. By Theorem 10.1 (which we assume to have been proved for $n-2$), $L_{2n-8}\Gamma^4(A,n-2)$ is a sum of copies of $A/2^{(2)}$. Since $A/2^{(2)}$ has no self-extension of degree one, there is no extension problem between the columns $p=-4$ and $p=-3$. The extension problem on the abutment can therefore be restated as a short exact sequence

$$0 \to L_{2n-8}\Gamma^4(A,n-2) \oplus A/2^{(2)} \to L_{2n}\Gamma^4(A,n) \to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \oplus A \otimes A/3^{(1)} \to 0.$$

But the only nontrivial extension of $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ by a functor of the form $(A/2^{(2)})^{\oplus k}$ is given by a functor of the form $\Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \oplus (A/2^{(2)})^{\oplus k-1}$. Thus we have only two possibilities for $L_{2n}\Gamma^4(A,n)$, namely:

(i) $L_{2n-8}\Gamma^4(A,n-2) \oplus \Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \oplus A \otimes A/3^{(1)}$.

(ii) $L_{2n-8}\Gamma^4(A,n-2) \oplus A/2^{(2)} \oplus \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \oplus A \otimes A/3^{(1)}$.

The difference between (i) and (ii) can be seen in the dimension of the $\mathbb{F}_2$–vector space $L_{2n}\Gamma^4(A,n) \otimes \mathbb{F}_2$. So we can use mod 2 reduction to determine which of the two possibilities is correct. Let $d_i(n)$ and $\delta_i(n)$ be the dimensions of $L_i\Gamma^4(A,n) \otimes \mathbb{F}_2$

and $L_i\Gamma^4_{\mathbb{F}_2}(A/2,n)\otimes\mathbb{F}_2$, respectively. By Proposition 10.4, $L_{2n-1}\Gamma^4(A,n)$ is an $\mathbb{F}_2$–vector space, hence $d_{2n-1}(n)=\dim({}_2L_{2n-1}\Gamma^4(A,n))$. Similarly, $d_{2n-9}(n-2)=\dim({}_2L_{2n-1}\Gamma^4(A,n))$. We have:

$$
\begin{aligned}
&d_{2n}(n)\\
&\quad=\delta_{2n}(n)-d_{2n-1}(n) && \text{by (10-3)}\\
&\quad=\delta_{2n-8}(n-2)+\dim\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})+\dim A/2^{(2)}-d_{2n-1}(n) && \text{by Lemma 10.3}\\
&\quad=\delta_{2n-8}(n-2)+\dim\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})+d_{2n-9}(n-2) && \text{by Proposition 10.4}\\
&\quad=d_{2n-8}(n-2)+\dim\Gamma^2_{\mathbb{F}_2}(A/2) && \text{by (10-4).}
\end{aligned}
$$

Thus the possibility (ii) is excluded for dimension reasons, so that (i) holds. □

**Proposition 10.9** *If $n$ is odd, then*

$$L_{2n}\Gamma^4(A,n)=L_{2n-8}\Gamma^4(A,n-2)\oplus\Lambda^2_{\mathbb{F}_2}(A/2^{(1)})\oplus A/2^{(2)}\oplus A\otimes A/3^{(1)}. \tag{10-17}$$

**Proof** Recall that there are only two possibilities for $E^\infty_{2n}$, namely:

(a) $E^\infty_{2n}=L_{2n-8}\Gamma^4(A,n-2)\oplus\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})\oplus A\otimes A/3^{(1)}$.

(b) $E^\infty_{2n}=L_{2n-8}\Gamma^4(A,n-2)\oplus A/2^{(1)}\oplus\Lambda^2_{\mathbb{F}_2}(A/2^{(1)})\oplus A\otimes A/3^{(1)}$.

We now list the possibilities for $L_{2n}\Gamma^4(A,n)$. By the same reasoning as in Proposition 10.8, one finds the following three possibilities, where $L_{2n-8}\Gamma^4(A,n-2)'$ denotes the expression $L_{2n-8}\Gamma^4(A,n-2)$ with one copy of $A/2^{(2)}$ deleted:

(i) $L_{2n-8}\Gamma^4(A,n-2)'\oplus\Gamma_{\mathbb{Z}}(A/2^{(1)})\oplus A\otimes A/3^{(1)}$.

(ii) $L_{2n-8}\Gamma^4(A,n-2)\oplus\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})\oplus A\otimes A/3^{(1)}$.

(iii) $L_{2n-8}\Gamma^4(A,n-2)\oplus A/2^{(2)}\oplus\Lambda^2_{\mathbb{F}_2}(A/2^{(1)})\oplus A\otimes A/3^{(1)}$.

To be specific, (i) and (ii) correspond to the possible reconstructions of $L_{2n}\Gamma^4(A,n)$ if (a) holds, and (ii) and (iii) correspond to the possible reconstructions if (b) holds.

We can exclude (i) for dimension reasons. Indeed, we can compute the dimension of $L_{2n}\Gamma^4_{\mathbb{F}_2}(A/2,n)\otimes\mathbb{F}_2$ as in the proof of Proposition 10.8. We find that $\dim L_{2n}\Gamma^4(A,n)\otimes\mathbb{F}_2=\dim E^\infty_{2n}\otimes\mathbb{F}_2$. By Lemma 10.6 this implies that $L_{2n}\Gamma^4(A,n)$ is (non-functorially) isomorphic to $E^\infty_{2n}$, hence it is an $\mathbb{F}_2$–vector space. Thus (i) does not hold.

On the other hand, we can exclude (ii) for functoriality reasons. Indeed, the universal coefficient theorem yields a surjective morphism of strict polynomial functors

$$L_{2n+1}\Gamma^4(A/2,n)\twoheadrightarrow {}_2L_{2n}\Gamma^4(A,n).$$

If (ii) holds, then we can compose this surjective morphism with the projection onto the summand $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ of $L_{2n}\Gamma^4(A,n)$ to get a surjective morphism

$$L_{2n+1}\Gamma^4(A/2,n) \twoheadrightarrow \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}). \tag{10-18}$$

But the source of this map was computed in Example 5.2. It is of the form

$$L_{2n+1}\Gamma^4(A/2,n) \simeq (A/2^{(1)} \otimes A/2^{(1)}) \oplus (A/2^{(2)})^{\oplus k}$$

for some value of $k$. By the Hom computations of Appendix A, there is no nonzero map from $A/2^{(2)}$ to $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ and the only nonzero morphism from $A/2^{(1)} \otimes A/2^{(1)}$ to $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ is not surjective. This contradicts the existence of the surjective morphism (10-18). Thus (ii) does not hold. It follows that (iii) holds. □

# 11 A conjectural description of the functors $L_i\Gamma^d_{\mathbb{Z}}(A,n)$

In this section, we return to the study of the derived functors $L_i\Gamma^d(A,n)$ for arbitrary abelian groups $A$. We therefore drop strict polynomial structures and consider the derived functors as genuine functors of the abelian group $A$. The combinatorics of weights, however, will still be present in the picture. We begin with a new description of the stable homology of Eilenberg–Mac Lane spaces, equivalent to Cartan's classical one. We then use this new parametrization of the stable homology groups in order to formulate a conjectural functorial description of the derived functors $L_*\Gamma^d(A,n)$ for all abelian groups $A$ and positive integers $n$ and $d$. We finally show that the computations of the present article as well as some computations of [8] agree with the conjecture.

## 11.1 A new description of the stable homology

We begin this section by quoting Cartan's computation of the stable homology:

$$H^{\mathrm{st}}_i(A) := \lim_n H_{n+i}(K(A,n),\mathbb{Z}) \simeq H_{n+i}(K(A,n),\mathbb{Z}) \quad \text{if } i<n.$$

To this purpose, we first recall Cartan's admissible words for the reader's convenience. Fix a prime number $p$. A $p$–admissible word is a non-empty word $\alpha$ of finite length formed with the letters $\phi_p$, $\gamma_p$ and $\sigma$. These correspond to the homological operations that he refers to as the transpotence, the $p^{\text{th}}$ divided power operation and the suspension, respectively. They must satisfy the following two conditions:

(1) The word $\alpha$ starts with the letter $\sigma$ or $\phi_p$.

(2) The number of letters $\sigma$ on the right of each letter $\gamma_p$ or $\phi_p$ in $\alpha$ is even.

A $p$–admissible word is of first type ("de première espèce") if it ends with a $\sigma$ and of second type ("de deuxième espèce") if it ends with a $\phi_p$ (words finishing on the right with the letter $\gamma_p$ will not be considered here). There are two basic integers associated to a $p$–admissible word $\alpha$:

- The degree of $\alpha$ is the integer $\deg\alpha$, defined recursively as follows. The degree of the empty word is zero, and
$$\deg(\phi_p\beta) = 2 + p\deg\beta, \quad \deg(\sigma\beta) = 1 + \deg\beta, \quad \deg(\gamma_p\beta) = p\deg\beta.$$
- The height of $\alpha$ is the integer $h(\alpha)$ corresponding to the number of letters $\sigma$ and $\phi_p$ in $\alpha$.

**Theorem 11.1** [9, Exposé 11, théorème 2] *There is a graded isomorphism, functorial with respect to the abelian group $A$:*

$$H_*^{\mathrm{st}}(A) \simeq A[0] \oplus \bigoplus_{p \text{ prime}} \Bigg( \bigoplus_{\alpha\in X_1(p)} A/p[\deg\alpha - h(\alpha)] \oplus \bigoplus_{\alpha\in X_2(p)} {}_pA[\deg\alpha - h(\alpha)] \Bigg),$$

*where $X_i(p)$ stands for the set of $p$–admissible words of $i^{th}$ type for $i = 1, 2$, starting on the left with the letters $\sigma\gamma_p$.*

**Remark 11.2** Cartan's proof of Theorem 11.1 is based on the integral computation of [9, Exposé 11, théorème 1]. However, it can also be deduced from the mod $p$ computations of [9, Exposés 9 and 10] (see also Betley [3]) and the universal coefficient theorem if one knows in advance that, for all primes $p$, the stable homology only contains $p$–torsion. A simple proof of the latter fact is contained in Dold and Puppe [11, Korollar 10.2].

We now propose a compact reformulation of Theorem 11.1. Denote by $\mathcal{A}$ the set of sequences of integers $\alpha = (t_1, \dots, t_m)$ (for some $m \geq 1$) such that $t_1 \geq \cdots \geq t_m > 0$. For every $\alpha \in \mathcal{A}$ let $o(\alpha)$ denote the number of distinct strictly positive integers $t_j$ in the sequence $\alpha = (t_1, \dots, t_m)$. For any abelian group $A$, we define an object $\mathcal{S}\mathrm{t}(A)$ of the derived category of abelian groups by the formula

$$\mathcal{S}\mathrm{t}(A) := \bigoplus_{\alpha=(t_1,\dots,t_m)\in\mathcal{A}} \bigoplus_{p \text{ prime}} A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} o(\alpha)}[2(p^{t_1} + \cdots + p^{t_m} - m)]. \tag{11-1}$$

Cartan's description of the integral stable homology $H_*^{\mathrm{st}}(A)$ of an Eilenberg–Mac Lane space $K(A,n)$ may then be rephrased as follows:

**Theorem 11.3** *There exists, functorially in the abelian group $A$, a graded isomorphism*

$$H_*^{\mathrm{st}}(A) \simeq H_*(A \oplus \mathcal{S}\mathrm{t}(A)). \tag{11-2}$$

**Proof** We observe that there is a bijection $\xi\colon X_1(p) \xrightarrow{\sim} X_2(p)$, where $\xi(\alpha)$ is obtained from $\alpha$ by replacing the last two letters $\sigma^2$ of $\alpha$ by the single letter $\phi_p$. This bijection preserves the degree and decreases the height by one, so that the $p$–primary part in the description of $H_*^{\mathrm{st}}(A)$ Theorem 11.1 can be rewritten as

$$\bigoplus_{\alpha\in X_1(p)} \big(A/p[\deg\alpha - h(\alpha)] \oplus {}_pA[\deg\alpha - h(\alpha)+1]\big). \tag{11-3}$$

We define an equivalence relation $\mathcal{R}$ on $X_1(p)$ as follows. For a word $\alpha \in X_1(p)$, its $\sigma^2\gamma_p$*–substitution* is the word of $X_1(p)$ obtained by replacing each occurrence of the letter $\phi_p$ by the group of letters $\sigma^2\gamma_p$. For example, the $\sigma^2\gamma_p$–substitution of $\sigma\gamma_p\phi_p\sigma^4\phi_p\sigma^2$ is the word $\sigma\gamma_p\sigma^2\gamma_p\sigma^6\gamma_p\sigma^2$. We say that two words of $X_1(p)$ are equivalent if they have the same $\sigma^2\gamma_p$–substitution. Let $\alpha$ be a $p$–admissible word beginning with the letter $\sigma\gamma_p$. We say that a word $\alpha$ is *restricted* if it begins on the left with $\sigma\gamma_p$ and is composed only of the letters $\sigma$ and $\gamma_p$ (ie no $\phi_p$ occurs in the word $\alpha$). We define $R(p)$ to be the subset of $X_1(p)$ consisting of those $p$–admissible words in $X_1(p)$ which are restricted. Each equivalence class in $X_1(p)$ contains exactly one restricted admissible word, so that we can rewrite the direct sum (11-3) as

$$\bigoplus_{\alpha\in R(p)}\ \bigoplus_{\beta\mathcal{R}\alpha} (A/p[\deg\beta - h(\beta)] \oplus {}_pA[\deg\beta - h(\beta)+1]). \tag{11-4}$$

We can replace the indexing set $R(p)$ in (11-4) by the set $\mathcal{A}$. Indeed, there is a bijection $\chi\colon R(p) \xrightarrow{\sim} \mathcal{A}$ defined as follows. Each restricted $p$–admissible word has the form

$$\sigma\gamma_p \underbrace{(\sigma^2)\cdots(\sigma^2)}_{k_1 \text{ terms}} \gamma_p \underbrace{(\sigma^2)\cdots(\sigma^2)}_{k_2 \text{ terms}} \gamma_p \cdots \gamma_p \underbrace{(\sigma^2)\cdots(\sigma^2)}_{k_s \text{ terms}}, \tag{11-5}$$

where $s$ is the number of occurrences of $\gamma_p$ and the $k_i$ are nonnegative. Our bijection $\chi$ is defined by sending such a restricted $p$–admissible word to the sequence of integers

$$(\underbrace{s,\dots,s}_{k_s \text{ terms}}, \underbrace{s-1,\dots,s-1}_{k_{s-1} \text{ terms}}, \dots, \underbrace{1,\dots,1}_{k_1 \text{ terms}}). \tag{11-6}$$

If we define the degree of a sequence $(t_1,\dots,t_n) \in \mathcal{A}$ as the sum $2(p^{t_1}+\cdots+p^{t_n})$, and its height as $2n$, then the bijection $\chi$ preserves the degree and the height.

Let us now describe the $\mathcal{R}$–equivalence classes in $X_1(p)$. Let $C$ be one such equivalence class, containing a restricted admissible word $\alpha$ of the form (11-5). The elements in $C$ are all the words which can be obtained from $\alpha$ by substituting, for some groups of terms $\sigma^2\gamma_p$, the letter $\phi_p$. Observe that the number of groups of terms $\sigma^2\gamma_p$ available for such a substitution is equal to the number of positive $k_i$, and this is $o(\alpha)-1$ (where $o(\alpha)$ is the number of distinct positive integers in the associated sequence (11-6)). Thus

$C$ contains $2^{o(\alpha)-1}$ elements. Moreover, there are $\binom{o(\alpha)-1}{i}$ distinct words $\beta$ obtained from $\alpha$ by exactly $i$ substitutions and they all satisfy the conditions $\deg\beta = \deg\alpha$ and $h(\beta) = h(\alpha) - i$. As a consequence, (11-4) can be rewritten as a direct sum

$$\bigoplus_{\alpha\in\mathcal{A}} \bigoplus_{i=0}^{o(\alpha)-1} \big(A/p[\deg\alpha - h(\alpha) - i] \oplus {}_pA[\deg\alpha - h(\alpha) - i + 1]\big)^{\oplus\binom{o(\alpha)-1}{i}}. \tag{11-7}$$

Finally, the homology of the complex

$$A \overset{L}{\otimes} \mathbb{Z}/p$$

is the graded abelian group ${}_pA[1] \oplus A/p[0]$ and the object

$$\mathbb{Z}/p \overset{L}{\otimes} \mathbb{Z}/p$$

is isomorphic to $\mathbb{Z}/p[0]\oplus\mathbb{Z}/p[1]$ in the derived category. This determines by induction on $n$ a functorial isomorphism of graded abelian groups:

$$H_*(A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} n}) \simeq \bigoplus_{i=0}^{n-1} ({}_pA[1+i] \oplus A/p[i])^{\oplus\binom{n-1}{i}}. \tag{11-8}$$

The statement of Theorem 11.3 follows by combining (11-7) and (11-8). □

Just as one speaks of stable homology, one defines, following [11, Section 8.3], the stable derived functors

$$L_i^{\mathrm{st}} F(A) = \lim_n L_{n+i} F(A,n) \simeq L_{n+i} F(A,n) \quad \text{if } i < n.$$

As explained in Appendix B, there exists a graded isomorphism, natural with respect to the abelian group $A$,

$$H_*^{\mathrm{st}}(A) \simeq \bigoplus_{d\geq 0} L_*^{\mathrm{st}} S^d(A). \tag{11-9}$$

By décalage the (stable) derived functors of symmetric powers are isomorphic to the (stable) derived functors of the divided power functors. Hence, (11-9) can be used to obtain a description of $L_*^{\mathrm{st}}\Gamma^d(A)$. However, to obtain such a description, we have to separate the various summands. In other words, we have to determine which summands ${}_pA$ and $A/p$ contribute to which stable derived functors $L_*^{\mathrm{st}}\Gamma^d(A)$. To this purpose, we define yet another integer associated to $p$–admissible words:

Let $r(\alpha)$ be the number of occurrences of the letters $\phi_p$ and $\gamma_p$ in a $p$–admissible word $\alpha$. Then the weight $w(\alpha)$ is defined by $w(\alpha) = p^{r(\alpha)}$ if $\alpha$ is of first type, and $w(\alpha) = p^{r(\alpha)-1}$ if $\alpha$ is of second type.

As explained for example in Touzé [29, Section 10.2], the direct summand of $H^{\mathrm{st}}_*(A)$ corresponding to $L^{\mathrm{st}}_* S^d(A)$ is given by the admissible words of weight $d$. We have to translate this into the new indexation provided by Theorem 11.3. We obtain the following description of the stable derived functors of the functor $\Gamma^d(A)$:

**Proposition 11.4** *If the integer $d$ is not a prime power, then*

$$L^{\mathrm{st}}_i \Gamma^d(A) = 0 \quad \text{for all } i \geq 0.$$

*For any prime number $p$ and any integer $r > 0$,*

$$L^{\mathrm{st}}_i \Gamma^{p^r}(A) = H_i\Bigg( \bigoplus_{\alpha=(t_1,\dots,t_m)\in\mathcal{A},\, t_1=r} A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} o(\alpha)}[2(p^{t_2}+\cdots+p^{t_m}-m)]\Bigg).$$

**Proof** Returning to the proof of Theorem 11.3, we see that the bijection $\xi$ preserves the weights. Moreover, all the words in the same equivalence class in $X_1(p)$ also have the same weight. Thus all the terms corresponding to the homology of the summand

$$A \overset{L}{\otimes} \mathbb{Z}/p^{\otimes o(\alpha)}$$

have the same weight as $\alpha$. Now, if $\alpha$ is a restricted $p$–admissible word corresponding to the sequence $\chi(\alpha) = (t_1, \dots, t_n) \in \mathcal{A}$ then $w(\alpha) = p^{t_1}$. Proposition 11.4 follows. □

## 11.2 The conjecture

We observed in Proposition 11.4 that the stable derived functors of $\Gamma^d(A)$ may conveniently be repackaged by appropriately deriving those summands $A/p$ which most obviously occur in Cartan's computation, these being the summands which correspond to *restricted* admissible words, that is, the words which do not involve the transpotence operation $\phi_p$. Our contention in what follows is that a similar mechanism also works unstably. To be more specific, let $\mathcal{A}_{\leq m} \subset \mathcal{A}$ denote the subset containing all the sequences of length less than or equal to $m$. We can reformulate the definition of $\mathcal{A}_{\leq m}$ as

(11-10) $$\mathcal{A}_{\leq m} = \{(t_1, \dots, t_m) : t_1 \geq t_2 \geq \cdots \geq t_m \geq 0 \text{ and } t_1 \neq 0\}.$$

For every $\alpha \in \mathcal{A}_{\leq m}$ we still denote by $o(\alpha)$ the number of distinct strictly positive integers $t_j$ in the sequence $\alpha = (t_1, \dots, t_m)$. The $\mathcal{A}_{\leq m}$ form an increasing family of subsets which exhaust $\mathcal{A}$. Moreover, $\mathcal{A}_{\leq m}$ can be interpreted as an indexing set for those stable summands $A/p$ corresponding to restricted admissible words which already appear in the unstable homology of $K(A, 2m+1)$ or, equivalently, in the derived functors $L_*\Gamma(A, 2m-1)$.

Our conjecture asserts that the derived functors $L_*\Gamma(A,n)$ are filtered and that the associated graded pieces have the following description. If $n = 2m+1$ there is an isomorphism (functorial with respect to an arbitrary abelian group $A$)

$$\operatorname{gr}(L_*\Gamma(A,n)) \simeq \pi_*\Big(L\Lambda(A) \otimes \bigotimes_{p \text{ prime}} \bigotimes_{\alpha\in\mathcal{A}_{\leq m+1}} L\Lambda(A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} o(\alpha)})\Big) \tag{11-11}$$

and, similarly, if $n = 2m$ there is a functorial isomorphism

$$\operatorname{gr}(L_*\Gamma(A,n)) \simeq \pi_*\Big(L\Gamma(A) \otimes \bigotimes_{p \text{ prime}} \bigotimes_{\alpha\in\mathcal{A}_{\leq m}} L\Gamma(A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} o(\alpha)})\Big). \tag{11-12}$$

The two isomorphisms above do not preserve the homological grading. We will explain below how to introduce appropriate shifts of degrees on the right-hand side so as to obtain graded isomorphisms. For the moment, we keep things simple by discussing the ungraded version of the conjecture.

Both sides of (11-11) and (11-12) are equipped with weights and our conjectural isomorphisms preserve the weights. To be more specific, the weight is defined on the left-hand sides of (11-11) and (11-12) by viewing $\operatorname{gr}(L_*\Gamma^d(A,n))$ as the homogeneous summand of weight $d$. The weights on the right-hand sides of (11-11) and (11-12) are defined as follows. We view

$$L_*\Lambda^d(A) \quad \text{and} \quad L_*\Gamma^d(A)$$

as functors of weight $d$ and the expressions

$$L\Lambda^d(A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} o(\alpha)}) \quad \text{and} \quad L\Gamma^d(A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} o(\alpha)})$$

corresponding to a sequence $\alpha = (t_1, \dots)$ as functors of weight $p^{t_1}$ (as we did in Proposition 11.4). Finally, weights are additive with respect to tensor products.

To describe more concretely the homogeneous component of weight $d$ of the right-hand sides of isomorphisms (11-11) and (11-12), we introduce the following notation. Given a prime integer $p$, a nonnegative integer $m$, a nonnegative integer $d_0$ and a family of nonnegative integers $(d_\alpha)_{\alpha\in\mathcal{A}_{\leq m}}$ which is not identically zero, we denote by $d(d_0,(d_\alpha),m;p)$ the positive integer defined by

$$d(d_0,(d_\alpha),m;p) := d_0 + \sum_{\alpha\in\mathcal{A}_{\leq m}} d_\alpha p^{t_1(\alpha)}, \tag{11-13}$$

where $t_1(\alpha)$ denotes the first integer in the sequence $\alpha$, that is, $\alpha = (t_1(\alpha), \dots)$. The integer $d(d_0, (d_\alpha), m; p)$ is the weight of the following objects of the derived category:

$$\mathcal{E}(d_0, (d_\alpha), m; p) := L\Lambda^{d_0}(A) \otimes \bigotimes_{\alpha \in \mathcal{A}_{\leq m}} L\Lambda^{d_\alpha}(A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} o(\alpha)}), \tag{11-14}$$

$$\mathcal{D}(d_0, (d_\alpha), m; p) := L\Gamma^{d_0}(A) \otimes \bigotimes_{\alpha \in \mathcal{A}_{\leq m}} L\Gamma^{d_\alpha}(A \overset{L}{\otimes} \mathbb{Z}/p^{\overset{L}{\otimes} o(\alpha)}). \tag{11-15}$$

With these notations, the homogeneous component of weight $d$ of the right-hand side of the isomorphism (11-11) is given by the homotopy groups of the direct sum of $L\Lambda^d(A)$ and all the terms $\mathcal{E}(d_0, (d_\alpha), m+1; p)$ such that $d(d_0, (d_\alpha), m+1; p) = d$ for all nonnegative integers $d_0$, all families of integers $(d_\alpha)$ which are not identically zero and all prime integers $p$. The right-hand side of the isomorphism (11-12) has an obvious similar description.

Finally, we introduce suitable shifts in order to transform the isomorphisms (11-11) and (11-12) into graded isomorphisms. Given a prime integer $p$, nonnegative integers $m$ and $d_0$, and a family of nonnegative integers $(d_\alpha)_{\alpha \in \mathcal{A}_{\leq m}}$ which is not identically zero, we set

$$\ell(d_0, (d_\alpha), m; p) := (2m+1)d_0 + \sum_{\alpha \in \mathcal{A}_{\leq m}} \ell(\alpha; p) d_\alpha, \tag{11-16}$$

where the integer $\ell(\alpha; p)$ associated to a sequence $\alpha = (t_1, \dots, t_m)$ is given by

$$\ell(\alpha; p) := \begin{cases} 2p^{t_2} + \cdots + 2p^{t_m} + 1 & \text{if } m > 1, \\ 1 & \text{if } m = 1. \end{cases} \tag{11-17}$$

We also set

$$e(d_0, (d_\alpha), m; p) := \Big( \sum_{\alpha \in \mathcal{A}_{\leq m}} d_\alpha \Big) - d_0. \tag{11-18}$$

We may now state the graded version of our conjecture:

**Conjecture 11.5** *Let $s$ and $n$ be positive integers.*

(1) *Assume that $n = 2m+1$. Then there exists a filtration on $L_s\Gamma^d(A, n)$ such that the associated graded functor is isomorphic to the direct sum of the term*

$$L_{s-nd}\Lambda^d(A)$$

*together with the terms*

$$\pi_s\big(\mathcal{E}(d_0, (d_\alpha), m+1; p)[\ell(d_0, (d_\alpha), m+1; p)]\big)$$

*for all primes $p$, all nonnegative integers $d_0$ and all families of integers $(d_\alpha)_{\alpha\in\mathcal{A}_{\leq m+1}}$ which are not identically zero that satisfy $d(d_0,(d_\alpha),m;p)=d$.*

(2) *Similarly, if $n=2m$, there exists a filtration on $L_s\Gamma^d(A,n)$ such that the associated graded functor is isomorphic to the direct sum of the term*

$$L_{s-nd}\Gamma^d(A)$$

*together with the terms*

$$\pi_s\big(\mathcal{D}(d_0,(d_\alpha),m;p)[\ell(d_0,(d_\alpha),m;p)+e(d_0,(d_\alpha),m;p)]\big)$$

*for all primes $p$, all nonnegative integers $d_0$ and all families of integers $(d_\alpha)_{\alpha\in\mathcal{A}_{\leq m}}$ which are not identically zero that satisfy $d(d_0,(d_\alpha),m;p)=d$.*

The remainder of the present section is devoted to proving that the conjecture holds in a certain number of cases.

## 11.3 The cases $d=2$ and $d=3$ for all $A$ and all $n$

For $s,\ m\geq 1$, and $d=2,\ 3$, there is no filtration to consider and Conjecture 11.5 reduces to the natural isomorphisms

$$\begin{aligned}(11\text{-}19)\quad & L_s\Gamma^2(A,2m+1)\\ &\simeq \pi_s\Big(\bigoplus_{i=0}^{m}(A\overset{L}{\otimes}\mathbb{Z}/2[2m+2i+1])\oplus L\Lambda^2(A)[4m+2]\Big),\end{aligned}$$

$$(11\text{-}20)\qquad L_s\Gamma^2(A,2m)\simeq \pi_s\Big(\bigoplus_{i=0}^{m-1}(A\overset{L}{\otimes}\mathbb{Z}/2[2m+2i])\oplus L\Gamma^2(A)[4m]\Big),$$

$$\begin{aligned}(11\text{-}21)\quad L_s\Gamma^3(A,2m+1)\simeq \pi_s\Big(&\bigoplus_{i=0}^{m}\big(A\overset{L}{\otimes}\mathbb{Z}/3[2m+4i+1]\\ &\oplus A\overset{L}{\otimes}A\overset{L}{\otimes}\mathbb{Z}/2[4m+2i+2]\big)\oplus L\Lambda^3(A)[6m+3]\Big),\end{aligned}$$

$$\begin{aligned}(11\text{-}22)\quad L_s\Gamma^3(A,2m)\simeq \pi_s\Big(&\bigoplus_{i=0}^{m-1}\big(A\overset{L}{\otimes}\mathbb{Z}/3[2m+4i]\\ &\oplus A\overset{L}{\otimes}A\overset{L}{\otimes}\mathbb{Z}/2[4m+2i]\big)\oplus L\Gamma^3(A)[6m]\Big),\end{aligned}$$

consistently with the results in [8, Sections 4 and 5].

## 11.4 The case $d = 4$ for $A$ free and $n$ odd

We now proceed to prove that the conjecture agrees with our previous computations for $d = 4$, $A$ free and $n = 2m+1$ with $m \geq 0$. In that case the conjecture says that (up to a filtration) $L_*\Gamma^4(A,n)$ is isomorphic to the homology groups of the following complexes (i)–(vii). At the prime $p = 3$, there is a single sort of complex:

(i) $$A \overset{L}{\otimes} A \overset{L}{\otimes} \mathbb{Z}/3\,[4m+4k+2], \quad 0 \leq k \leq m.$$

At the prime $p = 2$, we have the five types of complexes

(ii) $$L\Lambda^2(A) \overset{L}{\otimes} A \overset{L}{\otimes} \mathbb{Z}/2\,[6m+2k+3], \quad 0 \leq k \leq m,$$

(iii) $$L\Lambda^2(A \overset{L}{\otimes} \mathbb{Z}/2)[4m+4k+2], \quad 0 \leq k \leq m,$$

(iv) $$A \overset{L}{\otimes} \mathbb{Z}/2 \overset{L}{\otimes} \mathbb{Z}/2\,[2m+6k+2l+1], \quad 0 \leq k+l \leq m,\ l \neq 0,$$

(v) $$A \overset{L}{\otimes} \mathbb{Z}/2\,[2m+6k+1], \quad 0 \leq k \leq m,$$

(vi) $$A \overset{L}{\otimes} A \overset{L}{\otimes} \mathbb{Z}/2 \overset{L}{\otimes} \mathbb{Z}/2\,[4m+2k+2l+2], \quad 0 \leq l < k \leq m,$$

together with the final term

(vii) $$L\Lambda^4(A)[8m+4].$$

To verify that these expressions coincide with our computations from Section 10, we proceed by induction on $m$. For $m = 0$, the formula for $L_*\Gamma^4(A,1)$ was described in Proposition 9.1. We filter the term $L_3\Gamma^4(A,1) = \Phi^4(A)$ in (9-4) and replace it here by the direct sum $\Lambda^2(A) \otimes A/2) \oplus \Gamma^2_{\mathbb{F}_2}(A/2)$. The result of Proposition 9.1 then coincides with the present $m = 0$ case, once we observe that $L_*\Lambda^2(A/2) \simeq \Lambda^2(A/2)[0] \oplus \Gamma^2_{\mathbb{F}_2}(A/2)[1]$ [8, Section 2.2; 2].

To prove that the formulas provided by the conjecture agree for a general $n = 2m+1$ with the computations of Theorem 10.1, it suffices to show that the additional summands predicted by the conjecture when passing from $L_*\Gamma^4(A,n-2)[8]$ to $L_*\Gamma^4(A,n)$ agree with those obtained in (10-1). Let us denote by $G(m)$ the sum of all the terms (i)–(vii). One verifies that $G(m) = G(m-1)[8] \oplus \Delta(m)$, where $\Delta(m)$ is given by

(11-23) $$\begin{aligned}\Delta(m) &= A \otimes A/3[2n] \oplus L\Lambda^2(A) \otimes A/2[3n] \oplus L\Lambda^2(A/2)[2n] \\ &\quad \oplus \Big(\bigoplus_{\ell=1}^{m} A \overset{L}{\otimes} \mathbb{Z}/2^{\overset{L}{\otimes}2}\Big) \oplus A/2[n] \oplus \Big(\bigoplus_{k=1}^{m} A \overset{L}{\otimes} A \overset{L}{\otimes} \mathbb{Z}/2^{\overset{L}{\otimes}2}[2n+2k]\Big).\end{aligned}$$

In $\Delta(m)$ we replace the expression $\mathbb{Z}/2^{\overset{L}{\otimes}2}$ by $\mathbb{Z}/2[0] \oplus \mathbb{Z}/2[1]$ and $L\Lambda^2(A/2)$ by $\Lambda^2(A/2)[0] \oplus \Gamma^2_{\mathbb{F}_2}(A/2)[1]$. Then $\Delta(m)$ coincides with the additional summands occurring when one passes, for $n$ odd, from $L_*\Gamma^4(A,n-2)[8]$ to $L_*\Gamma^4(A,n-2)$ in Theorem 10.2, provided we replace the summand $\Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}[3n]$ in (10-1) by the direct sum $\Lambda^2(A/2) \otimes A/2[3n] \oplus A/2 \otimes A/2[3n]$. This proves the conjecture for $d=4$, $A$ free and $n=2m+1$ odd.

## 11.5 The case $d=4$ for $A$ free and $n$ even

We now proceed to prove that the conjecture agrees with our previous computations for $d=4$, $A$ free and $n=2m$ with $m \geq 1$. It is straightforward to verify that the conjecture agrees for $m=1$ with the computation of Proposition 9.3, since we know that $L_*\Gamma^2(A/2) \simeq \Gamma^2_{\mathbb{Z}}(A/2)[0] \oplus \Gamma^2_{\mathbb{F}_2}(A/2)[1]$ [8; 2, Section 4] (there is no filtration to consider in this situation). The conjecture says that (up to a filtration) $L_*\Gamma^4(A,n)$ is isomorphic to the homology groups of the following complexes (a)–(g). At the prime $p=3$, there is a single sort of complex:

(a) $$A \overset{L}{\otimes} A \overset{L}{\otimes} \mathbb{Z}/3[4m+4k] \quad 0 \leq k \leq m-1.$$

For the prime $p=2$, we have the complexes

(b) $$L\Gamma^2(A) \overset{L}{\otimes} A \overset{L}{\otimes} \mathbb{Z}/2[6m+2k] \quad 0 \leq k \leq m-1,$$

(c) $$L\Gamma^2(A \overset{L}{\otimes} \mathbb{Z}/2)[4m+4k] \quad 0 \leq k \leq m-1,$$

(d) $$A \overset{L}{\otimes} \mathbb{Z}/2 \overset{L}{\otimes} \mathbb{Z}/2[2m+6k+2l] \quad 0 \leq k+l \leq m-1,\ l \neq 0,$$

(e) $$A \overset{L}{\otimes} \mathbb{Z}/2[2m+6k] \quad 0 \leq k \leq m-1,$$

(f) $$A \overset{L}{\otimes} A \overset{L}{\otimes} \mathbb{Z}/2 \overset{L}{\otimes} \mathbb{Z}/2[4m+2k+2l] \quad 0 \leq l < k \leq m-1,$$

together with the final term

(g) $$L\Gamma^4(A)[8m].$$

To prove that the formulas provided by the conjecture agree for $n=u2m$ with the computations of Theorem 10.1, it suffices to show, as above, that the additional summands predicted by the conjecture when passing from $L_*\Gamma^4(A,n-2)[8]$ to $L_*\Gamma^4(A,n)$ agree with those obtained in (10-2). Let us denote by $J(m)$ the sum of all the terms (a)–(g).

One verifies that $J(m) = J(m-1)[8] \oplus \Delta'(m)$, where

$$\begin{aligned}(11\text{-}24)\quad \Delta'(m) = A \otimes A/3[2n] \oplus \Gamma^2 A \otimes A/2[3n] \oplus L\Gamma^2(A/2)[2n] \\ \oplus \Big( \bigoplus_{l=1}^{m-1} A \overset{L}{\otimes} \mathbb{Z}/2 \overset{L}{\otimes} \mathbb{Z}/2[2n+2l] \Big) \oplus A/2 \\ \oplus \Big( \bigoplus_{k=1}^{m-1} A/2 \overset{L}{\otimes} A/2[4m+2k] \Big).\end{aligned}$$

We replace in $\Delta'(m)$ the summand $L\Gamma^2(A/2)$ by its value $\Gamma^2_{\mathbb{Z}}(A/2)[0] \oplus \Gamma^2_{\mathbb{F}_2}(A/2)[1]$ (as in [8, Section 2.2; 2, Section 4)] and, once more, replace

$$\mathbb{Z}/2 \overset{L}{\otimes} \mathbb{Z}/2$$

by $\mathbb{Z}/2[0] \oplus \mathbb{Z}/2[1]$. Then $\Delta'(m)$ coincides with the additional summands occurring when one passes, for $n$ even, from $L_*\Gamma^4(A, n-2)[8]$ to $L_*\Gamma^4(A, n-2)$ in Theorem 10.2. Note that in this $n$ even case, no summand in $\Delta'(m)$ requires any filtering.

## 11.6 The case $n = 1$ for $A$ free and all $d$

In the case $n = 1$ and $A$ free, the conjecture asserts that, up to a filtration, the $p$–primary part of $L_*\Gamma^d(A, 1)$ is isomorphic to

$$(11\text{-}25)\quad \pi_*\Big( \bigoplus_{(k_0,\dots,k_d)} \Lambda^{k_0}(A) \overset{L}{\otimes} L\Lambda^{k_1}(A/p) \overset{L}{\otimes} \cdots \overset{L}{\otimes} L\Lambda^{k_d}(A/p)[k_0 + \cdots + k_d] \Big),$$

where the sum runs over all sequences of nonnegative integers $(k_0, k_1, \dots, k_d)$ of length exactly $d+1$ satisfying $\sum k_i p^i = d$.

On the other hand, we have explicitly computed the derived functors $L_*\Gamma(A, 1)$ in Section 6. By Theorem 6.3 and Proposition 6.18, the $p$–primary part of $L_i\Gamma^d(A, 1)$ is concentrated in degrees $i < d$ and it is isomorphic (up to a filtration) in these degrees to the homogeneous component of degree $i$ and weight $d$ of the cycles of the tensor product of an exterior algebra with trivial differential and a family of Koszul algebras:

$$(11\text{-}26)\qquad \big(\Lambda_{\mathbb{F}_p}(A/p[1]), 0\big) \otimes \bigotimes_{r \geq 1} \big(\Lambda_{\mathbb{F}_p}(A/p^{(r)}[1]) \otimes \Gamma_{\mathbb{F}_p}(A/p^{(r)}[2]), d_{\mathrm{Kos}}\big).$$

We will now reformulate this result in a form closer to (11-25). For this, we consider the following modification of the Koszul algebra over $\mathbb{Z}$, namely the dg–$\mathcal{P}_{\mathbb{Z}}$–algebra $(\Gamma(A[2]) \otimes \Lambda(A[1]), p d_{\mathrm{Kos}})$ with the same underlying graded $\mathcal{P}_{\mathbb{Z}}$–algebra, but whose differential is the Koszul differential multiplied by $p$. We denote by $C^k(A)$ the complex of functors given by its homogeneous component of weight $k$.

**Lemma 11.6** *For $i < d$, the $p$–primary part of the functor $L_i\Gamma^d(A,1)$ is isomorphic to the homology of the complex*

$$\bigoplus_{(k_0,\dots,k_d)} (\Lambda^{k_0}(A)[k_0], 0) \otimes C^{k_1}(A) \otimes \cdots \otimes C^{k_d}(A),$$

*where the sum runs over all sequences of nonnegative integers $(k_0,\dots,k_d)$ satisfying $\sum k_i p^i = d$.*

**Remark 11.7** The isomorphism of Lemma 11.6 is really an isomorphism of functors, not an isomorphism of strict polynomial functors. For example, this isomorphism does not preserve the weights.

**Proof of Lemma 11.6** We break the proof into two steps:

**Step 1** Since we are interested in the homogeneous component of weight $d$ of (11-26), we can limit ourselves to the differential graded subalgebra of (11-26):

$$\big(\Lambda_{\mathbb{F}_p}(A/p[1]), 0\big) \otimes \bigotimes_{1\le r\le d} \big(\Lambda_{\mathbb{F}_p}(A/p^{(r)}[1]) \otimes \Gamma_{\mathbb{F}_p}(A/p^{(r)}[2]), d_{\mathrm{Kos}}\big). \tag{11-27}$$

By forgetting the strict polynomial structure, the strict polynomial functors $A/p^{(r)}$ become isomorphic to $A/p$ and the differential graded algebra (11-27) is functorially isomorphic to the differential graded algebra

$$\big(\Lambda_{\mathbb{F}_p}(A/p[1]), 0\big) \otimes \bigotimes_{1\le r\le d} \big(\Lambda_{\mathbb{F}_p}(A/p[1]) \otimes \Gamma_{\mathbb{F}_p}(A/p[2]), d_{\mathrm{Kos}}\big). \tag{11-28}$$

Moreover, under this isomorphism, the homogeneous summand of weight $d$ of (11-27) corresponds to the homogeneous summand of (11-28) supported by the subfunctors

$$\bigoplus \Lambda^{k_0}_{\mathbb{F}_p}(A/p[1]) \otimes \Lambda^{a_1}_{\mathbb{F}_p}(A/p) \otimes \Gamma^{b_1}_{\mathbb{F}_p}(A/p) \otimes \cdots \otimes \Lambda^{a_\ell}_{\mathbb{F}_p}(A/p) \otimes \Gamma^{b_\ell}_{\mathbb{F}_p}(A/p), \tag{11-29}$$

summing over all sequences $(k_0, a_1, b_1, \dots, a_\ell, b_\ell)$ that satisfy $k_0 + \sum (a_i + b_i) p^i = d$.

**Step 2** We claim that the subalgebra of cycles of positive degree of the functorial graded algebra (11-28) is isomorphic to the homology algebra of

$$(\Lambda_{\mathbb{Z}}(A[1]), 0) \otimes \bigotimes_{1\le r\le d} \big(\Lambda_{\mathbb{Z}}(A[1]) \otimes \Gamma_{\mathbb{Z}}(A[2]), p d_{\mathrm{Kos}}\big) \tag{11-30}$$

and we claim that the isomorphism sends the kernels of the summand (11-29) of the differential graded algebra (11-28) isomorphically to the homology of the summand of the differential graded algebra (11-30)

$$\bigoplus_{(k_0,\dots,k_d)} (\Lambda^{k_0}(A[1]),0)\otimes C^{k_1}(A)\otimes\cdots\otimes C^{k_d}(A),$$

where the sum runs over all sequences of nonnegative integers $(k_0,\dots,k_d)$ satisfying $\sum k_i p^i = d$. The statement of Lemma 11.6 follows from this claim, so to finish the proof of Lemma 11.6 we only have to justify our claim.

Koszul algebras are acyclic in positive degrees by Proposition 6.6. Hence, the positive-degree homology of the differential graded algebra (11-30) is equal to the mod $p$ reduction of the algebra formed by the cycles of positive degree of

$$(\Lambda_{\mathbb{Z}}(A[1]),0)\otimes \bigotimes_{1\le r\le d} (\Lambda_{\mathbb{Z}}(A[1])\otimes\Gamma_{\mathbb{Z}}(A[2]), d_{\mathrm{Kos}}).$$

The latter is isomorphic to the algebra formed by the cycles of positive degree of the algebra (11-28). This justifies our claim. □

It follows from the next lemma that the graded functor (11-25) can also be rewritten as the homology of the complex of Lemma 11.6. Recall that $C^k(A)$ is the homogeneous part of weight $k$ of $(\Gamma(A[2])\otimes\Lambda(A[1]), pd_{\mathrm{Kos}})$, hence its desuspension $C^k(A)[-k]$ is the homogeneous part of weight $k$ of $(\Gamma(A[1])\otimes\Lambda(A[0]), pd_{\mathrm{Kos}})$.

**Lemma 11.8** *Let $A$ be a free abelian group, and let $k$ be a positive integer. The normalized chains of $L\Lambda^k(A/p)$ are naturally isomorphic to the complex $C^k(A)[-k]$.*

**Proof** We have $L\Lambda(A/p) = \Lambda(K(A \xrightarrow{\times p} A))$. The functor $K$ is explicit and we readily check that, for all complexes $C_1 \xrightarrow{\partial} C_0$, the normalized chains of the simplicial object $\Lambda(K(C_1 \xrightarrow{\partial} C_0))$ is the complex whose degree-$n$ component is

$$\Lambda^{>0}(C_1)^{\otimes n}\otimes\Lambda(C_0),$$

where $\Lambda^{>0}(C_i)$ stands for the augmentation ideal of the exterior algebra and whose differential maps an element $x_1\otimes\cdots\otimes x_n\otimes y$ to the sum

$$\sum_{i=1}^{n-1}(-1)^i x_1\otimes\cdots\otimes x_ix_{i+1}\otimes\cdots\otimes x_n\otimes y + (-1)^n x_1\otimes\cdots\otimes x_{n-1}\otimes\Lambda(\partial)(x_n)y.$$

Consider the normalized chains $\mathcal{N}\Lambda(A/p)$ of $L\Lambda(A/p)$ as a differential graded $\mathcal{P}_{\mathbb{Z}}$–algebra. Its homogeneous component of weight $k$ yields the normalized chains

$\mathcal{N}\Lambda^k(A/p)$ of $L\Lambda^k(A/p)$. These normalized chains contain $C^k(A)[-k]$ as a subcomplex. Indeed the inclusion is given by viewing the object of degree $i$ of $C^k(A)[-k]$, that is, the functor $\Gamma^i(A)\otimes\Lambda^{k-i}(A)$, as a subfunctor of $(\Lambda^1(A))^{\otimes i}\otimes\Lambda^{k-i}(A)$ by the canonical inclusion of invariants into the tensor product. One readily checks that the differential of $C^k(A)[-k]$ coincides with the restriction of the differential of $\mathcal{N}\Lambda^k(A/p)$.

To finish the proof, it remains to show that the inclusion of complexes

$$C^k(A)[-k]\hookrightarrow\mathcal{N}\Lambda^k(A/p) \tag{11-31}$$

is a quasi-isomorphism. For this, we filter both complexes by the weight of the exterior power on the right, that is, the term $F_s(C^k(A)[-k])$ of the filtration is the subcomplex of $C_k(A)$ supported by the $\Gamma^i(A)\otimes\Lambda^{k-i}(A)$ for $k-i\geq s$, and the term $F_s(\mathcal{N}\Lambda^k(A/p))$ is the subcomplex of $\mathcal{N}\Lambda^k(A/p)$ supported by the $\Lambda^{i_1}(A)\otimes\cdots\otimes\Lambda^{i_n}(A)\otimes\Lambda^{k-\sum i_j}(A)$ for $k-\sum i_j\geq s$. Both filtrations have finite length and the inclusion of complexes preserves the filtrations, whence a morphism

$$\mathrm{gr}(C^k(A)[-k])\to\mathrm{gr}(\mathcal{N}\Lambda^k(A/p)). \tag{11-32}$$

Since the filtrations are finite, the fact that (11-31) is a quasi-isomorphism will follow from the fact that (11-32) is. But we readily check that $\mathrm{gr}(C^k(A)[-k])$ is equal to the complex $\bigoplus_{i=0}^k\Gamma^i(A)\otimes\Lambda^k(A,0)$ with zero differential, that $\mathrm{gr}(\mathcal{N}\Lambda^k(A/p))$ is equal to the complex $\bigoplus_{i=0}^k\mathcal{N}\Lambda^i(A,1)\otimes(\Lambda^k(A),0)$, and that the morphism (11-32) is simply the morphism constructed from the quasi-isomorphisms $\Gamma^i(A)[i]\hookrightarrow\mathcal{N}\Lambda^i(A,1)$. This proves that (11-32) is a quasi-isomorphism, hence that (11-31) is a quasi-isomorphism. □

The results of Lemma 11.6 and 11.8 together prove:

**Proposition 11.9** *The conjecture holds for $n=1$ and $A$ a free abelian group.*

# Appendix A: Some computations of Hom and $\mathrm{Ext}^1$ in functor categories

In this appendix, we review some elementary computations of Hom and $\mathrm{Ext}^1$ in functor categories. The functor categories which we consider are the following:

1. The category $\mathcal{F}_{\mathbb{Z}}$ of functors from free abelian groups of finite type to abelian groups.
2. The category $\mathcal{P}_{\Bbbk}$ of strict polynomial functors defined over a field $\Bbbk$.
3. The category $\mathcal{P}_{\mathbb{Z}}$ of strict polynomial functors defined over $\mathbb{Z}$.

These categories are related by exact faithful functors

$$\mathcal{P}_{\mathbb{F}_p} \to \mathcal{P}_{\mathbb{Z}} \quad \text{and} \quad \mathcal{P}_{\mathbb{Z}} \to \mathcal{F}_{\mathbb{Z}}.$$

To be more specific, the first one sends a functor $F \in \mathcal{P}_{\mathbb{F}_p}$ to the functor $F \circ I/p$, where $I/p$ denotes the functor sending a free abelian group of finite type $A$ to the $\mathbb{F}_p$–vector space $A \otimes_{\mathbb{Z}} \mathbb{Z}/p = A/p$. The second one is the forgetful functor from strict polynomial functors to ordinary functors. Since they are exact, these functors induce morphisms on the level of Ext groups. In particular, the computation of a Hom or $\operatorname{Ext}^1$ in any of these three categories gives us partial information regarding the Hom or the $\operatorname{Ext}^1$ in the other two. We will be more precise about this below.

## A.1 General techniques for Ext* computations

We first recall some techniques from standard homological algebra which are efficient when computing Ext groups in the functor categories that we are considering (we refer the reader for more details and further techniques for computing Ext groups to Friedlander and Suslin [14], Franjou, Friedlander, Scorichenko and Suslin [13], Touzé [29] and Pirashvili [23]). First of all, by the Yoneda lemma, the functors $P^B(A) = \mathbb{Z}\operatorname{Hom}_{\mathbb{Z}}(B, A)$ with $A$ as the variable and a free finitely generated abelian group $B$ as a parameter are projective generators in $\mathcal{F}_{\mathbb{Z}}$ and we have, for any functor $F$,

(A-1)
$$\operatorname{Ext}^i_{\mathcal{F}_{\mathbb{Z}}}(P^B, F) = \begin{cases} F(B) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Similarly, the functors $\Gamma^{d,N}_R(M) = \Gamma^d_R(\operatorname{Hom}_R(N, M))$ with $M$ as the variable and a projective finitely generated $R$–module $N$ as a parameter provide a family of projective generators of the categories $\mathcal{P}_R$ of strict polynomial functors over a commutative ring $R$. For any strict polynomial functor $F$ in $\mathcal{P}_R$ we have a formula analogous to (A-1), where $F^d$ denotes the homogeneous component of weight $d$ of $F$:

(A-2)
$$\operatorname{Ext}^i_{\mathcal{P}_R}(\Gamma^{d,N}_R, F) = \begin{cases} F^d(N) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Compared to the category $\mathcal{F}_{\mathbb{Z}}$, the categories of strict polynomial functors have the pleasant additional feature that functors are equipped with weights and that, for any pair of homogeneous strict polynomial functors $F$ and $G$ of distinct weights, we have

$$\operatorname{Ext}^*_{\mathcal{P}_R}(F, G) = 0.$$

Among the classical techniques for computing Ext groups in our categories, one of the most important ones is the sum–diagonal adjunction, which we will now recall. For all functors $F$, $G$ and $H$ in $\mathcal{F}_{\mathbb{Z}}$, there are isomorphisms

$$\mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(H, F\otimes G) \simeq \mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}(2)}(H^{\boxplus}, F\boxtimes G), \tag{A-3}$$

$$\mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(F\otimes G, H) \simeq \mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}(2)}(F\boxtimes G, H^{\boxplus}), \tag{A-4}$$

where the terms on the right-hand side denote Ext groups computed in the category $\mathcal{F}_{\mathbb{Z}}(2)$ of *bifunctors*, between the bifunctors

$$H^{\boxplus}\colon (A,B)\mapsto H(A\oplus B) \quad \text{and} \quad F\boxtimes G\colon (A,B)\mapsto F(A)\otimes G(B).$$

In many cases (for example if $H=\Gamma^2_{\mathbb{Z}}$) one can express $H^{\boxplus}$ as a direct sum of functors of the form $H_1\boxtimes H_2$. Such bifunctors are sometimes called *of separable type*. Thus (A-4) leads us to consider extension groups of the form

$$\mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2, F\boxtimes G).$$

There are several situations in which we can actually compute such Ext groups. For example, these extension groups vanish if $H_2$ is a constant functor and $G(0)=0$. Together with the sum–diagonal adjunction, this yields the following fundamental vanishing lemma:

**Lemma A.1** [22] *Let $H$ be an additive functor and let $F$ and $G$ be a pair of functors satisfying $F(0)=G(0)=0$. Then we have*

$$\mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(H, F\otimes G) = 0 = \mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(F\otimes G, H).$$

The following proposition gives another example for which we can compute Ext groups between functors of separable type.

**Proposition A.2** (Künneth formulas) *Let $F$, $G\in\mathcal{F}_{\mathbb{Z}}$ be a pair of functors with values in $\mathbb{F}_p$–vector spaces and let $H_1$, $H_2\in\mathcal{F}_{\mathbb{Z}}$. Assume that $H_1(A)$ is a free abelian group for all $A$. Then there is an isomorphism*

$$\mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(H_1, F)\otimes \mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(H_2, G) \simeq \mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2, F\boxtimes G).$$

*Assume instead that, for all $A$, both $H_1(A)$ and $H_2(A)$ are $\mathbb{F}_p$–vector spaces, and denote by $E^*$ the graded $\mathbb{F}_p$–vector space*

$$E^* := \mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(H_1, F)\otimes \mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(H_2, G).$$

*Then there is an isomorphism $\mathrm{Ext}^0_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2, F\boxtimes G)\simeq E^0$ and a long exact sequence of $\mathbb{F}_p$–vector spaces:*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2, F\boxtimes G)\longrightarrow E^1 \longrightarrow \mathrm{Ext}^0_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2, F\boxtimes G)\\ &\xrightarrow{\partial} \mathrm{Ext}^2_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2, F\boxtimes G)\longrightarrow E^2 \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2, F\boxtimes G)\xrightarrow{\partial}\cdots. \end{aligned}$$

**Proof** In this proof we use the concise notation $[F,G]$ for Hom groups between $F$ and $G$ (the Hom groups are meant to be in the category of functors or of bifunctors, according to the context). Let $P^X$ and $P^Y$ be standard projectives of $\mathcal{F}_{\mathbb{Z}}$. The bifunctor $P^X\boxtimes P^Y$ is a projective object of $\mathcal{F}_{\mathbb{Z}}(2)$ and the Yoneda lemma yields an isomorphism

$$[P^X\boxtimes P^Y, F\boxtimes G]\simeq F(X)\otimes G(Y).$$

Moreover, the canonical morphism

$$[P^X,F]\otimes[P^Y,G]\to[P^X\boxtimes P^Y,F\boxtimes G] \tag{A-5}$$

can be identified via the Yoneda lemma with the identity morphism of $F(X)\otimes G(Y)$. In particular, the map (A-5) is an isomorphism. Let $P_{1\bullet}$ and $P_{2\bullet}$ be projective resolutions of $H_1$ and $H_2$, respectively. By (A-5), there is an isomorphism of complexes

$$[P_{1\bullet},F]\otimes[P_{2\bullet},G]\simeq[P_{1\bullet}\boxtimes P_{2\bullet},F\boxtimes G]. \tag{A-6}$$

The left-hand side of (A-6) is a tensor product of two complexes of $\mathbb{F}_p$–vector spaces. Hence, by the Künneth formula, its homology is isomorphic to

$$\mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(H_1,F)\otimes\mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}}(H_2,G).$$

Assume that $H_1(A)$ is a free abelian group for all $A$. Then $P_{1\bullet}\boxtimes P_{2\bullet}$ is a projective resolution of $H_1\boxtimes H_2$, hence the homology of the right-hand side of (A-6) is isomorphic to $\mathrm{Ext}^*_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2,F\boxtimes G)$. We thus obtain the first assertion of Proposition A.2.

Assume instead that $H_1(A)$ and $H_2(A)$ are $\mathbb{F}_p$–vector spaces for all $A$. Then $P_{1\bullet}\boxtimes P_{2\bullet}$ is a complex of projectives whose homology is equal to $H_1\boxtimes H_2$ in degrees 0 and 1 and is trivial elsewhere. It follows that there is a hypercohomology spectral sequence [30, Section 5.7.9] with second page

$$E_2^{p,q}=\begin{cases}\mathrm{Ext}^p_{\mathcal{F}_{\mathbb{Z}}(2)}(H_1\boxtimes H_2,F\boxtimes G) & \text{if } q=1,2,\\ 0 & \text{if } q\neq 1,2,\end{cases}$$

and differential $d_2\colon E_2^{p,q}\to E_2^{p+2,q-1}$, and which converges to the homology of the right-hand side of (A-6). This implies the second statement in Proposition A.2. □

The sum–diagonal adjunction works exactly in the same way for strict polynomial functors over any commutative ring $R$. The isomorphism (A-4) remains valid when $\mathcal{F}_{\mathbb{Z}}$ is replaced by $\mathcal{P}_R$. For extension groups between strict polynomial bifunctors of separable type, the Künneth formulas of Proposition A.2 remain valid if $\mathcal{F}_{\mathbb{Z}}$ is replaced by $\mathcal{P}_{\mathbb{Z}}$. The vanishing lemma, Lemma A.1, also holds. If we are interested in strict polynomial functors defined over a field $\Bbbk$ the situation is even nicer: in that case, we always have an isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(H_1, F) \otimes \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(H_2, G) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(H_1 \boxtimes H_2, F \boxtimes G).$$

## A.2 Some Hom computations

The following elementary lemma allows us to compare Hom groups between the functor categories under consideration.

**Lemma A.3** *The functor $\mathcal{P}_{\mathbb{F}_p} \to \mathcal{P}_{\mathbb{Z}}$ defined by precomposition by $I/p$ is full and faithful. The forgetful functor $\mathcal{P}_{\mathbb{Z}} \to \mathcal{F}_{\mathbb{Z}}$ is faithful.*

**Proof** It is clear that both functors are faithful. The only thing that we have to prove is the following isomorphism, for all $F$, $G \in \mathcal{P}_{\mathbb{F}_p}$:

$$\mathrm{Hom}_{\mathcal{P}_{\mathbb{F}_p}}(F, G) \simeq \mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(F \circ I/p, G \circ I/p) \tag{A-7}$$

By left exactness of Hom, the proof reduces to the case where $F$ is a standard projective, ie $F = \Gamma^{d,U}_{\mathbb{F}_p}$, with $U = \mathbb{F}_p^n$. Now observe that for all $A$ we have

$$\Gamma^{d,U}_{\mathbb{F}_p}(A/p) = \Gamma^{d,B}_{\mathbb{Z}}(A) \otimes \mathbb{F}_p,$$

with $B = \mathbb{Z}^n$. Thus we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{P}_{\mathbb{F}_p}}(\Gamma^{d,U}_{\mathbb{F}_p}, G) & & \longrightarrow & & \mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^{d,U}_{\mathbb{F}_p} \circ I/p, G \circ I/p) \\
{\scriptstyle (1)}\downarrow \simeq & & & & \downarrow \simeq \\
G^d(U) & \xrightarrow{\simeq} & G^d(B/p) & \xleftarrow[(2)]{\simeq} & \mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^{d,B}_{\mathbb{Z}}, G \circ I/p)
\end{array}$$

where $G^d$ is the homogeneous component of weight $d$ of $G$, the maps (1) and (2) are provided by the Yoneda lemma and the vertical map on the right is induced by the canonical projection $\Gamma^{d,B}_{\mathbb{Z}}(A) \to \Gamma^{d,U}_{\mathbb{F}_p}(A/p)$. Hence the isomorphism (A-7) holds. □

The forgetful functor $\mathcal{P}_{\mathbb{Z}} \to \mathcal{F}_{\mathbb{Z}}$ is not full in general. For example, there is no nonzero morphism of strict polynomial functors from $I/p \circ I^{(1)}$ to $I/p$, because these strict polynomial functors are homogeneous of different weights. However, the forgetful functor sends both of these to the same ordinary functor $I/p$ and the identity morphism is a nonzero morphism $I/p \to I/p$. The following lemma is proved in a manner similar to Lemma A.3 (reduce to the case where $F$ is a standard projective and then use the Yoneda lemma).

| $G(V)$ \ $F(V)$ | $\Gamma^{p^r}_{\Bbbk}(V)$ | $V^{\otimes p^r}$ | $V^{(r)}$ | $S^{p^r}(V)$ | $\Lambda^{p^r}(V)$ |
|---|---|---|---|---|---|
| $\Gamma^{p^r}_{\Bbbk}(V)$ | $\Bbbk$ | $\Bbbk$ | $0$ | $\Bbbk$ | $\begin{cases} 0 & \text{if } p\neq 2 \\ \Bbbk & \text{if } p=2 \end{cases}$ |
| $V^{\otimes p^r}$ | $\Bbbk$ | $\Bbbk^{\lvert\Sigma_{p^r}\rvert}$ | $0$ | $\Bbbk$ | $\Bbbk$ |
| $V^{(r)}$ | $\Bbbk$ | $0$ | $\Bbbk$ | $0$ | $0$ |
| $S^{p^r}(V)$ | $\Bbbk$ | $\Bbbk$ | $\Bbbk$ | $\Bbbk$ | $0$ |
| $\Lambda^{p^r}(V)$ | $0$ | $\Bbbk$ | $0$ | $\begin{cases} 0 & \text{if } p\neq 2 \\ \Bbbk & \text{if } p=2 \end{cases}$ | $\Bbbk$ |

Table 3: Some values of $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F,G)$ for functors $F$ and $G$ of weight $p^r$ with $r>0$, where $\Bbbk$ is a field of prime characteristic $p$.

**Lemma A.4** *Let $\Bbbk$ be a field. For all strict polynomial functors $F$ and $G$, precomposition by the Frobenius twist $I^{(1)}$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F,G) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F\circ I^{(1)}, G\circ I^{(1)}).$$

Let $\Bbbk$ be a field. Table 3 gathers some elementary Hom computations in $\mathcal{P}_{\Bbbk}$, which follow from the left exactness of Hom and the techniques recalled in Section A.1. By Lemma A.3, these also provide Hom computations in $\mathcal{P}_{\mathbb{Z}}$. One can also verify, using the techniques recalled in Section A.1, that the computations of $\mathrm{Hom}_{\mathcal{F}_{\mathbb{Z}}}(F\circ I/p, G\circ I/p)$ for the functors $F$ and $G$ listed in Table 3 give the same result as in $\mathcal{P}_{\mathbb{Z}}$.

## A.3 Some $\mathrm{Ext}^1$ computations

**A.3.1 Computations in $\mathcal{P}_{\Bbbk}$, with $\Bbbk$ a field of positive characteristic** The results given here are all well known as special cases of more general statements — see eg [27; 29] — but we give here some self-contained and elementary proofs.

**Lemma A.5** *Let $\Bbbk$ be a field of positive characteristic and consider $F$, $G\in\mathcal{P}_{\Bbbk}$ with finite-dimensional values. Precomposition by the Frobenius twist induces an isomorphism*

$$\mathrm{Ext}^1_{\mathcal{P}_{\Bbbk}}(F,G) \xrightarrow{\simeq} \mathrm{Ext}^1_{\mathcal{P}_{\Bbbk}}(F\circ I^{(1)}, G\circ I^{(1)}).$$

**Proof** There is a short exact sequence $0\to K_F\to P_F\to F\to 0$, where $P_F$ is a direct sum of standard projectives (ie of functors of the form $\Gamma^{d,U}_{\Bbbk}$). Similarly, there is a short exact sequence $0\to G\to J_G\to Q_G\to 0$, where $J_G$ is a product of functors of the form $S^d_{\Bbbk,U'}\colon V\mapsto S^d_{\Bbbk}(U'\otimes V)$. By considering the long exact sequences of Ext associated to these short exact sequences, we can reduce the proof to showing that $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Gamma^{d,U}_\Bbbk \circ I^{(1)}, S^d_{\Bbbk,U'} \circ I^{(1)})$ is zero. By the exponential formula for divided powers, we can decompose $\Gamma^{d,U}_\Bbbk$ as a direct sum of functors of the form $\Gamma^{d_1}_\Bbbk \otimes \cdots \otimes \Gamma^{d_k}_\Bbbk$ with $\sum d_j = d$ and decompose $S^d_{\Bbbk,U}$ in a similar manner. By the sum–diagonal adjunction and the Künneth formula, the vanishing of

$$\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Gamma^{d,U}_\Bbbk \circ I^{(1)}, S^d_{\Bbbk,U'} \circ I^{(1)})$$

will then follow from the vanishing of $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Gamma^e_\Bbbk \circ I^{(1)}, S^e_\Bbbk \circ I^{(1)})$ for all $e \le d$, which we will now prove. We first prove the vanishing of

$$\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Gamma^e_\Bbbk \circ I^{(1)}, S^e_\Bbbk \circ I^{(1)})$$

for $e = p^r$ with $r \ge 0$. Consider the presentation of $\Gamma^{p^r}_\Bbbk \circ I^{(1)}$ as in Lemma 4.8 and let $K$ be the kernel of the projection

$$\Gamma^{p^{r+1}}_\Bbbk \xrightarrow{\pi} \Gamma^{p^r}_\Bbbk \circ I^{(1)}.$$

Since $\Gamma^{p^{r+1}}_\Bbbk$ is projective, we have an epimorphism

$$\mathrm{Hom}_{\mathcal{P}_\Bbbk}(K, \Gamma^{p^r}_\Bbbk \circ I^{(1)}) \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Gamma^{p^r}_\Bbbk \circ I^{(1)}, S^{p^r}_\Bbbk \circ I^{(1)})$$

whose left-hand term embeds into $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(\Gamma^{p^{r+1}-1}_\Bbbk \otimes \Gamma^1_\Bbbk, S^{p^r}_\Bbbk \circ I^{(1)})$, which is zero by sum–diagonal adjunction. Now we prove the vanishing of $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Gamma^e_\Bbbk \circ I^{(1)}, S^e_\Bbbk \circ I^{(1)})$ for arbitrary $e$. Let $e = e_k p^k + \cdots + e_0$ be the $p$–adic decomposition of $e$. Then the canonical inclusion

$$\Gamma^e_\Bbbk(V) \hookrightarrow \bigotimes_{i=0}^{k} (\Gamma^{p^i}_\Bbbk(V))^{\otimes e_i} =: H(V)$$

admits a retract, provided by the multiplication in the divided power algebra. Thus $\Gamma^e_\Bbbk$ is a direct summand of $H$, so that $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Gamma^e_\Bbbk \circ I^{(1)}, S^e_\Bbbk \circ I^{(1)})$ is a direct summand of $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(H \circ I^{(1)}), S^e_\Bbbk \circ I^{(1)})$. The latter group is zero by sum–diagonal adjunction. □

**Proposition A.6** *Let $\Bbbk$ be a field of characteristic 2. Any extension of degree one in $\mathcal{P}_\Bbbk$ between two functors of the form $\bigotimes_{i=1}^n (\Gamma^{d_i}_\Bbbk \circ I^{(r_i)})$ is trivial.*

**Proof** By iterated use of the sum–diagonal adjunction and the Künneth formula, the proof reduces to showing that

$$\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(F, G) = 0 \tag{A-8}$$

for $F(V) = \Gamma^d_\Bbbk \circ I^{(r)}$ and $G = \Gamma^e_\Bbbk \circ I^{(s)}$. Since there are no nontrivial extensions between functors with different weights, we can assume that $d2^r = e2^s$. By Lemma A.5, precomposition by the Frobenius twist induces an isomorphism on the level of $\mathrm{Ext}^1$, so the proof reduces to showing (A-8) when one of the integers $r$ or $s$ is equal to zero. If $r=0$, then (A-8) holds by projectivity of $F$. Hence it suffices to prove (A-8) for $F=\Gamma^d_{\Bbbk}\circ I^{(r)}$ and $G=\Gamma^{dp^r}_{\Bbbk}$ with $r>0$. Let $\otimes^{dp^r}$ denote the functor $V\mapsto V^{\otimes dp^r}$. By sum–diagonal adjunction and the Künneth formula, $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,\otimes^{dp^r})=0$. Hence

$$\mathrm{Ext}^1_{\mathcal{P}_{\Bbbk}}(F,G)\simeq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F,C), \tag{A-9}$$

where $C$ is the cokernel of the canonical inclusion $\Gamma^{dp^r}_{\Bbbk}\to\otimes^{dp^r}$. Since $C$ embeds into $\Lambda^2_{\Bbbk}\otimes(\otimes^{p^r-2})$ and $\Gamma^{dp^r}_{\Bbbk}$ surjects onto $F$, the right-hand side of (A-9) embeds into $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\Gamma^{dp^r}_{\Bbbk},\Lambda^2_{\Bbbk}\otimes(\otimes^{p^r-2}))$. The latter group is trivial by (A-2). □

There is a more general statement than Proposition A.6 over fields of odd characteristic, whose proof is completely similar.

**Proposition A.7** *Let $\Bbbk$ be a field of odd characteristic $p$. Any degree-one extension in $\mathcal{P}_{\Bbbk}$ between functors of the form $\bigotimes_{i=1}^n(\Gamma^{d_i}\circ I^{(r_i)})\otimes\bigotimes_{j=1}^m(\Lambda^{e_j}\circ I^{(s_j)})$ is trivial.*

**Remark A.8** Proposition A.7 does not hold when $\Bbbk$ is a field of characteristic $p=2$. For example $\mathrm{Ext}^1_{\mathcal{P}_{\Bbbk}}(I^{(1)},\Lambda^2)$ is one-dimensional, generated by the extension

$$0\to\Lambda^2\to\Gamma^2\to I^{(1)}\to 0,$$

where the map $\Lambda^2(V)\to\Gamma^2(V)$ sends $x\wedge y$ to $x\cdot y$ (and is only well defined if $p=2$). This nontrivial extension shows up in many computations and explains why our results take on different forms, depending on the parity of the characteristic (or of the torsion over $\mathbb{Z}$).

### A.3.2 Computations in $\mathcal{P}_{\mathbb{Z}}$

The following lemma is a formal consequence of the fact that precomposition by the functor $I/p$ yields a full and faithful functor $\mathcal{P}_{\mathbb{F}_p}\to\mathcal{P}_{\mathbb{Z}}$.

**Lemma A.9** *For all $F$, $G\in\mathcal{P}_{\mathbb{F}_p}$, precomposition by $I/p$ yields an injective map*

$$\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{F}_p}}(F,G)\hookrightarrow\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(F\circ I/p,G\circ I/p).$$

We cannot expect that precomposition by $I/p$ induces an isomorphism in general. For example, we have $\Gamma^d_{\mathbb{F}_p}(A/p)=\Gamma^d_{\mathbb{Z}}(A)\otimes\mathbb{F}_p$, so we have a short exact sequence in $\mathcal{P}_{\mathbb{Z}}$:

$$0\to\Gamma^d_{\mathbb{Z}}\to\Gamma^d_{\mathbb{Z}}\to\Gamma^d_{\mathbb{F}_p}\circ I/p\to 0$$

By taking the Ext long exact sequence associated to the short exact sequence, we obtain that, for all $G\in\mathcal{P}_{\mathbb{F}_p,d}$,

$$\mathrm{Ext}^i_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^d_{\mathbb{F}_p}\circ I/p,G\circ I/p)=\begin{cases}G(\mathbb{Z}/p) & \text{if } i=0,1,\\ 0 & \text{if } i>2.\end{cases}$$

On the other hand, by projectivity of $\Gamma^d_{\mathbb{F}_p}$ in $\mathcal{P}_{\mathbb{F}_p}$, there is no nonzero extension of $\Gamma^d_{\mathbb{F}_p}$ by $G$ in $\mathcal{P}_{\mathbb{F}_p}$. We now describe some explicit elementary computations in $\mathcal{P}_{\mathbb{Z}}$. As in the main body of the article, we abuse notations and denote the functors by their values to avoid cumbersome notations. For example, we write $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ rather than $\Gamma^2_{\mathbb{F}_2} \circ I^{(1)} \circ I/2$.

**Lemma A.10** *Let $p$ be a prime number. Then $\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(A/p^{(r)}, A/p^{(r)}) = 0$ for all positive integers $r$.*

**Proof** There is an exact sequence

$$\bigoplus_{\substack{k+\ell=p^r\\ k,\ell>0}} \Gamma^k_{\mathbb{Z}}(A) \otimes \Gamma^\ell_{\mathbb{Z}}(A) \to \Gamma^{p^r}_{\mathbb{Z}}(A) \to A/p^{(r)} \to 0,$$

where the left-hand map is induced by the multiplication. Let $K(A)$ be the kernel of the map $\Gamma^{p^r}_{\mathbb{Z}}(A) \to A/p^{(r)}$. Applying the functor $\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{Z}}}(-, A/p^{(r)})$ to the short exact sequence

$$0 \to K(A) \to \Gamma^{p^r}_{\mathbb{Z}}(A) \to A/p^{(r)} \to 0$$

yields an isomorphism

$$\mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(K(A), A/p^{(r)}) \simeq \mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(A/p^{(r)}, A/p^{(r)}). \tag{A-10}$$

Since $\bigoplus_{k+\ell=p^r,k,\ell>0} \Gamma^k_{\mathbb{Z}}(A)\otimes\Gamma^\ell_{\mathbb{Z}}(A)$ surjects onto $K(A)$, it follows that the left-hand side of (A-10) embeds into a direct sum of terms $\mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^k_{\mathbb{Z}}(A) \otimes \Gamma^\ell_{\mathbb{Z}}(A), A/p^{(r)})$. But these expressions are trivial by Lemma A.1, whence the result. □

We now compute the extensions of $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ by $A/2^{(2)}$. The cokernel of the map $\Gamma^3_{\mathbb{Z}}(A) \otimes A \to \Gamma^4_{\mathbb{Z}}(A)$ induced by the multiplication of the divided powers algebra is a homogeneous strict polynomial functor of weight 4, whose image under the forgetful functor $\mathcal{P}_{\mathbb{Z}} \to \mathcal{F}_{\mathbb{Z}}$ is the ordinary functor $\Gamma^2_{\mathbb{Z}}(A/2)$. We therefore will denote this cokernel by $\Gamma^2_{\mathbb{Z}}(A/2^{(1)})$. The following computation is used in Proposition 9.4:

**Lemma A.11** *There is an isomorphism $\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}), A/2^{(2)}) \simeq \mathbb{Z}/2$. The non-split extension is*

$$0 \to A/2^{(2)} \to \Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \to 0,$$

*where the maps $A/2^{(2)} \to \Gamma^2_{\mathbb{Z}}(A/2^{(1)})$ and $\Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ are the unique nonzero morphisms between these strict polynomial functors.*

**Proof** Consider the commutative diagram:

$$\begin{array}{ccccccccc}
 & & \Gamma^3_{\mathbb{Z}}(A)\otimes A\oplus\Gamma^2_{\mathbb{Z}}(A)\otimes\Gamma^2_{\mathbb{Z}}(A) & \xrightarrow{(2,0)} & \Gamma^3_{\mathbb{Z}}(A)\otimes A & \longrightarrow & \Gamma^3_{\mathbb{F}_2}(A/2)\otimes A/2 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma^4_{\mathbb{Z}}(A) & \xrightarrow{\times 2} & \Gamma^4_{\mathbb{Z}}(A) & \longrightarrow & \Gamma^4_{\mathbb{F}_2}(A/2) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/2^{(2)} & \overset{(a)}{\dashrightarrow} & \Gamma^2_{\mathbb{Z}}(A/2^{(1)}) & \overset{(b)}{\dashrightarrow} & \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In this diagram, the upper vertical arrows are induced by the multiplication in the divided power algebra and the rows and columns are exact. The two dashed arrows are produced by the universal property of cokernels. It is easy to compute that the arrow $(a)$ is injective when $A=\mathbb{Z}$, hence it is injective for all $A$ by additivity of $A/2^{(2)}$. Since the middle row is exact, an elementary diagram chase then shows that the bottom row also is. We have thus obtained an extension of $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ by $A/2^{(2)}$. It is non-split because the middle term $\Gamma^2_{\mathbb{Z}}(A/2^{(1)})$ has 4–torsion.

We know that $\mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(A/2^{(2)}, A/2^{(2)})=\mathbb{Z}/2$ and $\mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(A/2^{(2)}, \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}))=0$. The left exactness of the functor Hom therefore implies that the dashed arrow $(a)$ is the unique nonzero morphism from $A/2^{(2)}$ to $\Gamma^2_{\mathbb{Z}}(A/2^{(1)})$. To prove that the dashed arrow $(b)$ is also characterized as the unique nonzero morphism available, we make use of the fact that $\Gamma^4(A)$ surjects onto $\Gamma^2_{\mathbb{Z}}(A/2^{(1)})$ and that there is only one nonzero morphism from $\Gamma^4(A)$ to $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$.

Finally, let us compute the group $\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}), A/2^{(2)})$. This is an $\mathbb{F}_2$–vector space and we will now show that it is of dimension one. We already know that its dimension is at least $1$. The short exact sequence

$$0\to\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})\to A/2^{\otimes 2}\to\Lambda^2_{\mathbb{F}_2}(A/2^{(1)})\to 0$$

therefore induces an isomorphism

$$\mathbb{Z}/2=\mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}), A/2^{(2)})\simeq\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(\Lambda^2_{\mathbb{F}_2}(A/2^{(1)}), A/2^{(2)}).$$

By Lemma A.10, the long exact sequence of Ext's associated to the short exact sequence $0\to\Lambda^2_{\mathbb{F}_2}(A/2^{(1)})\to\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})\to A/2^{(2)}\to 0$ yields an injective map

$$\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}), A/2^{(2)})\hookrightarrow\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(\Lambda^2_{\mathbb{F}_2}(A/2^{(1)}), A/2^{(2)}).$$

It follows that $\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}), A/2^{(2)})$ has dimension exactly one, as asserted. □

| $G(A)$ | $F(A)$: $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ | $F(A)$: $A/2^{(1)} \otimes A/2^{(1)}$ | $F(A)$: $\Gamma^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)} = \Gamma^2_{\mathbb{Z}}(A) \otimes A/2^{(1)}$ |
|---|---|---|---|
| $A/2^{(2)}$ | $\mathbb{Z}/2$ | 0 | 0 |
| $A/2^{(1)} \otimes A/2^{(1)}$ | 0 | 0 | 0 |
| $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ | 0 | 0 | 0 |
| $\Lambda^2_{\mathbb{F}_2}(A/2^{(1)})$ | 0 | 0 | 0 |
| $\Gamma^2_{\mathbb{Z}}(A/2^{(1)})$ | 0 | 0 | 0 |
| $\Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2^{\oplus 2}$ | $\mathbb{Z}/2$ |

Table 4: Some computations of $\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(F,G)$ for functors $F$ and $G$ of weight 4.

The following lemma is used in Proposition 9.2. The middle term in the unique non-split extension of $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ by $\Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}$ which is provided by this computation is the functor which is denoted $\Phi^4(A)$ in Proposition 7.2 and elsewhere in the text.

**Lemma A.12** *There is an isomorphism*

$$\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2^{(1)}), \Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2^{(1)}) \simeq \mathbb{Z}/2.$$

**Proof** The $\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{Z}}}(A/2^{(r+1)}, -)$ long exact sequence associated to the short exact sequence

$$0 \to \Lambda^2_{\mathbb{F}_2}(A/2^{(r)}) \to A/2^{(r)\otimes 2} \to S^2_{\mathbb{F}_2}(A/2^{(r)}) \to 0$$

yields an isomorphism

$$\mathbb{Z}/2 \simeq \mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(A/2^{(r+1)}, S^2_{\mathbb{F}_2}(A/2^{(r)})) \simeq \mathrm{Ext}^1_{\mathcal{P}_{\mathbb{Z}}}(A/2^{(r+1)}, \Lambda^2_{\mathbb{F}_2}(A/2^{(r)})).$$

The result of Lemma A.12 follows from this assertion and Lemma A.10 by using the sum–diagonal adjunction and Proposition A.2. □

Table 4 collects the results of Lemmas A.11 and A.12 and some other easy computations obtained with the same techniques. Some of these computations were used in Section 10.

**A.3.3 Examples of computations in $\mathcal{F}_{\mathbb{Z}}$** One can sometimes compute $\mathrm{Ext}^1$ groups in $\mathcal{F}_{\mathbb{Z}}$ by methods close to those which we used in $\mathcal{P}_{\mathbb{Z}}$. For example, the proof of Lemma A.11 carries over without change in $\mathcal{F}_{\mathbb{Z}}$, so that we obtain the following result:

**Lemma A.13** *There is an isomorphism* $\mathbb{Z}/2 \simeq \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2), \Lambda^2_{\mathbb{F}_2}(A/2) \otimes A/2)$.

The reasoning of the proof of Lemma A.11 has to be slightly modified in $\mathcal{F}_{\mathbb{Z}}$. Since $\Gamma^2_{\mathbb{Z}}(A/2)$ has 4–torsion, the forgetful functor $\mathcal{P}_{\mathbb{Z}} \to \mathcal{F}_{\mathbb{Z}}$ sends the extension

$$0 \to A/2^{(2)} \to \Gamma^2_{\mathbb{Z}}(A/2^{(1)}) \to \Gamma^2_{\mathbb{F}_2}(A/2^{(1)}) \to 0$$

to a non-split extension in $\mathcal{F}_{\mathbb{Z}}$. But the forgetful functor sends $A/2^{(2)}$ to the ordinary functor $A/2$, and $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}}}(A/2, A/2) = \mathbb{Z}/2$ (the middle term in the nontrivial extension being the non-split extension $A/4$). Thus, the self-extensions of $A/2$ come into play. Reasoning as in the proof of Lemma A.11, we obtain a short exact sequence of $\mathbb{F}_2$–vector spaces

$$0 \to \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}}}(A/2, A/2) \to \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2), A/2) \to \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}}}(\Lambda^2_{\mathbb{F}_2}(A/2), A/2) \to 0.$$

This leads to the following result:

**Lemma A.14** $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}}}(\Gamma^2_{\mathbb{F}_2}(A/2), A/2) = \mathbb{Z}/2^{\oplus 2}$.

# Appendix B: The integral homology of $K(A,n)$ for $A$ a free abelian group

In this appendix, we translate some of the computations of the derived functors of the divided power functors $\Gamma^d(A)$ achieved in this article in terms of the integral homology of Eilenberg–Mac Lane spaces. We begin with a short survey on the relation between the functors $L_*\Gamma^d(A,n)$ and the homology of $K(A,n)$. Then we present a table giving a functorial description of the groups $H_{n+i}(K(A,n);\mathbb{Z})$ in low degrees.

## B.1 The homology of $K(A,n)$ and the derived functors $L_*\Gamma(A,n)$

As proved by Dold and Puppe [11, Satz 4.16], there exist isomorphisms

$$\bigoplus_{d\geq 0} L_j S^d(B,n) \simeq H_j(K(B,n);\mathbb{Z}) \tag{B-1}$$

for all abelian groups $B$. We will now address the question of the naturality in $B$ of these isomorphisms, whose definition relies on the construction of a Moore space $M(B,n)$. Assume that there exists a full subcategory $\mathcal{C} \subset \mathcal{A}b$ and a functor

$$M(-,n)\colon \mathcal{C} \to \mathrm{Ho}(\mathrm{Top}_*). \tag{B-2}$$

One can then verify, by going through the proof of [11, Satz 4.16], that the isomorphism (B-1) is natural in the object $B$ of $\mathcal{C}$. It is quite obvious how to construct such a functor (B-2) when $\mathcal{C}$ is the category of free abelian groups, since one easily proves that the functor $H_n(-;\mathbb{Z})$ induces an isomorphism $[M(A,n), M(A',n)]_* \simeq \mathrm{Hom}_{\mathbb{Z}}(A, A')$ whenever $A$ and $A'$ are free. It follows that the isomorphism (B-1) is natural in $B$ whenever $B$ is free. The isomorphism (B-1), together with the décalage isomorphisms (3-9), reduces the computation of the homology of $K(A,n)$ for $A$ free to that of the derived functors of $\Gamma^d(A)$. For $n = 1$, the functorial isomorphism (B-1) can thus be rewritten as the well-known isomorphism

$$\Lambda^j(A) \simeq H_j(K(A,1);\mathbb{Z}),$$

while for $n \geq 2$ one obtains, by double décalage, functorial isomorphisms

$$\bigoplus_{d\geq 0} L_{j-2d}\Gamma^d(A, n-2) \simeq H_j(K(A,n);\mathbb{Z}). \tag{B-3}$$

The situation is more complicated for arbitrary abelian groups $B$, since it is known that there can be no functor (B-2) when $\mathcal{C} = \mathcal{A}b$. Indeed, if $n \geq 2$ and $B$ is an abelian group endowed with a free resolution $0 \to K \to L \to B \to 0$, we can interpret this resolution as the image by $H_n$ of a cofiber sequence $M(K,n) \to M(L,n) \to M(B,n)$. The associated Barratt–Puppe sequence determines a short exact sequence of groups

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(B, \pi_{n+1}M(B',n)) \to [M(B,n), M(B',n)]_* \xrightarrow{H_n} \mathrm{Hom}_{\mathbb{Z}}(B, B') \to 0.$$

The existence of a functorial Moore space construction on the category of all abelian groups would imply that this sequence always splits. This however is known to be false, since, for example, $[M(\mathbb{Z}/2), M(\mathbb{Z}/2)]_* \simeq \mathbb{Z}/4$.

It follows from the previous discussion that if one considers arbitrary abelian groups $B$, the isomorphism (B-1) must be replaced by the second quadrant spectral sequence

$$E^1_{p,q} = L_{p+q}S^{-p}(B,n) \Longrightarrow H_{p+q}(K(B,n);\mathbb{Z}), \tag{B-4}$$

which, by [7], degenerates at $E^1$. As a result, the expressions involving in this context the derived functors of the functors $S^k(B)$ (or equivalently by décalage of $\Gamma^k(B)$) only describe the abutment $H_{p+q}(K(B,n);\mathbb{Z})$ up to a filtration, since they live in $E^1 = E^\infty$:

$$\bigoplus_{d\geq 0} L_{j-2d}\Gamma^d(B, n-2) \simeq \bigoplus_{d\geq 0} L_j S^d(B,n) \simeq \bigoplus_{d\geq 0} \mathrm{gr}_d(H_j(K(B,n);\mathbb{Z})). \tag{B-5}$$

In the stable range, that is for the homology groups $H_{i+n}(K(B,n);\mathbb{Z})$ with $i < n$, the filtration splits. Indeed, in the stable range these homology groups are direct sums of copies of the functors ${}_pB$ (the $p$–torsion subgroup of $B$) and $B/p$ (the mod $p$ reduction of $B$), for prime integers $p$. This follows, for example, from Cartan's computation of the stable homology of Eilenberg–Mac Lane spaces [9, Exposé 11, théorème 2]. Such functors are simple functors, hence they cannot be nontrivially filtered. The filtration is usually not split in non-stable degrees. This is already visible in the simplest case, that of the homology group $H_3(K(B,1);\mathbb{Z})$, in other words the third homology group of the abelian group $B$. In that case, the spectral sequence (B-4) reduces to the functorial short exact sequence

$$0 \longrightarrow \Lambda^3(B) \longrightarrow H_3(B) \longrightarrow L_1\Lambda^2(B,0) \longrightarrow 0. \tag{B-6}$$

It is shown in [21, Corollary 3.1] that this exact sequence cannot be functorially split.

## B.2 The homology of $K(A\,,n)$ in low degrees for $A$ free

Table 5 gives a functorial description of the groups $H_{n+i}(K(A,n);\mathbb{Z})$ as a functor of the free abelian group $A$ for all integers $n$ and $i$ such that $1 \le n \le 11$ and $0 \le i \le 10$. The convention in lines $n = 1$ to $11$ of the table is that each empty box is actually filled with a copy of the expression in the lowest non-empty box above it. These are therefore all filled with additive groups $A/p$, since each of these terms arises by suspension from the one immediately above it. Every expression on the line "adm." displays Cartan's labelling of the stable groups in the corresponding column in terms of his admissible sequences and its associated prime. In the line above this one, labelled $\mathcal{T}$, we display (for those admissible words which we called restricted in Section 11.1) the corresponding labelling by decreasing sequences of integers mentioned in Theorem 11.3. The order in which the items in each box in these two lines are listed reflects the order in which the additive functors to which they correspond occurred in the lines above it.

The table is obtained from our computations of derived functors of divided powers and the functorial isomorphism (B-3). Indeed, since the functors $L_{j-2d}\Gamma^d(A,n-2)$ vanish for $j-2d \le n-2$, we need only consider these functors for $d \le 6$. The required values the derived functors for $d \le 4$ were obtained in Sections 8–10 above. For $d = 5,\ 6$, it is only necessary to know the values of the derived functors $L_r\Gamma^d_{\mathbb{Z}}(A,n-2)$ for $r \le 2$. The latter can easily be obtained, either by a partial analysis of the maximal filtration of $\Gamma^d_{\mathbb{Z}}(A)$ for such values of $d$, as provided in Example 8.6 for $d \le 4$, or by reliance for those values of $r$ on the mod $p$ reduction method described in Section 6. This second method reduces the problem to that of a functorial computation of certain derived functors of $L_*\Gamma^d_{\mathbb{F}_p}(V,n)$ for an $\mathbb{F}_p$–vector space $V$, a question which we discussed in Section 5.

The table shows that the homology groups within the range of values of the pair $(n,i)$ considered can be expressed functorially as direct sums of tensor and exterior powers of the elementary functors $A$ and $A/p$ for primes $p \le 5$ together with divided power functors $\Gamma^i_{\mathbb{Z}}(A)$ and $\Gamma^2_{\mathbb{Z}}(A/2)$. The only exceptions to this rule are the occurrences

| $n$ \ $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | $A$ | $\Lambda^2(A)$ | $\Lambda^3(A)$ | $\Lambda^4(A)$ | $\Lambda^5(A)$ | $\Lambda^6(A)$ | $\Lambda^7(A)$ | $\Lambda^8(A)$ |
| 2 | | 0 | $\Gamma^2(A)$ | 0 | $\Gamma^3(A)$ | 0 | $\Gamma^4(A)$ | 0 |
| 3 | | | $A/2$ | $\Lambda^2(A)$ | $A/3$ | $A\otimes A/2$ | $\Lambda^3(A)\oplus A/2$ | $A\otimes A/3\oplus\Lambda^2(A/2)$ |
| 4 | | | | 0 | $\Gamma^2(A)\oplus A/3$ | 0 | $A\otimes A/2\oplus A/2$ | 0 |
| 5 | | | | | $A/2\oplus A/3$ | $\Lambda^2(A)$ | $A/2$ | $A\otimes A/2$ |
| 6 | | | | | | 0 | $\Gamma^2(A)\oplus A/2$ | 0 |
| 7 | | | | | | | $A/2\oplus A/2$ | $\Lambda^2(A)$ |
| 8 | | | | | | | | 0 |
| 9 | | | | | | | | |
| $\mathcal{T}$ | | | $\{1;2\}$ | | $\{1,1;2\},\{1;3\}$ | | $\{1,1,1;2\}$, $\{2;2\}$ | |
| adm. | | | $(2;2)$ | | $(4;2),(4;3)$ | | $(6;2),(4,2;2)$ | |

| $n$ | 8 | 9 | 10 |
|---|---|---|---|
| 1 | $\Lambda^9(A)$ | $\Lambda^{10}(A)$ | $\Lambda^{11}(A)$ |
| 2 | $\Gamma^5(A)$ | 0 | $\Gamma^6(A)$ |
| 3 | $\Phi^4(A)\oplus A/5$ | $\Lambda^4(A)\oplus(A\otimes A/2)$ | $\Lambda^2(A)\otimes A/3\oplus A\otimes\Lambda^2(A/2)$ |
| 4 | $\Gamma^3(A)\oplus A/5$ $\oplus A\otimes A/3\oplus\Gamma^2(A/2)$ | $\Gamma^2_{\mathbb{F}_2}(A/2)$ | $\Gamma^2(A)\otimes A/2\oplus A\otimes A/2$ |
| 5 | $A/3\oplus A/5\oplus A/2$ | $\Lambda^2(A/2)\oplus A/2$ $\oplus(A\otimes A/2)\oplus(A\otimes A/3)$ | $\Lambda^3(A)\oplus\Gamma^2_{\mathbb{F}_2}(A/2)$ |
| 6 | $(A\otimes A/2)\oplus A/3\oplus A/5\oplus A/2$ | $A/2$ | $\Gamma^2(A/2)\oplus(A\otimes A/3)$ |
| 7 | $A/3\oplus A/5\oplus A/2$ | $A/2\oplus(A\otimes A/2)$ | $A/2$ |
| 8 | $\Gamma^2(A)\oplus A/3\oplus A/5\oplus A/2$ | $A/2$ | $A/2\oplus(A\otimes A/2)$ |
| 9 | $A/2\oplus A/3\oplus A/5\oplus A/2$ | $A/2\oplus\Lambda^2(A)$ | $A/2$ |
| 10 | | $A/2$ | $\Gamma^2(A)\oplus A/2$ |
| 11 | | | $A/2\oplus A/2$ |
| $\mathcal{T}$ | $\{1,1,1,1;2\},\{1,1;3\}$, $\{1;5\},\{2,1;2\}$ | | $\{1,1,1,1,1;2\},\{2,1,1;2\}$ |
| adm. | $(8;2),(8;3),(8;5),(6,2;2)$ | $(6,3;2)$ | $(10;2),(8,2;2)$ |

Table 5: A functorial description of the groups $H_{n+i}(K(A,n);\mathbb{Z})$ as a functor of the free abelian group $A$ for $n$ and $i$ with $1\le n\le 11$ and $0\le i\le 10$.

of the direct summands $\Gamma^2_{\mathbb{F}_2}(A/2)$ in $H_{13}(K(A,4);\mathbb{Z})$ and $H_{15}(K(A,5);\mathbb{Z})$ and of the new functor $\Phi^4(A)$ in $H_{11}(K(A,3);\mathbb{Z})$, for which we have provided a number of descriptions in the main body of our text. The fact that the primes 2, 3 and 5 seem to play a special role here is simply due to our chosen range of values for the integers $n$ and $i$. As these values increase the functors $A/p$ will occur for additional primes $p$.

While we obtain in this way a complete understanding of the homology of $K(A,n)$ for $A$ free in the range mentioned above, we wish to draw the reader's attention to the fact that the situation is more complicated when one no longer restricts oneself to functors on the category of free abelian groups $A$. Additional functors of $A$ occur in the general case, whose values are torsion groups. The simplest of these functors are the additive functors

$$ {}_pA := \ker(p\colon A \to A). $$

Immediately after these, one encounters the functors $\Omega(A)$ and $R(A)$, which Eilenberg and Mac Lane introduced in their foundational text [12, Sections 13 and 22]. These would now be denoted $L_1\Lambda^2(A,0)$ and $L_1\Gamma^2_{\mathbb{Z}}(A,0)$ (or simply $L_1\Lambda^2(A)$ and $L_1\Gamma^2_{\mathbb{Z}}(A)$), respectively, and they would indeed appear in a table similar to ours for a general $A$, as the derived versions of the functors $\Lambda^2(A)$ and $\Gamma^2(A)$, which are to be found in positions $(n,i)=(1,1)$ and $(n,i)=(2,2)$ of our table, respectively. The décalage morphisms imply that, for $A$ non-free, these functors contribute to the groups $H_*(K(A,1);\mathbb{Z})$ and $H_*(K(A,2);\mathbb{Z})$, respectively, so that the first two lines of our table below are already much more complicated in that case.

In this table we can already observe many of the phenomena discussed in the text. As we mentioned above, the first of these is the lowest occurrence of the periodic phenomenon represented by the functor $\Phi^4(A)$, which is the only new functor within our range of values. The pair of functors $\Gamma^2_{\mathbb{Z}}(A/2)$ for the values 4 and 6 of $n$ are also noteworthy, as they produce some 4–torsion in the homology. Finally, it will be seen that the two 2–torsion expressions $\Gamma^2_{\mathbb{F}_2}(A/2)$, while isomorphic, actually correspond to two distinct situations. Indeed they may be thought of in the spirit of Conjecture 11.5 as the first derived functors of $\Gamma^2(A/2)$ and $\Lambda^2(A/2)$, respectively. This is reflected in the different behaviour of these two homology groups under suspension. While the functor $\Gamma^2_{\mathbb{F}_2}(A/2)$ that lives in bidegree $(n,i)=(4,9)$ suspends to an $A/2$ in $H^{\mathrm{st}}_9(K(A);\mathbb{Z})$, as one would expect, this is not the case for the $\Gamma^2_{\mathbb{F}_2}(A/2)$ in bidegree $(5,10)$. One verifies by the long exact sequence of [11, Korollar 6.11] that the suspension sends this element of $H_{15}(K(A,5);\mathbb{Z})$ to the summand $\Gamma^2_{\mathbb{Z}}(A/2)$ of $H_{16}(K(A,6);\mathbb{Z})$ by the unique nontrivial transformation from $\Gamma^2_{\mathbb{F}_2}(A/2)$ to $\Gamma^2_{\mathbb{Z}}(A/2)$; in other words, the composite map

$$ \Gamma^2_{\mathbb{F}_2}(A/2) \xrightarrow{V} A/2 \xrightarrow{F} S^2(A/2) \longrightarrow \Gamma^2_{\mathbb{Z}}(A/2), $$

where the maps $V$ and $F$ are the Verschiebung and Frobenius maps, respectively. Since the image of this map is decomposable in $\Gamma^2_{\mathbb{Z}}(A/2)$, it follows that an additional suspension sends this image to zero in $H_{17}(K(A,7);\mathbb{Z})$. In particular, this functor $\Gamma^2_{\mathbb{F}_2}(A/2)$ does not produce an additional stable summand $A/2$ in $H^{\mathrm{st}}_{10}(A;\mathbb{Z})$.

# Appendix C: The derived functors of $\Gamma^4(A,n)$ for $A$ free and $1 \le n \le 4$

Table 6 provides a complete description of the derived functors of $\Gamma^4(A,n)$ for $n \le 4$. In particular, the functors $L_{n+i}\Gamma^4(A,n)$ are trivial for $1 \le i \le 4$ and $i > 12$.

| $i$ \ $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | $A/2^{(2)}$ | $A/2^{(2)}$ | $A/2^{(2)}$ | $A/2^{(2)}$ |
| 1 | $\Lambda^2(A/2^{(1)}) \oplus A\otimes A/3^{(1)}$ | 0 | 0 | 0 |
| 2 | $\Phi^4(A)$ | $\Gamma^2(A/2^{(1)}) \oplus A\otimes A/3^{(1)}$ | $A/2^{(2)}$ | $A/2^{(2)}$ |
| 3 | $\Lambda^4(A)$ | $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ | $\Lambda^2(A/2^{(1)})\oplus A/2^{(2)} \oplus A\otimes A/3^{(1)}$ | $A/2^{(2)}$ |
| 4 | 0 | $\Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)}$ | $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ | $\Gamma^2(A/2^{(1)})\oplus A\otimes A/3^{(1)}$ |
| 5 | 0 | 0 | $A/2^{(1)}\otimes A/2^{(1)}$ | $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ |
| 6 | 0 | $\Gamma^4(A)$ | $\Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)} \oplus A/2^{(2)}$ | $A/2^{(1)}\otimes A/2^{(1)}\oplus A/2^{(2)}$ |
| 7 | 0 | 0 | $\Lambda^2(A/2^{(1)}) \oplus A\otimes A/3^{(1)}$ | $A/2^{(1)}\otimes A/2^{(1)}$ |
| 8 | 0 | 0 | $\Phi^4(A)$ | $\Gamma^2(A/2^{(1)})\oplus\Gamma^2(A)\otimes A/2^{(1)} \oplus A\otimes A/3^{(1)}$ |
| 9 | 0 | 0 | $\Lambda^4(A)$ | $\Gamma^2_{\mathbb{F}_2}(A/2^{(1)})$ |
| 10 | 0 | 0 | 0 | $\Gamma^2_{\mathbb{F}_2}(A/2)\otimes A/2^{(1)}$ |
| 11 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | $\Gamma^4(A)$ |

Table 6: The derived functors $L_{n+i}\Gamma^4(A,n)$ for $A$ free and $1 \le n \le 4$.

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