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Stable pair invariants on Calabi–Yau threefolds containing $\mathbb{P}^2$

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We relate Pandharipande–Thomas stable pair invariants on Calabi–Yau 3–folds containing the projective plane with those on the derived equivalent orbifolds via the wall-crossing method. The difference is described by generalized Donaldson–Thomas invariants counting semistable sheaves on the local projective plane, whose generating series form theta-type series for indefinite lattices. Our result also derives non-trivial constraints among stable pair invariants on such Calabi–Yau 3–folds caused by a Seidel–Thomas twist.

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1 Introduction

1.1 Motivation

It is an important subject to count algebraic curves on Calabi–Yau 3–folds, or more generally on CY3 orbifolds,[1] in connection with string theory. So far at least three curve counting theories have been proposed and studied: Gromov–Witten (GW) theory (see Behrend [4]), Donaldson–Thomas (DT) theory (see Thomas [50] and Maulik, Nekrasov, Okounkov and Pandharipande [37]) and Pandharipande–Thomas (PT) stable pair theory (see Pandharipande and Thomas [42]). It was conjectured, and proved in many cases, that these theories are equivalent: the equivalence of DT and PT theories was proved in Bridgeland [10], Toda [53] and Stoppa and Thomas [49] using Hall algebras, and the equivalence of GW and PT theories was proved by Pandharipande and Pixton [41] for many Calabi–Yau 3–folds, including quintic 3–folds, using degenerations and torus localizations.

On the other hand, the derived category of coherent sheaves $D^b \operatorname{Coh}(X)$ on a Calabi–Yau 3–fold X is also an important mathematical subject, due to its role in Kontsevich's homological mirror symmetry conjecture [30]. It was suggested by Pandharipande and

[1] In this paper, an orbifold means a smooth Deligne–Mumford stack.

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Thomas [42] that the derived category also plays a crucial role in curve counting, as their stable pair invariants count two-term complexes

$$(\mathcal{O}_X \xrightarrow{s} F) \in D^b \operatorname{Coh}(X),$$

where F is a pure one-dimensional sheaf and s is surjective in dimension one. In this paper, we are concerned with how symmetries in the derived categories affect stable pair invariants. More precisely, we are interested in the following questions:

Question 1.1 (i) How are stable pair invariants on two Calabi–Yau 3–folds or orbifolds related, if they have equivalent derived categories?

(ii) How are stable pair invariants on a Calabi–Yau 3–fold constrained, due to the presence of non-trivial autoequivalences of the derived category?

The purpose of this paper is to study Question 1.1 for stable pair invariants on Calabi–Yau 3–folds X which contain $\mathbb{P}^2$, and their derived equivalent CY3 orbifolds $\mathcal{Y}$. Our results include new progress on Question 1.1: (i) the relation of stable pair invariants on X and $\mathcal{Y}$, where $\mathcal{Y}$ does *not* satisfy the hard Lefschetz (HL) condition[2] (ii) constraints of stable pair invariants on X caused by a Seidel–Thomas twist [48]. The relation of our work with existing works will be discussed in Section 1.3.

1.2 Main result

Let X be a smooth projective Calabi–Yau 3–fold which contains a divisor

$$\mathbb{P}^2 \cong D \subset X.$$

We have two phenomena related to (i) and (ii) in Question 1.1:

(i) The divisor D is contracted by a birational morphism $f\colon X \to Y$ to an orbifold singularity with type $\frac{1}{3}(1,1,1)$. The associated smooth Deligne–Mumford stack $\mathcal{Y} \to Y$ is derived equivalent to X:

$$\Phi\colon D^b \operatorname{Coh}(\mathcal{Y}) \xrightarrow{\sim} D^b \operatorname{Coh}(X).$$

(ii) The object $\mathcal{O}_D$ is a spherical object in $D^b \operatorname{Coh}(X)$, and we have the associated autoequivalence, called a *Seidel–Thomas twist* [48]:

$$\mathrm{ST}_{\mathcal{O}_D}\colon D^b \operatorname{Coh}(X) \xrightarrow{\sim} D^b \operatorname{Coh}(X).$$

[2] The HL condition on a CY3 orbifold is equivalent to the crepant resolution of its coarse moduli space having at most one-dimensional fibers.

Contrary to the 3–fold flop case as in Toda [56] and Calabrese [17], curves on $\mathcal{Y}$ and X may be transformed to objects with two-dimensional supports under the equivalences Φ and $\mathrm{ST}_{\mathcal{O}_D}$, respectively. In order to deal with this issue, we also involve generalized DT invariants (see Joyce and Song [25] and Kontsevich and Soibelman [31])

$$\mathrm{DT}(r, c, m) \in \mathbb{Q} \tag{1}$$

on the non-compact Calabi–Yau 3–fold $\pi\colon \omega_{\mathbb{P}^2} \to \mathbb{P}^2$. The invariant (1) counts semistable sheaves E on $\omega_{\mathbb{P}^2}$ satisfying

$$\operatorname{rank}(\pi_* E) = r, \quad c_1(\pi_* E) = c, \quad \mathrm{ch}_2(\pi_* E) = m.$$

The following is a rough statement of our main result:

Theorem 1.2 (Theorem 5.11, Theorem 5.20) *Assuming Conjecture 1.3 below, we have the following:*

(i) *The stable pair invariants on $\mathcal{Y}$ are described as explicit polynomials of stable pair invariants on X and generalized DT invariants (1) on $\omega_{\mathbb{P}^2}$.*

(ii) *If there is $\mathcal{L} \in \operatorname{Pic}(X)$ with $\mathcal{L}|_D \cong \mathcal{O}_D(1)$, then there exist explicit polynomial relations among stable pair invariants on X and generalized DT invariants (1) on $\omega_{\mathbb{P}^2}$ caused by $\mathrm{ST}_{\mathcal{O}_D} \circ \otimes \mathcal{L}$.*

The result of Theorem 1.2(i) in particular derives a recursion formula for stable pair invariants on X with curve classes proportional to $[l]$ for a line $l \subset D$ (in other words, stable pair invariants on $\omega_{\mathbb{P}^2}$), whose coefficients involve the invariants (1) (see Corollary 5.14). The result of Theorem 1.2(ii) implies a stronger statement: the stable pair invariants on X with curve classes β satisfying $D \cdot \beta < 0$ are described in terms of those with curve classes $\beta - c[l]$ for $c > 0$, with coefficients involving (1) (see Remark 5.21).

In the previous paper [57], the author proved a recursion formula for the generating series of the invariants (1) with $r > 0$ in terms of theta-type series for indefinite lattices. It is also possible to describe the invariants (1) with $r = 0, c > 0$ in terms of stable pair invariants on X with curve classes proportional to $[l]$ (see Lemma 3.15). These results imply that, in principle, one can compute the relations between stable pair invariants concerning Question 1.1 for the derived equivalences Φ and $\mathrm{ST}_{\mathcal{O}_D}$. The resulting formulas in Theorems 5.11 and 5.20 are complicated, and we leave it to a future work to give a more conceptual understanding of our result.

We should mention that the result of Theorem 1.2 is still conditional on the following conjecture, which was also assumed in the author's previous work [56].

Conjecture 1.3 *Let $\mathcal{M}$ be the moduli stack of objects $E \in D^b \operatorname{Coh}(X)$ satisfying $\operatorname{Ext}^{<0}(E,E)=0$. For $[E]\in\mathcal{M}$, let G be a maximal reductive subgroup of $\operatorname{Aut}(E)$. Then there is a G–invariant analytic open neighborhood V of 0 in $\operatorname{Ext}^1(E,E)$, a G–invariant holomorphic function $f\colon V\to\mathbb{C}$ with $f(0)=df|_0=0$, and a smooth morphism of complex-analytic stacks*

$$\Phi\colon [\{df=0\}/G]\to\mathcal{M}$$

of relative dimension $\dim\operatorname{Aut}(E)-\dim G$.

This conjecture has been a technical obstruction to generalizing Joyce and Song's wall-crossing formula for DT invariants [25] for coherent sheaves to the derived category. It was proved for $E\in\operatorname{Coh}(X)$ by Joyce and Song [25], and announced by Behrend and Getzler. There is more recent progress toward it, which will be reviewed in the next subsection. Without assuming Conjecture 1.3, we can prove the Euler characteristic version of Theorem 1.2 (ie results for the naive Euler characteristics of stable pair moduli spaces), as stated in Section 5.7.

1.3 Related works

In Toda [56] and Calabrese [17], the flop transformation formula of stable pair invariants was obtained from the categorical viewpoint, giving an answer to Question 1.1(i) for birational Calabi–Yau 3–folds. In the orbifold case, let Y be a Calabi–Yau 3–fold with Gorenstein quotient singularities and $X\to Y$ its crepant resolution. Under the HL condition on the associated Deligne–Mumford stack $\mathcal{Y}\to Y$, Bryan, Cadman and Young [13] formulated a conjectural relationship between DT invariants on X and those on $\mathcal{Y}$. Combined with the DT/PT correspondence (see Bridgeland [10], Toda [53] and Stoppa and Thomas [49]) on X, and Bayer's announced work on it for CY3 orbifolds with HL condition, we have a conjectural answer to Question 1.1(i) in this situation. The conjecture in [13] is still open, but some progress toward it is obtained in Calabrese [18], Bryan and Steinberg [15] and Ross [45].

In the above HL case, the resulting formula should be described by a product formula for the generating series of stable pair invariants. In our situation of Theorem 1.2, the stack $\mathcal{Y}$ does not satisfy the HL condition, and it seems unlikely that the results are formulated as product formulas for the generating series. From the categorical viewpoint, the main difference from the HL case is the non-triviality of the Euler pairings between objects supported on the fibers of $X\to Y$. Due to this non-triviality, the combinatorics of the wall-crossing becomes complicated, and it seems hard to understand the result in terms of the generating series. In any case, we hope that the result of Theorem 1.2 would give a hint toward a generalization of the conjecture in Bryan, Cadman and Young [13] without the HL condition.

There exist few works concerning Question 1.1(ii) so far. We can say that the rationality of the generating series of stable pair invariants, conjectured in Pandharipande and Thomas [42] and proved in Toda [54] and Bridgeland [10], is interpreted to be an answer to Question 1.1(ii) for the derived dualizing functor. Also, the automorphic property of sheaf-counting invariants on local K3 surfaces under Hodge isometries, together with product expansion of the generating series of stable pair invariants on them (see Toda [55]) in terms of the former invariants, is interpreted to be an answer to Question 1.1(ii) for autoequivalences of K3 surfaces; see Toda [51; 55]. The result of Theorem 1.2(ii) provides a further example of such phenomena.

In GW theory, an analogue of Question 1.1(i) has been one of the central themes. Since birational Calabi–Yau 3–folds or orbifolds should be derived equivalent (see Bridgeland [7], Bridgeland, King and Reid [11] and Kawamata [27]), Question 1.1(i) for GW theory is related to the analytic continuation problem for quantum cohomologies discussed in Ruan [47], Bryan and Graber [14] and Coates, Iritani and Tseng [19]. Also, we expect that Question 1.1(ii) is related to the modularity problem for partition functions of GW invariants, as the action of autoequivalences on the derived category should correspond to the monodromy action under the mirror symmetry. We refer to Okounkov and Pandharipande [40] and Milanov, Ruan and Shen [38] for works on modularity in GW theory.

In recent years, we have seen progress toward an algebraic version of Conjecture 1.3 using derived algebraic geometry. By the work of Pantev, Toën, Vaquie and Vezzosi [44], the stack $\mathcal{M}$ is shown to be a derived stack with a (-1)–shifted symplectic structure. Using this fact, Ben-Bassat, Brav, Bussi and Joyce [6] showed that $\mathcal{M}$ has Zariski-locally an atlas which is written as a critical locus of a certain algebraic function. Still, this is not enough to conclude Conjecture 1.3. However, under the assumption that $\mathcal{M}$ is Zariski-locally written as a quotient stack of the form $[S/\operatorname{GL}_n(\mathbb{C})]$ for some complex scheme S, Bussi [16, Theorem 4.3] showed a result which is very similar to Conjecture 1.3. Indeed, her result implies relevant Behrend function identities for objects in $\mathcal{M}$, which are enough for our applications. At this moment, the author does not know how to eliminate the local quotient stack assumption, nor prove it in the situations we are interested in.

1.4 Ideas behind the proof of Theorem 1.2

Theorem 1.2 follows from a wall-crossing argument in the space of weak stability conditions, as in the author's previous papers [53; 56; 55]. In order to explain the

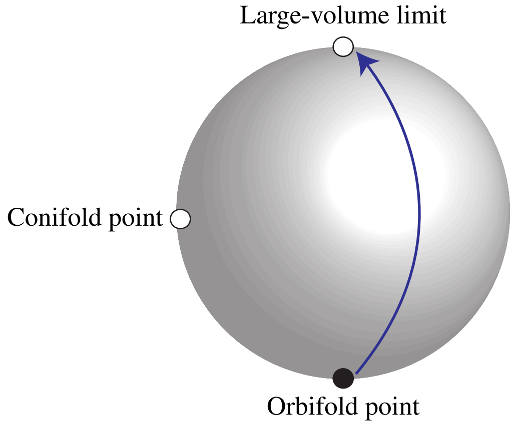

Figure 1: The space of stability conditions on $\omega_{\mathbb{P}^2}$.

argument, we first recall Bayer and Macrì's description of the space $\operatorname{Stab}(\omega_{\mathbb{P}^2})$ of Bridgeland stability conditions on $D^b \operatorname{Coh}(\omega_{\mathbb{P}^2})$ in [1]. They showed that the double quotient stack of $\operatorname{Stab}(\omega_{\mathbb{P}^2})$ by the actions of $\operatorname{Aut} D^b \operatorname{Coh}(\omega_{\mathbb{P}^2})$ and the additive group $\mathbb{C}$ contains the parameter space of the mirror family of $\omega_{\mathbb{P}^2}$. The latter space has three special points: the large volume limit, a conifold point and an orbifold point (see Figure 1). Near the large volume limit, the semistable objects consist of (essentially) Gieseker semistable sheaves on $\omega_{\mathbb{P}^2}$. At the orbifold point, the semistable objects consist of representations of the McKay quiver under the derived McKay correspondence; see Bridgeland, King and Reid [11]. By taking a path connecting the orbifold point with the large volume limit, one can relate representations of the McKay quiver with semistable sheaves on $\omega_{\mathbb{P}^2}$ by wall-crossing phenomena: there is a finite number of walls on this path such that the sets of semistable objects are constant on the interval, but jump at walls.

Let us return to our global situation. In the situation of Theorem 1.2, we define the triangulated category

$$\mathcal{D}_{X/Y} := \langle \mathcal{O}_X, D^b \operatorname{Coh}_{\leq 1}(X/Y) \rangle_{\mathrm{tr}}.$$

Here $\operatorname{Coh}_{\leq 1}(X/Y)$ is the category of coherent sheaves on X which are at most one-dimensional outside D. Our strategy is to construct a path similar to Figure 1 in the space of weak stability conditions on $\mathcal{D}_{X/Y}$,

$$\sigma_t \in \operatorname{Stab}_{\Gamma_\bullet}(\mathcal{D}_{X/Y}), \quad t \in \mathbb{R}_{\geq 0}.$$

This one-parameter family is an analogue of the path in Figure 1, ie $\lim_{t \to \infty} \sigma_t$ corresponds to the large volume limit, and σ_0 corresponds to the orbifold point. We show that the rank-one σ_t–stable objects for $t \gg 1$ consist of objects of the form

$$\mathcal{O}_X(rD) \otimes (\mathcal{O}_X \to F) \tag{2}$$

for $r \in \mathbb{Z}$ and a stable pair $(\mathcal{O}_X \to F)$ on X. We also show that the rank-one σ_0–stable objects consist of objects of the form

$$\Phi(\mathcal{O}_{\mathcal{Y}} \to F) \tag{3}$$

for a stable pair $(\mathcal{O}_{\mathcal{Y}} \to F)$ on $\mathcal{Y}$. Then we can relate the objects (2) and (3) by wall-crossing phenomena. If we assume Conjecture 1.3, then Joyce and Song's wall-crossing formula [25] applies in our setting. It relates stable pair invariants on $\mathcal{Y}$ with those on X together with the invariants (1), giving Theorem 1.2(i).

We now explain the idea of Theorem 1.2(ii). It follows from a general principle explained in Toda [58, Section 1]. In general, suppose that there is a stability condition τ on the derived category of a Calabi–Yau 3–fold which has a symmetric property with respect to an autoequivalence Θ in a certain sense. In [58], such a stability condition τ was called *Gepner-type* with respect to Θ. Let $\mathrm{DT}_\tau(v)$ be the DT-type invariant (if it exists) counting τ–semistable objects with numerical class v. The Gepner-type property of τ would yield

$$\mathrm{DT}_\tau(v) = \mathrm{DT}_\tau(\Theta_* v). \tag{4}$$

On the other hand, one may relate both sides of (4) with classical DT invariants counting sheaves or curves by wall-crossing. Combined with the identity (4), one may obtain non-trivial constraints among classical DT invariants caused by Θ.

In Figure 1, the orbifold point is known to be Gepner-type with respect to $\Theta = \mathrm{ST}_{\mathcal{O}_D} \circ \otimes \mathcal{L}$. Since the weak stability condition σ_0 on $\mathcal{D}_{X/Y}$ is an analogue of the orbifold point, one expects that the above general philosophy may be applied to obtain constraints among stable pair invariants on X caused by Θ. In our situation, the equivalence Θ does not preserve $\mathcal{D}_{X/Y}$, so σ_0 is not Gepner-type in a strict sense. However, one can prove that Θ takes σ_0–stable objects to similar stable objects in another triangulated category

$$\mathcal{D}^{\mathcal{L}}_{X/Y} := \langle \mathcal{L}, D^b \operatorname{Coh}_{\le 1}(X/Y) \rangle_{\mathrm{tr}}.$$

Namely, there also exists a one-parameter family $\sigma_t^{\mathcal{L}}$ of weak stability conditions on $\mathcal{D}^{\mathcal{L}}_{X/Y}$ such that σ_0–stable objects and $\sigma_0^{\mathcal{L}}$–stable objects coincide under the equivalence Θ. We then apply the similar wall-crossing formula in $\mathcal{D}^{\mathcal{L}}_{X/Y}$ from $\sigma_0^{\mathcal{L}}$ to $\sigma_t^{\mathcal{L}}$ for $t \gg 1$. It implies another description of stable pair invariants on $\mathcal{Y}$ in terms of those on X and the invariants (1). By comparing it with the result of Theorem 1.2(i), we obtain the constraints in Theorem 1.2(ii).

1.5 Plan of the paper

In Section 2, we recall derived equivalences concerning Calabi–Yau 3–folds containing $\mathbb{P}^2$, and fix some notation. In Section 3, we recall stable pair invariants, generalized DT invariants on local $\mathbb{P}^2$, and their properties. In Section 4, we construct a one-parameter family of weak stability conditions on the triangulated category $\mathcal{D}_{X/Y}$. In Section 5, we describe the wall-crossing phenomena with respect to our weak stability conditions, and prove Theorem 1.2.

1.6 Notation and convention

In this paper, all the varieties or stacks are defined over $\mathbb{C}$. For a d–dimensional variety X, we denote by $H^*(X,\mathbb{Q})$ the even part of the singular cohomology of X, and write its elements as $(v^0, v^1, \dots, v^d)$ for $v^i \in H^{2i}(X,\mathbb{Q})$. We sometimes omit $\mathbb{Q}$ and just write $H^{2i}(X,\mathbb{Q})$, $H_{2i}(X,\mathbb{Q})$ as $H^{2i}(X)$, $H_{2i}(X)$. For a triangulated category $\mathcal{D}$ and a set of objects S in $\mathcal{D}$, we denote by $\langle S\rangle_{\mathrm{tr}}$ the triangulated closure, ie the smallest triangulated category of $\mathcal{D}$ which contains S. Also $\langle S\rangle_{\mathrm{ex}}$ is the extension closure, ie the smallest extension-closed subcategory in $\mathcal{D}$ which contains S. For a Deligne–Mumford stack $\mathcal{Y}$, we denote by $\mathrm{Coh}(\mathcal{Y})$ the abelian category of coherent sheaves on $\mathcal{Y}$. For $d \in \mathbb{Z}$, we denote by $\mathrm{Coh}_{\leq d}(\mathcal{Y}) \subset \mathrm{Coh}(\mathcal{Y})$ the subcategory of objects $E \in \mathrm{Coh}(\mathcal{Y})$ satisfying $\dim \mathrm{Supp}(E) \leq d$. If $d = 0$, we write the subscript "≤ 0" just as "0", eg we write $\mathrm{Coh}_{\leq 0}(X)$ as $\mathrm{Coh}_0(X)$, etc. For $E \in D^b\,\mathrm{Coh}(\mathcal{Y})$, we denote by $\mathcal{H}^i(E) \in \mathrm{Coh}(\mathcal{Y})$ its i^{th} cohomology.

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2 The derived category of Calabi–Yau 3–folds containing $\mathbb{P}^2$

2.1 Geometry of Calabi–Yau 3–folds containing $\mathbb{P}^2$

Let X be a smooth projective Calabi–Yau 3–fold, ie

$$K_X = 0, \quad H^1(X, \mathcal{O}_X) = 0.$$

We always assume that there is a closed embedding

$$i\colon \mathbb{P}^2 \hookrightarrow X,$$

whose image we denote by D. There exist several examples of such Calabi–Yau 3–folds, as follows:

Example 2.1 Let X be the hypersurface in $\mathbb{P}^3 \times \mathbb{P}^1$ given by

$$X = \left\{ y_1 \left(\sum_{i=1}^{4} x_i^4 \right) + y_2 \prod_{i=1}^{4} x_i = 0 \right\}.$$

Here $x_i, 1 \le i \le 4$ are homogeneous coordinates of $\mathbb{P}^3$ and $y_i, 1 \le i \le 2$ are those of $\mathbb{P}^1$. Then X is a smooth Calabi–Yau 3–fold which contains planes $(y_1 = x_i = 0)$ for each $1 \le i \le 4$.

Example 2.2 Let $S_1 \subset \mathbb{P}^3$ be a plane, $S_7 \subset \mathbb{P}^3$ a smooth hypersurface of degree seven, such that the divisor $S_1 + S_7$ is a normal crossing divisor. Let $Z \to \mathbb{P}^3$ be the double cover branched along $S_1 + S_7$. Then Z has A_1–singularities along the pullback of $S_1 \cap S_7$. Let

$$X \to Z$$

be the blow-up along the singular locus. Then X is a smooth Calabi–Yau 3–fold, and the pullback of S_1 contains $\mathbb{P}^2$ as an irreducible component. See [26, Section 4.2].

Example 2.3 Let E be the elliptic curve which admits an automorphism of order 3. The group $G = \mathbb{Z}/3\mathbb{Z}$ acts on $E \times E \times E$ diagonally. Then the crepant resolution

$$X \to (E \times E \times E)/G$$

is a smooth Calabi–Yau 3–fold which contains 27 planes. See [3, Section 2].

Since $K_X = 0$, we have $\omega_D \cong \mathcal{O}_D(D)$ by the adjunction. Therefore, for a line $l \subset D$, we have

$$D \cdot l = -3, \quad D^2 = -3[l], \quad D^3 = 9.$$

If H is an ample divisor on X, the divisor $3H + (H \cdot l)D$ is obviously nef and big on X. By the basepoint-free theorem (see [29, Theorem 3.3]), some multiple of it gives a birational morphism

$$f\colon X \to Y \tag{5}$$

which contracts a divisor $D \subset X$ to a point $p \in Y$. It is well-known that[3]

$$\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[\![x,y,z]\!]^G. \tag{6}$$

Here $G := \mathbb{Z}/3\mathbb{Z}$ acts on $\mathbb{C}[\![x,y,z]\!]$ via the weight $(1,1,1)$. Since $p \in Y$ is a quotient singularity, we have the associated smooth Deligne–Mumford stack[4]

$$g\colon \mathcal{Y} \to Y \tag{7}$$

whose coarse moduli space is isomorphic to Y. The diagram

$$\begin{tikzcd} & X \times_Y \mathcal{Y} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dotted, "\phi"] \arrow[dr, "f"'] & & \mathcal{Y} \arrow[dl, "g"] \\ & Y & \end{tikzcd} \tag{8}$$

is pulled back via $\operatorname{Spec} \widehat{\mathcal{O}}_{Y,p} \to Y$ to the pullback via $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathbb{C}^3/G,0} \to \mathbb{C}^3/G$ of the standard McKay diagram[5] of local $\mathbb{P}^2$:

$$\begin{tikzcd} & {[\mathrm{Bl}_0 \mathbb{C}^3/G]} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ U = \omega_{\mathbb{P}^2} \arrow[rr, dotted, "\phi"] \arrow[dr, "f"'] & & {[\mathbb{C}^3/G]} \arrow[dl, "g"] \\ & \mathbb{C}^3/G. & \end{tikzcd} \tag{9}$$

Here $\mathrm{Bl}_0 \mathbb{C}^3 \to \mathbb{C}^3$ is the blow-up at the origin, which admits the G–action since G fixes the origin. Also

$$\pi\colon U = \omega_{\mathbb{P}^2} \to \mathbb{P}^2 \tag{10}$$

is the total space of the canonical line bundle of $\mathbb{P}^2$, which is the coarse moduli space of the quotient stack $[\mathrm{Bl}_0 \mathbb{C}^3/G]$.

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[3]For example, the argument of [39, (3.3.5)] shows the isomorphism (6).

[4]We refer to [27, Definition 2.1] for the construction of $\mathcal{Y}$. We note that Y satisfies the condition $(*)$ in [27, Definition 2.1] since $p \in Y$ is a quotient singularity.

[5]By abuse of notation, in the diagram (9), we use the same notation for the morphisms used in the diagram (8).

2.2 Derived equivalence

Since the diagram (9) is toric, we can apply [27, Theorem 4.2] (also see [11]) to show the derived equivalence

$$\Phi := \boldsymbol{R}p_*\boldsymbol{L}q^*\colon D^b\,\mathrm{Coh}(\mathcal{Y}) \xrightarrow{\sim} D^b\,\mathrm{Coh}(X). \tag{11}$$

Let ρ_j be the one-dimensional representation of G with weight j. The objects

$$S_j := \rho_{-j}\otimes\mathcal{O}_0 \in \mathrm{Coh}_G(\mathbb{C}^3)$$

naturally define the objects $S_j \in \mathrm{Coh}(\mathcal{Y})$. Here $\mathrm{Coh}_G(\mathbb{C}^3)$ is the category of G–equivariant coherent sheaves on $\mathbb{C}^3$, which coincides with the category of coherent sheaves on $[\mathbb{C}^3/G]$. Also, we set

$$T_0 := \mathcal{O}_D, \quad T_1 := \Omega_D(1)[1], \quad T_2 := \mathcal{O}_D(-1)[2]. \tag{12}$$

We have the following lemma.

Lemma 2.4 *We have* $\Phi(\mathcal{O}_{\mathcal{Y}}) \cong \mathcal{O}_X$ *and* $\Phi(S_j)\cong T_j$ *for* $0\le j\le 2$.

Proof It is enough to prove the same claim for the local derived equivalence

$$\Phi = \boldsymbol{R}p_*\boldsymbol{L}q^*\colon D^b\,\mathrm{Coh}_G(\mathbb{C}^3) \xrightarrow{\sim} D^b\,\mathrm{Coh}(\omega_{\mathbb{P}^2}).$$

Let $\mathcal{E}$ be the vector bundle on $U=\omega_{\mathbb{P}^2}$ given by

$$\mathcal{E} = \mathcal{O}_U\oplus\mathcal{O}_U(1)\oplus\mathcal{O}_U(2).$$

It is well-known that we have the derived equivalence (see [8])

$$\boldsymbol{R}\,\mathrm{Hom}(\mathcal{E},-)\colon D^b\,\mathrm{Coh}(U)\xrightarrow{\sim} D^b\,\mathrm{mod}(B), \tag{13}$$

where B is the non-commutative algebra defined by $\mathrm{End}(\mathcal{E})$. The algebra B is the path algebra of a quiver with three vertices and some relations. Under the equivalence (13), the objects T_0, T_1, T_2 are sent to the simple objects corresponding to the three vertices. Therefore, the isomorphism $\Phi(S_j)\cong T_j$ follows if we show that

$$\boldsymbol{R}\,\mathrm{Hom}(\mathcal{O}_U(k),\Phi(S_j)) = \mathbb{C}^{\delta_{jk}} \tag{14}$$

for $0\le j,k\le 2$. Let $p'\colon \mathrm{Bl}_0(\mathbb{C}^3)\to U$ be the Galois G–cover. Then we have

$$p'_*\mathcal{O}_{\mathrm{Bl}_0(\mathbb{C}^3)} \cong \mathcal{O}_U\oplus\mathcal{O}_U(1)\otimes\rho_1\oplus\mathcal{O}_U(2)\otimes\rho_2.$$

It follows that

$$\Phi(\mathcal{O}_{\mathbb{C}^3}\otimes\rho_{-k}) \cong (\mathcal{O}_U\otimes\rho_{-k}\oplus\mathcal{O}_U(1)\otimes\rho_{1-k}\oplus\mathcal{O}_U(2)\otimes\rho_{2-k})^G \cong \mathcal{O}_U(k) \tag{15}$$

for $0 \le k \le 2$. Therefore (14) holds. The isomorphism $\Phi(\mathcal{O}_{\mathcal{Y}}) \cong \mathcal{O}_X$ also follows from the isomorphism (15) for $k = 0$. □

We define the abelian category

$$\mathrm{Coh}_{\le d}(X/Y) := \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}\, f_* E \le d\}. \tag{16}$$

By the construction of Φ, we have the commutative diagram[6]

$$\begin{tikzcd} D^b \mathrm{Coh}_0(\mathcal{Y}) \arrow[r, hook] \arrow[d, "\Phi"] & D^b \mathrm{Coh}_{\le 1}(\mathcal{Y}) \arrow[r, hook] \arrow[d, "\Phi"] & D^b \mathrm{Coh}(\mathcal{Y}) \arrow[d, "\Phi"] \\ D^b \mathrm{Coh}_0(X/Y) \arrow[r, hook] & D^b \mathrm{Coh}_{\le 1}(X/Y) \arrow[r, hook] & D^b \mathrm{Coh}(X) \end{tikzcd} \tag{17}$$

where each vertical arrow is an equivalence. Moreover, the definition of the equivalence (11) easily implies that the following diagram also commutes:

$$\begin{tikzcd} D^b \mathrm{Coh}(\mathcal{Y}) \arrow[rr, "\Phi"] \arrow[dr, "\boldsymbol{R} g_*"'] & & D^b \mathrm{Coh}(X) \arrow[dl, "\boldsymbol{R} f_*"] \\ & D^b \mathrm{Coh}(Y). & \end{tikzcd}$$

2.3 Numerical Grothendieck groups

Let $\mathcal{D}$ be a $\mathbb{C}$–linear triangulated category satisfying

$$\sum_{i \in \mathbb{Z}} \dim \mathrm{Hom}(E, F[i]) < \infty$$

for all $E, F \in \mathcal{D}$. Under this condition, the following Euler pairing is well-defined:

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathrm{Hom}(E, F[i]). \tag{18}$$

This pairing descends to the pairing on the Grothendieck group $K(\mathcal{D})$ of $\mathcal{D}$. Two elements $E_1, E_2 \in K(\mathcal{D})$ are called *numerically equivalent* if we have $\chi(E_1, F) = \chi(E_2, F)$ for any $F \in K(\mathcal{D})$. The *numerical Grothendieck group* $N(\mathcal{D})$ of $\mathcal{D}$ is defined to be the group of numerical equivalence classes of $K(\mathcal{D})$. By definition, the Euler pairing (18) descends to a perfect pairing on $N(\mathcal{D})$.

[6]It is well-known that, for a variety Y and $d \in \mathbb{Z}_{\ge 0}$, the category $D^b \mathrm{Coh}_{\le d}(Y)$ is equivalent to the full subcategory of $D^b \mathrm{Coh}(Y)$ whose objects have cohomologies in $\mathrm{Coh}_{\le d}(Y)$. This fact follows from [46, Lemma 7.40].

Let X, $\mathcal{Y}$ be as in the previous subsections. We set

$$N(X) := N(D^b\operatorname{Coh}(X)), \quad N(\mathcal{Y}) := N(D^b\operatorname{Coh}(\mathcal{Y})).$$

The equivalence (11) induces the isomorphism

$$\Phi_*\colon N(\mathcal{Y}) \xrightarrow{\sim} N(X). \tag{19}$$

Since X is a smooth projective 3–fold, it satisfies the Hodge conjecture by the hard Lefschetz theorem. Together with the Riemann–Roch theorem, the Chern character map from $K(X)$ descends to the injective homomorphism

$$\mathrm{ch}\colon N(X) \hookrightarrow H^*(X, \mathbb{Q}). \tag{20}$$

In particular, both $N(X)$ and $N(\mathcal{Y})$ are finitely generated free abelian groups. The Euler paring (18) on $N(X)$ is described as

$$\begin{aligned}\chi(E,F) = \sum_{j=0}^{3}(-1)^j \operatorname{ch}_j(E)\operatorname{ch}_{3-j}(F) \\ + \frac{c_2(X)}{12}\big(\operatorname{ch}_0(E)\operatorname{ch}_1(F) - \operatorname{ch}_1(E)\operatorname{ch}_0(F)\big).\end{aligned} \tag{21}$$

We set

$$\begin{aligned} N_{\le d}(\mathcal{Y}) &:= \operatorname{Im}(K(\operatorname{Coh}_{\le d}(\mathcal{Y})) \to N(\mathcal{Y})), \\ N_{\le d}(X/Y) &:= \operatorname{Im}(K(\operatorname{Coh}_{\le d}(X/Y)) \to N(X)). \end{aligned}$$

By the diagram (17), we have the commutative diagram

$$\begin{tikzcd} \bigoplus_{j=0}^{2}\mathbb{Z}[S_j] \arrow[r, hook, "\cong"] \arrow[d, "\cong"', "{[S_j]\mapsto[T_j]}"] & N_0(\mathcal{Y}) \arrow[r, hook] \arrow[d, "\cong"', "\Phi_*"] & N_{\le 1}(\mathcal{Y}) \arrow[r, hook] \arrow[d, "\cong"', "\Phi_*"] & N(\mathcal{Y}) \arrow[d, "\cong"', "\Phi_*"] \\ K(\mathbb{P}^2) \arrow[r, "i_*", "\cong"'] \arrow[d, "\mathrm{ch}"] & N_0(X/Y) \arrow[r, hook] & N_{\le 1}(X/Y) \arrow[r, hook] \arrow[d, "\mathrm{ch}"] & N(X) \arrow[d, "\mathrm{ch}"] \\ H^*(\mathbb{P}^2,\mathbb{Q}) \arrow[rr, "i_\sharp"] & & \mathbb{Q}[D]\oplus H^{\ge 4}(X,\mathbb{Q}) \arrow[r, hook] & H^*(X,\mathbb{Q}) \end{tikzcd}$$

Remark 2.5 It is well-known that $K(\mathbb{P}^2) \cong \mathbb{Z}^3$, and the Euler paring on it is perfect. This fact easily shows that the map $i_*\colon K(\mathbb{P}^2) \to N_0(X/Y)$ is an isomorphism.

In what follows, we fix the isomorphism

$$\mathbb{Q}^{\oplus 3} \xrightarrow{\cong} H^*(\mathbb{P}^2, \mathbb{Q}), \quad (r, c, m) \mapsto (r, c\cdot h, m\cdot h^2), \tag{22}$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, and write elements of $H^*(\mathbb{P}^2, \mathbb{Q})$ as $(r, c, m) \in \mathbb{Q}^3$ via (22). In this notation, the map $i_\sharp$ is given by

$$i_\sharp(r, c, m) = \big(0, r[D], \big(\tfrac{3}{2}r + c\big)[l], \tfrac{3}{2}r + \tfrac{3}{2}c + m\big) \tag{23}$$

by the Grothendieck–Riemann–Roch theorem.

2.4 Seidel–Thomas twist

The object $\mathcal{O}_D \in \mathrm{Coh}(X)$ is a spherical object, ie

$$\mathrm{Ext}^i_X(\mathcal{O}_D, \mathcal{O}_D) = \begin{cases} \mathbb{C} & i = 0, 3, \\ 0 & i \neq 0, 3. \end{cases}$$

By [48], there is the associated autoequivalence

$$\mathrm{ST}_{\mathcal{O}_D}\colon D^b\,\mathrm{Coh}(X) \xrightarrow{\sim} D^b\,\mathrm{Coh}(X),$$

called the *Seidel–Thomas twist*. For any $E \in D^b\,\mathrm{Coh}(X)$, the object $\mathrm{ST}_{\mathcal{O}_D}(E)$ fits into the distinguished triangle

$$\boldsymbol{R}\,\mathrm{Hom}(\mathcal{O}_D, E) \otimes \mathcal{O}_D \to E \to \mathrm{ST}_{\mathcal{O}_D}(E). \tag{24}$$

We now assume that there is a line bundle $\mathcal{L}$ on X satisfying the condition

$$i^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(1). \tag{25}$$

Remark 2.6 Note that the image of the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ contains $\mathcal{O}_X(D)|_D \cong \mathcal{O}_D(3)$. Hence the existence of a line bundle $\mathcal{L}$ satisfying (25) is equivalent to the existence of a line bundle $\mathcal{L}'$ such that $\mathcal{L}'|_D \cong \mathcal{O}_D(a)$ for some $a \in \mathbb{Z}$ which is coprime to 3. For example, such a line bundle $\mathcal{L}$ exists in Examples 2.1, 2.2, 2.3. However, we are not sure about the existence of such a line bundle $\mathcal{L}$ in general. The assumption on the existence of $\mathcal{L}$ will be used in Section 5.6.

The isomorphism (15) for $k = 1$ shows that the object

$$\mathcal{L}^\dagger := \Phi^{-1}(\mathcal{L}) \tag{26}$$

is also a line bundle on $\mathcal{Y}$. We set

$$\Theta := \mathrm{ST}_{\mathcal{O}_D} \circ \otimes \mathcal{L}\colon D^b\,\mathrm{Coh}(X) \xrightarrow{\sim} D^b\,\mathrm{Coh}(X). \tag{27}$$

Lemma 2.7 *The equivalence*

$$\Theta^\dagger := \Phi^{-1} \circ \Theta \circ \Phi\colon D^b\,\mathrm{Coh}(\mathcal{Y}) \to D^b\,\mathrm{Coh}(\mathcal{Y}) \tag{28}$$

restricts to the autoequivalence of $\mathrm{Coh}(\mathcal{Y})$ *given by* $- \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{L}^\dagger$.

Proof We have the local autoequivalence

$$\Theta^{\dagger}\colon D^b\,\mathrm{Coh}_G(\mathbb{C}^3) \to D^b\,\mathrm{Coh}_G(\mathbb{C}^3) \tag{29}$$

constructed in the same way as (28), replacing $\mathcal{L}$ by $\mathcal{O}_U(1)$. By the isomorphism (15) for $k=1$, it is enough to prove that the functor (29) is isomorphic to tensoring by $\mathcal{O}_{\mathbb{C}^3} \otimes \rho_{-1}$. A direct computation easily shows that

$$\Theta(\mathcal{O}_U) \cong \mathcal{O}_U(1), \quad \Theta(\mathcal{O}_U(1)) \cong \mathcal{O}_U(2), \quad \Theta(\mathcal{O}_U(2)) \cong \mathcal{O}_U.$$

Here $\Theta = \mathrm{ST}_{\mathcal{O}_{\mathbb{P}^2}} \circ \otimes\, \mathcal{O}_U(1)$ by abuse of notation. This implies that

$$\Theta^{\dagger}(\mathcal{O}_{\mathbb{C}^3} \otimes \rho_{-j}) \cong \mathcal{O}_{\mathbb{C}^3} \otimes \rho_{-j-1}$$

for $j \in \mathbb{Z}/3\mathbb{Z}$. Hence the functor (29) is isomorphic to tensoring by $\mathcal{O}_{\mathbb{C}^3} \otimes \rho_{-1}$ on the objects $\mathcal{O}_{\mathbb{C}^3} \otimes \rho_{-j}$. Since the objects $\mathcal{O}_{\mathbb{C}^3} \otimes \rho_{-j}$ for $0 \le j \le 2$ are projective objects of $\mathrm{Coh}_G(\mathbb{C}^3)$ which generate $\mathrm{Coh}_G(\mathbb{C}^3)$, we obtain the result. $\square$

Let $\mathrm{Coh}_{\le d}(X/Y)$ be the abelian category defined by (16). By Lemma 2.7, we have the commutative diagram

$$\begin{tikzcd} \mathrm{Coh}_{\le d}(\mathcal{Y}) \arrow[r, hook] \arrow[d, "\otimes \mathcal{L}^{\dagger}"] & D^b\,\mathrm{Coh}_{\le d}(\mathcal{Y}) \arrow[r, "\Phi"] \arrow[d, "\otimes \mathcal{L}^{\dagger}"] & D^b\,\mathrm{Coh}_{\le d}(X/Y) \arrow[d, "\Theta"] \\ \mathrm{Coh}_{\le d}(\mathcal{Y}) \arrow[r, hook] & D^b\,\mathrm{Coh}_{\le d}(\mathcal{Y}) \arrow[r, "\Phi"] & D^b\,\mathrm{Coh}_{\le d}(X/Y) \end{tikzcd} \tag{30}$$

where each vertical arrow is an equivalence. The equivalence (27) also induces the commutative diagram

$$\begin{tikzcd} N_{\le 1}(X/Y) \arrow[r, "\Theta_*"] \arrow[d, "\mathrm{ch}"'] & N_{\le 1}(X/Y) \arrow[d, "\mathrm{ch}"] \\ \mathbb{Q}[D] \oplus H^{\ge 4}(X,\mathbb{Q}) \arrow[r, "\Theta_\sharp"] & \mathbb{Q}[D] \oplus H^{\ge 4}(X,\mathbb{Q}). \end{tikzcd} \tag{31}$$

Using the triangle (24), the linear isomorphism $\Theta_\sharp$ is calculated as

$$\begin{aligned} &\Theta_\sharp(r[D],\beta,n) \\ &\quad = \Big(\big(\tfrac{5}{2}r + \beta D\big)[D], \beta + \big(\tfrac{13}{4}r + \tfrac{3}{2}\beta D\big)[l], n + \tfrac{11}{4}r + \tfrac{3}{2}\beta D + c_1(\mathcal{L})\beta\Big). \end{aligned} \tag{32}$$

3 Stable pairs and generalized DT invariants

In this section, we recall stable pairs, generalized DT invariants, and their properties.

3.1 Stable pair invariants

First we recall the definition of stable pairs on a Calabi–Yau 3–fold X introduced by Pandharipande and Thomas:

Definition 3.1 [42] A stable pair on X consists of data

$$s\colon \mathcal{O}_X \to F, \tag{33}$$

where $F \in \mathrm{Coh}_{\le 1}(X)$ is pure one-dimensional, and $\mathrm{Cok}(s) \in \mathrm{Coh}_0(X)$.

For $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, let

$$P_n(X, \beta)$$

be the moduli space of stable pairs (33) with $[F] = \beta$[7] and $\chi(F) = n$. The moduli space $P_n(X, \beta)$ is a projective scheme with a symmetric perfect obstruction theory. It is also regarded as a moduli space of two-term complexes $(\mathcal{O}_X \xrightarrow{s} F)$, satisfying the condition

$$\mathrm{ch}(\mathcal{O}_X \to F) = (1, 0, -\beta, -n).$$

Here $\mathcal{O}_X$ is located in degree zero, and $H_2(X)$ is identified with $H^4(X)$ by Poincaré duality. Let ν be the Behrend constructible function [5] on $P_n(X, \beta)$. The stable pair invariant $P_{n,\beta}(X)$ is defined by

$$P_{n,\beta}(X) := \int_{P_n(X,\beta)} \nu \, d\chi.$$

Here, for a constructible function ν on a variety M, we define

$$\int_M \nu \, d\chi := \sum_{m \in \mathbb{Z}} m \cdot \chi(\nu^{-1}(m)).$$

Remark 3.2 By the injectivity of (20), the map

$$N_{\le 1}(X) \to H_2(X, \mathbb{Z}) \oplus \mathbb{Z} \tag{34}$$

sending F to $([F], \chi(F))$ is injective. Therefore, if $P_n(X, \beta) \neq \varnothing$, we may write it as $P(X, \alpha)$, and $P_{n,\beta}(X)$ as $P_\alpha(X)$, for $\alpha \in N_{\le 1}(X)$ corresponding to (β, n) under (34).

[7] If we write $[F] = \beta$, it means that the fundamental homology class determined by F equals to β. By abuse of notation, we also use the notation $[F]$ for the class of F in the numerical Grothendieck group.

Remark 3.3 Suppose that X contains a divisor $D \cong \mathbb{P}^2$. For a line $l \subset D$, set β above to be $\beta = c[l]$ for $c > 0$. Then, for any stable pair $(\mathcal{O}_X \to F) \in P_n(X, c[l])$, the sheaf F is supported on D. Since the formal neighborhoods of $D \subset X$ and of the zero section $\mathbb{P}^2 \subset \omega_{\mathbb{P}^2}$ are isomorphic, the invariant $P_{n,c[l]}(X)$ coincides with the stable pair invariant on $\omega_{\mathbb{P}^2}$.

It is straightforward to generalize the notion of stable pairs to a CY3 orbifold $\mathcal{Y}$.

Definition 3.4 An orbifold stable pair on $\mathcal{Y}$ consists of data

$$s\colon \mathcal{O}_{\mathcal{Y}} \to F, \tag{35}$$

where $F \in \mathrm{Coh}_{\leq 1}(\mathcal{Y})$ is pure, ie $\mathrm{Hom}(\mathrm{Coh}_0(\mathcal{Y}), F) = 0$, and $\mathrm{Cok}(s) \in \mathrm{Coh}_0(\mathcal{Y})$.

Similarly to Remark 3.2, we denote by

$$P(\mathcal{Y}, \gamma)$$

for $\gamma \in N_{\leq 1}(\mathcal{Y})$ the moduli space of orbifold stable pairs (35) satisfying $[F] = \gamma$ in $N_{\leq 1}(\mathcal{Y})$. In this paper, we don't pursue the foundation of orbifold stable pair moduli spaces, eg a GIT construction of $P(\mathcal{Y}, \gamma)$. At least we have the following lemma, which is enough for our purpose.

Lemma 3.5 *The moduli space $P(\mathcal{Y}, \gamma)$ is a finite-type open algebraic subspace of the moduli space of simple objects in $D^b\,\mathrm{Coh}(\mathcal{Y})$.*

Proof The result is an immediate consequence of Proposition 5.2 and Proposition 5.4 proven later. □

Using the Behrend function ν on $P(\mathcal{Y}, \gamma)$, the orbifold stable pair invariant is defined by

$$P_\gamma(\mathcal{Y}) := \int_{P(\mathcal{Y},\gamma)} \nu \, d\chi. \tag{36}$$

Note that $P_\gamma(\mathcal{Y}) = 0$ for $0 \neq \gamma \in N_0(\mathcal{Y})$ by the definition of orbifold stable pairs.

3.2 Generalized DT invariants

We recall the construction of generalized DT invariants on a Calabi–Yau 3–fold X, following [25]. Let $\mathcal{C}oh(X)$ be the moduli stack of all the objects in $\mathrm{Coh}(X)$. The stack-theoretic Hall algebra $H(X)$ is $\mathbb{Q}$–spanned by the isomorphism classes of the symbols

$$[\rho\colon \mathcal{X} \to \mathcal{C}oh(X)], \tag{37}$$

where $\mathcal{X}$ is an Artin stack of finite type over $\mathbb{C}$ with affine geometric stabilizers and ρ is a 1–morphism. The relation is generated by

$$[\rho\colon \mathcal{X} \to \mathcal{C}oh(X)] \sim [\rho|_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{C}oh(X)] + [\rho|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{C}oh(X)], \tag{38}$$

where $\mathcal{Y} \subset \mathcal{X}$ is a closed substack and $\mathcal{U} := \mathcal{X} \setminus \mathcal{Y}$. There is an associative $*$–product on $H(X)$ based on the Ringel–Hall algebras (see [23, Section 5.1]). The unit is given by

$$1 = [\operatorname{Spec} \mathbb{C} \to \mathcal{C}oh(X)],$$

which corresponds to $0 \in \operatorname{Coh}(X)$. Also, there is a Lie subalgebra

$$H^{\mathrm{Lie}}(X) \subset H(X)$$

consisting of elements supported on *virtual indecomposable objects*. We refer to [23, Section 5.2] for the details of the definition of $H^{\mathrm{Lie}}(X)$.

Let $C(X)$ be the Lie algebra

$$C(X) := \bigoplus_{v \in N(X)} \mathbb{Q} \cdot c_v$$

with bracket given by

$$[c_{v_1}, c_{v_2}] := (-1)^{\chi(v_1, v_2)} \chi(v_1, v_2) c_{v_1+v_2}. \tag{39}$$

By [25, Theorem 5.12], there is a Lie algebra homomorphism

$$\Pi\colon H^{\mathrm{Lie}}(X) \to C(X) \tag{40}$$

such that if $\mathcal{X}$ is a $\mathbb{C}^*$–gerbe over an algebraic space $\mathcal{X}'$ we have

$$\Pi([\rho\colon \mathcal{X} \to \mathcal{C}oh_v(X)]) = -\Bigl(\sum_{k \in \mathbb{Z}} k \cdot \chi(\nu^{-1}(k))\Bigr) c_v.$$

Here $\mathcal{C}oh_v(X)$ is the stack of sheaves with numerical class v, ρ is an open immersion and ν is Behrend's constructible function on $\mathcal{X}'$.

Let H be an ample divisor on X. For $v \in N(X)$, let[8]

$$\mathcal{M}^{\mathrm{s(ss)}}(v) \subset \mathcal{C}oh(X) \tag{41}$$

be the open substack of H–Gieseker (semi)stable sheaves $E \in \operatorname{Coh}(X)$ satisfying $[E] = v$. The stack (41) determines the element

$$\delta(v) := [\mathcal{M}^{\mathrm{ss}}(v) \subset \mathcal{C}oh(X)] \in H(X).$$

[8] We omit H in the subscript of the moduli spaces and invariants, as its choice does not matter in the situation of Section 3.3.

This element also defines the element of $H^{\mathrm{Lie}}(X)$

$$\epsilon(v) := \sum_{\substack{k \geq 1, v_1 + \cdots + v_k = v \\ p(v_i) = p(v)}} \frac{(-1)^{k-1}}{k} \delta(v_1) * \cdots * \delta(v_k).$$

Here $p(v)$ is the reduced Hilbert polynomial of a sheaf E with $[E] = v$.

Definition 3.6 The generalized DT invariant $\mathrm{DT}(v) \in \mathbb{Q}$ is defined by the formula

$$\Pi(\epsilon(v)) = -\mathrm{DT}(v) \cdot c_v.$$

Remark 3.7 If $\mathcal{M}^{\mathrm{s}}(v) = \mathcal{M}^{\mathrm{ss}}(v)$, then they are $\mathbb{C}^*$–gerbes over a quasi-projective scheme $M^{\mathrm{s}}(v)$. In this case, the invariant $\mathrm{DT}(v)$ is written as

$$\mathrm{DT}(v) = \int_{M^{\mathrm{s}}(v)} \nu \, d\chi,$$

where ν is the Behrend function on $M^{\mathrm{s}}(v)$.

It has been expected that the above arguments generalize to the derived category setting. Namely, we expect that we can replace the category of coherent sheaves by the heart of a t-structure $\mathcal{A} \subset D^b \operatorname{Coh}(X)$, and Gieseker stability by Bridgeland stability [9] or weak stability [53]. The arguments are almost parallel, except that we need one technical result on the local description of the moduli stack of objects in the derived category, proven for coherent sheaves in [25, Theorem 5.3]. Conjecture 1.3 in the introduction is required to show the existence of a Lie algebra homomorphism (40) in the derived category setting.

3.3 Generalized DT invariants on local $\mathbb{P}^2$

The construction of generalized DT invariants also applies to the non-compact Calabi–Yau 3–fold $U = \omega_{\mathbb{P}^2}$. Let $\mathrm{Coh}_{\mathrm{c}}(U)$ be the category of coherent sheaves on U with compact supports, and set

$$\Lambda := \mathrm{Im}\big(\mathrm{ch}\colon \mathrm{Coh}(\mathbb{P}^2) \setminus \{0\} \to H^*(\mathbb{P}^2, \mathbb{Q})\big).$$

Under the isomorphism (22), we have

$$\Lambda \subset \left\{ (r, c, m) \in \mathbb{Z}^2 \oplus \tfrac{1}{2}\mathbb{Z} : \begin{array}{l} r > 0 \text{ or} \\ r = 0, c > 0 \text{ or} \\ r = c = 0, m > 0 \end{array} \right\}.$$

By replacing $\mathrm{Coh}(X)$ by $\mathrm{Coh}_c(U)$ in Section 3.2, we obtain the invariant

$$\mathrm{DT}(r,c,m) \in \mathbb{Q}$$

which counts Gieseker semistable sheaves[9] $F \in \mathrm{Coh}_c(U)$ satisfying

$$\mathrm{ch}(\pi_* F) = (r,c,m).$$

Here $(r,c,m) \in \Lambda$, and π is the projection (10). We also define $\mathrm{DT}(r,c,m)$ for elements $(r,c,m) \notin \Lambda$ by

$$\mathrm{DT}(r,c,m) := \begin{cases} \mathrm{DT}(-r,-c,-m) & (r,c,m) \in -\Lambda, \\ 0 & \pm(r,c,m) \notin \Lambda. \end{cases}$$

Remark 3.8 If $(r,c,m) \in -\Lambda$, then $\mathrm{DT}(r,c,m)$ counts objects $F[1] \in D^b(\mathrm{Coh}_0(U))$ for semistable sheaves $F \in \mathrm{Coh}_0(U)$ with $\mathrm{ch}(\pi_* F) = -(r,c,m)$.

Remark 3.9 By the Bogomolov inequality, the invariant $\mathrm{DT}(r,c,m)$ is non-zero only if $c^2 \geq 2rm$.

Remark 3.10 If $r > 0$, the invariant $\mathrm{DT}(r,c,m)$ coincides with the analogous invariant obtained by replacing Gieseker stability with slope stability. See [57, Lemma 2.10].

Remark 3.11 For a Gieseker semistable sheaf $F \in \mathrm{Coh}_c(U)$, the sheaf $F \otimes \mathcal{O}_U(\pm 1)$ is also Gieseker semistable. This implies

$$\mathrm{DT}(r,c,m) = \mathrm{DT}\big(r, c+r, m+c+\tfrac{1}{2}r\big). \tag{42}$$

We also have the following lemma:

Lemma 3.12 *We have the equality*

$$\mathrm{DT}(r,c,m) = \mathrm{DT}(r,-c,m). \tag{43}$$

Proof If $r = 0$ and $c > 0$, then the result follows since $F \mapsto \mathcal{E}xt^1_{\mathbb{P}^2}(F, \mathcal{O}_{\mathbb{P}^2})$ gives an isomorphism between $\mathcal{M}^{\mathrm{ss}}(0,c,m)$ and $\mathcal{M}^{\mathrm{ss}}(0,c,-m)$. (For example, see the proof of [43, Proposition 2.2].) Hence we may assume that $r > 0$. Let us set $\mu = c/r$ and[10]

$$\mathcal{B}_\mu := \left\langle F, \mathrm{Coh}_0(U)[-1] : \begin{array}{c} F \in \mathrm{Coh}_c(U) \text{ is slope} \\ \text{semistable with slope } \mu \end{array} \right\rangle_{\mathrm{ex}}.$$

[9] Indeed, such sheaves are scheme-theoretically supported on the zero section of π, if $r > 0$ or $r = 0$, $c > 0$. See [57, Lemma 2.3].

[10] The slope function on $F \in \mathrm{Coh}_c(U)$ is defined by that of $\pi_* F$. See Section 4.1.

Let $H(U)$ be the Hall algebra of $\mathrm{Coh}_c(U)$; it is $\mathbb{Q}$–spanned by the symbols (37), replacing $\mathcal{Coh}(X)$ by the stack $\mathcal{Coh}_c(U)$ of objects in $\mathrm{Coh}_c(U)$. We consider its completion

$$\widehat{H}(U) := \prod_{c^2 \geq 2rm} H_{(r,c,m)}(U),$$

where $H_{(r,c,m)}(U)$ is the subspace of $H(U)$ spanned by symbols (37) such that $\rho(x)$ for any $x \in \mathcal{X}$ corresponds to an object $E \in \mathrm{Coh}_c(U)$ with $\mathrm{ch}(\pi_* E) = (r, c, m)$. Similarly to Section 3.2, the moduli stacks of objects in $\mathcal{B}_\mu$, slope semistable sheaves in $\mathrm{Coh}_c(U)$ with slope μ, and objects in $\mathrm{Coh}_0(U)[-1]$ determine elements

$$\delta_{\mathcal{B}_\mu}, \delta_{\mathcal{M}_\mu}, \delta_0 \in \widehat{H}(U),$$

respectively. By the definition of $\mathcal{B}_\mu$, any object $E \in \mathcal{B}_\mu$ fits into an exact sequence

$$0 \to F \to E \to Q[-1] \to 0,$$

where $F \in \mathrm{Coh}_c(U)$ is a slope semistable sheaf with slope μ, and $Q \in \mathrm{Coh}_0(U)$. This exact sequence is unique up to an isomorphism, hence we have the following relation in $\widehat{H}(U)$:

$$(1 + \delta_{\mathcal{B}_\mu}) = (1 + \delta_{\mathcal{M}_\mu}) * (1 + \delta_0). \tag{44}$$

Let Ψ be the autoequivalence of $D^b \mathrm{Coh}_c(U)$, given by

$$\Psi(F) = \boldsymbol{R}\mathcal{H}om_U(F, \mathcal{O}_U(3))[1].$$

Then an object $F \in D^b \mathrm{Coh}_c(U)$ satisfies $\mathrm{ch}(\pi_* F) = (r, c, m)$ if and only if

$$\mathrm{ch}(\pi_* \Psi(F)) = (r, -c, m).$$

Moreover, the proof of [55, Lemma 9.1] shows that Ψ restricts to the equivalence between $\mathcal{B}_{-\mu}$ and $\mathcal{B}_\mu$. Replacing μ by $-\mu$ in (44) and applying Ψ_*, we obtain the relation

$$1 + \delta_{\mathcal{B}_\mu} = \Psi_*(1 + \delta_{\mathcal{B}_{-\mu}}) = (1 + \delta_0) * \Psi_*(1 + \delta_{\mathcal{M}_{-\mu}}). \tag{45}$$

We set $\epsilon_{\mathcal{M}_\mu} = \log(1 + \delta_{\mathcal{M}_\mu})$ and $\epsilon_0 = \log(1 + \delta_0)$. The relations (44), (45) imply that

$$\Psi_* \epsilon_{\mathcal{M}_{-\mu}} = \log\bigl(\exp(-\epsilon_0) * \exp(\epsilon_{\mathcal{M}_\mu}) * \exp(\epsilon_0)\bigr).$$

By the Baker–Campbell–Hausdorff formula, we obtain

$$\Psi_* \epsilon_{\mathcal{M}_{-\mu}} = \epsilon_{\mathcal{M}_\mu} + \begin{pmatrix} \text{multiple commutators} \\ \text{of } \epsilon_{\mathcal{M}_\mu} \text{ and } \epsilon_0 \end{pmatrix}. \tag{46}$$

Because $\chi(F, F') = 0$ for any $F, F' \in \mathcal{B}_\mu$, the multiple-commutator parts of (46) vanish after applying the integration map (40) for $H(U)$. By Remark 3.10, we obtain the desired identity (43). □

3.4 The invariants DT(r, c, m) for $r > 0$

In this subsection, we review the work of [57][11] on the invariants $\mathrm{DT}(r, c, m)$ with $r > 0$. For $r > 0$, we define the generating series

$$\mathrm{DT}(r, c) := \sum_{m} \mathrm{DT}(r, c, m)(-q^{\frac{1}{2r}})^{c^2-2rm}. \tag{47}$$

If the pair (r, c) is coprime, then the series (47) is the generating series of Euler numbers of moduli spaces of stable sheaves on $\mathbb{P}^2$ studied in the context of Vafa and Witten's S-duality conjecture [59]. We give some examples of the series (47).

Example 3.13 If $r = 1$, the work of Göttsche [20] on Hilbert schemes of points yields

$$\mathrm{DT}(1, c) = q^{\frac{\chi(S)}{24}} \eta(q)^{-\chi(S)}. \tag{48}$$

Here $\eta(q) = q^{1/24} \prod_{k \geq 1}(1 - q^k)$ is the Dedekind eta function.

Example 3.14 The case where $r = 2$ and $c = 1$ has been studied in several articles [28; 60; 61; 21; 35; 12; 32]. From these articles, one can show that

$$\mathrm{DT}(2, 1) = q^{\frac{1}{4}} \eta(q)^{-6} \frac{1}{\sum_{k \in \mathbb{Z}} q^{k^2}} \sum_{\substack{(a,b) \in (0,1/2)+\mathbb{Z}^2 \\ a > b > 0}} (2a - 6b) q^{a^2 - b^2}.$$

Proposition 3.6 of [55] shows that the series $\mathrm{DT}(r, c)$ satisfies a certain recursion formula in terms of modular forms and theta-type series for indefinite lattices. In the notation of that result, the recursion formula is

$$\begin{aligned} \mathrm{DT}(r, c) = & \sum_{\substack{k \geq 2,\ r_1, \ldots, r_k \in \mathbb{Z}_{\geq 1} \\ r_1 + \cdots + r_k = r}} \ \sum_{\substack{\bar{\beta}_i = (c_i, a_i) \in \mathrm{NS}_{<r_i}(\widehat{\mathbb{P}}^2) \\ 1 \leq i \leq k}} \ \sum_{G \in G(k)} \frac{(-1)^k}{2^{k-1}} \\ & \cdot U^{c,G}_{(r_1, \bar{\beta}_1), \ldots, (r_k, \bar{\beta}_k)}(q) \cdot \vartheta_{r,1-c}(q)^{-1} \prod_{i=1}^{k} \vartheta_{r_i, a_i}(q) \cdot \prod_{i=1}^{k} \mathrm{DT}(r_i, c_i). \end{aligned} \tag{49}$$

[11] More recently, Manschot [36] described a closed formula for the Betti numbers of moduli spaces of semistable sheaves on $\mathbb{P}^2$.

We briefly explain the notation used in (49). Let

$$\rho\colon \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$$

be the blow-up at a point and C the exceptional locus of ρ. We identity $\mathbb{Z}^2$ with $\mathrm{NS}(\widehat{\mathbb{P}}^2)$ via $(x,y) \mapsto x \cdot \rho^* h + y \cdot [C]$. The set $\mathrm{NS}_{<r}(\widehat{\mathbb{P}}^2)$ is a finite subset of $\mathrm{NS}(\widehat{\mathbb{P}}^2)$ given by

$$\mathrm{NS}_{<r}(\widehat{\mathbb{P}}^2) := \{(x,y) \in \mathbb{Z}^2 : 0 \le x < r,\ 0 \le y < r\}.$$

The set $G(k)$ is a finite set of certain oriented graphs, defined by

$$G(k) := \left\{ \begin{matrix} \text{connected and simply connected oriented graphs} \\ \text{with vertices } \{1,2,\dots,k\},\ i \to j \text{ implies } i < j \end{matrix} \right\}. \tag{50}$$

We also have the series (see [55, Definition 3.2])

$$U^{c,G}_{(r_1,\bar\beta_1),\dots,(r_k,\bar\beta_k)}(q) := \sum_{\substack{\beta_i \in \mathrm{NS}(\widehat{\mathbb{P}}^2),\ \beta_i \equiv \bar\beta_i \ (\mathrm{mod}\ r_i) \\ \beta_1 + \cdots + \beta_k = (c,1-c)}} U(\{(r_i,\beta_i)\}_{i=1}^m, H, F_+) \cdot \prod_{i \to j \text{ in } G} K_{\widehat{\mathbb{P}}^2}(r_j\beta_i - r_i\beta_j) q^{-\sum_{1 \le i < j \le k} \frac{(r_j\beta_i - r_i\beta_j)^2}{2rr_ir_j}}. \tag{51}$$

The coefficient $U(\{(r_i,\beta_i)\}_{i=1}^k, H, F_+) \in \mathbb{Q}$ is Joyce's combinatorial coefficient (see [24, Definition 4.4]), which is complicated but explicit. The series (51) is a sum of certain theta-type series for indefinite lattices which converges absolutely for $|q| < 1$ (see [57, Proposition 3.3]). Finally, the series $\vartheta_{r,a}(q)$ is a classical theta series

$$\vartheta_{r,a}(q) := \sum_{(k_1,\dots,k_{r-1}) \in (a/r,\dots,a/r) + \mathbb{Z}^{r-1}} q^{\sum_{1 \le i \le j \le r-1} k_ik_j}.$$

The three sums in (49) are finite sums, and $r_i \in \mathbb{Z}_{\ge 1}$ in the right-hand side of (49) satisfies $r_i < r$. By induction on r and the formula (48) for $r = 1$, we are in principle able to compute $\mathrm{DT}(r,c,m)$ for any $r \ge 1$ and c, m. By the convergence of the series (51), the series (47) also converges absolutely for $|q| < 1$.

3.5 The invariants $\mathrm{DT}(0,c,m)$ with $c > 0$

The $r = 0$ case was not treated in [55]. In this case, we can describe $\mathrm{DT}(0,c,m)$ with $c > 0$ in terms of stable pair invariants on X with curve classes proportional to $[l]$, ie stable pair invariants on $\omega_{\mathbb{P}^2}$ (see Remark 3.3). Using the results of [54; 10], we have the following lemma:

Lemma 3.15 *For $c > 0$ and $3c + 2m \neq 0$, we have the identity*

$$\mathrm{DT}(0,c,m) = \sum_{\substack{k\ge 1,(c_j,n_j)\in\mathbb{Z}^2,1\le j\le k\\ (c_1,n_1)+\cdots+(c_k,n_k)=(c,\frac{3}{2}c+m)}} \frac{2(-1)^{\frac{3}{2}c+m-k}}{(3c+2m)k}\left(\prod_{j=1}^{k} P_{n_j,c_j[l]}(X)-\prod_{j=1}^{k} P_{-n_j,c_j[l]}(X)\right).$$

Proof For $n \in \mathbb{Z}$ and $\beta \in H_2(X,\mathbb{Z})$, let $N_{n,\beta}(X) \in \mathbb{Q}$ be the generalized DT invariant on X given by[12]

$$N_{n,\beta}(X) := \mathrm{DT}(0,0,\beta,n).$$

Since any one-dimensional sheaf F on X with $[F] = c[l]$ on X is supported on D, we have the equality

$$\mathrm{DT}(0,c,m) = N_{\frac{3}{2}c+m,c[l]}(X). \tag{52}$$

By the results of [54; 10], we have the formula

$$1 + \sum_{n\in\mathbb{Z},c>0} P_{n,c[l]}(X)q^n t^c = \prod_{n>0,c>0} \exp\left((-1)^{n-1} N_{n,c[l]}(X)q^n t^c\right)^n \left(\sum_{n,c} L_{n,c[l]}(X)q^n t^c\right). \tag{53}$$

Here $L_{n,c[l]}(X) \in \mathbb{Q}$ is a certain invariant, which satisfies $L_{n,c[l]}(X) = L_{-n,c[l]}(X)$ and is zero for $|n| \gg 0$ for fixed c. By taking the logarithm of (53), replacing q by q^{-1} and taking the difference, we obtain the formula

$$\sum_{n\in\mathbb{Z},c>0} (-1)^{n-1} n N_{n,c[l]}(X)q^n t^c = \log\left(1 + \sum_{n\in\mathbb{Z},c>0} P_{n,c[l]}(X)q^n t^c\right) - \log\left(1 + \sum_{n\in\mathbb{Z},c>0} P_{n,c[l]}(X)q^{-n} t^c\right).$$

Here we have used the fact that $N_{n,c[l]}(X) = N_{-n,c[l]}(X)$ (see [54, Lemma 4.3 (i)]). Combined with (52), we obtain the desired identity. □

Remark 3.16 By (42), one can replace $\mathrm{DT}(0,c,m)$ with $\mathrm{DT}(0,c,m+kc)$ for $k \in \mathbb{Z}$ so that $3c + 2(m+kc) \neq 0$ holds. Then applying Lemma 3.15, one can describe $\mathrm{DT}(0,c,m)$ in terms of $P_{n,c[l]}(X)$ even if $3c + 2m = 0$.

[12] The invariant $N_{n,\beta}(X)$ does not depend on the choice of an ample divisor H. See [25, Theorem 6.16].

4 The space of weak stability conditions

Let X be a smooth projective Calabi–Yau 3–fold containing a divisor $D \cong \mathbb{P}^2$. In this section, we construct a one-parameter family of weak stability conditions on a triangulated category $\mathcal{D}_{X/Y}$ associated to the birational contraction (5).

4.1 Tilting of $\mathrm{Coh}_{\le d}(X/Y)$

For $0 \neq F \in \mathrm{Coh}(\mathbb{P}^2)$, let $\mu(F)$ be its slope, given by

$$\mu(F) := \frac{c_1(F)\cdot h}{\mathrm{rank}(F)} \in \mathbb{Q} \cup \{\infty\}.$$

Here $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ and $\mu(F) = \infty$ if $\mathrm{rank}(F) = 0$. The above slope function defines the notion of μ–semistable sheaves on $\mathbb{P}^2$ in the usual way.

Let $\mathrm{Coh}_{\le d}(X/Y)$ be the abelian subcategory of $\mathrm{Coh}(X)$ defined by (16). We define the pair of subcategories $(\mathcal{T}_{\le d}, \mathcal{F})$ on $\mathrm{Coh}_{\le d}(X/Y)$ in the following way:

$$\begin{aligned} \mathcal{T}_{\le d} &:= \left\langle \mathrm{Coh}_{\le d}(X), i_* T : \begin{array}{c} T \in \mathrm{Coh}(\mathbb{P}^2) \text{ is } \mu\text{–semistable} \\ \text{with } \mu(F) > -\frac{1}{2} \end{array} \right\rangle_{\mathrm{ex}}, \\ \mathcal{F} &:= \left\langle i_* F : \begin{array}{c} F \in \mathrm{Coh}(\mathbb{P}^2) \text{ is } \mu\text{–semistable} \\ \text{with } \mu(F) \le -\frac{1}{2} \end{array} \right\rangle_{\mathrm{ex}}. \end{aligned} \tag{54}$$

In what follows, we assume that $d \in \{0, 1\}$. We have the following lemma:

Lemma 4.1 *The subcategories* $(\mathcal{T}_{\le d}, \mathcal{F})$ *form a torsion pair of* $\mathrm{Coh}_{\le d}(X/Y)$*, ie* $\mathrm{Hom}(\mathcal{T}_{\le d}, \mathcal{F}) = 0$ *and any object* $E \in \mathrm{Coh}_{\le d}(X/Y)$ *fits into an exact sequence*

$$0 \to T \to E \to F \to 0 \tag{55}$$

for $T \in \mathcal{T}_{\le d}$, $F \in \mathcal{F}$.

Proof The condition $\mathrm{Hom}(\mathcal{T}_{\le d}, \mathcal{F}) = 0$ is obvious from the definition of $(\mathcal{T}_{\le d}, \mathcal{F})$. We check the condition (55). For $E \in \mathrm{Coh}_{\le d}(X/Y)$, there is an exact sequence of sheaves

$$0 \to T' \to E \to F' \to 0, \tag{56}$$

where $T' \in \mathrm{Coh}_{\le d}(X)$ and F' is pure two-dimensional (if it is non-zero) supported on D. Note that any pure two-dimensional semistable sheaf on X supported on D is scheme-theoretically supported on D (see [55, Lemma 2.3]). Therefore, by truncating the Harder–Narasimhan filtration of F', and combining the exact sequence (56), we obtain the desired exact sequence (55). □

By taking the tilting (see [22]) with respect to the torsion pair $(\mathcal{T}_{\le d}, \mathcal{F})$, we obtain the heart of a bounded t-structure on $D^b\operatorname{Coh}_{\le d}(X/Y)$:

$$\mathcal{B}_{\le d} := \langle \mathcal{F}[1], \mathcal{T}_{\le d}\rangle_{\mathrm{ex}} \subset D^b\operatorname{Coh}_{\le d}(X/Y).$$

Note that we have

$$\mathcal{B}_0 = \mathcal{B}_{\le 1} \cap D^b\operatorname{Coh}_0(X/Y).$$

4.2 Relation of $\mathcal{B}_{\le d}$ and $\operatorname{Coh}_{\le d}(\mathcal{Y})$

Let Φ be the derived equivalence (11). Note that $\Phi(\operatorname{Coh}_{\le d}(\mathcal{Y}))$ is the heart of a bounded t-structure on $D^b\operatorname{Coh}_{\le d}(X/Y)$ by the diagram (17). We set

$$\begin{aligned} \mathcal{T}^{\dagger} &:= \Phi(\operatorname{Coh}_{\le d}(\mathcal{Y})) \cap \mathcal{B}_{\le d}[1], \\ \mathcal{F}^{\dagger}_{\le d} &:= \Phi(\operatorname{Coh}_{\le d}(\mathcal{Y})) \cap \mathcal{B}_{\le d}. \end{aligned} \tag{57}$$

Note that $\mathcal{T}^{\dagger}$ does not depend on the choice of $d \in \{0,1\}$.

Proposition 4.2 *The pair $(\mathcal{T}^{\dagger}, \mathcal{F}^{\dagger}_{\le d})$ is a torsion pair of $\Phi(\operatorname{Coh}_{\le d}(\mathcal{Y}))$ such that*

$$\mathcal{B}_{\le d} = \langle \mathcal{F}^{\dagger}_{\le d}, \mathcal{T}^{\dagger}[-1]\rangle_{\mathrm{ex}}.$$

Proof Let $\mathcal{H}^i_{\mathcal{B}_{\le d}}(*)$ be the i^{th} cohomology functor on $D^b\operatorname{Coh}_{\le d}(X/Y)$ with respect to the t-structure with heart $\mathcal{B}_{\le d}$. The claim is equivalent to either one of the following conditions (see [2, Proposition 2.3.2]):

$$\mathcal{H}^i_{\mathcal{B}_{\le d}}(\Phi(F)) = 0, \quad i \neq -1, 0, \text{ for any } F \in \operatorname{Coh}_{\le d}(\mathcal{Y}), \tag{58}$$

$$\mathcal{H}^i(\Phi^{-1}(F)) = 0, \quad i \neq 0, 1, \text{ for any } F \in \mathcal{B}_{\le d}. \tag{59}$$

We first check (58) for $d=0$. Since $\operatorname{Coh}_0(\mathcal{Y})$ is the extension closure of $\mathcal{O}_x$ for $x \neq p$ and S_j with $j = 0,1,2$, we may assume that F is either one of the above objects. Obviously (58) is satisfied for $F = \mathcal{O}_x$ for $x \neq p$. We have $\Phi(S_j) = T_j$ by Lemma 2.4, and the definition of T_j in (12) yields

$$T_0 \in \mathcal{T}_0, \quad T_1 \in \mathcal{F}[1], \quad T_2 \in \mathcal{F}[2].$$

Therefore (58) is satisfied for $F = S_j$ with $0 \le j \le 2$.

Next we prove (59) for $d=1$. By definition, the category $\mathcal{B}_{\le 1}$ is the extension closure of objects in $\mathcal{B}_0$ and objects in $\operatorname{Coh}_{\le 1}(X)$. Since (58) holds for $d=0$, the condition (59) also holds for $d=0$. Therefore it is enough to show that the condition (59) holds for any $F \in \operatorname{Coh}_{\le 1}(X)$. Note that $\mathcal{H}^i(\Phi^{-1}(F))$ is supported on $p \in \mathcal{Y}$ for $i \neq 0$, and the subcategory of objects in $\operatorname{Coh}_0(\mathcal{Y})$ supported on p is generated by S_j for

$0 \le j \le 2$. Hence it is enough to check the vanishing of the following spaces for any $0 \le j \le 2$ and $k > 0$:

$$\mathrm{Hom}(S_j[k], \Phi^{-1}(F)) \cong \mathrm{Ext}_X^{-j-k}(T_j[-j], F), \tag{60}$$

$$\mathrm{Hom}(\Phi^{-1}(F), S_j[-k-1]) \cong \mathrm{Ext}_X^{j-k-1}(F, T_j[-j]). \tag{61}$$

Here we have used Lemma 2.4. Since $T_j[-j]$ is a sheaf, the space (60) obviously vanishes as $-j-k < 0$ and (61) also vanishes for the same reason except when $j = 2$ and $k = 1$. If $(j,k) = (2,1)$, the space (61) is $\mathrm{Hom}(F, T_2[-2])$, which also vanishes since $T_2[-2]$ is a pure two-dimensional sheaf. Therefore we obtain (59) for $d = 1$. □

We have the following corollary of this proposition:

Corollary 4.3 *For any pure one-dimensional* $F \in \mathrm{Coh}_{\le 1}(\mathcal{Y})$*, we have* $\Phi(F) \in \mathcal{B}_{\le 1}$.

Proof Since $F \in \mathrm{Coh}_{\le 1}(\mathcal{Y})$ is pure, we have $\mathrm{Hom}(\mathcal{T}^{\dagger}, \Phi(F)) = 0$. By Proposition 4.2, we have $\Phi(F) \in \mathcal{F}^{\dagger}_{\le 1} \subset \mathcal{B}_{\le 1}$. □

4.3 The abelian category $\mathcal{A}_{X/Y}$

We define the triangulated category $\mathcal{D}_{X/Y}$ in the following way:

$$\mathcal{D}_{X/Y} := \langle \mathcal{O}_X, D^b\,\mathrm{Coh}_{\le 1}(X/Y)\rangle_{\mathrm{tr}} \subset D^b\,\mathrm{Coh}(X).$$

This triangulated category plays a crucial role in our main purpose. Let $\mathcal{A}_{X/Y}$ be the subcategory of $\mathcal{D}_{X/Y}$ defined by

$$\mathcal{A}_{X/Y} := \langle \mathcal{O}_X, \mathcal{B}_{\le 1}[-1]\rangle_{\mathrm{ex}} \subset \mathcal{D}_{X/Y}.$$

We have the following lemma:

Lemma 4.4 *There is a bounded t-structure on* $\mathcal{D}_{X/Y}$ *whose heart is given by* $\mathcal{A}_{X/Y}$. *In particular,* $\mathcal{A}_{X/Y}$ *is an abelian category.*

Proof Let $\mathcal{F}'$ be the subcategory of $\mathrm{Coh}(X)$ defined by

$$\mathcal{F}' := \{E \in \mathrm{Coh}(X) : \mathrm{Hom}(\mathcal{T}_{\le 1}, E) = 0\}.$$

Since $\mathcal{T}_{\le 1} \subset \mathrm{Coh}(X)$ is closed under quotients and $\mathrm{Coh}(X)$ is noetherian, the pair $(\mathcal{T}_{\le 1}, \mathcal{F}')$ forms a torsion pair of $\mathrm{Coh}(X)$ (see [56, Lemma 2.15 (i)]). We set

$$\mathcal{A} := \langle \mathcal{F}', \mathcal{T}_{\le 1}[-1]\rangle_{\mathrm{ex}} \subset D^b\,\mathrm{Coh}(X). \tag{62}$$

Note that $\mathcal{A}$ is the heart of a bounded t-structure on $D^b \operatorname{Coh}(X)$, and

$$\mathcal{A} \cap D^b \operatorname{Coh}_{\le 1}(X/Y) = \mathcal{B}_{\le 1}[-1].$$

We apply Proposition 4.5 below for

$$\mathcal{D} = D^b \operatorname{Coh}(X), \quad L = \mathcal{O}_X, \quad \mathcal{D}' = D^b \operatorname{Coh}_{\le 1}(X/Y), \quad \mathcal{A}' = \mathcal{B}_{\le 1}[-1].$$

It is obvious that $\mathcal{A}'$ is closed under subobjects and quotients in $\mathcal{A}$. The condition (63) below is equivalent to the vanishings

$$\operatorname{Hom}(\mathcal{O}_X, \mathcal{F}) = 0, \quad \operatorname{Hom}(\mathcal{T}_{\le 1}[-1], \mathcal{O}_X) = 0.$$

The vanishing of $\operatorname{Hom}(\mathcal{O}_X, \mathcal{F})$ is equivalent to the vanishing of $\operatorname{Hom}(\mathcal{O}_X, i_* F)$ for any μ–semistable $F \in \operatorname{Coh}(\mathbb{P}^2)$ with $\mu(F) \le -\frac{1}{2}$. This is equivalent to the vanishing of $\operatorname{Hom}(\mathcal{O}_D, F)$, which follows from the μ–semistability of F and the inequalities

$$\mu(\mathcal{O}_D) = 0 > -\tfrac{1}{2} \ge \mu(F).$$

The vanishing of $\operatorname{Hom}(\mathcal{T}_{\le 1}[-1], \mathcal{O}_X)$ is equivalent to the vanishings

$$\operatorname{Hom}(G[-1], \mathcal{O}_X) = 0, \quad \operatorname{Hom}(i_* T[-1], \mathcal{O}_X) = 0$$

for any $G \in \operatorname{Coh}_{\le 1}(X)$ and μ–semistable $T \in \operatorname{Coh}(\mathbb{P}^2)$ with $\mu(T) > -\frac{1}{2}$. The vanishing of $\operatorname{Hom}(G[-1], \mathcal{O}_X)$ is equivalent to the vanishing of $H^2(X, G)$, which is obvious. The vanishing of $\operatorname{Hom}(i_* T[-1], \mathcal{O}_X)$ is equivalent to the vanishing of $\operatorname{Hom}(T, \omega_{\mathbb{P}^2})$, which follows from the μ–semistability of T and the inequalities

$$\mu(T) > -\tfrac{1}{2} > -3 = \mu(\omega_{\mathbb{P}^2}). \qquad \square$$

We have used the following result:

Proposition 4.5 [53, Proposition 3.6] *Let $\mathcal{D}$ be a $\mathbb{C}$–linear triangulated category and $\mathcal{A} \subset \mathcal{D}$ the heart of a bounded t-structure on $\mathcal{D}$. Take $L \in \mathcal{A}$ such that* $\operatorname{End}(L) = \mathbb{C}$ *and a full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ satisfying the following conditions:*

- *The category $\mathcal{A}' := \mathcal{A} \cap \mathcal{D}'$ is the heart of a bounded t-structure on $\mathcal{D}'$, which is closed under subobjects and quotients in $\mathcal{A}$.*
- *For any object $F \in \mathcal{A}'$, we have*

$$\operatorname{Hom}(L, F) = \operatorname{Hom}(F, L) = 0. \tag{63}$$

Let $\mathcal{D}_L \subset \mathcal{D}$ be the triangulated subcategory defined by $\mathcal{D}_L := \langle L, \mathcal{D}' \rangle_{\mathrm{tr}}$. Then $\mathcal{A}_L := \mathcal{D}_L \cap \mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}_L$, which satisfies $\mathcal{A}_L = \langle L, \mathcal{A}' \rangle_{\mathrm{ex}}$.

Let $\mathcal{C} \subset D^b \operatorname{Coh}(X)$ be the subcategory defined by

$$\mathcal{C} := \langle \mathcal{O}_X(rD), \operatorname{Coh}_{\le 1}(X)[-1] : r \in \mathbb{Z} \rangle_{\mathrm{ex}}.$$

We will use the following lemmas on the abelian category $\mathcal{A}_{X/Y}$.

Lemma 4.6 *The category* $\mathcal{C}$ *is a subcategory of* $\mathcal{A}_{X/Y}$*. In particular, for any* $r \in \mathbb{Z}$ *and a stable pair* $(\mathcal{O}_X \xrightarrow{s} F)$ *on* X*, we have*

$$\mathcal{O}_X(rD) \otimes (\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{A}_{X/Y}. \tag{64}$$

Proof It is enough to check that $\mathcal{O}_X(rD) \in \mathcal{A}_{X/Y}$ for any $r \in \mathbb{Z}$ to show the inclusion $\mathcal{C} \subset \mathcal{A}_{X/Y}$. Obviously this holds for $r=0$. If $r>0$, we have the distinguished triangle

$$\mathcal{O}_X((r-1)D) \to \mathcal{O}_X(rD) \to \mathcal{O}_D(-3r).$$

Since $\mathcal{O}_D(-3r) \in \mathcal{F} \subset \mathcal{A}_{X/Y}$, we have $\mathcal{O}_X(rD) \in \mathcal{A}_{X/Y}$ by induction on r. If $r<0$, we have the distinguished triangle

$$\mathcal{O}_D(-3r-3)[-1] \to \mathcal{O}_X(rD) \to \mathcal{O}_X((r+1)D).$$

Since $\mathcal{O}_D(-3r-3)[-1] \in \mathcal{T}_{\le 1}[-1] \subset \mathcal{A}_{X/Y}$, we have $\mathcal{O}_X(rD) \in \mathcal{A}_{X/Y}$ by induction on r. Hence $\mathcal{C} \subset \mathcal{A}_{X/Y}$ holds. By the distinguished triangle

$$\mathcal{O}_X(rD) \otimes F[-1] \to \mathcal{O}_X(rD) \otimes (\mathcal{O}_X \xrightarrow{s} F) \to \mathcal{O}_X(rD),$$

the object (64) is an object in $\mathcal{C}$, hence an object in $\mathcal{A}_{X/Y}$. □

Lemma 4.7 *For any* $E \in \mathcal{A}_{X/Y}$ *with* $\operatorname{rank}(E) = 1$*, there is a filtration* $E_1 \subset E_2 \subset E_3 = E$ *in* $\mathcal{A}_{X/Y}$ *such that*

$$E_1 \in \mathcal{F}, \quad E_2/E_1 \in \mathcal{C}, \quad E/E_2 \in \mathcal{T}_{\le 1}^{\mathrm{pure}}[-1]. \tag{65}$$

Here $\mathcal{T}_{\le 1}^{\mathrm{pure}}$ *is the subcategory of* $\mathcal{T}_{\le 1}$ *consisting of pure two-dimensional sheaves.*

Proof Recall that the category $\mathcal{A}_{X/Y}$ is a subcategory of $\mathcal{A}$ defined by (62). By the definition of $\mathcal{A}$, there is an exact sequence

$$0 \to F \to E \to T[-1] \to 0 \tag{66}$$

in $\mathcal{A}$ for $F \in \mathcal{F}'$ and $T \in \mathcal{T}_{\le 1}$. We have the exact sequences of sheaves

$$0 \to F' \to F \to F'' \to 0,$$
$$0 \to T' \to T \to T'' \to 0,$$

where F' is the torsion part of F and T' is the maximal subsheaf of T contained in $\operatorname{Coh}_{\le 1}(X)$. Since $E \in \mathcal{A}_{X/Y}$, the sheaf F' is supported on D, and satisfies

$\operatorname{Hom}(\mathcal{T}_{\le 1}, F') = 0$. It follows that $F' \in \mathcal{F}$, and also $T'' \in \mathcal{T}_{\le 1}^{\mathrm{pure}}$ holds by the construction of T'. The sheaf F'' is a torsion-free rank-one sheaf on X, whose determinant is trivial outside D. Hence F'' is written as $\mathcal{O}_X(rD) \otimes I_C$ for some $r \in \mathbb{Z}$ and a subscheme $C \subset X$ with $\dim C \le 1$. We set $E_1 = F'$ and E_2 to be the kernel of the composition in $\mathcal{A}$

$$E \twoheadrightarrow T[-1] \twoheadrightarrow T''[-1].$$

Then E_2/E_1 fits into the exact sequence in $\mathcal{A}$

$$0 \to F'' \to E_2/E_1 \to T'[-1] \to 0.$$

Since $F'', T'[-1] \in \mathcal{C}$, we have $E_2/E_1 \in \mathcal{C}$, showing that $E_\bullet$ is a desired filtration. □

We also have the following positivity lemma:

Lemma 4.8 *For any $E \in \mathcal{A}_{X/Y}$, we have $\operatorname{rank}(E) \ge 0$ and*

$$-2\operatorname{ch}_2(E) - \tfrac{2}{3} D \operatorname{ch}_1(E) \ge 0. \tag{67}$$

Here, for $\beta \in H^4(X)$, the inequality $\beta > 0$ means that it is a numerical class of a cycle on X with effective integral one.

Proof The first statement is obvious. For the second statement, note that $F \in \operatorname{Coh}(\mathbb{P}^2)$ with $\operatorname{ch}(F) = (r, c, m)$ satisfies

$$\operatorname{ch}_2(i_* F) + \tfrac{1}{3} D \operatorname{ch}_1(i_* F) = \left(\tfrac{1}{2} r + c\right)[l].$$

Therefore, the positivity of (67) follows from the construction of $\mathcal{A}_{X/Y}$. □

4.4 The abelian category $\mathcal{A}_{\mathcal{Y}}$

Let $\mathcal{Y}$ be the orbifold (7), which is derived equivalent to X. We define

$$\mathcal{D}_{\mathcal{Y}} := \langle \mathcal{O}_{\mathcal{Y}}, D^b \operatorname{Coh}_{\le 1}(\mathcal{Y}) \rangle_{\mathrm{tr}} \subset D^b \operatorname{Coh}(\mathcal{Y}).$$

By Lemma 2.4 and the diagram (17), the equivalence (11) restricts to the equivalence

$$\Phi\colon \mathcal{D}_{\mathcal{Y}} \xrightarrow{\sim} \mathcal{D}_{X/Y}.$$

We define the following subcategory of $\mathcal{D}_{\mathcal{Y}}$:

$$\mathcal{A}_{\mathcal{Y}} := \langle \mathcal{O}_{\mathcal{Y}}, \operatorname{Coh}_{\le 1}(\mathcal{Y})[-1] \rangle_{\mathrm{ex}} \subset \mathcal{D}_{\mathcal{Y}}.$$

Lemma 4.9 *There is a bounded t-structure on* $\mathcal{D}_{\mathcal{Y}}$ *whose heart is given by* $\mathcal{A}_{\mathcal{Y}}$.

Proof In [53, Lemma 3.5], the same statement was proved for non-orbifold Calabi–Yau 3–folds using Proposition 4.5. The same argument of [53, Lemma 3.5] applies without any modification. □

Lemma 4.10 *For an orbifold stable pair* $(\mathcal{O}_{\mathcal{Y}} \xrightarrow{s} F)$ *on* $\mathcal{Y}$, *we have*

$$\Phi(\mathcal{O}_{\mathcal{Y}} \xrightarrow{s} F) \in \mathcal{F}^{\sharp} := \Phi(\mathcal{A}_{\mathcal{Y}}) \cap \mathcal{A}_{X/Y}. \tag{68}$$

Proof Since F is pure one-dimensional, we have $\Phi(F)[-1] \in \mathcal{F}^{\sharp}$ by Corollary 4.3. Then the result follows from the distinguished triangle

$$\Phi(F)[-1] \to \Phi(\mathcal{O}_{\mathcal{Y}} \xrightarrow{s} F) \to \mathcal{O}_X. \qquad \square$$

The category $\mathcal{T}^{\dagger}$ defined by (57) is a subcategory of $\Phi(\mathrm{Coh}_{\leq 1}(\mathcal{Y}))$, hence we have $\mathcal{T}^{\dagger}[-1] \subset \Phi(\mathcal{A}_{\mathcal{Y}})$. The relationship between $\mathcal{A}_{X/Y}$ and $\mathcal{A}_{\mathcal{Y}}$ is given as follows:

Lemma 4.11 *The subcategory* $\mathcal{T}^{\dagger}[-1] \subset \Phi(\mathcal{A}_{\mathcal{Y}})$ *fits into a torsion pair* $(\mathcal{T}^{\dagger}[-1], \mathcal{F}^{\sharp})$ *on* $\Phi(\mathcal{A}_{\mathcal{Y}})$ *such that*

$$\mathcal{A}_{X/Y} = \langle \mathcal{F}^{\sharp}, \mathcal{T}^{\dagger}[-2] \rangle_{\mathrm{ex}}.$$

Proof By Proposition 4.2, we have

$$\Phi(\mathcal{A}_{\mathcal{Y}}) \subset \langle \mathcal{A}_{X/Y}[1], \mathcal{A}_{X/Y} \rangle_{\mathrm{ex}}.$$

Hence $\Phi(\mathcal{A}_{\mathcal{Y}})$ and $\mathcal{A}_{X/Y}$ are related by a tilting for some torsion pair of $\Phi(\mathcal{A}_{\mathcal{Y}})$. The free part is $\Phi(\mathcal{A}_{\mathcal{Y}}) \cap \mathcal{A}_{X/Y}$, which coincides with $\mathcal{F}^{\sharp}$ by its definition (68). The torsion part is

$$\Phi(\mathcal{A}_{\mathcal{Y}}) \cap \mathcal{A}_{X/Y}[1] = \Phi(\mathrm{Coh}_{\leq 1}(\mathcal{Y}))[-1] \cap \mathcal{B}_{\leq 1} = \mathcal{T}^{\dagger}[-1].$$

Hence we obtain the result. □

4.5 Weak stability conditions on $\mathcal{D}_{X/Y}$

We construct a one-parameter family of weak stability conditions on $\mathcal{D}_{X/Y}$ in the sense of [53], using the t-structures in the previous subsections. Let Γ be the free abelian group defined by

$$\Gamma := \mathrm{Im}(K(\mathcal{D}_{X/Y}) \to N(X)).$$

The map $\Gamma \to \mathbb{Z}$ sending F to $\mathrm{rank}(F)$ has a splitting by $1 \mapsto [\mathcal{O}_X]$. Hence we have

$$\Gamma \cong \mathbb{Z} \oplus N_{\leq 1}(X/Y), \tag{69}$$

and the natural map cl: $K(\mathcal{D}_{X/Y}) \to \Gamma$ given by $F \mapsto [F]$ is identified with the map

(70) $$\mathrm{cl}(F) = (\mathrm{rank}(F), [F] - \mathrm{rank}(F)[\mathcal{O}_X]).$$

We set $\Gamma_0 = N_0(X/Y)$ and $\Gamma_1 = N_{\leq 1}(X/Y)$, so that we have the filtration

(71) $$0 = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 = \Gamma.$$

The subquotients of this filtration are described by the isomorphisms (see Remark 2.5):

(72) $$\mathrm{rank}\colon \Gamma_2/\Gamma_1 \xrightarrow{\sim} \mathbb{Z}, \quad f_*\colon \Gamma_1/\Gamma_0 \xrightarrow{\sim} N_1(Y), \quad i_*\colon K(\mathbb{P}^2) \xrightarrow{\sim} \Gamma_0.$$

Here $N_1(Y) \subset H_2(Y, \mathbb{Z})$ is the subgroup generated by algebraic one-cycles on Y.

Let us take an element

(73) $$Z = \{Z_j\}_{j=0}^2 \in \prod_{j=0}^{2} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_j/\Gamma_{j-1}, \mathbb{C}).$$

For $v \in \Gamma$, we take the unique j such that $v \in \Gamma_j \setminus \Gamma_{j-1}$, and set

$$Z(v) := Z_j(\bar{v}) \in \mathbb{C}.$$

Here $\bar{v} \in \Gamma_j/\Gamma_{j-1}$ is the class of v in Γ_j/Γ_{j-1}. For $E \in \mathcal{D}_{X/Y}$, we write $Z(\mathrm{cl}(E))$ just as $Z(E)$ for simplicity.

Definition 4.12 [53] A *weak stability condition* on $\mathcal{D}_{X/Y}$ with respect to the filtration $\Gamma_\bullet$ is data $(Z, \mathcal{A})$, where Z is an element (73) and $\mathcal{A} \subset \mathcal{D}_{X/Y}$ is the heart of a bounded t-structure satisfying

(74) $$Z(\mathcal{A} \setminus \{0\}) \subset \mathbb{H} := \{r \exp(\pi\phi\sqrt{-1}) : r > 0,\ 0 < \phi \leq 1\}.$$

The data $(Z, \mathcal{A})$ should also satisfy other technical conditions, called the *Harder–Narasimhan property* and *support property* (see [53, Section 2]).

An object $E \in \mathcal{A}$ is called *Z–stable* or *Z–semistable* if, for any subobject $0 \neq F \subsetneq E$ in $\mathcal{A}$, we have the inequality

$$\arg Z(F) < \arg Z(E/F) \quad \text{or} \quad \arg Z(F) \leq \arg Z(E/F)$$

in $(0, \pi]$, respectively. Similarly to the space of Bridgeland stability conditions [9], the set of weak stability conditions $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_{X/Y})$ has a structure of a complex manifold such that the forgetting map

$$\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_{X/Y}) \to \prod_{j=0}^{2} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_j/\Gamma_{j-1}, \mathbb{C})$$

sending $(Z, \mathcal{A})$ to Z is a local homeomorphism.

We fix $\psi \in \left(\frac{\pi}{2}, \pi\right)$ and an ample divisor ω on Y. For $t \in \mathbb{R}$, we construct

$$Z_t = \{Z_{j,t}\}_{j=0}^{2} \in \prod_{j=0}^{2} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_j/\Gamma_{j-1}, \mathbb{C})$$

via the isomorphisms (72) in the following way:

$$\begin{aligned} &Z_{2,t}\colon \mathbb{Z} \to \mathbb{C}, && R \mapsto R \cdot \exp(\psi\sqrt{-1}), \\ &Z_{1,t}\colon N_1(Y) \to \mathbb{C}, && \beta \mapsto -(\beta \cdot \omega)\sqrt{-1}, \\ &Z_{0,t}\colon K(\mathbb{P}^2) \to \mathbb{C}, && F \mapsto \int_{\mathbb{P}^2} e^{h/2 - t\sqrt{-1}} \operatorname{ch}(F). \end{aligned} \tag{75}$$

Here $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$. For $(r, c, m) \in \mathbb{Q}^3$, we set

$$(\hat{r}, \hat{c}, \hat{m}) := \left(r, c + \tfrac{1}{2}r, m + \tfrac{1}{2}c + \tfrac{1}{8}r\right).$$

If we write $\operatorname{ch}(F) = (r, c, m)$ for $F \in K(\mathbb{P}^2)$ via (22), then we have $e^{h/2}\operatorname{ch}(F) = (\hat{r}, \hat{c}, \hat{m})$, and $Z_{0,t}(F)$ is written as

$$Z_{0,t}(F) = \hat{m} - \tfrac{1}{2}t^2\hat{r} - t\hat{c}\sqrt{-1}.$$

Lemma 4.13 *The data*

$$\sigma_t := (Z_t, \mathcal{A}_{X/Y}), \quad t > 0$$

determine a one-parameter family of weak stability conditions on $\mathcal{D}_{X/Y}$ *with respect to* $\Gamma_\bullet$.

Proof We check that (74) holds. For non-zero $E \in \mathcal{A}_{X/Y}$, suppose that $\operatorname{rank}(E) \neq 0$. Then $\operatorname{rank}(E) > 0$, $[E] \in \Gamma_2 \setminus \Gamma_1$ and

$$Z_t(E) = \operatorname{rank}(E) \cdot \exp(\psi\sqrt{-1}) \in \mathbb{H}.$$

If $\operatorname{rank}(E) = 0$, then we have $E \in \mathcal{B}_{\le 1}[-1]$. If furthermore $E \notin \mathcal{B}_0[-1]$, then $[E] \in \Gamma_1 \setminus \Gamma_0$ and $-f_*\operatorname{ch}_2(E)$ is a numerical class of an effective one-cycle on Y. Therefore

$$Z_t(E) = -f_*\operatorname{ch}_2(E) \cdot \omega\sqrt{-1} \in \mathbb{H}.$$

Finally, if $0 \neq E \in \mathcal{B}_0[-1]$, then we have $Z_t(E) = Z_{0,t}(E)$. The construction of Bridgeland stability conditions on local $\mathbb{P}^2$ in [1, Section 4] shows that $(Z_{0,t}, \mathcal{B}_0[-1])$ determines a Bridgeland stability condition on $D^b\operatorname{Coh}_0(X/Y)$. Hence we have $Z_{0,t}(E) \in \mathbb{H}$. The other properties (Harder–Narasimhan property, support property) are checked in a straightforward way. For example, the same argument of [55, Lemma 3.4] works. □

We now investigate the limiting point $\lim_{t\to+0}\sigma_t$ of the above weak stability conditions. The following lemma shows that the one-parameter family in Lemma 4.13 connects the large-volume limit point with the orbifold point, a similar picture obtained by Bayer and Macrì [1] for the space of Bridgeland stability conditions on local $\mathbb{P}^2$ (see Figure 1 in Section 1.4).

Lemma 4.14 *We have*

$$\sigma_0 := (Z_0, \Phi(\mathcal{A}_{\mathcal{Y}})) \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_{X/Y}),$$

and it coincides with $\lim_{t\to+0}\sigma_t$.

Proof Let T_j, $0\le j\le 2$, be the objects given by (12). A direct calculation shows that

$$\begin{aligned} Z_{0,t}(T_0[-1]) &= -\tfrac{1}{8}+\tfrac{1}{2}t^2+\tfrac{1}{2}t\sqrt{-1},\\ Z_{0,t}(T_1[-1]) &= -\tfrac{3}{4}-t^2,\\ Z_{0,t}(T_2[-1]) &= -\tfrac{1}{8}+\tfrac{1}{2}t^2-\tfrac{1}{2}t\sqrt{-1}. \end{aligned} \tag{76}$$

In particular, $\arg Z_{0,0}(T_j[-1])=\pi$. By Lemma 2.4, this implies that the pair

$$(Z_{0,0}, \Phi(\mathrm{Coh}_0(\mathcal{Y}))[-1])$$

is a Bridgeland stability condition on $D^b\,\mathrm{Coh}_0(X/Y)$. This fact together with the same argument of Lemma 4.13 shows that σ_0 is an element of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_{X/Y})$. By Lemma 4.11, $\Phi(\mathcal{A}_{\mathcal{Y}})$ and $\mathcal{A}_{X/Y}$ differ by a tilting, hence $\lim_{t\to+0}\sigma_t=\sigma_0$ follows from [53, Lemma 7.1]. □

5 Comparison of stable pair invariants

In this section, we relate rank-one σ_t–semistable objects for $t\gg 1$ with stable pairs on X, and those for $0<t\ll 1$ with orbifold stable pairs on $\mathcal{Y}$. We then apply the Joyce–Song wall-crossing formula to derive a relationship between stable pair invariants on X and those on $\mathcal{Y}$.

5.1 Moduli stacks of semistable objects

Let $\mathcal{M}$ be the moduli stack of objects $E\in D^b\,\mathrm{Coh}(X)$ satisfying the condition

$$\mathrm{Ext}^{<0}(E,E)=0.$$

By a result of Lieblich [33], the stack $\mathcal{M}$ is an Artin stack locally of finite type over $\mathbb{C}$. For $R\in\mathbb{Z}_{\ge 0}$, let

$$\mathcal{O}bj^{\le R}(\mathcal{A}_{X/Y})\subset\mathcal{M}$$

be the substack of objects $E \in \mathcal{A}_{X/Y}$ with $\operatorname{rank}(E) \le R$.

Lemma 5.1 *The stack* $\mathcal{O}bj^{\le 1}(\mathcal{A}_{X/Y})$ *is an open substack of* $\mathcal{M}$. *In particular, it is an Artin stack locally of finite type over* $\mathbb{C}$.

Proof The proof is similar to [53, Lemma 3.15], so we just give a brief explanation. Let $\mathcal{A}$ be the abelian category defined by (62). By the argument of [34], the torsion pair on $\operatorname{Coh}(X)$ which defines $\mathcal{A}$ forms a stack of torsion theories, which implies that the stack $\mathcal{O}bj(\mathcal{A})$ of objects in $\mathcal{A}$ is an open substack of $\mathcal{M}$. Therefore it is enough to show that the embedding

$$\mathcal{O}bj^{\le 1}(\mathcal{A}_{X/Y}) \subset \mathcal{O}bj(\mathcal{A})$$

is an open immersion. For $E \in \mathcal{A}_{X/Y}$ with $\operatorname{rank}(E) \le 1$, we have the exact sequence in $\mathcal{A}$

$$\mathcal{H}^0(E) \to E \to \mathcal{H}^1(E)[-1]$$

satisfying the following two conditions:

- The sheaf $\mathcal{H}^0(E)$ is torsion free on $X \setminus D$.
- The determinant line bundle $\det(E)$ is of the form $\mathcal{O}_X(rD)$ for some $r \in \mathbb{Z}$.

Conversely, if an object $E \in \mathcal{A}$ with $\operatorname{rank}(E) \le 1$ satisfies these two conditions, then we have $E \in \mathcal{A}_{X/Y}$. The openness of the former condition follows from the same spectral sequence argument of [53, Lemma 3.15, Step 1], and the latter condition is obviously open. □

For $t \in \mathbb{R}_{>0}$, $R \in \{0,1\}$ and $\alpha \in N_{\le 1}(X/Y)$, let

$$\mathcal{M}_t(R,\alpha) \subset \mathcal{O}bj^{\le 1}(\mathcal{A}_{X/Y}) \tag{77}$$

be the substack of Z_t–semistable objects $E \in \mathcal{A}_{X/Y}$ satisfying $\operatorname{cl}(E) = (R,\alpha)$, where cl is the map (70).

Proposition 5.2 *Suppose that*

$$(R,\alpha) \in (0, N_0(X/Y)) \quad \textit{or} \quad (1, N_{\le 1}(X/Y)).$$

(i) *The stack* $\mathcal{M}_t(R,\alpha)$ *is an Artin stack of finite type over* $\mathbb{C}$, *such that* (77) *is an open immersion.*

(ii) *There is a finite number of real numbers*

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = \infty$$

such that, for $t \in (t_{i-1}, t_i)$, *the stack* $\mathcal{M}_t(R,\alpha)$ *is constant and consists of* Z_t*–stable objects.*

Proof The proof follows from the same arguments in the author's previous papers [55; 52]. We don't repeat their details here, and just give a brief explanation in the $R=1$ case. In both (i) and (ii) we use Lemma 4.8 instead of [55, Lemma 2.10].

(i) Following the proof of [55, Lemma 4.13(ii)], we can show that the set of objects in $\mathcal{M}_t(1,\alpha)$ is bounded. Indeed, for any object $[E]\in\mathcal{M}_t(1,\alpha)$, one can take a filtration

$$E_1\subset E_2\subset E_3=E$$

as in Lemma 4.7. The object $E_2/E_1\in\mathcal{C}$ also admits a filtration

$$F_1\subset F_2\subset F_3=E_2/E_1$$

such that F_1 and F_3/F_2 are objects in $\mathrm{Coh}_{\le 1}(X)[-1]$, and $F_2/F_1=\mathcal{O}_X(rD)$. Using Lemma 4.8, the Bogomolov inequality on $\mathbb{P}^2$ and the Z_t–semistability of E, we can bound the numbers of Harder–Narasimhan factors and numerical classes of E_1, E_3/E_2, F_1, F_3/F_2. This implies the boundedness of objects in $\mathcal{M}_t(1,\alpha)$. The openness of (77) follows from the boundedness of semistable objects by the same proof of [52, Theorem 3.20].

(ii) We apply the same proof of [55, Proposition 9.7]. Let $A\in\mathcal{B}_{\le 1}[-1]$ be an object satisfying the following conditions:

- There are $t>0$, $[E]\in\mathcal{M}_t(1,\alpha)$ and an injection $A\hookrightarrow E$ or a surjection $E\twoheadrightarrow A$ in $\mathcal{A}_{X/Y}$.
- We have $Z_t(A)\in\mathbb{R}_{>0}\exp(\sqrt{-1}\psi)$.

Note that the second condition implies that $A\in\mathcal{B}_0[-1]$. By taking the filtration of A as in Lemma 4.7 and using the Bogomolov inequality on $\mathbb{P}^2$, one can show that the possible numerical classes for such A is a finite set. In particular, the possible $t\in\mathbb{R}$ is also a finite set, giving the desired result. □

The result of Proposition 5.2(ii) in particular shows that, for $t\in\mathbb{R}_{>0}\setminus\{t_1,\dots,t_{k-1}\}$, we have the $\mathbb{C}^*$–gerbe structure

$$\mathcal{M}_t(1,\alpha)\to M_t(1,\alpha) \tag{78}$$

for an algebraic space $M_t(R,\alpha)$ of finite type.

5.2 DT-type invariants

We define the DT-type invariants counting Z_t–semistable objects $E\in\mathcal{A}_{X/Y}$ with $\mathrm{rank}(E)\le 1$. We first define the rank-one invariants. Let us take

$$\alpha\in N_{\le 1}(X/Y),\quad t\in\mathbb{R}_{>0}\setminus\{t_1,\dots,t_{k-1}\},$$

where t_i is given in Proposition 5.2 for $\mathcal{M}_t(1,\alpha)$. We define

$$\mathrm{DT}_t(1,\alpha) := \int_{M_t(1,\alpha)} \nu \, d\chi.$$

Here $M_t(1,\alpha)$ is the coarse moduli space of $\mathcal{M}_t(1,\alpha)$ given in (78), and ν is the Behrend function on $M_t(1,\alpha)$.

In the rank-zero case, there may be strictly σ_t–semistable objects even for a general t. So we need to use the Hall algebra as in Section 3.2 to define the invariants. Let $H^0(\mathcal{A}_X)$ be the stack-theoretic Hall algebra of rank-zero objects in $\mathcal{A}_X$. As a $\mathbb{Q}$–vector space, it is spanned by isomorphism classes of symbols

$$[\rho\colon \mathcal{X} \to \mathcal{O}bj^0(\mathcal{A}_X)],$$

where $\mathcal{X}$ is an Artin stack of finite type with affine geometric stabilizers and ρ is a 1–morphism. The relation is generated by (38) after replacing $\mathcal{C}oh(X)$ by $\mathcal{O}bj^0(\mathcal{A}_X)$. We also have the associative $*$–product on $H^0(\mathcal{A}_X)$, similarly to the $*$–product on $H(X)$.

For $\alpha \in N_0(X/Y)$, the stack $\mathcal{M}_t(0,\alpha)$ determines the element

$$\delta_t(0,\alpha) := [\mathcal{M}_t(0,\alpha) \subset \mathcal{O}bj^0(\mathcal{A}_X)] \in H^0(\mathcal{A}_X).$$

We define the element $\epsilon_t(0,\alpha) \in H^0(\mathcal{A}_X)$ to be

$$\epsilon_t(0,\alpha) := \sum_{\substack{k\geq 1,\, \alpha_1,\dots,\alpha_k \in N_0(X/Y) \\ \arg Z_{0,t}(\alpha_i) = \arg Z_{0,t}(\alpha)}} \frac{(-1)^{k-1}}{k} \delta_t(0,\alpha_1) * \cdots * \delta_t(0,\alpha_k).$$

Similarly to (40), we have the integration map Π from the Lie algebra of elements supported on virtual indecomposable objects in $H^0(\mathcal{A}_X)$ to the Lie algebra

$$C^0(\mathcal{A}_X) = \bigoplus_{\alpha \in N_0(X/Y)} \mathbb{Q} \cdot c_\alpha$$

with bracket given by (39). The element $\epsilon_t(0,\alpha)$ is supported on virtual indecomposable objects, and the invariant $\mathrm{DT}_t(0,\alpha) \in \mathbb{Q}$ is defined by

$$\Pi(\epsilon_t(0,\alpha)) = -\mathrm{DT}_t(0,\alpha) \cdot c_\alpha.$$

It counts Z_t–semistable objects $E \in \mathcal{A}_X$ with $\mathrm{cl}(E) = (0,\alpha)$.

5.3 The invariants $\mathrm{DT}_t(R, \alpha)$ for $t \gg 1$ and $0 < t \ll 1$

Let us take $\alpha \in N_{\leq 1}(X/Y)$ and its Chern character $\mathrm{ch}(\alpha) \in \mathbb{Q}[D] \oplus H^{\geq 4}(X)$. Note that we can write $(1, \mathrm{ch}(\alpha)) \in H^*(X, \mathbb{Q})$ as

$$(1, \mathrm{ch}\,\alpha) = e^{rD}(1, 0, -\beta, -n)$$

for some $r \in \mathbb{Z}$, $\beta \in H^4(X)$ and $n \in H^6(X)$. By identifying $H^4(X)$ with $H_2(X)$ and $H^6(X)$ with $\mathbb{Q}$, we have the following proposition:

Proposition 5.3 *For $t \gg 0$, we have $M_t(1, \alpha) \cong P_n(X, \beta)$. In particular, we have $\mathrm{DT}_t(1, \alpha) = P_{n,\beta}(X)$ for $t \gg 0$.*

Proof We first construct a morphism

$$M_t(1, \alpha) \to P_n(X, \beta) \tag{79}$$

for $t \gg 0$. Let us take an object $[E] \in M_t(1, \alpha)$ and a filtration in $\mathcal{A}_{X/Y}$

$$E_1 \subset E_2 \subset E_3 = E$$

as in Lemma 4.7. If $E_1 \neq 0$, it contradicts the Z_t–stability of E for $t \gg 0$ as

$$\lim_{t \to \infty} \arg Z_t(E_1) = \pi > \psi = \arg Z_t(E).$$

Therefore $E_1 = 0$, and the same argument also shows $E_3/E_2 = 0$. It follows that $E \in \mathcal{C}$. We take the distinguished triangle

$$\mathcal{H}^0(E) \to E \to \mathcal{H}^1(E)[-1],$$

which is an exact sequence in $\mathcal{C}$. Note that $\mathcal{H}^0(E)$ is of the form $\mathcal{O}_X(rD) \otimes I_C$ for a subscheme $C \subset X$ with $\dim C \leq 1$ and $\mathcal{H}^1(E) \in \mathrm{Coh}_{\leq 1}(X)$. By the Z_t–stability of E for $t \gg 0$, the sheaf $\mathcal{H}^1(E)$ must be zero-dimensional, and the following holds:

$$\mathrm{Hom}(\mathrm{Coh}_0(X)[-1], E) = 0.$$

Therefore, applying [53, Lemma 3.11(iii)], we see that E is isomorphic to an object of the form

$$\mathcal{O}_X(rD) \otimes (\mathcal{O}_X \to F) \tag{80}$$

for some stable pair $(\mathcal{O}_X \to F) \in P_n(X, \beta)$. The morphism (79) is defined by sending E to $(\mathcal{O}_X \to F)$.

Conversely, we construct a morphism

$$P_n(X, \beta) \to M_t(1, \alpha) \tag{81}$$

for $t \gg 0$. For a stable pair $(\mathcal{O}_X \to F) \in P_n(X,\beta)$, the complex (80) is an object in $\mathcal{A}_{X/Y}$ by Lemma 4.6. We show that the object (80) is Z_t–stable for $t \gg 0$. Let us take an exact sequence in $\mathcal{A}_{X/Y}$

$$0 \to A \to \mathcal{O}_X(rD) \otimes (\mathcal{O}_X \to F) \to B \to 0,$$

with $A, B \neq 0$. Then A or B is an object in $\mathcal{B}_{\le 1}[-1]$. Suppose that $A \in \mathcal{B}_{\le 1}[-1]$. We show that $\arg Z_t(A) < \psi$ holds for $t \gg 0$. If $A \notin \mathcal{B}_0[-1]$, then $\arg Z_t(A) = \frac{\pi}{2}$, and the claim is obvious. We may assume that $A \in \mathcal{B}_0[-1]$. If $\mathcal{H}^0(A) \neq 0$, then it is a pure two-dimensional sheaf, which implies $\mathrm{Hom}(\mathcal{H}^0(A), \mathcal{C}) = 0$ by the definition of $\mathcal{C}$. This is a contradiction, as $\mathcal{H}^0(A)$ is a subobject of (80) in $\mathcal{A}_{X/Y}$. Hence $\mathcal{H}^0(A) = 0$ and $A \in \mathcal{T}_0[-1]$ holds. Then we have

$$\lim_{t \to \infty} \arg Z_t(A) = 0 < \psi.$$

A similar argument shows that $\arg Z_t(B) > \psi$ for $t \gg 0$ if $B \in \mathcal{B}_{\le 1}[-1]$, and we conclude that the object (80) is Z_t–stable for $t \gg 0$. The morphism (80) is defined by sending $(\mathcal{O}_X \to F)$ to the object (80). The morphisms (79), (81) are inverse to each other, hence they are isomorphisms. □

Proposition 5.4 *For* $0 < t \ll 1$*, we have* $M_t(1,\alpha) \cong P(\mathcal{Y}, -\Phi_*^{-1}(\alpha))$*. In particular, we have* $\mathrm{DT}_t(1,\alpha) = P_{-\Phi_*^{-1}(\alpha)}(\mathcal{Y})$*.*

Proof We first note that an object $E \in \mathcal{A}_{X/Y}$ with $\mathrm{rank}(E) = 1$ is Z_t–stable for $0 < t \ll 1$ if and only if $E \in \Phi(\mathcal{A}_{\mathcal{Y}})$ and it is Z_0–stable. This statement is an immediate consequence of Lemma 4.14, together with the fact that any rank-one Z_0–semistable object in $\Phi(\mathcal{A}_{\mathcal{Y}})$ is Z_0–stable. The latter fact holds since there is no rank-zero object $F \in \Phi(\mathcal{A}_{\mathcal{Y}})$ with $\arg Z_0(F) = \psi$. Therefore it is enough to show that a rank-one object $E \in \Phi(\mathcal{A}_{\mathcal{Y}})$ is Z_0–stable if and only if it is isomorphic to an object of the form

$$\Phi(\mathcal{O}_{\mathcal{Y}} \to F) \tag{82}$$

for an orbifold stable pair $(\mathcal{O}_{\mathcal{Y}} \to F)$ on $\mathcal{Y}$.

Let us take a Z_0–stable object $E = \Phi(G) \in \Phi(\mathcal{A}_{\mathcal{Y}})$ with $\mathrm{rank}(E) = 1$. By the definition of $\mathcal{A}_{\mathcal{Y}}$, there is a filtration in $\mathcal{A}_{\mathcal{Y}}$

$$G_1 \subset G_2 \subset G_3 = G \tag{83}$$

satisfying the following:

$$G_1 \in \mathrm{Coh}_{\le 1}(\mathcal{Y})[-1], \quad G_2/G_1 = \mathcal{O}_{\mathcal{Y}}, \quad G_3/G_2 \in \mathrm{Coh}_{\le 1}(\mathcal{Y})[-1].$$

By the Z_0–stability of E, we have $G_3/G_2 \in \mathrm{Coh}_0(\mathcal{Y})[-1]$, hence the sequence

$$0 \to \mathcal{O}_{\mathcal{Y}} \to G_3/G_1 \to G_3/G_2 \to 0$$

splits. This implies that we can replace the filtration (83) so that $G_3/G_2 = 0$ holds. Hence G is isomorphic to a two-term complex of the form $(\mathcal{O}_{\mathcal{Y}} \xrightarrow{s} F)$ with $F \in \mathrm{Coh}_{\le 1}(\mathcal{Y})$. By the Z_0–stability of E, we have

$$\mathrm{Hom}(\mathrm{Coh}_0(\mathcal{Y})[-1], E) = 0,$$

hence F must be a pure one-dimensional sheaf on $\mathcal{Y}$. The Z_0–stability of E also implies that the cokernel of s is zero-dimensional. It follows that $(\mathcal{O}_{\mathcal{Y}} \xrightarrow{s} F)$ is an orbifold stable pair, and E is isomorphic to an object of the form (82).

Conversely, let us take an object (82). We take an exact sequence in $\Phi(\mathcal{A}_{\mathcal{Y}})$

$$0 \to A \to \Phi(\mathcal{O}_{\mathcal{Y}} \to F) \to B \to 0$$

with $A, B \neq 0$. Note that either A or B is an object of $\Phi(\mathrm{Coh}_{\le 1}(\mathcal{Y}))[-1]$. Suppose that A is an object of $\Phi(\mathrm{Coh}_{\le 1}(\mathcal{Y}))[-1]$. Since $(\mathcal{O}_{\mathcal{Y}} \to F)$ is an orbifold stable pair, there is no non-zero morphism from an object in $\Phi(\mathrm{Coh}_0(\mathcal{Y}))[-1]$ to the object (82). Therefore $A \notin \Phi(\mathrm{Coh}_0(\mathcal{Y}))[-1]$, which implies that

$$\arg Z_0(A) = \tfrac{\pi}{2} < \psi.$$

Suppose that B is an object of $\Phi(\mathrm{Coh}_{\le 1}(\mathcal{Y}))[-1]$. Then we have $B \in \Phi(\mathrm{Coh}_0(\mathcal{Y}))[-1]$ as $(\mathcal{O}_{\mathcal{Y}} \to F)$ is an orbifold stable pair. Hence $\arg Z_0(B) > \psi$, and the object (82) is Z_0–stable. □

Finally, in this subsection, we investigate the rank-zero DT-type invariants:

Proposition 5.5 *Suppose that $\alpha \in N_0(X/Y)$ is written as $\alpha = -i_*\alpha_0$ for $\alpha_0 \in K(\mathbb{P}^2)$ with $\mathrm{ch}(\alpha_0) = (r, c, m)$. If $\widehat{c} = c + \frac{1}{2}r > 0$, we have the following equality for $t \gg 0$:*

$$\mathrm{DT}_t(0, \alpha) = \mathrm{DT}(r, c, m). \tag{84}$$

Proof First suppose that $r > 0$ or $r = 0, c > 0$. Then a well-known argument shows that an object $E \in \mathcal{B}_0[-1]$ with numerical class α is $Z_{0,t}$–semistable for $t \gg 0$ if and only if $E[1]$ is a Gieseker semistable sheaf. For example, the proof of this fact for K3 surfaces in [51, Proposition 6.4, Lemma 6.5] works without any major modification. Also, similarly to those results, this fact also holds for every $Z_{0,t}$–stable factor of E. By the definition of $\mathcal{B}_0$, the sheaf $E[1]$ is supported on D. Since the formal neighborhood of $D \subset X$ is isomorphic to the formal neighborhood of the zero section of $\omega_{\mathbb{P}^2} \to \mathbb{P}^2$, we obtain the identity (84).

Next suppose that $r < 0$. Then an object $E \in \mathcal{B}_0[-1]$ with numerical class α is $Z_{0,t}$–semistable for $t \gg 0$ if and only if $\mathbb{D}(E)[1]$ is a Gieseker semistable sheaf on $\mathbb{P}^2$, where $\mathbb{D}(-) = \boldsymbol{R}\mathcal{H}om(-, \mathcal{O}_X)$ is the dualizing functor. This fact also follows from a well-known argument: the object $\mathbb{D}(E)$ has numerical class $-i_*\alpha_1$ for $\mathrm{ch}(\alpha_1) = -e^{-3h}(r, -c, m)$, and the argument of [55, Proposition 9.5] shows that $\mathbb{D}(E)$ is also Bridgeland semistable near the large-volume limit in a tilted heart. Hence the above argument for the $r > 0$ case shows that $\mathbb{D}(E)[1]$ is Gieseker semistable. It is easy to see that the same fact also holds for every $Z_{0,t}$–stable factor of E. Therefore we have the identity

$$\mathrm{DT}_t(0, \alpha) = \mathrm{DT}(e^{-3h}(r, -c, m))$$

for $t \gg 0$. Then the desired identity follows from (42) and (43). □

5.4 Combinatorial coefficients

In this subsection, we recall Joyce's combinatorial coefficients, which appear in our wall-crossing formula, and give their explicit description. Although these coefficients are combinatorially complicated, they naturally appear from the Hall algebra identities coming from Harder–Narasimhan filtrations, and their inversion formulas. We refer to [24, Section 4] for the derivation of these coefficients from the identities in the Hall algebra.

For $d \in \{0, 1\}$, we define the following positive cone:

$$N_{\le d}^{+}(X/Y) := \mathrm{Im}\big(\mathcal{B}_{\le d} \to N_{\le d}(X/Y)\big) \setminus \{0\}.$$

We also define

$$K^{+}(\mathbb{P}^2) := \big\{E \in K(\mathbb{P}^2) : i_*E \in N_0^{+}(X/Y)\big\},$$
$$\Gamma^{+} := \mathbb{Z}_{\ge 0} \oplus \{-N_{\le 1}^{+}(X/Y) \cup \{0\}\} \setminus \{(0,0)\}.$$

The cone Γ^+ coincides with the image of non-zero objects in $\mathcal{A}_{X/Y}$ under the map (70). Hence, for $v \in \Gamma^+$, the argument $\arg Z_t(v) \in (0, \pi]$ is well-defined. For $v, v' \in \Gamma^+$, we write

$$Z_{\infty}(v) \succ Z_{\infty}(v') \quad \text{or} \quad Z_{+0}(v) \succ Z_{+0}(v')$$

if $\arg Z_t(v) > \arg Z_t(v')$ holds for $t \gg 0$ or $0 < t \ll 1$, respectively.

Definition 5.6 [24, Definition 4.2] For $v_1, \dots, v_k \in \Gamma^+$, suppose that either (85) or (86) holds for each $i = 1, \dots, k-1$:

(85) $\quad Z_{\infty}(v_i) \preceq Z_{\infty}(v_{i+1}) \quad \text{and} \quad Z_{+0}(v_1 + \cdots + v_i) \succ Z_{+0}(v_{i+1} + \cdots + v_k),$

(86) $\quad Z_{\infty}(v_i) \succ Z_{\infty}(v_{i+1}) \quad \text{and} \quad Z_{+0}(v_1 + \cdots + v_i) \preceq Z_{+0}(v_{i+1} + \cdots + v_k).$

Then define

$$S(\{v_1,\dots,v_k\}, Z_\infty, Z_{+0}) := (-1)^a, \tag{87}$$

where a is the number of $i=1,\dots,k-1$ satisfying (85). If neither (85) nor (86) holds for some i, we define $S(\{v_1,\dots,v_k\}, Z_\infty, Z_{+0}) = 0$.

Let $\iota(x)$ be the function on $\mathbb{R}$ defined by

$$\iota(x) := \begin{cases} 1 & x > 0, \\ -1 & x \le 0. \end{cases} \tag{88}$$

We can also write (87) in the following way:

$$\begin{aligned}\frac{1}{2^{k-1}} \lim_{\varepsilon\to+0} \prod_{i=1}^{k-1} \bigl\{ \iota\bigl(\arg Z_{1/\varepsilon}(v_i) - \arg Z_{1/\varepsilon}(v_{i+1})\bigr) \\ - \iota\bigl(\arg Z_\varepsilon(v_1+\dots+v_i) - \arg Z_\varepsilon(v_{i+1}+\dots+v_k)\bigr)\bigr\}.\end{aligned} \tag{89}$$

Definition 5.7 [24, Definition 4.4] For $v_1,\dots,v_k \in \Gamma^+$, we define

$$\begin{aligned}U(\{v_1,\dots,v_k\}, Z_\infty, Z_{+0}) := \sum_{1\le k''\le k'\le k} \sum_{\substack{\psi\colon \{1,\dots,k\}\to\{1,\dots,k'\} \\ \psi'\colon \{1,\dots,k'\}\to\{1,\dots,k''\}}} \\ \prod_{a=1}^{k''} S(\{v_i^\dagger\}_{i\in\psi'^{-1}(a)}, Z_\infty, Z_{+0}) \frac{(-1)^{k''-1}}{k''} \prod_{b=1}^{k'} \frac{1}{|\psi^{-1}(b)|!}.\end{aligned} \tag{90}$$

Here ψ, ψ', $v_i^\dagger$ are as follows:

- ψ and ψ' are non-decreasing surjective maps.
- For $1 \le i, j \le k$ with $\psi(i) = \psi(j)$, we have $Z_\infty(v_i) = Z_\infty(v_j)$.
- For $1 \le i, j \le k''$, we have

$$Z_{+0}\Bigl(\sum_{a\in\psi^{-1}\psi'^{-1}(i)} v_a\Bigr) = Z_{+0}\Bigl(\sum_{a\in\psi^{-1}\psi'^{-1}(j)} v_a\Bigr). \tag{91}$$

- The elements $v_i^\dagger \in \Gamma^+$ for $1\le i\le k'$ are defined to be

$$v_i^\dagger = \sum_{j\in\psi^{-1}(i)} v_j. \tag{92}$$

We also define the following more explicit function:

Definition 5.8 Suppose that $1 \le e \le k$ and $(r_j, c_j, m_j) \in \mathbb{Q}^3$ for $1 \le j \le k$, $j \ne e$ are given. If $\hat{c}_j = c_j + \frac{1}{2} r_j > 0$ holds for all $j \ne e$, we define

(93) $$U(\{(r_j, c_j, m_j)\}_{j \ne e}) := \lim_{\varepsilon \to +0} \sum_{\substack{\text{non-decreasing } \psi\colon \{1,\dots,k\} \twoheadrightarrow \{1,\dots,k'\} \\ e' := \psi(e), \psi^{-1}(e') = \{e\}, \psi(i) = \psi(j) \text{ implies } v_i \sim v_j}} \frac{1}{2^{k'-1}} \prod_{i=1}^{k'} \frac{1}{|\psi^{-1}(i)|!}$$
$$\times \prod_{i < e'-1} \left\{ \iota\left(\frac{r^{\dagger}_{i+1}}{c^{\dagger}_{i+1}} - \frac{r^{\dagger}_i}{c^{\dagger}_i} + \varepsilon\left(\frac{m^{\dagger}_i}{c^{\dagger}_i} - \frac{m^{\dagger}_{i+1}}{c^{\dagger}_{i+1}} \right) \right) - \iota\left(\sum_{j=1}^{i} m^{\dagger}_j \right) \right\}$$
$$\times \left\{ \iota(-r^{\dagger}_{e'-1}) - \iota\left(\sum_{j=1}^{e'-1} m^{\dagger}_j \right) \right\} \cdot \left\{ -\iota(-r^{\dagger}_{e'+1}) + \iota\left(\sum_{j=e'+1}^{k'} m^{\dagger}_j \right) \right\}$$
$$\times \prod_{i > e'} \left\{ \iota\left(\frac{r^{\dagger}_{i+1}}{c^{\dagger}_{i+1}} - \frac{r^{\dagger}_i}{c^{\dagger}_i} + \varepsilon\left(\frac{m^{\dagger}_i}{c^{\dagger}_i} - \frac{m^{\dagger}_{i+1}}{c^{\dagger}_{i+1}} \right) \right) + \iota\left(\sum_{j=i+1}^{k'} m^{\dagger}_j \right) \right\}.$$

Here ι is the function (88), $v_i \sim v_j$ means $v_i = a v_j$ for some $a \in \mathbb{Q}_{>0}$, and

$$(r^{\dagger}_i, c^{\dagger}_i, m^{\dagger}_i) := \sum_{j \in \psi^{-1}(i)} \left(r_j, c_j + \tfrac{1}{2} r_j, m_j + \tfrac{1}{2} c_j + \tfrac{1}{8} r_j \right), \quad 1 \le i \le k'.$$

In the following lemma, we relate (90) with the function (93):

Lemma 5.9 *For* $v_1, \dots, v_k \in \Gamma^+$*, suppose that there is* $1 \le e \le k$ *such that* $\operatorname{rank}(v_e) > 0$ *and* $\operatorname{rank}(v_j) = 0$ *for* $j \ne e$*.*

(i) *If (90) is non-zero, then* v_j *is written as* $(0, -i_* \alpha_j)$ *for* $\alpha_j \in K^+(\mathbb{P}^2)$ *satisfying* $\operatorname{ch}(\alpha_j) = (r_j, c_j, m_j)$ *with* $\hat{c}_j = c_j + \frac{1}{2} r_j > 0$*.*

(ii) *In the notation of (i), we have the identity*

(94) $$U(\{v_1, \dots, v_k\}, Z_\infty, Z_{+0}) = U(\{(r_j, c_j, m_j)\}_{j \ne e}).$$

Proof Note that $\lim_{t \to +0} \arg Z_t(v)$ is either 0 or $\frac{\pi}{2}$ or π for any $v \in \Gamma^+$ with $\operatorname{rank}(v) = 0$, while $\arg Z_t(v) = \psi$ for any $t > 0$ if $\operatorname{rank}(v) > 0$. This implies that the equality (91) never happens for $i \ne j$, hence we have $k'' = 1$. For the same reason, $Z_\infty(v_j) = Z_\infty(v_e)$ never happens for $j \ne e$, hence the map ψ in (90) should satisfy $\psi^{-1}\psi(e) = \{e\}$. Also, for rank-zero $v, v' \in \Gamma^+$, it is easy to see that $Z_\infty(v) = Z_\infty(v')$ is equivalent to the equality $v' = a v$ for some $a \in \mathbb{Q}_{>0}$.

(i) By the above observations, it is enough to prove (i) assuming $S(\{v_i\}_{i=1}^k, Z_\infty, Z_{+0})$ is non-zero. For $j \neq e$, we have $v_j \in N_{\le 1}(X/Y)$ as $\operatorname{rank}(v_j) = 0$. Suppose that $v_j \notin N_0(X/Y)$ for some $j \neq e$. If $j < e$, then for each $j \le j' < e$, we have

$$\tfrac{\pi}{2} = \arg Z_t(v_1 + \cdots + v_{j'}) < \arg Z_t(v_{j'+1} + \cdots + v_k) = \psi$$

for any $t > 0$. Hence $Z_\infty(v_{j'}) \succ Z_\infty(v_{j'+1})$ should hold. This implies that

$$\tfrac{\pi}{2} = \arg Z_t(v_j) > \arg Z_t(v_e) = \psi$$

for $t \gg 0$, which is a contradiction. A similar argument also leads to a contradiction if $j > e$. Hence we have $v_j \in N_0(X/Y)$ for any $j \neq e$, and can write $v_j = (0, -i_*\alpha_j)$ for some $\alpha_j \in K^+(\mathbb{P}^2)$. If we write $\operatorname{ch}(\alpha_j) = (r_j, c_j, m_j)$, then $\hat{c}_j \geq 0$ by the definition of $K^+(\mathbb{P}^2)$. If $\hat{c}_j = 0$ for some $j \neq e$, then $\arg Z_{0,t}(v_j) = \pi$ for any $t > 0$, and an argument similar to the one above leads to a contradiction. Therefore $\hat{c}_j > 0$ for any $j \neq e$.

(ii) Noting the observations before the proof of (i), the identity (94) is obtained by a direct computation of $S(\{v_i^\dagger\}_{i=1}^{k'}, Z_\infty, Z_{+0})$ through the description (89). For instance, if $i < e'$, one computes

$$\begin{aligned}
&\lim_{\varepsilon \to +0} \iota\big(\arg Z_\varepsilon(v_1^\dagger + \cdots + v_i^\dagger) - \arg Z_\varepsilon(v_{i+1}^\dagger + \cdots + v_k^\dagger)\big) \\
&\quad = \lim_{\varepsilon \to +0} \iota\left(\arg\left(\sum_{j=1}^{i} -m_j^\dagger + \tfrac{1}{2}\varepsilon^2 r_j^\dagger + \varepsilon c_j^\dagger \sqrt{-1}\right) - \psi\right) \\
&\quad = \iota\left(\sum_{j=1}^{i} m_j^\dagger\right).
\end{aligned}$$

The other values which appear in (89) can be computed in a similar way. □

We finally define the following function, which appears as a wall-crossing coefficient in the next subsection.

Definition 5.10 In the same situation of Definition 5.8, suppose that we are also given $r \in \mathbb{Z}$ and $\beta \in H_2(X)$. We define

$$\begin{aligned}
f(\{(r_j, c_j, m_j)\}_{j \neq e}, r, \beta) := \sum_{G \in G(k)} &\frac{1}{2^{k-1}} U(\{(r_j, c_j, m_j)\}_{j \neq e}) \\
&\times \prod_{\substack{a \to e \text{ or} \\ e \to a \text{ in } G}} \iota(e-a)\big(r_a + m_a + 3rc_a - r_a D\beta + \tfrac{3}{2}c_a + \tfrac{9}{2}rr_a + \tfrac{9}{2}r^2 r_a\big) \\
&\times \prod_{\substack{a \to b \text{ in } G \\ a,b \neq e}} 3(r_a c_b - r_b c_a).
\end{aligned}$$

Here $G(k)$ is the set of graphs defined by (50).

5.5 Wall-crossing formula

Combined with the results so far, we prove our main result.

Theorem 5.11 *Suppose that Conjecture 1.3 holds. Then for any* $\gamma \in N_{\leq 1}(\mathcal{Y})$*, we have the formula*

$$P_\gamma(\mathcal{Y}) = \sum_{(n,\beta)\in\mathbb{Z}\oplus H_2(X,\mathbb{Z})} \Bigg(\sum_{\substack{k\geq 1\\ 1\leq e\leq k}} \sum_{\substack{(r_j,c_j,2m_j)\in\mathbb{Z}^3,\ 1\leq j\leq k,\ j\neq e,\ r\in\mathbb{Z}\\ 2c_j+r_j>0,\ c_j^2\geq 2r_jm_j,\ \text{satisfying (96)}}} (-1)^{k-1+\sum_{a\neq e} r_a+m_a+rc_a-r_aD\beta+\frac{3}{2}c_a+\frac{rr_a}{2}+\frac{r^2r_a}{2}+\sum_{a<b,a,b\neq e} r_ac_b-r_bc_a} \times f(\{(r_j,c_j,m_j)\}_{j\neq e}, r, \beta)\cdot \prod_{j\neq e} \mathrm{DT}(r_j,c_j,m_j)\Bigg) P_{n,\beta}(X). \tag{95}$$

Here $f(\{(r_j,c_j,m_j)\}_{j\neq e}, r, \beta)$ *is given in Definition 5.10, and the condition (96) is*

$$(1,-\mathrm{ch}(\Phi_*\gamma)) = e^{rD}(1,0,-\beta,-n) - \sum_{j\neq e}\big(0, r_j[D], \big(\tfrac{3}{2}r_j+c_j\big)[l], \tfrac{3}{2}r_j+\tfrac{3}{2}c_j+m_j\big). \tag{96}$$

Proof Under Conjecture 1.3, we can apply the same wall-crossing formula in [25, Theorem 5.18] to the one-parameter family of weak stability conditions σ_t for $t\in\mathbb{R}_{>0}$. For any $\gamma\in N_{\leq 1}(\mathcal{Y})$, we obtain

$$\lim_{t\to+0}\mathrm{DT}_t(1,-\Phi_*\gamma) = \lim_{t\to\infty} \sum_{\substack{k\geq 1, v_1,\dots,v_k\in\Gamma^+\\ v_1+\cdots+v_k=(1,-\Phi_*\gamma)}} \sum_{G\in G(k)} \frac{(-1)^{k-1+\sum_{a<b}\chi(v_a,v_b)}}{2^{k-1}} \times U(\{v_1,\dots,v_k\}, Z_\infty, Z_{+0}) \prod_{\substack{a\to b\\ \text{in } G}} \chi(v_a,v_b) \prod_{j=1}^k \mathrm{DT}_t(v_j). \tag{97}$$

By Lemma 4.8, there is a unique $1\leq e\leq k$ such that $\mathrm{rank}(v_e)=1$ and $\mathrm{rank}(v_j)=0$ for $j\neq e$. Therefore, by Lemma 5.9, v_j is written as $(0,-i_*\alpha_j)$ for $\alpha_j\in K^+(\mathbb{P}^2)$ with $\mathrm{ch}(\alpha_j)=(r_j,c_j,m_j)$, $c_j+\frac{1}{2}r_j>0$. For $a,b\neq e$, we have

$$\chi(v_a,v_b) = 3(r_ac_b-r_bc_a)$$

by (21) and (23). Also, if we write $\mathrm{ch}(v_e)$ as $e^{rD}(1,0,-\beta,-n)$, then we have

$$\chi(v_a,v_e) = r_a+m_a+3rc_a-r_aD\beta+\tfrac{3}{2}c_a+\tfrac{9}{2}rr_a+\tfrac{9}{2}r^2r_a.$$

The equality $v_1+\cdots+v_k=(1,-\Phi_*\gamma)$ is equivalent to the equality of Chern characters

$$\mathrm{ch}(v_1)+\cdots+\mathrm{ch}(v_k)=(1,-\mathrm{ch}(\Phi_*\gamma)),$$

which coincides with the condition (96) by (23). Applying the above computations to (97), noting Remark 3.9 and using Propositions 5.3, 5.4, 5.5 and Lemma 5.9, we obtain the desired formula. □

Remark 5.12 By the finiteness of walls in Proposition 5.2, the sum in (95) must be a finite sum. It is also straightforward to check that, for a fixed $\gamma \in N_{\le 1}(\mathcal{Y})$, there is only a finite number of (n,β), $r\in\mathbb{Z}$, $k\ge 1$ and $\{(r_j,c_j,2m_j)\}_{j\neq e}\subset\mathbb{Z}^3$ satisfying (96) and the conditions

$$c_j+\tfrac{1}{2}r_j>0,\quad c_j^2\ge 2r_jm_j,\quad U(\{(r_j,c_j,m_j)\}_{j\neq e})\neq 0.$$

Remark 5.13 If $r\neq 0$, the invariant $\mathrm{DT}(r,c,m)$ is computed by the recursion formula (49). Also, the invariant $\mathrm{DT}(0,c,m)$ for $c\neq 0$ is described by polynomials of stable pair invariants on X by Lemma 3.15. Hence the formula (95) describes $\mathrm{PT}_\gamma(\mathcal{Y})$ in terms of polynomials of stable pair invariants on X, in principle.

If we restrict the curve class to be of the form $c[l]$ for $c>0$ and a line $l\subset D$, we obtain the following relationship between stable pair invariants and generalized DT invariants on local $\mathbb{P}^2$:

Corollary 5.14 *Suppose that Conjecture 1.3 holds. Then for any positive integer c and $n\in\mathbb{Z}$, we have the formula*

$$P_{n,c[l]}(X)=\sum_{\substack{(n',c')\in\mathbb{Z}^2\\ 0\le c'<c}}\Bigg(\sum_{\substack{k\ge 1\\ 1\le e\le k}}\sum_{\substack{(r_j,c_j,2m_j)\in\mathbb{Z}^3,\ 1\le j\le k,\ j\neq e,\ r:=\sum_{j\neq e}r_j\\ 2c_j+r_j>0,\ c_j^2\ge 2r_jm_j,\ \text{satisfying (99)}}}(-1)^{k+n-n'+rc'+rc+\frac{3}{2}r^3-\frac{1}{2}r+\sum_{a<b,a,b\neq e}r_ac_b-r_bc_a}$$
$$\times f(\{(r_j,c_j,m_j)\}_{j\neq e},r,c'[l])\cdot\prod_{j\neq e}\mathrm{DT}(r_j,c_j,m_j)\Bigg)P_{n',c'[l]}(X). \tag{98}$$

Here the condition (99) is

$$\begin{aligned} c&=c'+\tfrac{3}{2}r^2+r+\sum_{j\neq e}\big(\tfrac{1}{2}r_j+c_j\big),\\ n&=n'-\tfrac{3}{2}r^3+\big(\tfrac{3}{2}-3c'\big)r+\sum_{j\neq e}\big(\tfrac{3}{2}c_j+m_j\big).\end{aligned} \tag{99}$$

Proof We apply Theorem 5.11 for $\gamma \in N_0(\mathcal{Y})$ such that $\Phi_*\gamma \in N_0(X/Y)$ corresponds to $(c[l], n)$ under the map (34). Then $P_\gamma(\mathcal{Y}) = 0$ by the definition of orbifold stable pairs. In the right-hand side of (95), there is a unique term with $r = 0$ and $k = 1$ which gives $P_{n,c[l]}(X)$. The condition (96) determines r by $r = \sum_{j \neq e} r_j$, and the condition (96) for $\beta = c'[l]$ is equivalent to the condition (99), which implies $c' < c$. □

Remark 5.15 The formula (98) is a recursion formula for stable pair invariants on local $\mathbb{P}^2$ in terms of generalized DT invariants on it. By solving the recursion, one can in principle describe the stable pair invariants on local $\mathbb{P}^2$ in terms of generalized DT invariants on it with possibly non-zero rank.

Example 5.16 Let us consider the $c' = c - 1$ term of (98). Using the Bogomolov inequality in Remark 3.9, we see that there are three kinds of contributions:

(i) $k = 2$ and $\{(r_j, c_j, m_j)\}_{j \neq e} = \{(0, 1, n - n' - \frac{2}{3})\}$ with $n' < n - 1$.

(ii) $k = 2$ and $\{(r_j, c_j, m_j)\}_{j \neq e} = \{(-1, 1, \frac{1}{2})\}$.

(iii) $k = 3$ and $\{(r_j, c_j, m_j)\}_{j \neq e} = \{(1, 0, 0), (-1, 1, \frac{1}{2})\}$.

The corresponding invariants $\mathrm{DT}(r_j, c_j, m_j)$ are given by

$$\mathrm{DT}(0, 1, *) = 3, \quad \mathrm{DT}(1, 0, 0) = \mathrm{DT}\left(-1, 1, \tfrac{1}{2}\right) = 1. \tag{100}$$

By computing $f(\{(r_j, c_j, m_j)\}_{j \neq e})$, we obtain

$$\begin{aligned} P_{n,c[l]}(X) = &\\ &\sum_{n' < n-1} (-1)^{n-n'-1} 3(n - n') P_{n',(c-1)[l]}(X) + (-1)^{c-1} 3c P_{n-3c+2,(c-1)[l]}(X) \\ &\quad + (-9c^2 + 6c + 3) P_{n-1,(c-1)[l]}(X) + (\text{terms of } P_{n',c'[l]}(X) \text{ with } c' \leq c - 2). \end{aligned}$$

In particular, one obtains $P_{n,[l]}(X) = 3(-1)^{n-1} n$ from the above formula. This is compatible with the fact that $P_n(X, [l])$ is a $\mathbb{P}^{n-1}$–bundle over $\mathbb{P}^2$.

Example 5.17 In the $c' = c - 2$ term of (98), we see that the possible contributions of $\{(r_j, c_j, m_j)\}_{j \neq e}$ are as follows:

(i) $k = 2$ and $\{(0, 2, m), m \in \mathbb{Z}_{\geq 0}\}$, $\{(-1, 2, -2)\}$, $\{(-1, 2, -1)\}$.

(ii) $k = 3$ and

$$\begin{gathered} \left\{\left(0, 1, m - \tfrac{1}{2}\right), \left(0, 1, m' - \tfrac{1}{2}\right) : m, m' \in \mathbb{Z}_{>0}\right\}, \\ \left\{\left(0, 1, m - \tfrac{1}{2}\right), \left(-1, 1, -\tfrac{1}{2}\right) : m \in \mathbb{Z}_{>0}\right\}, \\ \{(-2, 2, 1), (1, 0, 0)\}, \{(-2, 2, 1), (2, 0, 0)\}, \left\{\left(-1, 1, -\tfrac{1}{2}\right), \left(1, 1, -\tfrac{1}{2}\right)\right\}, \\ \{(-1, 2, -2), (1, 0, 0)\}, \{(-1, 2, -1), (1, 0, 0)\}, \left\{\left(1, 1, \tfrac{1}{2}\right), \left(-1, 1, -\tfrac{1}{2}\right)\right\}. \end{gathered}$$

(iii) $k = 4$ and

$$\left\{\left(0,1,m-\tfrac{1}{2}\right),\left(-1,1,-\tfrac{1}{2}\right),(1,0,0) : m \in \mathbb{Z}_{\geq 0}\right\},$$
$$\left\{(1,0,0),\left(-1,1,-\tfrac{1}{2}\right),\left(-1,1,-\tfrac{1}{2}\right)\right\}.$$

(iv) $k = 5$ and $\left\{(1,0,0),(1,0,0),\left(-1,1,-\tfrac{1}{2}\right),\left(-1,1,-\tfrac{1}{2}\right)\right\}$.

Using (42), the invariants $\mathrm{DT}(r_j, c_j, m_j)$ are computed by (100), together with

$$\mathrm{DT}(2,0,0) = \tfrac{1}{4}, \quad \mathrm{DT}(1,0,-1) = 3, \quad \mathrm{DT}(0,2,0) = -6, \quad \mathrm{DT}(0,2,1) = -\tfrac{21}{4}.$$

The latter two invariants are computed using Lemma 3.15.

5.6 Constraints on stable pair invariants

In this subsection, we derive a non-trivial relationship among stable pair invariants induced by the Seidel–Thomas twist along $\mathcal{O}_D$. As in Section 2.4, we assume that there is an $\mathcal{L} \in \mathrm{Pic}(X)$ such that $i^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(1)$. Similarly to $\mathcal{D}_{X/Y}$ and $\mathcal{D}_{\mathcal{Y}}$, we define triangulated categories $\mathcal{D}^{\mathcal{L}}_{X/Y}, \mathcal{D}^{\mathcal{L}^\dagger}_{\mathcal{Y}}$ to be

$$\mathcal{D}^{\mathcal{L}}_{X/Y} := \langle \mathcal{L}, D^b \operatorname{Coh}_{\leq 1}(X/Y)\rangle_{\mathrm{tr}} \subset D^b \operatorname{Coh}(X),$$
$$\mathcal{D}^{\mathcal{L}^\dagger}_{\mathcal{Y}} := \langle \mathcal{L}^\dagger, D^b \operatorname{Coh}_{\leq 1}(\mathcal{Y})\rangle_{\mathrm{tr}} \subset D^b \operatorname{Coh}(\mathcal{Y}).$$

Here $\mathcal{L}^\dagger$ is the line bundle on $\mathcal{Y}$ given by $\Phi^{-1}(\mathcal{L})$, as we defined in (26). The existence of a line bundle $\mathcal{L}$ is required to define the categories $\mathcal{D}^{\mathcal{L}}_{X/Y}$ and $\mathcal{D}^{\mathcal{L}^\dagger}_{\mathcal{Y}}$. Note that the equivalence Φ restricts to the equivalence between $\mathcal{D}^{\mathcal{L}^\dagger}_{\mathcal{Y}}$ and $\mathcal{D}^{\mathcal{L}}_{X/Y}$. We also define

$$\mathcal{A}^{\mathcal{L}}_{X/Y} := \langle \mathcal{L}, \mathcal{B}_{\leq 1}[-1]\rangle_{\mathrm{ex}} \subset \mathcal{D}^{\mathcal{L}}_{X/Y},$$
$$\mathcal{A}^{\mathcal{L}^\dagger}_{\mathcal{Y}} := \langle \mathcal{L}^\dagger, \operatorname{Coh}_{\leq 1}(\mathcal{Y})[-1]\rangle_{\mathrm{ex}} \subset \mathcal{D}^{\mathcal{L}^\dagger}_{\mathcal{Y}}.$$

Lemma 5.18 *There are bounded t-structures on $\mathcal{D}^{\mathcal{L}}_{X/Y}$ and $\mathcal{D}^{\mathcal{L}^\dagger}_{\mathcal{Y}}$, whose hearts are $\mathcal{A}^{\mathcal{L}}_{X/Y}$ and $\mathcal{A}^{\mathcal{L}^\dagger}_{\mathcal{Y}}$, respectively.*

Proof For $\mathcal{A}^{\mathcal{L}}_{X/Y}$, we follow the same proof of Lemma 4.4. Applying Proposition 4.5 for

$$\mathcal{D} = D^b \operatorname{Coh}(X), \quad L = \mathcal{L}, \quad \mathcal{D}' = D^b \operatorname{Coh}_{\leq 1}(X/Y), \quad \mathcal{A}' = \mathcal{B}_{\leq 1}[-1],$$

it is enough to check the vanishings

$$\operatorname{Hom}(\mathcal{L}, i_*F) = 0, \quad \operatorname{Hom}(i_*T[-1], \mathcal{L}) = 0$$

for μ–semistable $F \in \operatorname{Coh}(\mathbb{P}^2)$ with $\mu(F) \leq -\frac{1}{2}$ and μ–semistable $T \in \operatorname{Coh}(\mathbb{P}^2)$ with $\mu(T) > -\frac{1}{2}$. Using $i^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(1)$, these vanishings hold by the same computation of Lemma 4.4. As for $\mathcal{A}_{\mathcal{Y}}^{\mathcal{L}^\dagger}$, the statement obviously follows from Lemma 4.9 since $\mathcal{A}_{\mathcal{Y}}^{\mathcal{L}^\dagger} = \mathcal{A}_{\mathcal{Y}} \otimes \mathcal{L}^\dagger$. □

We define the group homomorphism

$$\mathrm{cl}^{\mathcal{L}}\colon K(\mathcal{D}_{X/Y}^{\mathcal{L}}) \to \Gamma$$

via the isomorphism (69) in the following way:

$$\mathrm{cl}^{\mathcal{L}}(F) := (\mathrm{rank}(F), [F] - \mathrm{rank}(F)[\mathcal{L}]).$$

Let $\Gamma_\bullet$ be the filtration defined by (71), and Z_t the collection of group homomorphisms defined by (75). Similarly to Lemma 4.13 and Lemma 4.14, one can show that

$$\sigma_t^{\mathcal{L}} := (Z_t, \mathcal{A}_{X/Y}^{\mathcal{L}}), \quad t > 0$$

determine a one-parameter family of weak stability conditions on $\mathcal{D}_{X/Y}^{\mathcal{L}}$ with respect to $(\Gamma_\bullet, \mathrm{cl}^{\mathcal{L}})$ such that

$$\lim_{t \to +0} \sigma_t^{\mathcal{L}} = (Z_0, \Phi(\mathcal{A}_{\mathcal{Y}}^{\mathcal{L}^\dagger})).$$

Proposition 5.19 (i) *An object $F \in \mathcal{A}_{X/Y}^{\mathcal{L}}$ with $\mathrm{rank}(F) = 0$ is Z_t–semistable if and only if $F \in \mathcal{A}_{X/Y}$ and it is Z_t–semistable.*

(ii) *For $t \gg 1$, an object $E \in \mathcal{A}_{X/Y}^{\mathcal{L}}$ with $\mathrm{rank}(E) = 1$ is Z_t–stable for $t \gg 0$ if and only if E is isomorphic to an object*

$$\mathcal{O}_X(rD) \otimes \mathcal{L} \otimes (\mathcal{O}_X \to F) \tag{101}$$

for $r \in \mathbb{Z}$ and a stable pair $(\mathcal{O}_X \to F)$ on X.

(iii) *For $0 < t \ll 1$, an object $E \in \mathcal{A}_{X/Y}^{\mathcal{L}}$ with $\mathrm{rank}(E) = 1$ is Z_t–stable for $0 < t \ll 1$ if and only if E is isomorphic to an object*

$$\Phi(\mathcal{L}^\dagger \otimes (\mathcal{O}_{\mathcal{Y}} \to F)) \tag{102}$$

for an orbifold stable pair $(\mathcal{O}_{\mathcal{Y}} \to F)$.

Proof (i) is obvious. As for (ii), let $\mathcal{C}^{\mathcal{L}}$ be the subcategory of $D^b \operatorname{Coh}(X)$ defined by

$$\mathcal{C}^{\mathcal{L}} := \langle \mathcal{L}(rD), \mathrm{Coh}_{\le 1}(X)[-1] : r \in \mathbb{Z} \rangle_{\mathrm{ex}}.$$

Then we have the same statements of Lemma 4.6 and Lemma 4.7 after replacing $\mathcal{A}_{X/Y}$ and $\mathcal{C}$ by $\mathcal{A}_{X/Y}^{\mathcal{L}}$ and $\mathcal{C}^{\mathcal{L}}$, respectively. Also, objects of the form (101) are objects in $\mathcal{C}^{\mathcal{L}}$, and the identical argument of Proposition 5.3 shows that the objects (101) coincide with the set of rank-one Z_t–stable objects for $t \gg 0$ up to isomorphisms. The proof of (iii) is also identical to Proposition 5.4, after replacing $\mathcal{O}_{\mathcal{Y}}$ by $\mathcal{L}^\dagger$. □

Similarly to the definition of $f(\{(r_j,c_j,m_j)\}_{j\neq e},r,\beta)$, we set

$$g(\{(r_j,c_j,m_j)\}_{j\neq e},r,\beta) := \sum_{G\in G(k)} \frac{1}{2^{k-1}} U(\{(r_j,c_j,m_j)\}_{j\neq e}) \times \prod_{\substack{a\to e \text{ or} \\ e\to a \text{ in } G}} \iota(e-a)\big(m_a+\tfrac{1}{2}c_a+3rc_a-r_aD\beta+\tfrac{9}{2}rr_a+\tfrac{9}{2}r^2r_a\big) \times \prod_{\substack{a\to b \text{ in } G \\ a,b\neq e}} 3(r_ac_b-r_bc_a).$$

Theorem 5.20 *Suppose that Conjecture 1.3 holds. Then for any element $u\in\mathbb{Q}[D]\oplus H^{\geq 4}(X,\mathbb{Q})$, we have the formula*

$$\sum_{\substack{(n,\beta)\in \\ \mathbb{Z}\oplus H_2(X,\mathbb{Z})}} \Bigg(\sum_{\substack{k\geq 1 \\ 1\leq e\leq k}} \sum_{\substack{(r_j,c_j,2m_j)\in\mathbb{Z}^3,\ 1\leq j\leq k,\ j\neq e,\ r\in\mathbb{Z} \\ 2c_j+r_j>0,\ c_j^2\geq 2r_jm_j,\ \text{satisfying (104)}}} (-1)^{k+\sum_{j\neq e} r_j+m_j+rc_j-r_jD\beta+\frac{3}{2}c_j+\frac{rr_j}{2}+\frac{r^2r_a}{2}+\sum_{a<b,a,b\neq e} r_ac_b-r_bc_a} \times f(\{(r_j,c_j,m_j)\}_{j\neq e},r,\beta)\cdot\prod_{j\neq e}\mathrm{DT}(r_j,c_j,m_j)\Bigg) P_{n,\beta}(X) \tag{103}$$
$$= \sum_{\substack{(n,\beta)\in \\ \mathbb{Z}\oplus H_2(X,\mathbb{Z})}} \Bigg(\sum_{\substack{k\geq 1 \\ 1\leq e\leq k}} \sum_{\substack{(r_j,c_j,2m_j)\in\mathbb{Z}^3,\ 1\leq j\leq k,\ j\neq e,\ r\in\mathbb{Z} \\ 2c_j+r_j>0,\ c_j^2\geq 2r_jm_j,\ \text{satisfying (105)}}} (-1)^{k+\sum_{j\neq e} m_j+\frac{1}{2}c_j+rc_j+r_jD\beta+\frac{rr_j}{2}+\frac{r^2r_j}{2}+\sum_{a<b,a,b\neq e} r_ac_b-r_bc_a} \times g(\{(r_j,c_j,m_j)\}_{j\neq e},r,\beta)\cdot\prod_{j\neq e}\mathrm{DT}(r_j,c_j,m_j)\Bigg) P_{n,\beta}(X).$$

Here the conditions (104), (105) are

$$(1,-u) = e^{rD}(1,0,-\beta,-n) - \sum_{j\neq e}\big(0,r_j[D],\big(\tfrac{3}{2}r_j+c_j\big)[l],\tfrac{3}{2}r_j+\tfrac{3}{2}c_j+m_j\big), \tag{104}$$

$$e^{c_1(\mathcal{L})}-(0,\Theta_\sharp u) = e^{rD+c_1(\mathcal{L})}(1,0,-\beta,-n) - \sum_{j\neq e}\big(0,r_j[D],\big(\tfrac{3}{2}r_j+c_j\big)[l],\tfrac{3}{2}r_j+\tfrac{3}{2}c_j+m_j\big), \tag{105}$$

and $\Theta_\sharp$ is given by (32), ie

$$\Theta_\sharp(r[D],\beta,n) = \Big(\big(\tfrac{5}{2}r+\beta D\big)[D],\beta+\big(\tfrac{13}{4}r+\tfrac{3}{2}\beta D\big)[l],n+\tfrac{11}{4}r+\tfrac{3}{2}\beta D+c_1(\mathcal{L})\beta\Big).$$

Proof Similarly to the proof of Theorem 5.11, we apply the Joyce–Song wall-crossing formula to the one-parameter family of weak stability conditions $\sigma_t^{\mathcal{L}}$ for $t \in \mathbb{R}_{>0}$. For $t \gg 0$, the rank-one invariants count objects of the form (101), whose Chern characters are given by

$$\operatorname{ch}\big(\mathcal{O}_X(rD) \otimes \mathcal{L} \otimes (\mathcal{O}_X \to F)\big) = e^{rD+c_1(\mathcal{L})}(1, 0, -\beta, -n)$$

where $(\mathcal{O}_X \to F) \in P_n(X, \beta)$. For $0 < t \ll 1$, the rank-one invariants count objects of the form (102), whose Chern characters are given by

$$\begin{aligned}\operatorname{ch}\big(\Phi(\mathcal{L}^\dagger \otimes (\mathcal{O}_\mathcal{Y} \to F))\big) &= \operatorname{ch}\big(\mathcal{L} \to \Phi(\mathcal{L}^\dagger \otimes F)\big)\\ &= e^{c_1(\mathcal{L})} - \operatorname{ch}\Phi(\mathcal{L}^\dagger \otimes F)\\ &= e^{c_1(\mathcal{L})} - (0, \Theta_\sharp \operatorname{ch}(\Phi_*\gamma)),\end{aligned}$$

where $(\mathcal{O}_\mathcal{Y} \to F) \in P(\mathcal{Y}, \gamma)$. Here we have used the diagrams (30) and (31).

Let us take $v_a = -i_*\alpha_a \in N_0(X/Y)$ for $\alpha_a \in K^+(\mathbb{P}^2)$ with $\operatorname{ch}(\alpha_a) = (r_a, c_a, m_a)$ and $v_e \in N(X)$ with $\operatorname{ch}(v_e) = e^{rD+c_1(\mathcal{L})}(1, 0, -\beta, -n)$. Using (21), we have

$$\chi(v_a, v_e) = m_a + \tfrac{1}{2}c_a + 3rc_a - r_a D\beta + \tfrac{9}{2}rr_a + \tfrac{9}{2}r^2 r_a.$$

Therefore, for any $\gamma \in N_{\le 1}(\mathcal{Y})$, the wall-crossing formula for $\sigma_t^{\mathcal{L}}, t > 0$ is described as

$$P_\gamma(\mathcal{Y}) = \sum_{(n,\beta)\in\mathbb{Z}\oplus H_2(X,\mathbb{Z})} \Bigg(\sum_{\substack{k\ge 1\\ 1\le e\le k}} \sum_{\substack{(r_j,c_j,2m_j)\in\mathbb{Z}^3,\ 1\le j\le k,\ j\ne e,\ r\in\mathbb{Z}\\ 2c_j+r_j>0,\ c_j^2\ge 2r_jm_j,\ \text{satisfying (105) for } u=\operatorname{ch}(\Phi_*\gamma)}} (-1)^{k-1+\sum_{a\ne e} m_a+\frac{1}{2}c_a+rc_a+r_aD\beta+\frac{rr_a}{2}+\frac{r^2r_a}{2}+\sum_{a<b,a,b\ne e} r_ac_b-r_bc_a} \times g(\{(r_j, c_j, m_j)\}_{j\ne e}, r, \beta) \cdot \prod_{j\ne e} \operatorname{DT}(r_j, c_j, m_j)\Bigg) P_{n,\beta}(X).$$

Setting $u = \operatorname{ch}(\Phi_*\gamma)$ and comparing with Theorem 5.11, we obtain the desired identity. □

Remark 5.21 The relation (103) in Theorem 5.20 is not a tautological relation. Indeed let us take $u = (0, 0, \beta_0, n_0)$. Then the conditions (104), (105) imply

(106) $$\beta = \beta_0 - \tfrac{3}{2}r^2[l] - r[l] - \sum_{j\ne e}\big(\tfrac{1}{2}r_j + c_j\big)[l],$$

(107) $$\beta = \beta_0 + \tfrac{1}{2}\beta_0 D[l] - \tfrac{3}{2}r^2[l] - \sum_{j\ne e}\big(\tfrac{1}{2}r_j + c_j\big)[l],$$

respectively. Therefore, if $\beta_0 D < 0$, the relation (103) is of the form

$$P_{n_0,\beta_0}(X) + \sum_{n',0<c'} (\cdots) P_{n',\beta_0-c'[l]}(X) = \sum_{n'',0<c''} (\cdots) P_{n'',\beta_0-c''[l]}(X), \tag{108}$$

which is not tautological.

Example 5.22 Let $\bar{\beta}$ be a curve class on X such that $\bar{\beta} \cdot D = 1$ and $\bar{\beta} - [l]$ is not an effective curve class.[13] If we set $\beta_0 = \bar{\beta} + [l]$, then the equation (107) does not have a solution, and the only solution of (106) is $\beta = \bar{\beta}$ with $\{(r_j, c_j, m_j)\}_{j \neq e}$ given in Example 5.16. Then the relation (108) is computed as

$$P_{n,\bar{\beta}+[l]}(X) = \sum_{n'<n-1} (-1)^{n-n'-1} 3(n-n') P_{n',\bar{\beta}}(X) + 3P_{n-1,\bar{\beta}}(X) - 2P_{n,\bar{\beta}}(X).$$

5.7 Euler characteristic version

The results of Theorem 5.11, Corollary 5.14 and Theorem 5.20 are still conditional on Conjecture 1.3. If we use [24, Theorem 6.28] instead of [25, Theorem 5.18], we obtain the Euler characteristic version of the above results, ie similar results after replacing $P_{n,\beta}(X)$, $P_\gamma(\mathcal{Y})$ by the naive Euler characteristics $\chi(P_n(X,\beta))$, $\chi(P(\mathcal{Y},\gamma))$, without relying on any conjecture. The proofs are the same, and we only give their statements.

The following is an analogue of Theorem 5.11:

Theorem 5.23 *For any $\gamma \in N_{\leq 1}(\mathcal{Y})$, we have the formula*

$$\chi(P(\mathcal{Y},\gamma)) = \sum_{\substack{(n,\beta)\in \\ \mathbb{Z}\oplus H_2(X,\mathbb{Z})}} \Bigg(\sum_{\substack{k\geq 1 \\ 1\leq e\leq k}} \sum_{\substack{(r_j,c_j,2m_j)\in\mathbb{Z}^3,\ 1\leq j\leq k,\ j\neq e,\ r\in\mathbb{Z} \\ 2c_j+r_j>0,\ c_j^2\geq 2r_jm_j,\ \text{satisfying (96)}}} (-1)^{k-1+\sum_{j\neq e} r_j+c_j+2r_jm_j} \times f(\{(r_j,c_j,m_j)\}_{j\neq e}, r, \beta) \cdot \prod_{j\neq e} \mathrm{DT}(r_j,c_j,m_j) \Bigg) \chi(P_n(X,\beta)).$$

Remark 5.24 In Theorem 5.23, we have used the fact the Euler characteristic version of $\mathrm{DT}(r,c,m)$ differs from it only by a multiplication of $(-1)^{r+c+2rm+1}$ (see [57, Lemma 2.8]). This fact will be also used in Corollary 5.25 and Theorem 5.26 below.

The following is a corollary of Theorem 5.23, which is an analogue of Corollary 5.14:

[13] For instance, in Example 2.1, one can take $D = (y_1 = x_1 = 0)$ and $\bar{\beta} = [l']$ for a line l' in $(y_1 = x_2 = 0)$.

Corollary 5.25 *For any positive integer* c *and* $n \in \mathbb{Z}$, *we have the*

$$\chi(P_n(X, c[l])) = \sum_{\substack{(n',c') \in \mathbb{Z}^2 \\ 0 \le c' < c}} \Bigg(\sum_{\substack{k \ge 1 \\ 1 \le e \le k}} \sum_{\substack{(r_j, c_j, 2m_j) \in \mathbb{Z}^3,\ 1 \le j \le k,\ j \ne e,\ r := \sum_{j \ne e} r_j \\ 2c_j + r_j > 0,\ c_j^2 \ge 2r_j m_j,\ \text{satisfying (99)}}} (-1)^{k + \sum_{j \ne e} r_j + c_j + 2r_j m_j} \\ \times f(\{(r_j, c_j, m_j)\}_{j \ne e}, r, c'[l]) \cdot \prod_{j \ne e} \mathrm{DT}(r_j, c_j, m_j) \Bigg) \chi(P_{n'}(X, c'[l])).$$

Finally, we have an analogue of Theorem 5.20:

Theorem 5.26 *For any* $u \in \mathbb{Q}[D] \oplus H^{\ge 4}(X, \mathbb{Q})$, *we have the formula*

$$\sum_{\substack{(n,\beta) \in \\ \mathbb{Z} \oplus H_2(X,\mathbb{Z})}} \Bigg(\sum_{\substack{k \ge 1 \\ 1 \le e \le k}} \sum_{\substack{(r_j, c_j, 2m_j) \in \mathbb{Z}^3,\ 1 \le j \le k,\ j \ne e,\ r \in \mathbb{Z} \\ 2c_j + r_j > 0,\ c_j^2 \ge 2r_j m_j,\ \text{satisfying (104)}}} (-1)^{k - 1 + \sum_{j \ne e} r_j + c_j + 2r_j m_j} \\ \times f(\{(r_j, c_j, m_j)\}_{j \ne e}, r, \beta) \cdot \prod_{j \ne e} \mathrm{DT}(r_j, c_j, m_j) \Bigg) \chi(P_n(X, \beta))$$

$$= \sum_{\substack{(n,\beta) \in \\ \mathbb{Z} \oplus H_2(X,\mathbb{Z})}} \Bigg(\sum_{\substack{k \ge 1 \\ 1 \le e \le k}} \sum_{\substack{(r_j, c_j, 2m_j) \in \mathbb{Z}^3,\ 1 \le j \le k,\ j \ne e,\ r \in \mathbb{Z} \\ 2c_j + r_j > 0,\ c_j^2 \ge 2r_j m_j,\ \text{satisfying (105)}}} (-1)^{k - 1 + \sum_{j \ne e} r_j + c_j + 2r_j m_j} \\ \times g(\{(r_j, c_j, m_j)\}_{j \ne e}, r, \beta) \cdot \prod_{j \ne e} \mathrm{DT}(r_j, c_j, m_j) \Bigg) \chi(P_n(X, \beta)).$$

References

[1] **A Bayer**, **E Macrì**, *The space of stability conditions on the local projective plane*, Duke Math. J. 160 (2011) 263–322 MR2852118

[2] **A Bayer**, **E Macrì**, **Y Toda**, *Bridgeland stability conditions on threefolds, I: Bogomolov–Gieseker type inequalities*, J. Algebraic Geom. 23 (2014) 117–163 MR3121850

[3] **A Beauville**, *Some remarks on Kähler manifolds with* $c_1 = 0$, from: "Classification of algebraic and analytic manifolds", (K Ueno, editor), Progr. Math. 39, Birkhäuser, Boston (1983) 1–26 MR728605

[4] **K Behrend**, *Gromov–Witten invariants in algebraic geometry*, Invent. Math. 127 (1997) 601–617 MR1431140

[5] **K Behrend**, *Donaldson–Thomas type invariants via microlocal geometry*, Ann. of Math. 170 (2009) 1307–1338 MR2600874

[6] **O Ben-Bassat**, **C Brav**, **V Bussi**, **D Joyce**, *A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Topol. 19 (2015) 1287–1359 MR3352237

[7] **T Bridgeland**, *Flops and derived categories*, Invent. Math. 147 (2002) 613–632 MR1893007

[8] **T Bridgeland**, *t-structures on some local Calabi–Yau varieties*, J. Algebra 289 (2005) 453–483 MR2142382

[9] **T Bridgeland**, *Stability conditions on triangulated categories*, Ann. of Math. 166 (2007) 317–345 MR2373143

[10] **T Bridgeland**, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. 24 (2011) 969–998 MR2813335

[11] **T Bridgeland**, **A King**, **M Reid**, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. 14 (2001) 535–554 MR1824990

[12] **K Bringmann**, **J Manschot**, *From sheaves on $\mathbb{P}^2$ to a generalization of the Rademacher expansion*, Amer. J. Math. 135 (2013) 1039–1065 MR3086068

[13] **J Bryan**, **C Cadman**, **B Young**, *The orbifold topological vertex*, Adv. Math. 229 (2012) 531–595 MR2854183

[14] **J Bryan**, **T Graber**, *The crepant resolution conjecture*, from: "Algebraic geometry—Seattle 2005, Part 1", (D Abramovich, A Bertram, L Katzarkov, R Pandharipande, M Thaddeus, editors), Proc. Sympos. Pure Math. 80, Amer. Math. Soc. (2009) 23–42 MR2483931

[15] **J Bryan**, **D Steinberg**, *Curve counting invariants for crepant resolutions*, Trans. Amer. Math. Soc. 368 (2016) 1583–1619 MR3449219

[16] **V Bussi**, *Generalized Donaldson–Thomas theory over fields $\mathbb{K} \neq \mathbb{C}$*, preprint (2014) arXiv:1403.2403

[17] **J Calabrese**, *Donaldson–Thomas invariants and flops*, preprint (2014) arXiv:1111.1670

[18] **J Calabrese**, *On the crepant resolution conjecture for Donaldson–Thomas invariants*, J. Algebraic Geom. 25 (2016) 1–18 MR3419955

[19] **T Coates**, **H Iritani**, **H-H Tseng**, *Wall-crossings in toric Gromov–Witten theory, I: Crepant examples*, Geom. Topol. 13 (2009) 2675–2744 MR2529944

[20] **L Göttsche**, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. 286 (1990) 193–207 MR1032930

[21] **L Göttsche**, *Theta functions and Hodge numbers of moduli spaces of sheaves on rational surfaces*, Comm. Math. Phys. 206 (1999) 105–136 MR1736989

[22] **D Happel**, **I Reiten**, **S O Smalø**, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575, Amer. Math. Soc. (1996) MR1327209

[23] **D Joyce**, *Configurations in abelian categories, II: Ringel–Hall algebras*, Adv. Math. 210 (2007) 635–706 MR2303235

[24] **D Joyce**, *Configurations in abelian categories, IV: Invariants and changing stability conditions*, Adv. Math. 217 (2008) 125–204 MR2357325

[25] **D Joyce**, **Y Song**, *A theory of generalized Donaldson–Thomas invariants*, Mem. Amer. Math. Soc. 1020, Amer. Math. Soc. (2012) MR2951762

[26] **G Kapustka**, **M Kapustka**, *Primitive contractions of Calabi–Yau threefolds, I*, Comm. Algebra 37 (2009) 482–502 MR2493796

[27] **Y Kawamata**, *Log crepant birational maps and derived categories*, J. Math. Sci. Univ. Tokyo 12 (2005) 211–231 MR2150737

[28] **A A Klyachko**, *Moduli of vector bundles and numbers of classes*, Funktsional. Anal. i Prilozhen. 25 (1991) 81–83 MR1113131 In Russian; translation in Funct. Anal. Appl. 1 (1991), 67–69

[29] **J Kollár**, **S Mori**, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics 134, Cambridge Univ. Press (1998) MR1658959

[30] **M Kontsevich**, *Homological algebra of mirror symmetry*, from: "Proceedings of the International Congress of Mathematicians, Volume 1", (S D Chatterji, editor), Birkhäuser, Basel (1995) 120–139 MR1403918

[31] **M Kontsevich**, **Y Soibelman**, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, preprint (2008) `arXiv:0811.2435`

[32] **M Kool**, *Euler characteristics of moduli spaces of torsion free sheaves on toric surfaces*, Geom. Dedicata 176 (2015) 241–269 MR3347580

[33] **M Lieblich**, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. 15 (2006) 175–206 MR2177199

[34] **M Lieblich**, *The openness of tilted hearts* (2013) MR2998828 Appendix to D Arcara and A Bertram, "Bridgeland-stable moduli spaces for K–trivial surfaces", *J. Eur. Math. Soc.* 15 (2013), 1–38

[35] **J Manschot**, *The Betti numbers of the moduli space of stable sheaves of rank* 3 *on* $\mathbb{P}^2$, Lett. Math. Phys. 98 (2011) 65–78 MR2836429

[36] **J Manschot**, *Sheaves on* $\mathbb{P}^2$ *and generalized Appell functions*, preprint (2015) `arXiv:1407.7785`

[37] **D Maulik**, **N Nekrasov**, **A Okounkov**, **R Pandharipande**, *Gromov–Witten theory and Donaldson–Thomas theory, I*, Compos. Math. 142 (2006) 1263–1285 MR2264664

[38] **T Milanov**, **Y Ruan**, **Y Shen**, *Gromov–Witten theory and cycle-valued modular forms*, preprint (2012) `arXiv:1206.3879`

[39] **S Mori**, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. 116 (1982) 133–176 MR662120

[40] **A Okounkov**, **R Pandharipande**, *Virasoro constraints for target curves*, Invent. Math. 163 (2006) 47–108 MR2208418

[41] **R Pandharipande**, **A Pixton**, *Gromov–Witten/pairs correspondence for the quintic 3–fold*, preprint (2016) `arXiv:1206.5490v2`

[42] **R Pandharipande**, **R P Thomas**, *Curve counting via stable pairs in the derived category*, Invent. Math. 178 (2009) 407–447 MR2545686

[43] **R Pandharipande**, **R P Thomas**, *Stable pairs and BPS invariants*, J. Amer. Math. Soc. 23 (2010) 267–297 MR2552254

[44] **T Pantev**, **B Toën**, **M Vaquié**, **G Vezzosi**, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. 117 (2013) 271–328 MR3090262

[45] **D Ross**, *Donaldson–Thomas theory and resolutions of toric transverse A–Singularities*, preprint (2016) `arXiv:1409.7011`

[46] **R Rouquier**, *Dimensions of triangulated categories*, J. K-Theory 1 (2008) 193–256 MR2434186

[47] **Y Ruan**, *The cohomology ring of crepant resolutions of orbifolds*, from: "Gromov–Witten theory of spin curves and orbifolds", (T J Jarvis, T Kimura, A Vaintrob, editors), Contemp. Math. 403, Amer. Math. Soc. (2006) 117–126 MR2234886

[48] **P Seidel**, **R Thomas**, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. 108 (2001) 37–108 MR1831820

[49] **J Stoppa**, **R P Thomas**, *Hilbert schemes and stable pairs: GIT and derived category wall crossings*, Bull. Soc. Math. France 139 (2011) 297–339 MR2869309

[50] **R P Thomas**, *A holomorphic Casson invariant for Calabi–Yau 3–folds, and bundles on $K3$ fibrations*, J. Differential Geom. 54 (2000) 367–438 MR1818182

[51] **Y Toda**, *Moduli stacks and invariants of semistable objects on $K3$ surfaces*, Adv. Math. 217 (2008) 2736–2781 MR2397465

[52] **Y Toda**, *Limit stable objects on Calabi–Yau 3–folds*, Duke Math. J. 149 (2009) 157–208 MR2541209

[53] **Y Toda**, *Curve counting theories via stable objects, I: DT/PT correspondence*, J. Amer. Math. Soc. 23 (2010) 1119–1157 MR2669709

[54] **Y Toda**, *Generating functions of stable pair invariants via wall-crossings in derived categories*, from: "New developments in algebraic geometry, integrable systems and mirror symmetry", (M-H Saito, S Hosono, K Yoshioka, editors), Adv. Stud. Pure Math. 59, Math. Soc. Japan, Tokyo (2010) 389–434 MR2683216

[55] **Y Toda**, *Stable pairs on local $K3$ surfaces*, J. Differential Geom. 92 (2012) 285–371 MR2998674

[56] **Y Toda**, *Curve counting theories via stable objects, II: DT/ncDT flop formula*, J. Reine Angew. Math. 675 (2013) 1–51 MR3021446

[57] **Y Toda**, *Generalized Donaldson–Thomas invariants on the local projective plane*, preprint (2014) arXiv:1405.3366

[58] **Y Toda**, *Gepner type stability conditions on graded matrix factorizations*, Algebr. Geom. 1 (2014) 613–665 MR3296807

[59] **C Vafa**, **E Witten**, *A strong coupling test of S–duality*, Nuclear Phys. B 431 (1994) 3–77 MR1305096

[60] **K Yoshioka**, *The Betti numbers of the moduli space of stable sheaves of rank 2 on $\boldsymbol{P}^2$*, J. Reine Angew. Math. 453 (1994) 193–220 MR1285785

[61] **K Yoshioka**, *Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface*, Internat. J. Math. 7 (1996) 411–431 MR1395938

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