

## Quotient singularities, eta invariants, and self-dual metrics

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There are three main components to this article:

- (i) A formula for the  $\eta$ -invariant of the signature complex for any finite subgroup of  $\mathrm{SO}(4)$  acting freely on  $S^3$  is given. An application of this is a nonexistence result for Ricci-flat ALE metrics on certain spaces.
- (ii) A formula for the orbifold correction term that arises in the index of the self-dual deformation complex is proved for all finite subgroups of  $\mathrm{SO}(4)$  which act freely on  $S^3$ . Some applications of this formula to the realm of self-dual and scalar-flat Kähler metrics are also discussed.
- (iii) Two infinite families of scalar-flat anti-self-dual ALE spaces with groups at infinity not contained in  $\mathrm{U}(2)$  are constructed. Using these spaces, examples of self-dual metrics on  $n \# \mathbb{C}P^2$  are obtained for  $n \geq 3$ . These examples admit an  $S^1$ -action, but are not of LeBrun type.

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### 1 Introduction

The focus of this work is on questions arising from the study of four-dimensional spaces that have isolated singularities or noncompact ends which are modeled, respectively, on neighborhoods of the origin and of infinity of  $\mathbb{R}^4/\Gamma$ , where  $\Gamma \subset \mathrm{SO}(4)$  is a finite subgroup which acts freely on  $S^3$ .

In particular, we say that  $(M^4, g)$  is a *Riemannian orbifold with isolated singularities* if  $g$  is a smooth Riemannian metric away from a finite set of singular points, and at each singular point the metric is locally the quotient of a smooth  $\Gamma$ -invariant metric on  $B^4$  by some finite subgroup  $\Gamma \subset \mathrm{SO}(4)$  which acts freely on  $S^3$ . The group  $\Gamma$  is known as the *orbifold group* at that point.

Now, given such a compact orbifold with positive scalar curvature, the Green's function for the conformal Laplacian associated to any point  $p \in M$  is guaranteed to exist, which we denote by  $G_p$ . Then the noncompact space  $(M^4 \setminus \{p\}, G_p^2 g)$  is a complete scalar-flat orbifold with a coordinate system at infinity arising from inverted normal

coordinates in the metric  $g$  around  $p$ . This, which we call a *conformal blow-up*, motivates the following definition.

We say that a noncompact Riemannian orbifold  $(X^4, g)$  is *asymptotically locally Euclidean (ALE)* of order  $\tau$  if there exists a diffeomorphism

$$(1-1) \quad \psi: X^4 \setminus U \rightarrow (\mathbb{R}^4 \setminus \overline{B_R(0)})/\Gamma,$$

where  $U \subset X^4$  is compact and  $\Gamma \subset \text{SO}(4)$  is a finite subgroup of  $\text{SO}(4)$  which acts freely on  $S^3$ , satisfying  $(\psi_*g)_{ij} = \delta_{ij} + \mathcal{O}(r^{-\tau})$  and  $\partial^{|k|}(\psi_*g)_{ij} = \mathcal{O}(r^{-\tau-k})$ , for any partial derivative of order  $k$ , as  $r \rightarrow \infty$ , where  $r$  is the distance to some fixed basepoint. The group  $\Gamma$  is known as the *group at infinity*.

For such a space, let  $u: X^4 \rightarrow \mathbb{R}^+$  be a function satisfying  $u = \mathcal{O}(r^{-2})$  as  $r \rightarrow \infty$ . Then there is a compactification of the space  $(X^4, u^2g)$  to an orbifold, which we denote by  $(\widehat{X^4}, \widehat{g})$ . In general, this will only be a  $C^{1,\alpha}$ -orbifold. However, if the metric satisfies a condition known as anti-self-duality, which we discuss next, there exists a compactification to a  $C^\infty$ -orbifold with positive Yamabe invariant (see Tian and Viaclovsky [41] and Chen, LeBrun and Weber [9]).

On an oriented four-dimensional Riemannian manifold  $(M, g)$ , the Hodge star operator associated to the metric  $g$  acting on 2-forms satisfies  $*^2 = \text{Id}$  and, in turn, induces the decomposition  $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ , where  $\Lambda^2_\pm$  are, respectively, the  $\pm 1$  eigenspaces of  $*|_{\Lambda^2}$ . Viewing the Weyl tensor as an operator  $\mathcal{W}_g: \Lambda^2 \rightarrow \Lambda^2$ , this leads to the decomposition

$$(1-2) \quad \mathcal{W}_g = \mathcal{W}_g^+ + \mathcal{W}_g^-,$$

where  $\mathcal{W}_g^\pm = \Pi^\pm \circ \mathcal{W}_g \circ \Pi^\pm$ , with  $\Pi^\pm = (\text{Id} \pm *) / 2$  being the respective projection maps onto  $\Lambda^2_\pm$ . This decomposition is conformally invariant. The metric  $g$  is called *self-dual* if  $\mathcal{W}_g^- = 0$  and *anti-self-dual* if  $\mathcal{W}_g^+ = 0$ . It is important to note that by reversing orientation a self-dual metric becomes anti-self-dual and vice versa.

**Remark 1.1** The conformal compactification of an anti-self-dual ALE space, with group at infinity  $\Gamma$ , has the same orbifold group at the point of compactification as long as the orientation is reversed, in which case the metric is self-dual. Therefore, while our focus is on anti-self-dual ALE metrics, we will consider the self-dual orientation for compact orbifolds.

It is necessary to briefly introduce the classification of finite subgroups of  $\text{SO}(4)$  which act freely on  $S^3$  before stating our main results. A more thorough discussion is provided in Section 2A. These groups are given by the finite subgroups of  $\text{U}(2)$  which act freely on  $S^3$  and their orientation-reversed conjugates, by which we mean

that there is an orientation-reversing intertwining map between said groups. Given a group  $\Gamma \subset \text{SO}(4)$ , its orientation-reversed conjugate in  $\text{SO}(4)$  will be denoted by  $\bar{\Gamma}$ . In Table 1 below, we list all finite subgroups of  $\text{U}(2)$  which act freely on  $S^3$ . From this all desired subgroups of  $\text{SO}(4)$  can be understood. First, some remarks on notation:

- For  $m$  and  $n$  nonzero integers,  $L(m, n)$  denotes the cyclic group generated by

$$\begin{pmatrix} \exp(2\pi i/n) & 0 \\ 0 & \exp(2\pi i m/n) \end{pmatrix}.$$

- The map  $\phi: S^3 \times S^3 \rightarrow \text{SO}(4)$  is the standard double cover (2-2).
- The binary polyhedral groups (dihedral, tetrahedral, octahedral, and icosahedral) are denoted, respectively, by  $D_{4n}^*$ ,  $T^*$ ,  $O^*$ , and  $I^*$ .
- The index-2 diagonal subgroup of  $\phi(L(1, 4m) \times D_{4n}^*)$  and the index-3 diagonal subgroup of  $\phi(L(1, 6m) \times T^*)$  will be denoted by  $\mathfrak{J}_{m,n}^2$  and  $\mathfrak{J}_m^3$ , respectively.

| $\Gamma \subset \text{U}(2)$     | conditions               | order  |
|----------------------------------|--------------------------|--------|
| $L(m, n)$                        | $(m, n) = 1$             | $n$    |
| $\phi(L(1, 2m) \times D_{4n}^*)$ | $(m, 2n) = 1$            | $4mn$  |
| $\phi(L(1, 2m) \times T^*)$      | $(m, 6) = 1$             | $24m$  |
| $\phi(L(1, 2m) \times O^*)$      | $(m, 6) = 1$             | $48m$  |
| $\phi(L(1, 2m) \times I^*)$      | $(m, 30) = 1$            | $120m$ |
| $\mathfrak{J}_{m,n}^2$           | $(m, 2) = 2, (m, n) = 1$ | $4mn$  |
| $\mathfrak{J}_m^3$               | $(m, 6) = 3$             | $24m$  |

Table 1: Finite subgroups of  $\text{U}(2)$  which act freely on  $S^3$

**Remark 1.2** Often only noncyclic subgroups of  $\text{U}(2)$  will be considered. This excludes the cyclic groups  $\phi(L(1, 2m) \times D_4^*)$  and  $\mathfrak{J}_{m,1}^2$  (these are the  $n = 1$  cases).

We are now able to state the main results of this work. Although there is a relationship between the underlying ideas of their proofs, the results fall into three distinct categories and are separated accordingly.

### 1A Eta invariants and Einstein metrics

Let  $(M, g)$  be a compact orbifold with finitely many isolated singularities  $p_1, \dots, p_k$  having corresponding orbifold groups  $\Gamma_1, \dots, \Gamma_k \subset \text{SO}(4)$ . The orbifold signature

theorem gives the formula

$$(1-3) \quad \tau_{\text{top}}(M) = \tau_{\text{orb}}(M) - \sum_{i=1}^n \eta(S^3 / \Gamma_i),$$

where the quantity  $\tau_{\text{orb}}(M)$  is the orbifold signature defined by

$$(1-4) \quad \tau_{\text{orb}}(M) = \frac{1}{12\pi^2} \int_M (\|W^+\|^2 - \|W^-\|^2) dV_g,$$

and  $\eta(S^3 / \Gamma_i)$  is the  $\eta$ -invariant of the signature complex. Since  $\Gamma_i \subset \text{SO}(4)$  is a finite subgroup, this can be shown to be given by

$$(1-5) \quad \eta(S^3 / \Gamma_i) = \frac{1}{|\Gamma_i|} \sum_{\gamma \neq \text{Id} \in \Gamma_i} \cot(\frac{1}{2}r(\gamma)) \cot(\frac{1}{2}s(\gamma)),$$

where  $r(\gamma)$  and  $s(\gamma)$  denote the rotation numbers of  $\gamma \in \Gamma_i$ . The  $\eta$ -invariants of certain groups are known. For finite subgroups of  $\text{SU}(2)$ , they were computed directly in Nakajima [37] and Hitchin [21]. A formula for the  $\eta$ -invariant of cyclic groups  $L(q, p)$ , where  $q$  and  $p$  are relative prime integers satisfying  $1 \leq q < p$ , was proved in Ashikaga and Ishizaka [3]. This formula, which is stated in (3-1), is in terms of the  $k$  and  $e_i$  that arise in the modified Euclidean algorithm (2-6). It is also recovered easily from our work in Lock and Viaclovsky [34]. Lastly, a formula for the  $\eta$ -invariant of  $\phi(L(1, 2m) \times D_{4n}^*)$  was found in Wright [43]. However, there are still many finite subgroups of  $\text{SO}(4)$  for which the  $\eta$ -invariant was unknown, and our first result here is a formula for all possible cases. The proof is found in Section 3A.

**Theorem 1.3** *Let  $\Gamma \subset \text{SO}(4)$  be a noncyclic finite subgroup which acts freely on  $S^3$ . Then the  $\eta$ -invariant of the signature complex is given as follows:*

- For  $\Gamma \subset \text{U}(2)$ ,

$$(1-6) \quad \eta(S^3 / \Gamma) = \frac{2}{3} \left( \frac{2m^2 + 1}{|\Gamma|} \right) - 1 + \mathcal{A}_\Gamma,$$

where  $\mathcal{A}_\Gamma$  is a constant given in Table 2.

- For  $\Gamma \not\subset \text{U}(2)$ ,

$$(1-7) \quad \eta(S^3 / \Gamma) = -\eta(S^3 / \bar{\Gamma}),$$

where  $\eta(S^3 / \bar{\Gamma})$  is given by (1-6) since here  $\bar{\Gamma} \subset \text{U}(2)$ .

| $\Gamma \subset U(2)$            | $\mathcal{A}_\Gamma$                    | congruences/conditions  |
|----------------------------------|---|---|
| $\phi(L(1, 2m) \times D_{4n}^*)$ | $\eta(S^3/L(m, n))$                     | $(m, 2n) = 1$   |
| $\phi(L(1, 2m) \times T^*)$      | $\pm 4/9$                               | $m \equiv \pm 1 \pmod 6$  |
| $\phi(L(1, 2m) \times O^*)$      | $\pm 13/18$<br>$\pm 5/18$               | $m \equiv \pm 1 \pmod{12}$<br>$m \equiv \pm 5 \pmod{12}$  |
| $\phi(L(1, 2m) \times I^*)$      | $\pm 46/45$<br>$\pm 2/9$<br>$\pm 26/45$ | $m \equiv \pm 1 \pmod{30}$<br>$m \equiv \pm 7, \pm 13 \pmod{30}$<br>$m \equiv \pm 11 \pmod{30}$ |
| $\mathfrak{J}_{m,n}^2$           | $\eta(S^3/L(m, n))$                     | $(m, 2) = 2, (m, n) = 1$  |
| $\mathfrak{J}_m^3$               | 0                                       | $m \equiv 3 \pmod 6$  |

Table 2

**Remark 1.4** Notice that although the  $\mathcal{A}_\Gamma$  terms for the  $\eta$ -invariants of the subgroups  $\phi(L(1, 2m) \times D_{4n}^*)$  and  $\mathfrak{J}_{m,n}^2$  contain an  $\eta(S^3/L(m, n))$ , they can be computed algorithmically by using formula (3-1) for the  $\eta$ -invariant of a cyclic group.

We next give an application of Theorem 1.3. There is a well-known conjecture, due to Bando, Kasue and Nakajima [4], that the only simply connected Ricci-flat ALE metrics in dimension four are the hyperkähler ones. The following shows that the conjecture is true, provided one restricts to the diffeomorphism types of minimal resolutions of  $\mathbb{C}^2/\Gamma$  or any iterated blow-up thereof.

**Theorem 1.5** *Let  $\Gamma \subset U(2)$  be a finite subgroup which acts freely on  $S^3$ , and let  $X$  be diffeomorphic to the minimal resolution of  $\mathbb{C}^2/\Gamma$  or any iterated blow-up thereof. If  $g$  is a Ricci-flat ALE metric on  $X$ , then  $\Gamma \subset SU(2)$  and  $g$  is hyperkähler.*

This is proved in Section 3B by applying the Hitchin–Thorpe inequality for ALE metrics obtained by Nakajima, together with the result in Theorem 1.3. If one assumes that  $g$  is Kähler, the result is trivial, so we emphasize that we only make an assumption about the diffeomorphism type and do not assume  $g$  is Kähler.

It is interesting to note that there are non-simply connected Ricci-flat anti-self-dual ALE metrics. These spaces have been classified as cyclic quotients of Gibbons–Hawking multi-Eguchi–Hanson spaces. This was proved, with the assumption that the manifold is Kähler, in Şuvaina [40] and in general in Wright [43].

## 1B Self-dual deformations

If  $(M, g)$  is a self-dual four-manifold, then the local structure moduli space of self-dual conformal classes near  $g$  is controlled by the following elliptic complex known as the self-dual deformation complex:

$$(1-8) \quad \Gamma(T^*M) \xrightarrow{\mathcal{K}_g} \Gamma(S_0^2(T^*M)) \xrightarrow{\mathcal{D}_g} \Gamma(S_0^2(\Lambda_-^2)).$$

Here  $\mathcal{K}_g$  denotes the conformal Killing operator,  $S_0^2(T^*M)$  denotes traceless symmetric 2-tensors, and  $\mathcal{D}_g = (\mathcal{W}^-)'_g$  is the linearized anti-self-dual Weyl curvature operator.

If  $M$  is compact, then the index of this complex is given in terms of topological quantities via the Atiyah–Singer index theorem as

$$(1-9) \quad \text{Ind}(M, g) = \sum_{i=0}^2 (-1)^i \dim H_{\text{SD}}^i = \frac{1}{2}(15\chi_{\text{top}}(M) - 29\tau_{\text{top}}(M)),$$

where  $\chi_{\text{top}}(M)$  is the Euler characteristic,  $\tau_{\text{top}}(M)$  is the signature, and  $H_{\text{SD}}^i$  is the  $i^{\text{th}}$  cohomology group of (1-8), for  $i = 0, 1, 2$ .

In Lock and Viaclovsky [35], we discussed the deformation theory of certain scalar-flat Kähler ALE metrics. Unlike the scalar-flat Kähler condition, the anti-self-dual condition is conformally invariant, so we can transfer the deformation problem of anti-self-dual ALE spaces to their self-dual conformal compactifications. However, these conformal compactifications are orbifolds, upon which formula (1-9) does not necessarily hold as there are correction terms required arising from the singularities. Kawasaki [26] proved a version of the Atiyah–Singer index theorem for orbifolds, with the orbifold correction terms expressed as certain representation-theoretic quantities. In Lock and Viaclovsky [34], we explicitly determined this correction term in the case of cyclic quotient singularities. Our next result is a determination of the correction term for *all* finite subgroups  $\Gamma \subset \text{SO}(4)$  which act freely on  $S^3$ .

**Theorem 1.6** *Let  $(M, g)$  be a compact self-dual orbifold with a single orbifold point having orbifold group  $\Gamma$ , a noncyclic finite subgroup of  $\text{SO}(4)$ . Then the index of the self-dual deformation complex on  $(M, g)$  is given by*

$$\text{Ind}(M, g) = \frac{1}{2}(15\chi_{\text{top}}(M) - 29\tau_{\text{top}}(M)) + N(\Gamma),$$

where  $N(\Gamma)$  is given as follows:

- For  $\Gamma \subset \text{U}(2)$ ,

$$(1-10) \quad N(\Gamma) = -4b_\Gamma + \mathcal{B}_\Gamma,$$

where  $-b_\Gamma$ , given by (2-7), is the self-intersection number of the central rational curve in the minimal resolution of  $\mathbb{C}^2/\Gamma$  and  $B_\Gamma$  is a constant given in Table 4.

- For  $\Gamma \not\subset U(2)$ ,

$$(1-11) \quad N(\Gamma) = -N(\bar{\Gamma}) - \begin{cases} 6 & \text{if } \bar{\Gamma} \subset SU(2), \\ 7 & \text{if } \bar{\Gamma} \not\subset SU(2), \end{cases}$$

where  $N(\bar{\Gamma})$  is given by (1-10) since here  $\bar{\Gamma} \subset U(2)$ .

This is proved in Section 4A. Note that Theorem 1.6 generalizes easily to the case of orbifolds with any finite number of singularities. We also note that the second author [42] previously proved such an index formula for the binary polyhedral groups and their orientation-reversed conjugates, which is recovered here as well.

In [35], we proved the existence of scalar-flat Kähler ALE metrics on the minimal resolution of  $\mathbb{C}^2/\Gamma$  for all finite subgroups  $\Gamma \subset U(2)$ . When  $\Gamma \subset SU(2)$ , such metrics are necessarily hyperkähler and, from [35, Section 8], the anti-self-dual deformations are the same as the scalar-flat Kähler deformations. However, as an application of Theorem 1.6, we have the following, which shows the nonhyperkähler examples of these metrics have many more anti-self-dual deformations than scalar-flat Kähler deformations.

**Theorem 1.7** *Let  $g$  be a scalar-flat Kähler ALE metric on the minimal resolution  $X$  of  $\mathbb{C}^2/\Gamma$ , where  $\Gamma \subset U(2)$  is a finite subgroup which acts freely on  $S^3$ . Then the dimension of the moduli space of scalar-flat anti-self-dual ALE metrics near  $g$  is strictly larger than the dimension of the moduli space of scalar-flat Kähler ALE metrics near  $g$ , unless  $\Gamma \subset SU(2)$ , in which case there is equality.*

This is proved in Section 4B, where we also give a formula for the formal dimension of the moduli space of scalar-flat anti-self-dual metrics near  $g$ .

### 1C Self-dual constructions

In [42], Viaclovsky posed the question of existence of anti-self-dual ALE spaces with groups at infinity orientation-reversed conjugate to the binary polyhedral groups. The LeBrun negative mass metrics (see LeBrun [30]) are examples of scalar-flat Kähler, hence anti-self-dual, ALE spaces with groups at infinity orientation-reversed conjugate to cyclic subgroups of  $SU(2)$ . However, since the orientation-reversed conjugate groups to the binary polyhedral groups are not contained in  $U(2)$ , there cannot be scalar-flat Kähler ALE spaces with these groups at infinity. Therefore the natural question is that of the existence of anti-self-dual metrics. This question clearly extends to include the

orientation-reversed conjugate groups of all noncyclic finite subgroups of  $U(2)$  which act freely on  $S^3$ . We now give a partial answer to this question and use this to construct some new examples of self-dual metrics in Corollary 1.10.

**Theorem 1.8** *Let  $\Gamma_1 = \phi(L(1, 2m) \times D_{4n}^*)$  and  $\Gamma_2 = \mathfrak{J}_{m,n}^2$  with integers  $m, n$  as specified in Table 1 so the action on  $S^3$  is free. Then, for  $i = 1, 2$ , there exists a scalar-flat anti-self-dual ALE space  $(X_i, g_{X_i})$  with group at infinity  $\bar{\Gamma}_i$ , the orientation-reversed conjugate group to  $\Gamma_i$ , satisfying  $\pi_1(X_i) = \mathbb{Z}/2\mathbb{Z}$ . Furthermore, the  $g_{X_i}$  may be chosen to admit an isometric  $S^1$ -action.*

**Remark 1.9** It is still unknown whether there are such examples for the other noncyclic orientation-reversed conjugate subgroups which act freely on  $S^3$ , which is a very interesting question.

Let  $\Gamma_1$  and  $\Gamma_2$  be as in Theorem 1.8. For  $i = 1, 2$ , let  $(Y_i, g_{Y_i})$  denote the scalar-flat Kähler, hence anti-self-dual, ALE space with group at infinity  $\Gamma_i$ , obtained for the noncyclic ( $n > 1$ ) and cyclic ( $n = 1$ ) cases in, respectively, Lock and Viaclovsky [35] and Calderbank and Singer [8], and let  $(X_i, g_{X_i})$  denote the anti-self-dual ALE space with group at infinity  $\bar{\Gamma}_i$  obtained in Theorem 1.8 above. These can be compactified to self-dual orbifolds,  $(\hat{Y}_i, \hat{g}_{Y_i})$  and  $(\hat{X}_i, \hat{g}_{X_i})$ , and then attached via a self-dual orbifold gluing theorem. Although  $\pi_1(X_i) = \mathbb{Z}/2\mathbb{Z}$ , we will show that  $\pi_1(\hat{X}_i \# \hat{Y}_i) = \{1\}$  and thus have the following corollary.

**Corollary 1.10** *Let  $\Gamma_1 = \phi(L(1, 2m) \times D_{4n}^*)$  and  $\Gamma_2 = \mathfrak{J}_{m,n}^2$  with integers  $m, n$  as specified in Table 1 so the action on  $S^3$  is free. For  $i = 1, 2$ , define the integer*

$$(1-12) \quad \ell_i(m, n) = 3 + \begin{cases} k_{(n-m,n)} + k_{(m-n,m)} & \text{if } n > 1 \text{ and } m > 1, \\ m - 1 & \text{if } n = 1 \text{ and } m > 1, \\ n - 1 & \text{if } n > 1 \text{ and } m = 1, \\ 0 & \text{if } n = 1 \text{ and } m = 1, \end{cases}$$

where  $k_{(q,p)}$  denotes the length of the Hirzebruch–Jung modified Euclidean algorithm for  $(q, p)$  (see (2-6) for a description). Then, for  $i = 1, 2$ , there exist two distinct sequences of self-dual metrics on  $\ell_i(m, n) \# \mathbb{C}P^2$ , one limiting to an orbifold with a single orbifold point of type  $\Gamma_i$  and the other limiting to an orbifold with a single orbifold point of type  $\bar{\Gamma}_i$ . Furthermore, these examples may be chosen to admit a conformally isometric  $S^1$ -action, but they are not of LeBrun type.

Here, by *LeBrun-type* metrics, we mean the self-dual metrics on  $n \# \mathbb{C}P^2$  discovered by LeBrun [31]. The proofs of Theorem 1.8 and Corollary 1.10 are given in Section 5B.



**Remark 1.11** The  $m = n = 1$  case, which can only occur for  $\Gamma_1$ , is minimal here in the sense that  $\ell_1(1, 1) = 3$  is the smallest number of  $\mathbb{C}\mathbb{P}^2$ s on which we obtain a self-dual metric by this technique. The self-dual conformal classes we find in this case are contained in those classified by Honda [22, Theorem 1.1] and give examples of orbifold degeneration in Honda’s conformal classes. To the best of our knowledge, for all  $\ell_i(m, n) > 3$ , the sequences of metrics obtained in Corollary 1.10 are new examples of degeneration of self-dual metrics to orbifolds with these orbifold groups. Also, notice that for all  $\ell > 3$  there are multiple possibilities for  $m$  and  $n$  to obtain a self-dual metric on  $\ell \# \mathbb{C}\mathbb{P}^2$ . Since each such possibility limits to distinct orbifold metrics, the corresponding metrics on  $\ell \# \mathbb{C}\mathbb{P}^2$  are themselves distinct. For example,

$$\ell_1(1, 2) \# \mathbb{C}\mathbb{P}^2 = \ell_2(2, 1) \# \mathbb{C}\mathbb{P}^2 = 4 \# \mathbb{C}\mathbb{P}^2,$$

but the orbifold limits have singularities of types conjugate, respectively, to the groups  $\phi(L(1, 2) \times D_8^*)$  and  $\mathfrak{I}_{2,1}^2$ . It is an interesting question whether these self-dual metrics lie in the same component or distinct components of the moduli space of self-dual metrics on  $\ell \# \mathbb{C}\mathbb{P}^2$ .

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## 2 Background

### 2A Group actions and the Hopf fibration

It will be convenient to understand  $SO(4)$  in terms of quaternionic multiplication. We identify  $\mathbb{C}^2$  with the space of quaternions  $\mathbb{H}$  by

$$(2-1) \quad (z_1, z_2) \in \mathbb{C}^2 \longleftrightarrow z_1 + z_2 \hat{j} \in \mathbb{H},$$

and consider  $S^3$ , in the natural way, as the space of unit quaternions. It is well known that the map  $\phi: S^3 \times S^3 \rightarrow SO(4)$  defined by

$$(2-2) \quad \phi(a, b)(h) = a * h * \bar{b},$$

for  $a, b \in S^3$  and  $h \in \mathbb{H}$ , is a double cover. Right multiplication by unit quaternions gives  $SU(2)$ . Notice that this is just the restriction  $\phi|_{1 \times S^3}$ . The finite subgroups of  $SU(2)$  were found in [10] and are given by the cyclic groups  $L(-1, n)$ , for all

integers  $n$  greater than or equal to 1, and the binary polyhedral groups. Similarly, the restriction

$$(2-3) \quad \phi: S^1 \times S^3 \rightarrow \mathrm{U}(2),$$

where the  $S^1$  is given by  $e^{i\theta}$  for  $0 \leq \theta < 2\pi$ , is a double cover of  $\mathrm{U}(2)$ . The finite subgroups of  $\mathrm{U}(2)$  were classified in [15; 11]. Those which act freely on  $S^3$  were later classified in [38]. These are the groups given above in Table 1; we refer the reader to [35] for an explicit list of generators and a more thorough exposition.

Here, we are interested in all finite subgroups of  $\mathrm{SO}(4)$  which act freely on  $S^3$ , not just those in  $\mathrm{U}(2)$ . However, any such group is conjugate, in  $\mathrm{SO}(4)$ , to a subgroup of  $\phi(S^1 \times S^3)$  or  $\phi(S^3 \times S^1)$ , and moreover, these groups themselves are conjugate subgroups of  $\mathrm{O}(4)$  (see [38]). Since  $\phi(S^1 \times S^3) = \mathrm{U}(2)$ , we call the subgroups of  $\phi(S^3 \times S^1)$  the *orientation-reversed conjugate groups*. If  $\Gamma \subset \mathrm{U}(2)$ , we denote by  $\bar{\Gamma}$  its orientation-reversed counterpart, where  $\bar{\Gamma} \subset \phi(S^3 \times S^1) \subset \mathrm{SO}(4)$ . If  $\Gamma \subset \mathrm{U}(2)$  is finite, then it has a finite set of generators which can be written (not uniquely) as  $\{\phi(a_i, b_i)\}_{i=1, \dots, n}$  for some  $(a_i, b_i) \in S^1 \times S^3$ . Observe that, up to conjugation in  $\mathrm{SO}(4)$ , the orientation-reversed conjugate group  $\bar{\Gamma}$  would be generated by switching the left and right multiplication in the generators, ie, by the set  $\{\phi(b_i, a_i)\}_{i=1, \dots, n}$ .

A crucial step that underlies the results of this paper will be to consider quotients of certain orbifolds. The resulting spaces will have new singular points and it will be essential to understand their orbifold groups. To do this, it will be necessary to make use of the Hopf fibration. Given the standard embedding of  $S^3$  in  $\mathbb{C}^2$ , and writing  $S^2 = \mathbb{C} \cup \{\infty\}$ , the Hopf map  $\mathcal{H}: S^3 \rightarrow S^2$  is given by

$$(2-4) \quad \mathcal{H}(z_1, z_2) \mapsto z_1/z_2.$$

Observe that the Hopf fiber, over a general  $z \in S^2$ , is the  $S^1$  given by

$$(2-5) \quad e^{i\theta} (|z|^2 + 1)^{-\frac{1}{2}} (z, 1) = e^{i\theta} (|z|^2 + 1)^{-\frac{1}{2}} (z + \hat{j}) \in S^3.$$

Using (2-5) to examine this fibration under quaternionic multiplication, one finds the following. The Hopf fibration is preserved by all right multiplication, however it is only preserved under left multiplication by quaternions of the form  $e^{i\theta}$  and  $e^{i\theta} * \hat{j}$ . Thus, from (2-3), it is clear that all of  $\mathrm{U}(2)$  preserves the Hopf fiber structure. To find all other finite subgroups of  $\mathrm{SO}(4)$  which act freely on  $S^3$  and preserve the Hopf fibration, it is only necessary to examine those which are orientation-reversed conjugate to the finite subgroups of  $\mathrm{U}(2)$  listed in Table 1. Since the orientation-reversed conjugate groups are generated by switching left and right quaternionic multiplication of the generators of subgroups of  $\mathrm{U}(2)$ , it will be precisely those orientation-reversed conjugate to subgroups of  $\mathrm{U}(2)$  that have only elements of the form  $e^{i\theta}$  or  $e^{i\theta} * \hat{j}$  acting on the

right which preserve the Hopf fibration. In [35], we provided a list of these groups along with their generators, and referring to this, it is clear that the only noncyclic orientation-reversed conjugate groups that preserve the Hopf fiber structure are those given in Table 3.

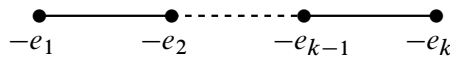
| noncyclic $\bar{\Gamma} \subset \text{SO}(4)$ | conditions               | generators  |
|---|--------------------------|---|
| $\overline{\phi(L(1, 2m) \times D_{4n}^*)}$   | $(m, 2n) = 1$            | $\phi(1, e^{\frac{\pi i}{m}}), \phi(e^{\frac{\pi i}{n}}, 1), \phi(\hat{j}, 1)$                    |
| $\tilde{\mathcal{J}}_{m,n}^2$                 | $(m, 2) = 2, (m, n) = 1$ | $\phi(1, e^{\frac{\pi i}{m}}), \phi(e^{\frac{\pi i}{n}}, 1), \phi(\hat{j}, e^{\frac{\pi i}{2m}})$ |

Table 3: Groups preserving the Hopf fibration but not contained in  $U(2)$

### 2B Minimal resolutions and ALE metrics

For all finite subgroups  $\Gamma \subset U(2)$  which act freely on  $S^3$ , Brieskorn described the minimal resolution of  $\mathbb{C}^2/\Gamma$  complex analytically [6]. Here, we will first describe the minimal resolution of cyclic quotients and then use this to provide a description for all possible cases.

Let  $\tilde{X}$  be the minimal resolution of  $\mathbb{C}^2/L(q, p)$ , where  $q$  and  $p$  are relative prime integers satisfying  $1 \leq q < p$ . The exceptional divisor of  $\tilde{X}$ , known as a Hirzebruch–Jung string, has intersection matrix



where  $k$  and the  $e_i$  are determined by the following Hirzebruch–Jung modified Euclidean algorithm:

$$\begin{aligned}
 p &= e_1q - a_1, \\
 q &= e_2a_1 - a_2, \\
 &\vdots \\
 a_{k-3} &= e_{k-1}a_{k-2} - 1, \\
 a_{k-2} &= e_k a_{k-1} = e_k,
 \end{aligned}
 \tag{2-6}$$

where the self-intersection numbers satisfy  $e_i \geq 2$  and where  $0 \leq a_i < a_{i-1}$  (see [20]). The integer  $k$  is called the *length* of the modified Euclidean algorithm. (These values can also be understood in terms of a continued fraction expansion of  $q/p$ .)

**Remark 2.1** We will often need to distinguish the length of the modified Euclidean algorithm for a particular pair of relatively prime integers satisfying  $1 \leq q < p$ , and therefore we denote this by  $k(q, p)$  when necessary.

For a noncyclic finite subgroup  $\Gamma \subset \mathrm{U}(2)$  which acts freely on  $S^3$ , the exceptional divisor of the minimal resolution of  $\mathbb{C}^2/\Gamma$  is a tree of rational curves with normal crossing singularities. There is a distinguished rational curve which intersects exactly three other rational curves. We refer to this as the *central rational curve*, and it has self-intersection number

$$(2-7) \quad -b_\Gamma = -2 - \frac{4m}{|\Gamma|} \left[ m - \left( m \bmod \frac{|\Gamma|}{4m} \right) \right],$$

where  $m$  corresponds to that of the group as in Table 1. Emanating from the central rational curve are three Hirzebruch–Jung strings corresponding to the singularities satisfying  $L(p_i - q_i, p_i) = \overline{L(q_i, p_i)}$ , for  $i = 1, 2, 3$ , where the  $L(q_i, p_i)$  are the cyclic singularities of the orbifold quotients of Theorem 2.2 specified for each group below.

In [35], we constructed scalar-flat Kähler ALE metrics on the minimal resolution of  $\mathbb{C}^2/\Gamma$  for all noncyclic finite subgroups  $\Gamma \subset \mathrm{U}(2)$  which act freely on  $S^3$ . The previously known examples of such spaces are for noncyclic finite subgroups  $\Gamma \subset \mathrm{SU}(2)$  (these are the binary polyhedral groups), for which Kronheimer obtained and classified hyperkähler metrics on these minimal resolutions [28; 29].

LeBrun [30] constructed a  $\mathrm{U}(2)$ -invariant scalar-flat Kähler ALE metric on the minimal resolution of  $\mathbb{C}^2/L(1, \ell)$  for all positive integers  $\ell$ . The  $\ell = 2$  case is the well-known Eguchi–Hanson metric [16]. The minimal resolution here is the total space of the bundle  $\mathcal{O}(-\ell)$  over  $\mathbb{C}\mathbb{P}^1$ . These are known as the LeBrun negative mass metrics and are denoted by  $(\mathcal{O}(-\ell), g_{\mathrm{LB}})$ .

In [8], Calderbank and Singer used the Joyce ansatz [25] to explicitly construct toric scalar-flat Kähler metrics on the minimal resolution of  $\mathbb{C}^2/L(q, p)$  for all relatively prime integers satisfying  $1 \leq q < p$ . When  $q = 1$  these are the LeBrun negative mass metrics, and when  $q = p - 1$  these are the toric multi-Eguchi–Hanson metrics (all monopole points lie on a common line) (see [19]).

## 2C Orbifold quotients

It will be essential to the work here to examine quotients of certain weighted projective spaces. These are defined in general as follows. For relatively prime integers which satisfy  $1 \leq r \leq q \leq p$ , the *weighted projective space*  $\mathbb{C}\mathbb{P}_{(r,q,p)}^2$  is  $S^5/S^1$ , where  $S^1$  acts by

$$(z_0, z_1, z_2) \mapsto (e^{ir\theta} z_0, e^{iq\theta} z_1, e^{ip\theta} z_2),$$

for  $0 \leq \theta < 2\pi$ . Each weighted projective space is a complex orbifold which admits a unique Bochner–Kähler metric that we refer to as the *canonical Bochner–Kähler*

metric (see [7] for existence and [13] for uniqueness). In four real dimensions, the Bochner tensor is precisely the anti-self-dual part of the Weyl tensor, so these metrics are self-dual Kähler.

Topologically, the conformal compactification of  $\mathcal{O}(-n)$  is the weighted projective space  $\mathbb{C}P^2_{(1,1,n)}$ , which has a singularity of type  $L(-1, n)$  at  $[0, 0, 1]$ , the point of compactification. In [12], Dabkowski and Lock proved an explicit Kähler conformal compactification of  $U(n)$ -invariant Kähler ALE spaces, ie, the conformal compactification of such spaces to Kähler orbifolds. (It is important to note that the resulting spaces are Kähler with respect to reverse-oriented complex structures.) Therefore, the LeBrun negative mass metric on  $\mathcal{O}(-n)$  has a conformal compactification to a Kähler metric on  $\widehat{\mathcal{O}}(-n) = \mathbb{C}P^2_{(1,1,n)}$ . Moreover, since the ALE metric is anti-self-dual, the compactified metric is self-dual, so this is necessarily the canonical Bochner–Kähler metric on  $\mathbb{C}P^2_{(1,1,n)}$ . Previously Joyce [24] proved that there is a quaternionic metric on  $\mathbb{C}P^2_{(1,1,n)}$  which must be conformal to  $(\mathcal{O}(-n), g_{LB})$ . However, from [12], we see that the canonical Bochner–Kähler metric on  $\mathbb{C}P^2_{(1,1,n)}$  is given explicitly by

$$(2-8) \quad g_{BK} = \frac{dr^2}{(1+r^2)(1+nr^2)} + \frac{r^2}{r^2+1} \left[ \sigma_1^2 + \sigma_2^2 + \left( \frac{1+nr^2}{r^2(1+r^2)} \right) \sigma_3^2 \right],$$

where  $\sigma_1, \sigma_2, \sigma_3$  is the usual left-invariant coframe on  $SU(2)$  and  $r = 0$  corresponds to  $[0, 0, 1]$ , the point of compactification. Also, the rational curve defined by

$$(2-9) \quad \Sigma := \{[z_0, z_1, 0] : z_0, z_1 \in \mathbb{C}\} \subset \mathbb{C}P^2_{(1,1,n)}$$

will frequently be considered, so we make a point of distinguishing it here. The corresponding rational curve in quotients of  $\mathbb{C}P^2_{(1,1,n)}$  will be denoted by  $\Sigma$  as well.

In [35], for each noncyclic finite subgroup  $\Gamma \subset U(2)$  which acts freely on  $S^3$ , we took a specific quotient of a certain  $(\mathcal{O}(-2m), g_{LB})$  to obtain a scalar-flat Kähler ALE orbifold with group at infinity  $\Gamma$  and with all singularities isolated and of cyclic type. The conformal compactification factor from [12] descends to compactify these quotients to self-dual Kähler orbifold quotients of  $(\mathbb{C}P^2_{(1,1,2m)}, g_{BK})$ . The following theorem is an immediate consequence of [35, Theorem 4.1].

**Theorem 2.2** *For each noncyclic finite subgroup  $\Gamma \subset U(2)$  which acts freely on  $S^3$ , the quotient  $(\mathbb{C}P^2_{(1,1,2m)}, g_{BK})/\Gamma$  is a self-dual Kähler orbifold with four isolated singularities — one at the point of compactification with orbifold group  $\Gamma$ , and three on the rational curve  $\Sigma$  with orbifold groups specified precisely as follows:*

| $\Gamma \subset U(2)$            | conditions               | orbifold groups             |
|----------------------------------|--------------------------|-----------------------------|
| $\phi(L(1, 2m) \times D_{4n}^*)$ | $(m, 2n) = 1$            | $L(1, 2), L(1, 2), L(m, n)$ |
| $\phi(L(1, 2m) \times T^*)$      | $(m, 6) = 1$             | $L(1, 2), L(m, 3), L(m, 3)$ |
| $\phi(L(1, 2m) \times O^*)$      | $(m, 6) = 1$             | $L(1, 2), L(m, 3), L(m, 4)$ |
| $\phi(L(1, 2m) \times I^*)$      | $(m, 30) = 1$            | $L(1, 2), L(m, 3), L(m, 5)$ |
| $\mathfrak{J}_{m,n}^2$           | $(m, 2) = 2, (m, n) = 1$ | $L(1, 2), L(1, 2), L(m, n)$ |
| $\mathfrak{J}_m^3$               | $(m, 6) = 3$             | $L(1, 2), L(1, 3), L(2, 3)$ |

**Remark 2.3** We write the orbifold groups of the three singularities on the rational curve  $\Sigma$  of the quotients  $(\mathbb{C}P^2_{(1,1,2m)}, g_{BK})/\Gamma$  in Theorem 2.2 above as  $L(q_i, p_i)$ , for  $i = 1, 2, 3$ , where  $q_i$  is chosen modulo  $p_i$  to satisfy  $1 \leq q_i < p_i$ .

### 3 Eta invariants and Einstein metrics

In this section, we derive the general formula for the  $\eta$ -invariant given in Theorem 1.3 and then prove the nonexistence result for Einstein metrics stated in Theorem 1.5.

#### 3A Proof of Theorem 1.3

Recall that, up to conjugation in  $SO(4)$ , the finite subgroups of  $SO(4)$  which act freely on  $S^3$  are the finite subgroups of  $U(2)$  which act freely on  $S^3$  and their orientation-reversed conjugates [38].

We first discuss the  $\eta$ -invariant of cyclic groups. For relatively prime integers which satisfy  $1 \leq q < p$ , the following formula is proved in [3; 34]:

$$(3-1) \quad \eta(S^3/L(q, p)) = \frac{1}{3} \left( \sum_{i=1}^k e_i + \frac{q^{-1;p} + q}{p} \right) - k,$$

where  $k$  and the  $e_i$  are as defined in (2-6), and  $q^{-1;p}$  denotes the inverse of  $q \pmod p$ .

**Remark 3.1** Let  $\Gamma \subset U(2)$  be a finite subgroup which acts freely on  $S^3$ , and let  $\bar{\Gamma} \subset SO(4)$  be its orientation-reversed conjugate. We have  $\eta(S^3/\Gamma) = -\eta(S^3/\bar{\Gamma})$ . Therefore, since all possible cyclic groups are orientation-preserving conjugate to some  $L(q, p)$ , and the  $\eta$ -invariants of these are given by (3-1), the theorem will follow from finding the  $\eta$ -invariants of all noncyclic finite subgroups of  $U(2)$  which act freely on  $S^3$ .

Since  $\tau_{\text{top}}(\mathbb{C}\mathbb{P}^2_{(1,1,2m)}, g_{\text{BK}}) = 1$ , from (1-3) and (3-1), observe that the orbifold signature is

$$(3-2) \quad \tau_{\text{orb}}(\mathbb{C}\mathbb{P}^2_{(1,1,2m)}) = \eta(1, 2m) + 1 = \frac{2m^2 + 1}{3m}.$$

Let  $\Gamma \subset \text{U}(2)$  be a finite subgroup which acts freely on  $S^3$  and consider  $\tilde{\Gamma}$ , where

$$(3-3) \quad \tilde{\Gamma} = \Gamma/L(1, 2m) \subset \text{U}(2)/L(1, 2m).$$

Clearly,  $\tilde{\Gamma}$  acts effectively on  $\mathbb{C}\mathbb{P}^2_{(1,1,2m)}$  and  $\mathbb{C}\mathbb{P}^2_{(1,1,2m)}/\tilde{\Gamma} = \mathbb{C}\mathbb{P}^2_{(1,1,2m)}/\Gamma$  is an orbifold with orbifold groups  $\Gamma$  at the point of compactification and  $L(q_i, p_i)$ , for  $i = 1, 2, 3$ , on the rational curve  $\Sigma$  as specified in Theorem 2.2. From the orbifold signature theorem (1-3), since

$$\tau_{\text{top}}(\mathbb{C}\mathbb{P}^2_{(1,1,2m)}/\tilde{\Gamma}) = 1 \quad \text{and} \quad \tau_{\text{orb}}(\mathbb{C}\mathbb{P}^2_{(1,1,2m)}/\tilde{\Gamma}) = \frac{1}{|\tilde{\Gamma}|} \left( \frac{2m^2 + 1}{3m} \right),$$

we find that

$$(3-4) \quad \eta(S^3/\Gamma) = \frac{1}{|\tilde{\Gamma}|} \left( \frac{2m^2 + 1}{3m} \right) - 1 - \sum_{i=1}^3 \eta(L(q_i, p_i)).$$

The proof is completed for each case by using the appropriate singularities as specified in Theorem 2.2 and then computing the corresponding cyclic  $\eta$ -invariants for the particular congruences of  $m$ .

For instance, when  $\Gamma = \phi(L(1, 2m) \times T^*)$ , observe from Theorem 2.2 that the singularities are of types  $L(1, 2)$ ,  $L(m, 3)$ , and  $L(m, 3)$ . Therefore

$$\begin{aligned} \eta(S^3/\phi(L(1, 2m) \times T^*)) &= \frac{1}{12} \left( \frac{2m^2 + 1}{3m} \right) - 1 - \eta(S^3/L(1, 2)) - 2\eta(S^3/L(m, 3)) \\ &= \frac{1}{12} \left( \frac{2m^2 + 1}{3m} \right) - 1 \pm \frac{4}{9} \quad \text{for } m \equiv \pm 1 \pmod{6}. \end{aligned}$$

The idea for the other cases is identical and the computations, which follow similarly, are omitted.

**Remark 3.2** The  $\eta$ -invariants of the binary polyhedral groups were known [37]. However, their direct computation is arduous and here they are recovered simply.

### 3B Proof of Theorem 1.5

We will use Theorem 1.3 along with the following ALE analogue of the Hitchin–Thorpe inequality due to Nakajima. Let  $(M, g)$  be a Ricci-flat ALE manifold with group at

infinity  $\Gamma$ , then

$$(3-5) \quad 2(\chi_{\text{top}}(M) - 1/|\Gamma|) \geq 3|\tau_{\text{top}}(M) - \eta(S^3/\Gamma)|,$$

with equality if and only if  $W^+$  or  $W^-$  vanishes identically [37] (see also [21]).

First, let  $X$  be diffeomorphic to the minimal resolution of  $\mathbb{C}^2/\Gamma$  for some finite subgroup  $\Gamma \subset \text{U}(2)$  which acts freely on  $S^3$ . For relatively prime integers satisfying the inequality  $1 \leq q < p$ , let  $k_{(q,p)}$  denote the length, and  $e_j^{(q,p)}$  the coefficients, of the modified Euclidean algorithm (2-6) for  $(q, p)$ . Then

$$(3-6) \quad \begin{aligned} \chi_{\text{top}}(X) &= 1 - \tau_{\text{top}}(X), \text{ and} \\ \tau_{\text{top}}(X) &= \begin{cases} -1 - \sum_{i=1}^3 k_{(p_i-q_i, p_i)} & \text{if } \Gamma \text{ is noncyclic,} \\ -k_{(q,p)} & \text{if } \Gamma = L(q, p), \end{cases} \end{aligned}$$

where the  $L(p_i - q_i, p_i)$ , for  $i = 1, 2, 3$ , are orientation-reversed orbifold groups of those given for  $\Gamma$  in Theorem 2.2. These orbifold groups are orientation-reversed conjugate to those on the quotient of weighted projective space appearing in the proof of Theorem 1.3, so rewriting (3-4) with respect to these groups here just reverses the sign of the  $\eta$ -invariants of the cyclic groups appearing in the sum.

For noncyclic  $\Gamma$  there is the term  $\sum_{i=1}^3 k_{(p_i-q_i, p_i)}$  in both  $\tau_{\text{top}}(X)$  and  $\eta(S^3/\Gamma)$ . This follows from (3-1), (3-4), and (3-6). Therefore, we compute that

$$\begin{aligned} &\tau_{\text{top}}(X) - \eta(S^3/\Gamma) \\ &= -\frac{1}{|\tilde{\Gamma}|} \left( \frac{2m^2 + 1}{3m} \right) - \frac{1}{3} \sum_{i=1}^3 \left( \sum_{j=1}^{k_{(p_i-q_i, p_i)}} e_j^{(p_i-q_i, p_i)} \right) - \frac{1}{3} \sum_{i=1}^3 \frac{(p_i - q_i)^{-1; p_i} + (p_i - q_i)}{p_i}. \end{aligned}$$

Since each summand here is positive, rewrite (3-5) as

$$(3-7) \quad \begin{aligned} &2 \sum_{i=1}^3 k_{(p_i-q_i, p_i)} + 4 - \frac{2}{|\Gamma|} \\ &\geq \frac{1}{|\tilde{\Gamma}|} \left( \frac{2m^2 + 1}{m} \right) + \sum_{i=1}^3 \left( \sum_{j=1}^{k_{(p_i-q_i, p_i)}} e_j^{(p_i-q_i, p_i)} \right) + \sum_{i=1}^3 \frac{(p_i - q_i)^{-1; p_i} + (p_i - q_i)}{p_i}. \end{aligned}$$

Observe that for any noncyclic  $\Gamma \subset \text{SU}(2)$  equality holds in (3-7), so any Ricci-flat metric in this case must also be anti-self-dual. Therefore, these are the hyperkähler ALE metrics classified by Kronheimer [28; 29]. Now we will show that (3-7) is violated for all noncyclic  $\Gamma \not\subset \text{SU}(2)$ . Recall, from (2-6), that all the  $e_j^{(p_i-q_i, p_i)}$  are at least 2. Since  $\Gamma \not\subset \text{SU}(2)$  is noncyclic, for each  $L(p_i - q_i, p_i)$  at least one of the  $e_j^{(p_i-q_i, p_i)}$



is strictly greater than 2, so the problem reduces to proving that the inequality

$$(3-8) \quad 1 - \frac{2}{|\Gamma|} \geq \frac{1}{|\tilde{\Gamma}|} \left( \frac{2m^2 + 1}{m} \right) + \sum_{i=1}^3 \frac{(p_i - q_i)^{-1;p_i} + (p_i - q_i)}{p_i}$$

is violated. Finally, observe that this is so since for all such groups at least one of the  $L(p_i - q_i, p_i)$  is  $L(1, 2)$ , for which

$$\frac{(p_i - q_i)^{-1;p_i} + (p_i - q_i)}{p_i} = 1.$$

If  $\Gamma = L(q, p)$  is cyclic, (3-5) reduces to

$$(3-9) \quad 2 \left( k_{(q,p)} + 1 - \frac{1}{p} \right) \geq \sum_{j=1}^{k_{(q,p)}} e_j + \frac{q^{-1;p} + q}{p}.$$

Clearly, this inequality holds if and only if  $\Gamma = L(-1, p)$ , in which case it holds with equality, so any Ricci-flat metric in this case must also be anti-self-dual. Therefore, these are the hyperkähler Gibbons–Hawking multi-Eguchi–Hanson metrics.

Finally, let  $X$  be diffeomorphic to the iterated blow-up of the minimal resolution of  $\mathbb{C}^2/\Gamma$ . In this case, (3-5) reduces to inequalities (3-7) and (3-8) with  $2\ell$  and  $3\ell$  added to the left- and right-hand sides of each, respectively, where  $\ell \geq 1$  is the number of blow-ups. Given the previous arguments, it is easy to see that (3-5) is always violated, and therefore no Ricci-flat metrics can exist.

## 4 Self-dual deformations

In Section 4A the proof of Theorem 1.6 is given. Then, using this along with our previous work in [34], we prove Theorem 1.7 in Section 4B.

### 4A Proof of Theorem 1.6

In [34], we proved an index formula for the anti-self-dual deformation complex on an orbifold with isolated cyclic singularities, which is easily adjusted to find an index formula for the self-dual deformation complex. As an intermediate step to this, we showed that if  $(M, g)$  is a compact self-dual orbifold with a finite number of isolated singularities  $p_1, \dots, p_n$  having corresponding orbifold groups  $\Gamma_1, \dots, \Gamma_n$ , where each  $\Gamma_i \subset \text{SO}(4)$  is any finite subgroup which acts freely on  $S^3$ , then the index of the self-dual deformation complex can be expressed by a topological quantity and a correction

term depending only on the  $\Gamma_i$  as follows:

$$(4-1) \quad \text{Ind}(M, g) = \frac{1}{2}(15\chi(M) - 29\tau(M)) + \sum_{i=1}^n N(\Gamma_i).$$

For  $\Gamma = L(q, p)$ , we proved that

$$(4-2) \quad N(L(q, p)) = \begin{cases} -\sum_{i=1}^k 4e_i + 12k - 10 & \text{if } q > 1, \\ -4p + 4 & \text{if } q = 1, \end{cases}$$

where  $k$  and the  $e_i$  are as in the modified Euclidean algorithm (2-6).

Recall that, up to conjugation in  $\text{SO}(4)$ , the set of finite subgroups of  $\text{SO}(4)$  which act freely on  $S^3$  are given by the finite subgroups of  $\text{U}(2)$  which act freely on  $S^3$  and their orientation-reversed conjugates [38]. From the following lemma, we see that it will be enough to find the correction terms for those subgroups of  $\text{U}(2)$ .

**Lemma 4.1** *Let  $\Gamma \subset \text{U}(2)$  be a noncyclic finite subgroup which acts freely on  $S^3$ , and let  $\bar{\Gamma} \subset \text{SO}(4)$  be its orientation-reversed conjugate group. Then the self-dual index correction term for  $\bar{\Gamma}$  is given in terms of that for  $\Gamma$  by*

$$N(\bar{\Gamma}) = -N(\Gamma) - \begin{cases} 7 & \text{if } \Gamma \not\subset \text{SU}(2), \\ 5 & \text{if } \Gamma \subset \text{SU}(2). \end{cases}$$

**Proof** Consider the quotient  $(S^4, g)/\Gamma$ , where  $g$  is the standard round metric and  $\Gamma$  acts around the  $x_5$ -axis. This is a self-dual orbifold with two singularities, one of type  $\Gamma$  and the other of type  $\bar{\Gamma}$ . It is well known that both  $H_{\text{SD}}^1$  and  $H_{\text{SD}}^2$  of the self-dual deformation complex vanish in this case, thus the index is given by  $\dim H_{\text{SD}}^0$ . From [36], we have that  $\dim H_{\text{SD}}^0 = 1$  if  $\Gamma \not\subset \text{SU}(2)$  and  $\Gamma$  is not cyclic, and  $\dim H_{\text{SD}}^0 = 3$  if  $\Gamma \subset \text{SU}(2)$  and  $\Gamma$  is not cyclic. Equating this with the index obtained from (4-1), since  $\chi_{\text{top}}(S^4/\Gamma) = 2$  and  $\tau_{\text{top}}(S^4/\Gamma) = 0$ , we find that

$$(4-3) \quad 8 + N(\Gamma) + N(\bar{\Gamma}) = \begin{cases} 1 & \text{if } \Gamma \not\subset \text{SU}(2), \\ 2 & \text{if } \Gamma \subset \text{SU}(2), \end{cases}$$

from which we can solve for  $N(\bar{\Gamma})$  to complete the proof. □

Therefore, to complete the proof of Theorem 1.6, it is only necessary to find the correction term for finite subgroups of  $\text{U}(2)$  which act freely on  $S^3$ , and the plan for the remainder of the proof is as follows. For any finite subgroup  $\Gamma \subset \text{U}(2)$  which acts freely on  $S^3$ , recall the orbifold quotients  $(\mathbb{C}\mathbb{P}_{(1,1,2m)}^2, g_{\text{BK}})/\Gamma$  from Theorem 2.2. In [23], Honda discovered the explicit form of the  $\text{U}(2)$ -action on  $H_{\text{SD}}^1$  of the self-dual deformation complex of  $(\mathbb{C}\mathbb{P}_{(1,1,2m)}^2, g_{\text{BK}})$ . Applying a representation-theoretic argument to this, we find the dimension of the space of invariant elements of  $H_{\text{SD}}^1$  under

the quotient by  $\Gamma$ . Since finding  $\dim H_{SD}^1$  of the self-dual deformation complex on the quotient  $(\mathbb{C}P_{(1,1,2m)}^2, g_{BK})/\Gamma$  is equivalent to finding the dimension of the space of invariant elements of  $H_{SD}^1$  on  $(\mathbb{C}P_{(1,1,2m)}^2, g_{BK})$  under the action of  $\Gamma$ , we then use this to recover the index. Finally we solve for  $N(\Gamma)$  in terms of the index, which at this point is known, and the correction terms of the cyclic quotient singularities, which are known from (4-2), that arise in the quotient.

In the following proposition we find the dimension of  $H_{SD}^1$  of  $(\mathbb{C}P_{(1,1,2m)}^2, g_{BK})/\Gamma$ , which we are able to give simply in terms of  $b_\Gamma$  (the negative of the self-intersection number of the central rational curve).

**Proposition 4.2** *Let  $\Gamma \subset U(2)$  be a noncyclic finite subgroup which acts freely on  $S^3$ . Then the dimension of the space of the first cohomology group of the self-dual deformation complex on  $(\mathbb{C}P_{(1,1,2m)}^2, g_{BK})/\Gamma$  is given by*

$$\dim H_{SD,\Gamma}^1 = 4b_\Gamma - C_\Gamma = \frac{16m}{|\Gamma|} \left[ m - \left( m \bmod \frac{|\Gamma|}{4m} \right) \right] + 8 - C_\Gamma,$$

where  $C_\Gamma$  is a constant given by the following table:

| $\Gamma \subset U(2)$   | $C_\Gamma$ | congruences                               |
|---|------------|---|
| $\phi(L(1, 2m) \times D_{4n}^*), \mathcal{I}_{m,n}^2$                   | 6          | $m \not\equiv 1 \pmod{ \Gamma /(4m)}$     |
|   | 8          | $m \equiv 1 \pmod{ \Gamma /(4m)}$         |
| $\phi(L(1, 2m) \times T^*), \phi(L(1, 2m) \times O^*), \mathcal{I}_m^3$ | 4          | $m \equiv -1 \pmod{ \Gamma /(4m)}$        |
|   | 6          | $m \not\equiv \pm 1 \pmod{ \Gamma /(4m)}$ |
|   | 8          | $m \equiv 1 \pmod{ \Gamma /(4m)}$         |
| $\phi(L(1, 2m) \times I^*)$   | 4          | $m \equiv 17, 23, 29 \pmod{30}$           |
|   | 6          | $m \equiv 7, 11, 13, 19 \pmod{30}$        |
|   | 8          | $m \equiv 1 \pmod{30}$                    |

**Proof** The space  $(\mathbb{C}P_{(1,1,2)}^2, g_{BK})$  is the Kähler conformal compactification of the Eguchi–Hanson metric on  $\mathcal{O}(-2)$  for which it is well known that  $\dim H_{SD}^1 = 0$ . This is the  $m = 1$  case. For  $m > 1$ , Honda showed that the complexification of  $H_{SD}^1$  of the self-dual deformation complex on  $(\mathbb{C}P_{(1,1,2m)}^2, g_{BK})$  is equivalent to

$$(4-4) \quad \rho \oplus \bar{\rho}, \quad \text{where } \rho = (S^{2m-2}(\mathbb{C}^2) \otimes \det) \oplus (S^{2m-4}(\mathbb{C}^2) \otimes \det^2)$$

as a representation space of  $U(2)$  (see [23]). The dimension of the space of invariant elements of  $H_{SD}^1$  under the action of  $\Gamma$  is equal to that under the action of any subgroup  $\Gamma' \subset \Gamma$  that has the same effective action as  $\Gamma$  and is given by

$$(4-5) \quad \dim H_{SD,\Gamma}^1 = \dim H_{SD,\Gamma'}^1 = \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} (\chi_\rho(\gamma) + \chi_{\bar{\rho}}(\gamma)) = \frac{2}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \chi_\rho(\gamma),$$

where  $\chi_\rho(\gamma)$  and  $\chi_{\bar{\rho}}(\gamma)$  denote the characters of  $\gamma$  with respect to each representation.

Since the eigenvalues of any  $\gamma \in U(2)$  can be written as  $\{z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}\}$  for some  $0 \leq \theta_1, \theta_2 < 2\pi$ , observe that

$$(4-6) \quad \chi_\rho(\gamma) = (z_1 z_2) \sum_{p=0}^{2m-2} z_1^{2m-2-p} z_2^p + (z_1 z_2)^2 \sum_{p=0}^{2m-4} z_1^{2m-4-p} z_2^p.$$

Clearly,  $\dim H_{SD}^1 = 2\chi_\rho(\pm Id) = 8m - 8$ .

In order to compute (4-5), we introduce a number-theoretic function and identity.

- For  $x \in \mathbb{R}$ , the sawtooth function is defined as

$$(4-7) \quad ((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases}$$

where  $[x]$  denotes the greatest integer less than  $x$ .

- For  $n, k$ , with  $k \geq 1$ , the following identity is due to Eisenstein (see [2]):

$$(4-8) \quad \sum_{j=1}^{n-1} \sin((2\pi k/n)j) \cot((\pi/n)j) = -2n((k/n)).$$

We first consider those groups  $\Gamma$  that are the image under  $\phi$  of the product of  $L(1, 2m)$  and a binary polyhedral group, and let  $\Gamma' \subset \Gamma$  be the subgroup that is the binary polyhedral group itself. Since each  $\Gamma'$  is a subgroup of  $SU(2)$ , all elements  $\gamma$  of  $\Gamma'$  have eigenvalues of the form  $\{z = e^{i\theta_\gamma}, z^{-1} = e^{-i\theta_\gamma}\}$ . Thus, for  $\gamma \neq \pm 1$ , the character (4-6) reduces to

$$(4-9) \quad \chi_\rho(\gamma) = \sum_{p=0}^{2m-2} z^{2m-2-2p} + \sum_{p=0}^{2m-4} z^{2m-4-2p} = 2 \sin((2m-2)\theta_\gamma) \cot(\theta_\gamma),$$

and therefore

$$(4-10) \quad \begin{aligned} \dim H_{SD,\Gamma}^1 &= \dim H_{SD,\Gamma'}^1 \\ &= \frac{4}{|\Gamma'|} \left[ (4m-4) + \sum_{\gamma \neq \pm Id \in \Gamma'} \sin((2m-2)\theta_\gamma) \cot(\theta_\gamma) \right]. \end{aligned}$$

Now  $\dim H_{SD,\Gamma'}^1$  will be found for each binary polyhedral group separately. We provide the eigenvalue decomposition of each group in the form of a list of sets of eigenvalues along with multiplicities, where the multiplicity of a particular set is the number of times the eigenvalues of that set appear in the set of all eigenvalues of the group. Grouping the eigenvalues as such will allow us to use (4-8) to compute (4-10). The eigenvalue

decompositions of the binary polyhedral groups are found easily by examining their well-known decomposition into conjugacy classes (see [39]).

$\Gamma' = D_{4n}^*$  The eigenvalue decomposition of the binary dihedral group is given by:

| set  | multiplicity |
|--|--------------|
| $S_1 = \{ \{ e^{\frac{\pi ik}{n}}, e^{-\frac{\pi ik}{n}} \} \}_{1 \leq k \leq 2n}$ | 1            |
| $S_2 = \{ \{ i, -i \} \}$  | $2n$         |

Summing the characters (4-9) of the elements of  $S_1$  that are not  $\pm Id$  gives two copies of the sum (4-8) where  $k = m - 1$ . The element of  $S_2$  does not contribute to the sum in (4-10) since  $((1/2)) = 0$ . Therefore

$$\begin{aligned} \dim H_{SD, D_{4n}^*}^1 &= \frac{1}{n} [(4m - 4) - 4n((m - 1)/n)] \\ &= \begin{cases} 4[(m - 1)/n] + 2 = 4b_{\phi(L(1,2m) \times D_{4n}^*)} - 6 & \text{if } m \not\equiv 1 \pmod{n}, \\ 4(m - 1)/n = 4b_{\phi(L(1,2m) \times D_{4n}^*)} - 8 & \text{if } m \equiv 1 \pmod{n}. \end{cases} \end{aligned}$$

$\Gamma' = T^*$  The eigenvalue decomposition of the binary tetrahedral group is given by:

| set  | multiplicity |
|--|--------------|
| $S_1 = \{ \{ 1, 1 \}, \{ -1, -1 \} \}$   | 1            |
| $S_2 = \{ \{ e^{\frac{\pi i}{3}}, e^{-\frac{\pi i}{3}} \}, \{ e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}} \} \}$ | 8            |
| $S_3 = \{ \{ i, -i \} \}$  | 6            |

Use (4-8) to sum the characters of the elements of  $S_2$  as given by (4-9). The element of  $S_3$  does not contribute to the sum in (4-10) since  $((1/2)) = 0$ . Therefore, adjusting these sums according to the appropriate multiplicities, we find that

$$\begin{aligned} \dim H_{SD, T^*}^1 &= \frac{1}{6} [(4m - 4) - 48((m - 1)/3)] \\ &= \begin{cases} \frac{4}{6}(m + 1) = 4b_{\phi(L(1,2m) \times T^*)} - 4 & \text{if } m \equiv 5 \pmod{6}, \\ \frac{4}{6}(m - 1) = 4b_{\phi(L(1,2m) \times T^*)} - 8 & \text{if } m \equiv 1 \pmod{6}. \end{cases} \end{aligned}$$

$\Gamma' = O^*$  The eigenvalue decomposition of the binary octahedral group is given by:

| set   | multiplicity |
|---|--------------|
| $S_1 = \{ \{ 1, 1 \}, \{ -1, -1 \} \}$  | 1            |
| $S_2 = \{ \{ e^{\frac{\pi i}{3}}, e^{-\frac{\pi i}{3}} \}, \{ e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}} \} \}$              | 8            |
| $S_3 = \{ \{ e^{\frac{\pi i}{4}}, e^{-\frac{\pi i}{4}} \}, \{ i, -i \}, \{ e^{\frac{3\pi i}{4}}, e^{-\frac{3\pi i}{4}} \} \}$ | 6            |
| $S_4 = \{ \{ i, -i \} \}$   | 12           |

Use (4-8) to sum the characters of the elements of  $S_2$  and  $S_3$  as given by (4-9). The element of  $S_4$  does not contribute to the sum in (4-10) since  $((1/2)) = 0$ . Therefore,

adjusting these sums according to the appropriate multiplicities, we find that

$$\dim H_{SD, O^*}^1 = \frac{1}{12} \left[ (4m - 4) - 48 \left( \left( \frac{1}{3}(m - 1) \right) \right) - 48 \left( \left( \frac{1}{4}(m - 1) \right) \right) \right]$$

$$= \begin{cases} \frac{1}{3}(m + 1) = b_{\phi(L(1,2m) \times O^*)} - 4 & \text{if } m \equiv 11 \pmod{12}, \\ \frac{1}{3}(m - 1) = b_{\phi(L(1,2m) \times O^*)} - 6 & \text{if } m \equiv 7 \pmod{12}, \\ \frac{1}{3}(m + 1) = b_{\phi(L(1,2m) \times O^*)} - 6 & \text{if } m \equiv 5 \pmod{12}, \\ \frac{1}{3}(m - 1) = b_{\phi(L(1,2m) \times O^*)} - 8 & \text{if } m \equiv 1 \pmod{12}. \end{cases}$$

$\Gamma' = I^*$  The eigenvalue decomposition of the binary octahedral group is given by:

| set  | multiplicity |
|--|--------------|
| $S_1 = \{\{1, 1\}, \{-1, -1\}\}$   | 1            |
| $S_2 = \{\{e^{\frac{\pi i}{3}}, e^{-\frac{\pi i}{3}}\}, \{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}\}$   | 20           |
| $S_3 = \{\{e^{\frac{\pi i}{5}}, e^{-\frac{\pi i}{5}}\}, \{e^{\frac{2\pi i}{5}}, e^{-\frac{2\pi i}{5}}\}, \{e^{\frac{3\pi i}{5}}, e^{-\frac{3\pi i}{5}}\}, \{e^{\frac{4\pi i}{5}}, e^{-\frac{4\pi i}{5}}\}\}$ | 12           |
| $S_4 = \{\{i, -i\}\}$  | 30           |

Use (4-8) to sum the characters of the elements of  $S_2$  and  $S_3$  as given by (4-9). The element of  $S_4$  does not contribute to the sum in (4-10) since  $((1/2)) = 0$ . Therefore, adjusting these sums according to the appropriate multiplicities, we find that

$$\dim H_{SD, I^*}^1 = \frac{1}{30} \left[ (4m - 4) - 120 \left( \left( \frac{1}{3}(m - 1) \right) \right) - 120 \left( \left( \frac{1}{5}(m - 1) \right) \right) \right]$$

$$= \begin{cases} \frac{2}{15}(m + 1) = 4b_{\phi(L(1,2m) \times I^*)} - 4 & \text{if } m \equiv 29 \pmod{30}, \\ \frac{2}{15}(m + 7) = 4b_{\phi(L(1,2m) \times I^*)} - 4 & \text{if } m \equiv 23 \pmod{30}, \\ \frac{2}{15}(m + 13) = 4b_{\phi(L(1,2m) \times I^*)} - 4 & \text{if } m \equiv 17 \pmod{30}, \\ \frac{2}{15}(m - 4) = 4b_{\phi(L(1,2m) \times I^*)} - 6 & \text{if } m \equiv 19 \pmod{30}, \\ \frac{2}{15}(m + 2) = 4b_{\phi(L(1,2m) \times I^*)} - 6 & \text{if } m \equiv 13 \pmod{30}, \\ \frac{2}{15}(m + 8) = 4b_{\phi(L(1,2m) \times I^*)} - 6 & \text{if } m \equiv 7 \pmod{30}, \\ \frac{2}{15}(m + 4) = 4b_{\phi(L(1,2m) \times I^*)} - 6 & \text{if } m \equiv 11 \pmod{30}, \\ \frac{2}{15}(m - 1) = 4b_{\phi(L(1,2m) \times I^*)} - 8 & \text{if } m \equiv 1 \pmod{30}. \end{cases}$$

For the  $\mathfrak{I}_{m,n}^2$  and  $\mathfrak{I}_m^3$  cases, we will compute (4-5) over the entire group  $\Gamma$  since there is not a clear subgroup to play the role of  $\Gamma'$  as above. Since these groups are not contained in  $SU(2)$ , not all  $\gamma \in \Gamma$  have eigenvalues of the form  $\{e^{i\theta_\gamma}, e^{-i\theta_\gamma}\}$ , and therefore (4-10) does not hold. In [35, Proposition 6.1], we performed an eigenvalue decomposition of these groups, which will be used to find  $\dim H_{SD, \Gamma}^1$ , and from which we know that all

elements of both groups have eigenvalues of the form  $\{e^{i(\theta_1+\theta_2)}, e^{i(\theta_1-\theta_2)}\}$ . Therefore, we compute

$$(4-11) \quad \chi_\rho(\gamma) = e^{i(2m\theta_1)} \left[ \sum_{p=0}^{2m-2} (e^{i\theta_2})^{2m-2-2p} + \sum_{p=0}^{2m-4} (e^{i\theta_2})^{2m-4-2p} \right]$$

$$= \begin{cases} e^{i(2m\theta_1)}(4m-4) & \text{if } \theta_2 = \ell\pi \text{ for } \ell \in \mathbb{Z}, \\ e^{i(2m\theta_1)}[2 \sin((2m-2)\theta_2) \cot(\theta_2)] & \text{if } \theta_2 \neq \ell\pi \text{ for } \ell \in \mathbb{Z}. \end{cases}$$

$\Gamma = \mathfrak{J}_{m,n}^2$  The eigenvalue decomposition of  $\mathfrak{J}_{m,n}^2$  is given by:

| set   | multiplicity |
|---|--------------|
| $S_1 = \{(-1)^\ell \{e^{\pi i(\frac{\ell}{m} + \frac{k}{n})}, e^{\pi i(\frac{\ell}{m} - \frac{k}{n})}\}\}_{0 \leq \ell \leq m-1 \text{ and } 0 \leq k \leq 2n-1}$         | 1            |
| $S_2 = \{(-1)^\ell \{e^{\pi i(\frac{2\ell+1}{2m} + \frac{1}{2})}, e^{\pi i(\frac{2\ell+1}{2m} - \frac{1}{2})}\}\}_{0 \leq \ell \leq m-1 \text{ and } 0 \leq k \leq 2n-1}$ | 1            |

Using (4-11), the characters are found to be:

$$\chi_\rho(\gamma \in S_1) = \begin{cases} 2 \sin\left(\frac{2\pi}{n} k(m-1)\right) \cot\left(\frac{\pi}{n} k\right) & \text{if } k \neq 0, k \neq n, \\ 4m-4 & \text{if } k = 0, k = n, \end{cases}$$

$$\chi_\rho(\gamma \in S_2) = 0 \quad \text{since } \theta_2 = \frac{\pi}{2}.$$

Then, using (4-8) to sum the characters, we find that

$$\dim H_{SD, \mathfrak{J}_{m,n}^2}^1 = \frac{1}{2mn} [2m \cdot (4m-4) - 8mn((m-1)/n)]$$

$$= \begin{cases} 4[(m-1)/n] + 2 = 4b_{\mathfrak{J}_{m,n}^2} - 6 & \text{if } m \not\equiv 1 \pmod n, \\ \frac{4}{n}(m-1) = 4b_{\mathfrak{J}_{m,n}^2} - 8 & \text{if } m \equiv 1 \pmod n. \end{cases}$$

$\Gamma = \mathfrak{J}_m^3$  The eigenvalue decomposition of  $\mathfrak{J}_m^3$  is given by:

| set   | multiplicity |
|---|--------------|
| $S_1 = \{\pm e^{\frac{\pi i r}{m}} \{1, 1\}\}_{0 \leq r < m}$   | 1            |
| $S_2 = \{\pm e^{\frac{\pi i r}{m}} \{i, -i\}\}_{0 \leq r < m}$  | 3            |
| $S_3 = \{\pm e^{\frac{\pi i(3r+1)}{3m}} \{e^{\frac{\pi i}{3}}, e^{-\frac{\pi i}{3}}\}\}_{0 \leq r < m}$   | 2            |
| $S_4 = \{\pm e^{\frac{\pi i(3r+1)}{3m}} \{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}\}_{0 \leq r < m}$ | 2            |
| $S_5 = \{\pm e^{\frac{\pi i(3r+2)}{3m}} \{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}\}_{0 \leq r < m}$ | 4            |

Using (4-11), the characters are found to be

$$\begin{aligned}
 \chi_\rho(\gamma \in S_1) &= 4m - 4, \\
 \chi_\rho(\gamma \in S_2) &= 0, \\
 \chi_\rho(\gamma \in S_3) &= 2e^{\frac{2\pi i}{3}} \sin(2(m-1)\pi/3) \cot(\pi/3) = -e^{\frac{2\pi i}{3}}, \\
 \chi_\rho(\gamma \in S_4) &= 2e^{\frac{2\pi i}{3}} \sin(2(m-1)2\pi/3) \cot(2\pi/3) = -e^{\frac{2\pi i}{3}}, \\
 \chi_\rho(\gamma \in S_5) &= 2e^{\frac{4\pi i}{3}} \sin(2(m-1)2\pi/3) \cot(2\pi/3) = -e^{\frac{4\pi i}{3}}.
 \end{aligned}
 \tag{4-12}$$

The evaluation of the last three characters follows from the equivalence  $m \equiv 0 \pmod 3$ . Since all eigenvalues contained in the same set have equal characters, by multiplying each character found in (4-12) by  $2m$ , which is the size of each set, and the multiplicity of the respective set, and summing, we find that

$$\begin{aligned}
 \dim H_{SD, \mathcal{J}_m^3}^1 &= \frac{1}{12m} [2m \cdot (4m - 4) - 8m(e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}})] = \frac{2}{3}m \\
 &= 4b_{\mathcal{J}_m^3} - 6. \quad \square
 \end{aligned}$$

Now we complete the proof of Theorem 1.6, by finding  $N(\Gamma)$ , the nontopological correction term for the index, for all finite noncyclic subgroups  $\Gamma$  of  $U(2)$ . The cohomology groups of the self-dual deformation complex on the quotient  $(\mathbb{C}P^2_{(1,1,2m)}, g_{BK})/\Gamma$  correspond to the invariant elements of the respective cohomology groups on the cover, which we denote by  $H_{SD, \Gamma}^i$  for  $i = 1, 2, 3$ . Given  $\dim H_{SD, \Gamma}^1$ , found in Proposition 4.2, it is only necessary to find  $\dim H_{SD, \Gamma}^0$  and  $\dim H_{SD, \Gamma}^2$  to recover the index.

The second cohomology group  $H_{SD}^2$  of the self-dual deformation complex for the Bochner–Kähler metric on the weighted projective space  $(\mathbb{C}P^2_{(1,1,2m)}, g_{BK})$  vanishes by [32, Theorem 4.2], and therefore in the quotient it vanishes as well. The cohomology group  $H_{SD, \Gamma}^0$  is given by the elements of  $H_{SD}^0$  on the cover that commute with the respective  $\Gamma$ . The  $S^1$  given by  $\phi(e^{i\theta}, 1)$  is contained in the centralizer of all the  $\Gamma$ , and it is easy to check that, for all noncyclic  $\Gamma$ , these are the only elements of  $H_{SD}^0$  which commute with the respective  $\Gamma$ , so  $\dim H_{SD, \Gamma}^0 = 1$ . Therefore, the index is

$$\text{Ind}((\mathbb{C}P^2_{(1,1,2m)}, g_{BK})/\Gamma) = 1 - \dim H_{SD, \Gamma}^1.
 \tag{4-13}$$

The quotient  $(\mathbb{C}P^2_{(1,1,2m)}, g_{BK})/\Gamma$  has four singularities with orbifold groups  $\Gamma$  and  $L(q_i, p_i)$ , for  $i = 1, 2, 3$ , as specified above in Theorem 2.2. Thus, from (4-1), the index is also given by

$$\text{Ind}((\mathbb{C}P^2_{(1,1,2m)}, g_{BK})/\Gamma) = \frac{1}{2}(15\chi_{\text{top}} - 29\tau_{\text{top}}) + \sum_{i=1}^3 N(L(q_i, p_i)) + N(\Gamma).
 \tag{4-14}$$



Equating (4-13) and (4-14), since  $\chi_{\text{top}} = 3$  and  $\tau_{\text{top}} = 1$ , we find that

$$(4-15) \quad N(\Gamma) = -7 - \dim H_{\text{SD},\Gamma}^1 - \sum_{i=1}^3 N(L(q_i, p_i)).$$

For each  $\Gamma$ , insert the corresponding  $\dim H_{\text{SD},\Gamma}^1$  and cyclic orbifold groups, as were found in Proposition 4.2 and Theorem 2.2, respectively. Finally, the proof is completed, for  $\Gamma \subset U(2)$ , by examining the conditions on the  $\Gamma$  and the respective possible congruences of  $m$ , and using these appropriately along with the known cyclic correction terms (4-2) to see that

$$(4-16) \quad N(\Gamma) = -4b_\Gamma + \mathcal{B}_\Gamma,$$

where the  $\mathcal{B}_\Gamma$  are constants given in Table 4.

| $\Gamma \subset U(2)$            | $\mathcal{B}_\Gamma$ | congruences                 |
|----------------------------------|----------------------|-----------------------------|
| $\phi(L(1, 2m) \times D_{4n}^*)$ | $7 - N(L(m, n))$     | $m \not\equiv 1 \pmod n$    |
|                                  | $5 + 4n$             | $m \equiv 1 \pmod n$        |
| $\phi(L(1, 2m) \times T^*)$      | 5                    | $m \equiv 5 \pmod 6$        |
|                                  | 21                   | $m \equiv 1 \pmod 6$        |
| $\phi(L(1, 2m) \times O^*)$      | 1                    | $m \equiv 11 \pmod{12}$     |
|                                  | 9                    | $m \equiv 7 \pmod{12}$      |
|                                  | 17                   | $m \equiv 5 \pmod{12}$      |
|                                  | 25                   | $m \equiv 1 \pmod{12}$      |
| $\phi(L(1, 2m) \times I^*)$      | -3                   | $m \equiv 29 \pmod{30}$     |
|                                  | 5                    | $m \equiv 19 \pmod{30}$     |
|                                  | 9                    | $m \equiv 17, 23 \pmod{30}$ |
|                                  | 17                   | $m \equiv 7, 13 \pmod{30}$  |
|                                  | 21                   | $m \equiv 11 \pmod{30}$     |
|                                  | 29                   | $m \equiv 1 \pmod{30}$      |
| $\mathfrak{J}_{m,n}^2$           | $7 - N(L(m, n))$     | $m \not\equiv 1 \pmod n$    |
|                                  | $5 + 4n$             | $m \equiv 1 \pmod n$        |
| $\mathfrak{J}_m^3$               | 13                   | $m \equiv 3 \pmod 6$        |

Table 4: Self-dual index correction terms for  $\Gamma \subset U(2)$

We compute  $N(\phi(L(1, 2m) \times T^*))$  as an example. From Theorem 2.2 we know that the singularities are of types  $L(1, 2)$ ,  $L(m, 3)$ , and  $L(m, 3)$ , so using (4-2) to find the correction terms for these singularities along with  $H_{\text{SD},\Gamma}^1$  as determined in

Proposition 4.2, observe that

$$\begin{aligned} N(\phi(L(1, 2m) \times T^*)) &= -7 - N(L(1, 2)) - 2N(L(m, 3)) - 4b_{\phi(L(1,2m) \times T^*)} + \begin{cases} 8 & \text{if } m \equiv 1 \pmod 6, \\ 6 & \text{if } m \equiv 5 \pmod 6 \end{cases} \\ &= -4b_{\phi(L(1,2m) \times T^*)} + \begin{cases} 21 & \text{if } m \equiv 1 \pmod 6, \\ 5 & \text{if } m \equiv 5 \pmod 6. \end{cases} \end{aligned}$$

The idea for the other cases is identical and the computations, which follow similarly, are omitted.

**Remark 4.3** Although the correction terms for  $\phi(L(1, 2m) \times D_{4n}^*)$  and  $\mathfrak{J}_{m,n}^2$  contain an  $N(L(m, n))$ , they are computed algorithmically via (4-2).

**Remark 4.4** The second author found the correction terms for the binary polyhedral groups [42]. These are recovered here as well in the  $m = 1$  case.

### 4B Proof of Theorem 1.7

Let  $(X, g)$  be a scalar-flat Kähler ALE metric on the minimal resolution of  $\mathbb{C}^2/\Gamma$  for some finite subgroup  $\Gamma \subset U(2)$  which acts freely on  $S^3$ . In [35], we showed that the dimension of infinitesimal scalar-flat Kähler deformations is at most

$$(4-17) \quad d_{\Gamma, \max} = 2 \left( \sum_{i=1}^{k_{\Gamma}} (e_i - 1) \right) + k_{\Gamma} - 3,$$

where  $-e_i$  is the self-intersection number of the  $k^{\text{th}}$  rational curve; hence, to prove Theorem 1.7, we will consider the self-dual conformal compactification  $(\hat{X}, \hat{g})$ , and show that the dimension of the moduli space of self-dual conformal structures near  $\hat{g}$  is greater than or equal to  $d_{\Gamma, \max}$ , with equality if and only if  $\Gamma \subset SU(2)$ . We separate the proof into two parts, one for  $\Gamma$  noncyclic and one for  $\Gamma$  cyclic. The underlying idea is the same for each case, but it is necessary to consider them separately because of the differences between the respective  $N(\Gamma)$ .

Let  $\Gamma$  be noncyclic. To find a convenient presentation of  $d_{\Gamma, \max}$ , consider the description of the minimal resolution of  $\mathbb{C}^2/\Gamma$  from Section 2B. Recall there is the central rational curve with self-intersection  $-b_{\Gamma}$  from which three Hirzebruch–Jung strings corresponding to the singularities  $L(p_1 - q_i, p_i)$ , for  $i = 1, 2, 3$ , emanate.

Letting  $E_{(p_i - q_i, p_i)}$  denote the sum  $\sum_{j=1}^{k_{(p_i - q_i, p_i)}} e_j^{(p_i - q_i, p_i)}$ , where the  $e_j^{(p_i - q_i, p_i)}$  are the coefficients appearing in the modified Euclidean algorithm for the pair  $(p_i - q_i, p_i)$ , we see that

$$(4-18) \quad d_{\Gamma, \max} = \sum_{i=1}^3 (2E_{(p_i - q_i, p_i)} - k_{(p_i - q_i, p_i)}) + 2b_{\Gamma} - 4.$$

Now we examine the index of the self-dual deformation complex on  $(\widehat{X}, \widehat{g})$ . Recall that  $\tau_{\text{top}}(\widehat{X}) = 1 + \sum_{i=1}^3 k_{(p_i - q_i, p_i)}$  and  $\chi_{\text{top}}(\widehat{X}) = \tau_{\text{top}}(\widehat{X}) + 2$ . Thus, since  $H_{\text{SD}}^2(\widehat{X})$  of the self-dual deformation complex vanishes [32, Theorem 4.2], the index is given by

$$(4-19) \quad \text{Ind}(\widehat{X}, \widehat{g}) = \dim H_{\text{SD}}^0 - \dim H_{\text{SD}}^1 = -7 \sum_{i=1}^3 k_{(p_i - q_i, p_i)} + 8 + N(\Gamma).$$

Using (4-15) for  $N(\Gamma)$  and Proposition 4.2 for  $H_{\text{SD}, \Gamma}^1$ , and since  $\dim H_{\text{SD}}^0 = 1$ , observe that

$$(4-20) \quad \dim H_{\text{SD}}^1 = 7 \sum_{i=1}^3 k_{(p_i - q_i, p_i)} + 4b_{\Gamma} - C_{\Gamma} + \sum_{i=1}^3 N(L(q_i, p_i)),$$

where the  $C_{\Gamma}$  are constants specified in Proposition 4.2. Although  $N(\Gamma)$  is given explicitly in Table 4, it is more useful to consider (4-15) here. We would like to understand each  $N(L(q_i, p_i))$  in terms of  $N(L(p_i - q_i, p_i))$  as to be able to compare (4-20) with  $d_{\Gamma, \max}$ . In [34], we showed that, for  $1 < q_i < p_i - 1$ ,

$$(4-21) \quad N(L(q_i, p_i)) = -N(L(p_i - q_i, p_i)) - 12 = 4E_{(p_i - q_i, p_i)} - 12k_{(p_i - q_i, p_i)} - 2,$$

and from (4-1) it is clear that  $N(L(1, p_i)) = -N(L(-1, p_i)) - 10$  for  $p_i > 2$ . Therefore, we find that

$$(4-22) \quad \dim H_{\text{SD}}^1 = \sum_{i=1}^3 (4E_{(p_i - q_i, p_i)} - 5k_{(p_i - q_i, p_i)}) + 4b_{\Gamma} - C_{\Gamma} - 2\kappa,$$

where  $\kappa$  is the number of singularities not of types  $L(\pm 1, p_i)$ ; note that  $\kappa \leq 1$  with equality only in the case that  $\Gamma = \phi(L(1, 2m) \times I^*)$  and  $m \equiv \pm 2 \pmod{5}$ . Using (2-7), observe that

$$(4-23) \quad \dim H_{\text{SD}}^1 - d_{\Gamma, \max} = \sum_{i=1}^3 (2E_{(p_i - q_i, p_i)} - 4k_{(p_i - q_i, p_i)}) + \frac{8m}{|\Gamma|} \left[ m - \left( m \bmod \frac{|\Gamma|}{4m} \right) \right] - C_{\Gamma} - 2\kappa + 8.$$

For  $\Gamma \subset \text{SU}(2)$ , it is clear from (4-23) that  $\dim H_{\text{SD}}^1 = d_{\Gamma, \max}$ . These are the hyperkähler metrics (see [35, Section 8]). For  $\Gamma \not\subset \text{SU}(2)$ , each  $E_{(p_i - q_i, p_i)}$  is greater than  $2k_{(p_i - q_i, p_i)}$  since at least one of the  $e_j^{(p_i - q_i, p_i)}$  is greater than 2. Also,  $C_{\Gamma} \leq 8$ , and in particular,  $C_{\Gamma} \leq 6$  for the  $\Gamma$  where  $\kappa = 1$ . Therefore, we see that

$$(4-24) \quad \dim H_{\text{SD}}^1 - d_{\Gamma, \max} \geq \sum_{i=1}^3 (2E_{(p_i - q_i, p_i)} - 4k_{(p_i - q_i, p_i)}) > 0.$$

Now let  $\Gamma = L(q, p)$ , for some relatively prime integers satisfying  $1 \leq q < p$ , so

$$(4-25) \quad d_{\Gamma, \max} = \sum_{j=1}^{k(q,p)} 2e_j - k(q,p) - 3,$$

where the  $e_j$  are as in the modified Euclidean algorithm for the pair  $(q, p)$ . Now we examine the index of the self-dual deformation complex on  $(\hat{X}, \hat{g})$ . Recall that  $\tau_{\text{top}}(\hat{X}) = k(q,p)$  and  $\chi_{\text{top}}(\hat{X}) = k(q,p) + 2$ . Once again, from [32, Theorem 4.2], we see that  $H_{\text{SD}}^2$  vanishes. Therefore

$$(4-26) \quad \text{Ind}(\hat{X}, \hat{g}) = \dim H_{\text{SD}}^0 - \dim H_{\text{SD}}^1 = -7k(q,p) + 15 + N(L(q, p)),$$

and using (4-2) we find that

$$(4-27) \quad \dim H_{\text{SD}}^1 = \dim H_{\text{SD}}^0 + \sum_{j=1}^{k(q,p)} 4e_j - 5k(q,p) - 5 - 2\kappa,$$

where  $\kappa = 1$  if  $q = 1$  and  $\kappa = 0$  otherwise. Observe that

$$(4-28) \quad \dim H_{\text{SD}}^1 - d_{\Gamma, \max} = \dim H_{\text{SD}}^0 + \sum_{j=1}^{k(q,p)} 2e_j - 4k(q,p) - 2 - 2\kappa.$$

For  $\Gamma = L(-1, p) \subset \text{SU}(2)$ , these are the hyperkähler multi-Eguchi–Hanson metrics, and it is well known that the dimension of the moduli space is  $3(p-2) = d_{\Gamma, \max}$ . For  $\Gamma \not\subset \text{SU}(2)$ , at least one  $e_j$  is greater than 2 and thus  $\dim H_{\text{SD}}^1 - d_{\Gamma, \max} > 0$ , which completes the proof.

## 5 Self-dual constructions

It is well known that self-dual orbifolds with complementary singularities, and both with vanishing  $H_{\text{SD}}^2$  of the self-dual deformation complex, can be glued together to obtain self-dual metrics on the connected sum (see [14; 17; 33; 27; 1]). The following theorem summarizes the results of these works.

**Theorem 5.1** *Let  $(M_1, [g_1])$  and  $(M_2, [g_2])$  be compact self-dual orbifolds which have complementary singularities, ie, the respective orbifold groups are orientation-reversed conjugate to each other. If the second cohomology group of the self-dual deformation complex on each orbifold vanishes, then the connect sum  $M_1 \# M_2$ , taken at the complementary singularities, admits self-dual metrics.*

Recall the list, given in Table 3, of the finite subgroups of  $SO(4)$  which act freely on  $S^3$ , preserve the Hopf fibration and are not contained in  $U(2)$ . Since these groups do not lie in  $U(2)$ , a scalar-flat Kähler ALE space with such a group at infinity cannot exist. Thus, as posed by the second author in [42], the natural question is that of the existence of scalar-flat anti-self-dual ALE metrics with these groups at infinity. Theorem 1.8 answers this question in the affirmative for these groups, which is proved below in Section 5A. Subsequently, Corollary 1.10 is proved in Section 5B.

### 5A Proof of Theorem 1.8

Let  $\overline{\Gamma}_1 = \overline{\phi(L(1, 2m) \times D_{4n}^*)}$  and  $\overline{\Gamma}_2 = \overline{\mathcal{J}_{m,n}^2}$ . Since both preserve the Hopf fiber structure, they act isometrically on the LeBrun negative mass metrics. First, for each of these groups, we will take the quotient of some appropriate negative mass metric to obtain an orbifold ALE space with the action of the entire group at infinity. The idea here is analogous to that of [35, Theorem 4.1]. Notice that both  $\overline{\Gamma}_1$  and  $\overline{\Gamma}_2$  contain the generator  $\phi(e^{\frac{\pi i}{n}}, 1)$ , and as such, respectively for each we take the quotient of  $(\mathcal{O}(-2n), g_{LB})$  by the subgroup

$$(5-1) \quad \overline{\Gamma}_i' = \begin{cases} \langle \phi(1, e^{\frac{\pi i}{m}}), \phi(\hat{j}, 1) \rangle & \text{if } i = 1, \\ \langle \phi(\hat{j}, e^{\frac{\pi i}{2m}}) \rangle & \text{if } i = 2. \end{cases}$$

Here, the action is induced by the usual action on  $\mathbb{C}^2$  and, in particular, on the  $\mathbb{C}P^1$  at the origin it descends via the Hopf map. Since  $\overline{\Gamma}_i' \not\subset U(2)$ , the Kählerian property is not preserved in the quotient, however, the anti-self-dual property is preserved.

Since  $\hat{j} * (z_1 + z_2 \hat{j}) = \bar{z}_1 \hat{j} - \bar{z}_2$ , observe that under the Hopf map,  $\phi(\hat{j}, 1)$  acts as

$$(5-2) \quad \mathcal{H}(\phi(\hat{j}, 1)(z_1, z_2)) = -\frac{\bar{z}_2}{z_1} \in S^2 = \mathbb{C} \cup \{\infty\},$$

which is the antipodal map. Also, notice  $\phi(\hat{j}, e^{\frac{\pi i}{2m}})^2 = \phi(-1, e^{\frac{\pi i}{m}})$  fixes points on the  $\mathbb{C}P^1$  at the origin that are identified by the antipodal map (the points  $\{0\}$  and  $\{\infty\}$  in  $S^2 = \mathbb{C} \cup \{\infty\}$ ). Therefore, similar to the work in [35, Theorem 4.1], we find that the quotients  $(\mathcal{O}(-2n), g_{LB})/\overline{\Gamma}_i'$ , for  $i = 1, 2$ , are anti-self dual ALE orbifolds with groups at infinity the respective  $\overline{\Gamma}_i$ , each having one singularity with orbifold group  $L(-n, m)$  on the  $\mathbb{R}P^2$  resulting from the quotient of the  $\mathbb{C}P^1$  at the origin by the antipodal map. Clearly, this space satisfies  $\pi_1(\mathcal{O}(-2n)/\overline{\Gamma}_i') = \mathbb{Z}/2\mathbb{Z}$  and  $\tau_{\text{top}}(\mathcal{O}(-2n)/\overline{\Gamma}_i') = 0$ . Notice that when  $m = 1$ , which can only occur for  $\Gamma_2$ , these are in fact smooth quotients and the proof is complete. Therefore, from here on we can assume  $m > 1$ .

We note that  $O(-2n)/\overline{\Gamma}_i'$  is diffeomorphic to a nonorientable bundle over  $\mathbb{R}P^2$ . These are classified by  $H^2(\mathbb{R}P^2, \mathbb{Z}_w)$ , where  $\mathbb{Z}_w$  is the system of local coefficients determined by the first Stiefel–Whitney class  $w$  of the bundle. It turns out that  $H^2(\mathbb{R}P^2, \mathbb{Z}_w)$  is isomorphic to  $\mathbb{Z}$ , and under a suitable choice of isomorphism, this bundle is mapped to the integer  $-n$  (see [5]).

The remainder of the proof will follow from the self-dual orbifold gluing of Theorem 5.1. Consider the self-dual conformal compactification  $(\widehat{\mathcal{O}}(-2n), \widehat{g}_{LB})/\overline{\Gamma}_i'$  which has two singularities — one with orbifold group  $L(n, m)$  on the  $\mathbb{R}P^2$  and the other with orbifold group  $\overline{\Gamma}_i$  at the point of compactification, for  $i = 1, 2$ . Let  $(\widehat{C}_{L(-n,m)}, \widehat{h})$  denote the compactification of a Calderbank–Singer metric with group  $L(-n, m)$  at infinity. The second cohomology group  $H^2_{SD}$  of the self-dual deformation complex of both orbifolds vanishes here by [32, Theorem 4.2]. Therefore, we can apply Theorem 5.1 to obtain a self-dual orbifold conformal structure  $[\widehat{g}_i]$  with positive Yamabe invariant on

$$(5-3) \quad \widehat{X}_i = \widehat{\mathcal{O}}(-2n)/\overline{\Gamma}_i' \# \widehat{C}_{L(-n,m)},$$

where the connected sum is taken at the  $L(n, m)$  orbifold point and the point of compactification, respectively, for  $i = 1, 2$ . The orbifold  $(\widehat{X}_i, [\widehat{g}_i])$  has one singularity with orbifold group  $\overline{\Gamma}_i$ . Thus, for  $i = 1, 2$ , taking the conformal blow-up at this point (since the Yamabe invariant is positive), we obtain a scalar-flat anti-self-dual ALE space  $(X_i, g_{X_i})$  with group at infinity  $\overline{\Gamma}_i$ , satisfying  $\pi_1(X_i) = \mathbb{Z}/2\mathbb{Z}$  and  $\tau_{top}(X_i) = -k_{(m-n,m)}$ . We note that it follows from the gluing theorem that the second cohomology group of the self-dual deformation complex  $H^2_{SD}(\widehat{X}_i)$  also vanishes for these spaces.

Finally, an equivariant version of the gluing theorem can in fact be used to ensure that the spaces  $(X_i, g_{X_i})$  admit an isometric  $S^1$ -action.

### 5B Proof of Corollary 1.10

Let  $\Gamma_1 = \phi(L(1, 2m) \times D_{4n}^*)$  and  $\Gamma_2 = \mathfrak{J}_{m,n}^2$  with  $m, n$  as specified in Table 1 so the action on  $S^3$  is free. For  $i = 1, 2$ , let  $(Y_i, g_{Y_i})$  denote the scalar-flat Kähler, hence anti-self-dual, ALE space with group at infinity  $\Gamma_i$ , obtained for the noncyclic ( $n > 1$ ) and cyclic ( $n = 1$ ) cases, respectively, in [35; 8], and let  $(X_i, g_{X_i})$  be the scalar-flat anti-self-dual ALE spaces with the orientation-reversed groups at infinity  $\overline{\Gamma}_1$  and  $\overline{\Gamma}_2$  of Theorem 1.8. Taking the conformal compactifications  $(\widehat{Y}_i, \widehat{g}_{Y_i})$  and  $(\widehat{X}_i, \widehat{g}_{X_i})$  with the self-dual orientation, as in Remark 1.1, we obtain self-dual orbifolds with orbifold groups  $\Gamma_i$  and  $\overline{\Gamma}_i$ , respectively. Since these are orientation-reversed conjugate, and because  $H^2_{SD}$  vanishes for each orbifold as pointed out above, we are once again able to use the self-dual orbifold gluing of Theorem 5.1 to obtain a self-dual conformal

structure on the orbifold connect sum  $\widehat{X}_i \# \widehat{Y}_i$ , where the connect sum is taken at the points of compactification.

It is clear that the signature of this space,  $\tau(\widehat{X}_i \# \widehat{Y}_i) = -\tau(X_i) - \tau(Y_i)$ , is dependent upon  $m$  and  $n$ , and to highlight this, we denote it by  $\ell_i(m, n)$ , for  $i = 1, 2$ . For any  $n$ ,  $\tau(X_i)$  is easy to find from the work of Section 5A, and in the noncyclic case ( $n > 1$ ),  $\tau(Y_i)$  is given in [35]. In the cyclic case ( $n = 1$ ), however, one finds that  $\Gamma_1$  and  $\Gamma_2$  are both orientation-preserving conjugate to  $L(2m + 1, 4m)$  for their respective  $m$ , and hence that  $\tau(Y_i) = -k_{(2m+1, 4m)} = -3$ . Therefore, we find that

$$(5-4) \quad \ell_i(m, n) := \tau(\widehat{X}_i \# \widehat{Y}_i) = 3 + \begin{cases} k_{(n-m, n)} + k_{(m-n, m)} & \text{if } n > 1 \text{ and } m > 1, \\ m - 1 & \text{if } n = 1 \text{ and } m > 1, \\ n - 1 & \text{if } n > 1 \text{ and } m = 1, \\ 0 & \text{if } n = 1 \text{ and } m = 1. \end{cases}$$

Note that since  $m$  must be even for  $\Gamma_2$ , the last two cases in (5-4) cannot occur for this group. Also, we distinguish the  $m = n = 1$  case for  $\Gamma_1$ , as  $(\ell_1(1, 1) = 3) \# \mathbb{C}P^2$  is the minimal number of  $\mathbb{C}P^2$ s on which a self-dual metric is obtained by this technique.

Now, we will show that  $\widehat{X}_i \# \widehat{Y}_i$  is, in fact, simply connected. Cover  $\widehat{X}_i \# \widehat{Y}_i$  with open sets  $U_{\widehat{X}_i}$  and  $U_{\widehat{Y}_i}$  containing the  $\widehat{X}_i$  and  $\widehat{Y}_i$  components of the connect sum, respectively, and so that  $U_{\widehat{X}_i} \cap U_{\widehat{Y}_i}$  deformation retracts to  $S^3/\Gamma_i$ . Recall that we have the homomorphisms of fundamental groups  $i_{\widehat{X}_i}: \pi_1(U_{\widehat{X}_i} \cap U_{\widehat{Y}_i}) \rightarrow \pi_1(U_{\widehat{X}_i})$  and  $i_{\widehat{Y}_i}: \pi_1(U_{\widehat{X}_i} \cap U_{\widehat{Y}_i}) \rightarrow \pi_1(U_{\widehat{Y}_i})$  induced from the respective inclusion maps. Observe that  $i_{\widehat{X}_i}$  is surjective since  $\pi_1(U_{\widehat{X}_i}) = \mathbb{Z}/2\mathbb{Z}$ , and because the antipodal map on  $S^2$  is induced by the action of a generator of  $\overline{\Gamma}_i$  under the Hopf map. Also, the map  $i_{\widehat{Y}_i}$  is clearly trivial since  $\pi_1(U_{\widehat{Y}_i}) = \{1\}$ . By the Seifert–van Kampen theorem,

$$(5-5) \quad \pi_1(\widehat{X}_i \# \widehat{Y}_i) = \pi_1(U_{\widehat{X}_i}) * \pi_1(U_{\widehat{Y}_i}) / N,$$

where  $*$  denotes the free product and  $N$  is the normal subgroup of  $\pi_1(U_{\widehat{X}_i}) * \pi_1(U_{\widehat{Y}_i})$  generated by  $i_X(\gamma)i_Y(\gamma)^{-1}$  for all  $\gamma \in \pi_1(U_{\widehat{X}_i} \cap U_{\widehat{Y}_i})$ . Therefore, given that  $i_{\widehat{X}_i}$  is surjective,  $i_{\widehat{Y}_i}$  is trivial, and  $\pi_1(U_{\widehat{Y}_i}) = \{1\}$ , as discussed above, we find that  $\pi_1(\widehat{X}_i \# \widehat{Y}_i) = \{1\}$ . By the results of Donaldson and Freedman,  $\widehat{X}_i \# \widehat{Y}_i$  is homeomorphic to  $\ell_i(m, n) \# \mathbb{C}P^2$  (see, for example, [18]). Finally, by taking sequences of conformal factors which uniformly scale one of the factors to have zero volume in the limit, the orbifold limiting statement follows immediately from this construction.

Again, an equivariant version of the gluing theorem can in fact be used to ensure that these examples admit a conformally isometric  $S^1$ -action. By examining this construction, it is clear to see that the  $S^1$ -action is not semifree as there is always an  $\mathbb{R}P^2$  with a  $\mathbb{Z}/2\mathbb{Z}$  stabilizer, and thus these metrics are distinct from those discovered by LeBrun in [31].

**Remark 5.2** The Donaldson–Freedman result only provides a homeomorphism, but we suspect that these manifolds are in fact *diffeomorphic* to  $\ell \# \mathbb{C}P^2$ .

## References

- [1] **A G Ache, J A Viaclovsky**, *Asymptotics of the self-dual deformation complex*, J. Geom. Anal. 25 (2015) 951–1000 MR
- [2] **T M Apostol**, *Modular functions and Dirichlet series in number theory*, 2nd edition, Graduate Texts in Mathematics 41, Springer, New York (1990) MR
- [3] **T Ashikaga, M Ishizaka**, *Another form of the reciprocity law of Dedekind sum* (2008) Available at <http://eprints3.math.sci.hokudai.ac.jp/1849/>
- [4] **S Bando, A Kasue, H Nakajima**, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. 97 (1989) 313–349 MR
- [5] **D Baraglia**, *Topological T-duality for general circle bundles*, Pure Appl. Math. Q. 10 (2014) 367–438 MR
- [6] **E Brieskorn**, *Rationale Singularitäten komplexer Flächen*, Invent. Math. 4 (1967/1968) 336–358 MR
- [7] **R L Bryant**, *Bochner–Kähler metrics*, J. Amer. Math. Soc. 14 (2001) 623–715 MR
- [8] **D M J Calderbank, M A Singer**, *Einstein metrics and complex singularities*, Invent. Math. 156 (2004) 405–443 MR
- [9] **X Chen, C LeBrun, B Weber**, *On conformally Kähler, Einstein manifolds*, J. Amer. Math. Soc. 21 (2008) 1137–1168 MR
- [10] **H S M Coxeter**, *The binary polyhedral groups, and other generalizations of the quaternion group*, Duke Math. J. 7 (1940) 367–379 MR
- [11] **H S M Coxeter**, *Regular complex polytopes*, 2nd edition, Cambridge Univ. Press (1991) MR
- [12] **M G Dabkowski, M T Lock**, *On Kähler conformal compactifications of  $U(n)$ -invariant ALE spaces*, Ann. Global Anal. Geom. 49 (2016) 73–85 MR
- [13] **L David, P Gauduchon**, *The Bochner-flat geometry of weighted projective spaces*, from: “Perspectives in Riemannian geometry”, (V Apostolov, A Dancer, N Hitchin, M Wang, editors), CRM Proc. Lecture Notes 40, Amer. Math. Soc., Providence, RI (2006) 109–156 MR
- [14] **S Donaldson, R Friedman**, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearity 2 (1989) 197–239 MR
- [15] **P Du Val**, *Homographies, quaternions and rotations*, Clarendon Press, Oxford (1964) MR



- [16] **T Eguchi, A J Hanson**, *Self-dual solutions to Euclidean gravity*, Ann. Physics 120 (1979) 82–106 MR
- [17] **A Floer**, *Self-dual conformal structures on  $l\mathbb{C}P^2$* , J. Differential Geom. 33 (1991) 551–573 MR
- [18] **D S Freed, K K Uhlenbeck**, *Instantons and four-manifolds*, 2nd edition, Mathematical Sciences Research Institute Publications 1, Springer, New York (1991) MR
- [19] **G W Gibbons, S W Hawking**, *Gravitational multi-instantons*, Physics Letters B 78 (1978) 430–432
- [20] **F Hirzebruch**, *Über vierdimensionale Riemannsche Flächenmehrfacher analytischer Funktionen von zwei komplexen Veränderlichen*, Math. Ann. 126 (1953) 1–22 MR
- [21] **N J Hitchin**, *Einstein metrics and the eta-invariant*, Boll. Un. Mat. Ital. B 11 (1997) 95–105 MR
- [22] **N Honda**, *Self-dual metrics and twenty-eight bitangents*, J. Differential Geom. 75 (2007) 175–258 MR
- [23] **N Honda**, *Deformation of LeBrun’s ALE metrics with negative mass*, Comm. Math. Phys. 322 (2013) 127–148 MR
- [24] **D Joyce**, *The hypercomplex quotient and the quaternionic quotient*, Math. Ann. 290 (1991) 323–340 MR
- [25] **DD Joyce**, *Explicit construction of self-dual 4-manifolds*, Duke Math. J. 77 (1995) 519–552 MR
- [26] **T Kawasaki**, *The index of elliptic operators over  $V$ -manifolds*, Nagoya Math. J. 84 (1981) 135–157 MR
- [27] **A Kovalev, M Singer**, *Gluing theorems for complete anti-self-dual spaces*, Geom. Funct. Anal. 11 (2001) 1229–1281 MR
- [28] **P B Kronheimer**, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. 29 (1989) 665–683 MR
- [29] **P B Kronheimer**, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. 29 (1989) 685–697 MR
- [30] **C LeBrun**, *Counter-examples to the generalized positive action conjecture*, Comm. Math. Phys. 118 (1988) 591–596 MR
- [31] **C LeBrun**, *Explicit self-dual metrics on  $\mathbb{C}P_2\#\dots\#\mathbb{C}P_2$* , J. Differential Geom. 34 (1991) 223–253 MR
- [32] **C LeBrun, B Maskit**, *On optimal 4-dimensional metrics*, J. Geom. Anal. 18 (2008) 537–564 MR
- [33] **C LeBrun, M Singer**, *A Kummer-type construction of self-dual 4-manifolds*, Math. Ann. 300 (1994) 165–180 MR

- [34] **MT Lock, J A Viaclovsky**, *Anti-self-dual orbifolds with cyclic quotient singularities*, J. Eur. Math. Soc. (JEMS) 17 (2015) 2805–2841 MR
- [35] **MT Lock, J A Viaclovsky**, *A smörgåsbord of scalar-flat Kähler ALE surfaces*, J. Reine Angew. Math. (2016) online publication June
- [36] **D McCullough**, *Isometries of elliptic 3-manifolds*, J. London Math. Soc. 65 (2002) 167–182 MR
- [37] **H Nakajima**, *Self-duality of ALE Ricci-flat 4-manifolds and positive mass theorem*, from: “Recent topics in differential and analytic geometry”, (T Ochiai, editor), Adv. Stud. Pure Math. 18, Academic Press, Boston (1990) 385–396 MR
- [38] **P Scott**, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983) 401–487 MR
- [39] **R Stekolshchik**, *Notes on Coxeter transformations and the McKay correspondence*, Springer, Berlin (2008) MR
- [40] **I Şuvaina**, *ALE Ricci-flat Kähler metrics and deformations of quotient surface singularities*, Ann. Global Anal. Geom. 41 (2012) 109–123 MR
- [41] **G Tian, J Viaclovsky**, *Moduli spaces of critical Riemannian metrics in dimension four*, Adv. Math. 196 (2005) 346–372 MR
- [42] **J A Viaclovsky**, *An index theorem on anti-self-dual orbifolds*, Int. Math. Res. Not. 2013 (2013) 3911–3930 MR
- [43] **E P Wright**, *Quotients of gravitational instantons*, Ann. Global Anal. Geom. 41 (2012) 91–108 MR

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