Geometry & Topology 20 (2016) 2433–2438

Erratum: Lipschitz connectivity and filling invariants in solvable groups and buildings

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This note corrects some omissions in Section 2 of the paper "Lipschitz connectivity and filling invariants in solvable groups and buildings" [Geom. Topol. 18 (2014) 2375–2417].

20E42, 20F65

In the course of writing this paper, I changed the statement of Theorem 1.3, but did not change the proof. This erratum presents the omitted proofs. Thanks to Moritz Gruber for noticing the omission.

This erratum replaces Lemmas 2.6–2.8 and the proof of Theorem 1.3.

Recall the original theorem:

Theorem 1.3 [4] *Suppose that $Z \subset X$ is a nonempty closed subset with metric given by the restriction of the metric of X. Suppose that X is a geodesic metric space such that the Assouad–Nagata dimension $\dim_{\mathrm{AN}}(X)$ of X is finite. Suppose that one of the following is true:*

- *Z is Lipschitz n–connected.*
- *X is Lipschitz n–connected, and if X_p, $p \in P$ are the connected components of $X \setminus Z$, then the sets $H_p = \partial X_p$ are Lipschitz n–connected with uniformly bounded implicit constant.*

Then Z is undistorted up to dimension $n+1$.

First, we note that the second condition implies the first condition:

Lemma 1 *Suppose that X is Lipschitz n–connected and that Z is a closed subset of X. Let X_p, $p \in P$ be the connected components of $X \setminus Z$ and suppose that the sets ∂X_p are Lipschitz n–connected with uniformly bounded implicit constant. Then Z is Lipschitz n–connected.*

Published: 15 September 2016 DOI: 10.2140/gt.2016.20.2433

Proof Suppose that $f\colon S^n \to Z$ is a Lipschitz map. We claim that there is a Lipschitz map $h\colon D^{n+1} \to Z$ that extends f and such that $\operatorname{Lip} h \lesssim \operatorname{Lip} f$. By the Lipschitz connectivity of X, there is a Lipschitz extension $g\colon D^{n+1} \to X$ such that $\operatorname{Lip} g \lesssim \operatorname{Lip} f$; if the image of g lies in Z, we're done. Otherwise, suppose that X_p is a connected component of $X \setminus Z$ and let $K_p = g^{-1}(X_p)$. Then g sends $\partial K_p \to \partial X_p$. The set ∂X_p is Lipschitz n–connected, so any Lipschitz map from a closed subset of D^{n+1} to ∂X_p can be extended to a Lipschitz map on all of D^{n+1} (see [1, Theorem 1.2], [2, Theorem 2], [3, Theorem 1.4]). We therefore construct a map $h_p\colon K_p \to \partial X_p$ such that h_p agrees with g on ∂K_p and $\operatorname{Lip} h_p \lesssim \operatorname{Lip} g$. Then

$$h(x) = \begin{cases} g(x) & \text{if } g(x) \in Z, \\ h_p(x) & \text{if } g(x) \in X_p \end{cases}$$

is an extension of f, and $\operatorname{Lip} h \lesssim \operatorname{Lip} f$. □

It thus suffices to prove the theorem in the case that Z is Lipschitz n–connected. We recall some notation from [4, Section 2]; in the following, $\epsilon > 0$ is a small number that will depend on the cycle to be filled.

- We cover X by a collection $\mathcal{D} = \mathcal{D}(\epsilon)$ of open sets D_k, $k \in K$ such that $\operatorname{diam} D_k \gtrsim \epsilon$ for all k. This cover can be broken into a "fine" portion consisting of a cover of Z by ϵ–balls and a "coarse" portion consisting of subsets of $X \setminus Z$ whose diameters are roughly proportional to their distance from Z. For each $D_k \in \mathcal{D}$, we define a 1–Lipschitz function $\tau_k\colon X \to \mathcal{R}$ such that $\tau_k \geq \epsilon$ on D_k, $\operatorname{diam} \operatorname{supp} \tau_k \sim \operatorname{diam} D_k$, and $\operatorname{supp} \tau_k$ intersects Z if and only if D_k intersects Z.
- $\Sigma = \Sigma(\epsilon)$ is a QC complex based on the nerve of the cover of X by the sets $\operatorname{supp} \tau_k$. We denote the vertices of Σ by v_k. Then $\dim \Sigma \leq 2 \dim_{\mathrm{AN}} X + 1$, and for any $k \in K$, the diameter of any simplex of Σ containing v_k is comparable to $\operatorname{diam} \operatorname{supp} \tau_k$.
- The map $g\colon X \to \Sigma$ has Lipschitz constant independent of ϵ. The map g is defined by normalizing the τ_k to obtain a partition of unity $g_k(x) = \tau_k(x)/\overline{\tau}(x)$, where $\overline{\tau}(x) = \sum_i \tau_i(x)$, then using the g_k as coordinate functions. In the proof of [4, Lemma 2.5], we showed that for each k, we have

  $$\operatorname{Lip}(g_k) \sim \operatorname{diam}(\operatorname{supp} \tau_k)^{-1}.$$

  Since $\operatorname{diam}(\operatorname{supp} \tau_k) \gtrsim \epsilon$, we have $\operatorname{Lip}(g_k) \lesssim \epsilon^{-1}$ for all k.

We can use the connectivity of Z to construct a map $h\colon \Sigma^{(n+1)} \to Z$ as in the proof of [3, Theorem 1.4].

Lemma 2 *There is a Lipschitz extension $h\colon \Sigma^{(n+1)} \to Z$ with Lipschitz constant independent of ϵ such that $d(h(g(z)), z) \lesssim \epsilon$ for every $z \in Z$.*

Proof For each vertex v_k of Σ, if D_k intersects Z, we choose $h(v_k) \in Z \cap D_k$. Otherwise, we let $h(v_k) \in Z$ be such that $d(h(v_k), \operatorname{supp} \tau_k) \le 2d(Z, \operatorname{supp} \tau_k)$. If two vertices v_k and $v_{k'}$ are connected by an edge e, then

$$\ell(e) \lesssim \operatorname{diam}(\operatorname{supp} \tau_k) + \operatorname{diam}(\operatorname{supp} \tau_{k'}),$$

and $\operatorname{supp} \tau_k$ intersects $\operatorname{supp} \tau_{k'}$. Thus,

$$\begin{aligned} d(h(v_k), h(v_{k'})) &\lesssim d(Z, \operatorname{supp} \tau_k) + \operatorname{diam}(\operatorname{supp} \tau_k) + \operatorname{diam}(\operatorname{supp} \tau_{k'}) + d(Z, \operatorname{supp} \tau_{k'}) \\ &\lesssim \ell(e), \end{aligned}$$

so h is Lipschitz on $\Sigma^{(0)}$. Since Z is Lipschitz n–connected and Σ is a QC complex, we can extend h to a Lipschitz map on $\Sigma^{(n+1)}$. Finally, if $z \in Z$, then $z \in D_k$ for some $k \in K$, and $g(z)$ is in the star of v_k. It follows that $d(g(z), v_k) \lesssim \epsilon$, and so

$$d(h(g(z)), z) \le \operatorname{Lip}(h) d(g(z), v_k) + d(h(v_k), z) \lesssim \epsilon + \operatorname{diam} D_k \lesssim \epsilon,$$

as desired. □

Suppose that $\alpha \in C_m^{\mathrm{Lip}}(Z)$ is a m–cycle and $m \le n$. In [4], we tried to construct a filling of α in Z by constructing a filling in X, sending that filling to Σ, then approximating it in $\Sigma^{(n+1)}$ and sending it back to Z. That is, we first use the Lipschitz connectivity of X to construct a chain β in X whose boundary is α. Its push-forward $g_\sharp(\beta)$ can be approximated by a simplicial chain P_β^0 so that ∂P_β^0 is a simplicial approximation of $g_\sharp(\alpha)$. Since $\operatorname{supp} P_\beta^0 \subset \Sigma^{(n+1)}$, the $(m+1)$–chain $h_\sharp(P_\beta^0)$ is a chain in Z and its boundary is ϵ–close to α. It remains to construct an annulus between $h_\sharp(\partial P_\beta^0)$ and α. In [4], there were some errors in this construction, and we will correct those issues here.

Theorem 1.3 will follow from the following lemma.

Lemma 3 *There is a $c_\alpha > 0$, depending on the number of simplices in α and their Lipschitz constants, such that for any $\epsilon > 0$, there are two annuli, $\gamma \in C_{m+1}^{\mathrm{Lip}}(\Sigma(\epsilon))$ and $\lambda \in C_{m+1}^{\mathrm{Lip}}(Z)$, and an m–cycle $\alpha' = \alpha'(\epsilon) \in C_m(\Sigma(\epsilon))$ such that*

$$\partial\gamma = g_\sharp(\alpha) - \alpha', \quad \operatorname{mass} \gamma \lesssim c_\alpha \epsilon,$$

$$\partial\lambda = \alpha - h_\sharp(\alpha'), \quad \operatorname{mass} \lambda \lesssim c_\alpha \epsilon.$$

Proof of Theorem 1.3 Given γ and λ, we construct a filling of α by letting $P_\beta \in C_{m+1}(\Sigma(\epsilon))$ be a simplicial approximation of $g_\sharp(\beta) - \gamma$. Since $\partial(g_\sharp(\beta) - \gamma)$ is already simplicial, we have

$$\partial P_\beta = \partial(g_\sharp(\beta) - \gamma) = \alpha'.$$

Then

$$\partial(\lambda + h_\sharp(P_\beta)) = \alpha,$$
$$\operatorname{mass}(\lambda + h_\sharp(P_\beta)) \lesssim \operatorname{mass}\beta + c_\alpha\epsilon.$$

Letting ϵ go to 0, we find that $\mathrm{FV}_Z(\alpha) \lesssim \mathrm{FV}_X(\alpha)$ as desired. □

It thus suffices to construct α', γ and λ as above.

Proof of Lemma 3 The cycle α' will be based on a subdivision of α. For any $\delta \in (0,1)$, a Euclidean m–simplex can be subdivided into roughly δ^{-m} simplices of diameter less than δ, so there is a $c_\alpha > 0$, depending on the number of simplices in α and their Lipschitz constants, such that for any $\delta \in (0,1)$, we can subdivide α into a sum $\sum_{i=1}^N \Delta_i$ of simplices, where $N \le c_\alpha\delta^{-m}$ and

$$\operatorname{diam}\Delta_i \le \operatorname{Lip}\Delta_i < \delta.$$

Let $L_g = \sup_k \operatorname{Lip}(g_k) \sim \epsilon^{-1}$ and let

$$\delta = \frac{1}{2(\dim(\Sigma)+1)L_g} \sim \epsilon.$$

Then $\alpha = \sum_{i=1}^N \Delta_i$, where $N \lesssim c_\alpha\epsilon^{-m}$.

We will construct the simplicial cycle α' by sending each vertex of each Δ_i to the nearest vertex of Σ. For each point $z \in Z$, let $k(z) \in K$ be an index that maximizes $g_k(z)$ and let $v(z) = v_{k(z)}$. We claim that if $z_{i,0},\dots,z_{i,m} \in Z$ are the vertices of Δ_i, then $v(z_{i,0}),\dots,v(z_{i,m})$ are the vertices of a simplex of Σ (possibly with duplicates). Since the g_k form a partition of unity with bounded multiplicity, we know that

$$g_{k(z_{i,j})}(z_{i,j}) \ge \frac{1}{\dim(\Sigma)+1}.$$

If $z \in \Delta_i$, then

$$g_{k(z_{i,j})}(z) \ge \frac{1}{\dim(\Sigma)+1} - L_g d(z_{i,j}, z) > 0, \tag{1}$$

so $g_{k(z_{i,j})}(z) > 0$ for all j, and $\{v(z_{i,0}),\dots,v(z_{i,m})\}$ is the vertex set of a simplex in Σ. We define α' to be the simplicial cycle

$$\alpha' = \sum_i \langle v(z_{i,0}),\dots,v(z_{i,m})\rangle.$$

This is a sum of at most N simplices, each with diameter on the order of ϵ, so

$$\operatorname{mass} \alpha' \lesssim N\epsilon^m \lesssim c_\alpha.$$

Next, we construct γ and λ. We construct γ from a straight-line homotopy between $g_\sharp(\alpha)$ and α'. Consider Σ as a subset of the infinite simplex $\Delta^K \subset \ell^2(K)$ with vertex set $\{v_k\}_{k\in K}$. Let Δ^m be the standard m–simplex $\Delta^m = \langle e_0, \dots, e_m\rangle$. We view Δ_i as a map $\Delta_i\colon \Delta^m \to Z$. Likewise, we write $\alpha' = \sum_i \Delta'_i$, where $\Delta'_i\colon \Delta^m \to \Sigma$ is the linear map such that $\Delta'_i(e_j) = v(z_{i,j})$.

Let $x \in \Delta^m$ and let $z = \Delta_i(x)$. We claim that $g(z)$ and $\Delta'_i(x)$ are both contained in the same simplex of Σ. For $s \in \Sigma$, let $\operatorname{supp} s$ be the vertex set of the minimal simplex containing s; then, by the definition of g,

$$\operatorname{supp} g(z) = \{v_k \mid g_k(z) > 0\},$$
$$\operatorname{supp} \Delta'_i(x) = \{v_{k(z_{i,0})}, \dots, v_{k(z_{i,m})}\}.$$

By (1), we have $g_{k(z_{i,j})}(z) > 0$ for all j, so $\operatorname{supp} \Delta'_i(x) \subset \operatorname{supp} g(z)$. Consequently, we can define a map

$$\overline{\Delta}_i\colon \Delta^m \times [0,1] \to \Sigma, \quad \overline{\Delta}_i(x,t) = tg(\Delta_i(x)) + (1-t)\Delta'_i(x).$$

Let $\gamma = \sum_i [\overline{\Delta}_i]$, where $[\overline{\Delta}_i]$ is the image of the fundamental class of $\Delta^m \times [0,1]$. Then $\partial\gamma = g_\sharp(\alpha) - \alpha'$ as desired. Furthermore, since $\operatorname{Lip} \Delta_i \lesssim \epsilon$ and $\operatorname{Lip} \Delta'_i \lesssim \epsilon$, we have $\operatorname{Lip} \overline{\Delta}_i \lesssim \epsilon$, and

$$\operatorname{mass} \gamma \lesssim N\epsilon^{m+1} \lesssim c_\alpha \epsilon.$$

To construct λ, we use the Lipschitz connectivity of Z. For each i, we have

$$d(\Delta_i, h\circ\Delta'_i) \lesssim \epsilon, \quad \operatorname{Lip}(\Delta_i) \lesssim \epsilon, \quad \operatorname{Lip}(h\circ\Delta'_i) \lesssim \epsilon,$$

so we can use the Lipschitz connectivity of Z to construct prisms $p_i\colon \Delta^m \times [0,1] \to Z$ such that

$$p_i|_{\Delta^m\times 0} = \Delta_i, \quad p_i|_{\Delta^m\times 1} = h\circ\Delta'_i \quad \text{and} \quad \operatorname{Lip}(p_i) \lesssim \epsilon.$$

Let $\lambda = \sum_i [p_i]$. If we are careful to match corresponding faces in neighboring simplices, then

$$\partial\lambda = \alpha - h_\sharp(\alpha'),$$
$$\operatorname{mass} \lambda \lesssim N\epsilon^{m+1} \lesssim c_\alpha \epsilon,$$

as desired. □

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Received: 18 May 2015