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Hyperbolic structures from Sol on pseudo-Anosov mapping tori

KENJI KOZAI

The invariant measured foliations of a pseudo-Anosov homeomorphism induce a natural (singular) Sol structure on mapping tori of surfaces with pseudo-Anosov monodromy. We show that when the pseudo-Anosov $\phi\colon S \to S$ has orientable foliations and does not have 1 as an eigenvalue of the induced cohomology action on the closed surface, then the Sol structure can be deformed to nearby cone hyperbolic structures, in the sense of projective structures. The cone angles can be chosen to be decreasing from multiples of 2π.

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1 Introduction

Let $S = S_{g,n}$ be a surface of genus g with n punctures such that $2g+n > 2$. Given a homeomorphism $\phi\colon S \to S$, we can define the mapping torus

$$M_\phi = \frac{S \times [0,1]}{(x,1) \sim (\phi(x),0)}.$$

Thurston's hyperbolization theorem [22] states that M_ϕ is hyperbolic if and only if ϕ is pseudo-Anosov. A pseudo-Anosov homeomorphism $\phi\colon S \to S$ has two transverse (possibly singular) foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ with transverse measures μ_s and μ_u, respectively, and a constant $\lambda > 1$ such that ϕ preserves $\mathcal{F}^s$ and $\mathcal{F}^u$ and scales the measures by λ^{-1} and λ. When S is not closed, the map ϕ induces a pseudo-Anosov map on the closed surface $\overline{S}$ of genus g, where the n punctures have been filled in. We will also call this map $\phi\colon \overline{S} \to \overline{S}$.

The measured foliations $(\mathcal{F}^s, \mu_s)$ and $(\mathcal{F}^u, \mu_u)$ endow S with a singular Euclidean metric. The corresponding suspension flow ϕ_t on M_ϕ, expanding the leaves of $\mathcal{F}^u$ by a factor of e^t and contracting the leaves of $\mathcal{F}^s$ by e^{-t}, has period $\log\lambda$, so that $\phi_{\log\lambda} = \phi$. One model for Sol geometry is to take $\mathbb{R}^3$ with the metric

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

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so the suspension flow can be viewed as an isometry of Sol translating the surface S in the z direction. The identification $(x, y, z + \log\lambda) \sim (\phi(x, y), z)$ then defines a singular Sol structure on M_ϕ, with singular locus Σ given by the orbits of the singular points and punctures of $\mathcal{F}^s$ and $\mathcal{F}^u$.

In the case where S is a punctured torus, Hodgson [14] studied how to deform representations of $\pi_1(M_\phi)$ near a representation corresponding to a projection of the Sol structure. Sol space contains embedded hyperbolic planes, and the representations studied in [14] correspond to projecting the 3–manifold onto a hyperbolic plane inside Sol, resulting in a reducible representation that gives M_ϕ the structure of a transversely hyperbolic foliation (recall that a representation $\rho\colon \pi_1(M_\phi) \to \mathrm{PSL}(2, \mathbb{C})$ is *irreducible* if the only subspaces of $\mathbb{C}^2$ that are invariant under ρ are trivial). Further results about deforming reducible representations to irreducible representations can be found in Frohman and Klassen [8], Heusener and Kroll [10] and Abdelghani and Lines [2]. Heusener, Porti and Suárez [12] have also shown that hyperbolic structures can be regenerated from Sol, constructing a path of nearby hyperbolic structures that collapse onto a circle, and rescaling the metric as it collapses to obtain the Sol metric on M_ϕ.

In the case where S is not the punctured torus, such a regeneration theorem is not known. In this paper, we utilize half-pipe (HP) geometry, studied by Danciger [4], to regenerate hyperbolic structures in a more general setting. In particular, we will prove the following result.

Theorem 6.3 *Let $\phi\colon S \to S$ be a pseudo-Anosov homeomorphism whose stable and unstable foliations, $\mathcal{F}^s$ and $\mathcal{F}^u$, are orientable and $\phi^*\colon H^1(\bar{S}) \to H^1(\bar{S})$ does not have 1 as an eigenvalue. Then there exists a family of singular hyperbolic structures on M_ϕ, smooth on the complement of Σ and with cone singularities along Σ, that degenerate to a transversely hyperbolic foliation. Furthermore, the* Sol *structure on M_ϕ is obtained as a rescaled limit, as projective structures, of the path of degenerating structures. Moreover, the cone angles can be chosen to be decreasing.*

The proof of Theorem 6.3 uses HP structures as an intermediate. We find a family of HP structures that collapse, such that rescaling the collapse in an appropriate manner yields Sol. The HP structures involved are built from a representation $\rho_0\colon \pi_1(M_\phi \setminus \Sigma) \to \mathrm{PSL}(2, \mathbb{C})$ arising from projecting the 3–dimensional Sol space to one of its embedded hyperbolic planes, along with a first-order deformation of the representation. The following result of Danciger is an application of the Ehresmann–Thurston principle:

Theorem [4, Proposition 3.6] *Let M_0 be a compact n–manifold with boundary and let M be a thickening of M_0 so that $M \setminus M_0$ is a collar neighborhood of ∂M_0.*

Suppose M has an HP *structure defined by the developing map D_{HP}, and holonomy representation σ_{HP}. Let X be either $\mathbb{H}^n$ or* AdS^n *and let $\rho_t\colon \pi_1(M_0) \to \mathrm{Isom}(X)$ be a family of representations compatible to first order at time $t = 0$ with σ_{HP}. Then we can construct a family of X–structures on M_0 with holonomy ρ_t for short time.*

As noted in [4], given an HP structure, the regeneration of a hyperbolic structure only requires that it exists on the level of representations. In Theorem 6.3, the conditions that the invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are orientable and that ϕ^* does not have 1 as an eigenvalue guarantee smoothness of the representation variety at ρ_0, so we can find a nearby family of representations ρ_t. We also do a simple computation to generalize Danciger's notion of infinitesimal cone angle to multiple components. This allows us to adapt the HP machinery to show that there are singular hyperbolic structures near the HP structures, which give the Sol structure as a rescaled limit. We will then show that the singular locus can be controlled so that the family of $\mathbb{H}^3$ structures are cone manifolds.

Outline In Section 2, we present an overview of geometric structures and infinitesimal deformations. Section 3 describes the collapsed structure as a metabelian representation and establishes the notation used in the following section. Section 4 proves smoothness of the representation variety at the metabelian representation, which is used in Section 5 to show that we can find nearby 3–dimensional hyperbolic structures via HP geometry. Section 6 analyzes the behavior of the singular locus to show that the singularities can be realized as cone singularities, providing the final step to Theorem 6.3.

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2 Background

Let X be a manifold and G be a group of analytic diffeomorphisms of X. We will study geometric structures on a manifold M through the framework of (X, G)–structures described by Ehresmann [5] and Thurston [21].

2.1 (X, G)–structures

An (X, G)–*structure* on a manifold M is a collection of charts $\{\psi_\alpha\colon U_\alpha \to X\}$, where the $\{U_\alpha\}$ are an open cover of M and the transition maps $\psi_\alpha \psi_\beta^{-1}$ are restrictions of elements $g_{\alpha\beta} \in G$.

In the context of this paper, we will take X to be (a subset of) $\mathbb{RP}^3$ and G to be (a subgroup of) $\mathrm{PGL}(4,\mathbb{R})$, with $\mathbb{H}^3$ and Sol being described as projective structures. An (X,G)–structure on M defines a developing map $D\colon \widetilde{M} \to X$ that is equivariant under the holonomy representation $\rho\colon \pi_1(M) \to X$.

A smooth family of (X,G)–structures on a manifold M can be described by a family of developing maps $D_t\colon \widetilde{M} \to X$ and corresponding holonomy representations $\rho_t\colon \pi_1(M) \to G$. Two families of (X,G)–structures D_t and F_t such that $D_0 = F_0$ are equivalent if there exists a smooth family g_t of elements in G and a smooth family of diffeomorphisms ϕ_t defined on all but a neighborhood of ∂M such that $D_t = g_t \circ F_t \circ \widetilde{\phi}_t$, where $\widetilde{\phi}_t$ is the lift of ϕ_t, $g_0 = 1$ and $\widetilde{\phi}_0$ is the identity. Such a deformation D_t is trivial if D_0 is equivalent to the family of structures $F_t = D_0$. In this case, the holonomy representations also differ by conjugation by a smooth family g_t, ie $\rho_t = g_t \rho_0 g_t^{-1}$.

We will study deformations of geometric structures through their representations. Let $R(\pi_1(M),G) = \mathrm{Hom}(\pi_1(M),G)$ be the variety of representations of $\pi_1(M)$ into G, $\mathcal{X}(\pi_1(M),G) = R(\pi_1(M),G)/\!/G$ be the character variety, where the quotient is the GIT quotient as G acts by conjugation, and let $\mathcal{D}(M,(X,G))$ be the space of (X,G)–structures on M up to the equivalence defined. The Ehresmann–Thurston principle states that, locally, deformations of geometric structures can be studied by their holonomy representations (see [9] for a proof of the theorem).

Theorem (Ehresmann–Thurston principle) *Let X be a manifold upon which a Lie group G acts transitively. Let M have a (X,G)–structure with holonomy representation $\rho\colon \pi_1(M) \to G$. For ρ' sufficiently near ρ in the space of representations* $\mathrm{Hom}(\pi_1(M),G)$*, there exists a nearby (X,G)–structure on M with holonomy representation ρ'.*

Given a smooth family of representations $\rho_t\colon \pi_1(M) \to G$, we can study the infinitesimal change in ρ_t at ρ_0, as in [14]. The derivative of the homomorphism condition $\rho_t(ab) = \rho_t(a)\rho_t(b)$ yields

$$\rho_t'(ab) = \rho_t'(a)\rho_t(b) + \rho_t(a)\rho_t'(b).$$

In order to normalize the derivative, we multiply on the right by $\rho_t(ab)^{-1}$ to translate back to the identity element to obtain

$$\rho_t'(ab)\rho_t(ab)^{-1} = \rho_t'(a)\rho_t(a)^{-1} + \rho_t(a)\rho_t'(b)\rho_t(b)^{-1}\rho_t(a)^{-1}.$$

The second term is defined to be

$$\mathrm{Ad}_{\rho_t(a)}(\rho_t'(b)\rho_t(b)^{-1}) = \rho_t(a)\rho_t'(b)\rho_t(b)^{-1}\rho_t(a)^{-1}.$$

The Lie algebra of G, denoted by $\mathfrak{g}$, turns into a $\pi_1(M)$–module, with $\pi_1(M)$ acting via Ad_{ρ_0}. Then a cocycle of $\pi_1(M)$ with coefficients in $\mathfrak{g}$ twisted by Ad_{ρ_0} is defined as a map $z\colon \pi_1(M) \to \mathfrak{g}$, where $z(\gamma) = \rho'(\gamma)\rho_0(\gamma)^{-1}$ and ρ' is the derivative evaluated at $t = 0$, such that the map z satisfies the cocycle condition

$$z(ab) = z(a) + \mathrm{Ad}_{\rho_0(a)}\, z(b). \tag{1}$$

The group of all maps satisfying the condition (1) is defined to be $Z^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}_{\rho_0}})$. Differentiating the triviality condition for representations $\rho_t = g_t \rho_0 g_t^{-1}$ yields the coboundary condition

$$z(\gamma) = u - \mathrm{Ad}_{\rho_0(\gamma)}\, u \tag{2}$$

for some $u \in \mathfrak{g}$. The set of cocycles satisfying (2) is defined to be $B^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}_{\rho_0}})$, the set of coboundaries of $\pi_1(M)$ with coefficients in $\mathfrak{g}$ twisted by Ad_{ρ_0}. Weil [23; 16] has noted that $Z^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}_{\rho_0}})$ contains the tangent space to $R(\pi_1(M), G)$ at ρ_0 as a subspace. Provided that we can show that the representation variety at ρ_0 is smooth, we can study the space of cocycles to determine the first-order behavior of deformations of a representation ρ_0.

2.2 Hyperbolic geometry

The hyperboloid model for $\mathbb{H}^3$ is described as a subspace of $\mathbb{R}^{1,3}$. Topologically, $\mathbb{R}^{1,3}$ is the space $\mathbb{R}^4$, but it is endowed with the Lorentzian metric

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

The hyperboloid model for hyperbolic 3–space is

$$\mathbb{H}^3 = \{\vec{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^{1,3} : \|\vec{x}\| = -1, x_1 > 0\},$$

with the metric induced by ds. The isometry group of $\mathbb{H}^3$ in the hyperboloid model is the identity component $\mathrm{SO}^+(1,3)$ of $\mathrm{SO}(1,3)$. Each point in the hyperboloid model intersects exactly one line through the origin in $\mathbb{R}^{1,3}$. Hence, we can also identify the hyperboloid with a subset of $\mathbb{RP}^3$, given by

$$\mathbb{H}^3 = \{[\vec{x}] = [x_1, x_2, x_3, x_4] \in \mathbb{RP}^3 : \|\vec{x}\| < 0\}.$$

There is a well-known method for taking an isometry of $\mathbb{H}^3$ from the upper half-space model (ie an element $A \in \mathrm{PSL}(2,\mathbb{C})$) to the corresponding isometry in the hyperboloid model (see for instance [1, page 66]). First, a point (x_1, x_2, x_3, x_4) from the hyperboloid model is identified with the matrix

$$P(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 + x_2 & x_3 + i x_4 \\ x_3 - i x_4 & x_1 - x_2 \end{bmatrix}.$$

Then A acts on the point (x_1, x_2, x_3, x_4) by

$$AP(x_1, x_2, x_3, x_4)A^*,$$

where A^* denotes the Hermitian transpose of A. This operation preserves $\det P = x_1^2 - x_2^2 - x_3^2 - x_4^2$, so it sends points of the hyperboloid in $\mathbb{R}^{1,3}$ to points of the hyperboloid. The corresponding isometry in the hyperboloid model is the element $A' \in \mathrm{SO}(1,3)$ so that

$$AP(x_1, x_2, x_3, x_4)A^* = P(A'(x_1, x_2, x_3, x_4)).$$

2.3 Sol geometry

Topologically, Sol is $\mathbb{R}^3$, with the metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$. In this model for Sol, one can see that by restricting to any plane $x = \text{constant}$, we obtain a 2–dimensional space that is isometric to the hyperbolic plane via the upper half-plane model. Restricting to the plane $y = \text{constant}$ also yields a space isometric to the hyperbolic plane as the lower half-plane model.

Sol also has an embedding into $\mathbb{RP}^3$ given by

$$(x, y, z) \mapsto [\cosh z, \sinh z, e^z x, e^{-z} y].$$

The image of this map gives Sol as the subspace

$$\mathrm{Sol} = \{[x_1, x_2, x_3, x_4] \in \mathbb{RP}^3 : -x_1^2 + x_2^2 < 0\}.$$

The group PGL(4) contains the identity component of the isometry group of Sol inside $\mathbb{RP}^3$ as elements of the form

$$\begin{bmatrix} \cosh c & \sinh c & 0 & 0 \\ \sinh c & \cosh c & 0 & 0 \\ ae^c & ae^c & 1 & 0 \\ be^{-c} & -be^{-c} & 0 & 1 \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$. Other components can be found by multiplying the diagonal 2×2 blocks by ± 1 or the upper left 2×2 block by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. A further treatment of Sol geometry can be found in [3].

2.4 HP geometry

There are also multiple copies of $\mathbb{H}^3$ lying inside $\mathbb{R}^4$. For each $s > 0$, we can take the hyperboloid

$$\mathbb{H}_s^3 = \{\vec{x} = (x_1, x_2, x_3, x_4) : -x_1^2 + x_2^2 + x_3^2 + s^2 x_4^2 = -1, x_1 > 0\},$$

and the subgroup G_s of $\mathrm{PGL}(4,\mathbb{R})$ preserving the form

$$-x_1^2+x_2^2+x_3^2+s^2x_4^2,$$

to obtain a space isometric to $\mathbb{H}^3$. The isometry to the usual hyperboloid model of $\mathbb{H}^3$ is given by the rescaling map

$$\mathfrak{r}_s=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&s^{-1}\end{bmatrix}.$$

Geometrically, we can think of the family of hyperboloids $\mathbb{H}^3_s$ as flattening out to $\mathbb{H}^2\times\mathbb{R}$ in $\mathbb{R}^4$. Taking the limit as $s\to 0$ yields a model for half-pipe geometry.

Danciger [4] studied degenerations of singular hyperbolic structures using the projective models. An appropriate rescaling of the degeneration yields half-pipe (HP) geometry, a transition geometry between hyperbolic geometry and anti-de Sitter (AdS) geometry.

Three-dimensional HP geometry HP^3, topologically, is $\mathbb{R}^3$. In terms of representations, it can be described as a rescaling of the collapse of the structure group from $\mathrm{SO}(1,3)$ to $\mathrm{SO}(1,2)$. Begin with a representation ρ_1 of $\pi_1(M)$ into $\mathrm{SO}(1,3)$, and describe the collapse of the manifold in the x_4 coordinate by a family of representations ρ_t, so that we end with a representation ρ_0 into $\mathrm{SO}(1,2)\subset\mathrm{SO}(1,3)$ of matrices of the form

$$\rho_0(\gamma)=\begin{bmatrix}A\in\mathrm{SO}(1,2)&0\\0&1\end{bmatrix}.$$

Conjugate the path of representations ρ_t degenerating in this matter by

$$\mathfrak{r}(t)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&t^{-1}\end{bmatrix}$$

and take the limit as $t\to 0$. This will yield a representation ρ_{HP} whose image lies in the set of matrices of $\mathrm{SO}(1,3)$ of the form

$$\lim_{t\to 0}\mathfrak{r}(t)\rho_t(\gamma)\mathfrak{r}(t)^{-1}=\begin{bmatrix}A\in\mathrm{SO}(1,2)&0\\\vec{v}^T&1\end{bmatrix}=\rho_{\mathrm{HP}}(\gamma),\tag{3}$$

where $\vec{v}^T$ is the transpose of a vector in $\mathbb{R}^3$. The vector $\vec{v}$ can be interpreted as an infinitesimal deformation of A into $\mathrm{SO}(1,3)$. A path of representations ρ_t satisfying (3) is said to be *compatible to first order* with ρ_{HP}. The map $\mathfrak{r}(t)$ takes the standard copy of $\mathbb{H}^3$ inside $\mathbb{R}^{1,3}$ to the isometric copy $\mathbb{H}^3_t$. As we take the limit $t \to 0$, we obtain HP^3 as

$$\mathrm{HP}^3 = \lim_{t\to 0} \mathbb{H}^3_t = \{(x_1, x_2, x_3, x_4) : -x_1^2 + x_2^2 + x_3^2 = -1, x_1 > 0\}.$$

As a subset of $\mathbb{RP}^3$, we can think of HP^3 as

$$\mathrm{HP}^3 = \{[x_1, x_2, x_3, x_4] : -x_1^2 + x_2^2 + x_3^2 < 0\}.$$

The structure group G_{HP} is the set of matrices of the form in (3).

A concrete description of $\vec{v}$ can be found by generalizing the isomorphism $\mathrm{SO}(1,3) \cong \mathrm{PSL}(2,\mathbb{C})$. Let κ_s be a non-zero element such that $\kappa_s^2 = -s^2$, and define an algebra $\mathcal{B}_s = \mathbb{R} + \mathbb{R}\kappa_s$ generated over $\mathbb{R}$ by 1 and κ_s. Furthermore, define a conjugation by

$$a + b\kappa_s \mapsto \overline{a + b\kappa_s} = a - b\kappa_s.$$

Then let A^* be the conjugate transpose of A.

We can define a map $P_s = \mathbb{H}^3_s \subset \mathbb{R}^{1,3} \to \mathrm{Herm}(2, \mathcal{B}_s)$ by

$$P_s(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 + x_2 & x_3 + \kappa_s x_4 \\ x_3 - \kappa_s x_4 & x_1 - x_2 \end{bmatrix},$$

where $\mathrm{Herm}(2, \mathcal{B}_s)$ is the set of 2×2 matrices with entries in $\mathcal{B}_s$ such that $A = A^*$. Then define the map $\mathrm{PSL}(2, \mathcal{B}_s) \to G_s$ by $A \mapsto A'$, where A' is the matrix that satisfies

$$A P_s(x_1, x_2, x_3, x_4) A^* = P(A'(x_1, x_2, x_3, x_4)).$$

When $s = 1$, this is the usual isometry from $\mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{SO}(1,3)$. Danciger proved the following:

Theorem [4, Propositions 4.15 and 4.19] *For $s > 0$, the map $\mathrm{PSL}(2, \mathcal{B}_s) \to G_s$ is an isomorphism. When $s = 0$, the map $\mathrm{PSL}(2, \mathcal{B}_0) \to G_0$ is an isomorphism onto the group of* HP *matrices.*

Moreover, in the case $s = 0$, we obtain a geometric interpretation for the vector $\vec{v}$ in (3). If we have a matrix in $\mathrm{PSL}(2, \mathcal{B}_0)$, we can write it as $A + B\kappa_0$, where A is symmetric and B is skew-symmetric. Similarly, we can write $P_0(x_1, x_2, x_3, x_4) = X + Y\kappa_0$, where

$$X = \begin{bmatrix} x_1 + x_2 & x_3 \\ x_3 & x_1 - x_2 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & x_4 \\ -x_4 & 0 \end{bmatrix}.$$

Then

$$(A + B\kappa_0)(X + Y\kappa_0)(A + B\kappa_0)^* = AXA^T + (BXA^T - AXB^T + AYA^T)\kappa_0.$$

In the map $\mathrm{PSL}(2, \mathcal{B}_0) \to G_0$, the symmetric part AXA^T determines the first three rows of the HP matrix, and the skew-symmetric part $(BXA^T - AXB^T + AYA^T)$ determines the bottom row of the HP matrix.

Lemma [4, Lemma 4.20] *Let $A + B\sigma$ have determinant ± 1. Then*

$$\det A = \det(A + B\sigma) = \pm 1 \quad \textit{and} \quad \operatorname{tr} BA^{-1} = 0.$$

In other words, B is in the tangent space at A of matrices of constant determinant ± 1.

Hence, when mapped into $\mathbb{RP}^3$, the symmetric part is the usual map $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{SO}(1, 2)$, and the bottom row of an HP matrix comes from the skew-symmetric part. The vector $\vec{v}$ in the HP matrix of (3) is an infinitesimal deformation of the $\mathrm{SO}(1, 2)$ matrix from the collapsed structure.

The key result about HP structures is that we can recover hyperbolic structures from them [4, Proposition 3.6]. Thus, if we can find an HP structure for M_ϕ and construct a transition at the level of representations, then we can deform it to nearby hyperbolic and AdS structures.

3 The metabelian representation

Let ϕ: $S \to S$ be a pseudo-Anosov homeomorphism with orientable invariant foliations $\mathcal{F}^s, \mathcal{F}^u$ with singular set $\sigma = \{s_0, s_1, \dots, s_n\}$ and transverse measures μ_s and μ_u. If S has a puncture p_0, then we can fill in the puncture by taking $\overline{S} = S \cup \{p_0\}$. Either the measured foliations extend smoothly to p_0, or p_0 is a singular point of the foliation. In either case, we simply include p_0 in the set σ, so we can simplify our analysis to the case where S is closed. The orientability assumption gives us some control over the eigenvalues of ϕ^*: $H^1(S) \to H^1(S)$. It also implies that the cone angles at the singular points in the singular Euclidean metric induced by the measured foliations are multiples of 2π — in particular, they are larger than 2π.

The following is a basic result about the eigenvalues of a pseudo-Anosov map; see [7; 17; 19].

Lemma 3.1 (cf McMullen [17, Theorem 5.3]) *Let ϕ be a pseudo-Anosov homeomorphism with dilatation factor λ. Suppose also that ϕ has orientable unstable and stable foliations $\mathcal{F}^u$ and $\mathcal{F}^s$. Then λ and λ^{-1} are simple eigenvalues of ϕ^*.*

Proof If $\mathcal{F}^u$ and $\mathcal{F}^s$ are orientable then their transverse measures μ_u, μ_s represent cohomology classes $\omega_\pm \in H^1(S)$. The fact that ϕ scales the invariant measures by $\lambda^{\pm 1}$ implies that $\phi^*(\omega_\pm) = \lambda^{\pm 1}\omega_\pm$, so that $\lambda^{\pm 1}$ are eigenvalues of ϕ^*.

Let $\omega \in H^1(S)$ be any cohomology class dual to a simple closed curve γ. Since ϕ is pseudo-Anosov, $\phi^{\pm n}(\gamma)$ limits to either $\mathcal{F}^u$ or $\mathcal{F}^s$. In particular,

$$\frac{(\phi^*)^{\pm n}\omega}{\lambda^{\pm n}} \to c\omega_\pm \tag{4}$$

for some $c \neq 0$. Since the classes ω dual to simple closed curves span $H^1(S)$, the eigenspaces for $\lambda^{\pm 1}$ are 1–dimensional. In fact, $\lambda^{\pm 1}$ must be simple eigenvalues by considering the Jordan canonical form. If there existed a generalized eigenvector ω such that $\phi^*\omega = \omega_\pm + \lambda^{\pm 1}\omega$, we would have $(\phi^*)^{\pm n}(\omega) = n\lambda^{\pm(n-1)}\omega_\pm + \lambda^{\pm n}\omega$, so that the condition in (4) is not satisfied. □

In addition to λ and λ^{-1} being simple eigenvalues, we also have that the corresponding eigenvectors come from the measures $\mathcal{F}^u$ and $\mathcal{F}^s$. In particular, if we take $\gamma_1, \gamma_2, \dots, \gamma_{2g}$ to be a basis for $H_1(S)$, then the eigenvector $\vec{e}_\lambda$ is given by

$$\vec{e}_\lambda = (\mu_u(\gamma_1), \mu_u(\gamma_2), \dots, \mu_u(\gamma_{2g}))^T,$$

where the transverse measure μ_u is taken to be a signed measure, ie $\mu_u(-\gamma) = -\mu_u(\gamma)$, if $-\gamma$ is the closed curve γ taken with the orientation opposite to that of $\mathcal{F}^u$. The eigenvector corresponding to λ^{-1} is given by

$$\vec{e}_{\lambda^{-1}} = (\mu_s(\gamma_1), \mu_s(\gamma_2), \dots, \mu_s(\gamma_{2g}))^T.$$

Choose a disk D that contains all of the points in σ, and fix a point on ∂D as the base point for $\pi_1(S \setminus \sigma)$. Let $\delta_1, \delta_2, \dots, \delta_n$ be generators of $\pi_1(S \setminus \sigma)$, so that each δ_i encircles exactly one singularity s_i, each δ_i lies entirely inside D, and the product $\delta_1\delta_2\cdots\delta_n$ is homotopic to the boundary ∂D.

Choose standard generators $\alpha_1, \alpha_2, \dots, \alpha_g$ and $\beta_1, \beta_2, \dots, \beta_g$ of $\pi_1(S)$ such that, for each i, (curves representing) α_i and β_i do not intersect ∂D except at the basepoint for π_1. We will also refer to these curves as

$$\gamma_i = \alpha_i, \quad \gamma_{g+i} = \beta_i, \quad \gamma_{2g+j} = \delta_j.$$

When convenient, we will use α_i, β_i, and δ_j to refer to their respective homology classes.

On the dual generators $\alpha_i^*, \beta_i^*, \delta_j^*$ of $H^1(S \setminus \sigma)$, the map ϕ^* has a block upper-triangular action: the first block on the diagonal corresponds to the action on the closed surface S, and the second block permutes the generators $\delta_1^*, \dots, \delta_n^*$ coming from the curves around the singular points. Strictly speaking, this matrix is a square matrix with dimensions one greater than the dimension of $H^1(S \setminus \sigma)$. There is one redundancy in the generators from the relation $\sum_{j=1}^n \delta_j = 0$ in homology. However, using the additional generator from the singularities makes the lower right block for ϕ^* easier to understand. When discussing $H^1(S \setminus \sigma)$ and ϕ^* in this section, we mean $H^1(S \setminus \sigma)$ with this additional generator and the action on $H^1(S \setminus \sigma)$ with the additional generator, respectively.

Using these generators for $\pi_1(S \setminus \sigma)$, we can present $\Gamma = \pi_1(N_\phi = M_\phi \setminus \Sigma)$ as follows: Γ is generated by the $\alpha_i, \beta_i, \delta_j$ and τ, subject to the relations

$$\tau\alpha_i\tau^{-1} = \phi(\alpha_i), \quad \tau\beta_i\tau^{-1} = \phi(\beta_i), \quad \tau\delta_j\tau^{-1} = w_j\delta_{k_j}w_j^{-1}, \quad \prod_{i=1}^{g}[\alpha_i, \beta_i] = \prod_{j=1}^{n}\delta_j,$$

where the w_j are words in the α_i, β_i and δ_j.

We start with the metabelian representation $\rho_0\colon \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ with

$$\rho_0(\gamma_i) = \begin{bmatrix} 1 & a_i = \mu_u(\gamma_i) \\ 0 & 1 \end{bmatrix},$$

where a_i is the signed length of γ_i in $\mathcal{F}^u$. Note that $a_i = 0$ for $2g < i \le n$. We also set

$$\rho_0(\tau) = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{bmatrix},$$

where τ is the generator in the S^1 direction of M_ϕ, and λ is the pseudo-Anosov dilatation factor of ϕ. There is a singular Sol structure on M_ϕ coming from the pseudo-Anosov action on $\mathcal{F}^u$ and $\mathcal{F}^s$, where $\mathcal{F}^u$ and $\mathcal{F}^s$ provide a singular Euclidean structure on the fibers of M_ϕ. Recall from Section 2.3 that Sol contains embedded hyperbolic planes as "vertical" planes. In the singular Sol structure on M_ϕ, these can be seen as products of a leaf of $\mathcal{F}^s$ with the S^1 direction. The metabelian representation ρ_0 is a projection of the singular Sol structure along the leaves of $\mathcal{F}^u$ onto one of these hyperbolic planes inside of Sol. Such a projection yields a *transversely hyperbolic foliation*; locally, M_ϕ can be viewed as an open subset of $\mathbb{H}^2 \times \mathbb{R}$, and the pseudometric is given by the metric on the $\mathbb{H}^2$ factor and ignoring the second factor.

4 Smoothness of the representation variety

The goal is to deform ρ_0 to a representation into $\mathrm{PSL}(2, \mathbb{C})$, and to realize the representation as the holonomy representation of an $(\mathbb{H}^3, \mathrm{PSL}(2, \mathbb{C}))$–structure on N. We consider $\rho_0 \in R(\pi_1(N_\phi), \mathrm{PSL}(2, \mathbb{R}))$ as the metabelian representation from the previous section. We begin by computing the dimension of the space of classes of twisted cocycles $z \in H^1(\pi_1(N_\phi), \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}_{\rho_0}})$.

Theorem 4.1 *Let ϕ be pseudo-Anosov with stable and unstable foliations which are orientable. Suppose also that $\phi^*\colon H^1(S) \to H^1(S)$ does not have 1 as an eigenvalue. Then $\dim H^1(\Gamma, \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}}) = k$, where k is the number of components of the boundary of N_ϕ.*

Proof Let $z \in Z^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$. Then z is determined by its values on $\gamma_1, \dots, \gamma_{2g+n}$, and τ, subject to the cocycle condition (1) imposed by the relations in Γ. These can be computed via the Fox calculus [16, Chapter 3]. Differentiating the relations

$$\tau\gamma_i\tau^{-1} = \phi(\gamma_i)$$

yields

$$\begin{aligned}
&\frac{\partial[\phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1}]}{\partial\gamma_i} = \frac{\partial\phi(\gamma_i)}{\partial\gamma_i} - \phi(\gamma_i)\tau\gamma_i^{-1} = \frac{\partial\phi(\gamma_i)}{\partial\gamma_i} - \tau,\\
&\frac{\partial[\phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1}]}{\partial\gamma_j} = \frac{\partial\phi(\gamma_i)}{\partial\gamma_j}, \quad i \neq j,\\
&\frac{\partial[\phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1}]}{\partial\tau} = \phi(\gamma_i) - \phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1} = \phi(\gamma_i) - 1.
\end{aligned} \tag{5}$$

Choosing the basis

$$e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

for $\mathfrak{sl}(2,\mathbb{C})$, the values $z(\gamma_i)$ can be expressed in coordinates (x_i, y_i, z_i), where $z(\gamma_i)$ is the matrix

$$z(\gamma_i) = \begin{bmatrix} y_i & x_i \\ z_i & -y_i \end{bmatrix},$$

and we similarly let $z(\tau)$ be given in the coordinates (x_0, y_0, z_0). We note that by using the coboundary condition from (2) we can compute the set of coboundaries $B^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ as the set of cocycles z' satisfying

$$z'(\gamma_i) = \begin{bmatrix} -a_i z & 2a_i y + a_i^2 z \\ 0 & a_i z \end{bmatrix} \quad \text{and} \quad z'(\tau) = \begin{bmatrix} 0 & x - \lambda x \\ z - \lambda^{-1} z & 0 \end{bmatrix},$$

where $x, y, z \in \mathbb{C}$ parametrize $B^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$. In particular, adding the appropriate coboundary z' to z, we can set $x_0 = z_0 = 0$. To simplify the calculation somewhat, we will assume that $z(\tau)$ has the form

$$z(\tau) = \begin{bmatrix} y_0 & 0 \\ 0 & -y_0 \end{bmatrix}.$$

We first note that if W is a word in the γ_i, then $\rho(W) = \left[\begin{smallmatrix} 1 & A \\ 0 & 1 \end{smallmatrix}\right]$ for some real number A. Then, under the chosen basis for $\mathfrak{sl}(2,\mathbb{C})$, $\mathrm{Ad}_{\rho_0(W)}$ acts by

$$\begin{bmatrix} 1 & -2A & -A^2 \\ 0 & 1 & A \\ 0 & 0 & 1 \end{bmatrix}.$$

We obtain one term from $\partial\phi(\gamma_i)/\partial\gamma_j$ for each instance of γ_j in $\phi(\gamma_i)$ (with a negative sign if γ_j^{-1} appears), and each term is a word in the γ_j.

Similarly, we can compute that $\mathrm{Ad}_{\rho_0(\tau)}$ acts on $\mathfrak{sl}(2,\mathbb{C})$ via

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}.$$

We see that $Z^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ is determined, as in [13], by a subset of vectors $\vec{v} = (x_1, \dots, x_{2g+n}, y_0, y_1, \dots, y_{2g+n}, z_1, \dots, z_{2g+n})^T$ such that $R\vec{v} = 0$, where R decomposes into blocks

$$R = \begin{bmatrix} \mathring{\phi}^* - \lambda I & -2\lambda\vec{a} & K & C \\ 0 & 0 & \mathring{\phi}^* - I & D \\ 0 & 0 & 0 & \mathring{\phi}^* - \lambda^{-1} I \end{bmatrix}, \quad \text{where } \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{2g+n} \end{pmatrix}.$$

Here the zeros represent block matrices of the appropriate sizes, and $\mathring{\phi}^*\colon H^1(S \setminus \sigma) \to H^1(S \setminus \sigma)$ is the $(2g+n) \times (2g+n)$ matrix describing the cohomology action induced by ϕ, which can be written as a block matrix

$$\begin{bmatrix} \phi^* & * \\ 0 & P \end{bmatrix},$$

where $P = (p_{ij})$ is a permutation matrix denoting the permutation of the singularities in σ by ϕ. In particular, if $\tau\delta_j\tau^{-1} = w_j\delta_{k_j}w_j^{-1}$, then $p_{jk_j} = 1$. By Lemma 3.1, $\phi^* - \lambda I$ and $\phi^* - \lambda^{-1} I$ have 1–dimensional kernel. Furthermore, since ϕ^* does not have 1 as an eigenvalue, the dimension of the kernel of $\mathring{\phi}^* - I$ is equal to the number of disjoint cycles of the permutation of the punctures. But a cycle in the permutation corresponds to a single boundary component of N_ϕ. Hence, the kernel of R has dimension at most $2 + k + 1$, where the additional 1 comes from the $(2g+n+1)^{\text{st}}$ column of R, and

$$k = \#\text{ of components of } \Sigma = \#\text{ of components of } \partial N.$$

Now consider the upper left portion of the matrix R, which we will call U:

$$U = \begin{bmatrix} \mathring{\phi}^* - \lambda I & -2\lambda\vec{a} & K \\ 0 & 0 & \mathring{\phi}^* - I \end{bmatrix}.$$

If $\operatorname{null}(R) > 2 + k$, then we must have that $\operatorname{null}(U) > k + 1$.

Since λ is a simple eigenvalue of ϕ^* and $(a_1, \dots, a_{2g})^T$ is a corresponding eigenvector for λ, $(a_1, \dots, a_{2g})^T$ is not in the image of $\phi^* - \lambda I$. Hence, for any $\vec{y}$ in the kernel of $\phi^* - I$, there is a unique y_0 such that $K\vec{y} - y_0(a_1, \dots, a_{2g})^T$ is in the image of $\phi^* - \lambda I$. Therefore $\operatorname{null}(U) = k + 1$

Hence $\operatorname{null}(R) = 2 + k$. However, the solution arising from the kernel of $\mathring{\phi}^* - \lambda I$ is the eigenvector

$$\vec{v} = (a_1, \dots, a_{2g+n}, 0, \dots, 0, 0, \dots, 0)^T,$$

which is a coboundary. So we have that $\dim H^1(\Gamma, \mathfrak{sl}(2, \mathbb{R})_{\mathrm{Ad}_{\rho_0}}) \le k + 1$. Finally, there is one further redundancy since

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] = \prod_{j=1}^{n} \delta_j.$$

From the $\mathring{\phi}^* - I$ block, we can see that $y_{2g+1}, \dots, y_{2g+n}$ can be freely chosen as long as $y_{2g+j} = y_{2g+k_j}$ whenever $\tau\delta_j\tau^{-1} = w_j\delta_{k_j}w_j^{-1}$. Hence, the upper left (= lower right) entry of $z\big(\prod_{j=1}^{n} \delta_j\big)$ can be freely chosen to be any quantity

$$y_{2g+1} + y_{2g+2} + \dots y_{2g+n}. \tag{6}$$

The relation $\prod_{i=1}^{g} [\alpha_i, \beta_i] = \prod_{j=1}^{n} \delta_j$ forces the sum in (6) to be a fixed quantity coming from the upper left entry of $\prod_{i=1}^{g} [\alpha_i, \beta_i]$, which has no dependence on y_{2g+j}, for $1 \le j \le n$.

Therefore, the relation drops the dimension of the space of cocycles by 1, and

$$\dim H^1(\Gamma, \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}_{\rho_0}}) = k. \qquad \square$$

In order to show that $R(\pi_1(N_\phi), \mathrm{PSL}(2, \mathbb{C}))$ is smooth at ρ_0, following [13; 11] we define a *formal deformation* of $\rho\colon \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$ for a fixed 3–manifold M to be a homomorphism $\rho_\infty\colon \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C}[\![t]\!])$ of the form

$$\rho_\infty(\gamma) = \pm \exp\Big(\sum_{i=1}^{\infty} t^i u_i(\gamma)\Big)\rho(\gamma)$$

where $u_i\colon \pi_1(M) \to \mathfrak{sl}(2, \mathbb{C})$ are elements of $C^1(\pi_1(M), \mathfrak{sl}(2, \mathbb{C})_{\mathrm{Ad}_\rho})$, and evaluating ρ_∞ at $t = 0$ yields ρ. If ρ_∞ is a homomorphism modulo t^{j+1}, we say that ρ_∞

is *a formal deformation up to order j*. A cocycle $u_1 \in Z^1(\pi_1(M), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_\rho})$ is *formally integrable* if there is a formal deformation of ρ with leading term u_1. In [13] it is shown that, given a deformation of order j, there is an obstruction class $\zeta_{j+1} \in H^2(\pi_1(M), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_\rho})$ to extending to a deformation of order $j+1$:

Proposition 4.2 [13, Proposition 3.1] *Let* $\rho \in R(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ *and* $u_i \in C^1(\pi_1(M), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_\rho})$, $1 \le i \le j$ *be given. If*

$$\rho_j(\gamma) = \exp\Big(\sum_{i=1}^{j} t^i u_i(\gamma)\Big)\rho(\gamma)$$

is a homomorphism into $\mathrm{PSL}(2, \mathbb{C}[\![t]\!])$ *modulo* t^{j+1}, *then there exists an obstruction class* $\zeta_{j+1}^{(u_1,\dots,u_k)} \in H^2(\pi_1(M), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_\rho})$ *such that:*

(1) *There is a cochain* $u_{j+1}\colon \pi_1(M) \to \mathfrak{sl}(2,\mathbb{C})$ *such that*

$$\rho_{j+1}(\gamma) = \exp\Big(\sum_{i=1}^{j+1} t^i u_i(\gamma)\Big)\rho(\gamma)$$

is a homomorphism modulo t^{j+2} *if and only if* $\zeta_{j+1} = 0$.

(2) *The obstruction* ζ_{j+1} *is natural, ie if* f *is a homomorphism then* $f^*\rho_j := \rho_j \circ f$ *is also a homomorphism modulo* t^{j+1}, *and*

$$f^*(\zeta_{j+1}^{(u_1,\dots,u_j)}) = \zeta_{j+1}^{(f^*u_1,\dots,f^*u_j)}.$$

We will denote the inclusion map by $i\colon \partial M \to M$.

Lemma 4.3 *Let* M *be a 3–manifold with torus boundary components* $\partial M = \bigsqcup_{i=1}^k T_i$. *Let* $\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ *be a non-abelian representation such that* $\rho(\pi_1(T_i))$ *contains a non-parabolic element for each component* T_i *of* ∂M. *If*

$$\dim H^1(\pi_1(M), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_\rho}) = k,$$

where k *is the number of components of* ∂M, *then* $i^*\colon H^2(M, \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_\rho}) \to H^2(\partial M, \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_\rho})$ *is injective.*

Proof We have the cohomology exact sequence for the pair $(M, \partial M)$:

$$H^1(M,\partial M) \longrightarrow H^1(M) \xrightarrow{\alpha} H^1(\partial M) \xrightarrow{\beta} H^2(M,\partial M) \longrightarrow H^2(M) \xrightarrow{i^*} H^2(\partial M) \longrightarrow H^3(M,\partial M) \longrightarrow \cdots,$$

where all cohomology groups are taken to be with the twisted coefficients $\mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_\rho}$. A standard Poincaré duality argument [13; 15; 20] gives that α has half-dimensional image. For a torus T,

$$\dim H^1(\pi_1(T),\mathfrak{sl}(2,\mathbb{C}))=2,$$

as long as $\rho(\pi_1(T))$ contains a hyperbolic element [20]. Hence, α is injective. Since β is dual to α under Poincaré duality, then β is surjective. This implies that i^* is injective. □

We apply this lemma to the metabelian representation ρ_0 to conclude that the representation variety is smooth at ρ_0.

Theorem 4.4 *The metabelian representation* $\rho_0\colon \pi_1(N_\phi)\to \mathrm{PSL}(2,\mathbb{C})$ *is a smooth point of* $R(\pi_1(N_\phi),\mathrm{PSL}(2,\mathbb{C}))$, *with local dimension* $k+3$.

Proof We begin by showing that every cocycle in $Z^1(\pi_1(N_\phi),\mathfrak{sl}(2,\mathbb{C}))$ is integrable. Suppose we have $u_1,\dots,u_j\colon \pi_1(N_\phi)\to\mathfrak{sl}(2,\mathbb{C})$ such that

$$\rho_j(\gamma)=\exp\Big(\sum_{i=1}^{j} t^i u_i(\gamma)\Big)\rho(\gamma)$$

is a homomorphism modulo t^{j+1}. We have that $\partial N_\phi=\bigsqcup_{i=1}^k T_i$ is a disjoint union of tori, and the restriction $\rho_j|_{\pi_1(T_i)}$ to $\pi_1(T_i)$ is also a formal deformation of order j. We have that $\rho_0(T_i)$ contains a non-parabolic element, namely $\rho_0(\tau)$, or a translate. Then the restriction of ρ_0 to $\pi_1(T_i)$ is a smooth point of the representation variety $R(\pi_1(T_i),\mathrm{PSL}(2,\mathbb{C}))$. Hence $\rho_j|_{\pi_1(T_i)}$ extends to a formal deformation of order $j+1$ by the formal implicit function theorem (see [13, Lemma 3.7]). This implies that the restriction of $\zeta_{j+1}^{(u_1,\dots,u_j)}$ to each component $H^2(T_i)<H^2(\partial N_\phi)$ vanishes.

Since

$$H^2(\partial N_\phi)=\bigoplus_{i=1}^{k} H^2(T_i)$$

we have

$$i^*\zeta_{k+1}^{(u_1,\dots,u_k)}=\zeta_{k+1}^{(i^*u_1,\dots,i^*u_k)}=0.$$

We have shown in Theorem 4.1 that $H^1(\pi_1(N_\phi),\mathrm{PSL}(2,\mathbb{C}))$ has dimension k. The injectivity of i^* implies that $\zeta_{k+1}^{(u_1,\dots,u_k)}=0$.

Applying [13, Proposition 3.6] to the formal deformation ρ_∞ results in a convergent deformation. Hence, ρ_0 is a smooth point of the representation variety. As ρ_0 is non-abelian, we have that $\dim B^1(\pi_1(N_\phi),\mathrm{PSL}(2,\mathbb{C}))=3$, so that the dimension of $R(\pi_1(N_\phi),\mathrm{PSL}(2,\mathbb{R}))$ is $k+3$. □

5 Singular hyperbolic structures

In this section, we will use the smoothness result from Theorem 4.4 to find representations that are near the Sol representation. In order to realize the representations as geometric structures, we will need the Ehresmann–Thurston principle [21].

Theorem (Ehresmann–Thurston principle) *Let X be a manifold upon which a Lie group G acts transitively. Let M have a (X,G)–structure with holonomy representation $\rho\colon \pi_1(M) \to G$. For ρ' sufficiently near ρ in the space of representations* $\mathrm{Hom}(\pi_1(M), G)$*, there exists a nearby (X,G)–structure on M with holonomy representation ρ'.*

To utilize the Ehresmann–Thurston principle, we will need to realize all of our structure groups as subgroups of $\mathrm{PGL}(4,\mathbb{R})$. We first study the process by which Sol can be seen as a limit of $\mathrm{HP} = \mathrm{HP}^3$.

Given $s > 0$, we let $\mathfrak{r}_1(s)$ be the rescaling map

$$\mathfrak{r}_1(s) = \begin{bmatrix} \frac{1}{2}(s+s^{-1}) & \frac{1}{2}(s-s^{-1}) & 0 & 0 \\ \frac{1}{2}(s-s^{-1}) & \frac{1}{2}(s+s^{-1}) & 0 & 0 \\ 0 & 0 & 0 & -s \\ 0 & 0 & s^{-1} & 0 \end{bmatrix}.$$

Then $\mathfrak{r}_1(s)$ takes HP to

$$\mathrm{HP}_s = \{[x_1, x_2, x_3, x_4] : -x_1^2 + x_2^2 + s^2 x_4^2 < 0\},$$

which we think of as a copy of HP under a projective change of coordinates. Conjugating G_{HP} by $\mathfrak{r}_1(s)$ gives the structure group G_{HP_s} of HP_s. Regular HP geometry is given by the case $s = 1$. Taking the limit as $s \to 0$ gives the subset

$$\mathrm{HP}_0 = \{[x_1, x_2, x_3, x_4] : -x_1^2 + x_2^2 < 0\}$$

of $\mathbb{RP}^3$. Notice that this is exactly the image of the embedding of Sol into $\mathbb{RP}^3$. We will use this fact to obtain a geometric transition at the level of representations, and apply the Ehresmann–Thurston principle to obtain corresponding developing maps. The map $\mathfrak{r}_1(s)$ can be thought of as the composition of three maps: the first a hyperbolic translation by $\log s$, which causes the x_3 and x_4 coordinates to converge to 0 in the projective sense; followed by a rescaling to recover those coordinates; and then a change of coordinates between x_3 and x_4 to obtain the correct form for Sol. Hence, this can be thought of as a further collapse onto a one-dimensional space, followed by a rescaling. In order to insure that the developing maps behave correctly, we will use the following lemma.

Lemma 5.1 [4, Lemma 3.7] *Let K be a compact set and let $F_t\colon K \to \mathbb{RP}^3$ be any continuous family of functions. Suppose $F_0(K)$ is contained in $\mathbb{X}_s$. Then there is an $\epsilon > 0$ such that $|t| < \epsilon$ and $|r - s| < \epsilon$ implies that $F_t(K)$ is contained in $\mathbb{X}_r$.*

Now we can prove the following result.

Theorem 5.2 *Let $\phi\colon S \to S$ be a pseudo-Anosov homeomorphism whose stable and unstable foliations, $\mathcal{F}^s$ and $\mathcal{F}^u$, are orientable and ϕ^* does not have 1 as an eigenvalue. Then there exists a family of singular hyperbolic structures on M_ϕ, smooth on the complement of Σ, that degenerate to a transversely hyperbolic foliation. Furthermore, the* Sol *structure on M_ϕ is obtained as a rescaled limit, as projective structures, of the path of degenerating structures.*

Proof From the proof of Theorem 4.1, we can find a cocycle

$$z \in Z^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{R})_{\mathrm{Ad}_{\rho_0}})$$

corresponding to $\mathcal{F}^s$. The simple eigenvalue λ^{-1} of $\mathring{\phi}^*$ has corresponding eigenvector coming from $b_1 = \mu_s(\gamma_1), \dots, b_{2g+n} = \mu_s(\gamma_{2g+n})$. More specifically, ϕ^* does not have 1 as an eigenvalue, so we can solve

$$(\mathring{\phi}^* - I)\begin{pmatrix} y_1 \\ \vdots \\ y_{2g+n}\end{pmatrix} = -D\begin{pmatrix} b_1 \\ \vdots \\ b_{2g+n}\end{pmatrix}, \tag{7}$$

where $D_{2g\times 2g}$ is the restriction of D to the upper left $2g \times 2g$ entries.

Finally, since λ is a simple eigenvalue of $\mathring{\phi}^*$, we can also solve

$$(\mathring{\phi}^* - \lambda I)\begin{pmatrix} x_1 \\ \vdots \\ x_{2g+n}\end{pmatrix} - 2\lambda\begin{pmatrix} a_1 \\ \vdots \\ a_{2g+n}\end{pmatrix} y_0 = -K\begin{pmatrix} y_1 \\ \vdots \\ y_{2g+n}\end{pmatrix} - C\begin{pmatrix} b_1 \\ \vdots \\ b_{2g+n}\end{pmatrix}. \tag{8}$$

Now we will use the above cocycle, which has the form

$$z(\gamma_i) = \begin{bmatrix} y_i & x_i \\ b_i & -y_i \end{bmatrix}, \quad z(\tau) = \begin{bmatrix} y_0 & 0 \\ 0 & -y_0 \end{bmatrix}.$$

The representation ρ_0 and the cocycle z are converted into an HP representation, using the description of G_{HP} given in Section 2.4. In particular, ρ_0 and z are combined to form a representation of $\pi_1(N_\phi)$ into $\mathrm{PSL}(2,\mathcal{B}_0)$ by $\gamma \mapsto \rho_0(\gamma)+z(\gamma)\rho_0(\gamma)\kappa_0$; then use the isomorphism from $\mathrm{PSL}(2,\mathcal{B}_0)$ to $G_0 = G_{\mathrm{HP}}$ to obtain

$$\rho_{\mathrm{HP}}(\gamma_i) = \begin{bmatrix} 1+\frac{1}{2}a_i^2 & -\frac{1}{2}a_i^2 & a_i & 0 \\ \frac{1}{2}a_i^2 & 1-\frac{1}{2}a_i^2 & a_i & 0 \\ a_i & -a_i & 1 & 0 \\ -b_i - a_i^2 b_i + 2a_i y_i + x_i & -b_i + a_i^2 b_i - 2a_i y_i - x_i & 2y_i - 2a_i b_i & 1 \end{bmatrix},$$

$$\rho_{\mathrm{HP}}(\tau) = \begin{bmatrix} \frac{1}{2}(\lambda+\lambda^{-1}) & \frac{1}{2}(\lambda-\lambda^{-1}) & 0 & 0 \\ \frac{1}{2}(\lambda-\lambda^{-1}) & \frac{1}{2}(\lambda+\lambda^{-1}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2y_0 & 1 \end{bmatrix}.$$

Conjugating the HP representation by

$$\mathfrak{r}_1(s) = \begin{bmatrix} \frac{1}{2}(s+s^{-1}) & \frac{1}{2}(s-s^{-1}) & 0 & 0 \\ \frac{1}{2}(s-s^{-1}) & \frac{1}{2}(s+s^{-1}) & 0 & 0 \\ 0 & 0 & 0 & -s \\ 0 & 0 & s^{-1} & 0 \end{bmatrix}$$

and taking $s \to 0$ gives the Sol representation

$$\rho_{\mathrm{Sol}}(\gamma_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_i & b_i & 1 & 0 \\ a_i & -a_i & 0 & 1 \end{bmatrix}, \quad \rho_{\mathrm{Sol}}(\tau) = \begin{bmatrix} \frac{1}{2}(\lambda+\lambda^{-1}) & \frac{1}{2}(\lambda-\lambda^{-1}) & 0 & 0 \\ \frac{1}{2}(\lambda-\lambda^{-1}) & \frac{1}{2}(\lambda+\lambda^{-1}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the Sol representation is obtained as a rescaled limit of a family of HP representations, with the rescaling limit given by $\mathfrak{r}_1(s)$.

The structure groups for HP, $\mathbb{H}^3$ and Sol can be written as subgroups of $\mathrm{PGL}(4,\mathbb{R})$, giving them $(\mathbb{RP}^3, \mathrm{PGL}(4,\mathbb{R}))$–structures. Since the Sol representation, as a representation into $\mathrm{PGL}(4,\mathbb{R})$, comes from an actual Sol structure on N_ϕ, then by the Ehresmann–Thurston principle, for small s, the conjugates $\mathfrak{r}_1(s)\rho_{\mathrm{HP}}\mathfrak{r}_1(s)^{-1}$ are holonomy representations for real projective structures, with developing maps D_s.

Moreover, the Sol structure can be thought of as a $(\mathrm{HP}_0, G_{\mathrm{HP}_0})$ structure, and applying Lemma 5.1 with $\mathbb{X} = \mathrm{HP}$ to D_s and a compact fundamental domain for N_ϕ, we see that for sufficiently small s, the projective structures from the Ehresmann–Thurston principle correspond to HP_s structures, which are rescaled HP structures.

Fix such an $s = s_0$, and consider the underlying HP structure. Since ρ_0 is a smooth point of $R(\pi_1(N_\phi), \mathrm{PSL}(2,\mathbb{C}))$, by work of Danciger [4, Proposition 3.6], there exists a family of hyperbolic structures on N_ϕ, given by their holonomy representations

$\rho_t\colon \pi_1(N_\phi) \to \mathrm{SO}(1,3)$, such that at $t=0$ we obtain the $\mathrm{SO}(1,3)$ version of the representation ρ_0. Furthermore, conjugating ρ_t by $\mathfrak{r}(t)$ yields ρ_{HP}.

For a fixed s,

$$\mathfrak{r}_1(s)\mathfrak{r}(t)\rho_t\mathfrak{r}(t)^{-1}\mathfrak{r}_1(s)^{-1}$$

limits to $\mathfrak{r}_1(s)\rho_{\mathrm{HP}}\mathfrak{r}_1(s)^{-1}$. So taking the diagonal path

$$\mathfrak{r}_1(t)\mathfrak{r}(t)\rho_t\mathfrak{r}(t)^{-1}\mathfrak{r}_1(t)^{-1}$$

yields a rescaling of ρ_t that limits to the Sol structure. □

Note that the cocycle z has the form

$$z(\gamma_i) = \begin{bmatrix} y_i & x_i \\ b_i & -y_i \end{bmatrix},$$

where $b_i = \mu_s(\gamma_i)$. In particular, the deformation of ρ_0 contains the information of $\mathcal{F}^s$. The deformation from the upper-triangular representation ρ_0, which is a projection parallel to $\mathcal{F}^u$ onto a leaf of $\mathcal{F}^s$, behaves like a deformation in a direction transverse to $\mathcal{F}^s$.

6 Behavior of the singular locus

Theorem 5.2 gives a family of hyperbolic structures on $M_\phi \setminus \Sigma$. In general, the singular locus Σ may not remain as cone singularities. In this section, we will show that it is possible to control the singularities so that we obtain a family of nearby cone manifolds.

The manifold $N_\phi = M_\phi \setminus \Sigma$ has torus boundary components, $\partial N_\phi = \bigsqcup_{i=1}^k T_i$. Let m_i be a meridian curve for T_i, and l_i a longitudinal curve. There is a model for a torus T degenerating to the HP structure described by the representation

$$\rho_{\mathrm{HP}}(m) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \omega & 1 \end{bmatrix}, \quad \rho_{\mathrm{HP}}(l) = \begin{bmatrix} \cosh d & \sinh d & 0 & 0 \\ \sinh d & \cosh d & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \mu & \pm 1 \end{bmatrix},$$

which is given in [4]. In particular, take the family of representations into $\mathrm{SO}(1,3)$ such that

$$\rho_t(m) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega t & -\sin\omega t \\ 0 & 0 & \sin\omega t & \cos\omega t \end{bmatrix}, \quad \rho_t(l) = \begin{bmatrix} \cosh d & \sinh d & 0 & 0 \\ \sinh d & \cosh d & 0 & 0 \\ 0 & 0 & \pm\cos\mu t & -\sin\mu t \\ 0 & 0 & \sin\mu t & \pm\cos\mu t \end{bmatrix}.$$

Then, conjugating by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^{-1} \end{bmatrix}$$

and taking the limit as $t \to 0$ yields $\rho_{\mathrm{HP}}(m)$ and $\rho_{\mathrm{HP}}(l)$. Thus, ω, which is called the *infinitesimal rotation* in [4], describes the infinitesimal change in the cone angle about that component of the singularity.

In the case that Σ has multiple components, as in our case, we can modify the computation. From the construction of ρ_0, we can see that each $\rho_0(l_i)$ is a hyperbolic translation with an axis in $\mathbb{H}^2$ having a common endpoint at infinity. Specifically, they all differ from

$$\rho_0(\tau) = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{bmatrix}$$

by a parabolic element. Namely, there exists some parabolic of the form

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{R}),$$

taking $\rho_0(\tau)$ to $\rho_0(l_i)$. If this is deformed by the infinitesimal isometry

$$\begin{bmatrix} y & x \\ b & -y \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{R}),$$

the deformation is encapsulated by the HP matrix

$$\begin{bmatrix} 1+\frac{1}{2}a^2 & -\frac{1}{2}a^2 & a & 0 \\ \frac{1}{2}a^2 & 1-\frac{1}{2}a^2 & a & 0 \\ a & -a & 1 & 0 \\ -b-a^2b+2ay+x & -b+a^2b-2ay-x & 2y-2ab & 1 \end{bmatrix},$$

which is the $\mathrm{PGL}(4, \mathbb{R})$ form of the $\mathrm{PSL}(2, \mathcal{B}_0)$ element

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} y & x \\ b & -y \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \kappa_0.$$

Then, for a general singularity, the representation ρ_{HP} should be such that $\rho_{\mathrm{HP}}(m_i)$ and $\rho_{\mathrm{HP}}(l_i)$ are conjugates of $\rho_{\mathrm{HP}}(m)$ and $\rho_{\mathrm{HP}}(l)$, with the conjugating matrix being of the above type. This gives the general form

$$
(9)\quad
\begin{aligned}
\rho_{\mathrm{HP}}(m_i) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a\omega & a\omega & \omega & 1 \end{bmatrix}, \\
\rho_{\mathrm{HP}}(l_i) &= \begin{bmatrix} \mp a^2 + (1+a^2)C & \pm a^2 - a^2 C + S & a(1-e^d) & 0 \\ \mp a^2 + a^2 C + S & \pm a^2 + C - a^2 C & a(1-e^d) & 0 \\ a(\mp 1 + e^{-d}) & a(\pm 1 - e^{-d}) & \pm 1 & 0 \\ f_1 & f_2 & 2ab(e^d \mp 1) + \mu & \pm 1 \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
C &= \cosh d, \\
S &= \sinh d, \\
f_1 &= -a\mu - (b + 2a^2 b - 2ay)(e^d \pm 1) + x(e^{-d} \mp 1), \\
f_2 &= a\mu + (2a^2 b - 2ay - b)(e^d \mp 1) - x(e^{-d} \mp 1).
\end{aligned}
$$

The curves $\delta_j = \gamma_{2g+j}$ are meridians of the boundary tori, so we verify that $\rho_{\mathrm{HP}}(\delta_j)$ agrees with the description of $\rho_{\mathrm{HP}}(m_i)$. From our computation of $\rho_{\mathrm{HP}}(\gamma_{2g+j})$, we notice that $a_{2g+j} = b_{2g+j} = 0$ since the signed length of δ_j around any singular point of the foliation is 0, so

$$
\rho_{\mathrm{HP}}(\delta_j) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_{2g+j} & -x_{2g+j} & 2y_{2g+j} & 1 \end{bmatrix}.
$$

Hence, the infinitesimal rotation is given by $\omega = 2y_{2g+j}$, where the y_{2g+j} can be chosen freely as long as they are the same for singular points in the same orbit of ϕ. It remains to show that $x_{2g+j} = -a\omega = -2ay_{2g+j}$, where a is the amount of parabolic translation that takes the axis between 0 and infinity to the axis given by the orbit of the singular point s_j.

Suppose that m is the order of the orbit of singular points that contains the singularity encircled by δ_j. Then, $\phi^m(\delta_j) = v_j \delta_j v_j^{-1}$ for some word $v_j \in \pi_1(S \setminus \sigma)$. Noting that

$$
\rho_0(\delta_j) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_0(v_j) = \begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix}
$$

for some real number A, the twisted cocycle condition yields, by using (5) with $\gamma_{2g+j} = \delta_j$, that

$$\begin{bmatrix} y_{2g+j} & \lambda^m x_{2g+j} \\ 0 & -y_{2g+j} \end{bmatrix} = \begin{bmatrix} y_{2g+j} & x_{2g+j} - 2y_{2g+j}A \\ 0 & -y_{2g+j} \end{bmatrix}. \tag{10}$$

This follows because $b_{2g+j} = 0$.

In addition to

$$\tau^m \delta_j \tau^{-m} = \phi^m(\delta_j) = v_j \delta_j v_j^{-1},$$

we have that

$$\tau^m l_j \tau^{-m} = v_j l_j v_j^{-1}.$$

As previously noted, $\rho_0(l_j)$ is conjugate to $\rho_0(\tau)^m$ by the parabolic element $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$. This yields

$$\rho_0(l_j) = \begin{bmatrix} \sqrt{\lambda}^m & a(-\sqrt{\lambda}^m + \sqrt{\lambda}^{-m}) \\ 0 & \sqrt{\lambda}^{-m} \end{bmatrix}.$$

From the relation $\tau^m l_j \tau^{-m} = v_j l_j v_j^{-1}$, we obtain that

$$\begin{bmatrix} \sqrt{\lambda}^m & a\lambda^m(-\sqrt{\lambda}^m + \sqrt{\lambda}^{-m}) \\ 0 & \sqrt{\lambda}^{-m} \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda}^m & (A+a)(-\sqrt{\lambda}^m + \sqrt{\lambda}^{-m}) \\ 0 & \sqrt{\lambda}^{-m} \end{bmatrix}.$$

This yields $A = \lambda^m a - a = a(\lambda^m - 1)$. The cocycle condition from (10) yields $x_{2g+j}(\lambda^m - 1) = -2y_{2g+j}a(\lambda^m - 1)$, which is exactly the desired condition $x_{2g+j} = -2ay_{2g+j}$. A similar computation can be used to find the parameters x and b, with b equaling the μ_s distance between τ and l_i. The longitudinal curves $\rho_{\mathrm{HP}}(l_i)$ are conjugates of multiples of $\rho_{\mathrm{HP}}(\tau)$. Since $\rho_{\mathrm{HP}}(\tau)$ has the form stipulated in (9) for $\rho_{\mathrm{HP}}(l_i)$, we have first-order compatibility of the HP representation with representations of cone singularities. From the previous computation of $\rho_{\mathrm{HP}}(\tau)$, we can see that $d = m \log \lambda$ and $\mu = 2my_0$.

In order to show that the components of the singular locus remain as cone singularities, we will additionally need to show that the subset of structures where the meridian curves remain elliptic is smooth so that the first-order compatibility can be realized by a path of structures on N_ϕ. The proof generalizes [4, Lemma 4.25] to multiple components.

Lemma 6.1 *The subset of $H^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ corresponding to singular hyperbolic structures near ρ_0 such that $\rho_t(m_i)$ remains elliptic has real dimension k.*

Proof The complex dimension of $H^1(\pi_1(T_i), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$, where T_i is a boundary component homeomorphic to a torus, is 2, given by the differentials $dl(l_i)$ and $dl(m_i)$ of the lengths $l(l_i)$ and $l(m_i)$. The subspace of $H^1(\pi_1(T_i), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ where $\rho(m_i)$ remains elliptic as it is deformed by a cocycle has real dimension equal to 3.

For each torus component T_i, by the Poincaré duality argument, the real dimension of the image of $H^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ in $H^1(\pi_1(T_i), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ is 2. Moreover, from the computation of the space of cocycles, we can pick an element $z \in H^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ with y_{2g+i} arbitrarily large, so that $z(m_i)$ increases translation length. Thus, the image is transverse to the subset of $H^1(\pi_1(T_i), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ where $\rho(m_i)$ remains elliptic. Noting that ∂N_ϕ is a disjoint union $\bigsqcup T_i$, the subset of the image of $H^1(\pi_1(N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ in $H^1(\pi_1(\partial N_\phi), \mathfrak{sl}(2,\mathbb{C})_{\mathrm{Ad}_{\rho_0}})$ has real dimension k. □

Lemma 6.1, along with Theorem 5.2, tells us that the we can choose a family of hyperbolic structures on N_ϕ near the Sol structure on N_ϕ such that the restriction of the corresponding representations to the boundary tori agree with representations of the models for cone singularities. After a finite number of applications of [4, Propositions 4.3 and 4.10], once on each component of Σ, we conclude that the representations can be realized as actual hyperbolic cone structures. We restate those propositions here.

Proposition [4, Proposition 4.3] *Let M be a manifold with a projective structure on $N = M \setminus \Sigma$ with cone-like singularities along $\Sigma = \{\gamma\}$. Let B be a small neighborhood of a point $p \in \Sigma$, with $\Sigma_B = \Sigma \cap B$. Then:*

1. *The developing map D on $\widetilde{B\setminus\Sigma_B}$ extends to the universal branched cover $\widetilde{B} = \widetilde{B\setminus\Sigma_B} \cup \Sigma_B$ of B branched over Σ_B.*
2. *D maps Σ_B diffeomorphically onto an interval of a line $\mathfrak{L}$ in $\mathbb{RP}^3$.*
3. *The holonomy $\rho(\pi_1(B \setminus \Sigma_B))$ point-wise fixes $\mathfrak{L}$.*

Proposition [4, Proposition 4.10] *Suppose $\rho_t\colon \pi_1(M) \to \mathrm{PGL}(4,\mathbb{R})$ is a path of representations such that:*

1. *ρ_0 is the holonomy representation of a projective structure on $N = M \setminus \Sigma$ with cone-like singularities along $\Sigma = \{\gamma\}$, and $\mathfrak{L}$ is the line in $\mathbb{RP}^3$ fixed by $\rho_0(\pi_1(\partial M))$.*
2. *$\rho_t(m)$ point-wise fixes a line $\mathfrak{L}_t$ with $\mathfrak{L}_t \to \mathfrak{L}$.*

Then, for all t sufficient small, ρ_t is the holonomy representation for a projective structure on N with cone-like singularities along Σ.

A computation of the commutator $\rho_{\mathrm{HP}}([\alpha_i, \beta_i])$ yields a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -f & f & g & 1 \end{bmatrix},$$

in (9), where

$$f = a_{g+i}^2 b_i + 2a_{g+i} y_i - a_i^2 b_{g+i} - 2a_i y_{g+i}, \quad g = -2a_{g+i} b_i + 2a_i b_{g+i}.$$

Therefore, the product of the commutators $\rho_{\mathrm{HP}}\big(\prod_{i=1}^{g}[\alpha_i, \beta_i]\big)$ also has this form. In the case where $\gamma_{2g+j} = \delta_j$, we also have that

$$\rho_{\mathrm{HP}}(\delta_j) = \rho_{\mathrm{HP}}(\gamma_{2g+j}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_{2g+j} & -x_{2g+j} & 2y_{2g+j} & 1 \end{bmatrix}.$$

Note that $y_{2g+j} = y_{2g+j'}$ if δ_j and $\delta_{j'}$ belong in the same cycle of the permutation (ie they are meridians for the same component of Σ). In other words, we have cone-type singularities that develop in the singular hyperbolic structure, and for each component of Σ, there is freedom in choosing the infinitesimal cone angle about that component. Moreover, the commutator/singularities relation

$$\prod_{i=1}^{g}[\alpha_i, \beta_i] = \prod_{j=1}^{n} \delta_j$$

says that the sum of the infinitesimal cone angles about each component, weighted by the number of singularities in the permutation for that component, must equal some quantity ω_{tot} determined by the loop $\prod_{i=1}^{g}[\alpha_i, \beta_i]$ that encircles all of the singularities.

Lemma 6.2 *The total infinitesimal cone angle ω_{tot} is non-zero.*

Proof A straight-forward computation shows that the $\omega = \omega_{\mathrm{tot}}$ entry in the commutator $\rho_{\mathrm{HP}}([\alpha_i, \beta_i])$ is given by $2(a_i b_{g+i} - a_{g+i} b_i)$. Hence, the ω entry in the product

$$\rho_{\mathrm{HP}}\left(\prod_{i=1}^{g}[\alpha_i, \beta_i]\right)$$

is the negative of the algebraic intersection pairing $\hat{\imath}(\vec{e}_\lambda, \vec{e}_{\lambda^{-1}})$. We note that the algebraic intersection is a symplectic form on $H^1(S)$.

Suppose $\vec{e}_\mu$ is an eigenvector of ϕ^* with eigenvalue $\mu \neq \lambda$. Then

$$\hat{i}(\vec{e}_\mu, \vec{e}_{\lambda^{-1}}) = \hat{i}(\phi^*\vec{e}_\mu, \phi^*\vec{e}_{\lambda^{-1}}) = \mu\lambda^{-1}\hat{i}(\vec{e}_\mu, \vec{e}_{\lambda^{-1}}).$$

Since $\mu \neq \lambda$, this means that $\hat{i}(\vec{e}_\mu, \vec{e}_{\lambda^{-1}}) = 0$.

If $\vec{e}_{\mu,p}$ is a generalized eigenvector such that $(\phi^* - \mu I)^p \vec{e}_{\mu,p} = 0$, then we induct on p. Notice that $\phi^*\vec{e}_{\mu,p} = \mu\vec{e}_{\mu,p} + c\vec{e}_{\mu,p-1}$, where $(\phi^* - \mu I)^{p-1}\vec{e}_{\mu,p-1} = 0$. Hence, if $\hat{i}(\vec{e}_{\mu,p-1}, \vec{e}_{\lambda^{-1}}) = 0$, then it must be that $\hat{i}(\vec{e}_{\mu,p}, \vec{e}_{\lambda^{-1}}) = 0$ as well, since

$$\hat{i}(\vec{e}_{\mu,p}, \vec{e}_{\lambda^{-1}}) = \hat{i}(\phi^*\vec{e}_{\mu,p}, \phi^*\vec{e}_{\lambda^{-1}}) = \mu\lambda^{-1}\hat{i}(\vec{e}_{\mu,p}, \vec{e}_{\lambda^{-1}}).$$

The generalized eigenvectors of ϕ^* span $\mathbb{R}^{2g}$ and λ is a simple eigenvalue, so that means that if $\hat{i}(\vec{e}_\lambda, \vec{e}_{\lambda^{-1}}) = 0$, then $\hat{i}(\vec{u}, \vec{e}_{\lambda^{-1}}) = 0$ for all $\vec{u} \in \mathbb{R}^{2g}$, contradicting the non-degeneracy condition for symplectic forms. □

We can now prove Theorem 6.3.

Theorem 6.3 *Let $\phi\colon S \to S$ be a pseudo-Anosov homeomorphism whose stable and unstable foliations, $\mathcal{F}^s$ and $\mathcal{F}^u$, are orientable and $\phi^*\colon H^1(\overline{S}) \to H^1(\overline{S})$ does not have 1 as an eigenvalue. Then there exists a family of singular hyperbolic structures on M_ϕ, smooth on the complement of Σ and with cone singularities along Σ, that degenerate to a transversely hyperbolic foliation. The degeneration can be rescaled so that the path of rescaled structures limits to the singular Sol structure on M_ϕ, as projective structures. Moreover, the cone angles can be chosen to be decreasing.*

Proof Lemma 6.1 and Theorem 5.2 imply that there exists a family of hyperbolic structures on N_ϕ near the Sol structure on N_ϕ such that the meridian and longitudinal curves of the boundary tori have the form in (9).

Apply Proposition 4.3 and Proposition 4.10 from [4] on one component γ of Σ to show that $M_\phi \setminus (\Sigma \setminus \gamma)$ has a projective structure with holonomy ρ_t with cone-like singularities along γ for sufficiently small t. Proceed inductively on each component of Σ.

Lemma 6.2 implies that the infinitesimal cone angles of each boundary component can be chosen to be negative, so that the cone angles are all decreasing. The total infinitesimal cone angle ω_{tot} is non-zero, and the proof of Lemma 6.2 shows that it is the negative of $\hat{i}(\vec{e}_\lambda, \vec{e}_{\lambda^{-1}})$, and taking a positive orientation for $\{\vec{e}_\lambda, \vec{e}_{\lambda^{-1}}\}$ leads to $\omega_{\text{tot}} < 0$. □

The results of [4] also imply that there are nearby AdS structures that collapse to the same transversely hyperbolic foliation, such that a similar rescaling gives the HP structure. The generalizations made here to those results can also easily be made for AdS structures, so there are also nearby AdS structures with tachyon (cone-like) singularities.

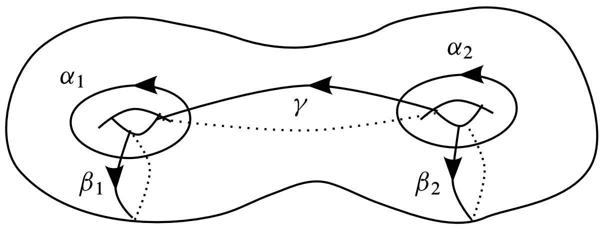

Figure 1: The curves $\alpha_1, \alpha_2, \beta_1, \beta_2$ which form the basis for $H_1(S)$, and γ.

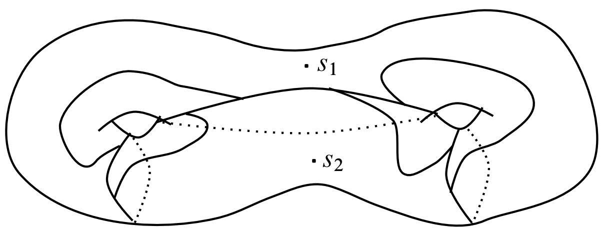

Figure 2: A train track for $\mathcal{F}^u$.

7 Genus-two example

We will compute the representations and parameters to find the deformation in a genus-two example. Begin with the curves $\alpha_1, \alpha_2, \beta_1, \beta_2$, which form the symplectic basis for $H_1(S)$. We begin with left Dehn twists $T_{\beta_1}, T_{\beta_2}, T_\gamma$ along β_1, β_2 and γ, followed by right Dehn twists $T_{\alpha_1}^{-1}, T_{\alpha_2}^{-1}$ along α_1 and α_2. Since the disjoint sets of curves $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2, \gamma\}$ fill, the resulting homeomorphism $\phi\colon S \to S$ is pseudo-Anosov (see [18] or [6, page 398]).

The stable and unstable foliations are orientable with two singular points of cone angle 4π, one in each of the two components of $S \setminus \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma\}$. A train track for $\mathcal{F}^u$ is shown in Figure 2, and we can verify that the foliations are orientable with two singularities s_1 and s_2.

The induced action on cohomology, with the generators $\alpha_1, \alpha_2, \beta_1, \beta_2$ and puncture curves δ_1, δ_2, is

$$\mathring{\phi}^* = \begin{bmatrix} 3 & -1 & -2 & 1 & -1 & 0 \\ -1 & 3 & 1 & -2 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

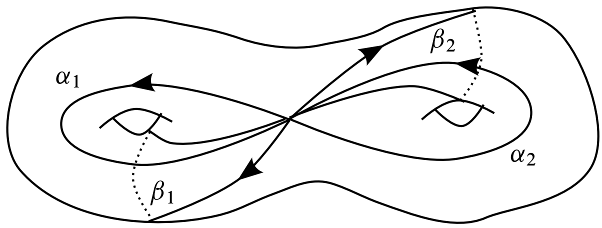

Figure 3: Generators for $\pi_1(S)$.

The matrix has largest eigenvalue $\lambda_1 = \frac{1}{2}(5+\sqrt{21})$. The other eigenvalue $\lambda_2 > 1$ is given by $\lambda_2 = \frac{1}{2}(3+\sqrt{5})$. The eigenvectors of ϕ^* for λ_1 and λ_1^{-1} are

$$\vec{e}_{\lambda_1} = \big(\tfrac{3+\sqrt{21}}{2}, -\tfrac{3+\sqrt{21}}{2}, -1, 1, 0, 0\big)^T \quad \text{and} \quad \vec{e}_{\lambda_1^{-1}} = \big(-\tfrac{\sqrt{21}-3}{2}, \tfrac{\sqrt{21}-3}{2}, -1, 1, 0, 0\big)^T.$$

We have a choice for $\vec{e}_{\lambda_1^{-1}}$ as it is only unique up to scale. We make the choice that is consistent with the orientation of the embedding of Sol into $\mathbb{R}^4$. In particular, in the standard embedding, the x–coordinate is contracted and the y–coordinate is expanded. Our choice for $\vec{e}_{\lambda_1}$ and $\vec{e}_{\lambda_1^{-1}}$ has the same orientation in the singular flat metric on S.

Thus, we obtain the parameters

$$\begin{aligned} a_1 &= -a_2 = \tfrac{1}{2}(3+\sqrt{21}), \\ a_3 &= -a_4 = -1, \\ b_1 &= -b_2 = -\tfrac{1}{2}(\sqrt{21}-3), \\ b_3 &= -b_4 = -1. \end{aligned}$$

Fix a basepoint and choose representatives for $\alpha_1, \alpha_2, \beta_1, \beta_2$ in $\pi_1(S)$, which we will also call $\alpha_1, \alpha_2, \beta_1, \beta_2$ (see Figure 3). In addition, taking generators δ_1 and δ_2 for loops around the singularities s_1 and s_2, we have the following action of ϕ on $\pi_1(S \setminus \sigma)$:

$$\begin{aligned} \phi(\alpha_1) &= \alpha_1\beta_1^{-1}\delta_1^{-1}\alpha_2\beta_2\alpha_2^{-2}\alpha_1^2\beta_1^{-1}, \\ \phi(\alpha_2) &= \alpha_2^2\beta_2^{-1}\alpha_2^2\beta_2^{-1}\alpha_2^{-1}\delta_1\beta_1\alpha_1^{-1}, \\ \phi(\beta_1) &= \beta_1\alpha_1^{-1}, \\ \phi(\beta_2) &= \alpha_1\beta_1^{-1}\delta_1^{-1}\alpha_2\beta_2\alpha_2^{-2}\beta_2\alpha_2\beta_2^{-1}\alpha_2^{-1}\delta_1^{-1}\beta_1\alpha_1^{-1}, \\ \phi(\delta_1) &= \delta_1, \\ \phi(\delta_2) &= \alpha_2\beta_2\alpha_2^{-2}\alpha_1\beta_1^{-1}\delta_1^{-1}\delta_2\delta_1\beta_1\alpha_1^{-1}\alpha_2^2\beta_2^{-1}\alpha_2^{-1}, \end{aligned}$$

with $a_5 = a_6 = b_5 = b_6 = 0$.

Thus, we have that

$$D = \begin{bmatrix} 11+2\sqrt{21} & \frac{-9-3\sqrt{21}}{2} & \frac{-17-3\sqrt{21}}{2} & 7+2\sqrt{21} & \frac{-13-3\sqrt{21}}{2} & 0 \\ \frac{3+\sqrt{21}}{2} & 15-2\sqrt{21} & \frac{-3-\sqrt{21}}{2} & \frac{13+\sqrt{21}}{2} & \frac{-5-\sqrt{21}}{2} & 0 \\ \frac{3+\sqrt{21}}{2} & 0 & \frac{-3-\sqrt{21}}{2} & 0 & 0 & 0 \\ \frac{5+\sqrt{21}}{2} & 1 & \frac{-5-\sqrt{21}}{2} & -1 & \frac{5+\sqrt{21}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5-\sqrt{21} \end{bmatrix},$$

$$C = \begin{bmatrix} -62-13\sqrt{21} & \frac{125+5\sqrt{21}}{2} & \frac{101+21\sqrt{21}}{2} & -133-28\sqrt{21} & \frac{77+17\sqrt{21}}{2} & 0 \\ \frac{15+3\sqrt{21}}{2} & -103-20\sqrt{21} & \frac{-15-3\sqrt{21}}{2} & \frac{59+9\sqrt{21}}{2} & \frac{-23-5\sqrt{21}}{2} & 0 \\ \frac{15+3\sqrt{21}}{2} & 0 & \frac{-15-3\sqrt{21}}{2} & 0 & 0 & 0 \\ \frac{13+3\sqrt{21}}{2} & -4-\sqrt{21} & \frac{-13-3\sqrt{21}}{2} & 19-4\sqrt{21} & \frac{23+5\sqrt{21}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 23+5\sqrt{21} \end{bmatrix}$$

and $K = -2D$. From this, we calculate from (7) that

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = -(\phi^* - I)^{-1} \left(D_{4\times 4} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} + \begin{pmatrix} -y_5 \\ y_5 \\ 0 \\ -2y_5 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2}(-3+\sqrt{21}) \\ \frac{1}{2}(-3+\sqrt{21}) - 2y_5 \\ -13+5\sqrt{21} - \frac{1}{3}y_5 \\ \frac{1}{2}(-53+17\sqrt{21}) - \frac{1}{3}5y_5 \end{pmatrix}$$

and y_5 and y_6 are free. The span of $\phi^* - \lambda_1 I$ is generated by the first three columns, so we can take $x_4 = 0$ (taking $x_4 \neq 0$ would change the solution by a co-boundary). We then compute the other x_i and y_0 from (8), yielding

$$\begin{aligned} x_1 &= \tfrac{1}{42}(-18312 + 887\sqrt{21}) + \tfrac{1}{42}(-3353 + 1121\sqrt{21})y_5, \\ x_2 &= \tfrac{1}{84}(-2835 + 2573\sqrt{21}) + \tfrac{1}{84}(-812 + 40\sqrt{21})y_5, \\ x_3 &= \tfrac{1}{6}(-2166 + 615\sqrt{21}) + \tfrac{1}{6}(-853 + 169\sqrt{21})y_5, \\ x_4 &= 0, \\ x_5 &= 0, \\ x_6 &= \tfrac{1}{3}(6 + 2\sqrt{21})y_6, \\ y_0 &= \tfrac{1}{84}(7119 - 1552\sqrt{21}) + \tfrac{1}{84}(1183 - 267\sqrt{21})y_5. \end{aligned}$$

The ω entry in the commutator $\rho_{\mathrm{HP}}([\alpha_i, \beta_i])$ is computed to be $2(a_i b_{2+i} - a_{2+i} b_i)$. Hence, the total infinitesimal cone angle ω_{tot} is equal to $-4\sqrt{21}$. The infinitesimal cone angles about the two boundary components should add up to $\omega_{\mathrm{tot}} = -4\sqrt{21}$, and the individual infinitesimal cone angles can be chosen so that the cone angles about both singularities are decreasing towards 2π. By scaling the b_i by a positive scalar, it is also possible to change ω_{tot} to any negative number.

8 Discussion

The hypotheses in Theorem 6.3 are satisfied by pseudo-Anosov maps on the punctured torus, so the result includes the previously known case for the punctured torus. There exist examples of pseudo-Anosov maps for other hyperbolic surfaces that satisfy the conditions in the theorem.

For an arbitrary pseudo-Anosov ϕ, the induced map ϕ^* has 1 as an eigenvalue if and only if the mapping torus M_ϕ has first Betti number > 1. If ϕ^* does not have 1 as an eigenvalue but the invariant foliations are not orientable, one can take an orientation cover for the foliation and lift the pseudo-Anosov to the cover. However, this may introduce additional eigenvalues for the lifted map. These conditions are needed to prove Theorem 4.1 in order to guarantee that an infinitesimal deformation can be realized by a smooth path of deformed structures for small time, but it would be interesting to know if the deformation can be carried out even when the smoothness condition is not satisfied.

The result in Theorem 6.3 is local; we can find a deformation of the cone angles for small time. It would be of further interest to know whether the deformation can be carried out all the way to the complete structure on M_ϕ. This would give a direct connection between the hyperbolic structure on fibered manifolds and the combinatorial properties of the pseudo-Anosov monodromy.

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Department of Mathematics, University of California, Berkeley
970 Evans Hall #3840, Berkeley, CA 94720-3840, USA

kozai@math.berkeley.edu

http://math.berkeley.edu/~kozai/

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