

An arithmetic Zariski 4–tuple of twelve lines

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Using the invariant developed by E Artal, V Florens and the author, we differentiate four arrangements with the same combinatorial information but in different deformation classes. From these arrangements, we construct four other arrangements such that there is no orientation-preserving homeomorphism between them. Furthermore, some pairs of arrangements among this 4–tuple form new arithmetic Zariski pairs, ie a pair of arrangements conjugate in a number field with the same combinatorial information but with different embedding topology in $\mathbb{C}\mathbb{P}^2$.

32S22; 32Q55, 54F65

1 Introduction

The study of the relation between topology and algebraic geometry was initiated by F Klein and H Poincaré at the beginning of the twentieth century. It was known by the work of O Zariski [21; 22; 23] that the topology of the embedding of an algebraic curve C in the complex projective plane $\mathbb{C}\mathbb{P}^2$ is not determined by the local topology of its singular points. Indeed, he used the fundamental group of their *complement* $E_{C_i} = \mathbb{C}\mathbb{P}^2 \setminus C_i$ as a strong topological invariant to prove that two sextics C_1, C_2 have the same *combinatorial information* but different *topological types*. Such a pair of curves was called a Zariski pair by E Artal [1]. There are many examples of Zariski pairs (or larger k –tuples) that have been discovered (for examples see E Artal, JI Cogolludo and H Tokunaga [5], P Cassou-Noguès, C Eyral and M Oka [8], A Degtyarev [10], M Oka [16] or I Shimada [19]), while essentially two Zariski pairs of *line arrangements* are known. The first is a pair of arrangements with complex equations constructed by G Rybnikov [18; 4], and the second is a pair of complexified real arrangements obtained by E Artal, J Carmona, JI Cogolludo and MA Marco [3]. The proof of the former is done using the lower central series of the fundamental group, and the latter using the braid monodromy. Computers were used in both proofs. This small number of examples for line arrangements, and the routine use of a computer, show the difficulty in understanding what characterizes the topological type of an arrangement.

In the present paper, new counterexamples to the combinatoriality of the topological type of arrangements are explicitly constructed. These arrangements are defined

in the 10th cyclotomic field, and their equations are connected by the action of an element of the Galois group of $\mathbb{Q}(\zeta_{10})$. These new pairs are arithmetic Zariski pairs; in particular, their fundamental groups have the same profinite completion (ie the same finite quotients). In contrast with Rybnikov in [18], and as E Artal, J Carmona, JI Cogolludo and M A Marco in [3], the major part of the computations needed to single out these arrangements are doable by hand.

In [6], E Artal, V Florens and the author construct a new topological invariant of line arrangements $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ based on the inclusion map of the *boundary manifold* (that is, the boundary of a regular neighborhood of the arrangement) in the complement. This invariant depends on a character of the fundamental group of the complement and a special cycle in the incidence graph of the arrangement; and it can be computed directly from the *wiring diagram* of the arrangement. In this paper, we use this invariant to single out four arrangements (two pairs of complex conjugate arrangements) with the same combinatorial structure but lying in different deformation classes (ie an oriented and ordered Zariski 4–tuple). Combinatorially, they contain eleven lines, four points of multiplicity four, six triple points and some double points. Then, to delete all automorphisms of the combinatorics, we add a twelfth line to these arrangements (as in [3]). Thus we construct new Zariski pairs.

Currently, we do not know if they have isomorphic fundamental groups. However, the invariant $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ allows us to compute the quasi-projective part of the characteristic varieties; more precisely, to determine the quasi-projective depth of the character ξ (see E Artal [2] and the author [13]). Unfortunately, in the present case they are equal. Furthermore, the combinatorial structure of the arrangements satisfies the hypotheses of Dimca, Ibadula and Măcnic [11], so the projective part of the characteristic varieties is combinatorially determined. Thus the question of the combinatoriality of the characteristic varieties is still open.

In Section 2, we give usual definitions and define arrangements $\mathcal{N}^+, \mathcal{N}^-, \mathcal{M}^+$ and \mathcal{M}^- forming the oriented and ordered Zariski 4–tuple. In Section 3, we apply a classical argument to these four arrangements to construct the new examples of arithmetic Zariski pairs. The last section is divided into two parts. In the first one, we recall the construction and the definition of the invariant $\mathcal{I}(\mathcal{A}, \xi, \gamma)$; in the second one, we give the wiring diagrams of \mathcal{N}^+ and \mathcal{M}^+ required to compute the invariant together with the character ξ and the cycle γ allowing us to distinguish them. Then we compute the invariant for the four arrangements, and thus we prove that $(\mathcal{N}^+, \mathcal{N}^-, \mathcal{M}^+, \mathcal{M}^-)$ forms an oriented and ordered Zariski 4–tuple.

Acknowledgments This result is an improvement of a result obtained during my Ph D, supervised by E Artal and V Florens. Thanks to them for their supervision. I would

also like to thank M A Marco Buzunáriz for his list of combinatorics, which was the starting point for this research into Zariski pairs.

The author is partially supported by the JSPS-MAE Sakura Program.

2 The arrangements

In this section, a brief recall on combinatorics and realization is given (see P Orlik and H Terao [17] for definitions of classical objects), together with the description of the arrangements allowing us to construct the new examples of Zariski pairs.

2.1 The combinatorics

Definition 2.1 A *combinatorics* is a pair $\mathcal{C} = (\mathcal{L}, \mathcal{P})$, where \mathcal{L} is a finite set and \mathcal{P} a subset of the power set of \mathcal{L} , satisfying the conditions:

- $\#P \geq 2$ for all $P \in \mathcal{P}$.
- For any $L_1, L_2 \in \mathcal{L}$ with $L_1 \neq L_2$, there exists a unique $P \in \mathcal{P}$ such that $L_1, L_2 \in P$.

An *ordered combinatorics* \mathcal{C} is a combinatorics where \mathcal{L} is an ordered set. The elements of \mathcal{P} are called *points* and the one of \mathcal{L} *lines*.

Definition 2.2 Let $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ be a combinatorics. An *automorphism* of \mathcal{C} is a permutation of \mathcal{L} preserving \mathcal{P} . The set of such automorphisms is the *automorphism group* of the combinatorics \mathcal{C} .

The combinatorics can be encoded in the incidence graph, which is a subgraph of the Hasse diagram.

Definition 2.3 The *incidence graph* $\Gamma_{\mathcal{C}}$ of a combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ is a non-oriented bipartite graph where the set of vertices $V(\mathcal{C})$ is decomposed in two disjoint sets

$$V_{\mathcal{P}}(\mathcal{C}) = \{v_P \mid P \in \mathcal{P}\} \quad \text{and} \quad V_{\mathcal{L}}(\mathcal{C}) = \{v_L \mid L \in \mathcal{L}\}.$$

An edge of $\Gamma_{\mathcal{C}}$ joins $v_L \in V_{\mathcal{L}}(\mathcal{C})$ to $v_P \in V_{\mathcal{P}}(\mathcal{C})$ if and only if $L \in P$.

Remark 2.4 The automorphism group of \mathcal{C} is isomorphic to the group of automorphism of $\Gamma_{\mathcal{C}}$ respecting the structure of bipartite graph, ie preserving both $V_{\mathcal{P}}(\mathcal{C})$ and $V_{\mathcal{L}}(\mathcal{C})$ setwise. Generally it is smaller than the automorphism group of the graph.

The starting point to construct and to detect the new example of Zariski pairs is the combinatorics $\mathcal{K} = (\mathcal{L}, \mathcal{P})$ (obtained from a study of the combinatorics with 11 lines) defined by $\mathcal{L} = \{L_1, \dots, L_{11}\}$ and

$$\mathcal{P} = \left\{ \begin{array}{l} \{1, 2\}, \{1, 3, 5, 7\}, \{1, 4, 6, 8\}, \{1, 9\}, \{1, 10, 11\}, \{2, 3, 6, 9\}, \{2, 4, 5, 10\}, \\ \{2, 7, 11\}, \{2, 8\}, \{3, 4\}, \{3, 8, 11\}, \{3, 10\}, \{4, 7\}, \{4, 9, 11\}, \{5, 6\}, \\ \{5, 8, 9\}, \{5, 11\}, \{6, 7, 10\}, \{6, 11\}, \{7, 8\}, \{7, 9\}, \{8, 10\}, \{9, 10\} \end{array} \right\},$$

where we have written i in place of L_i in the sets comprising \mathcal{P} .

Proposition 2.5 *The automorphism group of \mathcal{K} is cyclic of order 4, and is generated by the permutation*

$$\sigma = (1\ 3\ 2\ 4)(5\ 6)(7\ 9\ 10\ 8).$$

Proof Let ϕ be an automorphism of \mathcal{K} . The line L_{11} is the only one containing four triple points, thus it is fixed by the ϕ . Since L_5 and L_6 are the only ones intersecting L_{11} in a double point, they are in distinct ϕ -orbits from the other lines. Similarly, the lines L_1, L_2, L_3 and L_4 contain two double points, one triple point and two quadruple points, thus they are in ϕ -orbits distinct from those of the other lines. The same argument work for the lines L_7, L_8, L_9 and L_{10} . Thus the decomposition of $\{1, \dots, 11\}$ in ϕ -orbits is a sub-decomposition of

$$(2-1) \quad \{1, 2, 3, 4\} \sqcup \{5, 6\} \sqcup \{7, 8, 9, 10\} \sqcup \{11\}.$$

We decompose the following in two parts.

First, we assume that $\phi(5) = 6$ and $\phi(6) = 5$. Then $\sigma(\{5, 8, 9\}) = \{6, 7, 10\}$ and thus $\phi(\{8, 9\}) = \{7, 10\}$.

- (1) If $\phi(8) = 7$ and $\phi(9) = 10$ then $\phi(\{3, 8, 11\}) = \{2, 7, 11\}$ and thus $\phi(3) = 2$. Using this and decomposition (2-1), we have $\phi(\{3, 10\}) = \{2, 8\}$ and then $\phi(10) = 8$. To finish, using the points $\{1, 9\}, \{2, 8\}, \{3, 10\}$ and $\{4, 7\}$ and decomposition (2-1), we obtain that

$$\phi = (1\ 3\ 2\ 4)(5\ 6)(7\ 9\ 10\ 8).$$

- (2) If $\phi(8) = 10$ and $\phi(9) = 7$ then in the same way we obtain that

$$\phi = (4\ 2\ 3\ 1)(5\ 6)(8\ 10\ 9\ 7).$$

Second, we assume that $\phi(5) = 5$ and $\phi(6) = 6$. Then the four quadruple points and decomposition (2-1) imply that $\phi(\{7, 10\}) = \{7, 10\}$ and $\phi(\{8, 9\}) = \{8, 9\}$.

- (1) If $\phi_{\{7,8,9,10\}} = \text{Id}$ then the sets $\{1, 9\}$, $\{2, 8\}$, $\{3, 10\}$, $\{4, 7\}$ and decomposition (2-1) imply that

$$\phi = \text{Id}.$$

- (2) If $\phi_{\{7,10\}} = \text{Id}$, $\phi(8) = 9$ and $\phi(9) = 8$ then the points $\{1, 9\}$, $\{2, 8\}$, $\{3, 10\}$ and $\{4, 7\}$ and decomposition (2-1) imply that $\phi(2) = 1$. Thus we have that $\phi(\{2, 7, 11\}) = \{1, 10, 11\}$ and then $\phi(7) = 10$, which is impossible.
- (3) If $\phi_{\{8,9\}} = \text{Id}$, $\phi(7) = 10$ and $\phi(10) = 7$ then in the same way we also obtain a contradiction.
- (4) If $\phi(7) = 10$, $\phi(8) = 9$, $\phi(9) = 8$ and $\phi(10) = 7$ then the points $\{1, 9\}$, $\{2, 8\}$, $\{3, 10\}$, $\{4, 7\}$ and decomposition (2-1) imply that

$$\phi = (1\ 2)(3\ 4)(7\ 10)(8\ 9).$$

We obtain that the automorphism group of \mathcal{K} is the cyclic group generated by the permutation $\sigma = (1\ 3\ 2\ 4)(5\ 6)(7\ 9\ 10\ 8)$. □

Remark 2.6 The set \mathcal{P} of points of \mathcal{K} is decomposed into eight orbits by the action of its automorphism group:

- the four points of multiplicity 4,
- the four points of multiplicity 3 of L_{11} ,
- the two other points of multiplicity 3,
- two orbits with two double points,
- two orbits with four double points,
- a single isolated orbit composed of the intersection point of L_5 and L_6 ,

and the set \mathcal{L} of lines of \mathcal{K} is decomposed into four orbits:

$$\mathcal{L} = \{L_1, L_2, L_3, L_4\} \sqcup \{L_5, L_6\} \sqcup \{L_7, L_8, L_9, L_{10}\} \sqcup \{L_{11}\}.$$

2.2 Complex realizations

The following definitions are given over the field of the complex numbers.

Definition 2.7 Let $\mathcal{A} = \{L_1, \dots, L_n\}$ be a line arrangement. The combinatorics of \mathcal{A} is the poset of all the intersections of the elements in \mathcal{A} , with respect to reverse inclusion.

Remark 2.8 The combinatorics of \mathcal{A} encodes the information of which singular point is on which line.

Definition 2.9 Let \mathcal{C} be a combinatorics. A complex line arrangement \mathcal{A} of $\mathbb{C}\mathbb{P}^2$ is a *realization* of \mathcal{C} if its combinatorics agrees with \mathcal{C} . An *ordered realization* of an ordered combinatorics is defined accordingly.

Notation If \mathcal{A} is a realization of a combinatorics \mathcal{C} , then the incidence graph is also denoted by $\Gamma_{\mathcal{A}}$.

Example The incidence graph of a generic arrangement with three lines is the cyclic graph with six vertices. Its automorphism group is the dihedral group D_3 .

Using the fact that three lines are concurrent if and only if the determinant of their coefficients is null, it is simple to verify the following proposition.

Proposition 2.10 *The arrangements defined by the following equations admit \mathcal{K} as combinatorics:*

$$\begin{aligned} L_1 : z = 0, & & L_2 : x + y - z = 0, \\ L_3 : x = 0, & & L_4 : y = 0, \\ L_5 : x - z = 0, & & L_6 : y - z = 0, \\ L_7 : -\alpha^3 x + z = 0, & & L_8 : y - \alpha z = 0, \\ L_9 : (\alpha - 1)x - y + z = 0, & & L_{10} : -\alpha(\alpha - 1)x + y + \alpha(\alpha - 1)z = 0, \\ L_{11} : -\alpha(\alpha - 1)x + y - \alpha z = 0, & & \end{aligned}$$

where α is a root of the 10th cyclotomic polynomial $\Phi_{10}(X) = X^4 - X^3 + X^2 - X + 1$.

We denote by \mathcal{N}^+ and \mathcal{M}^+ the arrangements for which $\alpha \simeq -0.31 + 0.95i$ and $\alpha \simeq 0.81 + 0.59i$, respectively, and by \mathcal{N}^- and \mathcal{M}^- their complex conjugate arrangements.

Remark 2.11 The end of this paper (see [Theorem 2.14](#) and [Section 4.2](#)) will prove that \mathcal{N}^+ , \mathcal{M}^+ , \mathcal{N}^- and \mathcal{M}^- are representatives of the four connected components of the order moduli space; see [\[3\]](#). Thus, these connected components admit representatives with complex equations in the ring of polynomials over the 10th cyclotomic field. Their equations are linked by an element of the Galois group of $\mathbb{Q}(\zeta_{10})$.

Definition 2.12 The *topological type* of an arrangement \mathcal{A} is the homeomorphism type of the pair $(\mathbb{C}\mathbb{P}^2, \mathcal{A})$. If the homeomorphism preserves the orientation of $\mathbb{C}\mathbb{P}^2$, then we have *oriented topological type*; and it is *ordered*, if \mathcal{A} is ordered and the homeomorphisms preserve this order.

Remark 2.13 If two ordered arrangements with the same combinatorics have different oriented and ordered topological type then they are in distinct ambient isotopy classes. The MacLane arrangements [15] are the first such examples.

With these definitions, we can state the main results of the paper.

Theorem 2.14 *There is no homeomorphism preserving both orientation and order between any two pairs among $(\mathbb{CP}^2, \mathcal{N}^+)$, $(\mathbb{CP}^2, \mathcal{N}^-)$, $(\mathbb{CP}^2, \mathcal{M}^+)$ and $(\mathbb{CP}^2, \mathcal{M}^-)$.*

Remark 2.15 Complex conjugation induces an orientation-reversing homeomorphism between $(\mathbb{CP}^2, \mathcal{N}^+)$ and $(\mathbb{CP}^2, \mathcal{N}^-)$, and also between $(\mathbb{CP}^2, \mathcal{M}^+)$ and $(\mathbb{CP}^2, \mathcal{M}^-)$.

Corollary 2.16 *There is no order-preserving homeomorphism between $(\mathbb{CP}^2, \mathcal{N}^+)$ and $(\mathbb{CP}^2, \mathcal{M}^+)$, or between $(\mathbb{CP}^2, \mathcal{N}^+)$ and $(\mathbb{CP}^2, \mathcal{M}^-)$, or between $(\mathbb{CP}^2, \mathcal{N}^-)$ and $(\mathbb{CP}^2, \mathcal{M}^+)$, or between $(\mathbb{CP}^2, \mathcal{N}^-)$ and $(\mathbb{CP}^2, \mathcal{M}^-)$.*

The proofs of both results are presented in Section 4.2.

3 Zariski pairs

The principal problem which appears while working with the previous combinatorics \mathcal{K} is that its automorphism group is not trivial. Indeed, this group is cyclic of order four and is generated by the permutation (see Proposition 2.5)

$$\sigma = (1\ 3\ 2\ 4)(5\ 6)(7\ 9\ 10\ 8).$$

This automorphism induces the automorphism $a \mapsto -a^2$ in the Galois group of the 10th cyclotomic field $\mathbb{Q}(\zeta_{10})$, where a is a primitive root of unity. The change of variables $(x, y, z) \mapsto (z, x + y - z, y)$ realizes this automorphism of the combinatorics. It cyclically permutes the arrangements \mathcal{N}^+ , \mathcal{M}^+ , \mathcal{N}^- and \mathcal{M}^- .

To remove the hypothesis “order-preserving” in Corollary 2.16, we use the same argument as in [3]: we add lines to the previous combinatorics to reduce the automorphism group of the combinatorics to the trivial group. Let us consider the combinatorics $\mathfrak{K} = (\mathfrak{L}, \mathfrak{P})$ obtained from \mathcal{K} by adding a line L_{12} at \mathcal{L} passing through the point $L_1 \cap L_3 \cap L_5 \cap L_7$ and generic with the other lines, that is $\mathfrak{L} = \{L_1, \dots, L_{12}\}$ and

$$\mathfrak{P} = \left\{ \begin{array}{l} \{1, 2\}, \{1, 3, 5, 7, 12\}, \{1, 4, 6, 8\}, \{1, 9\}, \{1, 10, 11\}, \{2, 3, 6, 9\}, \\ \{2, 4, 5, 10\}, \{2, 7, 11\}, \{2, 8\}, \{2, 12\}, \{3, 4\}, \{3, 8, 11\}, \{3, 10\}, \{4, 7\}, \\ \{4, 9, 11\}, \{4, 12\}, \{5, 6\}, \{5, 8, 9\}, \{5, 11\}, \{6, 7, 10\}, \{6, 11\}, \{6, 12\}, \\ \{7, 8\}, \{7, 9\}, \{8, 10\}, \{8, 12\}, \{9, 10\}, \{9, 12\}, \{10, 12\}, \{11, 12\} \end{array} \right\}.$$

It admits four realizations, denoted by \mathfrak{N}^+ , \mathfrak{N}^- , \mathfrak{M}^+ and \mathfrak{M}^- (in accordance with the realizations of \mathcal{K}).

Proposition 3.1 *The automorphism group of the combinatorics \mathfrak{K} is trivial.*

Proof By construction, the line L_{12} is the only line containing the point of multiplicity 5 and only double points otherwise. Then it is fixed by all automorphisms. Thus an automorphism of \mathfrak{K} fixes the unique point of multiplicity 5. But in \mathcal{K} the action of the automorphism group cyclically permutes the points of multiplicity 4. Since one of them was transformed into the unique point of multiplicity 5, then the automorphism group of \mathfrak{K} is trivial. \square

Theorem 3.2 *There is no homeomorphism between $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^+)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^+)$.*

Proof By [Corollary 2.16](#), there is no order-preserving homeomorphism between the pairs $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^+)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^+)$. Assume that there is a homeomorphism between $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^+)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^+)$ that does not preserve the order. Then it induces a non-trivial automorphism of the combinatorics \mathfrak{K} , which is impossible according to [Proposition 3.1](#). \square

Corollary 3.3 *There is no homeomorphism between $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^+)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^+)$ or $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^-)$, or $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^-)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^+)$ or $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^-)$.*

If the lines added to trivialize the automorphism of the combinatorics are conjugated in $\mathbb{Q}(\zeta_{10})$ then the Zariski pairs obtained are arithmetic pairs. In particular, their fundamental groups have the same profinite completion (ie the same finite quotients). But if the lines are not conjugated in $\mathbb{Q}(\zeta_{10})$ then the pairs obtained are not arithmetic.

Lemma 3.4 *There is no orientation-preserving homeomorphism between $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^+)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^-)$ or between $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^+)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^-)$.*

Proof By [Theorem 2.14](#), there is no homeomorphism preserving both orientation and order between the pairs $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^+)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^-)$. But, by construction, there is no non-trivial automorphism of the combinatorics \mathfrak{K} . Then there is no orientation-preserving homeomorphism from $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^+)$ to $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^-)$. \square

Corollary 3.5 *There is no orientation-preserving homeomorphism between any two pairs among $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^+)$, $(\mathbb{C}\mathbb{P}^2, \mathfrak{N}^-)$, $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^+)$ and $(\mathbb{C}\mathbb{P}^2, \mathfrak{M}^-)$.*

Proof This is a consequence of [Corollary 3.3](#) and [Lemma 3.4](#). \square

4 Oriented and ordered topological types

This section is the mathematical cornerstone of the paper. It contains (in Section 4.2) the distinction of the different ambient isotopy classes of the arrangements $\mathcal{N}^+, \mathcal{N}^-, \mathcal{M}^+$ and \mathcal{M}^- previously constructed, and then the proof that they form an oriented and ordered Zariski 4-tuple (Theorem 2.14).

4.1 The invariant $\mathcal{I}(\mathcal{A}, \xi, \gamma)$

4.1.1 An inner-cyclic arrangement For more details on the construction and for the computation of the invariant $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ see [6]. We denote by $E_{\mathcal{A}}$ the complement of $\mathcal{A} = \{L_1, \dots, L_n\}$ in $\mathbb{C}\mathbb{P}^2$, and let $\alpha_i \in H_1(E_{\mathcal{A}})$ be the homological meridian associated with the line $L_i \in \mathcal{A}$. Remark that the set $\{\alpha_2, \dots, \alpha_n\}$ generates $H_1(E_{\mathcal{A}})$; indeed, the meridians satisfy the relation $\alpha_1 + \dots + \alpha_n = 0$. A *character* on an arrangement \mathcal{A} is a group homomorphism

$$\xi: H_1(E_{\mathcal{A}}) \rightarrow \mathbb{C}^*,$$

with $\prod_{L_i \in \mathcal{A}} \xi(\alpha_i) = 1$ to respect the previous relation.

Definition 4.1 Let \mathcal{A} be an arrangement, ξ be a character on \mathcal{A} and γ be a cycle of $\Gamma_{\mathcal{A}}$. The triplet $(\mathcal{A}, \xi, \gamma)$ is an *inner-cyclic arrangement* if:

- (1) For all $v_{L_i} \in V_{\mathcal{L}}(\mathcal{C})$, if $v_{L_i} \in \gamma$, then $\xi(\alpha_i) = 1$.
- (2) For all $v_P \in V_P(\mathcal{C})$, if $v_P \in \gamma$, then $\xi(\alpha_i) = 1$ for all $L_i \ni P$.
- (3) For all $v_L \in \gamma$, if $P \in L$ then $\prod_{L_i \in P} \xi(\alpha_i) = 1$.

Remark 4.2 Suppose that \mathcal{A} and \mathcal{A}' are two realizations of the same combinatorics (ie there is an isomorphism $\phi: \mathcal{C}_{\mathcal{A}} \xrightarrow{\sim} \mathcal{C}_{\mathcal{A}'}$ of ordered combinatorics). If ξ is a character on $E_{\mathcal{A}}$, then it induces on $E_{\mathcal{A}'}$ a character ξ' defined by $\xi' \circ \phi = \xi$. Furthermore, if $(\mathcal{A}, \xi, \gamma)$ is an inner-cyclic arrangement, then $(\mathcal{A}', \xi', \gamma')$ is an inner-cyclic arrangement too, where γ' is the cycle of $\Gamma_{\mathcal{A}'}$ obtained from γ by ϕ . In other words, the existence of a character ξ and a cycle γ such that $(\mathcal{A}, \xi, \gamma)$ is an inner-cyclic arrangement is determined by the combinatorics of \mathcal{A} .

By the previous remark, we can define a character on the combinatorics \mathcal{K} and consider it on $\mathcal{N}^+, \mathcal{N}^-, \mathcal{M}^+$ and \mathcal{M}^- . Let ξ be such a character, defined by

$$\xi: (L_1, \dots, L_{11}) \mapsto (\zeta, \zeta^4, \zeta^3, \zeta^2, 1, 1, \zeta, \zeta^2, \zeta^3, \zeta^4, 1),$$

where ζ is a primitive fifth root of unity. Let $\gamma_{(5,6,11)}$ be the cycle of $H_1(\Gamma_{\mathcal{K}})$ defined by

$$v_{L_5} \longrightarrow v_{P_{5,6}} \longrightarrow v_{L_6} \longrightarrow v_{P_{6,11}} \longrightarrow v_{L_{11}} \longrightarrow v_{P_{11,5}}.$$

Proposition 4.3 *The triplets $(\mathcal{N}^+, \xi, \gamma_{(5,6,11)})$, $(\mathcal{N}^-, \xi, \gamma_{(5,6,11)})$, $(\mathcal{M}^+, \xi, \gamma_{(5,6,11)})$ and $(\mathcal{M}^-, \xi, \gamma_{(5,6,11)})$ are inner-cyclic arrangements.*

The proof of this proposition is straightforward, because the three conditions of Definition 4.1 are combinatorial.

4.1.2 Construction of the invariant The boundary manifold $B_{\mathcal{A}}$ is the common boundary of a regular neighborhood

$$\text{Tub}(\mathcal{A}) = \left(\bigcup B(P) \right) \cup \left(\bigcup \text{Tub}(L) \right)$$

of the exterior $\mathbb{C}P^2 \setminus \overset{\circ}{\text{Tub}}(\mathcal{A})$ and the arrangement \mathcal{A} (where the $B(P)$ are 4-balls centered at the singular points of \mathcal{A}). Up to homotopy type, there is a natural projection $\text{Tub}(\mathcal{A}) \xrightarrow{\rho} \Gamma_{\mathcal{A}}$, which induces an isomorphism ρ_* on the first homology groups. A holed neighborhood $\text{Tub}(\gamma)$ associated with γ is a submanifold of $\text{Tub}(\mathcal{A})$ of the form

$$\text{Tub}(\gamma) = \text{Tub}(\mathcal{A}) \setminus \left[\left(\bigcup_{v_L \notin \gamma} \text{Tub}(L) \right) \cup \left(\bigcup_{v_P \notin \gamma} B(P) \right) \right].$$

A nearby cycle $\tilde{\gamma} \in H_1(B_{\mathcal{A}})$ associated with a cycle $\gamma \in H_1(\Gamma_{\mathcal{A}})$ is defined as a path in $\partial(\text{Tub}(\gamma))$ isotopic to $\rho_*^{-1}(\gamma)$ in $\text{Tub}(\mathcal{A})$.

We denote by $i: B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$ the inclusion map of the boundary manifold in the complement. Let \mathcal{A} be a realization of \mathcal{C} and ξ be a torsion character on \mathcal{A} . We consider the map

$$\chi_{(\mathcal{A}, \xi)}: H_1(B_{\mathcal{A}}) \xrightarrow{i_*} H_1(E_{\mathcal{A}}) \xrightarrow{\xi} \mathbb{C}^*.$$

If $(\mathcal{A}, \xi, \gamma)$ is inner-cyclic and $\tilde{\gamma}$ is a nearby cycle associated with γ , then the value of $\chi_{(\mathcal{A}, \xi)}(\tilde{\gamma})$ is independent of the choice of the nearby cycle $\tilde{\gamma}$; see [6, Lemma 2.2]. Thus we define

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) = \chi_{(\mathcal{A}, \xi)}(\tilde{\gamma}),$$

where $\tilde{\gamma} \in H_1(B_{\mathcal{A}})$ is any nearby cycle associated with γ .

Theorem 4.4 [6] *Let \mathcal{A} and \mathcal{A}' be two ordered realizations of an ordered combinatorics \mathcal{C} . If $(\mathcal{A}, \xi, \gamma)$ and $(\mathcal{A}', \xi, \gamma)$ are two inner-cyclic arrangements with the same oriented and ordered topological type, then*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) = \mathcal{I}(\mathcal{A}', \xi, \gamma).$$

4.2 Computation of the invariant

4.2.1 Braided wiring diagrams The invariant $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ can be computed from the *braided wiring diagram* of the arrangement. It is a singular braid associated with the arrangement (for more details see [7; 9]) and it is defined as follows: Consider a line $L \in \mathcal{A}$ as the line at infinity, and let $\mathcal{A}^{\text{aff}} \subset \mathbb{C}^2 \simeq \mathbb{CP}^2 \setminus \{L\}$ be the associated affine arrangement. Let $p: \mathbb{C}^2 \rightarrow \mathbb{C}$ be a generic projection for the arrangement \mathcal{A}^{aff} (ie no line of \mathcal{A}^{aff} is a fiber of p). Let $\nu: [0, 1] \rightarrow \mathbb{C}$ be a smooth path containing the images of the singular points of \mathcal{A}^{aff} by p (or continuous and piecewise smooth if the images of the singularities are in the smooth part of ν).

Definition 4.5 The *braided wiring diagram* of \mathcal{A} associated with ν and p is defined by

$$W_{\mathcal{A}} = \{(x, y) \in \mathcal{A}^{\text{aff}} \mid p(x, y) \in \nu([0, 1])\}.$$

The trace $\omega_i = W_{\mathcal{A}} \cap L_i$ is called the *wire* associated to the line L_i .

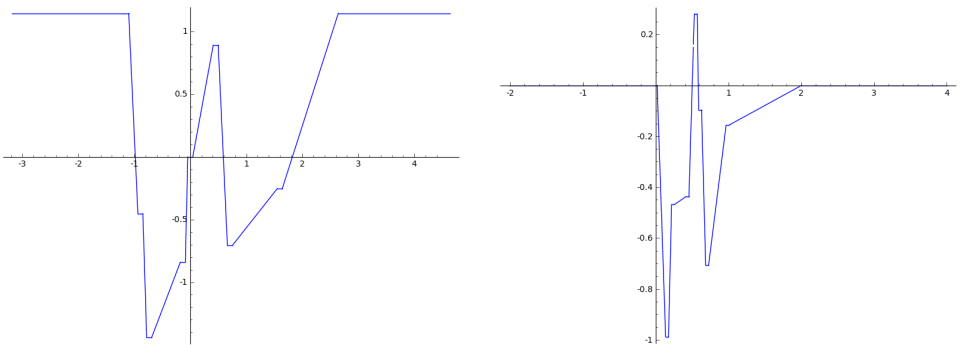


Figure 1: Paths ν used for the computation of the wiring diagrams. Left: the path $\nu_{\mathcal{N}^+}$. Right: the path $\nu_{\mathcal{M}^+}$.

From the equations of \mathcal{N}^+ and \mathcal{M}^+ , we compute their wiring diagrams. To use the result on the computation of $i_*: \pi_1(B_{\mathcal{A}}) \rightarrow \pi_1(E_{\mathcal{A}})$ developed in [12] by Florens, Marco and the author, we choose a line supporting the cycle γ as the line at infinity. With the change of variables $x \mapsto \lambda x$ and $z \mapsto \lambda x - z$, where $\lambda = e^{i\pi/4}$, the line L_5 is considered as the line at infinity for the projection $p: [x : y : z] \mapsto (x/z, y/z)$. Note that with this change of variables the lines L_1, L_3 and L_7 are vertical (ie fibers of the projection), so the projection is not generic. Nevertheless, we can draw the wiring diagram of $\mathcal{A} \setminus \{L_1, L_3, L_7\}$, and adding these lines as vertical ones (see Figures 2 and 3) and obtain a non-generic braided wiring diagram. The paths $\nu_{\mathcal{N}^+}$ and $\nu_{\mathcal{M}^+}$ considered to obtain these diagrams are pictured in Figure 1. All these computations were done using Sage [20]. The source [14] can be downloaded from the author’s website.

Remark 4.6 For smaller and clearer pictures, some braids are simplified (using Reidemeister moves) in the non-generic wiring diagrams of \mathcal{N}^+ and \mathcal{M}^+ pictured in Figures 2 and 3 .

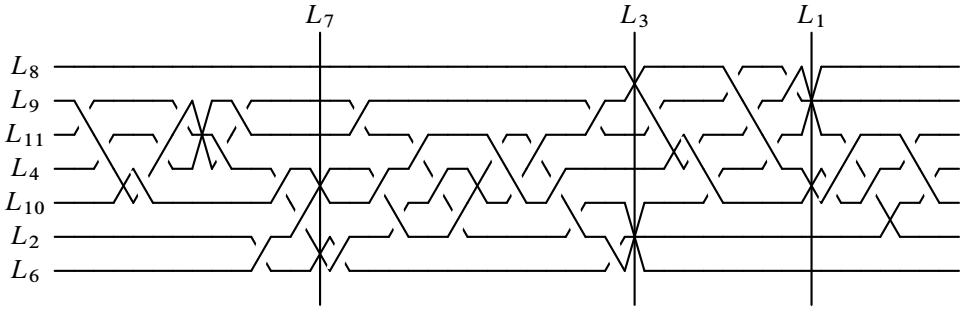


Figure 2: Non-generic braided wiring diagram of \mathcal{N}^+ .

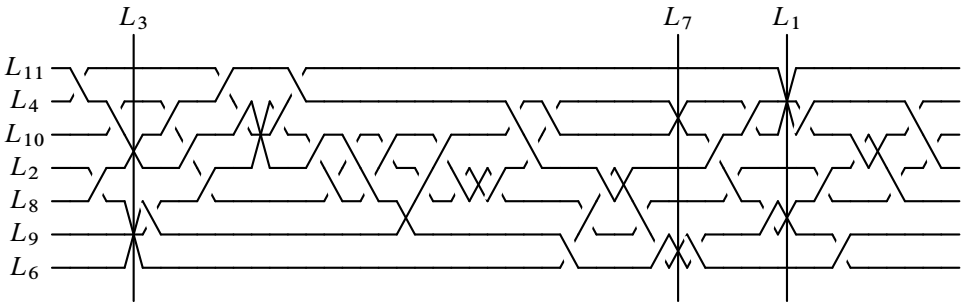


Figure 3: Non-generic braided wiring diagram of \mathcal{M}^+ .

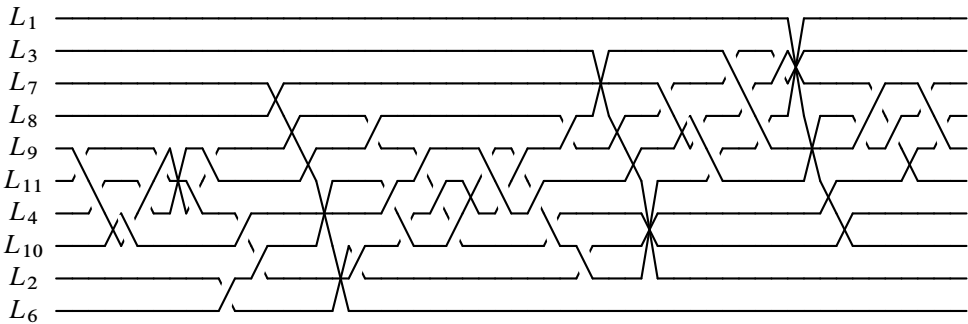


Figure 4: Generic braided wiring diagram of \mathcal{N}^+ .

To obtain the wiring diagrams, we slightly modify the center of the projection p , such that it is always on L_5 and distinct (but very close) to the intersection of the lines

L_1, L_3, L_5 and L_7 . For example, a generic braided wiring diagram of \mathcal{N}^+ is pictured in Figure 4; the perturbation applied to obtain this wiring diagram is such that the three vertical lines have a very negative slope and are still parallel (since their intersection point is still on the line at infinity).

4.2.2 Method to compute the invariant Let $\mathcal{A} = \{D_1, \dots, D_n\}$ be a line arrangement, ξ be a character and $\gamma_{(r,s,t)}$ be the cycle defined by

$$vD_r \longrightarrow vP_{r,s} \longrightarrow vD_s \longrightarrow vP_{s,t} \longrightarrow vD_t \longrightarrow vP_{r,t}$$

We assume that $(\mathcal{A}, \xi, \gamma_{(r,s,t)})$ is an inner-cyclic arrangement. Let $W_{\mathcal{A}}$ be a wiring diagram of \mathcal{A} such that the line L_r is considered (in $W_{\mathcal{A}}$) as the line at infinity.

To compute the invariant $\mathcal{I}(\mathcal{A}, \xi, \gamma_{(r,s,t)})$, we consider the singular braid formed by the part of $W_{\mathcal{A}}$ from the left of the diagram to the intersection point of L_s and L_t (excluding this point). Then, we construct a usual braid $\sigma_{(r,s,t)}^{\mathcal{A}} \in \mathbb{B}_{n-1}$ by replacing each singular point with a positive local half-twist, as illustrated in Figure 5.

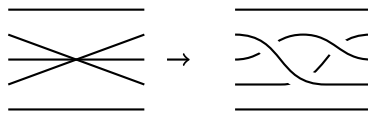


Figure 5: Construction of the braid $\sigma_{(r,s,t)}^{\mathcal{A}}$.

Remark 4.7 The braid $\sigma_{(r,s,t)}^{\mathcal{A}}$ is, in fact, the conjugating braid of any expression of the braid monodromy associated with $L_s \cap L_t$.

Finally [6, Proposition 4.3] implies that the invariant is the image under ξ of

$$(4-1) \quad \sum_{i=0}^n a_{i,s}(\sigma_{(r,s,t)}^{\mathcal{A}})\alpha_i - \sum_{i=0}^n a_{i,t}(\sigma_{(r,s,t)}^{\mathcal{A}})\alpha_i,$$

where $a_{i,j}(\sigma_{(r,s,t)}^{\mathcal{A}})$ counts with sign how many times the string associated with D_i crosses over the string associated with D_j in $\sigma_{(r,s,t)}^{\mathcal{A}}$. The braid $\sigma_{(r,s,t)}^{\mathcal{A}}$ is oriented from left to right, and the sign of the crossing is illustrated in Figure 6. For more details on the computation of the invariant see [6, Section 4].

Remark 4.8 By convention, $a_{i,i} = 0$ for all i .

Remark 4.9 Since D_r is a fiber of p , no line can cross over it. Thus, we need not consider this line in the previous computation; the value of the invariant is determined by what happens to D_s and D_t .



Figure 6: Sign of the crossing in $\sigma_{(r,s,t)}^A$. Left: positive crossing. Right: negative crossing.

4.2.3 Computation for \mathcal{N}^+ and \mathcal{M}^+ To apply the previous algorithm to \mathcal{N}^+ and \mathcal{M}^+ , we take $D_r = L_5$, $D_s = L_6$ and $D_t = L_{11}$; the character ξ considered is

$$\xi: (L_1, \dots, L_{11}) \mapsto (\zeta, \zeta^4, \zeta^3, \zeta^2, 1, 1, \zeta, \zeta^2, \zeta^3, \zeta^4, 1).$$

By Proposition 4.3, $(\mathcal{N}^+, \xi, \gamma_{(5,6,11)})$ and $(\mathcal{M}^+, \xi, \gamma_{(5,6,11)})$ are inner-cyclic arrangements. In the following, we give full details for the computation of $\mathcal{I}(\mathcal{N}^+, \xi, \gamma_{(5,6,11)})$.

- For \mathcal{N}^+ : the braid $\sigma_{(5,6,11)}^{\mathcal{N}^+}$ (pictured in Figure 7) is obtained from its generic braided wiring diagram (see Figure 4).

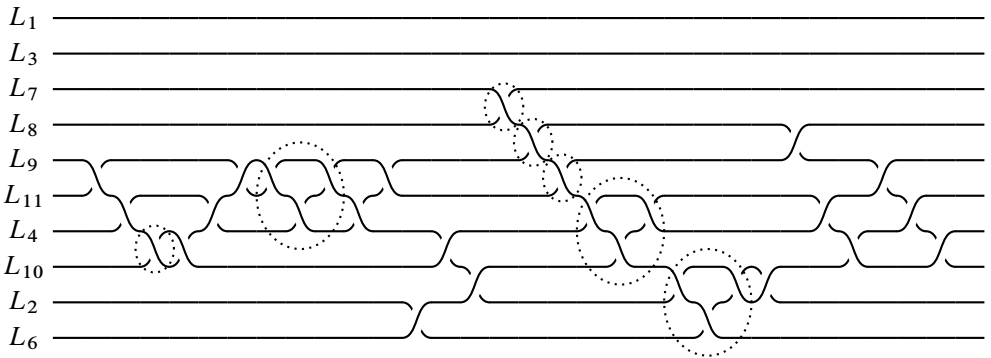


Figure 7: The braid $\sigma_{(5,6,11)}^{\mathcal{N}^+}$.

Remark 4.10 The circled crossings indicate the locations where singular points of $W_{\mathcal{N}^+}$ become local half-twists.

To determine the value of (4-1) we proceed in a two-fold manner. Firstly, we add (with sign) the meridians of the lines crossing over wire 6. Secondly, we subtract (with sign) the meridians of the lines crossing over wire 11.

Wire 6 is over-crossed:

- twice by wire 10 (once positively and once negatively),
- once positively by wire 7.

Wire 11 is over-crossed:

- three times by wire 9 (twice positively and once negatively),
- once negatively by wire 10,
- twice by wire 6 (one positively and one negatively),
- once positively by wire 7.

Thus, we obtain that

$$\mathcal{I}(\mathcal{N}^+, \xi, \gamma_{(5,6,11)}) = \xi((\alpha_7) - (\alpha_9 - \alpha_{10} + \alpha_7)) = \zeta.$$

- For \mathcal{M}^+ : from a perturbation of the non-generic braided wiring diagram pictured in Figure 3, we construct the braid $\sigma_{(5,6,11)}^{\mathcal{M}^+}$. It is pictured in Figure 8.

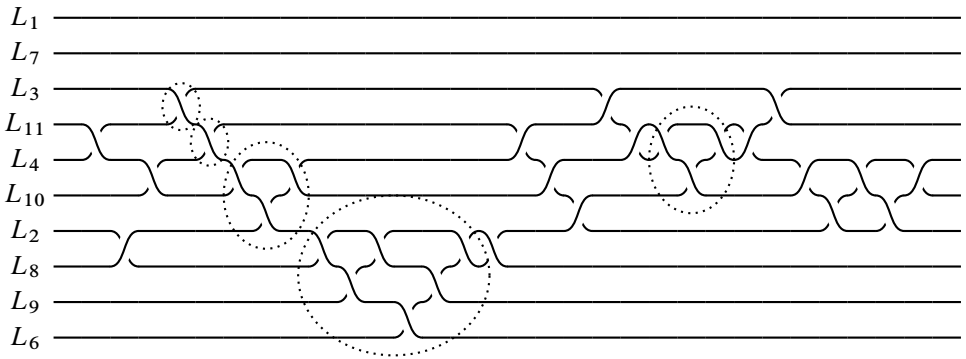


Figure 8: The braid $\sigma_{(5,6,11)}^{\mathcal{M}^+}$.

- Remark 4.11**
- (1) The perturbation applied to obtain the generic braided wiring diagram from the non-generic one pictured in Figure 3 is such that the three vertical lines have very negative slope.
 - (2) The circled crossings indicate the locations where singular points of $W_{\mathcal{M}^+}$ become local half-twists.

After computation, we obtain that

$$\mathcal{I}(\mathcal{M}^+, \xi, \gamma_{(5,6,11)}) = \xi((\alpha_3 + \alpha_9 + \alpha_2) - (\alpha_3)) = \zeta^2.$$

By [6, Proposition 2.5] we know that taking the invariant commutes with complex conjugation, so

$$\mathcal{I}(\mathcal{N}^-, \xi, \gamma_{(5,6,11)}) = \bar{\zeta} = \zeta^4 \quad \text{and} \quad \mathcal{I}(\mathcal{M}^-, \xi, \gamma_{(5,6,11)}) = \bar{\zeta}^2 = \zeta^3.$$

By Theorem 4.4, we have proved Theorem 2.14. Thus, to delete the “orientation-preserving” condition of Theorem 4.4, we consider the following lemma.

Lemma 4.12 *Let \mathcal{A}_1 and \mathcal{A}_2 be two arrangements with the same combinatorics and such that there is no homeomorphism preserving both orientation and order between $(\mathbb{C}\mathbb{P}^2, \mathcal{A}_1)$ and $(\mathbb{C}\mathbb{P}^2, \mathcal{A}_2)$. If there is no orientation-preserving homeomorphism between \mathcal{A}_2 and the complex conjugate of \mathcal{A}_1 then there is no order-preserving homeomorphism between $(\mathbb{C}\mathbb{P}^2, \mathcal{A}_1)$ and $(\mathbb{C}\mathbb{P}^2, \mathcal{A}_2)$.*

This is a consequence of [3, Theorem 4.19] (see also [13, Theorem 6.4.8] for a complete proof). Applying this lemma to [Theorem 2.14](#) (ie to the pairs $(\mathbb{C}\mathbb{P}^2, \mathcal{N}^\pm)$ and $(\mathbb{C}\mathbb{P}^2, \mathcal{M}^\pm)$), we obtain [Corollary 2.16](#).

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Received: 9 November 2014
Revised: 22 March 2015

