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On the Hodge conjecture for q–complete manifolds

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A complex manifold X of dimension n is said to be q–complete for some $q \in \{1,\dots,n\}$ if it admits a smooth exhaustion function whose Levi form has at least $n-q+1$ positive eigenvalues at every point; thus, 1–complete manifolds are Stein manifolds. Such an X is necessarily noncompact and its highest-dimensional a priori nontrivial cohomology group is $H^{n+q-1}(X;\mathbb{Z})$. In this paper we show that if $q<n$, $n+q-1$ is even, and X has finite topology, then every cohomology class in $H^{n+q-1}(X;\mathbb{Z})$ is Poincaré dual to an analytic cycle in X consisting of proper holomorphic images of the ball. This holds in particular for the complement $X=\mathbb{CP}^n\setminus A$ of any complex projective manifold A defined by $q<n$ independent equations. If X has infinite topology, then the same holds for elements of the group $\mathscr{H}^{n+q-1}(X;\mathbb{Z})=\lim_j H^{n+q-1}(M_j;\mathbb{Z})$, where $\{M_j\}_{j\in\mathbb{N}}$ is an exhaustion of X by compact smoothly bounded domains. Finally, we provide an example of a quasi-projective manifold with a cohomology class which is analytic but not algebraic.

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1 Introduction

Every irreducible p–dimensional closed complex subvariety Z in a compact complex manifold X defines an integral homology class $[Z]\in H_{2p}(X;\mathbb{Z})$ (see [5]). A finite linear combination $Z=\sum_j n_j Z_j$ of such subvarieties with integer coefficients is an *analytic cycle* in X, and the corresponding homology class

$$[Z]=\sum_j n_j[Z_j]\in H_{2p}(X;\mathbb{Z})$$

is an *analytic homology class*. A cohomology class $u\in H^{2k}(X;\mathbb{Z})$ is said to be (complex) *analytic* if it is Poincaré dual to an analytic homology class $z\in H_{2p}(X;\mathbb{Z})$ with $p=\dim X-k$. The same notions can be considered with rational coefficients $n_j\in\mathbb{Q}$. If X is compact Kähler then $H^{2k}(X;\mathbb{C})=\bigoplus_{i+j=2k}H^{i,j}(X)$ and the image in $H^{2k}(X;\mathbb{C})$ of any analytic cohomology class belongs to $H^{k,k}(X)$. The *Hodge conjecture* [37] states that every rational class $u\in H^{2k}(X;\mathbb{Q})\cap H^{k,k}(X)$ of

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type (k,k) of a compact projective manifold is analytic, hence algebraic by Chow's theorem. (Hodge's conjecture for integer coefficients fails both in the projective and in the Stein case; see Atiyah and Hirzebruch [5] and Grothendieck [33].) Although the Hodge conjecture spawned a great number of works (see Atiyah and Hirzebruch [5], Buhštaber [9], Cornalba and Griffiths [11], Deligne [12], Demailly [14], Soulé and Voisin [48], Totaro [50] and the references therein), it remains open.

The problem of representing even-dimensional cohomology classes by analytic cycles is very interesting and highly nontrivial also on nonclosed complex manifolds. Assume first that M is a compact complex manifold of dimension n with boundary ∂M. In view of the Poincaré–Lefschetz duality

$$H^k(M;G) \cong H_{2n-k}(M,\partial M;G), \quad k = 0,1,\dots,2n, \tag{1}$$

which holds for every abelian group G (see [42, Theorem 7.7, page 227] or [49, Theorem 20, page 298]), the relevant question is whether every cohomology class in $H^{2k}(M;\mathbb{Z})$ (or in $H^{2k}(M;\mathbb{Q})$) can be represented by a *relative analytic cycle* consisting of closed complex subvarieties of M with boundaries in ∂M.

Our first main result is the following. We use the notion of q–completeness due to Grauert [29; 30]; see below for the precise definition. The group $H^{n+q-1}(M;\mathbb{Z})$ appearing in Theorem 1.1 is the top-dimensional a priori nontrivial cohomology group of a q–complete manifold (see Equation (2)).

Theorem 1.1 *Let X be a complex manifold of dimension $n > 1$ and $M \subset X$ be a compact q–complete domain for some $q \in \{1,\dots,n-1\}$. If the number $n+q-1$ is even, then every cohomology class in $H^{n+q-1}(M;\mathbb{Z})$ is Poincaré dual to an analytic cycle $Z = \sum_j n_j Z_j$ of complex dimension $p = (n-q+1)/2$ with integer coefficients, where each Z_j is an embedded complex submanifold of M (immersed with normal crossings if $q = 1$) with smooth boundary $\partial Z_j \subset \partial M$.*

If we allow the components Z_j of the cycle to have boundaries in a collar around ∂M, or to be proper in the interior $M \backslash \partial M = \mathring{M}$ of M, then the analytic cycle representing a class in $H^{n+q-1}(M;\mathbb{Z})$ can be chosen to consist of holomorphic images of the unit ball in $\mathbb{C}^p$ (see Theorem 1.2 and Corollaries 1.3 and 6.3). We also prove the corresponding result for q–complete manifolds without boundary, possibly with infinite topology; see Theorem 1.4 and Remark 1.6. In the latter case the components of the cycle are properly immersed images of the open ball in $\mathbb{C}^p$. In particular, our analytic cycles are purely transcendental even if X is quasi-projective.

Before proceeding, we recall the notions of q–convexity and q–completeness. (See Grauert [29; 30]. A different version of these properties was introduced by Henkin and Leiterer [36, Definition 4.3]; here we use Grauert's original definitions.)

Let X be a complex manifold of dimension $n > 1$. The *Levi form* of a smooth function $\rho\colon X \to \mathbb{R}$ is the hermitian quadratic form on the holomorphic tangent bundle TX which is given in any local holomorphic coordinates $z = (z_1, \dots, z_n)$ on X by

$$L_\rho(z)(w) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k} w_j \overline{w}_k, \quad w = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

Let $q \in \mathbb{N}$ be an integer. The function $\rho\colon X \to \mathbb{R}$ is said to be *q–convex* on an open set $\Omega \subset X$ if its Levi form L_ρ has at least $n - q + 1$ positive eigenvalues (hence at most $q - 1$ negative or zero eigenvalues) at every point of Ω. The manifold X is *q–convex* if it admits a smooth exhaustion function $\rho\colon X \to \mathbb{R}$ which is q–convex on $X \setminus K$ for some compact set $K \subset X$; X is *q–complete* if ρ can be chosen q–convex on all of X. Note that a 1–convex function is a strongly plurisubharmonic function and a 1–complete manifold is a *Stein manifold* (see Grauert [28]). Every complex manifold of dimension n is trivially q–complete for any $q > n$, and is n–complete if it has no compact connected components (see Ohsawa [44] and Demailly [13]). In particular, manifolds considered in this paper are never compact. A closed complex submanifold Y of a q–convex (resp. q–complete) complex manifold X is also q–convex (resp. q–complete). It follows that a q–complete manifold does not contain any compact complex submanifold of dimension $\geq q$ and this bound is sharp in general (see Example 2 in Section 7).

A compact domain $M \subset X$ with smooth boundary $\partial M = M \setminus \mathring{M}$ is said to be q–complete if $M = \{\rho \leq 0\}$, where ρ is a q–convex function on a neighborhood of M with $d\rho \neq 0$ on $\partial M = \{\rho = 0\}$. If $\rho\colon X \to \mathbb{R}$ is a q–convex exhaustion function then for any regular value c of ρ the sublevel set $\{\rho \leq c\}$ is a q–complete domain in X. A 1–complete domain is a Stein domain with strongly pseudoconvex boundary.

The most interesting examples of q–convex manifolds for $q > 1$ arise as complements of complex subvarieties. For instance, the complement $\mathbb{CP}^n \setminus A$ of any compact projective submanifold $A \subset \mathbb{CP}^n$ of complex codimension q is q–convex (see Barth [7]). The same holds for the complement of any compact complex submanifold with Griffiths positive normal bundle in an arbitrary compact complex manifold (see Schneider [46]). In general the complement $\mathbb{CP}^n \setminus A$ is not q–complete but is $(2q-1)$–complete (see Peternell [45]); if however A is defined by q global equations in $\mathbb{CP}^n$ (ie a complete intersection) then $\mathbb{CP}^n \setminus A$ is q–complete. For example, $\mathbb{CP}^n \setminus \mathbb{CP}^{n-q}$ is q–complete for any pair of integers $1 \leq q \leq n$. More generally, if Y is a compact complex manifold, $L \to Y$ is a positive holomorphic line bundle, and $s_1, \dots, s_q\colon Y \to L$ are holomorphic sections whose common zero set $A = \{s_1 = 0, \dots, s_q = 0\}$ has codimension q in Y, then $Y \setminus A$ is q–complete (see Andreotti and Norguet [3]). A more complete list of examples and properties of q–convex and q–complete manifolds can be found in the surveys by Colţoiu [10] and Grauert [30].

The group $H^{n+q-1}(M;\mathbb{Z})$ appearing in Theorem 1.1 is the top-dimensional a priori nontrivial cohomology group of a q–complete manifold. Indeed, q–convexity of a function is a stable property in the fine $\mathscr{C}^2$ Whitney topology, so every q–complete manifold admits a q–convex *Morse* exhaustion function. The Morse index of any critical point of such a function is $\leq n+q-1$ (see Forstnerič [25, page 91] for the quadratic normal form), so it follows from Morse theory that a q–complete manifold M of dimension n is a handlebody with handles of indices at most $n+q-1$. In particular we have

$$H^k(M;G)=0 \qquad \forall k>n+q-1, \tag{2}$$

for any abelian group G.

Before discussing further results, we compare Theorem 1.1 with the known results in the literature, indicate why the standard proofs do not apply in our situation, and outline the method that we introduce to address this problem.

There are relatively few results concerning the Hodge conjecture for noncompact manifolds. For a Stein manifold X the Hodge conjecture holds for all cohomology groups $H^{2k}(X;\mathbb{Q})$ with rational coefficients, but it fails in general for integer coefficients; see Atiyah and Hirzebruch [5], Buhštaber [9], and Cornalba and Griffiths [11]. (It follows from Oka's theorem on complex line bundles that the Hodge conjecture holds for the lowest-dimensional integral cohomology group $H^2(X;\mathbb{Z})$; see Kodaira and Spencer [41].) Atiyah and Hirzebruch [4] showed that on any complex manifold X, a necessary condition for a cohomology class $z\in H^{2k}(X;\mathbb{Z})$ to be analytic is that z lie in the kernels of all differentials of the Atiyah–Hirzebruch spectral sequence associated to X. Any given cohomology class z satisfies this necessary condition after multiplication by some integer $N(z)\in\mathbb{N}$. An explicit expression for $N(z)$ was computed by Buhštaber [9]. It follows from his result that if z belongs to the top-dimensional a priori nontrivial cohomology group of a q–complete manifold, then $N(z)=1$. This explains why in the present article we do not need to consider cohomologies with rational coefficients.

On a Stein manifold the necessary condition of Atiyah and Hirzebruch is also sufficient (see Cornalba and Griffiths [11] for a detailed proof). Our results show that the same is true for the top-dimensional cohomology group of a q–complete manifold. However, this necessary condition is not sufficient in general. Examples to this effect, for degree 4 Hodge classes on certain projective hypersurfaces in $\mathbb{CP}^4$, were provided by Kollár [6, Lemma, page 134]. (See also the papers by Totaro [50] and Soulé and Voisin [48, Section 2] for more information.) Kollár's example, together with Theorem 1.4, provides an example of a quasi-projective threefold X with a *torsion* cohomology class in $H^4(X;\mathbb{Z})$ which is analytic (represented by a cycle consisting of properly embedded holomorphic discs) but is not algebraic (see Example 5 in Section 7).

Differentials of the Atiyah–Hirzebruch spectral sequence annihilate any non-torsion element of a cohomology group. From this point of view it is natural to give an example of a q–complete manifold X with a nontrivial torsion part in the top cohomology group $H^{n+q-1}(X;\mathbb{Z})$. Here is an example with $n=2$ and $q=1$, ie a Stein surface. This is a special case of the examples provided by Proposition 7.1.

Example 1 Consider a smooth complex curve C in $\mathbb{CP}^2$ of degree $d>1$ and genus $g=(d-1)(d-2)/2$. Its complement $X=\mathbb{CP}^2\setminus C$ is a Stein surface and we have that

$$H^2(X;\mathbb{Z})=\mathbb{Z}_d\oplus\mathbb{Z}^{\beta_1}=\mathbb{Z}_d\oplus\mathbb{Z}^{2g},$$

where $\beta_1=2-\chi(C)=2g$ is the first Betti number of C and $\mathbb{Z}_d=\mathbb{Z}/d\mathbb{Z}$.

By contrast, the top absolute homology group $H_{n+q-1}(X;\mathbb{Z})$ is always free. In the Stein case this is a classical result of Andreotti and Frankel [1]; see also Andreotti and Narasimhan [2] and Hamm [34] for Stein spaces. The result was generalized to the q–complete case by Sorani [47] and Hamm [35].

The proofs of the Hodge Conjecture on Stein manifolds, given in [5; 9; 11], proceed by representing even-dimensional cohomology classes by Chern classes of complex vector bundles. The Oka–Grauert principle [27] implies that every complex vector bundle over a Stein manifold admits a compatible holomorphic vector bundle structure. (See also [25].) The zero set of a generically chosen holomorphic section of such a bundle is an analytic cycle that is Poincaré dual to the Chern class of the bundle. A similar approach is used on compact Kähler manifolds with ample holomorphic line bundles. Analytic cycles obtained in this way are given by holomorphic equations, so one has no information on the complex structure of their irreducible components.

The Oka–Grauert principle fails in general on q–convex manifolds for $q>1$. In the present paper we introduce a completely different method which relies on the technique of constructing proper holomorphic maps, immersions and embeddings of strongly pseudoconvex Stein domains to q–complete manifolds, due to Drinovec Drnovšek and Forstnerič [19; 21] (see Theorem 3.3 in Section 3). We work with holomorphic maps from specific strongly pseudoconvex domains into M, inductively stretching their boundaries towards the boundary ∂M. This technique does not rely on the function theory of the target manifold, but only of the source strongly pseudoconvex domain where solutions of the $\overline{\partial}$–problem are readily available. This allows us to keep the same complex structure on these domains during the entire process. At every critical point of maximal index $n+q-1$ of the exhaustion function on M, a new component of the analytic cycle may appear, given as a cross-section of the corresponding handle (see Proposition 2.1). This cross-section can be realized in local holomorphic coordinates as the intersection of a $\mathbb{C}$–linear subspace $L \subset \mathbb{C}^n$ of complex dimension $p = (n-q+1)/2$ with a thin round tube around the core E of the handle. Such an intersection is an ellipsoid in $\mathbb{C}^p$ (the image of the unit ball in $\mathbb{C}^p$ under an $\mathbb{R}$–linear automorphism). However, it turns out that the precise choice of a domain in L will not be important as long as it is small enough, strongly pseudoconvex, and it contains the intersection point $E \cap L$. In particular, we are free to choose a ball in $L \cong \mathbb{C}^p$. This leads to analytic cycles (representing a given cohomology class) whose irreducible components are embedded or immersed copies of the ball $\mathbb{B}^p \subset \mathbb{C}^p$.

Due to rigidity phenomena in Cauchy–Riemann geometry one can not push the boundaries of immersed balls in our cycle exactly into the boundary of M. We choose instead an interior collar $A \subset M$ around the boundary ∂M; that is, a compact neighborhood of ∂M in M, homeomorphic to $\partial M \times [0,1]$, with $\partial M = \partial M \times \{0\} \subset \mathring{A}$. Then the complement $N = \overline{M \setminus A}$ is a compact manifold with boundary in $\mathring{M}$ that is homeomorphic to M. Since the inclusions $N \hookrightarrow M$ and $\partial M \hookrightarrow A$ are homotopy equivalences, they induce isomorphisms

$$H^k(M;G) \cong H^k(N;G), \quad H_k(M,\partial M;G) \cong H_k(M,A;G) \tag{3}$$

for any abelian group G (see Lemma 2.5). By Poincaré–Lefschetz duality we have

$$H^k(N;G) \cong H_{2n-k}(M, M \setminus N; G) \tag{4}$$

(see [42, Proposition 6.4, page 221]). From (3) and (4) it follows that

$$H^k(M;G) \cong H_{2n-k}(M,A;G);$$

that is, cohomology classes of M can be represented by cycles with boundaries in a collar around ∂M. In light of this, we have the following version of Theorem 1.1.

Theorem 1.2 *Let M be a compact q–complete domain in a complex manifold X of dimension n, where $q \in \{1,\dots,n-1\}$, and let $A \subset M$ be a collar around ∂M. If the number $n+q-1$ is even, then every class in $H^{n+q-1}(M;\mathbb{Z})$ is represented by a finite analytic cycle $Z = \sum_j n_j Z_j$ of complex dimension $p = (n-q+1)/2$ with integer coefficients, where each Z_j is an embedded complex submanifold of M with smooth boundary $\partial Z_j \subset A$ (immersed with normal crossings if $q = 1$) that is biholomorphic to the ball $\mathbb{B}^p \subset \mathbb{C}^p$.*

When $n = 2$ and $q = 1$ (that is, when M is a strongly pseudoconvex domain in a Stein surface), we get cycles consisting of analytic discs, ie holomorphic images of the unit disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Since one can always push the boundary of a holomorphic disc into the boundary of a strongly pseudoconvex domain (see eg Forstnerič and Globevnik [26]), we obtain the following corollary to Theorem 1.2. (See also Jöricke [39], especially Corollary 3 on page 78.)

Corollary 1.3 *If M is a strongly pseudoconvex Stein domain of dimension 2 then every class in $H^2(M;\mathbb{Z})$ is represented by an analytic cycle whose irreducible components are properly immersed discs with normal crossings that are smooth up to the boundary.*

We now explain a version of Theorem 1.1 for q–complete manifolds X without boundary, possibly with infinite topology.

Let $\rho\colon X \to \mathbb{R}$ be a q–convex Morse exhaustion function. Choose an exhaustion $M_1 \subset M_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} M_j = X$, where each $M_j = \{x \in X : \rho(x) \le c_j\}$ is a regular sublevel set and there is at most one critical point of ρ in each difference $\mathring{M}_j \setminus M_{j-1}$. The inclusion $M_j \hookrightarrow M_{j+1}$ induces a homomorphism $H^k(M_{j+1};G) \to H^k(M_j;G)$ and we have a well defined inverse limit

$$\mathcal{H}^k(X;G) = \lim_{j} H^k(M_j;G). \tag{5}$$

This definition is due to Atiyah and Hirzebruch [5]; see Section 5 for details. There is a natural surjective homomorphism $H^k(X;G) \to \mathcal{H}^k(X;G)$ from the singular cohomology whose kernel can be described by means of the first derived functor of the inverse limit. This kernel is trivial when G is a field (eg when $G = \mathbb{Q}$), and also when $G = \mathbb{Z}$ and the homology group $H_{k-1}(X)$ is not too bad (see Section 6).

An irreducible closed subvariety Z of dimension p in X with small singular locus defines a cohomology class in $H^{2n-2p}(X;\mathbb{Z})$. (If Z is smooth, this is the Thom class of Z in X made absolute; see Section 5 for the general case.) In this setting we say that the corresponding cohomology class in $\mathcal{H}^{2n-2p}(X;\mathbb{Z})$ is an analytic class represented by Z.

Theorem 1.4 *Let X be a complex manifold of dimension $n > 1$ which is q–complete for some $q \in \{1,\dots,n-1\}$. If the number $n+q-1 \ge 2$ is even then every class in $\mathcal{H}^{n+q-1}(X;\mathbb{Z})$ is represented by an analytic cycle $\sum_r n_r Z_r$, where each Z_r is a properly embedded (immersed with normal crossings if $q = 1$) complex submanifold of X biholomorphic to the ball of dimension $p = (n-q+1)/2$.*

Remark 1.5 The cycle $\sum_r n_r Z_r$ in Theorem 1.4 is infinite (but locally finite) in general; it can be chosen finite if X admits a q–convex exhaustion function with finitely many critical points. If $H_{n+q-2}(X;\mathbb{Z})$ is the direct sum of a free abelian group and a torsion abelian group (for example, if it is finitely generated), then the natural morphism $H^{n+q-1}(X;\mathbb{Z}) \to \mathcal{H}^{n+q-1}(X;\mathbb{Z})$ is an isomorphism (cf Proposition 6.2) and hence Theorem 1.4 applies to every cohomology class in $H^{n+q-1}(X;\mathbb{Z})$.

Remark 1.6 Our proof will show that, in the absence of critical points of index $> m$ for some even integer $0 \le m \le n+q-1$, Theorems 1.1, 1.2, and 1.4 hold for the top-dimensional nontrivial cohomology group H^m. In particular, if X is an odd-dimensional Stein manifold which is subcritical, in the sense that it admits a strongly plurisubharmonic exhaustion function $\rho\colon X \to \mathbb{R}$ without critical points of index $n = \dim X$, then Theorem 1.4 holds for the group $H^{n-1}(X;\mathbb{Z})$.

Our method also applies to analytic cycles of lower dimension, but these are Poincaré dual to higher-dimensional cohomology classes which are trivial in view of (2). On the other hand, it does not work for analytic cycles of real dimension $> n-q+1$, and hence for cohomology groups of dimension $< n+q-1$, because we are unable to push the boundaries of such cycles across the critical points of index $n+q-1$ of a q–convex exhaustion function. This difficulty is not only apparent as shown by examples in [21], and is further demonstrated by the fact that even in the Stein case ($q=1$) the analogues of Theorems 1.1 and 1.4 for integer coefficients fail in general according to Buhštaber [9]. It is a challenging problem to give a proof of the corresponding results from [9; 11] for lower-dimensional cohomology groups with rational coefficients avoiding the use of Chern classes and the Oka–Grauert principle. By finding such a proof one might hope to answer the following question.

Problem 1.7 *Assume that X is an n–dimensional complex manifold which is q–complete for some $q \in \{1,\dots,n-1\}$. Does the Hodge conjecture hold for the cohomology groups $H^{2k}(X;\mathbb{Q})$ when $2 \le 2k < n+q-1$?*

The analogue of the Hodge conjecture has also been considered in the category of symplectic manifolds. Donaldson [16; 15] constructed symplectic submanifolds of any even codimension in a given compact symplectic manifold (X,ω) of dimension $2n \ge 4$. In particular, he showed that if the cohomology class $[\omega/2\pi] \in H^2(X;\mathbb{R})$ admits a lift to an integral class $h \in H^2(X;\mathbb{Z})$, then for any sufficiently large integer $k \in \mathbb{N}$ the Poincaré dual of kh in $H_{2n-2}(X;\mathbb{Z})$ can be represented by a compact symplectic submanifold of real codimension two in X.

Problem 1.8 *Is it possible to represent certain cohomology classes of noncompact symplectic manifolds by noncompact cycles consisting of proper symplectic submanifolds?*

2 Topological preliminaries

This section reviews the necessary topological background. In particular, Proposition 2.1 gives a precise description of the effect of a handle attachment on the relative homology (and, by Poincaré–Lefschetz duality, on the cohomology) of a compact manifold with boundary. Although this is standard, we need very precise geometric information on the cycles generating the relative homology group. For this reason, and lacking a precise reference, we provide a detailed proof.

Denote by $\mathbb{B}^k$ the closed ball in $\mathbb{R}^k$ and by $\mathbb{S}^{k-1} = \partial\mathbb{B}^k$ the $(k-1)$–sphere. Let M be an orientable closed n–manifold with boundary ∂M, and let

$$\phi\colon \mathbb{S}^{k-1} \times \mathbb{B}^{n-k} \hookrightarrow \partial M \tag{6}$$

be the attaching map of a k–handle $H = \mathbb{B}^k \times \mathbb{B}^{n-k}$. We assume that ϕ is a homeomorphism onto its image. Set $N = M \cup_\phi H$; this is a compact manifold with boundary

$$\partial N = (\partial M \setminus \operatorname{im}\phi) \cup (\mathbb{B}^k \times \mathbb{S}^{n-k-1}).$$

Proposition 2.1 *Assume that $(M, \partial M)$ is a compact manifold with boundary, and let N be obtained by adding a handle of index k to M by an attaching map ϕ as in* (6). *Assume that for some $j \in \{1, \dots, n-k\}$ the group $H_j(M, \partial M)$ can be realized by a collection $\mathscr{C}$ of geometric j–cycles in $(M, \overline{\partial M \setminus \operatorname{im}\phi})$. Then $H_j(N, \partial N)$ can be realized by the cycles in $\mathscr{C}$ prolonged by inclusion, and, if $j = n-k$, possibly by an additional relative disc in any fiber of H (viewed as a j–disc bundle over $\mathbb{B}^k$).*

Before proving the proposition we recall some preliminary material.

Given pairs of topological spaces $A \subset M$ and $B \subset N$, the notation $f\colon (M, A) \to (N, B)$ means that $f\colon M \to N$ is a continuous map satisfying $f(A) \subset B$. A similar notation is used for maps of triads; thus $f\colon (M; A_1, A_2) \to (N; B_1, B_2)$ means that $f\colon M \to N$ is a continuous map satisfying $f(A_i) \subset B_i$ for $i = 1, 2$.

By $\frown$ and $\smile$ we denote the cap and the cup product on cohomology, respectively. For general background on homology and cohomology we refer to Bredon [8] or to Spanier [49].

We recall the following naturality property of the cap product (see Spanier [49, Section 5.6.16]). This holds for an arbitrary coefficient group which we omit from the notation.

Lemma 2.2 *Let $f\colon M \to N$ map a subset $A_1 \subset M$ to B_1 and $A_2 \subset M$ to B_2, that is, $f\colon (M; A_1, A_2) \to (N; B_1, B_2)$ is a map of triads. Let $u \in H^q(N, B_1)$ and $z \in H_n(M, A_1 \cup A_2)$. Let $f_1\colon (M, A_1) \to (N, B_1)$, $f_2\colon (M, A_2) \to (N, B_2)$ and $\bar{f}\colon (M, A_1 \cup A_2) \to (N, B_1 \cup B_2)$ be maps defined by f. Then the relation*

$$f_{2*}\left(f_1^* u \frown z\right) = u \frown \bar{f}_* z$$

holds in $H_{n-q}(N, B_2)$.

In other words, Lemma 2.2 renders the following diagram commutative:

$$\begin{tikzcd}
H^q(N, B_1) \arrow[r, "f_1^*"] \arrow[d, "u \mapsto u \frown \bar{f}_*(z)"'] & H^q(M, A_1) \arrow[d, "v \mapsto v \frown z"] \\
H_{n-q}(N, B_2) & H_{n-q}(M, A_2) \arrow[l, "f_{2*}"']
\end{tikzcd}$$

Let $f\colon (M; \operatorname{im}\phi, \overline{\partial M \setminus \operatorname{im}\phi}) \to (N; H, \partial N)$ and $J\colon (N; \varnothing, \partial N) \to (N; H, \partial N)$ be inclusions of triads, and let $\bar{f}\colon (M, \partial M) \to (N, H \cup \partial N)$ and $\bar{J}\colon (N, \partial N) \to (N, H \cup \partial N)$ be the associated maps of pairs. We choose compatible orientation classes $\mu \in H_n(M, \partial M)$ and $\nu \in H_n(N, \partial N)$ in the sense that $\bar{f}_*(\mu) = \bar{J}_*(\nu) \in H_n(N, H \cup \partial N)$. Consider the following diagram where the horizontal arrows are induced by inclusions and the vertical arrows by cap-products:

$$\begin{tikzcd}
H^q(N) \arrow[d, "\frown \nu"] & H^q(N, H) \arrow[l] \arrow[r] \arrow[d, "\frown \bar{J}_*(\nu)"] & H^q(M, \operatorname{im}\phi) \arrow[r, "i^*"] \arrow[d, "\frown \mu"] & H^q(M) \arrow[d, "\frown \mu"'] \\
H_{n-q}(N, \partial N) \arrow[r] & H_{n-q}(N, \partial N) & H_{n-q}(M, \overline{\partial M \setminus \operatorname{im}\phi}) \arrow[l] \arrow[r, "\iota_*"] & H_{n-q}(M, \partial M)
\end{tikzcd}$$

Lemma 2.3 *For any q, the above diagram is commutative. All the arrows, with the possible exception of i^* and ι_*, are isomorphisms when $q > 0$.*

Proof Lemma 2.2 implies that the left square is commutative by virtue of the map of triads J, and that the middle square is commutative by virtue of f. The right square is induced by Poincaré–Lefschetz duality for the decomposition $\partial M = \overline{\partial M \setminus \operatorname{im}\phi} \cup \operatorname{im}\phi$ (see [8, Section VI.9, Problems]); all cap-products with μ and ν are duality isomorphisms. The arrow $H^q(N, H) \to H^q(N)$ is an isomorphism when $q > 0$ because

H is contractible. By commutativity it follows first that also the cap-product with $\overline{f}_*(\mu) = \overline{J}_*(\nu)$ is an isomorphism, and second that the central bottom horizontal arrow is an isomorphism. Finally, the arrow $H^q(N, H) \to H^q(M, \operatorname{im}\phi)$ is an isomorphism because the quotients N/H and $M/\operatorname{im}\phi$ are evidently homeomorphic. □

Proof of Proposition 2.1 The morphism ι_* in the above diagram sits in the exact sequence of the triad $(M; \operatorname{im}\phi, \overline{\partial M \setminus \operatorname{im}\phi})$ as follows:

$$(7)\qquad H_j(\operatorname{im}\phi, \partial(\operatorname{im}\phi)) \to H_j(M, \overline{\partial M \setminus \operatorname{im}\phi}) \xrightarrow{\iota_*} H_j(M, \partial M) \\ \to H_{j-1}(\operatorname{im}\phi, \partial(\operatorname{im}\phi)).$$

Here, the pair $(\operatorname{im}\phi, \partial(\operatorname{im}\phi))$ is seen to be the product of pairs $\mathbb{S}^{k-1} \times (\mathbb{B}^{n-k}, \mathbb{S}^{n-k-1})$.

If $j < n-k$, both endgroups in (7) vanish and therefore, ι_* is an isomorphism.

If $j = n-k$, the sequence (7) reads

$$(8)\qquad \mathbb{Z} \to H_j(M, \overline{\partial M \setminus \operatorname{im}\phi}) \xrightarrow{\iota_*} H_j(M, \partial M) \to 0.$$

Here, the group of integers $\mathbb{Z}$ is generated by any relative disc $D_\zeta = (\{\zeta\} \times \mathbb{B}^j, \{\zeta\} \times \mathbb{S}^{j-1})$ where $\zeta \in \mathbb{S}^{n-j-1} = \mathbb{S}^{k-1}$. If ι_* is not an isomorphism, then, clearly, if we add to $\mathscr{C}$ a relative disc D_ζ (which may generate a torsion element in $H_j(M, \overline{\partial M \setminus \operatorname{im}\phi}) \cong H_j(N, \partial N)$), we obtain a collection of geometric generators for $H_j(M, \overline{\partial M \setminus \operatorname{im}\phi})$ which can be prolonged by inclusion to a set of geometric generators for $H_j(N, \partial N)$. We remark that in $H_j(N, \partial N)$, the relative discs D_ζ are homologous to the relative discs in the fibers of the handle H. □

Remark 2.4 The assumption in Proposition 2.1, that the group $H_j(M, \partial M)$ can be realized by a collection of geometric j–cycles in $(M, \overline{\partial M \setminus \operatorname{im}\phi})$, is always satisfied when $j \le n-k$. Indeed, the boundary sphere $\mathbb{S}^{k-1} = \partial\mathbb{B}^k$ of the core k–disc $\mathbb{B}^k$ of the handle has dimension $k-1$, while the boundaries of relative cycles representing the homology classes in $H_j(M, \partial M)$ have dimension $j-1$. We can choose these cycles so that only their boundaries intersect ∂M. Since $(j-1)+(k-1) \le (n-k-1)+(k-1) < n-1 = \dim \partial M$, a general position argument shows that the (finitely many) generators of $H_j(M, \partial M)$ can be represented by relative cycles in $(M, \partial M)$ whose boundaries avoid the attaching sphere $\phi(\mathbb{S}^{k-1}) \subset \partial M$. By choosing the transverse disc of the handle to be sufficiently thin we can ensure that these cycles avoid the attaching set $\operatorname{im}\phi$ of the handle.

Lemma 2.5 *Let M be a manifold and $A \subset M$ a collar of the boundary ∂M. Then the inclusion-induced morphisms $H_q(M, \partial M) \to H_q(M, A)$ are isomorphisms for all q.*

Proof Since the inclusion $\partial M \hookrightarrow A$ is a homotopy equivalence, this follows from naturality of the long homology exact sequence of the pair in conjunction with the 5–lemma. □

Let (X, A) be a pair of topological spaces. In the following proposition we use $C_p(X, A)$ for the group of relative (singular) p–chains with integer coefficients. Let K be an oriented compact p–manifold in X with $\partial K \subset A$. There is an orientation p–chain in $C_p(K, \partial K)$ which may be viewed also as a p–chain in $C_p(X, A)$. We denote the latter p–chain simply by $(K, \partial K)$. We say that the corresponding element in $H_p(X, A)$ is *represented* by $(K, \partial K)$.

Proposition 2.6 *Let M be an oriented manifold with boundary ∂M. Assume that the manifold M' is obtained from M by adding an exterior collar $A \cong \partial M \times [0, 1]$, so that $\partial M \equiv \partial M \times \{0\}$ and $\partial M' = \partial M \times \{1\}$. Let K be an oriented compact p–manifold in M' whose boundary ∂K is contained in $\partial M \times (0, 1]$. Furthermore, assume that K meets ∂M transversely. Then the geometric intersection of K with M yields a finite collection $\{L_i\}$ of connected oriented p–manifolds with $\partial L_i \subset \partial M$ so that the sum $\sum_i (L_i, \partial L_i)$ is a chain in $C_p(M, \partial M)$, homologous to $(K, \partial K)$ in $C_p(M', A)$. Consequently, if an element z' of $H_p(M', A)$ is represented by a linear combination of p–manifolds satisfying the conditions on K, there is an element z of $H_p(M, \partial M)$ which is also represented by a linear combination of p–manifolds and is mapped to z' under the isomorphism $H_p(M, \partial M) \to H_p(M', A)$.*

Proof As the compact manifold K meets ∂M transversely (and $\partial K \cap \partial M = \varnothing$), the intersection $K \cap \partial M$ is the union of a finite collection $\mathscr{C}$ of closed $(p-1)$–manifolds. Note that the connected components of $K' = K \setminus \partial M$ are contained either in M or in A. Their closures are connected submanifolds whose boundary components are in $\mathscr{C}$. Let $\{L_i\}$ denote the collection of closures of components of K' that are contained in M, and let $\{L'_j\}$ denote the collection of closures of components of K' that are contained in A. Assume that all are oriented compatibly with K. Tautologically, $\sum_i (L_i, \partial L_i)$ and $\sum_j (L'_j, \partial L'_j)$ can be viewed as chains in $C_p(M', A)$ whose sum is homologous to $(K, \partial K)$. On the other hand, as the L'_j are entirely in A, the chain $\sum_j (L'_j, \partial L'_j)$ is clearly nullhomologous in $H_p(M', A)$. Hence $(K, \partial K)$ is homologous to $\sum_i (L_i, \partial L_i)$, and as the latter forms a chain in $C_p(M, \partial M)$, the assertion has been proved. □

3 Analytic and geometric preliminaries

We adopt the usual convention that a map is holomorphic on a closed subset of a complex manifold if it is holomorphic on an open neighborhood of that set. When talking about a homotopy of such maps, it is understood that the neighborhood is independent of the parameter. By a *compact domain* (in a manifold) we shall always mean a compact set with smooth boundary.

We shall need the following transversality theorem, which follows easily from known results.

Theorem 3.1 *Assume that V is a compact strongly pseudoconvex domain with $\mathscr{C}^2$ boundary in a Stein manifold S, X is a complex manifold, and $M \subset X$ is a smooth submanifold. Let $r \in \{0, 1, 2, \dots, \infty\}$. Every map $f\colon V \to X$ of class $\mathscr{C}^r$ which is holomorphic in the interior $\mathring{V}$ can be approximated arbitrarily closely in the $\mathscr{C}^r(V, X)$ topology by holomorphic maps $\widetilde{f}\colon \widetilde{V} \to X$, defined on an open neighborhood $\widetilde{V} \subset S$ of V (depending on $\widetilde{f}$), such that $\widetilde{f}$ is transverse to M.*

Proof By [20, Theorem 1.2] we can approximate the map $f\colon V \to X$ arbitrarily closely in the $\mathscr{C}^r(V, X)$ topology by a holomorphic map $f_1\colon \Omega \to X$ on an open Stein neighborhood $\Omega \subset S$ of V. Pick a compact $\mathcal{O}(\Omega)$–convex subset $K \subset \Omega$ with $V \subset \mathring{K}$. By Kaliman and Zaidenberg [40] we can approximate f_1 as closely as desired, uniformly on K, by a holomorphic map $\widetilde{f}\colon \widetilde{V} \to X$ on an open neighborhood $\widetilde{V}$ of K such that $\widetilde{f}$ is transverse to M. (See also [25, Theorem 7.8.12, page 321]. In the cited sources this transversality theorem is stated for the case when M is a complex submanifold, or a Whitney stratified complex subvariety of X, but the proofs also apply to smooth submanifolds; see [51].) □

Remark 3.2 We also have the corresponding jet transversality theorem: the map $\widetilde{f}$ in Theorem 3.1 can be chosen such that its r–jet extension $j^r f\colon \widetilde{V} \to \mathscr{J}^r(\widetilde{V}, X)$ is transverse to a given smooth submanifold M of the complex manifold $\mathscr{J}^r(\widetilde{V}, X)$ of r–jets of holomorphic maps $\widetilde{V} \to X$. If X is an Oka manifold, then the jet transversality theorem holds for holomorphic maps $S \to X$ from an arbitrary Stein manifold S to X; see [24] or [25, Sec. 7.8]. We shall not need these additions.

Given a function $\rho\colon X \to \mathbb{R}$, we shall use the notation

$$X_{\rho<a} = \{\boldsymbol{x} \in X : \rho(\boldsymbol{x}) < a\}, \qquad X_{\rho>a} = \{\boldsymbol{x} \in X : \rho(\boldsymbol{x}) > a\},$$
$$X_{a<\rho<b} = \{\boldsymbol{x} \in X : a < \rho(\boldsymbol{x}) < b\}, \quad X_{\rho=a} = \{\boldsymbol{x} \in X : \rho(\boldsymbol{x}) = a\},$$

and similarly for the weak inequalities. If ρ is an exhaustion function on X then the sets $X_{\rho\le b}$ and $X_{a\le\rho\le b} = X_{\rho\le b} \setminus X_{\rho<a}$ are compact for every pair $a, b \in \mathbb{R}$.

The following is the main result of the paper [21], stated in the precise way that suits our present purposes. (The special case when V is a bordered Riemann surface is treated in [19]. The case when X is a domain in $\mathbb{C}^n$ is due to Dor [17].) This is the key analytic ingredient in the proof of our main results.

Theorem 3.3 [21, Theorem 1.1] *Let X be a complex manifold of dimension $n > 1$, $\rho\colon X \to \mathbb{R}$ a smooth exhaustion function, and* dist *a distance function on X inducing the manifold topology. Assume that for some pair of numbers $a < b$, the restriction of ρ to $X_{a\le\rho\le b}$ is a q–convex Morse function. Let V be a compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold S of dimension $p = \dim S$, and let $f_0\colon V \to X$ be a holomorphic map such that $f_0(V) \subset X_{\rho<b}$ and $f_0(\partial V) \subset X_{a<\rho<b}$ (see Figure 1). Suppose that at least one of the following two conditions holds:*

(a) $r := n - q + 1 \ge 2p$.

(b) $r > p$ *and ρ has no critical points of index $> 2(n-p)$ in $X_{a<\rho<b}$.*

Given a compact set $K \subset \mathring{V}$ and numbers $\gamma \in (a,b)$ and $\epsilon > 0$, there is a homotopy $f_t\colon V \to X$ $(t \in [0,1])$ of holomorphic maps satisfying the following properties:

(i) $f_t(V) \subset X_{\rho<b}$ *for all $s \in V$ and $t \in [0,1]$.*

(ii) $f_1(\partial V) \subset X_{\gamma<\rho<b}$ *(see Figure 1).*

(iii) $\sup_{s\in K} \operatorname{dist}(f_t(s), f_0(s)) < \epsilon$ *for all $t \in [0,1]$.*

(iv) $\rho(f_t(s)) > \rho(f_0(s)) - \epsilon$ *for all $s \in V$ and $t \in [0,1]$.*

If $2p < n$ then the map f_1 can be chosen to be an embedding, and if $2p = n$ then it can be chosen to be an immersion with simple double points (normal crossings).

If ρ is Morse and q–convex on $X_{\rho>a}$ and either of conditions (a) or (b) holds on $X_{\rho>a}$, then f_0 can be approximated uniformly on compacts in $\mathring{V}$ by proper holomorphic maps $\widetilde{f}\colon \mathring{V} \to X$ (embeddings if $2p < n$, immersions with normal crossings if $2p = n$).

Recall that the number $r = n - q + 1 \le n$, appearing in conditions (a) and (b), is a pointwise lower bound on the number of positive eigenvalues of the Levi form of ρ.

Theorem 3.3 is illustrated in Figure 1 which shows the initial and the final image of the domain V in X. The geometry at a critical point of ρ is illustrated in Figure 3 below which will be explained in the sequel. A few comments about the proof are in order, especially since we shall use not just the result itself, but also some of the key steps in the proof.

In the *noncritical case*, ie when ρ has no critical values in $[a,b]$, Theorem 3.3 holds under the condition that $r > p$. More precisely, the Levi form of ρ must have at least p positive eigenvalues in directions tangent to the level sets of ρ; the radial direction

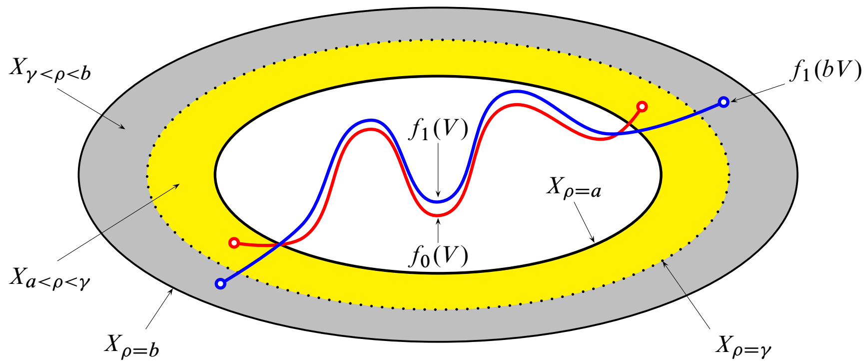

Figure 1: Theorem 3.3.

is irrelevant in this problem. The construction of a new map f_1 with boundary in $X_{\gamma<\rho<b}$ is achieved in finitely many steps by successively lifting small portions of the boundary of the image of V in X to higher level sets of ρ (see property (ii) in Theorem 3.3), paying attention to remain within $X_{\rho<b}$ (property (i)) and not to drop very much anywhere (property (iv)). Every step of the deformation is first carried out in a local chart on X by pushing the image in a suitable p–dimensional direction, tangent to the level set of ρ, on which the Levi form of ρ is strictly positive. Special holomorphic peaking functions constructed by Dor [17] are used for this purpose. The deformation is globalized by the method of gluing holomorphic sprays, developed in [19].

The second part of condition (b) is used in the following way. When trying to push the (image in X of the) boundary of V across a critical point of index k of ρ, we must be able to ensure that $f(\partial V)$ (which is of real dimension $2p-1$) avoids the real k–dimensional stable manifold of the critical point. By a general position argument this is possible if $(2p-1)+k<2n$, which is equivalent to $k\le 2(n-p)$.

Condition (a), that $n-q+1\ge 2p$, is equivalent to $n+q-1=2n-(n-q+1)\le 2(n-p)$. Since Morse indices of a q–convex function are $\le n+q-1$, this implies condition (b).

The last statement in Theorem 3.3 (on the existence of proper maps) follows from the first one by a standard recursive procedure. By combining the general position argument and approximating sufficiently well at every step, we can also ensure that in the limit we obtain a proper holomorphic embedding (if $2p<n$) or immersion (if $2p=n$). All this is very standard and well known at least since Whitney's classical work on immersions and embeddings. In view of [20, Theorem 1.2] the analogous result also holds if the initial map $f_0\colon V\to X$ is merely continuous on V and holomorphic on $\mathring{V}=V\setminus\partial V$.

The reader will need a certain amount of familiarity with the proof of Theorem 3.3. To this end, we recall in some detail the geometry of critical points of q–convex functions. Our main source is [21, Sections 2 and 3], where the reader can find further details. (This is also available in [25, Sections 3.9–3.10].)

By Sard's lemma we may assume that a and b are regular values of ρ. After a small deformation of ρ we may assume that its critical points in $X_{a<\rho<b}$ lie on different level sets. Let $\boldsymbol{x}_0$ be any critical point and k_0 its Morse index; hence $k_0 \le n+q-1$. Set $s=q-1$, so $r+s=n$. By [21, Lemma 2.1, page 9] (or [25, Lemma 3.9.4, page 91]) there exist

(i) a holomorphic coordinate map $z=(\xi,w)\colon U\to\mathbb{C}^r\times\mathbb{C}^s=\mathbb{C}^n$ on an open neighborhood $U\subset X$ of $\boldsymbol{x}_0$, with $z(\boldsymbol{x}_0)=0$,

(ii) an $\mathbb{R}$–linear change of coordinates $\psi(z)=\psi(\xi,w)=(\xi+l(w),g(w))$ on $\mathbb{C}^r\times\mathbb{C}^s$,

(iii) integers $k\in\{0,\dots,r\}$ and $m\in\{0,1,\dots,2s\}$ with $k+m=k_0$, and

(iv) a quadratic q–convex function $\mathbb{C}^n\cong\mathbb{C}^r\times\mathbb{R}^{2s}\to\mathbb{R}$ of the form

$$\widetilde{\rho}(\zeta,u)=-|x'|^2-|u'|^2+|x''|^2+|u''|^2+\sum_{j=1}^{r}\lambda_j y_j^2, \tag{9}$$

where $\zeta=(\zeta',\zeta'')\in\mathbb{C}^k\times\mathbb{C}^{r-k}$, $\zeta'=x'+\mathrm{i}y'\in\mathbb{C}^k$, $\zeta''=x''+\mathrm{i}y''\in\mathbb{C}^{r-k}$, $\lambda_j>1$ for $j=1,\dots,k$, $\lambda_j\ge 1$ for $j=k+1,\dots,r$, and $u=(u',u'')\in\mathbb{R}^m\times\mathbb{R}^{2s-m}$,

such that, setting

$$\phi(\boldsymbol{x})=\psi(z(\boldsymbol{x}))=(\zeta(\boldsymbol{x}),u(\boldsymbol{x}))\in\mathbb{C}^n,\quad \boldsymbol{x}\in U,$$

we have

$$\rho(\boldsymbol{x})=\rho(\boldsymbol{x}_0)+\widetilde{\rho}(\phi(\boldsymbol{x}))+o(|\phi(\boldsymbol{x})|^2),\quad \boldsymbol{x}\in U. \tag{10}$$

The function $\widetilde{\rho}$ of Equation (9) has a critical point of Morse index k_0 at the origin (due to the term $-|x'|^2-|u'|^2$) and no other critical points. For every fixed $u\in\mathbb{R}^{2s}$, the function $\mathbb{C}^r\ni\zeta\mapsto\widetilde{\rho}(\zeta,u)$ is strongly plurisubharmonic, so $\widetilde{\rho}$ is q–convex on $\mathbb{C}^r\times\mathbb{C}^s=\mathbb{C}^n$. Since the $\mathbb{R}$–linear map ψ preserves the foliation $u=\mathit{const}$ and is $\mathbb{C}$–linear on each leaf $\mathbb{C}^r\times\{u\}$, the function $\widetilde{\rho}\circ\psi$ is also q–convex on $\mathbb{C}^n$.

By a small deformation of ρ in a neighborhood of the critical point $\boldsymbol{x}_0$ we may assume that the remainder term in Equation (10) vanishes (cf [21, Lemma 2.1, page 9]); a critical point with this property is said to be *nice*. Hence we may work in the sequel with q–convex functions with nice critical points.

Assume without loss of generality that $\rho(\boldsymbol{x}_0)=0$. By shrinking U around the point $\boldsymbol{x}_0$ we may assume that ϕ maps U into a polydisc $P\subset\mathbb{C}^r\times\mathbb{R}^{2s}$ around the origin. Write

$$\widetilde{\rho}(x+\mathrm{i}y,u)=-|x'|^2-|u'|^2+Q(y,x'',u''),$$

where $\zeta=x+\mathrm{i}y\in\mathbb{C}^r$, $u\in\mathbb{R}^{2s}$ and

$$Q(y,x'',u'')=\sum_{j=1}^{r}\lambda_j y_j^2+|x''|^2+|u''|^2. \tag{11}$$

Pick a number $c_0\in(0,1)$ small enough such that ρ has no critical points other than $\boldsymbol{x}_0$ in the layer $X_{-c_0\le\rho\le 3c_0}\Subset X_{a<\rho<b}$, and we have

$$\{(x+\mathrm{i}y,u)\in\mathbb{C}^r\times\mathbb{R}^{2s}\colon |x'|^2+|u'|^2\le c_0,\ Q(y,x'',u'')\le 4c_0\}\ \subset\ P.$$

Consider the set

$$\widetilde{E}=\{(x+\mathrm{i}y,u)\in\mathbb{C}^r\times\mathbb{R}^{2s}\colon |x'|^2+|u'|^2\le c_0,\ y=0,\ x''=0,\ u''=0\}. \tag{12}$$

Its preimage $E=\phi^{-1}(\widetilde{E})\subset U\subset X$ is an embedded real analytic disc of dimension $k+m=k_0$ (the Morse index of ρ at $\boldsymbol{x}_0$) which is attached to the domain $X_{\rho\le -c_0}$ along the sphere $\mathbb{S}^{k_0-1}\cong\partial E\subset X_{\rho=-c_0}$. (In the metric on U inherited from the Euclidean metric on $\mathbb{C}^n$ by the coordinate map $\phi\colon U\to P\subset\mathbb{C}^n$, E is the stable manifold of $\boldsymbol{x}_0$ for the gradient flow of ρ.)

Let $\lambda=\min\{\lambda_1,\dots,\lambda_k\}>1$. Pick a number t_0 with $0<t_0<(1-1/\lambda)^2c_0$.

By [23, Lemma 6.7, page 178] (or [25, Lemma 3.10.1, page 92]) there exists a smooth convex increasing function $h\colon\mathbb{R}\to[0,+\infty)$ enjoying the following properties:

(i) $h(t)=0$ for $t\le t_0$.

(ii) For $t\ge c_0$ we have $h(t)=t-t_1$, where $t_1=c_0-h(c_0)\in(t_0,c_0)$.

(iii) For $t_0\le t\le c_0$ we have $t-t_1\le h(t)\le t-t_0$.

(iv) For all $t\in\mathbb{R}$ we have $0\le\dot{h}(t)\le 1$ and $2t\ddot{h}(t)+\dot{h}(t)<\lambda$.

The graph of h is shown in [21, Figure 2, page 11] and [23, Figure 3.4, page 93]; we reproduce it here for the convenience of the reader (see Figure 2).

With Q as in (11), we consider the smooth function $\widetilde{\tau}\colon\mathbb{C}^n\cong\mathbb{C}^r\times\mathbb{R}^{2s}\to\mathbb{R}$ given by

$$\widetilde{\tau}(\zeta,u)=-h\big(|x'|^2+|u'|^2\big)+Q(y,x'',u''). \tag{13}$$

Using the properties of h, it is easily verified that the function $\mathbb{C}^r\ni\zeta\mapsto\widetilde{\tau}(\zeta,u)$ is strongly plurisubharmonic for every fixed $u\in\mathbb{R}^{2s}$, so $\widetilde{\tau}$ is q–convex on $\mathbb{C}^n$. By the same argument as above (due to the special form of the $\mathbb{R}$–linear map ψ),

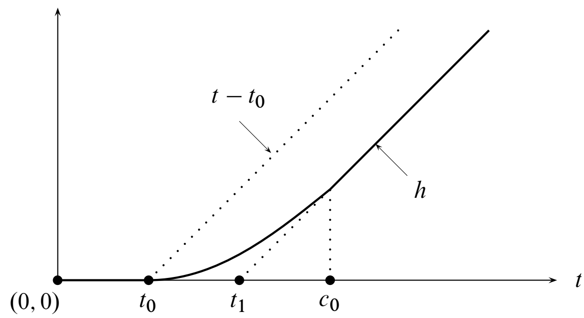

Figure 2: The function h.

the composition $\widetilde{\tau}\circ\psi$ has the same property and hence is also q–convex on $\mathbb{C}^n$. Furthermore, 0 is a critical value with critical locus

$$\{|x'|^2+|u'|^2\le t_0,\ x''=0,\ y=0,\ u''=0\},$$

and $\widetilde{\tau}$ has no critical values in $(0,+\infty)$. (See [21, Lemma 3.1, page 10] or [25, Lemma 3.10.1, page 92] for the details.)

Let $\phi=\psi\circ z\colon U\to\mathbb{C}^n$ be as above. Define the function $\tau\colon X_{\rho\le 3c_0}\to\mathbb{R}$ by

$$\tau=\begin{cases}\widetilde{\tau}\circ\phi=\widetilde{\tau}\circ\psi\circ z & \text{on } U\cap X_{\rho\le 3c_0},\\ \rho+t_1 & \text{on } X_{\rho\le 3c_0}\setminus U.\end{cases}\tag{14}$$

It is easily seen that τ is well defined and enjoys the following properties:

(α) $E\cup X_{\rho\le -c_0}\subset X_{\tau\le 0}\subset E\cup X_{\rho\le -t_0}$.

(β) $X_{\rho\le c_0}\subset X_{\tau\le 2c_0}\subset X_{\rho<3c_0}$.

(γ) τ is q–convex on the set $X_{-c_0\le\rho\le 3c_0}$.

(δ) τ has no critical values in the interval $(0,2c_0]$. (The level set $X_{\tau=0}$ is critical.)

(See [21, Lemma 3.1, page 10] or [25, Lemma 3.10.1, page 92] for details; for the strongly pseudoconvex case see also [23, Lemma 6.7, page 178].)

The critical locus of τ equals $\{|x'|^2+|u'|^2\le t_0,\ x''=0,\ y=0,\ u''=0\}\subset E$, and τ vanishes on this set. For every $c\in(0,2c_0]$ the sublevel set

$$\Omega_c=\{x\in X_{\rho<3c_0}:\tau(x)\le c\}\Subset X_{\rho<3c_0}\tag{15}$$

is a smoothly bounded q–complete domain. As τ has no critical values in $(0,2c_0]$, these domains are diffeomorphic to each other. If $c>0$ is chosen sufficiently small such that $c-t_1<0$ (the number $t_1>0$ was defined in property (ii) of h), then Ω_c is obtained by attaching to the subcritical sublevel set $X_{\rho\le c-t_1}$ a handle of index k_0

corresponding to the critical point of ρ at $\boldsymbol{x}_0$. Hence every domain Ω_c for $c \in (0, 2c_0]$ is diffeomorphic to the sublevel set $X_{\rho\le c}$ of ρ which contains the critical point $\boldsymbol{x}_0$ in its interior (since $c>0$). The inclusion $E \cup X_{\rho\le c-t_1} \hookrightarrow \Omega_c$ is a homotopy equivalence since $E\cup X_{\rho\le c-t_1}$ is a strong deformation retract of Ω_c. From the second case in (14) we also see that

$$\Omega_c \setminus U = X_{\rho\le c-t_1} \setminus U.$$

The sets Ω_c are illustrated in Figure 3 (reproduced from [21, Figure 1, page 10] and [25, Figure 3.5, page 94]).

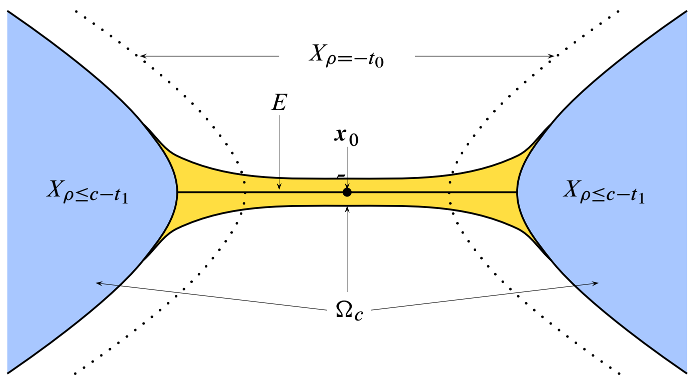

Figure 3: The set $\Omega_c = X_{\tau\le c}$.

Another observation will be crucial for our purposes. On the set $U\cap\{|x'|^2+|u'|^2\le t_0\}$ we have $h(x'(\boldsymbol{x}), u'(\boldsymbol{x}))=0$, and hence $\tau(\boldsymbol{x}) = Q(y(\boldsymbol{x}), x''(\boldsymbol{x}), u''(\boldsymbol{x}))$. Therefore

$$\Omega_c\cap U\cap\{|x'|^2+|u'|^2\le t_0\} = U\cap\{|x'|^2+|u'|^2\le t_0,\ Q\le c\}$$

is a tube around E which decreases to the disc $E\cap\{|x'|^2+|u'|^2\le t_0\}$ as $c\searrow 0$.

Note that $\Sigma := \{z\in\mathbb{C}^n : y=0,\ x''=0,\ u''=0\}$ is the $\mathbb{R}$–linear subspace containing the disc $\widetilde{E}$ of (12). If $\Lambda\subset\mathbb{C}^n$ is any $\mathbb{R}$–linear subspace of dimension $\dim_{\mathbb{R}}\Lambda = 2n-\dim_{\mathbb{R}}\Sigma = 2n-k_0$ intersecting Σ transversely at the origin, then for small enough $c\in(0,2c_0)$ the intersection $\Lambda\cap\{Q\le c\}$ is an ellipsoid in Λ. If $2n-k_0=2p$ is even, we can pick Λ such that $\widetilde{\Lambda} = \psi^{-1}(\Lambda)\subset\mathbb{C}^n$ is a p–dimensional $\mathbb{C}$–linear subspace of $\mathbb{C}^n$. The set

$$V_c := \Lambda\cap\{Q\le c\}$$

for small $c>0$ is an ellipsoid, and $\widetilde{V}_c := \psi^{-1}(V_c)$ is an ellipsoid in $\widetilde{\Lambda}\cong\mathbb{C}^p$. The set

$$Z_c := \{\boldsymbol{x}\in\Omega_c\cap U : \phi(\boldsymbol{x})\in V_c\} = \{\boldsymbol{x}\in\Omega_c\cap U : z(\boldsymbol{x})\in\widetilde{V}_c\} \tag{16}$$

is then a closed complex submanifold of $\Omega_c = X_{\tau\le c}$ with boundary $\partial Z_c \subset \partial\Omega_c = X_{\tau=c}$, which is biholomorphic to an ellipsoid in $\widetilde{\Lambda} \cong \mathbb{C}^p$ via the holomorphic coordinate map $z\colon U \to \mathbb{C}^n$. The submanifold Z_c is a holomorphic cross-section of the handle that was attached to the sublevel set $X_{\rho\le c-t_1}$ in order to get the domain Ω_c.

It is now possible to explain how to lift the boundary of the image variety $f(V) \subset X$ across a critical level of the q–convex function ρ. We do this in the following section in the context of proving Theorem 4.1.

4 Proof of Theorems 1.1 and 1.2

We first explain how Theorem 3.3 and its proof (see Section 3 above) imply the following result. Theorems 1.1 and 1.2 will follow easily from Theorem 4.1.

Theorem 4.1 *(Hypotheses as in Theorem 3.3) Assume that for some $\gamma_0 \in (a,b)$ the function ρ has no critical values on $[a,\gamma_0]$, and every element of the relative homology group $H_{2p}(X_{\rho<\gamma_0}, X_{a<\rho<\gamma_0};\mathbb{Z})$ can be represented by a finite analytic cycle $\sum_i n_i Z_i$, where every Z_i is a holomorphic image of a strongly pseudoconvex Stein domain V_i. Then for any $\gamma_1 \in (\gamma_0, b)$, each element of the homology group $H_{2p}(X_{\rho<b}, X_{\gamma_1<\rho<b};\mathbb{Z})$ can also be represented by a finite analytic cycle of the same form. If in addition the elements of $H_{2p}(X_{\rho<\gamma_0}, X_{a<\rho<\gamma_0};\mathbb{Z})$ are representable by cycles as above in which every domain V_i is the ball $\mathbb{B}^p \subset \mathbb{C}^p$, then the same is true for the group $H_{2p}(X_{\rho<b}, X_{\gamma_1<\rho<b};\mathbb{Z})$.*

Proof Let us first consider the *noncritical case* when ρ has no critical values in $[a,b]$. As there is no change in topology of the sublevel sets of ρ, the relative homology groups $H_j(X_{\rho\le t}, X_{\rho=t};\mathbb{Z})$ are isomorphic to each other for all $t \in [a,b]$ and all $j \in \mathbb{Z}_+$. Furthermore, for any pair of numbers α, β with $a \le \alpha < \beta \le b$, the set $X_{\alpha\le\rho\le\beta}$ is an interior collar around the boundary $X_{\rho=\beta}$ of the manifold with boundary $X_{\rho\le\beta}$, and also an exterior collar of $X_{\rho=\alpha} = \partial(X_{\rho\le\alpha})$. By Lemma 2.5 and excision we thus get an isomorphism

$$H_j(X_{\rho<\gamma_0}, X_{a<\rho<\gamma_0};\mathbb{Z}) \cong H_j(X_{\rho<b}, X_{\gamma_1<\rho<b};\mathbb{Z}) \tag{17}$$

for every $\gamma_1 \in (\gamma_0, b)$ and $j \in \mathbb{Z}_+$. Assume that V is a strongly pseudoconvex Stein domain of dimension p and $f_0\colon V \to X$ is a holomorphic map with range in $X_{\rho<\gamma_0}$ such that $f_0(\partial V) \subset X_{a<\rho<\gamma_0}$. Then $f_0(V)$ represents an element of the group $H_{2p}(X_{\rho<\gamma_0}, X_{a<\rho<\gamma_0};\mathbb{Z})$. Assuming that $p < n-q+1$, Theorem 3.3 tells us that f_0 is homotopic to another holomorphic map $f_1\colon V \to X$, whose image $f_1(V)$ represents the same element in $H_{2p}(X_{\rho<b}, X_{\gamma_1<\rho<b}; \mathbb{Z})$ under the isomorphism of (17). This holds in particular if $2p = n-q+1$, the case of interest to us.

It follows that we can lift an analytic p–cycle $\sum n_i f_i(V_i)$ in $(X_{\rho<\gamma_0}, X_{a<\rho<\gamma_0})$, with $p < n-q+1$, to an analytic p–cycle $\sum n_i \widetilde{f}_i(V_i)$ in $(X_{\rho<b}, X_{\gamma_1<\rho<b})$ such that these two cycles represent the same homology class under the isomorphism of (17) with $j = 2p$. In the new cycle we use the same domains V_i and weights n_i, only the maps change.

This completes the analysis of the noncritical case.

Assume now that ρ has critical points in $X_{a\le\rho\le b}$. We may assume that a and b are regular values, so ρ (being Morse) has only finitely many critical points in $X_{a<\rho<b}$. By a small deformation of ρ we may assume that these points lie on different level sets of ρ and each of them is nice, in the sense that ρ can be represented in the normal form of (10) without the remainder term. It suffices to prove Theorem 4.1 in the case when ρ has only one critical point in $X_{a<\rho<b}$; the general case then follows by a finite induction.

Thus, let $\boldsymbol{x}_0 \in X_{a<\rho<b}$ be the unique (nice) critical point of ρ in $X_{a\le\rho\le b}$. We may assume that $\rho(\boldsymbol{x}_0) = 0$. By the inductive hypothesis there exists a number $\gamma_0 \in (a, 0)$ such that every homology class in $H_{2p}(X_{\rho<\gamma_0}, X_{a<\rho<\gamma_0}; \mathbb{Z})$ is represented by an analytic cycle consisting of images of strongly pseudoconvex Stein domains.

We shall use the notation from the previous section; this pertains in particular to the positive numbers $c_0, t_0, t_1 > 0$, the coordinate map $\phi = \psi \circ z \colon U \to P \subset \mathbb{C}^n$ on a neighborhood $U \subset X$ of $\boldsymbol{x}_0$, the core $E = \phi^{-1}(\widetilde{E})$ of the handle (12) at $\boldsymbol{x}_0$, the function τ of (14), its sublevel sets $\Omega_c = X_{\tau\le c}$ in (15), and the ellipsoids $Z_c \subset \Omega_c$ of (16). We may assume that $\gamma_0 < -t_0 < 0$ and $0 < 3c_0 < b$.

It suffices to deal separately with each of the subvarieties in the given cycle. Let $f_0(V) \subset X_{\rho<\gamma_0}$ be such a subvariety, with $f_0(\partial V) \subset X_{a<\rho<\gamma_0}$. By the noncritical case, applied on $[a, -t_0/2]$, we can represent the homology class

$$[f_0(V)] \in H_{2p}(X_{\rho<\gamma_0}, X_{a<\rho<\gamma_0}; \mathbb{Z})$$

by a subvariety $\widetilde{f}_0(V) \subset X_{\rho<-t_0/2}$, with $\widetilde{f}_0(\partial V) \subset X_{-t_0<\rho<-t_0/2}$, such that

$$[\widetilde{f}_0(V)] \in H_{2p}(X_{\rho<-t_0/2}, X_{-t_0<\rho<-t_0/2}; \mathbb{Z}) \cong H_{2p}(X_{\rho<\gamma_0}, X_{a<\rho<\gamma_0}; \mathbb{Z}).$$

As $2p \le n-q+1$, the general position argument (cf Theorem 3.1) allows us to deform $\widetilde{f}_0$ slightly to ensure that $\widetilde{f}_0(\partial V) \cap E = \varnothing$, so we have $\widetilde{f}_0(\partial V) \subset X_{-t_0<\rho<-t_0/2} \setminus E$.

Recall that $\{\tau = 0\} \cap X_{\rho\ge -t_0} = E \cap X_{\rho\ge -t_0}$ (see property (α) below (14)). Hence for $c > 0$ small enough, we have $\widetilde{f}_0(\partial V) \subset X_{\tau>c}$, so $\widetilde{f}_0(V)$ also determines a homology class in $H_{2p}(X_{\tau<c_1}, X_{c<\tau<c_1};\mathbb{Z})$ for some constant c_1 satisfying $c<c_1<2c_0$. The domain $X_{\tau\le c_1}$ is diffeomorphic to $X_{\rho\le c_1}$, ie it is the manifold obtained by attaching to $X_{\rho\le -t_0}$ a handle of index k_0, representing the change of topology at the critical point $\boldsymbol{x}_0$. The homology class $[\widetilde{f}_0(V)]\in H_{2p}(X_{\tau<c_1}, X_{c<\tau<c_1};\mathbb{Z})$ is simply the prolongation of the same class in $H_{2p}(X_{\rho<-t_0/2}, X_{-t_0<\rho<-t_0/2};\mathbb{Z})$ in the sense of Proposition 2.1.

By using the noncritical case with the function τ of (14), we lift the boundary of the variety $\widetilde{f}_0(V)$ into the layer $X_{c_0<\rho<3c_0}$ above the critical level $\rho(\boldsymbol{x}_0)=0$; see conditions (α) and (β) following (14). Next, we use the noncritical case, this time again with the function ρ, to lift the boundary of the variety, obtained in the previous substep, closer towards the level set $X_{\rho=b}$. The result of these two modifications is that we replaced $\widetilde{f}_0(V)$ by a homologous analytic cycle $f_1(V)\subset X_{\rho<b}$ satisfying $f_1(\partial V)\subset X_{\gamma_1<\rho<b}$, where γ_1 is any number with $3c_0<\gamma_1<b$.

This shows that all analytic p–cycles with $2p\le n-q+1$, coming from below the critical level of ρ at $\boldsymbol{x}_0$, survive the passage of the critical level $\rho=\rho(\boldsymbol{x}_0)$ with the same normalizing strongly pseudoconvex domains.

In the subcritical case $2p<n-q+1$ this completes the proof since the relative homology group $H_{2p}(X_{\rho<t}, X_{\rho=t};\mathbb{Z})$ does not change as t passes the value $\rho(\boldsymbol{x}_0)$. The same is true if $2p=n-q+1$ and the Morse index of $\boldsymbol{x}_0$ is $<n+q-1$.

If $2p=n-q+1$, a critical point $\boldsymbol{x}_0$ of maximal Morse index $n+q-1$ of ρ may give birth to an additional generator of the relative homology (cf Proposition 2.1). As we have seen in Section 3, this new generator can be represented by a properly embedded complex ellipsoid $Z_c\subset X_{\tau\le c}$ with $\partial Z_c\subset X_{\tau=c}$, as in (16). We now explain how to replace this ellipsoid Z_c by a ball in the subsequent construction.

Pick a closed domain $B\subset Z_c$, biholomorphic to the closed ball $\overline{\mathbb{B}}^p\subset\mathbb{C}^p$, which contains the (unique) intersection point $\{\boldsymbol{x}_0\}=Z_c\cap E$ in its relative interior. We have $B=g(\overline{\mathbb{B}}^p)$ for a holomorphic embedding $g\colon \overline{\mathbb{B}}^p\to X$. There is a constant $c'\in(0,c)$ such that $Z_c\cap X_{\tau\le c'}$ is contained in the relative interior of B. (See Figure 4.)

The function τ of (14) has no critical values in the interval $(0,2c_0]$ by property (δ), so we see by the same argument as above that Z_c and B determine the same element in the relative homology group $H_{2p}(X_{\tau<2c_0}, X_{c'<\tau<2c_0};\mathbb{Z})$.

Applying the noncritical case, first with the function τ (to push the boundary of the ball B into a ρ–layer above the critical level) and subsequently with the function ρ, we can replace this new component by another holomorphic immersion (embedding if $2p<n$) of $(\overline{\mathbb{B}}^p, \partial\mathbb{B}^p)$ into $(X_{\rho<b}, X_{\gamma_1<\rho<b})$. This proves Theorem 4.1. □

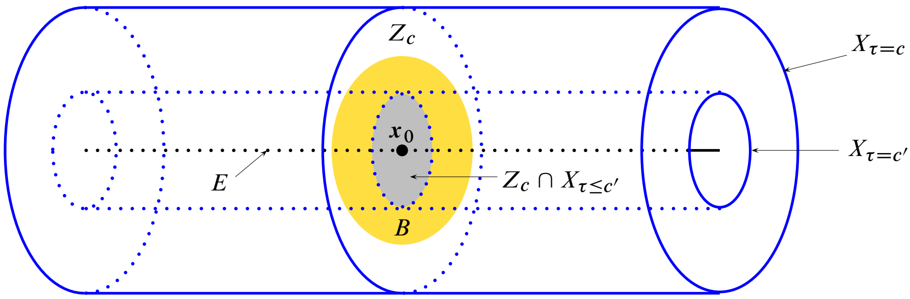

Figure 4: Crossection Z_c of the tube $X_{\tau\le c}$.

Proof of Theorem 1.2 Choose a q–convex Morse function ρ on a neighborhood of M in X such that $M=\{\rho\le 0\}$ and $d\rho\ne 0$ on ∂M. Pick numbers $a<\gamma_0<\gamma_1<b=0$ such that $\gamma_0<\min_M \rho<\gamma_1$, and γ_1 is close enough to 0 so that ρ has no critical values on $[\gamma_1,0]$. Then $X_{\gamma_1\le\rho\le 0}$ is a collar around $\partial M=X_{\rho=0}$, and by choosing γ_1 sufficiently close to 0 we may assume that it is contained in the given collar A. Since $X_{\rho<\gamma_0}=\varnothing$, the hypotheses of Theorem 4.1 are trivially satisfied. The result now follows directly from Theorem 4.1. $\square$

Proof of Theorem 1.1 Let M be a compact q–complete domain in a complex manifold X. Let $A\subset X$ be an interior collar and $B\subset X$ be an exterior collar around ∂M, and set $\widetilde M=M\cup B$. For any $j\in\mathbb{N}$ we have natural isomorphisms

$$H_j(M,\partial M;\mathbb{Z})\cong H_j(M,A;\mathbb{Z})\cong H_j(\widetilde M,B;\mathbb{Z}).$$

Let $2p=n-q+1$. By Theorem 1.2, every homology class $z\in H_{2p}(M,\partial M;\mathbb{Z})$ is represented by an analytic cycle $Z=\sum_i n_i Z_i$, where each $Z_i=f_i(\overline{\mathbb{B}}^p)$ is a holomorphic image of the ball $\overline{\mathbb{B}}^p\subset\mathbb{C}^p$ and $\partial Z_i=f_i(\partial\mathbb{B}^p)\subset A$. By the noncritical case of Theorem 4.1 we can replace Z by a homologous analytic cycle $\widetilde Z=\sum_i n_i\widetilde Z_i$ in $(\widetilde M,B)$, where $\widetilde Z_i=\widetilde f_i(\overline{\mathbb{B}}^p)$ for some holomorphic map

$$\widetilde f_i\colon \overline{\mathbb{B}}^p\to\widetilde M,\qquad \widetilde f_i(\partial\mathbb{B}^p)\subset B\setminus\partial M.$$

In view of Theorem 3.1 we can assume that each $\widetilde f_i$ is transverse to ∂M. By Proposition 2.6, a suitable integral linear combination of the intersections of the components of $\widetilde Z$ with M gives an analytic cycle in $(M,\partial M)$ which determines the homology class $z\in H_{2p}(M,\partial M;\mathbb{Z})$. The cycle in $(M,\partial M)$ obtained in this way has the form $\sum_{i,j} n_{i,j}\widetilde f_i(W_{i,j})$, where $W_{i,j}$ are smoothly bounded connected domains in $\mathbb{B}^p$. This proves Theorem 1.1. $\square$

5 Proof of Theorem 1.4

In this section we prove Theorem 1.4 which pertains to arbitrary q–complete manifolds, possibly of infinite topology. We begin with some preparations.

Let X be a q–complete manifold without boundary, and let ρ: $X \to \mathbb{R}$ be a q–convex Morse exhaustion function. Pick an exhaustion $M_1 \subset M_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} M_j = X$, where each M_j is a regular sublevel set of ρ and there is at most one critical point of ρ in each difference $\mathring{M}_j \setminus M_{j-1}$. Recall the definition $\mathcal{H}^k(X;G) = \lim_j H^k(M_j;G)$ in (5) and note that the inverse limit does not depend on the particular choice of ρ or the sublevel sets. The easiest way to see this is to define $\mathcal{H}^k(X;G)$, in analogy with Atiyah and Hirzebruch [5], as the inverse limit of groups $H^k(M;G)$ where M ranges over all compact subdomains of X. A particular exhaustion gives rise to a countable cofinal subset; hence the two inverse limits are isomorphic.

From now on, homology and cohomology will be taken with integer coefficients; we drop coefficients from the notation.

Let X have complex dimension n and let Z be a closed analytic subspace of complex dimension $p = n - k = (n - q + 1)/2$. We assume that the singular subspace Z_s of Z is either empty (ie Z is a submanifold) or discrete. (These are the only cases that we need to consider for our purposes.) We define the cohomology class dual to Z as follows. (See Atiyah and Hirzebruch [5, Section 5], and also Douady [18, Section V.A].) As $Z' = Z - Z_s$ is a closed submanifold of $X' = X - Z_s$, it admits a (smooth) tubular neighborhood, say W', in X'. There is a Thom class in $H^{2k}(W', W' - Z')$ induced by the canonical orientation of the complex normal bundle, and consequently, by excision, the corresponding class in $H^{2k}(X', X' - Z')$. By restriction, we obtain a class in $H^{2k}(X')$ and hence a class $\zeta' \in \mathcal{H}^{2k}(X')$. By [5, Lemma 5.3], the natural "restriction" morphism $\mathcal{H}^{2k}(X) \to \mathcal{H}^{2k}(X')$ is an isomorphism and we get an element $\xi \in \mathcal{H}^{2k}(X)$. The element ξ will be referred to as the cohomology class dual to Z. More generally, let $\{Z_r \mid r\}$ be a countable, locally finite collection of closed analytic subspaces of X whose singular sets are either empty or discrete. Let $\xi_r \in \mathcal{H}^{2k}(X)$ denote the cohomology class dual to Z_r and let n_r be integers. Local finiteness ensures that the possibly infinite sum $\sum_r n_r \xi_r$ makes sense in $\mathcal{H}^{2k}(X)$. To make this precise, consider the canonical projections

$$P^i\colon \mathcal{H}^{2k}(X) \to H^{2k}(M_i). \tag{18}$$

By local finiteness, $P^i(\xi_r)$ is nontrivial for at most finitely many r, and hence $\sum_r n_r P^i(\xi_r)$ is an honest element of $H^{2k}(M_i)$. The sequence $\{\sum_r n_r P^i(\xi_r) \mid i\}$ then defines an element of the inverse limit $\lim_i H^{2k}(M_i) = \mathcal{H}^{2k}(X)$ which we denote by $\sum_r n_r \xi_r$.

We propose the following definition which is a slight generalization of the definition in [5, Section 5.D].

Definition 5.1 The cohomology class $\sum_r n_r \xi_r \in \mathcal{H}^{2k}(X)$ described above is complex analytic, and it is dual to the complex analytic cycle $\sum_r n_r Z_r$.

The situation for a compact domain M with boundary is as follows. Let Z be an embedded complex submanifold of M with smooth boundary, and assume that Z meets ∂M transversely in ∂Z. As above, we assume that Z has complex dimension $p = n-k$. Thus, the pair $(Z, \partial Z)$ defines a homology class $[Z, \partial Z]$ in $H_{2n-2k}(M, \partial M)$. Then the cohomology class $x \in H^{2k}(M)$ that corresponds to $[Z, \partial Z]$ under Poincaré duality is exactly the restriction of the Thom class of Z in M. Precisely, let W be a tubular neighborhood of Z in M. By the transversality assumption, $W \cap \partial M$ is a tubular neighborhood of ∂Z in ∂M. Thus there is a Thom class $\tau_Z^W \in H^{2k}(W, W \setminus Z)$ induced by the canonical orientation of the normal bundle. The image of τ_Z^W under the zig-zag composite

$$H^{2k}(W, W \setminus Z) \xleftarrow[\cong]{\text{excision}} H^{2k}(M, M \setminus Z) \xrightarrow{\text{restriction}} H^{2k}(M)$$

equals x. (See Bredon [8, page 371].)

Proof of Theorem 1.4 Let $\{M_i\}$ be an exhaustion of X as above. Take $x \in \mathcal{H}^{2k}(X)$, let $x_i = P^i(x) \in H^{2k}(M_i)$ be the "restrictions" of x to M_i, and let

$$z_i \in H_{2n-2k}(X, X \setminus M_i) \cong H_{2n-2k}(M_i, \partial M_i)$$

be the dual homology classes.

We summarize the consequences of Theorems 1.2, 1.1, and 4.1 and their proofs. Each homology group $H_{2n-2k}(X, X \setminus M_i)$ has a distinguished set of generators

$$\{\zeta_r^i \mid 1 \le r \le \lambda_i\},$$

where $0 \le \lambda_{i-1} \le \lambda_i \le \lambda_{i-1} + 1$, and

$$\iota_i \colon H_{2n-2k}(X, X \setminus M_i) \to H_{2n-2k}(X, X \setminus M_{i-1})$$

maps each ζ_r^i to ζ_r^{i-1}, for $r \le \lambda_{i-1}$. If $\lambda_i > \lambda_{i-1}$ (ie when passing a critical point of the highest index), then $\iota_i(\zeta_{\lambda_i}^i) = 0$. Since $\iota_i(z_i) = z_{i-1}$ for all i, we can use (8) to express

$$z_i = \sum_{r=1}^{\lambda_i} n_r^i \zeta_r^i,$$

where $n_r^i = n_r^{i-1}$ for all $r \le \lambda_{i-1}$. In particular, it makes sense to define

$$n_r = \lim_{i\to\infty} n_r^i.$$

We describe and make use of the geometric representatives of ζ_r^i constructed above. Assume that $r = \lambda_i$, and $\zeta_r^i \in H_{2n-2k}(X, X \setminus M_i)$ is an additional generator. Then $\zeta_r^i = [f(V), f(\partial V)]$, where $V = \overline{\mathbb{B}}^p$ is the ball in $\mathbb{C}^p$ and $f = f^i\colon (V, \partial V) \to (X, X \setminus M_i)$ is a holomorphic embedding (immersion with normal crossings when $q = 1$). This gives rise to an inverse sequence of liftings

$$f^j\colon (V, \partial V) \to (X, X \setminus M_j)$$

such that $[f^j(V), f^j(\partial V)] = \zeta_r^j$ for all $j > i$. In addition, we may arrange that f^j and f^{j-1} are arbitrarily close on the compact subset $\{v \in V : d(v, bV) \ge 1/j\}$ of V. Finally, we may achieve by Theorem 3.1 that each f^j is transverse to ∂M_j.

By choosing f^j close enough to f^{j-1} for each j, we can make the sequence $\{f^j\}$ converge uniformly on compact subsets of $\mathring{V} = \mathbb{B}^p$ to a holomorphic embedding $\phi\colon \mathring{V} \to X$ that is transverse to all ∂M_j. For each $j \ge i$ the geometric intersection of the image $\phi(\mathring{V})$ with M_j yields a cycle $[\phi(\mathring{V}) \cap M_j, \phi(\mathring{V}) \cap \partial M_j] \in H_{2n-2k}(M_j, \partial M_j)$ homologous to that obtained by intersecting $f^j(V)$ with M_j. That in turn corresponds to $[f^j(V), f^j(\partial V)] = \zeta_r^j$ under the isomorphism $H_{2n-2k}(M_j, \partial M_j) \cong H_{2n-2k}(X, X \setminus M_j)$.

We let Z_r denote the image $\phi(\mathring{V})$. By doing this for each additional generator we obtain our collection of submanifolds $\mathcal{Z} = \{Z_r \mid 1 \le r \le \sup_j \lambda_j\}$. Note that this collection is locally finite in X (this is because the new generator which may appear at any of the critical points of maximal index does not enter into any of the previous subdomains during the subsequent lifting procedure, cf Theorem 3.3 and the proof of Theorem 4.1), and also that the cardinality of $\mathcal{Z}$ is precisely the number of critical points of the highest index. Let $\xi_r \in \mathcal{H}^{2k}(X)$ be the cohomology class dual to Z_r. To complete the proof we need to show that $x = \sum_r n_r \xi_r$. To this end, fix some r and assume, as above, that Z_r has resulted from an additional generator ζ_r^i; thus $r = \lambda_i > \lambda_{i-1}$. Write $i = i(r)$ in this context. Let W be a tubular neighborhood of Z_r in X. By transversality, we may construct W so that each intersection $W_j = W \cap M_j$ is either empty or a tubular neighborhood of $Z_r \cap M_j$ in M_j; the latter for all $j \ge i$. Assume that $W \cap M_j$ is nonempty. Recall that P^j is the canonical projection (18). Naturality of the Thom class guarantees that $P^j(\xi_r) \in H^{2k}(M_j)$ is the Poincaré dual of $[Z_r \cap M_j, Z_r \cap \partial M_j] \in H_{2n-2k}(M_j, \partial M_j)$. On the other hand, $[Z_r \cap M_j, Z_r \cap \partial M_j]$ corresponds to $\zeta_r^j \in H_{2n-2k}(X, X \setminus M_j)$, as explained above. Thus, $P^j(\xi_r)$ corresponds to ζ_r^j for $j \ge i(r)$ and to 0 for $j < i(r)$. Consequently,

$P^j(\sum_r n_r\xi_r) = P^j(\sum_{r:i(r)\le j} n_r\xi_r)$ corresponds precisely to $\sum_{r=1}^{\lambda_j} n_r^j \zeta_r^j = z_j$ for all j. Thus, $P^j(\sum_r n_r\xi_r) = x_j$ for all j, which implies $x = \sum_r n_r\xi_r$, as claimed.

This completes the proof for $q > 1$. The straightforward adjustments necessary to prove the case $q = 1$ will be left to the reader. □

6 Equality of $\mathcal{H}^k(X;G)$ and $H^k(X;G)$

Let X be a noncompact smooth manifold without boundary. As is well-known in homological algebra (see eg Milnor [43]), we have the short exact sequence

$$0 \to \lim_i{}^1 H^{k-1}(M_i;G) \to H^k(X;G) \to \lim_i H^k(M_i;G) = \mathcal{H}^k(X;G) \to 0. \tag{19}$$

Here, $\lim^1$ denotes the first (right) derived functor of the inverse limit. For inverse limits of abelian groups, it can be described as follows. If $\cdots \longrightarrow A_3 \xrightarrow{p_3} A_2 \xrightarrow{p_2} A_1$ is an inverse sequence of abelian groups, also called a tower of abelian groups, then the collection of the "bonding" morphisms $p_i\colon A_i \to A_{i-1}$ gives rise to a morphism $P\colon \prod_{i=1}^\infty A_i \to \prod_{i=1}^\infty A_i$. The kernel of $1-P$ is the inverse limit $\lim_i A_i$, and the cokernel can be taken as the definition of $\lim_i{}^1 A_i$.

We discuss two particular cases.

If $\{A_i\}$ is a tower of finite-dimensional vector spaces over some field, then $\lim^1 A_i$ is trivial (see Weibel [52, Exercise 3.5.2]). Thus if G is a field, $\lim_i{}^1 H^{k-1}(M_i;G)$ vanishes and $H^k(X;G) \to \mathcal{H}^k(X;G)$ is an isomorphism.

If $\{A_i\}$ is a tower of finitely generated abelian groups, then $\lim_i{}^1 A_i$ is a divisible abelian group of the form $\mathrm{Ext}(B,\mathbb{Z})$ where B is a countable torsion-free abelian group. It follows that if nontrivial, it is uncountable. See Jensen [38, Théorème 2.7] for a more precise description; we deal here with the case of our interest where $A_i = H^k(M_i;\mathbb{Z})$. First we need a lemma.

Lemma 6.1 *Let $0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$ be a short exact sequence of towers of abelian groups. Then there is a natural six-term exact sequence*

$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i \to \lim_i{}^1 A_i \to \lim_i{}^1 B_i \to \lim_i{}^1 C_i \to 0.$$

The proof is an easy consequence of the above definition of $\lim^1$ and the snake lemma.

Proposition 6.2 *Let $\{M_i\}$ be an exhaustion of X and let T denote the torsion subgroup of $H_{k-1}(X)$. Then $\lim_i{}^1 H^{k-1}(M_i;\mathbb{Z})$ is isomorphic to $\mathrm{Ext}(H_{k-1}(X)/T,\mathbb{Z})$. Hence if $\mathrm{Ext}(H_{k-1}(X)/T,\mathbb{Z}) = 0$, then the natural morphism $H^k(X;\mathbb{Z}) \to \mathcal{H}^k(X;\mathbb{Z})$ is an isomorphism.*

Proof Universal coefficients yield a short exact sequence of towers

$$0 \to \{\mathrm{Ext}(H_{k-2}M_i, \mathbb{Z}) \mid i\} \to \{H^{k-1}(M_i; \mathbb{Z}) \mid i\} \to \{\mathrm{Hom}(H_{k-1}M_i, \mathbb{Z}) \mid i\} \to 0.$$

As $\{\mathrm{Ext}(H_{k-2}M_i, \mathbb{Z})\}$ is a tower of finite groups, Lemma 6.1 and Jensen [38, Proposition 2.3] yield an isomorphism $\lim_i{}^1 H^{k-1}(M_i; \mathbb{Z}) \cong \lim_i{}^1 \mathrm{Hom}(H_{k-1}M_i, \mathbb{Z})$.

Set $T_i = T(H_{k-1}M_i)$ and note that $\mathrm{Hom}(H_{k-1}M_i, \mathbb{Z})$ is naturally isomorphic to $\mathrm{Hom}(H_{k-1}M_i/T_i, \mathbb{Z})$. As $H_{k-1}M_i/T_i$ is free, the group $\lim_i{}^1 \mathrm{Hom}(H_{k-1}M_i, \mathbb{Z})$ is isomorphic to $\mathrm{Ext}(\mathrm{colim}_i(H_{k-1}M_i/T_i), \mathbb{Z})$ (see [38, page 16]). In turn, we have that $\mathrm{colim}_i(H_{k-1}M_i/T_i)$ is isomorphic to $H_{k-1}X/T$, where $T = T(H_{k-1}X)$. □

The following immediate corollary to Proposition 6.2 is an addendum to Theorem 1.4 in Section 1.

Corollary 6.3 *Let X be an n–dimensional q–complete manifold with $n+q-1 = 2k$ even. Denote by T the torsion subgroup of $H_{n+q-2}(X)$. If $\mathrm{Ext}\,(H_{n+q-2}(X)/T, \mathbb{Z}) = 0$, then Theorem 1.4 applies to every cohomology class in $H^{n+q-1}(X; \mathbb{Z})$. This holds in particular if $H_{n+q-2}(X; \mathbb{Z})$ is the direct sum of a free abelian group and a torsion abelian group; for instance, if it is finitely generated.*

7 Examples

In this section we give a few examples which illustrate the scope of our results.

Recall that, if X is a complex manifold of dimension n and $\rho\colon X \to \mathbb{R}$ is a smooth exhaustion function which is q–convex on the set $X_{\rho>c}$ for some $c \in \mathbb{R}$, then any compact complex subvariety A of X of dimension $\dim A \geq q$ is contained in the sublevel set $X_{\rho\leq c}$. This is an easy consequence of the maximum principle for plurisubharmonic functions. In particular, a q–complete manifold does not contain any compact complex subvarieties of dimension $\geq q$; this bound is sharp, as shown by the following example.

Example 2 The manifold $X = \mathbb{CP}^n \setminus \mathbb{CP}^{n-q}$ is q–complete for any pair of integers $1 \leq q \leq n$. More generally, if $A \subset \mathbb{CP}^n$ is a closed projective manifold of complex codimension q, then $X = \mathbb{CP}^n \setminus A$ is q–convex, and is q–complete if A is a complete intersection. (See Barth [7] and Peternell [45].) Such a manifold X contains homologically nontrivial compact complex submanifolds of any dimension from 1 up to $q-1$ (for example, projective linear subspaces avoiding A), but every compact complex subvariety of dimension $\geq q$ in $\mathbb{CP}^n$ intersects A by Bezout's theorem.

We begin with an example showing that, in some cases, top-dimensional cohomology classes may be represented both by compact and also by noncompact analytic cycles.

Example 3 Assume that X^n is a q–complete manifold for some $q > 1$. A compact complex submanifold $C \subset X$ of complex dimension $p \in \{1, \dots, q-1\}$ represents a homology class $[C] \in H_{2p}(X, \mathbb{Z})$, and hence, by Poincaré–Lefschetz duality, a cohomology class $u \in H^{2n-2p}(X; \mathbb{Z})$. If $2p = n - q + 1$ then, by Theorem 1.4, the class u is also represented by an analytic cycle consisting of noncompact closed p–dimensional subvarieties of X. The equality $2p = n - q + 1$ holds for some $p \in \{1, \dots, q-1\}$ if and only if n and q are of different parity and $2 \le n-q+1 \le 2q-2$; equivalently,

$$3 \le q+1 \le n \le 3q-3, \quad p = (n-q+1)/2. \tag{20}$$

For any triple of integers (n, p, q) satisfying (20) we get a nontrivial example by taking $X = \mathbb{CP}^n \setminus \mathbb{CP}^{n-q}$ and $C \cong \mathbb{CP}^p \subset X$. In this case the class $[C] \in H_{2p}(X, \mathbb{Z}) \cong H_{2p}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$ is a generator of the corresponding homology group.

We now focus on examples where the Hodge representation of top-dimensional cohomology classes by compact analytic cycles is impossible, but our results give representation by noncompact analytic cycles.

Example 4 Let $A \subset \mathbb{CP}^n$ be a smooth complete intersection defined by q independent holomorphic equations; then $X = \mathbb{CP}^n \setminus A$ is q–complete (see Example 2). Assume that $n+q-1$ is even. By Theorem 1.4 and Remark 1.5 every element of $H^{n+q-1}(X; \mathbb{Z})$ can be represented by a noncompact analytic cycle of complex dimension $p = (n-q+1)/2$. However, if $p \ge q$ then a nonzero element of $H^{n+q-1}(X; \mathbb{Z})$ can not be represented by a compact analytic p–cycle since every such cycle intersects A in view of Bezout's theorem (or by observing that a q–complete manifold does not contain any compact complex subvarieties of dimension $\ge q$). The inequality $p \ge q$ is equivalent to $n + 1 \ge 3q$. The lowest dimensional non-Stein example arises when $p = q = 2$ and $n = 5$; in this case $X = \mathbb{CP}^5 \setminus A$, where A is a 3–fold defined by 2 independent equations. By the above argument any nonzero element of $H^6(X; \mathbb{Z})$ is analytic (represented by a complex 2–cycle), but it can not be represented by a compact cycle.

With X as in Example 4, the top-dimensional cohomology group $H^{n+q-1}(X; \mathbb{Z})$ can be quite big and it can contain torsion as shown by the following proposition.

Proposition 7.1 *Let $A \subset \mathbb{CP}^n$ be a smooth complete intersection of codimension q and set $X = \mathbb{CP}^n \setminus A$. Assume that $m := \dim A = n - q \ge 1$ is odd. Then*

$$H^{n+q-1}(X; \mathbb{Z}) = \mathbb{Z}_d \oplus \mathbb{Z}^{\beta_m},$$

where d is the degree of A and $\beta_m = m + 1 - \chi(A)$ is the m^{th} Betti number of A.

The Euler characteristic $\chi(A)$ can be computed easily from the multidegree of A by the Hirzebruch formula. For example, if A is the intersection of two quintics in $\mathbb{CP}^5$ then its third Betti number is 4504, so we have $H^6(\mathbb{CP}^5 \setminus A) \cong \mathbb{Z}_{25} \oplus \mathbb{Z}^{4504}$.

Proof We omit the coefficient group $\mathbb{Z}$ in the calculations. Poincaré–Lefschetz duality gives $H^{n+q-1}(X) = H_{n-q+1}(\mathbb{CP}^n, A) = H_{m+1}(\mathbb{CP}^n, A)$. The latter group can be computed from the exact homology sequence of the pair $i\colon A \hookrightarrow \mathbb{CP}^n$,

$$H_{m+1}(A) \xrightarrow{i_*} H_{m+1}(\mathbb{CP}^n) \longrightarrow H_{m+1}(\mathbb{CP}^n, A) \xrightarrow{\partial} H_m(A) \longrightarrow H_m(\mathbb{CP}^n).$$

Since m is odd, we have $H_{m+1}(\mathbb{CP}^n) = \mathbb{Z}$ and $H_m(\mathbb{CP}^n) = 0$. The Lefschetz hyperplane theorem yields (abstract) isomorphisms $H^k(A;\mathbb{Z}) \cong H^k(\mathbb{CP}^n;\mathbb{Z})$ for $k \neq m$, $k \leq 2m$. In our case, with $k = m+1$ even, both groups equal $\mathbb{Z}$ and the map $\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}$ is multiplication by d. Moreover, $H^m(A;\mathbb{Z})$ is also free, $H^m(A;\mathbb{Z}) = \mathbb{Z}^{\beta_m}$. The result follows from the short exact sequence

$$0 \to \mathbb{Z}_d \to H_{m+1}(\mathbb{CP}^n, A) \to \mathbb{Z}^{\beta_m} \to 0. \qquad \square$$

Remark 7.2 It may be interesting to observe that the free part of $H^{n+q-1}(X;\mathbb{Z})$ in Proposition 7.1 is not representable by compact cycles, even smoothly. To see this, let Z be an oriented closed smooth submanifold of X of dimension $n-q+1$. By compactness, there is a smooth tubular neighborhood W of Z in $\mathbb{CP}^n$ that is contained in X. The cohomology class dual to Z is the image of the Thom class of the normal bundle under the composite

$$H^{n+q-1}(W, W \setminus Z;\mathbb{Z}) \xleftarrow{\text{excision}} H^{n+q-1}(X, X \setminus Z;\mathbb{Z}) \xrightarrow{\text{restriction}} H^{n+q-1}(X;\mathbb{Z}).$$

But a class in $H^{n+q-1}(X, X \setminus Z;\mathbb{Z})$ excises back to one in $H^{n+q-1}(\mathbb{CP}^n, \mathbb{CP}^n \setminus Z;\mathbb{Z})$ and therefore the restriction morphism $H^{n+q-1}(X, X\setminus Z;\mathbb{Z}) \to H^{n+q-1}(X;\mathbb{Z})$ factors through $H^{n+q-1}(\mathbb{CP}^n;\mathbb{Z}) \to H^{n+q-1}(X;\mathbb{Z})$. It follows from the proof of Proposition 7.1 and duality (see [31, page 55]) that the image of the latter is precisely the torsion subgroup of $H^{n+q-1}(X;\mathbb{Z})$, and misses elements of infinite order completely.

The following example was provided by a referee, for which we are most grateful.

Example 5 Let p and d be natural numbers such that p is relatively prime to 6 and d is divisible by p^3. Assume that Y is a very general smooth hypersurface of degree d in $\mathbb{CP}^4$. Kollár (see [6, Lemma, page 134] and also [48]) has shown that any algebraic curve $C \subset Y$ has degree divisible by p, from which it follows that the generator α of $H^4(Y;\mathbb{Z}) \cong \mathbb{Z}$ is not algebraic (and hence not analytic by GAGA). However, the multiple $d\alpha$ is algebraic.

Let P be a generic 2–plane and let C be the intersection of Y and P; the degree of C equals d (hence C renders $d\alpha$ algebraic). The complement $X = Y \setminus C$ is a closed submanifold of the 2–complete manifold $\mathbb{CP}^4 \setminus P$ and as such also 2–complete; $H^4(X;\mathbb{Z})$ is the top nontrivial cohomology group.

As in the proof of Proposition 7.1, we have $H^4(X;\mathbb{Z}) = H^4(Y \setminus C;\mathbb{Z}) \cong H_2(Y,C;\mathbb{Z})$ by duality, and the homology exact sequence of the pair (Y,C) yields $H^4(X;\mathbb{Z}) \cong \mathbb{Z}_d \oplus \mathbb{Z}^{\beta_1}$, where β_1 is the first Betti number of C; the inclusion-induced morphism $H_2(C;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$ corresponds to multiplication by d on $\mathbb{Z}$.

Now by Theorem 1.4, every element of $H^4(X;\mathbb{Z})$ is analytic and can be represented by a 1–dimensional cycle consisting of proper holomorphic discs in X whose boundary cluster sets lie on C. However, the generator of the torsion subgroup $\mathbb{Z}_d$ of $H^4(X;\mathbb{Z})$ is not represented by a compact algebraic cycle. Namely, in Y that cycle would be homologous to a $(1+kd)$–multiple of the generator of $H_2(Y;\mathbb{Z})$ (for some integer k) by virtue of the exact sequence $H_2(C;\mathbb{Z}) \to H_2(Y;\mathbb{Z}) \to H_2(Y,C;\mathbb{Z})$; a contradiction since every closed curve in Y has degree divisible by p.

We thus have an example of a quasi-projective threefold with a torsion cohomology class which is analytic but not algebraic. Unlike in Example 4, the obstruction can not be explained by dimension reasons since a generic closed curve in Y avoids C.

In this connection we mention the Griffiths–Harris conjecture [32], which predicts that for a general hypersurface Y of degree $d \ge 6$ in $\mathbb{CP}^4$, the degree of every curve in Y is divisible by d. If this holds then in the above example no torsion element of $H^4(Y \setminus C;\mathbb{Z})$ can be represented compactly. On the other hand, one sees as in Remark 7.2 that elements of infinite order are not representable by compact cycles even smoothly.

We conclude by considering a couple of examples of the form $X = Y \times \mathbb{CP}^1$ with Y Stein, so X is 2–complete. Let Y denote the Stein surface

$$Y = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : z_0^2 + z_1^2 + z_2^2 \neq 0\}. \tag{21}$$

For $k \in \mathbb{N}$ let Y^k denote the k^{th} Cartesian power of Y; thus Y^k is a Stein manifold of complex dimension $2k$. For every pair of integers $1 \le j \le k$, the Chern class $c_j(Y^k) \in H^{2j}(Y^k;\mathbb{Z})$ is the nonzero element of the group $H^{2j}(Y^k;\mathbb{Z}) \cong \mathbb{Z}_2$ (cf Forster [22, Proposition 3]). In particular, $c_1(Y) \neq 0$ and the higher Chern classes of Y vanish. (Compare with Example 1.)

Example 6 Consider the 3–manifold $X = Y \times \mathbb{CP}^1$, where Y is given by (21). Clearly X is 2–complete and has finite topology. The Künneth theorem yields

$$H^4(X;\mathbb{Z}) \cong H^2(Y;\mathbb{Z}) \otimes H^2(\mathbb{CP}^1;\mathbb{Z}) \cong \mathbb{Z}_2.$$

The group $H^4(X;\mathbb{Z})$ is generated by the cohomological cross product $c_1(Y)\times\xi$, where $\xi\in H^2(\mathbb{CP}^1;\mathbb{Z})$ is the first Chern class of the tautological line bundle. By Theorem 1.4 and Remark 1.5, the nonzero element of $H^4(X;\mathbb{Z})$ can be represented by a noncompact analytic 1–cycle. However, it can not be represented by a compact analytic 1–cycle. Indeed, every compact analytic 1–cycle in X equals $\sum_j\{y_j\}\times\mathbb{CP}^1$ for some $y_j\in Y$, and is homologous to a multiple of $\{y_0\}\times\mathbb{CP}^1 =: Z$ for any $y_0\in Y$. It is easily seen by standard arguments that the cohomology class in $H^4(X;\mathbb{Z})$ dual to Z (in the sense of Section 5) is trivial.

Example 7 Let Y be the surface in (21) and set $X=Y^2\times\mathbb{CP}^1$. We have that $n=\dim X=5$, $q=2$ (ie X is 2–complete), and $n+q-1=6$. The Künneth theorem yields

$$H^6(X;\mathbb{Z})\cong H^2(Y;\mathbb{Z})\otimes H^2(Y;\mathbb{Z})\otimes H^2(\mathbb{CP}^1;\mathbb{Z})\cong\mathbb{Z}_2.$$

It is generated by the cross product $c_1(Y)\times c_1(Y)\times\xi$, where $\xi\in H^2(\mathbb{CP}^1;\mathbb{Z})$ is the first Chern class of the tautological line bundle. By Theorem 1.4 and Remark 1.5, the nonzero element of $H^6(X;\mathbb{Z})$ can be represented by a noncompact analytic 2–cycle in X. However, since X is 2–complete, it does not admit any compact analytic 2–cycles.

Similar results hold for the manifolds $X_{k,m}=Y^k\times\mathbb{CP}^m$ for higher values of $k,m\in\mathbb{N}$.

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References

[1] **A Andreotti**, **T Frankel**, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. 69 (1959) 713–717 MR0177422

[2] **A Andreotti**, **R Narasimhan**, *A topological property of Runge pairs*, Ann. of Math. 76 (1962) 499–509 MR0140714

[3] **A Andreotti**, **F Norguet**, *Cycles of algebraic manifolds and $\partial\bar\partial$–cohomology*, Ann. Scuola Norm. Sup. Pisa 25 (1971) 59–114 MR0288314

[4] **M F Atiyah**, **F Hirzebruch**, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math. 3, Amer. Math. Soc. (1961) 7–38 MR0139181

[5] **M F Atiyah**, **F Hirzebruch**, *Analytic cycles on complex manifolds*, Topology 1 (1962) 25–45 MR0145560

[6] **E Ballico**, **F Catanese**, **C Ciliberto** (editors), *Classification of irregular varieties: minimal models and abelian varieties*, Lecture Notes in Mathematics 1515, Springer, Berlin (1992) MR1180332

[7] **W Barth**, *Der Abstand von einer algebraischen Mannigfaltigkeit im komplex-projektiven Raum*, Math. Ann. 187 (1970) 150–162 MR0268181

[8] **G E Bredon**, *Topology and geometry*, Graduate Texts in Mathematics 139, Springer, New York (1993) MR1224675

[9] **V M Buhštaber**, *Modules of differentials of the Atiyah–Hirzebruch spectral sequence, II*, Mat. Sb. 83 (1970) 61–76 MR0300284 In Russian; translated in Math. USSR-Sb. 12 (1970) 59–75

[10] **M Colţoiu**, *q–convexity: a survey*, from: "Complex analysis and geometry", (V Ancona, E Ballico, R M Mirò-Roig, A Silva, editors), Pitman Res. Notes Math. Ser. 366, Longman, Harlow (1997) 83–93 MR1477441

[11] **M Cornalba**, **P Griffiths**, *Analytic cycles and vector bundles on non-compact algebraic varieties*, Invent. Math. 28 (1975) 1–106 MR0367263

[12] **P Deligne**, *The Hodge conjecture*, from: "The millennium prize problems", (J Carlson, A Jaffe, A Wiles, editors), Clay Math. Inst., Cambridge, MA (2006) 45–53 MR2238273

[13] **J-P Demailly**, *Cohomology of q–convex spaces in top degrees*, Math. Z. 204 (1990) 283–295 MR1055992

[14] **J-P Demailly**, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics 1, International Press, Somerville, MA (2012) MR2978333

[15] **S Donaldson**, *On the existence of symplectic submanifolds*, from: "Contact and symplectic geometry", (C B Thomas, editor), Publ. Newton Inst. 8, Cambridge Univ. Press (1996) 307–310 MR1432468

[16] **S K Donaldson**, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. 44 (1996) 666–705 MR1438190

[17] **A Dor**, *Immersions and embeddings in domains of holomorphy*, Trans. Amer. Math. Soc. 347 (1995) 2813–2849 MR1282885

[18] **A Douady**, *Cycles analytiques (d'après M F Atiyah et F Hirzebruch)*, from: "Séminaire Bourbaki 1961/1962 (Exposé 223)", W. A. Benjamin, Amsterdam (1966) MR1611206 Reprinted as pp 5–26 in Séminaire Bourbaki 7, Soc. Math. France, Paris, 1995

[19] **B Drinovec Drnovšek**, **F Forstnerič**, *Holomorphic curves in complex spaces*, Duke Math. J. 139 (2007) 203–253 MR2352132

[20] **B Drinovec-Drnovšek**, **F Forstnerič**, *Approximation of holomorphic mappings on strongly pseudoconvex domains*, Forum Math. 20 (2008) 817–840 MR2445119

[21] **B Drinovec Drnovšek**, **F Forstnerič**, *Strongly pseudoconvex domains as subvarieties of complex manifolds*, Amer. J. Math. 132 (2010) 331–360 MR2654777

[22] **O Forster**, *Plongements des variétés de Stein*, Comment. Math. Helv. 45 (1970) 170–184 MR0269880

[23] **F Forstnerič**, *Noncritical holomorphic functions on Stein manifolds*, Acta Math. 191 (2003) 143–189 MR2051397

[24] **F Forstnerič**, *Holomorphic flexibility properties of complex manifolds*, Amer. J. Math. 128 (2006) 239–270 MR2197073

[25] **F Forstnerič**, *Stein manifolds and holomorphic mappings: the homotopy principle in complex analysis*, Ergeb. Math. Grenzgeb. 56, Springer, Heidelberg (2011) MR2975791

[26] **F Forstnerič**, **J Globevnik**, *Discs in pseudoconvex domains*, Comment. Math. Helv. 67 (1992) 129–145 MR1144617

[27] **H Grauert**, *Analytische Faserungen über holomorph-vollständigen Räumen*, Math. Ann. 135 (1958) 263–273 MR0098199

[28] **H Grauert**, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. 68 (1958) 460–472 MR0098847

[29] **H Grauert**, *Une notion de dimension cohomologique dans la théorie des espaces complexes*, Bull. Soc. Math. France 87 (1959) 341–350 MR1505124

[30] **H Grauert**, *Theory of q–convexity and q–concavity*, from: "Several complex variables, VII", (H Grauert, T Peternell, R Remmert, editors), Encyclopaedia Math. Sci. 74, Springer, Berlin (1994) 259–284 MR1326623

[31] **M L Green**, *Infinitesimal methods in Hodge theory*, from: "Algebraic cycles and Hodge theory", (A Albano, F Bardelli, editors), Lecture Notes in Math. 1594, Springer, Berlin (1994) 1–92 MR1335239

[32] **P Griffiths**, **J Harris**, *On the Noether–Lefschetz theorem and some remarks on codimension-two cycles*, Math. Ann. 271 (1985) 31–51 MR779603

[33] **A Grothendieck**, *Hodge's general conjecture is false for trivial reasons*, Topology 8 (1969) 299–303 MR0252404

[34] **H A Hamm**, *Zur Homotopietyp Steinscher Räume*, J. Reine Angew. Math. 338 (1983) 121–135 MR684017

[35] **H A Hamm**, *Zum Homotopietyp q–vollständiger Räume*, J. Reine Angew. Math. 364 (1986) 1–9 MR817633

[36] **G M Henkin**, **J Leiterer**, *Andreotti–Grauert theory by integral formulas*, Progress in Mathematics 74, Birkhäuser, Boston (1988) MR986248

[37] **W V D Hodge**, *The topological invariants of algebraic varieties*, from: "Proceedings of the International Congress of Mathematicians, I", Amer. Math. Soc. (1952) 182–192 MR0046075

[38] **C U Jensen**, *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*, Lecture Notes in Mathematics 254, Springer, Berlin (1972) MR0407091

[39] **B Jöricke**, *Envelopes of holomorphy and holomorphic discs*, Invent. Math. 178 (2009) 73–118 MR2534093

[40] **S Kaliman**, **M Zaĭdenberg**, *A tranversality theorem for holomorphic mappings and stability of Eisenman–Kobayashi measures*, Trans. Amer. Math. Soc. 348 (1996) 661–672 MR1321580

[41] **K Kodaira**, **D C Spencer**, *Divisor class groups on algebraic varieties*, Proc. Nat. Acad. Sci. USA 39 (1953) 872–877 MR0063122

[42] **W S Massey**, *Singular homology theory*, Graduate Texts in Mathematics 70, Springer, New York (1980) MR569059

[43] **J Milnor**, *On axiomatic homology theory*, Pacific J. Math. 12 (1962) 337–341 MR0159327

[44] **T Ohsawa**, *Completeness of noncompact analytic spaces*, Publ. Res. Inst. Math. Sci. 20 (1984) 683–692 MR759689

[45] **M Peternell**, *q–completeness of subsets in complex projective space*, Math. Z. 195 (1987) 443–450 MR895316

[46] **M Schneider**, *Über eine Vermutung von Hartshorne*, Math. Ann. 201 (1973) 221–229 MR0357858

[47] **G Sorani**, *Omologia degli spazi q–pseudoconvessi*, Ann. Scuola Norm. Sup. Pisa 16 (1962) 299–304 MR0157005

[48] **C Soulé**, **C Voisin**, *Torsion cohomology classes and algebraic cycles on complex projective manifolds*, Adv. Math. 198 (2005) 107–127 MR2183252

[49] **E H Spanier**, *Algebraic topology*, Springer, New York (1981) MR666554 Corrected reprint

[50] **B Totaro**, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc. 10 (1997) 467–493 MR1423033

[51] **S Trivedi**, *Stratified transversality of holomorphic maps*, Internat. J. Math. 24 (2013) 1350106, 12 MR3158578

[52] **C A Weibel**, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge Univ. Press (1994) MR1269324

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