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On the uniqueness of the contact structure approximating a foliation

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According to a theorem of Eliashberg and Thurston, a C^2–foliation on a closed 3–manifold can be C^0–approximated by contact structures unless all leaves of the foliation are spheres. Examples on the 3–torus show that every neighbourhood of a foliation can contain nondiffeomorphic contact structures.

In this paper we show uniqueness up to isotopy of the contact structure in a small neighbourhood of the foliation when the foliation has no torus leaf and is not a foliation without holonomy on parabolic torus bundles over the circle. This allows us to associate invariants from contact topology to foliations. As an application we show that the space of taut foliations in a given homotopy class of plane fields is not connected in general.

53D10, 57R30, 57R17

1 Introduction and results

The purpose of this paper is to determine which foliations on closed 3–manifolds have the property that all positive contact structures in a sufficiently small neighbourhood of the foliation are isotopic (for definitions and basic results see Section 2A). According to the following theorem of Y Eliashberg and W Thurston, most foliations can be approximated by contact structures:

Theorem 1.1 [12] *Let $\mathcal{F}$ be an oriented C^2–foliation by surfaces on a closed oriented 3–manifold. If $\mathcal{F}$ is not isomorphic to a foliation by spheres on $S^2 \times S^1$, then every C^0–neighbourhood of $\mathcal{F}$ contains a positive contact structure.*

It can be shown quite easily [12] that the foliation by the first factor on $M = S^2 \times S^1$ cannot be approximated by a contact structure, ie there is a C^0–neighbourhood of the foliation which does not contain a contact structure.

Theorem 1.1 provides a first link between foliations and contact structures. Before the appearance of [12] these fields developed independently. The approximation theorem allows one to obtain potentially interesting contact structures from construction of

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foliations. For example, the work of D Gabai [17] on constructions of foliations from sutured manifold decompositions provides a rich source of interesting contact structures. Via this construction there is a connection between sutured manifolds and gauge theory; see P Kronheimer and T Mrowka [40]. The most prominent application of this circle of ideas is in the proof of the property P conjecture by Kronheimer and Mrowka [41].

In view of Theorem 1.1 it is natural to ask to what extent the foliation determines the contact structures up to isotopy in sufficiently small neighbourhoods. The following well-known example shows that the isotopy type of a contact structure in a small neighbourhood of a foliation is not completely determined by the foliation.

Example 1.2 Let $\mathcal{F}$ be the foliation of $T^2 \times S^1 = \mathbb{R}^3/\mathbb{Z}^3$ by tori corresponding to the first factor. Then for $0 \neq \varepsilon \to 0$ and $k \neq 0$ the contact planes ξ_k defined by the 1–forms

$$\alpha_{k,\varepsilon} := dt + \varepsilon(\cos(2\pi k t)\, dx_1 - \sin(2\pi k t)\, dx_2)$$

converge to $T\mathcal{F}$. Different ε yield isotopic contact structures which we therefore denote by ξ_k. According to Y Kanda [39], the contact structure ξ_k is isotopic to ξ_l if and only if $k = l$. They are distinguished by their Giroux torsion (we review the definition of this invariant in Definition 2.34).

However, the question of whether or not torus leaves are the only source of ambiguity was raised by V Colin as [8, Question 5.9]. Also, K Honda, W Kazez and G Matić suggested [37, second paragraph on page 306] that

> "*...contact topology may ultimately be a discrete version of foliation theory.*"

One piece of evidence for this is the following theorem of Honda, Kazez and Matić:

Theorem 1.3 [37] *Let ψ be an orientation-preserving pseudo-Anosov diffeomorphism of a hyperbolic surface and M_ψ the surface fibration over the circle with monodromy ψ.*

There is a unique tight contact structure ξ on M_ψ such that $\langle e(\xi), [\Sigma]\rangle = 2-2g$. In particular, there is a C^0–neighbourhood U of the foliation by fibres of $M_\psi \to S^1$ in the space of plane fields such that $\xi \cong \xi'$ for all pairs of positive contact structures $\xi, \xi' \in U$.

The following theorem answers Colin's question affirmatively up to a small set of exceptions. It can be viewed as confirmation of the above remark from [37].

Theorem 1.4 *Let $\mathcal{F}$ be a coorientable C^2–foliation (or a C^2–confoliation) on a closed oriented 3–manifold satisfying the following conditions:*

(i) *$\mathcal{F}$ has no closed leaf of genus $g \leq 1$,*

(ii) *$\mathcal{F}$ is not a foliation by planes,*

(iii) *$\mathcal{F}$ is not a foliation by cylinders.*

Then there is a C^0–neighbourhood U of $\mathcal{F}$ in the space of plane fields and a contact structure ξ in U such that every positive contact structure in U is isotopic to ξ.

The case that a leaf of $\mathcal{F}$ is a sphere is excluded since the Reeb stability theorem (under the orientation assumptions at hand) implies that a foliation with a spherical leaf is a foliation by spheres on $S^2 \times S^1$. According to a theorem of H Rosenberg [51], C^2–foliations by planes exist only on the 3–torus. Later, G Hector [31] proved that foliations by cylinders exist only on parabolic T^2–bundles over the circle. This shows that the foliations in (ii)–(iii) of Theorem 1.4 are very special. Thus torus leaves are essentially the only source of nonuniqueness of the isotopy classes of contact structure which are sufficiently close to a given confoliation. Foliations which satisfy the assumptions of Theorem 1.4 will be called *atoral* and the isotopy class of positive contact structures in the neighbourhood of Theorem 1.6 *approximates* $\mathcal{F}$.

When a foliation has torus leaves, then every neighbourhood contains nonisotopic contact structures distinguished by their Giroux torsion. If the torus leaves satisfy a certain stability condition, then the Giroux torsion is the only source of ambiguity of the contact structures in small neighbourhoods of $\mathcal{F}$. In order to state the corresponding theorem we need the following definition.

Definition 1.5 Two contact structures ξ', ξ'' are stably equivalent with respect to a finite collection of pairwise disjoint embedded tori if the following conditions hold:

(i) It is possible to isotope the tori and to choose a contact form α such that the restriction of α to the isotoped tori is closed (such tori are called *pre-Lagrangian*).

(ii) ξ' and ξ'' become isotopic after inserting a contact structure

$$\big(T^2 \times [0,1], \ker\big(\cos(2\pi k(t+t_0))\, dx_1 - \sin(2\pi k(t+t_0))\, dx_2\big)\big)$$

with suitable parameters $k > 0$ and $t_0 \in \mathbb{R}$ along the pre-Lagrangian tori.

Theorem 1.6 *If the foliation (or confoliation) $\mathcal{F}$ satisfies only the weaker hypothesis*

(i′) *all torus leaves of ξ have attractive holonomy*

and the assumptions (ii)–(iii) *of Theorem 1.4, then there is a C^0–neighbourhood U of $\mathcal{F}$ such that any two contact structures ξ', ξ'' in U are stably equivalent with respect to the torus leaves of $\mathcal{F}$.*

As explained in Section 6, condition (i′) can be generalized somewhat further. We explain the details in Section 6A. Examples show that foliations with torus leaves violating (i′) do not satisfy the conclusion in the theorem above. However, the examples known to the author in which this happens are rather special. In view of potential applications of Theorem 1.6, the characterization of those foliations with torus leaves which violate (i′) but still satisfy the conclusion of Theorem 1.6 is an interesting open problem (see for example Proposition 9.13).

Theorem 1.4 allows us to associate invariants from contact topology (for example the contact invariant from Heegaard Floer theory) to atoral foliations. Combining Theorem 1.1 and Theorem 1.4, we obtain:

Theorem 9.3 *Let $\mathcal{F}_t$, $t \in [0,1]$, be a C^0–continuous family of atoral C^2–foliations. Then the positive contact structures ξ_0 and ξ_1 approximating $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively, are isotopic.*

This provides an obstruction to finding a path of atoral foliations connecting two atoral foliations. This is of interest since the work of H Eynard-Bontemps [14; 15] shows that two atoral foliations are homotopic through foliations as soon as the two foliations are homotopic as plane fields. The foliations in the homotopy constructed by Eynard-Bontemps contain Reeb components and therefore violate the hypothesis of Theorem 1.4.

In many interesting cases the class of atoral foliations on a manifold coincides with the class of taut foliations. In Example 9.5 we show that the Brieskorn homology sphere $\Sigma(2,3,11)$ has a taut foliation $\mathcal{F}$ that is not homotopic to the foliation $\overline{\mathcal{F}}$ (this is $\mathcal{F}$ with the opposite coorientation) through foliations without Reeb components, although $\mathcal{F}$ and $\overline{\mathcal{F}}$ are homotopic as oriented plane fields. After this article appeared in preprint form J Bowden [3] gave other examples of this kind.

Other applications of Theorem 1.4 and Theorem 1.6 can be found in Section 9C.

Finally, let us note that when it is possible to prove a parametric version of Theorem 1.4 without too much additional difficulty, then we will do so. The parametric versions are Theorem 4.2 and Theorem 5.1; they cover foliations with holonomy which do not have closed leaves.

1A Some ideas for the proof of the uniqueness result

The main tools used in this paper stem from Colin [6], Eliashberg and Thurston [12], E Giroux [24] and Honda, Kazez and Matić [37].

Just like the proof of Theorem 1.1, the proof of the uniqueness theorem deals with minimal sets and the rest of the manifold in separate steps. These steps are treated in different order in the proofs of Theorem 1.1 and Theorem 1.4. First, we fix a pair of neighbourhoods $\widehat{N} \supset N$ of the set of closed leaves and particular curves with linear holonomy (Sacksteder curves; see Section 5A) and we choose a C^0–neighbourhood of $\mathcal{F}$ such that the restriction of every contact structure in the C^0–neighbourhood to $\widehat{N}$ is tight. Given two contact structures ξ, ξ' in an even smaller neighbourhood of $\mathcal{F}$, we first deform ξ so that the resulting contact structure $\widehat{\xi}$ coincides with ξ' outside of N and so that the contact structures remain tight on $\widehat{N}$ throughout the deformation. Then we use classification results for tight contact structures in order to show that $\widehat{\xi}$ and ξ' are isotopic on $\widehat{N}$. A somewhat different procedure has to be used when $\mathcal{F}$ is a foliation without holonomy.

The first step follows the structure of the proof of the following theorem of Colin:

Theorem 4.1 [6] *Let ξ be a contact structure on the closed 3–manifold M. Then there is a C^0–neighbourhood of ξ in the space of smooth plane fields so that every contact structure in U is isotopic to ξ.*

Since we start with a confoliation and not with a contact structure several modifications are needed. As in the proof of Theorem 4.1 one starts with a polyhedral decomposition of M, and the main modification of the proof of Theorem 4.1 concerns extensions of the polyhedra which lead to controlled modifications of the characteristic foliation on the boundary.

The contact structures $\xi, \widehat{\xi}, \xi'$ are transverse to a rank 1–foliation on the tubular neighbourhood $\widehat{N}$. This can be used to show that the restrictions of $\widehat{\xi}, \xi'$ to $\widehat{N}$ are tight. We then want to appeal to classification results for tight contact structures. In the case when a connected component of $\widehat{N}$ is a solid torus and the characteristic foliation on the boundary has exactly two nondegenerate closed leaves, the contact structure is uniquely determined up to isotopy. If a connected component of $\widehat{N}$ is the tubular neighbourhood of a closed leaf of $\mathcal{F}$, then the contact structure on $\widehat{N} \cong \Sigma \times [-1,1]$ is not uniquely determined by the properties of $\widehat{\xi}, \xi'$ we have mentioned so far.

If Σ is a closed leaf then the Euler class $e(\mathcal{F})$ of $\mathcal{F}$ satisfies the extremal condition

$$\langle e(\mathcal{F}), [\Sigma] \rangle = \pm(2-2g), \tag{1-1}$$

where g is the genus of Σ and $g \geq 2$ since in the presence of a spherical leaf there is nothing to prove and the case of torus leaves is excluded. We will assume in the following that (1-1) holds with the plus sign on the right-hand side. (In the opposite case one has to interchange positive/negative singularities and attractive/repulsive closed leaves).

Following ideas in Giroux [24], we show that tight contact structures on $\Sigma \times [-1, 1]$ can be distinguished using sheets of the movie of characteristic foliations on $\Sigma_t := \Sigma \times \{t\}$ which contain attractive closed leaves of the characteristic foliation on $\Sigma_{\pm 1}$. A sheet A is an embedded submanifold in M such that the characteristic foliation $A(\xi)$ is a nonsingular foliation by circles; more details can be found in Section 3. The sheets we will consider are formed by closed leaves of the characteristic foliations $\Sigma_t(\xi)$ and of simple closed curves formed by positive elliptic singularities and stable leaves of positive hyperbolic singularities. (If ξ is sufficiently close to $\mathcal{F}$, then $\Sigma_t(\xi)$ has no negative singularities.)

When the genus of Σ is larger than 1, we show in Section 7 using the pre-Lagrangian extension lemma from Section 3C1 that a tight contact structure on $\widehat{N}$ is uniquely determined by its restriction to $\widehat{N} \setminus N$ when it is sufficiently close to $\mathcal{F}$. Then it follows that $\widehat{\xi}$ is isotopic to ξ' and therefore ξ is isotopic to ξ'.

If Σ is a closed surface of genus ≥ 2 we rely on classification results for tight contact structures from [37]. If Σ is a torus, then we can use the more complete classification of tight contact structures on $T^2 \times [-1, 1]$ in the form given in [24] to obtain Theorem 1.6

One of the most important points in the proof of these theorems is to ensure that there is no sheet connecting the two boundary components of $\widehat{N}$. This is done by choosing the neighbourhood of $\mathcal{F}$ in the space of plane fields properly. In particular, all plane fields are transverse to the foliation on $\widehat{N} \cong \Sigma \times [-1, 1]$ induced by the second factor.

Because of the position of the contact plane field with respect to the parts of sheets consisting of attractive closed leaves of the characteristic foliation on level surfaces Σ_t, restrictions on the C^0–distance between the contact structures and $\mathcal{F}$ lead to restrictions on the position of sheets. This is illustrated in Figure 1.

The figures on the left-hand side of Figure 1 show the intersection of $\mathcal{F}$ with an annulus that is transverse to the line field $\widehat{N}(\mathcal{F})$ when Σ is a stable (upper part of Figure 1) or an unstable (lower part) torus leaf. In each case the right-hand side shows the intersection of the same annulus with a sheet of a contact structure which could arise when $\mathcal{F}$ is approximated by a contact structure ξ. The thickened arcs correspond to those parts of sheets where $\Sigma_t \cap A$ is an attractive closed leaf of $\Sigma_t(\xi)$ and the straight segments correspond to the contact planes.

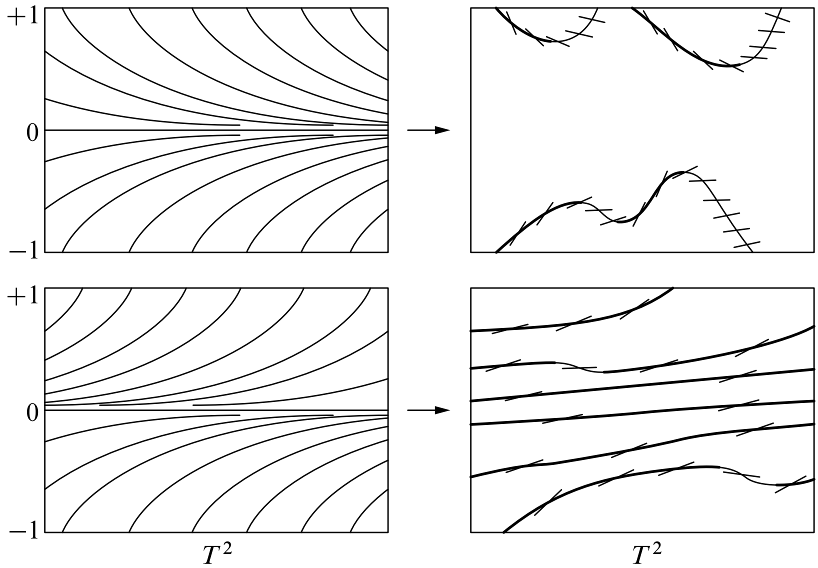

Figure 1: Stable/unstable torus leaves and sheets of nearby contact structures

A difference between the case of torus leaves and the case of surfaces of higher genus is that in the case of surfaces of higher genus an embedding of an annulus connecting the two boundary components of $\widehat{N} \cong \Sigma \times [-1, 1]$ is determined up to isotopy by the boundary curves, while this is not true if the leaf is a torus. If a torus leaf is stable then one can still choose the neighbourhood U of $\mathcal{F}$ such that there are no sheets connecting the two boundary components of $\partial \widehat{N}$. If the torus is not stable, then it may happen that no such neighbourhood exists.

As we have already mentioned, foliations without holonomy have to be treated in a different fashion. Recall from Eliashberg and Thurston [12] that foliations without holonomy can be C^0–approximated by fibrations. The most delicate part of the proof of Theorem 1.4 for foliations without holonomy is to find a fibration which approximates the foliation well enough so that one can exclude the appearance of sheets which intersect every fibre of the fibration for contact structures close to $\mathcal{F}$. These approximations are constructed in Section 8C using a theorem of Dirichlet about Diophantine approximations of real numbers.

1B Organization of the paper

This paper consists of nine sections. The author hopes that the results of this paper are relevant for people interested in contact structures or foliations. In order to make it more accessible we have included most of the relevant definitions and basic theorems in

Section 2. However, we are very brief and we only prove statements which we did not find in the literature. Also, some results (like Sacksteder's theorem) from the theory of foliations are stated in the section where they are used. In Section 3 we review Giroux's theory of movies of contact structures from [24] (some of this material can be found in Geiges [18]). Again, most results we prove are modifications of theorems in [24] or results which are probably well-known but which we did not find in the literature in the required form. An exception is the pre-Lagrangian extension lemma in Section 3C1, which is a new result.

The proofs of Theorem 1.4 and Theorem 1.6 are contained in Sections 4–8. The author hopes that by dealing with increasingly more difficult situations in separate sections the proofs become more transparent than a proof covering all possible types of minimal sets at once.

- Section 4 deals with the case of transitive confoliations and its main purpose is to extend the proof of Theorem 4.1 using ribbons. This technique will be used in all subsequent cases, except in the case of foliations without holonomy.
- Section 5 contains a proof of the uniqueness theorem for confoliations which are not foliations without holonomy and have no closed leaves. In this section we also show how the subsequent proofs for foliations carry over to the confoliated case.
- Section 6 contains a proof of Theorem 1.6 in the case when there are no closed leaves of higher genus.
- Section 7 completes the proofs of Theorem 1.4 and Theorem 1.6 for confoliations which are not foliations without holonomy.
- Section 8 contains the proof of Theorem 1.4 when $\mathcal{F}$ is a foliation without holonomy. We also discuss which torus bundles satisfy the conclusion of Theorem 1.6.

Finally, Section 9 contains a discussion of applications of the uniqueness result and examples where the approximating contact structure is not well defined. In particular, we show that neighbourhoods of foliations by planes and foliations by cylinders contain many nonisotopic contact structures with vanishing Giroux torsion.

At the beginning of Sections 4.2–8 we give an informal outline of the main difficulties arising in that section. The goal is to help the reader understand what certain arguments and lemmas might be used for before they are applied in a formal way. Also, this is a long paper and Section 1A as well as the introductions of the individual sections can hopefully serve as a reminder for a reader who has read parts of the paper and is returning to the paper at a later time. This author hopes that the benefits of these outlines outweigh the confusion their informality might sometimes create.

Notation The standard notation for the characteristic foliation of a contact structure ξ on a surface Σ is $\xi\Sigma$ (as in [24]). This is convenient when there is only one contact structure and families of surfaces. When there are families of surfaces and families of contact structures then I prefer to write $\Sigma_s(\xi_1)$ instead of $\xi_1\Sigma_s$, for example. The convention used here will be $\Sigma(\xi)$.

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2 Preliminaries

Sections 2A–2C contain basic definitions from contact topology and the theory of foliations in order to make this text more accessible. In Section 2D we review the relevant classification results for tight contact structures.

2A Contact structures, foliations and confoliations

In this paper M will always denote a closed connected oriented 3–manifold. We fix an auxiliary Riemannian metric on M. In this section we give some standard definitions and fix some conventions used throughout this paper. We start with the definition of a foliation. Usually, foliations are defined in terms of a foliated atlas. For our purposes the following definition is more convenient.

Definition 2.1 A C^k–smooth, $k \geq 1$, *foliation* $\mathcal{F}$ on M is a plane field such that $\mathcal{F} = \ker(\alpha)$ for a locally defined C^k–smooth 1–form α and $\alpha \wedge d\alpha \equiv 0$.

By the theorem of Frobenius and [49, Proposition 1.0.2] this is equivalent to the standard definition of a foliation of codimension 1 when $k \geq 1$. When $k = 0$ it is not even true in general that a foliation defined by an atlas corresponds to a subbundle of the tangent bundle of M. But since we will only be interested in C^2–foliations, we do not have to discuss this (more information can be found in A Candel and L Conlon [4]).

Given a foliation $\mathcal{F}$, there is a collection of immersed hypersurfaces which are everywhere tangent to $\mathcal{F}$; a maximal connected hypersurface with this property is a *leaf* of $\mathcal{F}$. We will often confuse the collection of leaves with the corresponding plane field.

Definition 2.2 A positive *contact structure* on a 3–manifold is a C^1–plane field ξ such that every 1–form α defined on an open set V with $\ker(\alpha) = \xi|_V$ satisfies $\alpha \wedge d\alpha > 0$. Negative contact structures are defined by requiring $\alpha \wedge d\alpha < 0$. A positive *confoliation* ξ is a C^1–smooth plane field on M such that $\alpha \wedge d\alpha \geq 0$ for every 1–form defining ξ on an open set.

Note that if $\alpha \wedge d\alpha > 0$ holds somewhere, then the same is true for every other 1–form defining the same distribution. All plane fields in this paper will be oriented subbundles of TM, so we can assign an Euler class to each foliation, contact structure or confoliation. We consider two plane fields to be different when they coincide but have opposite orientations. If ξ is an oriented plane field, then $\overline{\xi}$ denotes the same plane field with its orientation reversed.

The condition that ξ is a positive confoliation has the following geometric interpretations:

- Fix a vector field X tangent to ξ and a disc D transverse to the flow lines of X. The disc is oriented such that its orientation followed by the orientation of X is the orientation of M. We denote the flow of X by φ_t. Then the line field $TD \cap \varphi_{-t*}(\xi)$ rotates clockwise with nonnegative speed as t increases.
- Let ξ be a positive confoliation on $D^2 \times \mathbb{R}$ which is transverse to the second factor and complete as a connection. Then the parallel transport $h\colon \mathbb{R} \to \mathbb{R}$ along ∂D^2 satisfies

$$h(x) \leq x \quad \text{for all } x \in \mathbb{R}. \tag{2-1}$$

Theorem 1.1 and the second interpretation implies that the closure of the space of positive contact structures in the space of C^1–plane fields with respect to the C^0–distance is exactly the space of positive confoliations.

The following terminology is borrowed from contact topology.

Definition 2.3 Let ξ be a smooth plane field on M. A piecewise smooth curve γ in M is *Legendrian* if it is tangent to ξ.

Definition 2.4 If ξ is a confoliation, then the open set

$$H(\xi) = \{x \in M \mid \xi \text{ is a positive contact structure on a neighbourhood of } x\}$$

is the *contact region* of ξ. We say that ξ is *transitive* if for every point of M there is a Legendrian curve which connects x and $H(\xi)$. The *fully foliated set* of a confoliation consists of those points which are not connected to $H(\xi)$ by a Legendrian curve.

The fully foliated set of a confoliation is a closed subset of M containing immersed hypersurfaces everywhere tangent to ξ. We will refer to these hypersurfaces as leaves. The theorems from foliation theory which we shall use later carry over to fully foliated sets of confoliations.

Definition 2.5 Let $\mathcal{F}$ be a foliation on M. A subset $X \subset M$ is called *minimal* if

(i) X is closed,

(ii) X is a nonempty union of leaves of $\mathcal{F}$, and

(iii) X contains no proper subset satisfying (i) and (ii).

If M is compact and carries a foliation, then there are minimal sets and the topological closure of a leaf contains a minimal set. Moreover, every minimal set X of a foliation belongs to one of three categories. In order to describe them we fix a point $p \in X$ and short interval I transverse to $\mathcal{F}$ containing p. These are the possibilities:

- X is a closed leaf. Then $X \cap I$ is a discrete set.
- $X = M$. Then every leaf is dense and $X \cap I = I$.
- X is an exceptional minimal set, ie $X \cap I$ is a Cantor set (so no point of $X \cap I$ is isolated and $X \cap I$ is nowhere dense).

This is true for foliations of codimension one regardless of the smoothness of the foliation [4]. It also holds for the fully foliated set of a confoliation.

If L is an integral surface of a confoliation ξ (ie a surface tangent to ξ) and $\gamma\colon S^1 \to L$ a smooth map, then the holonomy along ξ is defined as follows: Fix an immersed annulus

$$\varphi\colon S^1 \times (-\delta, \delta) \to M$$

transverse to ξ such that $\varphi(z, 0) = \gamma(z)$. The characteristic foliation on this annulus has a closed leaf, namely $\gamma(S^1)$, and the Poincaré return map h_φ defined by parallel transport along the oriented curve γ is well defined on a neighbourhood of 0 in $(-\delta, \delta)$. The conjugacy class h_γ of the germ of h_φ depends only on γ; it is independent of φ. In particular, it makes sense to speak of attractive and repulsive holonomy (ie $|h_\gamma(x)| < |x|$ etc) or of fixed points on both sides of γ in the annulus.

Definition 2.6 The conjugacy class of this germ is the *holonomy* of ξ along γ, and ξ has *nontrivial linear holonomy* along γ if $h'_\gamma(0) \neq 1$. A foliation $\mathcal{F}$ is a *foliation without holonomy* if h_γ is the germ of the identity for all closed curves γ tangent to $\mathcal{F}$.

If ξ is a foliation, then the holonomy depends only on the free homotopy class of γ in the integral surface. If ξ is not a foliation, then we have at least the following lemma.

Lemma 2.7 *Let L be an integral surface of a confoliation. If h_γ has nontrivial linear holonomy, then the same is true for all curves which are freely homotopic to γ. Also, if Σ is a closed surface and γ is a nonseparating simple closed curve with attractive holonomy, then every curve isotopic to γ also has attractive holonomy.*

Proof The first statement is proved in [12, Lemma 1.3.17] when Σ is a closed surface. The general situation will be relevant when we consider applications of Sacksteder's theorem. Let γ' be a closed curve which is freely isotopic to γ. Then the cycle $\gamma \cap (-\gamma')$ bounds either an annulus or a collection of discs contained in L. Depending on the side of γ on which a disc from the collection or the annulus lies one applies (2-1) to the confoliation on one side of L to conclude that the holonomy of γ pushed across the disc is decreasing more above/below L than γ. If $\gamma \cup (-\gamma')$ bounds an annulus one adds a Legendrian curve λ connecting γ and γ' so that $\gamma \cup \lambda \cup (-\gamma') \cup (-\lambda)$ bounds a disc. The holonomy along the boundary is weakly decreasing by (2-1). It follows that the linear holonomy of ξ along γ and γ' has the same value.

The second statement also follows almost immediately from (2-1): Let γ, γ' be isotopic such that γ has attractive holonomy. Consider a covering of Σ such that lifts of γ remain closed but γ, γ' have disjoint lifts $\widetilde{\gamma}, \widetilde{\gamma}'$ and the annulus $\widetilde{A}$ between the two lifts has $(-\widetilde{\gamma} \cup \widetilde{\gamma}')$ as its oriented boundary. Such a covering exists because γ is nonseparating in Σ.

As above we pick an embedded arc $\lambda \subset \widetilde{A}$ connecting $\widetilde{\gamma}(0), \widetilde{\gamma}'(0)$. Then (2-1) applied to the disc bounding the concatenation of $\lambda, \widetilde{\gamma}', (-\lambda), (-\widetilde{\gamma})$ and the pulled back confoliation on a tubular neighbourhood $\Sigma \times (-\delta, \delta)$ of Σ such that the second factor is transverse to ξ implies
$$h_{\gamma'}(x) \leq h_\lambda \circ h_\gamma \circ h_\lambda^{-1}(x)$$
since $h_\gamma = h_{\widetilde{\gamma}}$ and $h_{\gamma'} = h_{\widetilde{\gamma}'}$. Also, we use the obvious definition of the holonomy h_λ along arcs. This implies the claim for $x > 0$. An analogous argument proves the claim for $x < 0$. □

It is easy to construct confoliations on neighbourhoods of surfaces such that the second conclusion of the lemma does not hold for a separating curve (when attractive is defined by strict inequalities). Note also that for the first part of the previous lemma we needed that the confoliation is defined on both sides of L.

Recall the Reeb stability theorem. It can be found eg in [4]. The last part in the statement below is a consequence of the usual Reeb stability theorem (compare [12, Proposition 1.3.7]).

Theorem 2.8 (Reeb stability) *Let $\mathcal{F}$ be a foliation and Σ a leaf of $\mathcal{F}$ diffeomorphic to a sphere. Then $\mathcal{F}$ is diffeomorphic to the foliation by the first factor on $S^2 \times S^1$. Every confoliation transverse to the fibres of $S^2 \times S^1 \to S^2$ is diffeomorphic to a foliation by spheres.*

This theorem holds with minimal smoothness assumptions on the foliation and is also true for confoliations. Since the product foliation on $S^2 \times S^1$ cannot be approximated by contact structures, spherical leaves do not play any role in the uniqueness problem.

2B Gray's theorem, surfaces in contact manifolds, convexity

In the proof of Theorem 1.4 will use Gray's theorem:

Theorem 2.9 *Let ξ_t be a smooth family of smooth contact structures on a closed manifold M. Then there is an isotopy ψ_t of M so that $\psi_{t*}(\xi_t) = \xi_0$ for all t.*

The proof of this theorem is based on Moser's method, which is described eg in [18]. Theorem 2.9 holds in the relative case (ie if ξ_t is constant on some domain, the resulting isotopy is then the identity on that domain) and it also works with parameters. By Gray's theorem, in order to prove Theorem 1.4, it suffices to find a neighbourhood of ξ so that for every pair of contact structures in that neighbourhood there is a family of contact structures interpolating between them.

The Moser method is omnipresent in all results producing contact isotopies, eg the theory of convex surfaces outlined below.

2B1 Characteristic foliations and their singular points Let Σ be an oriented surface embedded in a contact manifold. If Σ has boundary, then the boundary will be assumed to be Legendrian.

Definition 2.10 The *characteristic foliation* $\Sigma(\xi)$ on Σ is determined by the singular line field $\xi \cap T\Sigma$ on Σ; the singularities are points where $\xi(x) = T_x\Sigma$. A singularity is *positive* if $\xi_x = T_x\Sigma$ as oriented vector spaces, otherwise the singularity is *negative*. If $\Sigma(\xi)$ is one-dimensional at $x \in \Sigma$, then $\Sigma(\xi)(x)$ is oriented so that this orientation followed by the coorientation of ξ coincides with the orientation of the surface. An isolated singular point of the characteristic foliation is *elliptic* or *hyperbolic* if its index is $+1$ or -1, respectively.

The fact that ξ is a contact structure has strong consequences for the characteristic foliation on a small neighbourhood of the singularities of $\Sigma(\xi)$. Recall that according to Giroux [21] the divergence of a singular point of $\Sigma(\xi)$ never vanishes, its sign is well defined and coincides with the sign of the singularity. As the next lemma shows, this is the only property which distinguishes characteristic foliations of contact structures from general singular foliations:

Lemma 2.11 *Let $\mathcal{G}$ be a singular foliation on Σ such that there is a defining form α with $d\alpha \neq 0$ at all singular points. Then there is a contact structure ξ on $\Sigma \times (-1,1)$ such that $\Sigma_0(\xi) = \mathcal{G}$. This contact structure is unique up to an isotopy on a small neighbourhood of Σ_0 and the isotopy is tangent to the characteristic foliation on Σ_0.*

The following lemma shows that the dynamical properties of the characteristic foliation are quite restricted near isolated singular points. A part of this lemma can be found in [24, page 629].

Lemma 2.12 *Let $p \in \Sigma$ be an isolated singular point of the characteristic foliation $\Sigma(\xi)$. Then the index of p equals -1, 0 or $+1$ and the characteristic foliation on a neighbourhood of p is topologically conjugate to a neighbourhood of a hyperbolic, simply degenerate or elliptic singularity, respectively.*

Proof We assume that p is positive. Choose local coordinates x_1, x_2 on Σ around p with $x_1(p) = x_2(p) = 0$ and a 1–form defining α on a small neighbourhood of p so that there is a vector field V on Σ near p such that

$$i_V(d\alpha|_\Sigma) = \alpha|_\Sigma.$$

Since α is a contact form, $d\alpha$ is an area form on Σ near p and the vector field V is well defined near p. In terms of x_1, x_2,

$$V(x_1,x_2) = \begin{pmatrix} a_{11}(x_1,x_2) & a_{12}(x_1,x_2) \\ a_{21}(x_1,x_2) & a_{22}(x_1,x_2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + o(\|(x_1,x_2)\|)$$

for smooth functions $a_{ij}(x_1,x_2)$ with $a_{11}(0,0) + a_{22}(0,0) > 0$ because the divergence of V is positive at p. Hence the eigenvalues of $A = ((a_{ij}(p))_{i,j})$ are either both real and at least one of them is positive or both eigenvalues are complex with positive real part.

Unless 0 is an eigenvalue of A the singularity is nondegenerate and the index depends only on the sign of $\det(A)$. If 0 is an eigenvalue of A, then p is degenerate and by the centre manifold theorem (see eg [28, Theorem 3.2.1]) there is a 1–dimensional unstable manifold (uniquely defined and of class C^r) tangent to the eigenspace of the nonvanishing eigenvalue and a centre manifold Z (not necessarily unique and only of class C^{r-1}) tangent to the kernel of A. Both submanifolds are invariant under the flow of V.

The index of the singularity is now completely determined by the nature of the isolated zero at p of the restriction of V to Z. The index of p is -1 or 1 if p is an attractive or repelling singularity of $V|_Z$, respectively. If the singularity is attractive on one side while it is repelling on the other, then the index of p is 0.

This also shows that an isolated singularity with index ± 1 of the characteristic foliation of a contact structure is topologically equivalent to a nondegenerate singularity with the same index. If the index is 0, then the unstable manifold of the singularity decomposes a neighbourhood of p into two parts, one half-space is filled with integral curves of V whose α–limit set is p while the other half looks like the corresponding half-space of a hyperbolic singularity and the centre manifold is unique on that side. $\square$

Let p be a singularity of $\Sigma(\xi)$ and $U \subset \Sigma$ a neighbourhood of p such that $d\alpha|_U$ is an area form. Assume that p is a positive singularity of index -1 or 0. Then p has a stable leaf. Choose a point $x \in U$ on a stable leaf and a point $y \in U$ on the strong unstable manifold of p. We fix half-open intervals σ_x and σ_y containing x and y, respectively, such that there are leaves of the characteristic foliation in U connecting points in the interior of σ_x to σ_y (see Figure 2).

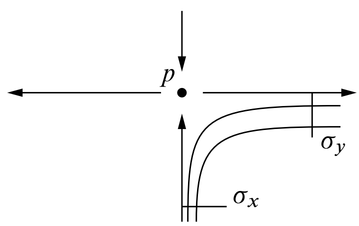

Figure 2: Holonomy near a positive singularity p with index -1

The positive divergence of p has consequences for the map φ from σ_x to σ_y defined by following the leaves of the characteristic foliation:

Lemma 2.13 *For all $K \in \mathbb{R}$ there is a neighbourhood of x in σ_x such that*

$$\varphi'(q) > K$$

for all $q \neq x$ in that neighbourhood.

Proof Let X be the vector field satisfying $i_X d\beta = \beta$, where $\alpha = dt + \beta$. Then the time-t flow ψ_t expands β exponentially in t, ie $\psi_t^* \beta = e^t \beta$. As q approaches x the time the flow takes to move q to $\varphi(q)$ goes to infinity because p is a singularity. This implies the claim. $\square$

As for closures of leaves in foliations, there is a classification of limit sets of leaves of singular foliations on surfaces provided that the foliation has finitely many singular points. According to [46, Theorem 2.6.1], the α–/ω–limit set of a leaf γ is one of the following:

- The leaf γ itself when γ is periodic.
- A singular point.
- A closed leaf of the foliation or a cycle consisting of a chain of stable leaves of singular points.
- A quasi-minimal set. These are α–/ω–limit sets of leaves containing a nontrivial recurrent leaf, ie a nonperiodic leaf $\widehat{\gamma}$ which is dense inside the quasi-minimal set.

2B2 Convexity In this section, we review the notion of convexity. The material presented here was developed by Giroux in [21]. Since the notion of convexity is standard in contact topology by now we will be very brief.

Definition 2.14 Let $\Sigma \subset (M,\xi)$ be an oriented surface in a contact manifold such that $\partial\Sigma$ is Legendrian. Then Σ is *convex* if there is a contact vector field transverse to Σ.

Building on [48], Giroux showed that convexity can be achieved by C^∞–small perturbations of Σ when Σ is closed. If Σ has Legendrian boundary, then according to Kanda [39], the same statement holds (at least for C^0–small perturbations fixing the boundary) if the twisting number of ξ along $\partial\Sigma$ is not positive. When Σ is convex, a lot of information about the contact structure near Σ is contained in the *dividing set* Γ. In order to define it we fix a contact vector field X transverse to Σ. Then

$$\Gamma = \{x \in \Sigma \mid X \in \xi_x\}.$$

It turns out that Γ is always a submanifold transverse to $\Sigma(\xi)$ whose isotopy type does not depend on the choice of X. Moreover, whether or not a surface is convex can be determined using only the characteristic foliation on Σ.

Lemma 2.15 (Giroux [21]) *Σ is convex if and only if it has a decomposition into two subsurfaces Σ^+, Σ^- with boundary such that the boundary $\partial\Sigma^+ = \partial\Sigma^-$ which is not part of $\partial\Sigma$ is transverse to $\Sigma(\xi)$ and there are defining forms for the singular foliation α_+ on Σ^+ and α_- on Σ^- such that $d\alpha_+ > 0$ and $d\alpha_- < 0$.*

In this case the dividing set is isotopic to the closure of the parts of $\partial\Sigma_\pm$ which are not contained in $\partial\Sigma$.

In other words, the dividing set of a convex surface separates the surface into two domains such that the characteristic foliation on each part is tangent to a Liouville vector field associated to an exact area form.

Given a closed convex surface one can compute the evaluation of the Euler class on that surface via the formula

$$\langle e(\xi), [\Sigma]\rangle = \chi(\Sigma^+) - \chi(\Sigma^-), \tag{2-2}$$

a fact which is attributed to Kanda [39]. The following lemma shows that characteristic foliations on convex surfaces can be manipulated effectively.

Definition 2.16 Let Σ be a compact oriented surface and Γ a collection of pairwise disjoint simple closed curves and arcs which are transverse to the boundary separating Σ into two surfaces with boundary Σ^+, Σ^-. A singular foliation $\mathcal{G}$ on Σ is *adapted* to Γ (or $\mathcal{G}$ is *divided* by Γ) if

(i) $\mathcal{G}$ is transverse to Γ, the boundary of Σ consists of leaves and singularities of $\mathcal{G}$, and

(ii) there are defining forms $\alpha_\pm$ for $\mathcal{G}$ on $\Sigma^\pm$ such that $d\alpha_+ > 0$ and $d\alpha_- < 0$.

Lemma 2.17 (Giroux [21]) *Let $\Sigma \subset (M,\xi)$ be a convex surface, X a transverse contact vector field, Γ the associated dividing set and $\mathcal{G}$ a singular foliation adapted to Γ. Then there is an isotopy $\varphi_s\colon \Sigma \to M$, $s \in [0,1]$, such that φ_0 is the inclusion and $\varphi_{1*}(\mathcal{G})$ is the characteristic foliation on $\varphi_1(\Sigma)$. Moreover, $\varphi_s(\Sigma)$ is transverse to X for all s and the characteristic foliation on $\varphi_s(\Sigma)$ is divided by*

$$\big\{\varphi_s(\Sigma) \cap \{x \in M \mid X \in \xi(x)\}\big\}.$$

A somewhat stronger version of this statement is Lemma 3.3. An immediate consequence of this lemma is the Legendrian realization principle [39; 34].

Lemma 2.18 *Let $\Sigma \subset (M,\xi)$ be a convex surface, Γ a dividing set and $C \subset \Sigma$ a simple closed curve transverse to Γ such that every connected component of $\overline{\Sigma \setminus C}$ meets Γ. Then there is an isotopy as in Lemma 2.17 such that $\varphi_1(C)$ is a Legendrian curve in $\varphi_1(\Sigma)$.*

A basic tool for controlling modifications of the dividing set on a given convex surface used in particular in the work of K Honda and J Etnyre is the attachment of bypasses.

Definition 2.19 (Honda [34]) A *bypass* for the convex surface $\Sigma \subset (M,\xi)$ is an oriented half-disc D which is embedded (except that the two corners of D may coincide) with the following properties:

(i) $\partial D = \gamma_1 \cup \gamma_2$ is the union of two smooth Legendrian arcs such that γ_1 is contained in Σ and intersects the dividing set Γ of Σ transversely in exactly
- three points, two of which are the endpoints of γ_1, or
- two points, if the endpoints of γ_1 coincide.

The bypass is called *singular* in the latter case.

(ii) The interior of D and the interior of γ_2 are disjoint from Σ.

(iii) All singular points of $D(\xi)$ along γ_2 are positive. Apart from $\gamma_1 \cap \gamma_2$ there is exactly one more singularity of $D(\xi)$ on γ_1. It is negative elliptic.

One boundary component Σ' of a neighbourhood of $D \cup \Sigma$ can be chosen such that Σ' is convex, diffeomorphic to Σ and the dividing set on Σ' is obtained from the dividing set on Σ by the operation shown in Figure 3 in Section 3A3. In that figure the dividing set is dashed and γ_1 is the diagonal arc in the left-most figure. For more information about bypasses and their applications we refer the reader to the papers [34; 35; 37] by Honda and the references therein.

2B3 Basins of attractive orbits The notion of a Legendrian polygon was introduced by Eliashberg in [10] as a framework to describe the closure of all leaves of the characteristic foliation of a contact structure on a generic surface which come from a fixed positive singularity. The situation for confoliations is slightly more complicated; this is discussed in [58]. Here we recall the terminology in a lengthy definition without restricting to unions of leaves coming from a particular singular point.

Definition 2.20 A *Legendrian polygon* (Q, V, α) on an oriented surface in a contact manifold (M, ξ) is a smooth immersion

$$\alpha\colon Q \setminus V \to \Sigma$$

such that Q is an oriented surface with boundary and corners, $V \subset \partial Q$ is a finite subset, α is an orientation-preserving embedding on the interior of Q, and each segment of $\partial Q \setminus V$ is mapped to a Legendrian arc. Smooth pieces of ∂Q are mapped to smooth Legendrian curves of $\Sigma(\xi)$. The points of V are called *virtual vertices* and corners correspond to singular points of the characteristic foliation.

For $v \in V$ we require that the two segments of ∂Q ending at v are mapped by α to two leaves of $\Sigma(\xi)$ such that the corresponding orientations of the two segments of $\partial Q \setminus V$ both point away or towards v. In the first and the second cases the images of these segments have the same α–limit set and ω–limit set, respectively, denoted γ_v, and γ_v is not a singularity of $\Sigma(\xi)$.

The preimage of a singularity p of $\Sigma(\xi)$ is a *pseudovertex* if it has index 0 or -1 and a neighbourhood (in ∂Q) of the preimage is mapped to two stable or two unstable leaves of p. Corners of ∂Q are mapped to singular points. If a corner is mapped to a hyperbolic singularity p of index 0 or -1, then one adjacent segment of ∂Q is mapped to a stable leaf while the other adjacent segment is mapped to an unstable leaf of p.

Let $\beta \subset \Sigma$ be a nondegenerate attractive closed orbit of $\Sigma(\xi)$. The following definitions (except the notions upper/lower) and lemmas also apply when β is a piecewise smooth closed curve consisting of negative singularities and unstable leaves of negative singularities.

Definition 2.21 Let B_β be the union of all leaves of $\Sigma(\xi)$ whose ω–limit set is β and which accumulate on β from a fixed side. This set is a *basin* of β. We say that B_β is the *upper* basin if it lies on the side of β in U determined by the coorientation of ξ; B_β is the *lower* basin if it lies on the opposite side.

The proof of the following lemma is completely analogous to the proof of [58, Lemma 3.2]. The assumptions made before the corresponding result [58, Lemma 3.4] about the nondegeneracy of singular points are not necessary and were made in order to facilitate the presentation (see also Lemma 2.12).

Lemma 2.22 *Let B_β be a basin of an attractive closed leaf of $\Sigma(\xi)$. There is a Legendrian polygon (Q, V, α) on Σ with $Q = [0,1] \times S^1$ such that*

(i) $\alpha(\{0\} \times S^1) = \beta$, *and*

(ii) $\alpha(Q \setminus V) \cup \bigcup_{v \in V} \gamma_v = \overline{B}_\beta$.

We say that the Legendrian polygon *covers* the basin.

2C Properties of contact structures and foliations

In this section we summarize definitions concerning geometric properties of foliations, contact structures and confoliations.

Definition 2.23 A foliation $\mathcal{F}$ is *taut* if for every leaf L of $\mathcal{F}$ there is a closed curve transverse to $\mathcal{F}$ which intersects L. A *Reeb component* is a foliation on $S^1 \times D^2$ such that the boundary is a leaf. An oriented foliation of $S^1 \times [0,1]$ by lines is a two-dimensional Reeb component if the boundary curves are leaves which are oriented in opposite directions. A foliation is *minimal* if every leaf is dense.

In Figure 16 in Section 6 one can see a pair of two-dimensional Reeb components.

The following definition of tight confoliations is an extension of the usual definition of tightness for contact structures as introduced in [12].

Definition 2.24 An *overtwisted disc* in a confoliated manifold is an embedded closed disc D such that ∂D is Legendrian and all singularities of $D(\xi)$ on ∂D have the same sign. A contact structure is *tight* if there is no overtwisted disc, otherwise it is *overtwisted*. A contact structure is *universally tight* if the pullback of ξ to the universal covering $\widetilde{M}$ of M is tight.

A confoliation is called *tight* if for every overtwisted disc D there is an integral D' of ξ with the following properties:

(i) D' is a disc, $\partial D' = \partial D$, and

(ii) $\langle e(\xi), [D \cup D'] \rangle = 0$, where $e(\xi) \in H^2(M;\mathbb{Z})$ is the Euler class of ξ and D, D' are oriented in such a way that their union is also oriented.

This definition interpolates between tight contact structures (in this situation there are no integral discs D') and foliations without Reeb components. Tight contact structures (and foliations without Reeb components) satisfy the Thurston–Bennequin inequalities. In order to state them, let Σ be an oriented compact embedded surface (whose boundary is positively transverse to the plane field ξ). Then

$$\begin{aligned} \langle e(\xi), [\Sigma] \rangle &= 0 && \text{if } \Sigma \cong S^2, \\ |\langle e(\xi), [\Sigma] \rangle| &\leq -\chi(\Sigma) && \text{if } \Sigma \not\cong S^2 \text{ is closed}, \\ -\langle e(\xi), [\Sigma] \rangle &\leq -\chi(\Sigma) && \text{if } \partial\Sigma \neq \varnothing, \end{aligned} \tag{2-3}$$

where the left-hand side of the last inequality is the obstruction for the extension of the trivialization of ξ along the boundary of Σ given by $\Sigma(\xi)$ to the interior. As shown in [58], the Thurston–Bennequin inequalities for tight confoliations do not hold in general, while they are always satisfied for s-tight confoliations. Recall that $H(\xi)$ is the open set where the confoliation ξ is a contact structure.

Definition 2.25 Let ξ be a confoliation on M. An *overtwisted star* on a compact embedded surface $\Sigma \subset M$ is a Legendrian polygon $\alpha\colon D^2 \setminus V \to \Sigma \subset M$, where $V \subset \partial D$ is a finite set of points such that the following hold:

- $\alpha(\partial D \setminus V)$ is a union of Legendrian arcs such that for all $v \in V$ the image of the two arcs approaching v on ∂D have the same ω–limit set γ_v when the arcs are oriented towards v, and $\gamma_v \cap H(\xi) = \varnothing$.
- All singularities of $\Sigma(\xi)$ on $\alpha(\partial D \setminus V)$ have the same sign, and this sign is opposite to all singularities in $\alpha(\mathring{D})$.

A tight confoliation without overtwisted stars is *s-tight*.

Tight contact structures are considered to be much more interesting than overtwisted contact structures because of the following classification result of Eliashberg [9].

Theorem 2.26 *For ξ_0 a contact structure on a closed manifold M with an overtwisted disc D, let $\mathrm{Cont}(M, D, \xi_0)$ be the space of contact structures which have D as an overtwisted disc and are homotopic as plane fields to ξ_0 relative to D.*

The space $\mathrm{Cont}(M, D, \xi_0)$ is weakly contractible. In particular, two overtwisted contact structures are isotopic if and only if they are homotopic as plane fields.

Moreover, there are interesting analogies between taut foliations and symplectically fillable contact structures, and between foliations with Reeb components and overtwisted contact structures.

It is easy to show that tightness implies s-tightness in the context of the following theorem.

Theorem 2.27 (Eliashberg and Thurston [12]) *Let ξ be a confoliation on $\mathbb{R}^3$ which is transverse to the fibres of the projection $\mathbb{R}^3 \to \mathbb{R}^2$ and complete as a connection of this bundle. Then ξ is tight (and s-tight).*

Example 2.28 The 1–form $dz + f(x,y,z)\,dy$ defines a contact structure if $\partial f/\partial x > 0$; it defines a confoliation if $\partial f/\partial x \geq 0$. A simple case when ξ is a complete connection of the bundle

$$\mathbb{R}^3 \to \mathbb{R}^2, \quad (x,y,z) \mapsto (x,y)$$

is when f is an affine or bounded function.

Usually, tightness of a contact structure is shown by either embedding the contact manifold into a contact structure which is already known to be tight, or by using symplectic fillings or gluing theorems (eg from [5]).

Definition 2.29 Let M be a closed oriented manifold and ξ a confoliation. A symplectic manifold (X, ω) is a *weak symplectic filling* of (M, ξ) if

(i) $M = \partial X$ as oriented manifolds (where X is oriented by $\omega \wedge \omega$ and the outward normal first convention is used to orient the boundary), and

(ii) $\omega|_\xi$ is a symplectic vector bundle.

The following theorem is due to M Gromov (for the case when ξ is a confoliation see [58]).

Theorem 2.30 *If a contact manifold (M,ξ) admits a weak symplectic filling, then it is tight.*

This criterion is used in [12] to show the following result about contact structures approximating taut foliations.

Theorem 2.31 *Every contact structure which is sufficiently C^0–close to a taut foliation is universally tight.*

In general a symplectically fillable contact structure does not have to be universally tight but at least there is a very efficient criterion to decide whether or not there is a universally tight neighbourhood of a convex surface.

Lemma 2.32 (Giroux's criterion) *Let Σ be a convex surface in a contact manifold (M,ξ) and Γ its dividing set. If $\Sigma \cong S^2$, then we require that Γ is connected, otherwise we ask that no component of Γ bounds a disc in Σ.*

Then Σ has a neighbourhood so that the restriction of ξ to that neighbourhood is universally tight.

This lemma applies to the case when Σ is a sphere in a contact manifold such that $\Sigma(\xi)$ has exactly two singular points and all leaves of $\Sigma(\xi)$ connect the two singular points. Such a sphere is automatically convex. The following corollary of Giroux's criterion can be found in [37].

Corollary 2.33 *Let ξ be a tight oriented contact structure near an oriented closed surface Σ with positive genus such that $\langle e(\xi),[\Sigma]\rangle = \pm\chi(\Sigma)$ and Σ is convex. Then Σ^- or Σ^+ is a nonempty union of annuli.*

Proof No component of the dividing set of Σ bounds a smooth disc. Hence all components of Σ^+ and Σ^- have nonpositive Euler characteristic. The claim is now an easy consequence of the equalities

$$\langle e(\xi),[\Sigma]\rangle = \chi(\Sigma^+) - \chi(\Sigma^-) \quad \text{and} \quad \chi(\Sigma) = \chi(\Sigma^+) + \chi(\Sigma^-). \qquad \square$$

According to the Thurston–Bennequin inequalities (2-3) the situation considered in the corollary corresponds to the maximal possible absolute value of the evaluation of the Euler class on a closed surface in a tight contact manifold or a foliation without Reeb components. A contact structure ξ on $\Sigma \times [0,1]$ will be called *extremal* if $|\langle e(\xi),[\Sigma]\rangle| = -\chi(\Sigma)$, where Σ is an oriented closed surface.

There is another invariant associated to contact structures which can distinguish diffeomorphism classes of tight contact structures.

Definition 2.34 Let (M,ξ) be a contact manifold. For every positive integer n we consider the contact structures

$$\xi_n = \ker\big(\cos(2\pi nt)\,dx_1 - \sin(2\pi nt)\,dx_2\big)$$

on $T^2 \times [0,1]$. The *Giroux torsion* of (M,ξ) is

$$\sup\big\{m \mid m=0 \text{ or } m\in\mathbb{N}^+ \text{and there is a contact embedding } (T^2\times[0,1],\xi_m)\to(M,\xi)\big\}.$$

In the previous definition one can specify the isotopy class of the embedding of $T = T \times \{0\}$. If such an embedding is specified (eg by a torus leaf of a foliation) then we will sometimes refer to the Giroux torsion along T. The contact structures ξ_k from Example 1.2 have Giroux torsion $k-1$ and are hence distinguished by this invariant.

2D Classification results for tight contact structures

There are several classification results for tight contact structures up to isotopy relative to the boundary that we shall use. They concern B^3, $S^1 \times D^2$, $T^2 \times [0,1]$ and $\Sigma \times [0,1]$, where Σ is a surface with genus $g \geq 2$. The following result is fundamental. It is usually stated in a weaker form covering only the connectedness of the space of contact structures. The version given here is implicitly contained in Eliashberg [10, Theorem 2.4.2] and stated in Giroux [22].

Theorem 2.35 *The space of positive tight contact structures on $(B^3, \partial B^3)$ which induce a fixed characteristic foliation $\mathcal{G}_0$ on ∂D is weakly contractible. It is nonempty if and only if $\mathcal{G}_0$ admits a taming function.*

Taming functions and their construction are the main tool that we use in the proof of Theorem 2.35. For the definition of taming functions and other material we refer to [10] and [58]. Admittedly, there are no proofs with parameters in these references. However, note that by Lemma 2.12 there are taming functions on neighbourhoods of degenerate isolated singularities of characteristic foliations. Moreover, the space of functions taming a fixed characteristic foliation on a sphere is contractible [10, Remark 4.4.4]. Taming functions increase along leaves of the characteristic foliation and they can be constructed parametrically provided that the contact structures are tight.

Finally, the proof of Theorem 2.35 also shows that for a family of characteristic foliations $\mathcal{G}_s$ on ∂D there is a family of contact structures ξ_s on B^3 such that $(\partial B^3)(\xi_s) = \mathcal{G}_s$ for all s provided that there is a family of taming functions for the foliation $\mathcal{G}_s$. Also, this family of extensions is unique up to homotopy.

2D1 Contact structures on solid and thickened tori A lot of information about the classification of tight contact structures on the solid torus up to isotopy can be found in [34; 24]. We will only need the following simple case but we give a parametric version.

Theorem 2.36 *Let ξ be a tight contact structure on $N = D^2 \times S^1$ with convex boundary such that the dividing set has exactly two connected components and the intersection number of each component with a meridional disc is ± 1.*

Then the space of positive tight contact structures on N which coincide with ξ near ∂N is weakly contractible.

Proof By Lemma 2.17 we may assume that $\partial N(\xi)$ has the following properties:

(i) There are two cancelling pairs of singularities, one negative and the other one positive. There are no closed orbits and no connections between hyperbolic singularities.

(ii) Both unstable leaves of the positive hyperbolic singularity are connected to the negative elliptic singularity and their union bounds a meridional disc D in N. The union of both stable leaves of the negative hyperbolic singularities also bounds a meridional disc D' and $D'(\xi)$ is convex with respect to ξ.

Let S be a compact manifold and let ξ_s, $s \in S$, be a smooth family of tight contact structures on N with $\xi_s = \xi$ near ∂N. We will construct a family of contact structures ξ_s' with $\xi_s' = \xi$ near ∂N such that the characteristic foliation on D' is constant while $D(\xi_s) = D(\xi_s')$ for all $s \in S$. Then by Theorem 2.35 applied to the ball with (set-theoretic) boundary $\partial N \cup D'$, the family ξ_s' of contact structures is homotopic to the constant family ξ_τ for a fixed $\tau \in S$, ie the family ξ_s, $s \in S$, is homotopic to a constant family.

We now construct the family ξ_s', $s \in S$. The properties of ξ near ∂N imply that we can fix two curves γ_1, γ_2 on ∂N transverse to $\partial N(\xi)$ separating ∂D from $\partial D'$. The contact structure ξ naturally extends to a slightly thicker solid torus N'. We choose two smooth embedded spheres Σ_1, Σ_2 such that

- Σ_i contains a neighbourhood of γ_i in ∂N for $i = 1, 2$,
- $\Sigma_1 \cap \Sigma_2 = D \cup D'$, and
- $\Sigma_i \setminus (D \cup D')$ does not meet the interior of N.

The question whether or not a given singular foliation on Σ_i which is transverse to γ_i admits a taming function depends only on the characteristic foliations on the discs $\Sigma_i \setminus \gamma_i$. Hence the singular foliations on Σ_i given by ξ on $\Sigma_i \setminus D$ and by $D(\xi_s)$

on D admit taming functions and by Theorem 2.35 there is a family of tight contact structures on each of the balls bounded by these spheres. Together they form a family of tight contact structures ξ'_s on N. By construction $D(\xi'_s) = D(\xi)$. □

The theorem below contains the information from [24, Theorem 4.4] about the classification of tight contact structures on the thickened torus $T^2 \times [-1, 1]$ that we are going to use. (The theorem is stated in a way which can be easily translated to the terminology developed in [24]. This terminology is explained in Section 3A below.)

Theorem 2.37 (Giroux) *Let $\mathcal{G}_{\pm 1}$ be two foliations on T^2 which have exactly $n_{\pm 1} > 0$ attractive closed leaves such that there is no Reeb component of dimension* 2 *and all closed leaves are nondegenerate.*

If the slopes of the closed leaves of $\mathcal{G}_{\pm 1}$ are different, then for a given integer $k \geq 0$ there is a contact structure ξ on $T^2 \times [-1, 1]$ whose Giroux torsion is k and

(i) $T_{\pm 1}(\xi) = \mathcal{G}_{\pm 1}$,

(ii) *for all $t \in [-1, 1]$ all singularities of the characteristic foliation on $T^2 \times \{t\}$ are positive, and*

(iii) *there is an embedded torus $T' \subset T^2 \times (-1, 1)$ isotopic to T_0 such that $T'(\xi)$ is a foliation by closed leaves.*

This contact structure is uniquely determined by these properties.

If the slopes of the closed leaves of $\mathcal{G}_{\pm 1}$ coincide then the same statement holds except that for $k = 0$, the contact structure is I–invariant and there is no torus satisfying (iii).

We will see later in Section 3B3 that a contact structure satisfying the assumptions (i)–(iii) of the theorem is automatically universally tight. Theorem 2.37 is then obtained from [24, Theorem 4.4] using Remark 3.15. We will apply the uniqueness part of Theorem 2.37; the case of I–invariant contact structures will not occur.

2D2 Contact structures on $\Sigma \times [-1, 1]$ with $g(\Sigma) \geq 2$ Before we can state the main classification result about tight contact structures on $N = \Sigma \times [-1, 1]$ with $|\langle e(\xi), [\Sigma]\rangle| = -\chi(\Sigma)$ and convex boundary, we need to recall the notion of the *relative Euler class* from [37]. We will choose the orientation of ξ such that $\langle e(\xi), [\Sigma]\rangle = \chi(\Sigma)$, ie Σ_t^- is a nonempty union of disjoint annuli whenever Σ_t is convex. The dividing set of $\Sigma_{\pm 1}$ will be denoted by $\Gamma_{\pm 1}$.

Let $\beta \subset \Sigma_i$, $i = \pm 1$, be a closed curve. We say that β is *primped* if the following conditions hold:

- β is nonisolating in Σ_i, ie β is transverse to Γ_i, and every boundary component of $\overline{\Sigma_i \setminus (\beta \cup \Gamma_i)}$ meets Γ_i.
- The intersection $\beta \cap \Sigma_i^-$ consists only of arcs each of which does not separate an annulus in Σ_i^- into two connected components.

Clearly every curve is isotopic to a primped curve. According to the Legendrian realization principle (Lemma 2.18) there is a C^0–small isotopy of Σ_i through convex surfaces so that β is a Legendrian curve on the isotoped surface. In order to define the relative Euler class $\widetilde{e}(\xi)$ on $[\beta \times I] \in H_2(N, \partial N, \mathbb{Z})$ we proceed as follows: Isotope $\beta \times \{\pm 1\}$ to primped Legendrian curves in $\Sigma_{\pm 1}$. Then consider an annulus A bounded by the two Legendrian curves (since Σ is not T^2 this annulus is uniquely determined up to isotopy relative to the boundary). After a small perturbation we may assume that the annulus is convex. In analogy to (2-2) one defines

$$\langle \widetilde{e}(\xi), [\beta \times I] \rangle := \chi(A^+) - \chi(A^-). \tag{2-4}$$

The following is shown in [37]:

Proposition 2.38 *The relative Euler class $\widetilde{e}(\xi) \in H^2(N, \partial N; \mathbb{Z})$ is well defined and extends the Euler class $e(\xi)$ viewed as a homomorphism $e(\xi)\colon H_2(N;\mathbb{Z}) \to \mathbb{Z}$ to $H_2(N, \partial N; \mathbb{Z})$.*

Now we can state [37, Theorem 1.1]:

Theorem 2.39 (Honda, Kazez and Matić) *Let Σ be a closed oriented surface of genus $g \geq 2$ and $\mathcal{G}_{\pm 1}$ singular foliations on $\Sigma \times \{\pm 1\}$ such that $\mathcal{G}_{\pm 1}$ is adapted to a dividing set $\Gamma_{\pm 1}$ consisting of exactly two nonseparating closed curves bounding an annulus such that*

$$\chi(\Sigma_{-1}^+) - \chi(\Sigma_{-1}^-) = \chi(\Sigma_1^+) - \chi(\Sigma_1^-).$$

If Γ_{-1} and Γ_1 are not isotopic, then there are exactly four isotopy classes of tight contact structures ξ on N so that $\Sigma_{\pm 1}(\xi) = \mathcal{G}_{\pm 1}$. They are distinguished by the relative Euler class $\widetilde{e}(\xi)$ which takes the values

$$\mathrm{PD}(\widetilde{e}(\xi)) = \pm \gamma_{-1} \pm \gamma_{+1} \in H_1(N, \mathbb{Z})$$

where $\gamma_{\pm 1}$ is a connected component of $\Gamma_{\pm 1}$.

If Γ_{-1} and Γ_1 are isotopic, then there are exactly five isotopy classes of tight contact structures on N inducing the given characteristic foliation on ∂N. Three of these contact structures satisfy $\mathrm{PD}(\widetilde{e}(\xi)) = 0$, while the two remaining isotopy classes satisfy

$$\mathrm{PD}(\widetilde{e}(\xi)) = \pm 2\gamma_{-1} = \pm 2\gamma_1 \in H_1(N, \mathbb{Z}).$$

In all of the above cases the tight contact structures are universally tight.

We still need to explain how to distinguish tight contact structures with $\mathrm{PD}(\widetilde{e}(\xi)) = 0$. This is done by embedding properties. The following definition of a basic slice is not quite the same as [37, Definition 5.12], but the two definitions are equivalent by Theorem 2.39 (and the setup used in [37] to analyze what is called the base case in [37, page 323]).

Definition 2.40 A *basic slice* is a tight contact structure on $\Sigma \times [-1, 1]$ such that

(i) Σ_{-1} and Σ_1 satisfy all assumptions of Theorem 2.39,

(ii) γ_{-1} and γ_1 intersect exactly once, and

(iii) $\mathrm{PD}(\widetilde{e}(\xi)) = \pm(\gamma_{-1} - \gamma_1)$ when γ_{-1}, γ_1 are oriented so that $\gamma_1 \cdot \gamma_{-1} = 1$.

As in [37], we denote a basic slice by $[\![\gamma_{-1}, \gamma_1; \pm(\gamma_{-1} - \gamma_1)]\!]$ depending on the value of the relative Euler class.

The definition of a basic slice is independent of the orientation of γ_{-1}, γ_1 satisfying $\gamma_1 \cdot \gamma_{-1} = 1$.

Proposition 2.41 (Honda, Kazez and Matić [37]) *Let ξ be a tight contact structure on N such that $\Gamma_{-1} = \Gamma_1 = 2\gamma$. Then ξ is isotopic to a vertically invariant contact structure if and only if there is no embedding of a basic slice $[\![\gamma, \gamma'; \pm(\gamma - \gamma')]\!]$. There are two tight contact structures ξ_+, ξ_- such that there are contact embeddings*

$$[\![\gamma, \gamma'; +(\gamma - \gamma')]\!] \to (N, \xi_+),$$
$$[\![\gamma, \gamma'; -(\gamma - \gamma')]\!] \to (N, \xi_-)$$

mapping the boundary component Σ_0 of the basic slice to Σ_{-1}, while there are no contact embeddings

$$[\![\gamma, \gamma'; -(\gamma - \gamma')]\!] \to (N, \xi_+),$$
$$[\![\gamma, \gamma'; +(\gamma - \gamma')]\!] \to (N, \xi_-)$$

with the same property.

The relative Euler class behaves well when $\Sigma \times [-1, +1]$ is decomposed along Σ_0 provided that the contact structure on $\Sigma \times [-1, 1]$ is tight and $\Sigma_{\pm 1}, \Sigma_0$ are convex such that the dividing set consists of two connected nonseparating curves. As in [37, Theorem 6.1], this is best expressed as follows:

$$\mathrm{PD}(\widetilde{e}(\xi|_{\Sigma \times [-1,0]})) + \mathrm{PD}(\widetilde{e}(\xi|_{\Sigma \times [0,1]})) = \mathrm{PD}(\widetilde{e}(\xi|_{\Sigma \times [-1,1]})). \tag{2-5}$$

3 Movies and their properties

In this section we explain some of the material in Giroux's work [24] about families of characteristic foliations of positive contact structures on $\Sigma \times [-1,1]$, where Σ is a closed oriented surface. Parts of this material can also be found in [18]. Our main omission is that we do not discuss the turnaround locus (*lieu de retournement* in [24]). The results from Section 3A and Section 3B were used by Giroux to obtain a classification of tight contact structures on torus bundles and lens spaces. Some of the results proved in Section 3A are probably folklore (like Lemma 3.16) but we did not find a good reference. We will apply some of the techniques of Giroux to contact structures on $\Sigma \times [-1,1]$ with convex boundary: The main result from Section 3C1 is new and will be used in combination with Theorem 2.39 in the proof of Theorem 1.4.

3A Movies associated to contact structures

Let ξ be a contact structure and $\Sigma \subset M$ an embedded oriented surface with Legendrian boundary such that there is a tubular neighbourhood $\Sigma \times [-1,1]$ such that the boundary of $\Sigma_t := \Sigma \times \{t\}$ is Legendrian.

According to the (strong) Thom transversality theorem we may assume that all singular points of the characteristic foliation $\Sigma(\xi) = T\Sigma \cap \xi$ on Σ, ie points $p \in \Sigma$ where $T_p\Sigma = \xi(p)$, are isolated. This remains true for all surfaces appearing in compact finite-dimensional families.

Fix a tubular neighbourhood $\Sigma \times [-1,1]$ of Σ so that ξ is positive with respect to the product orientation of $\Sigma \times [-1,1]$. We fix a 1–form α defining ξ on $\Sigma \times [-1,1]$ so that

$$\alpha = \lambda_t + u_t\, dt, \tag{3-1}$$

where λ_t vanishes on $\partial\Sigma_t$ and u_t is a smooth function. In particular, λ_t defines the characteristic foliation of ξ on $\Sigma_t = \Sigma \times \{t\}, t \in [-1,1]$. This family will be referred to as a *movie* of ξ.

The contact condition $\alpha \wedge d\alpha > 0$ is equivalent to

$$u_t\, d\lambda_t + \lambda_t \wedge (du_t - \dot{\lambda}_t) > 0. \tag{3-2}$$

It has implications for the singular foliations appearing in the movie; see Lemma 3.7 and Lemma 3.10 below. Recall also that the contact condition implies that the divergences at positive and negative singular points of $\Sigma_t(\xi)$ are positive and negative, respectively. Although it is not clear which families of singular foliations are movies associated to a positive contact structure, there is the following uniqueness result (Lemma 2.1 of [24]).

Lemma 3.1 *Let ξ_0 and ξ_1 be two positive contact structures on $\Sigma\times[-1,1]$ such that $\Sigma_t(\xi_0)=\Sigma_t(\xi_1)$ for all $t\in[-1,1]$. Then ξ_0 and ξ_1 are isotopic.*

Proof The movie of a positive contact structure determines the family λ_t up to multiplication with a nowhere-vanishing function when we consider only 1–forms λ_t coming from a defining form as in (3-1). The set of functions u_t satisfying (3-2) for a given family of 1–forms λ_t is convex. Hence the lemma follows from Gray's theorem. □

There are a few situations when it is easy to show that a given family of singular foliations is the movie of a positive contact structure.

Lemma 3.2 *Let $\mathcal{G}_t$, $t\in[-1,1]$, be a family of singular foliations on Σ such that there is a continuous family of curves Γ_t dividing $\mathcal{G}_t$, ie for every t there is a smooth function v_t on Σ such that*

$$v_t\,d\lambda_t+\lambda_t\wedge dv_t>0. \tag{3-3}$$

Then there is a contact structure ξ on $\Sigma\times[-1,1]$ such that $\Sigma_t(\xi)=\mathcal{G}_t$.

Proof Notice that we made no assumption concerning the dependence of v_t on t. But the set of functions v_t satisfying (3-3) for a given family of 1–forms λ_t on Σ is convex. By compactness of the interval we can replace v_t by a family of smooth functions with the same properties as v_t which in addition depends smoothly on t. We use the notation v_t for the new family. Then for k sufficiently large, the 1–form

$$kv_t\,dt+\lambda_t$$

on $\Sigma\times[-1,1]$ is a contact form with the desired properties. □

The following lemma is the corresponding uniqueness result.

Lemma 3.3 [24, Lemma 2.7] *Let ξ_0,ξ_1 be two contact structures on $\Sigma\times[-1,1]$ which coincide near the boundary such that the characteristic foliations $\Sigma_t(\xi_0)$ and $\Sigma_t(\xi_1)$ are divided by Γ_t, where Γ_t varies continuously with $t\in[-1,1]$.*

Then ξ_0 and ξ_1 are isotopic relative to the boundary.

Proof The contact structures ξ_i, $i=0,1$, are defined by 1–forms $u_t^i\,dt+\lambda_t^i$. Since Γ_t divides the characteristic foliations $\Sigma_t(\xi_i)$ there is a family of functions v_t^i for $i=0,1$ such that

- 0 is a regular value of v_t^i and $\Gamma_t=(v_t^i)^{-1}(0)$, and
- $v_t^i\,d\lambda_t+\lambda_t^i\wedge dv_t^i>0$.

Since we can multiply λ_t^1 by the nowhere-vanishing function $h = v_t^0/v_t^1$, we may assume that $v_t^0 = v_t^1 =: v_t$. Then for every positive constant k the family of 1–forms

$$(su_t^i + (1-s)kv_t)\,dt + \lambda_t^i, \quad s \in [0,1],\ i = 0,1$$

on $\Sigma \times [-1,1]$ is a family of contact forms. Moreover, for k sufficiently large the 1–forms

$$kv_t\,dt + s\lambda_t^0 + (1-s)\lambda_t^1, \quad s \in [0,1]$$

on $\Sigma \times [-1,1]$ are contact. The conclusion follows again from Gray's theorem. □

We will need the following slight modification of Lemma 3.3 which gives a relative version of the previous lemmas.

Lemma 3.4 *Let ξ, ξ' be contact structures on $\Sigma \times [-1,1]$ so that there is a family of compact subsurfaces $F_t \subset \Sigma_t$ such that*

(i) *$\Sigma_t(\xi) = \Sigma_t(\xi')$ outside of F_t,*

(ii) *∂F_t is transverse to $\Sigma_t(\xi)$, and*

(iii) *there are contact forms α, α' defining ξ, ξ' such that $d\alpha|_{F_t} > 0$ and $d\alpha'|_{F_t} > 0$.*

Then ξ and ξ' are isotopic.

Moreover, given a contact structure ξ defined by α, a family of domains F_t with properties (ii)–(iii) *and a smooth family λ_t of 1–forms such that $d\lambda_t > 0$ on F_t with $\alpha|_{F_t} = \lambda_t$ on a neighbourhood of the boundary of*

$$\bigcup_{-1 \le t \le 1} F_t,$$

there is a contact structure ξ' on $\Sigma_t \times [-1,1]$ which coincides with ξ outside of $\bigcup_t F_t$ whose characteristic foliation is defined by λ_t inside of F_t.

Proof We begin with the existence part. We may assume that there is a domain with boundary $F \subset \Sigma$ so that $F_t = F \times \{t\}$ and the characteristic foliations of ξ near ∂F_t are independent of t. Fix a collar C of ∂F. The part of ∂C in the complement of F will be denoted by γ_{out}, and we set $\gamma_{\text{in}} = \partial C \setminus \gamma_{\text{out}}$.

Since ∂F_t is transverse to $\Sigma_t(\xi)$ we may choose C so thin that every leaf of $C(\xi)$ connects two boundary components of C and $\alpha|_{C_t \cap F_t} = \lambda_t$. Without loss of generality, assume that $C_t(\xi)$ is constant. By (ii)–(iii), the characteristic foliations point out of F_t along the boundary.

The angle between ξ and F_t along γ_{out} is bounded away from 0 by a constant ν. Since $d\lambda_t > 0$ the 1–form $\lambda_t + k\,dt$ defines a positive contact structure on $F \times [-1,1]$. If k is sufficiently large, then the angle between F_t and $\xi' = \ker(\lambda_t + k\,dt)$ is smaller than ν along γ_{in}. Since $C(\xi)$ is a product foliation, the contact structure ξ' defined inside of $F \times [-1,1]$ can be extended to a contact structure on $\Sigma \times [-1,1]$ by twisting along the leaves of $C(\xi)$.

The proof that the resulting contact structure ξ' on $\Sigma \times [-1,1]$ is isotopic to ξ is analogous to the proof of Lemma 3.3. $\square$

Before we proceed with manipulations of movies we state the result of an explicit computation we will use later.

Assume that ξ is a contact structure on a family of annuli A_t, $t \in [0,1]$, defined by $\alpha = dt + \lambda_t$, such that the characteristic foliation is transverse to the boundary of A_t for all t and points outwards. The contact condition implies that $d_A\lambda_t + \dot{\lambda}_t \wedge \lambda_t$ is an area form (here d_A is the exterior differential on the annuli).

Let X be a vector field on $A \times [0,1]$ without zeroes tangent to the characteristic foliation but pointing in the opposite direction and let φ_s be the flow of X defined for $s \geq 0$. The functions f_X and h_X are defined by

$$\begin{aligned} i_X d_A\lambda_t &= h_X \lambda_t, \\ i_X(d_A\lambda_t + \dot{\lambda}_t \wedge \lambda_t) &= f_X \lambda_t. \end{aligned}$$

By the contact condition f_X is bounded away from 0 and negative. Then

$$\varphi_s^*\alpha = \exp\left(\int_0^s \varphi_\sigma^* h_X \, d\sigma\right)\left(\lambda_t + \exp\left(-\int_0^s \varphi_\sigma^* f_X \, d\sigma\right) dt\right). \tag{3-4}$$

This shows that the contact structure defined by $\varphi_s^*\alpha$ converges to the tangent planes of the annulus (this is probably familiar from [21]).

3A1 Elimination of singularities The following lemma is a translation of [24, Lemma 2.15]. It is a version of the elimination lemma from [10] which also controls the characteristic foliation not only on one surface Σ but also on nearby surfaces.

Lemma 3.5 *Let ξ be a contact structure on $\Sigma \times [-1,1]$ such that $\Sigma_0(\xi)$ has two singular points e_0, h_0 which have the same sign (we will assume it is negative) and are connected by a leaf of $\Sigma_0(\xi)$. Moreover, for $-1 < t < 1$, assume that there are singularities e_t, h_t of $\Sigma_t(\xi)$ such that h_t is connected to e_t by a continuous family of leaves C_t of $\Sigma_t(\xi)$. We fix $0 < \delta < 1$ and a neighbourhood U of $\bigcup_{|t| \leq \delta} \overline{C}_t$ such that e_t, h_t are the only singular points of $\Sigma_t(\xi)$ inside $\Sigma_t \cap U$.*

(i) *There is an isotopy of* $\Sigma\times[-1,1]$ *with support inside* U *such that the characteristic foliation of the isotoped contact structure has no singular points in* $\Sigma_t\cap U$ *for* $|t|\le\delta$.

(ii) *Given a family of points* $a'_t\in\Sigma_t\setminus\overline{C}_t$ *connected to* e_t *by a leaf of* $(\Sigma_t\cap U)(\xi)$ *and another family* b'_t *of points on the unstable leaf of* b_t *which is opposite to* C_t, *the isotopy in* (i) *can be chosen so that, for the isotoped contact structure, the leaf of the characteristic foliation on* Σ_t *through* b'_t *passes arbitrarily close to* a'_t.

(iii) *Denote by* G_t *the union of all segments of leaves of* $\Sigma_t(\xi)$ *with both endpoints in* U. *Assume that for a given compact set* $\Lambda\subset(-1,1)$ *the restriction of the characteristic foliation on* Σ_t *to* G_t *can be defined by a* 1*–form with nowhere-vanishing exterior derivative. Then the isotopy from* (i)–(ii) *can be chosen such that the isotoped copy of* Σ_t *is convex whenever* Σ_t *is convex*

Pairs of singular points of $\Sigma(\xi)$ like e_0,h_0 in the above lemma will be referred to as a *cancelling pair*.

We will use a partial converse of this result which allows us to create cancelling pairs of singular points. On a single surface it is possible to introduce a cancelling pair of singularities without any restriction. But by Lemma 3.10 we cannot arbitrarily prescribe the limit set of the (un-)stable leaf of the hyperbolic singularity which does not connect the elliptic singularity of the pair.

3A2 Closed leaves in movies In this section we discuss a result from Giroux's paper [24] about closed leaves of characteristic foliations.

First, let $\gamma\subset\Sigma_t$ be a nondegenerate closed leaf of the characteristic foliation. Let $V\subset\Sigma_t$ be a tubular neighbourhood of γ such that ∂V is transverse to $\Sigma_t(\xi)$ and γ is the unique closed leaf of $\Sigma_t(\xi)$ in V. Since γ is nondegenerate, the characteristic foliation is transverse to ∂V and points either outwards along both components or inwards along both components. Because the characteristic foliation depends smoothly on t the same is true for $\Sigma_{t'}$ for t' sufficiently close to t. Moreover, $\Sigma_{t'}(\xi)$ has a unique closed leaf. The union of these leaves is a smooth submanifold transverse to Σ_t.

Now let $\gamma\subset\Sigma$ be a degenerate closed leaf and φ the germ of the holonomy along γ with respect to a fixed segment σ through γ transverse to the characteristic foliation. We assume that the degeneracy of γ is finite, ie $\varphi^{(k)}(0)\neq 1$ for some $k\in\{1,2,\dots\}$ and $\varphi^{(j)}(0)=1$ for $j=0,\dots,k-1$. This is a C^∞–generic property.

We first discuss the case when k is even. Depending on the sign of $\varphi^{(k)}(0)$ there are two possibilities.

Definition 3.6 We say that γ is *positive* or *negative* if the holonomy of γ is repulsive or attractive, respectively, on the side of γ given by the coorientation of ξ while the behaviour on the other side is the opposite.

The following lemma can be found by combining [24, Lemma 2.12 and following remark].

Lemma 3.7 *Let γ be a positive degenerate closed orbit of $\Sigma_\tau(\xi)$. Then for $t > \tau$ and t close to τ there is a pair of nondegenerate closed leaves of $\Sigma_t(\xi)$ close to γ. One of these orbits is repulsive while the other one is attractive. For $t < \tau$ there is no closed leaf of Σ_t near γ.*

For negative degenerate orbits, the situation is opposite.

Thus positive and negative degenerate closed leaves indicate the birth and death, respectively, of a pair of closed leaves of the characteristic foliation.

The case when k is odd is simpler. Following the proof of [24, Lemma 2.12], we obtain:

Lemma 3.8 *If $\gamma \subset \Sigma_\tau$ has odd degeneracy, then the characteristic foliation on a surface Σ_t sufficiently close to Σ_τ has a single closed leaf γ_t near γ, and γ_t is attractive or repulsive if and only if the same is true for γ and γ_t is nondegenerate for $t \neq \tau$. The union of the closed leaves γ_t is a smooth embedded surface in M.*

So, to summarize this section, if γ is a closed leaf $\Sigma_t(\xi)$, then the union of nearby closed leaves on nearby surfaces Σ is a smooth submanifold.

3A3 Retrograde saddle-saddle connections Let ξ be a contact structure on $\Sigma \times [-1, 1]$.

Definition 3.9 A stable leaf η of a positive hyperbolic singularity of the characteristic foliation on Σ_0 which coincides with the unstable leaf of a negative hyperbolic singularity is a *retrograde saddle-saddle connection*.

The fact that ξ is a positive contact structure has consequences for the characteristic foliation on Σ_t when there is a retrograde saddle-saddle connection in $\Sigma_0(\xi)$. In the following lemma the words *over* and *under* refer to the coorientation of the leaves of the characteristic foliation.

Lemma 3.10 (Giroux [24]) *A retrograde saddle-saddle connection η on Σ_0 implies that for $t<0$ (resp. $t>0$) sufficiently close to 0, the stable leaf η of the positive hyperbolic singularity passes under (resp. over) the unstable leaf of the negative hyperbolic singularity.*

A retrograde saddle-saddle connection is depicted in Figure 3. If the surface Σ_t is convex for $t \neq 0$ then a bypass attachment along the thickened arc in the leftmost figure has the same effect on the dividing set (represented by dashed curves) as the retrograde saddle-saddle connection.

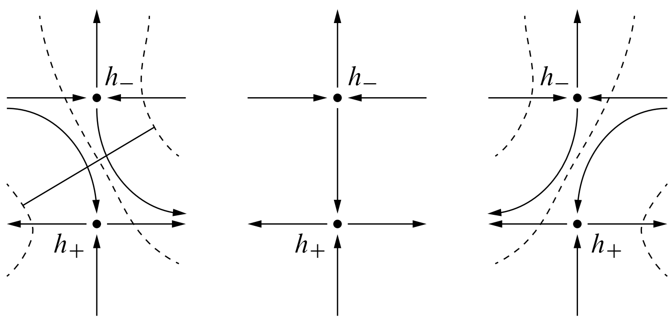

Figure 3: Retrograde saddle-saddle connection

Later we will want to reduce the number of retrograde saddle-saddle connections. In some situations, the classification of tight contact structures on the ball can be used for this.

Lemma 3.11 *Let ξ be a tight contact structure on $\Sigma \times [-1,1]$ such that the characteristic foliation on Σ_0 has a single retrograde saddle-saddle-connection η between a positive hyperbolic singularity h_+ and a negative hyperbolic singularity h_-.*

Assume that both unstable leaves of h_+ connect to the same negative elliptic singularity e_-, the retrograde saddle-saddle connection is the only unstable leaf of a negative hyperbolic singularity of $\Sigma_0(\xi)$ ending at e_- and Σ_t is convex for all $t \neq 0$. Then there is a contact structure ξ' isotopic to ξ with the following properties:

- *$\Sigma_t(\xi')$ is convex for all $t \in [-1,1]$. In particular, there are no retrograde saddle-saddle connections.*
- *ξ' coincides with ξ outside of a tubular neighbourhood of the union of the unstable leaves of $\Sigma_0(\xi)$.*

The situation considered in this lemma is depicted in Figure 4. The dashed lines represent the dividing set. Lemma 3.11 is tailored to one application. The assumptions imply that the dividing set on $\Sigma_t(\xi)$ does not change when one passes from $t<0$ to $t>0$. Therefore the lemma is a consequence of the discussion of trivial bypasses in [36, Lemma 2.10].

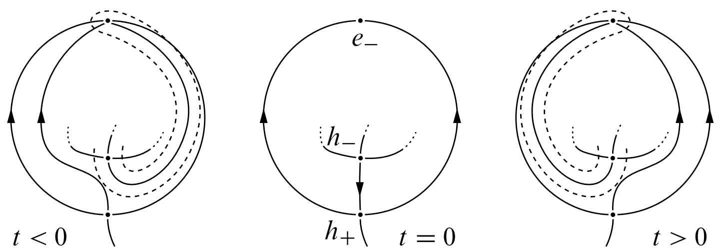

Figure 4: The movie considered in Lemma 3.11

Let us describe how the movie of the contact structure with a retrograde saddle-saddle-connection as in Lemma 3.11 can be replaced by a movie which is convex at all levels.

Let $V \subset \Sigma_0$ be a tubular neighbourhood of the two unstable leaves of h_+ whose boundary is transverse to $\Sigma_0(\xi)$. After adding pairs of cancelling positive singularities and a small perturbation on the complement of a neighbourhood of the retrograde saddle-saddle connection if necessary, we may assume that the basin of V is compact. Then $\Sigma_t(\xi)$ and V_t have the same properties as V for $t \in [-\delta, \delta]$ close to 0. Now replace the movie $V_t(\xi)$ by a family of singular foliations obtained by rotating the characteristic foliation in the annulus V clockwise starting with $\Sigma_{-\delta}(\xi)$ so that as t increases no stable leaf of h_+ meets the point where the unstable leaf of h_- enters V_t and $V_\delta \subset \Sigma_\delta(\xi)$ makes a full twist. This is possible because there is exactly one unstable leaf of a negative singularity entering V.

The singular foliations obtained in this way all admit dividing sets. By Lemma 3.2 the movie of singular foliations is the movie of a contact structure ξ' on $\Sigma \times [-\delta, \delta]$ which coincides with ξ on the boundary.

The assumption that e_- is a negative elliptic singularity which is not connected to any negative hyperbolic singularity can be replaced by the following assumption: both unstable leaves of h_+ end on the same connected component e_- of the graph formed by negative elliptic singularities, attractive closed leaves and negative hyperbolic singularities together with their unstable leaves, and e_- is a closed tree. The annulus V above then has to contain the entire tree.

3A4 Sheets of movies Consider a contact structure on $\Sigma\times[-1,1]$ and the movie of characteristic foliations $\Sigma_t(\xi)$ with t varying in $[-1,1]$.

Definition 3.12 A *sheet* of the movie is a smooth embedded surface $A\subset\Sigma\times[-1,1]$ such that every connected component of $A\cap\Sigma_t$ is a smooth Legendrian curve and all singularities of $\Sigma_t(\xi)$ on the curve have the same sign.

These surfaces play a very important role in [24]. In that paper, a sheet is referred to as *feuille* while the collection of all sheets is called *feuillage*. By definition, A is foliated by circles. Therefore A is either a torus, an annulus, a Klein bottle, or a Möbius band. In this paper A will always by orientable, so A will be either a torus or an annulus. The following lemma is part of [24, Lemma 3.17]. It implies that a sheet A is really foliated by closed Legendrian curves:

Lemma 3.13 *Every sheet A is a pre-Lagrangian surface, ie $A(\xi)$ is a nonsingular foliation by closed Legendrian curves.*

Averaging the contact form using a flow which is tangent to $A(\xi)$ and periodic, we obtain a contact form α whose restriction to A is closed. Then $A\subset M$ is a Lagrangian submanifold of the symplectization $(M\times\mathbb{R},d(e^t\alpha))$ of (M,α). Moreover, A is either tangent to Σ_t along a given closed Legendrian curve $\beta\subset\Sigma_t\cap A$ or A is everywhere transverse to Σ_t along β. More precisely, A is tangent to Σ_t along a Legendrian curve $\beta\subset A$ if and only if β is a degenerate closed leaf of $\Sigma_t(\xi)$.

We consider the situation when ξ is transverse to $\mathcal{I}$, the foliation given by the second factor of $\Sigma\times[-1,1]$. In this case the behaviour of a sheet relative to the product decomposition $\Sigma\times[-1,1]$ is subject to restrictions which we now describe.

Let α be a contact form which is closed on A and $\beta\subset\Sigma_t\cap A$ a nondegenerate attractive closed leaf. Because α is a contact form the 1–form $(d\alpha)(\dot\beta,\cdot)$ is nonvanishing. Moreover, $d\alpha(\dot\beta,X)$ is positive when $(\dot\beta,X)$ is an oriented basis of ξ (by the contact condition and because of the fact that α coorients ξ). Since β is attractive the 2–form $d\alpha$ is a negative area form on $T\Sigma|_\beta$. By Lemma 3.13 $d\alpha(\dot\beta,\cdot)$ vanishes on A and both ξ and Σ are cooriented by $\mathcal{I}$. Therefore the lines

$$\mathcal{I}_p,\quad \xi_p/\dot\beta,\quad T_pA/\dot\beta,\quad T_p\Sigma_t/\dot\beta \tag{3-5}$$

appear in this order in the projective line $\mathbb{P}(T_pM/\dot\beta)$. Hence the slope of ξ is steeper at $p\in\beta$ than the slope of A in that point (we interpret the second factor in $\Sigma\times[-1,1]$ as height). This is shown in Figure 1 where the parts of sheets consisting of attractive closed leaves of $\Sigma_t(\xi)$ are thickened. If β is repulsive, then $\xi(p)$ is closer to $T_p\Sigma_t$

than T_pA. Moreover, if A is tangent to $\mathcal{I}$ at a point $p \in \Sigma_t$, then the closed leaf γ_p of T_pA is a Legendrian curve on Σ_t which is either closed and repelling or which contains some positive singularities, and after elimination of these singularities we again obtain a closed repelling leaf.

Proposition 3.14 *Let ξ be a contact structure on $\Sigma \times [-1,1]$ which is transverse to and cooriented by the second factor and β a closed attractive leaf of $\Sigma_t(\xi)$.*

Then β is contained in a sheet $A(\beta)$ of the movie $\Sigma_t(\xi)$ consisting of nondegenerate attractive closed leaves of $\Sigma_t(\xi)$. The restriction of the projection $\mathrm{pr}\colon \Sigma \times [-1,1] \to \Sigma$ *to this sheet is a submersion. As t increases the image of $\Sigma_t \cap A$ moves in the direction opposite to the direction determined by the coorientation of ξ.*

Proof Let β be an attractive closed leaf of $\Sigma_t(\xi)$. Then this curve is part of a sheet $A(\beta)$ by the implicit function theorem and this also implies that $A(\beta)$ is transverse to the surfaces Σ_t. By (3-5) all vectors in the tangent space $TA(\beta)$ which are not tangent to β lie in those connected components of $TM \setminus (\xi \cup T\Sigma_t)$ which do not contain elements of $T\mathcal{I}$. Since pr_* is an isomorphism on $T\Sigma_t$ and $\xi|_\beta$ it is an isomorphism on TA, too. $\square$

Hence sheets are transverse to Σ_t away from degenerate closed leaves and they are transverse to $\mathcal{I}$ along attractive pieces. According to Lemma 3.7 the locus where sheets are not transverse to Σ_t corresponds to degenerate closed leaves of $\Sigma_t(\xi)$.

Remark 3.15 Proposition 3.14 has important consequences for how sheets are embedded into $\Sigma \times [-1,1]$ when the contact structure is transverse to the foliation $\mathcal{I}$ given by the second factor.

Let β be a nondegenerate attractive closed curve in Σ_{-1} (the following discussion for $\beta \subset \Sigma_{+1}$ is completely analogous). If one moves on the sheet $A(\beta)$ containing β, then the t–coordinate increases until either a degenerate closed orbit or Σ_{+1} is reached. We will consider only the first case. Moreover, we assume that this degenerate leaf is of birth-death type since otherwise the sheet simply continues.

By Lemma 3.7 the degenerate closed orbit is negative and after we cross the degenerate closed leaf the sheet consists of repulsive closed orbits of Σ_t and the t–coordinate decreases as we move on A away from β.

Along the part of $A(\beta)$ which consists of either closed repulsive leaves of $\Sigma_t(\xi)$ or of a graph consisting of unstable leaves of positive hyperbolic singularities connected to positive elliptic singularities such that the graph is diffeomorphic to a circle, the

t–coordinate decreases until a degenerate closed orbit of birth-death type or Σ_{-1} is reached unless the sheet simply ends in a level surface Σ_t (this happens for example if an elliptic singularity of the movie lying on A forms a cancelling pair with a hyperbolic singularity which does not lie on A such that these two singularities merge on Σ_t). Now assume that a degenerate orbit in $\Sigma_{t'}$ is reached on $A(\beta)$ and the orientation of this orbit is opposite to the orientation of β. Then the t–coordinate increases as we move on the sheet $A(\beta)$ away from β, but the part of the sheet consisting of attractive closed leaves of $\Sigma_t(\xi)$ is now trapped inside a solid torus bounded by the sheet and an annulus in $\Sigma_{t'}$. Therefore the sheet $A(\beta)$ reaches the highest level

$$t(A(\beta)) = \sup\{t \in [-1,1] \mid A(\beta) \cap \Sigma_t \neq \varnothing\}$$

along an attractive closed leaf of $\Sigma_{t(A(\beta))}(\xi)$ which is attractive or degenerate, and the orientation of this leaf coincides with the orientation of β provided that the supremum above is actually attained. The next thing we show is that this is always the case.

The situation described here is depicted in Figure 5 (which contains some notation and a dashed line that will be explained later). The parts of $A(\beta)$ which consist of attractive leaves of $\Sigma_t(\xi)$ are thickened.

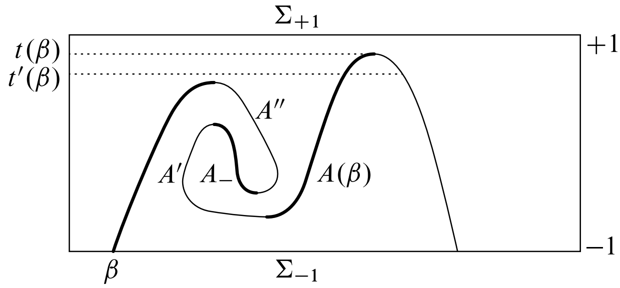

Figure 5: Sheet in $\Sigma \times [-1,1]$ containing $\beta \subset \Sigma_{-1}$

The following lemma shows that degenerate closed orbits are the *only* way in which attractive closed orbits of a movie in a contact manifold appear or disappear (the lemma is wrong when ξ is just a plane field) when ξ is transverse to the second factor of $\Sigma \times [-1,1]$. This also provides a natural way to compactify sheets consisting of attractive closed leaves of $\Sigma_s(\xi)$.

Lemma 3.16 *Let A be a sheet consisting of closed attractive leaves of $\Sigma_t(\xi)$ such that $\Sigma_t \cap A$ is nonempty for $t \in [-1,b)$.*

The annulus $A \subset \Sigma \times [-1,b)$ can be compactified by adding a closed leaf γ_b of the characteristic foliation of Σ_b. The holonomy of γ_b is attractive on the side determined by the coorientation of ξ. If $A \subset \Sigma \times (b,1]$, then the holonomy of γ_b is attractive on the side opposite to the coorientation of ξ.

Proof We consider the case $A \subset \Sigma \times [-1, b)$. The set $\mathcal{L} = \bar{A} \cap \Sigma_b$ is nonempty, closed and saturated (ie a union of leaves of the characteristic foliation).

If $\mathcal{L}$ contains a nontrivial recurrent leaf ρ, ie a nonclosed leaf which accumulates on itself, then we can find a closed curve τ on Σ_b transverse to $\Sigma_b(\xi)$ through a given point of ρ. Because ρ is recurrent this leaf intersects τ infinitely many times and all intersection points are transverse with the same sign when τ is oriented. Then the intersection number of $\beta_t = A \cap \Sigma_t$ with τ is unbounded as t approaches b. But this is absurd since the homology class of $\beta_t \subset \Sigma_t \cong \Sigma$ is constant. Therefore $\mathcal{L}$ does not contain a nontrivial recurrent leaf.

Now assume that $\mathcal{L}$ contains a degenerate closed leaf as a proper subset (there could be a chain of degenerate closed leaves connected by leaves of the characteristic foliation). Then every degenerate closed leaf is positive because otherwise two closed leaves of $\Sigma_t(\xi)$ would intersect A for $t < b$ by Lemma 3.7. Figure 6 depicts a configuration with one positive and one negative degenerate orbit.

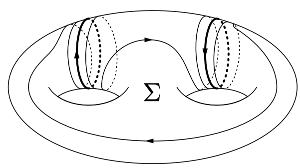

Figure 6: Impossible limit configuration

But when all degenerate closed leaves are positive then using the leaves of $\Sigma_b(\xi)$ which connect the degenerate closed orbits in $\mathcal{L}$ one can construct a closed curve τ transverse to $\Sigma_b(\xi)$ which intersects the degenerate closed leaves. This leads to the same contradiction as above. Thus $\mathcal{L}$ is a closed attractive leaf, or it is degenerate, or $\mathcal{L}$ contains a cycle. If $\mathcal{L}$ is an attractive leaf we compactify A by adding it. Also, if $\mathcal{L}$ is degenerate, then it has to be negative by Lemma 3.7 and serves as a natural compactification of A.

In order to finish the proof we have to exclude the possibility that $\mathcal{L}$ contains a cycle λ consisting of stable/unstable leaves of singularities of index 0 or -1. For this recall that all singularities of $\Sigma_t(\xi)$ are positive. Therefore the holonomy of the characteristic foliation is strongly repelling (this is the property described in Lemma 2.13) when one passes from a stable leaf to an unstable leaf of the characteristic foliation.

We choose short transversals σ_i, $i \in \mathbb{Z}_m$, of $\Sigma_b(\xi)$, one for each stable leaf in the cycle λ, in cyclic order and intersecting $\mathcal{L}$ exactly once (as the segments σ_x, σ_y in Figure 2 in Section 2B). Let $x_i = \sigma_i \cap \rho$. The holonomy of $\Sigma_b(\xi)$ along λ is defined on the side where the attractive closed leaves in Σ_t accumulate on λ. Thus we obtain homeomorphisms φ_i from a half-open interval in σ_i to a half-open interval in σ_{i+1} (the boundary points of the half-open intervals are $x_i = \sigma_i \cap \lambda$). These homeomorphisms are smooth away from x_i, and $\varphi_i'(x_i) = \infty$ for all $i \in \mathbb{Z}_m$ by Lemma 2.13.

Consider the attractive leaf $\beta_t = A \cap \Sigma_t$ for $t < b$ close to b and the holonomy diffeomorphisms ψ_t it induces on open sets of τ_i to open sets of τ_{i+1}. Let $y_{t,i} := \tau_i \cap \beta_t$. Since β_t is attractive,

$$\psi_{t,1}'(y_{t,1}) \cdot \dots \cdot \psi_{t,m}'(y_{t,m}) < 1.$$

Therefore we may (after choosing sequences and subsequences) assume without loss of generality that $\psi_{t,1}'(y_1) < 1$. But on the side of λ where the holonomy of $\Sigma_b(\xi)$ is defined $\psi_{t,1}$ converges uniformly to φ_1. This contradicts the fact that φ_1 is very repelling.

So $\mathcal{L}$ is either an attractive closed leaf or a negative degenerate orbit of $\Sigma_b(\xi)$. □

According to Lemma 3.7 one can extend the sheet beyond $\bar{A} \cap \Sigma_b$. Whenever there are conditions which ensure that there is a closed repulsive leaf or union of stable leaves of singular points of the characteristic foliation such that this union is the boundary of the basin of $A \cap \Sigma_t$, we will assume that this circle is smooth and we extend the sheet we are considering as far as possible. A condition which often ensures that sheets can be extended easily is that

- the contact structure is tight, and
- the basin of $A \cap \Sigma_t$ is contained in an annulus bounded by attractive closed leaves of $\Sigma_t(\xi)$.

Finally, we fix some terminology. We could say that a connected sheet is maximal if it is not a proper subset of a connected sheet. The problem with this definition is that leaves of characteristic foliations in a smooth sheet A can contain singularities (all of the same sign). Therefore, a smooth Legendrian curve in $A \cap \Sigma_t$ can be the limit of a family of nonsmooth Legendrian curves (the nonsmooth points are elliptic singularities of the characteristic foliation) which would naturally extend the sheet if they were smooth. However, the nonsmoothness of the curves can be easily corrected using for example Lemma 3.4.

Definition 3.17 A connected sheet A is *maximal* if it is not a proper subset of a smooth connected sheet and no component of ∂A is the limit of nonsmooth Legendrian curves in Σ_t such that all singularities have the same sign.

3A5 Simplifying the dynamics of characteristic foliations in movies Let Σ be a closed surface with positive genus $g \geq 1$. The purpose of this section is to describe how contact structures on $\Sigma \times [-1,1]$ can be isotoped so that the characteristic foliations on Σ_t have relatively simple dynamical properties when Σ_t is not convex with respect to the isotoped contact structure.

Definition 3.18 A surface Σ in a contact manifold has the *Poincaré-Bendixson property* if $\Sigma(\xi)$ has no nontrivial recurrent orbits.

If Σ has the Poincaré–Bendixson property and $\Sigma(\xi)$ has only finitely many singularities, then according to [46] all limit sets of leaves of $\Sigma(\xi)$ are

- closed leaves, or
- singular points, or
- cycles formed by singularities and leaves connecting them.

An embedded closed surface in a contact manifold has this property after a C^∞–generic perturbation. The point of [24, Lemma 2.10] is to ensure this property for all those surfaces $\Sigma_t \subset \Sigma \times [-1,1]$ which are not convex. We are going to use the following simple refinement of that lemma.

Lemma 3.19 *Let ξ be a contact structure on $N = \Sigma \times [-1,1]$ such that the boundary surfaces are convex. Then there is an isotopy of ξ relative to the boundary such that after the isotopy Σ_t has the Poincaré–Bendixson property for all $t \in [-1,1]$ for which Σ_t is not convex.*

If there is a sheet $A(\beta)$ such that one boundary component β_+ of $A(\beta)$ is contained in Σ_1 while the other boundary component β_- is contained in Σ_{-1} and β_+, β_- are nondegenerate and both attractive or both repelling, then the isotopy can be chosen to preserve the sheet $A(\beta)$.

Proof The proof follows Giroux's proof of [24, Lemma 2.10] closely. We summarize the required changes and the main idea. Let us first recall that the Poincaré–Bendixson theorem (see [30, page 154]) states that a singular foliation on the plane or the sphere has no nontrivial recurrent orbits.

For concreteness we assume that β_+ and β_- are both attractive. Then there is a family of annuli $P_t \subset \Sigma_t$ containing $\Sigma_t \cap A(\beta)$ (this intersection may have several connected components) such that ∂P_t is transverse to $\Sigma_t(\xi)$. We chose the identification $N \cong \Sigma \times [-1,1]$ such that $P_t = P \subset \Sigma$ is constant. Now fix a graph F so that

(i) $F \cup P$ is planar,

(ii) the complement $\Sigma \setminus F \cup P$ is also planar, and

(iii) F is nonisolating in Σ_{+1} and in Σ_{-1}.

Then F can be realized as graph consisting of Legendrian curves, negative elliptic and positive hyperbolic singularities. There is a positive number δ such that all surfaces Σ_t with $t \in [-1, -1+3\delta] \cup [1-3\delta, 1]$ are convex. Using Lemma 3.3 and the usual proof of the Legendrian realization principle (Lemma 2.18) we can now isotope ξ near the boundary of N so that

- the isotopy is supported in the interval determined by $|t-1| \leq 3\delta$,
- the characteristic foliation on Σ_t is constant for $t \in [-1+\delta, -1+2\delta]$ and $t \in [1-2\delta, 1-\delta]$, and
- F is a leaf of the characteristic foliation on Σ_t for $t \in [-1+\delta, -1+2\delta]$ and $t \in [1-2\delta, 1-\delta]$,

where all surfaces Σ_t are convex while keeping ξ constant on ∂N so that for suitable small real numbers $\delta_\pm > 0$, the graph F is realized as a Legendrian graph in $\Sigma_{1-\delta_+}$ and $\Sigma_{-1+\delta_-}$. Clearly, this can be done without changing anything near $A(\beta)$.

A thickening of F combined with P is a planar subsurface F_{in} of Σ whose complement is also planar. In addition, choosing the thickening appropriately, we may assume that the characteristic foliation on $\Sigma_{1-\delta_+}$ and $\Sigma_{-1+\delta_-}$ is transverse to ∂F^{in}. Let F^{out} be the complement of F^{in} with a collar of the boundary removed. The collar is chosen so that, following leaves of the characteristic foliation on the collar, one gets a retraction of the collar onto $\partial F^{\mathrm{out}}$. The characteristic foliations point out of F^{out} and into F^{in} for $t \in [-1+\delta, -1+2\delta] \cup [1-2\delta, 1-\delta]$.

Now choose a strictly monotone function $g\colon [0,1] \to [1-2\delta, 1]$ such that $g = \mathrm{id}$ on $[1-\delta, 1]$. Pick an isotopy ϕ_τ of N which translates along leaves of $\mathcal{I}$ such that

$$\begin{aligned} \phi_1(F_t^{\mathrm{in}}) &= F_{g(t)}^{\mathrm{in}} && \text{for } t \geq 0, \\ \phi_1(F_t^{\mathrm{out}}) &= F_{-g(-t)}^{\mathrm{out}} && \text{for } t \leq 0. \end{aligned}$$

The contact structure $\widehat{\xi} = \phi_{1*}^{-1}(\xi)$ has the desired properties: for $t \in [-1+\delta, 1-\delta]$ there are no nontrivial recurrent orbits in $\Sigma_t(\widehat{\xi})$ by the Poincaré–Bendixson theorem, and Σ_t is convex with respect to $\widehat{\xi} = \xi$ when $t \in [-1, -1+\delta]$ or $t \in [1-\delta, 1]$. □

Of course, Lemma 3.19 also holds in the presence of several sheets with the same properties as $A(\beta)$. The proof above implies that the resulting contact structure can be assumed to be C^1–generic with respect to the surfaces Σ_t, ie we can make genericity assumptions concerning for example the nature of connections between hyperbolic singularities.

3B Manipulations and properties of sheets

In this section we explain how to manipulate sheets and circumstances under which it is possible to find overtwisted discs from certain configurations of sheets.

3B1 Simplifying sheets The next lemma is part of the proof of [24, Proposition 3.22]. It allows us to isotope ξ so that the sheet contains fewer degenerate closed curves after the isotopy. For example, the part $A' \cup A_- \cup A''$ of $A(\beta)$ in Figure 5 can be replaced by collection of attractive closed leaves of $\Sigma_t(\xi)$.

Lemma 3.20 *Let $A \subset (\Sigma \times [-1,1], \xi)$ be a sheet such that A is the union of three sheets A', A_-, A'' with the following properties:*

(i) *$\gamma' = A' \cap A_-$ and $\gamma'' = A'' \cap A_-$ are degenerate closed orbits with parallel orientations.*

(ii) *$A_- \cap \Sigma_t$ is a smooth attractive Legendrian curve unless $t = t_{\min}$ or $t = t_{\max}$, with*

$$t_{\min} = \min\{t \in [-1,1] \mid A_- \cap \Sigma_t \neq \varnothing\},$$
$$t_{\max} = \max\{t \in [-1,1] \mid A_- \cap \Sigma_t \neq \varnothing\}.$$

(iii) *For all $t \in (t_{\min}, t_{\max})$ there is a compact annulus $S_t \subset \Sigma_t$ whose unoriented boundary consists of $A_- \cap \Sigma_t$ and $A' \cap \Sigma_t$ such that S_t intersects no other sheets of ξ.*

Then there is a family of contact structures ξ_s, $s \in [0,1]$, with $\xi_0 = \xi$ which is constant near ∂A such that, after the deformation, there is a new sheet A_1 which coincides with A_0 near the boundary where $A_1 \cap \Sigma_t$ is either empty or an attractive closed leaf.

The next lemma shows that a given degenerate closed leaf of birth-death type can be replaced by a retrograde saddle-saddle connection. However, without additional assumptions it is not possible to exclude the formation of degenerate closed leaves passing through a given neighbourhood of the original degenerate leaf.

Lemma 3.21 *Let ξ be a contact structure on $\Sigma \times [-1,1]$ and $\gamma \subset \Sigma_t$ with $t \in (-1,1)$ a degenerate closed orbit which is attractive on one side while it is repelling on the other side. Then there is a contact structure ξ' which is isotopic to ξ, coincides with ξ outside of an arbitrarily small neighbourhood of Σ_t and γ is replaced by a retrograde saddle-saddle connection such that there is no degenerate closed leaf of $\Sigma_t(\xi')$ in a neighbourhood of γ.*

Proof We consider the case when the degenerate closed orbit is negative. We use the following model contact structure on $A\times[-\delta,\delta], \delta>0$, where $A=S^1\times[-1,1]$ is an annulus. On A_0 we fix the following singular foliation:

(i) Both boundary components are parallel nonsingular Legendrian curves, one of which is repelling and the other one attractive. They are both nondegenerate.

(ii) There are four nondegenerate singular points $e_\pm, h_\pm$. Here $e_\pm$ and $h_\pm$ are elliptic and hyperbolic points, respectively, with the $\pm$ indicating the sign of the divergence.

(iii) There is a retrograde saddle-saddle connection starting at h_- and ending at h_+ such that for $t>0$ the stable leaf which participates in the retrograde saddle-saddle connection comes from the boundary of the annulus.

(iv) The remaining stable leaf of h_+ comes from e_+, the remaining unstable leaf of h_- ends at e_-.

(v) One stable leaf of h_- comes from the repulsive boundary component, the other from e_+. The unstable leaves of h_+ connect h_+ to e_- and to the attractive boundary component.

By Lemma 2.11 this singular foliation is the characteristic foliation on A_0 of a contact structure on $A\times[-\delta,\delta]$. By Lemma 3.10 the stable leaf of h_+ in $A_t(\xi)$ which participates in the retrograde connection comes from e_+ when $t<0$ and from the repulsive boundary for $t\neq 0$. The only nonconvex level is A_0.

By Lemma 3.5 we can eliminate h_+, e_+ and h_-, e_- in A_0 and since there is a unique leaf connecting the hyperbolic singularity $h_\pm$ to $e_\pm$ there is a unique way to eliminate the singularities. Outside of a neighbourhood of 0, either there is a pair of parallel closed leaves in the interior of the annulus, or all leaves of the characteristic foliation (except the boundary leaves) start at one boundary component and go to the other. (Part (iii) of that lemma can be used to arrange that, away from a neighbourhood of A_0, there are exactly two or zero closed leaves in the interior of the annulus.)

After the elimination, the contact structure is transverse to a rank-1 foliation transverse to A_s, $s\in(-1,1)$. According to Theorem 2.27 the contact structure is tight and there are only three sheets: two at the boundary and one in the interior of the annuli. After a deformation of the interior sheet there is exactly one negative degenerate orbit at exactly one level.

This proves the claim in the model case. In order to deal with the general case note that the degenerate leaf is part of a sheet (as explained in Section 3A2). The isotopy used in Lemma 3.20 to reduce the number of connected components of the intersection of a sheet with surfaces in a product neighbourhood can be reverted so as to increase this number (this is called folding; see also [34, Figure 19]). For example, the sheet in Figure 5 in Section 3A4 can be obtained by folding a sheet which intersects each surface at most twice.

After folding the sheets outside of a small neighbourhood of the degenerate closed leaf we want to eliminate, and isotoping the folded part of the sheet to the surface containing the degenerate leaf, we apply the construction of the model case. Then we undo the folding using Lemma 3.20 to get the desired result. □

The above lemmas allow us to arrange that the set of instances when Σ_t is not convex is discrete and the only cause of nonconvexity is the presence of retrograde saddle-saddle connections.

Lemma 3.22 *Let ξ be a contact structure on $\Sigma \times [-1,1]$ with convex boundary and $A_1,\dots,A_n$ a collection of sheets consisting of attractive closed leaves of Σ_t connecting the two boundary components of $\Sigma \times [-1,1]$.*

There is a contact structure ξ_{PB} isotopic to ξ relative to the boundary and $A_1 \cup \dots \cup A_n$ such that for all $t \in [-1,1]$ when $\Sigma_t(\xi_{\mathrm{PB}})$ is not convex there is a single retrograde saddle-saddle connection. The number of t with Σ_t not convex with respect to ξ_{PB} is finite. Moreover, the upper basin of $A_i \cap \Sigma_t$ is compact for all t where $\Sigma_t(\xi_{\mathrm{PB}})$ is not convex.

Proof We assume that $n = 1$ and abbreviate $A_1 = A$. First, we arrange that the characteristic foliation on Σ_t has no nontrivial recurrent leaves at nonconvex levels.

This can be done by applying Lemma 3.19 to $\Sigma \times [-1,1]$ with respect to the sheet A. Recall that in the proof of Lemma 3.19 we arranged that, for nonconvex levels, Σ_t is decomposed into two planar regions such that $\Sigma_t(\xi'_{\mathrm{PB}})$ is transverse to the boundary of the regions. Since the latter condition is open, we may impose that the movie $\Sigma_t(\xi'_{\mathrm{PB}})$ is generic.

The planarity of the regions also implies that if η is a degenerate closed orbit of $\Sigma_t(\xi'_{\mathrm{PB}})$ for $t \in [-1,1]$ then there is no sequence of closed orbits η_i of $\Sigma_{t_i}(\xi'_{\mathrm{PB}})$ whose limit contains η: If such a sequence would exist, then there would be a closed orbit η_i whose intersection number with η is positive. But η and η' have to be contained in the same region from the proof of Lemma 3.19 since they intersect and they are transverse to the boundary of the regions from the proof of Lemma 3.19. But two closed curves contained in a planar region have vanishing intersection number. Hence the set of levels Σ_t with $t \in [-1,1]$ containing a degenerate closed orbit is discrete and hence finite.

Using Lemma 3.21 we isotope the contact structure ξ'_{PB} on the union of neighbourhoods of degenerate closed orbits such that after the isotopy we obtain a contact structure ξ_{PB} where there are no degenerate closed leaves in the movie $\Sigma_t(\xi_{\mathrm{PB}})$ for $t \in [-1,1]$. This amounts to introducing cancelling pairs of singularities inside the regions from the proof of Lemma 3.19. Hence this operation does not affect the Poincaré–Bendixson property.

So at all nonconvex levels $\Sigma_t(\xi_{\mathrm{PB}})$, $t \in [-1,1]$, has a retrograde saddle-saddle connection and by genericity we can arrange that each level contains at most one connection between saddle points. Moreover, each retrograde saddle-saddle connection is isolated because of Lemma 3.10 because the stable leaves of h_- and unstable leaves of h_+ of the singularities participating in such a connection are rigidly attached to some stable limit set of the characteristic foliation for levels close to the level where the retrograde saddle-saddle connection occurs. Thus ξ_{PB} has the desired properties except maybe the compactness of the basin.

Let $t \in [-1,1]$ be such that Σ_t is not convex with respect to ξ_{PB}. If the upper basin of $A \cap \Sigma_t$ is not compact, then we can introduce cancelling pairs of singularities along all closed leaves and cycles of $\Sigma_t(\xi_{\mathrm{PB}})$ (the signs of the singularities have to be chosen in such a way that we do not introduce retrograde saddle-saddle connections when we place the singularities on cycles). Then the upper basin of $A \cap \Sigma_t$ is compact (also on nearby levels). □

3B2 Overtwisted discs from compressible sheets Giroux's criterion (Lemma 2.32) implies that a convex oriented surface Σ has a tight neighbourhood if and only if no component of the dividing set bounds a disc, unless Σ is a sphere. In this section we give a criterion for finding overtwisted discs from sheets with particular properties. The following lemma is essentially [24, Lemma 3.34].

Lemma 3.23 *Let ξ be a contact structure and $\Sigma \times [-1,1]$ such that there is a sheet A with the following properties:*

- *A bounds a solid torus $S^1 \times D^2$ in the interior of $\Sigma \times [-1,1]$. Let*
$$t_{\min} = \min\{t \in [-1,1] \mid A \cap \Sigma_t \neq \varnothing\},$$
$$t_{\max} = \max\{t \in [-1,1] \mid A \cap \Sigma_t \neq \varnothing\}.$$
- *For one level surface Σ_t with $t_{\min} < t < t_{\max}$ there is a repulsive closed leaf α_t of $\Sigma_t(\xi)$ which is disjoint from the solid torus and isotopic to one of the curves $A \cap \Sigma_t$ so that the annulus bounded by α_t and the attractive leaf of $\Sigma_t(\xi)$ in $A \cap \Sigma_t$ contains no other closed leaf of $\Sigma_t(\xi)$.*

Then ξ is overtwisted.

Proof Without loss of generality we assume α_t is nondegenerate and $t=0$. Then α_0 is part of a sheet $A(\alpha_0)$ which intersects nearby surfaces in repulsive curves close to α_0. Using Lemma 3.5 we can arrange that the annulus bounded by α_0 and a connected component of $A(\alpha_0)\cap\Sigma_0$ does not contain a singular point of the characteristic foliation. Furthermore, we create a cancelling pair of negative singularities e,h on Σ_t such that all leaves which end at e come from the repulsive closed curve $A\cap\Sigma_t$ except one unstable leaf of h. Just like the closed repulsive leaf on the other side of $A\cap\Sigma_t$, this configuration persists in nearby levels, and by applying Lemma 3.20 to A we may assume that for all $t\in[t_{\min},t_{\max}]$ there is a cancelling pair of negative singularities and a nondegenerate repulsive closed leaf parallel to A, and by Lemma 3.4 we can replace these repulsive leaves by circles in Σ_t consisting of positive singularities of the isotoped contact structure. As in [24], we achieve the following conditions:

- $A\cap\Sigma_\tau$ is either empty, connected or has two connected components, and $A\cap\Sigma_\tau$ contains no singularities of the characteristic foliation except when $\tau=0$ and $A\cap\Sigma_0$ consists of two circles of singularities (one negative, the other positive).
- When $A\cap\Sigma_\tau$ consists of two connected components they have parallel orientations, so the two components do not bound a Reeb component.

We will find an overtwisted disc in a surface consisting of one arc in Σ_τ with $\tau\in J$, where J is a closed interval containing $[t_{\min},t_{\max}]$ in its interior.

(1) The first piece σ_1 of the boundary of a surface containing D consists of the family of negative elliptic singular points e_τ of $\Sigma_\tau(\xi)$ which contains e, and τ is contained in a closed interval J which is slightly larger than $[t_{\min},t_{\max}]$. When σ_1 is oriented from top to bottom it is positively transverse to ξ.

(2) The upper endpoint of σ_1 is connected to $A(\alpha_0)$ by a Legendrian arc in a surface sightly above $\Sigma_{t_{\max}}$. Let λ_4 be one of the Legendrian curves connecting e_τ to A'.
Similarly, the lower endpoint of σ_1 is connected to $A(\alpha_0)$ by a Legendrian curve λ_2, and we now use the orientation opposite to the orientation of λ_2 viewed as a leaf of the characteristic foliation.

(3) For each point e_τ between the two endpoints of σ_1 we choose an arc in Σ_τ which connects e_τ to $A(\alpha_0)\cap\Sigma_\tau$ such that the part of the arc which is transverse to the characteristic foliation is connected (this part may be empty). If we orient all arcs so that they point to the negative elliptic singularity, then the arc is never tangent and anti-parallel to the characteristic foliation on Σ_τ, except in Σ_0 where the arc is Legendrian.

(4) σ_3 is an arc in $A(\alpha_0)$ consisting of the endpoints of the arcs we have just picked. We orient σ_3 from bottom to top. Then σ_3 is positively transverse to ξ.

The concatenation σ of $\sigma_1, \lambda_2, \sigma_3, \lambda_4$ is a piecewise smooth curve whose smooth segments are positively transverse to ξ or Legendrian, and σ bounds D. We orient D so that $\sigma = \partial D$. From the properties of A and the characteristic foliations on Σ_τ it follows that all singularities of the characteristic foliation on D are positive except one point in the interior of $D \cap A$. Moreover, $D \cap A$ is a circle in D such that the characteristic foliation on D points inwards. Since all singular points of the characteristic foliation on D which do not lie in the disc bounded by $D \cap A$ are positive, the basin formed by all flow lines whose ω–limit set is inside the disc bounded by $A \cap D$ is well-defined and it yields an overtwisted disc. □

The second condition of Lemma 3.23 can be achieved using the Legendrian realization principle (Lemma 2.18) if the curves $A \cap \Sigma_t$ are nonseparating and the genus is at least two. If $\Sigma \cong T^2$, then the results in [24] show that the presence of a sheet bounding a solid torus without any further assumptions does not suffice to produce an overtwisted disc (the corresponding contact structures on $T^2 \times [0, 1]$ are tight, but virtually overtwisted).

3B3 Transverse contact structures on $\Sigma \times [-1, 1]$ The purpose of this section is to prove that contact structures on $\Sigma \times [-1, 1]$ which are transverse to the second factor are tight when the boundary does not have a neighbourhood with an obvious overtwisted disc. For this and for other purposes we give an efficient construction of contact structures transverse to the foliation $\mathcal{I}$ given by the second factor of $\Sigma \times [-1, 1]$. We shall assume that the genus of the underlying surface is at least two, the case of tori is simpler.

The construction of contact structures transverse to $\mathcal{I}$ is explained in the following example, which yields a contact structure on $\Sigma \times \mathbb{R}$ that is a complete connection of the $\mathbb{R}$–bundle because it is periodic with respect to a translation of the second factor.

Example 3.24 Let Σ be an oriented surface of genus $g \geq 2$. We fix two nonseparating oriented disjoint closed curves γ', γ'' and we choose four singular foliations $\mathcal{F}_1, \dots, \mathcal{F}_4$ on Σ such that all singularities have positive divergence as follows:

(1) γ' is a closed attractive leaf of $\mathcal{F}_1$ with nondegenerate holonomy, γ'' is a curve with attractive holonomy on one side and repulsive holonomy on the other side such that the degenerate closed leaf marks the birth of a pair of parallel closed leaves on surfaces Σ_t in $\Sigma \times (-\varepsilon, \varepsilon)$ which lie above Σ_0 (see Lemma 3.7). All leaves of the characteristic foliation whose α–limit set is γ'' accumulate on γ' (except γ'' itself, of course) and, except for the degenerate closed orbit, $\mathcal{F}_1$ is of Morse–Smale type. By Lemma 2.11, there is a contact structure ξ_1 on $\Sigma \times [-1, 1]$ such that $\mathcal{F}_1 = \Sigma_0(\xi_1)$ and the only nonconvex level is Σ_0.

(2) γ'' is a closed attractive leaf of $\mathcal{F}_2$ with nondegenerate holonomy and γ' is a closed degenerate leaf such that this marks the disappearance of a pair of closed leaves on Σ_s of the contact structure ξ_2 on $\Sigma\times[-1,1]$ determined by $\mathcal{F}_2=\Sigma_0(\xi_2)$. All leaves of $\mathcal{F}_2$ which come from γ' accumulate on γ'' (except γ'). So for $s<0$ there are two attractive closed leaves on Σ_s while there is only one for $s>0$.

(3) γ'' is an attractive closed leaf of $\mathcal{F}_3$ while γ' is a degenerate closed leaf, but in contrast to $\mathcal{F}_2$ it now marks the birth of a pair of nondegenerate closed leaves. Again, all leaves whose α–limit set is γ' have γ'' as their ω–limit set (again, except γ'). The corresponding contact structure on $\Sigma\times[-1,1]$ is called ξ_3. For $s<0$ there is only one attractive closed leaf on Σ_s isotopic to γ'' while for $s>0$ there are two such leaves, one of them isotopic to γ' and the other isotopic to γ''.

(4) γ' is a closed attractive leaf of $\mathcal{F}_4$ and γ'' is a degenerate closed leaf which marks the cancellation of a pair of nondegenerate closed leaves of ξ_4. Again, we require that all leaves of $\mathcal{F}_4$ coming from γ'' to have γ' as their ω–limit set.

For each contact structure ξ_i, $i=1,\dots,4$, on $\Sigma\times I_i\cong\Sigma\times[-1,1]$, the surface Σ_t is convex except when $t=0$. In order to glue the two pieces such that the resulting contact structure has no negative singularity, a little bit of care is needed since the condition (3-5) concerning the position of tangent spaces of sheets consisting of attractive closed leaves, the contact planes along these sheets, the tangent spaces of the surfaces and the vertical direction has to be satisfied. Isotoping the foliation $\mathcal{F}_2$ so that the attractive closed leaves lie on the side of γ' and γ'' which is opposite to the side determined by the coorientation of the leaves in the surface, we can glue the two contact structures ξ_1' and ξ_2' (which are restrictions of ξ_1 and the (isotoped) contact structure ξ_2, respectively) to $\Sigma\times I_1$ and $\Sigma\times I_2$, respectively, using Lemma 3.3, such that the resulting contact structure ξ_{12} on $\Sigma\times(I_1\cup I_2)$ is transverse to the second factor.

Similarly, one can now combine isotoped versions of ξ_{12},ξ_3 and ξ_4 to obtain a contact structure on $\Sigma\times\left(\bigcup_i I_i\right)\cong\Sigma\times[0,1]$ transverse to the second factor such that the contact structure ξ near Σ_0 coincides with the contact structure on Σ_1 when we use the second factor to identify these levels.

In order to obtain a contact structure on $\Sigma\times\mathbb{R}$ which is transverse to the second factor and complete when viewed as a connection it suffices to glue infinitely many copies together.

Lemma 3.25 *Let Σ be a closed surface of genus $g\geq 1$ and ξ a contact structure on $\Sigma\times[-1,1]$ transverse to the fibres of the projection $\Sigma\times[-1,1]\to\Sigma$ such that $\Sigma_{\pm 1}$ is convex and no component of the dividing set bounds a disc. Then ξ is universally tight.*

Proof Since ξ is transverse to the foliation $\mathcal{I}$ defined by the second factor in $\Sigma\times[-1,1]$, it is automatically extremal, ie $|\langle e(\xi),[\Sigma]\rangle| = 2g-2$, and we coorient ξ using the second factor. In particular, $\langle e(\xi),[\Sigma]\rangle = 2-2g$. By assumption, no component of the dividing set on $\Sigma_{\pm 1}$ bounds a disc. Hence the dividing curves on each of the surfaces $\Sigma_{\pm 1}$ come in pairs, each pair bounding an annulus containing a closed attractive leaf.

The idea of the proof is to embed $(\Sigma\times[-1,1],\xi)$ into $(\Sigma\times\mathbb{R},\widehat{\xi})$ so that $\widehat{\xi}$ is transverse to the foliation $\widehat{\mathcal{I}}$ corresponding to the $\mathbb{R}$–factor and $\widehat{\xi}$ is a complete connection on $\Sigma\times\mathbb{R}\to\Sigma$. Theorem 2.27 then implies that $\widehat{\xi}$ is universally tight, and hence the same is true for ξ (the embedding of $\Sigma\times[-1,1]$ maps Σ_0 to $\Sigma\times\{0\}\subset\Sigma\times\mathbb{R}$).

We attach layers of contact structures obtained as in Example 3.24 in order to successively reduce the number of connected components of the dividing set and to arrange that in the end the only attractive closed curve is nonseparating in Σ. Using Lemma 3.3 we can modify the characteristic foliations so that at each step of the elimination no new attractive closed curves appear. Some care is needed when we want to eliminate a component which separates the surface into two pieces. In this situation one first introduces a nonseparating closed repulsive curve using Lemma 2.18. Using the folding procedure we obtain a contact structure with an attractive closed leaf isotopic to the repulsive curve.

We end up with a contact structure on $\Sigma\times[-2,2]$ which is transverse to the second factor, has convex boundary and the characteristic foliation on the boundary has exactly one nonseparating attractive curve. We then attach infinitely many layers obtained in Example 3.24. □

Remark 3.26 The condition that no component of the dividing set of $\Sigma_{\pm 1}$ bounds a disc clearly cannot be omitted. However, if there is one component γ of the dividing set which bounds a disc, then we consider the case that D_γ contains no other component of the dividing set.

Then there is an attractive closed leaf β bounding a larger disc D_β containing D_γ in its interior since the interior of D_γ necessarily contains a singular point which is positive by transversality. Now consider the basin of all leaves of $\Sigma_{\pm 1}(\xi)$ which leave D_γ through γ. The closure of the basin may not contain any singularities at all (since they would have the opposite sign as the singularities inside the disc).

Therefore the basin has Legendrian boundary and is again a disc. The boundary is an attractive closed orbit.

3C Boundary elementary contact structures

Let Σ be a closed oriented surface of positive genus g and ξ a contact structure on $N = \Sigma \times [0,1]$. For our purposes it suffices to consider only the case when ∂N is convex. We require that the contact structure is extremal in the sense that

$$\chi(\Sigma) = 2 - 2g = \langle e(\xi), [\Sigma] \rangle, \tag{3-6}$$

where $e(\xi)$ is the Euler class of ξ viewed as an oriented vector bundle. The Thurston–Bennequin inequalities (2-3) imply that the left-hand side of (3-6) cannot be bigger than the right-hand side provided that ξ is tight.

When Σ has Legendrian boundary and $\Sigma(\xi)$ has no singular points on $\partial\Sigma$, the Thurston–Bennequin inequality (2-3) for closed surfaces remains true (this can be seen by doubling the surface). In this situation it therefore still makes sense to speak about extremal contact structures, and again Σ^- is a union of annuli when Σ is convex for an extremal contact structure ξ.

From now on we assume that ξ is tight. By Corollary 2.33 the surface Σ^- is then the union of annuli whenever Σ is convex. Each such annulus contains a Legendrian curve which is the ω–limit set of all leaves entering the annulus. Let β denote such a curve. We will sometimes refer to such curves as *sinks*.

The following definition is an adaptation of [24, Definition 3.14] for our situation.

Definition 3.27 A contact structure is *boundary elementary* with respect to the product decomposition $\Sigma \times [0,1]$ of N if for each annulus of Σ_i^- containing the sink β, $i = 0, 1$, there is an annulus $A(\beta)$ which is foliated by Legendrian curves in $A(\beta) \cap \Sigma_t$ so that $\beta \subset \partial A(\beta) \subset \partial N$.

Compared to Giroux's definition in [24] of elementary contact structures there are two differences:

- (i) If ξ is elementary in the sense of [24], then this has consequences for all closed leaves of characteristic foliation on $(\partial N)(\xi)$. Definition 3.27 requires only the existence of some repulsive closed leaves of the characteristic foliation on ∂N.
- (ii) Definition 3.27 does not put restrictions on the characteristic foliation of all surfaces Σ_t in the interior of N.

Given a contact structure on N, we will need to isotope ξ so that it becomes boundary elementary. This is relatively easy to achieve when $\Sigma = T$ is a torus because if $T_t(\xi)$ intersects a sheet A in a homotopically nontrivial curve, then by the Poincaré–Bendixson theorem $T_t(\xi)$ has no nontrivial recurrent leaf since the complement of $A \cap T$ is planar.

3C1 The pre-Lagrangian extension lemma The following lemma will be the main tool for the extension of pre-Lagrangian surfaces.

Lemma 3.28 *Let ξ be a contact structure on $N=\Sigma\times[-1,1]$ such that $\Sigma_{\pm 1}$ is convex and $\partial\Sigma_t$ is an attractive Legendrian curve for all t. Assume that A is a sheet and the following conditions are satisfied:*

- (i) *ξ satisfies the extremal condition* (3-6) (*and the paragraph following this equation*).
- (ii) *A is transverse to Σ_t for all $t\in[-1,1]$ and $\beta_t = A\cap\Sigma_t$ is a nonseparating curve. All sheets which meet $\Sigma^-_{\pm 1}$ connect Σ_{-1} and Σ_1.*
- (iii) *β_t is either a closed attractive leaf or contains only negative singularities.*
- (iv) *For $i=\pm 1$ the characteristic foliation $\Sigma_i(\xi)$ has a repulsive closed leaf β'_i parallel to $A\cap\Sigma_i$. Moreover β'_{+1} and β'_{-1} lie on the same side of A.*
- (v) *The maximal sheet $A'_{\pm 1}$ containing $\beta'_{\pm 1}$ does not connect $\beta'_{\pm 1}$ to a repulsive leaf in the same boundary component of N.*

Then ξ is isotopic to a contact structure $\widehat{\xi}$ such that the isotopy is the identity near the boundary and A and there is a sheet $\widehat{A}$ connecting β'_{-1} and β'_1.

The proof of this lemma is rather lengthy and will be given in Section 3C2. Our main application of Lemma 3.28 is the following result which we will refer to as the pre-Lagrangian extension lemma.

Lemma 3.29 *Let ξ be an extremal contact structure on $N=\Sigma\times[-1,1]$ such that the boundary is convex, $\partial\Sigma_t$ is an attractive Legendrian curve, ξ is transverse to the foliation $\mathcal{I}$ corresponding to the second factor and there is a pair of isotopic closed leaves β,β' of $\Sigma_{-1}(\xi)$ such that β is attractive and β' is repulsive and the following conditions are satisfied:*

- (i) *The maximal sheet containing β does not connect the two boundary components of Σ.*
- (ii) *β' is not part of a properly embedded sheet in N.*
- (iii) *β' lies on the side of β opposite to the coorientation of ξ.*
- (iv) *For all other attractive closed leaves α of $\Sigma_{-1}(\xi)$ or $\Sigma_{+1}(\xi)$, the maximal sheet containing α connects the two boundary components of N.*
- (v) *No sheet meets the interior of the annulus bounded by $-\beta\cup\beta'$.*

Then there is a contact structure $\widehat{\xi}$ isotopic to ξ relative to the boundary such that the sheet $\widehat{A}(\beta)$ containing β is properly embedded and $\partial\widehat{A}(\beta)=-\beta\cup\beta'$.

There is an analogous lemma when $\beta \subset \Sigma_1$. In that case β' is supposed to lie on the side determined by the coorientation of ξ (in requirement (iii) above). The dashed line in Figure 5 in Section 3A4 corresponds to an extension of the sheet $A(\beta)$ where $\beta \subset \Sigma_{-1}$.

Proof Since $A(\beta)$ does not connect the two boundary components of N we have

$$t(\beta) = \max\{t \in [-1,1] \mid A(\beta) \cap \Sigma_t \neq \varnothing\} < 1.$$

By Lemma 3.16 and Lemma 3.7, $A(\beta) \cap \Sigma_{t(\beta)}$ is a closed degenerate leaf and there is a level $t'(\beta)$ such that for $t'(\beta) \le t < t(\beta)$ the characteristic foliation on Σ_t contains a closed attractive leaf $\beta_t = A(\beta) \cap \Sigma_t$ and a closed repulsive leaf β'' parallel to β which lies on the side of β_t opposite to the coorientation of the contact structure. (So the pairs β, β' and β_t, β''_t are isotopic.) Note that β_t is isotopic to β as oriented curves since there are no negative singularities.

After applying Lemma 3.20 to all sheets, we may assume that the restriction of ξ to $\Sigma \times [-1, t'(\beta)]$ satisfies the hypothesis of Lemma 3.28. Then we obtain the desired contact structure $\widehat{\xi}$. $\square$

3C2 The proof of Lemma 3.28 Before we start with the proof we fix some notation: for fixed t let Γ_+ be the graph on Σ_t formed by positive singularities and stable leaves of positive singularities of $\Sigma_t(\xi)$ together with repulsive closed leaves, and let Γ_- be the graph formed by negative singularities and unstable leaves of negative singularities and attractive closed leaves. Thus Γ_+, Γ_- is *not* the dividing set of any surface (we want to use the same notation as in [24]).

By the extremal condition (3-6) the connected components of Γ_- are either nonclosed trees or homeomorphic to one circle with finitely many (eventually nonclosed) trees attached to the circle. In order to simplify the presentation we assume that every repulsive closed leaf appearing in a path in Γ_+ is replaced by a pair of positive cancelling singularities such that both unstable leaves of the new positive hyperbolic singularity come from the new elliptic singularity and the path passes through the new elliptic singularity.

In the proof below we assume that ξ is tight. If that proof does not work for a given overtwisted contact structure, then this is because there is an overtwisted disc in the complement of the sheets $A(\beta)$ and $A(\alpha)$ for α an attractive closed leaf of $\Sigma_{-1}(\xi)$. Then the classification of overtwisted contact structures (Theorem 2.26) implies all claims of the lemma.

Proof of Lemma 3.28 Before we begin the construction note that by Lemma 3.20 we may assume that every sheet connecting a component of Σ_{-1}^- to a component of Σ_1^- is transverse to Σ_t for all t. Furthermore, when we refer to the basin of β_t we mean the basin lying on the same side of β_t as $\beta'_{\pm 1}$.

If $\Sigma_t(\xi)$ is convex for *all* $t \in [-1,1]$, then by Lemma 3.4 or Lemma 3.3 we can isotope ξ without changing A so that the characteristic foliation of the isotoped contact structure on Σ_t has a closed repulsive leaf parallel to $\beta_t = A \cap \Sigma_t$ for all $t \in [-1,1]$. The collection of these repulsive leaves then provides the desired sheet A'.

If Σ_t is not convex for all t we will arrange that the basin of $A(\beta \cap \Sigma_t)$ never contains a negative singularity and construct the desired sheet. All this will be achieved in three steps: In Step 1, we arrange that there are only finitely many nonconvex levels and after that, in Step 2, we deal with each nonconvex level individually. In Step 3 we apply Lemma 3.4 to construct the desired extension of $A(\beta)$.

Step 1 According to Lemma 3.22, ξ is isotopic to a contact structure ξ_{PB} (the subscript PB refers to the Poincaré–Bendixson property) with the following properties:

- There are only finitely many levels $t \in [-1,1]$ where $\Sigma_t(\xi_{\mathrm{PB}})$ is not convex. Moreover, at all these levels a single retrograde saddle-saddle connection is responsible for the nonconvexity and $\Sigma_t(\xi_{\mathrm{PB}})$ has no nontrivial recurrent leaves.
- The basin of β_t is compact for all nonconvex levels.

If $\Sigma_t(\xi_{\mathrm{PB}})$ contains a retrograde saddle-saddle connection, then we denote the negative and positive singularities participating in the retrograde connection by h_- and h_+, respectively. These singularities persist on nearby surfaces and we will denote these singularities also by $h_\pm$.

Step 2 Let t be a nonconvex level such that one of the singular points h_+ or h_- of $\Sigma_t(\xi_{\mathrm{PB}})$ is contained in the closure of the basin of β_t. We will isotope ξ_{PB} without creating new nonconvex levels such that Σ_t becomes convex or the retrograde saddle-saddle connection does not interact with the basin of β_t.

In the following we assume that all negative hyperbolic singularities of $\Sigma_\tau(\xi_{\mathrm{PB}})$ except h_- have been eliminated for τ close to t. Hence $\Sigma_t(\xi)$ has exactly two negative singularities (one of them is h_-, the other one is elliptic).

Case A Both stable leaves of h_- are contained in the boundary of the basin, ie h_- is a pseudovertex of the basin of β_τ for $\tau \neq t$ sufficiently close to t, and the isotopy type of the dividing set of Σ_τ does not change when τ crosses t.

If h_- is a pseudovertex of the basin of β_t, then we apply Lemma 3.11: Let ν', ν'' denote the unstable leaves of h_+. Let Γ'_- and Γ''_- be the connected components of Γ_-

containing the ω–limit sets of ν' and ν'', respectively. This configuration is shown in Figure 7.

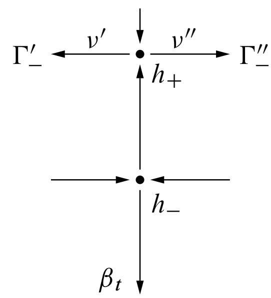

Figure 7: Configuration in Case A

Because ξ_{PB} is extremal and tight, both Γ'_- and Γ''_- are trees since otherwise $\Sigma^-_\tau(\xi_{\mathrm{PB}})$ would have components which are not diffeomorphic to annuli for $\tau > t$ or $\tau < t$ by Lemma 3.10. For the same reason $\Gamma'_- = \Gamma''_-$ because otherwise the dividing set of $\Sigma^-_\tau(\xi_{\mathrm{PB}})$ would contain a component bounding a disc for τ close to t. Hence we can indeed apply Lemma 3.11 so that the number of nonconvex levels is reduced by one.

Now consider the case when the hyperbolic singularity h_- is not a pseudovertex of the basin of β_t. Then the unstable leaf connecting h_- to β_τ for $\tau \neq 0$ is the unstable leaf of h_- which participates in the retrograde saddle-saddle connection when $\tau = t$. So h_+ is part of the basin of β_t and both unstable leaves of h_+ accumulate on β_t (from the side containing the basin under consideration).

In this situation, both unstable leaves of h_+ accumulate on β_t from the same side and the region bounded by the two unstable leaves of h_+ contains a positive elliptic singularity e_+ because ξ_{PB} is tight. Thus we can eliminate the singular points e_+, h_+ of $\Sigma_t(\xi_{\mathrm{PB}})$ and on nearby surfaces.

In this way we have reduced the number of nonconvex levels by one. In particular, we did not lose the properties of the movie (like the Poincaré–Bendixson property, compactness of basins at nonconvex levels or the discreteness of nonconvex levels all of which are nonconvex due to the presence of retrograde saddle-saddle connections).

Case B Only one unstable leaf of h_- is contained in the closure of the basin while the other one is not. Then h_- is a corner of the basin and one unstable leaf of h_- is connected to a positive hyperbolic singularity. Since generically there is at most one saddle-saddle connection h_+ is a pseudovertex of the basin of β_t.

Let Γ'_+ be the connected component of Γ_+ containing h_+ and Γ'_- the connected component of Γ_- containing h_-. The unstable leaf of h_- which does not participate in the retrograde saddle-saddle connection will be denoted by η. The unstable leaves of h_+ are denoted by ν, ν' and ν accumulates on β_t (see the left part of Figure 8). In order to obtain closed graphs we remove the retrograde saddle-saddle connection between h_- and h_+ from $\Gamma'_\pm$. As before, Γ'_- has to be a tree.

Assume that ν accumulates on β_t while the other unstable leaf ν' of h_+ does not accumulate on β_t from the same side as ν (if that happens we are in the situation of the second part of Case A). There are two subcases depending on whether the ω–limit set of ν' is contained in Γ'_- or not.

Case B1 The easier case is when the ω–limit set of ν' is not contained in Γ'_-. As in Case A the isotopy type of the dividing set of $\Sigma_\tau(\xi_{PB})$ does not change when τ passes t. Therefore we can again apply Lemma 3.5 to both singularities in Γ'_- without losing the properties mentioned at the end of Case A. Thus we simply eliminated one nonconvex level by removing the negative hyperbolic singularity participating in the retrograde connection.

Case B2 The much more intricate case is when the ω–limit sets of η and ν' are both contained in Γ'_-. In this situation, a pair of dividing curves appears (resp. disappears) as τ crosses t and the corresponding component of Σ^-_τ splits off (resp. merges) with the component of Σ^-_τ containing $A \cap \Sigma_\tau$.

Case B2.1 There is a simple path c in Γ'_+ with the following properties:

(i) The unstable leaves of the positive hyperbolic singularities on c lying on the same side of c as the stable leaf of h_+ which is connected to Γ'_- are also connected to Γ'_-.

(ii) The ω–limit set $\widehat{\beta}_t$ of the other unstable of $\widehat{h}_+$ is either different from β_t or this unstable leaf accumulates on β_t from the other side than the unstable leaf ν of h_+.

(iii) $\widehat{h}_+$ is the only hyperbolic singularity on c with this property.

This configuration is schematically depicted in Figure 8, where c is the thickened curve.

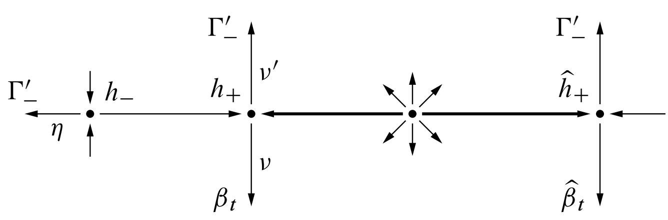

Figure 8: Configuration in Case B2.1

Using Lemma 3.5 (in particular part (ii) of that lemma) starting at the elliptic singularity closest to $\widehat{h}_+$ we can eliminate the hyperbolic/elliptic singularities along c so that the retrograde connection between h_- and the no-longer present h_+ is replaced by a retrograde saddle-saddle connection of h_- with $\widehat{h}_+$.

The benefit for us is that the appearance/disappearance of the component of Σ_τ^- containing h_- at the nonconvex level $\tau = t$ has no longer anything to do with the basin of β_t. Now the component of Σ_τ^- containing h_- splits off from or merges with a component of Σ_τ^- different from the one containing $\Sigma_\tau^- \cap A$ or this happens on the side of β_τ opposite to the side under consideration. The construction does not affect the properties mentioned at the end of Case A.

Case B2.2 If there is no path with the properties of c in Case B2.1, then the basin of Γ'_+, ie the closure of leaves of the characteristic foliation whose α–limit set is contained in Γ'_+, is a subsurface S with two boundary components and corners. One boundary component is β_t while the other boundary component contains Γ'_- and h_- is a corner. (Recall that by genericity we may assume that for all τ, the characteristic foliation on Σ_τ has at most one saddle-saddle connection.) Also, if Γ'_- were contained in the interior of the closure of the basin of Γ'_+ then the current assumption (nonexistence of a path like c from Case B2.1) would imply that β_t bounds a subsurface of Σ_t. But β_t is nonseparating.

Let $\Sigma_\tau^-(h_-)$ be the connected component of Σ_τ^- which contains h_- for τ close to t for the relatively open subinterval $I(h_-) \subset (-1,1)$ where this region does not contain β_τ.

The boundary points of $I(h_-)$ correspond to nonconvex levels and we eliminate *all* negative singularities from the characteristic foliation of ξ_{PB} on these levels. According to Lemma 3.16 the closure of A' in $\Sigma \times [-1,1]$ is obtained by adding degenerate orbits to A'. (In order to obtain a smooth sheet we have to make sure that the graph formed by unstable leaves of negative singularities and elliptic singularities in $\Sigma_\tau^-(h_-)$ with $\tau \in I(h_-)$ is smooth, but this can be achieved by modifications in neighbourhoods of elliptic singularities in $\Sigma_\tau^-(h_-)$.)

While in the situation of Case B2.1 it was not possible in general to prevent the formation of nontrivial recurrent orbits or an infinite number of degenerate closed leaves, now the fact that no leaf of the characteristic foliation can enter the surface S through β_t implies that we do not lose the Poincaré–Bendixson property at nonconvex levels and only one degenerate closed leaf appeared. (At most one annulus in Σ^- degenerates at a given nonconvex level.)

Our present goal is to isotope the contact structure so that A' disappears completely. There are two possibilities:

(1) Both boundary components of A' meet one of the surfaces Σ_τ with $-1 < \tau < 1$ and the degenerate closed leaves of the characteristic foliations at these levels are parallel

(2) Like (1), except that the degenerate closed leaves are anti-parallel.

It will turn out that the second possibility contradicts the tightness of ξ. But first, we deal with the first two cases (which can be treated simultaneously) using an inductive procedure.

There are two extreme situations, namely $\chi(S) = 0$ (ie S is an annulus) and $\chi(S) = \chi(\Sigma)$ (then $\Sigma \setminus S$ is an annulus), and the intermediate cases $-2 \geq \chi(S) \geq \chi(S) + 2$.

The case $\chi(S) = 0$ is straightforward: β_τ and $A' \cap \Sigma_\tau$ with $\tau \in I(h_-)$ bound an annulus $S_\tau \subset \Sigma_t$ such that both boundary curves are attractive closed curves (except in the boundary levels of A'). After eliminating all superfluous negative/positive singularities in S_t we obtain a family of repulsive closed curves separating the two boundary components. This allows us to eliminate A' completely using Lemma 3.20. By this procedure we have reduced the number of nonconvex levels.

The case $\chi(S) = \chi(\Sigma)$ can be treated in the same fashion when one considers $S' = \Sigma_t \setminus \mathring{S}$ or, in a more indirect fashion, inductively as the nonextremal cases.

In order to treat the case when $\chi(S)$ is not $\chi(\Sigma)$ or 0, we note that we may assume by induction (the induction starts with $\chi(S) = 0$) that the lemma was already proved for surfaces with attractive Legendrian boundary of lower genus. This is possible since neither the lower or the upper basin of $A' \cap \Sigma_t$ can contain β_t. Thus we may cut $\Sigma \times [-1, 1]$ along $A(\beta) \cap (\Sigma \times [-1, 1])$. Then the pre-Lagrangian extension lemma can be applied to A'. Using Lemma 3.20 A' can be eliminated completely or moved out of $\Sigma \times [-1, 1]$. We have thus reduced the number of nonconvex levels in $[-1, 1]$ and at each such level there is only a retrograde saddle-saddle connection.

As we shall see, we encounter only case (1) from above when the contact structure is tight. Hence after finitely many steps we have eliminated all negative singularities in the closure of the basin of β_t at nonconvex levels without losing the Poincaré–Bendixson property at nonconvex levels and the nonconvexity is again due only to retrograde saddle-saddle connections. The compactness of the basin at these levels is now easily arranged.

We now show that the case (2) from above does not occur when ξ is tight. Again this is by induction on the genus. We can cut $\Sigma \times [-1, 1]$ along A and we eliminate the negative singularities near both ends of A' thus replacing the retrograde saddle-saddle connections by degenerate closed leaves of characteristic foliations. Then we can extend the pre-Lagrangian surface A' to a pre-Lagrangian torus bounding a solid torus.

Now we apply the pre-Lagrangian extension lemma to the basin of the attractive curves in A' on the side opposite to S. Again this surface has lower genus and therefore we can find a pre-Lagrangian surface parallel to A' which consists of repulsive closed curves in Σ_t. But then the contact structure is overtwisted according to Lemma 3.23.

This shows that case (2) does not occur and after an isotopy of ξ we may assume that the lower of β_t does not contain a negative singular point at nonconvex levels. Moreover, there are still only finitely many nonconvex levels in $[-1,1]$ all of which correspond to retrograde saddle-saddle connections. The contact structure obtained after these isotopies is still denoted by ξ_{PB}.

Step 3 We now construct the desired pre-Lagrangian extension of $A(\beta)$. For all t such that $\Sigma_t(\xi_{\mathrm{PB}})$ is not convex the boundary of the closure of the basin of β_t does not contain a negative singularity. Let $V_t \subset \Sigma_t$ be a collar of β_t lying in the basin (covered by $(Q_t = S^1 \times [0,1], V_t = \varnothing, \alpha_t))$ such that $\Sigma_t(\xi_{\mathrm{PB}})$ is transverse to $\partial V_t \setminus \beta_t$ and all leaves of the characteristic foliation entering V_t accumulate on β_t.

Fix a domain $F_t \subset \Sigma_t$ containing a neighbourhood of the basin of β_t with V_t removed such that Lemma 3.4 can be applied to F_t. Such a neighbourhood exists because $\alpha(S^1 \times \{1\})$ contains only positive singularities and no stable leaf of a positive hyperbolic singularity in $\alpha(S^1 \times \{1\})$ comes from a negative singularity by construction. We modify the contact structure on neighbourhoods of F_t to obtain the attracting closed curves parallel to β_t near levels where ξ_{PB} is not convex. Once we have dealt with nonconvex levels we apply Lemma 3.3 to obtain the desired contact structure $\widehat{\xi}$.

In this last step we have used again that β and hence also β_t is nonseparating (as in Lemma 2.18). □

4 Transitive confoliations

In this section we prove Theorem 1.4 for transitive confoliations. Since a parametric version is not much more difficult we present that version. We fix once and for all a Riemannian metric on M which we use to define the C^k–topologies on the space of plane fields on M.

Before we state Colin's result recall that according to Gray's theorem every contact structure ξ on a closed manifold M has a C^1–neighbourhood such that every contact structure in that neighbourhood is isotopic to ξ. This follows from the fact that the contact condition is open in the C^1–topology. In particular, the contact structures interpolating between some contact structure ξ' in the C^1–neighbourhood and ξ can be chosen inside that neighbourhood. Colin has shown the following stability theorem for C^0–neighbourhoods.

Theorem 4.1 [6] *Let ξ be a contact structure on the closed 3–manifold M. Then there is a C^0–neighbourhood U of ξ in the space of smooth plane fields so that every contact structure in U is isotopic to ξ.*

The family of contact structures constructed in the proof of this theorem does not necessarily stay in U. We now extend this theorem further to the case when ξ is a transitive confoliation.

Theorem 4.2 *Let ξ be a transitive confoliation on a closed manifold M. Then there is a C^0–neighbourhood U of ξ such that the space of positive contact structures in U is weakly contractible in the space of all contact structures on M.*

We now give a very rough outline of the main difficulty of the proof in the nonparametric case. Let ξ be a transitive confoliation. We choose a decomposition of M into polyhedra which are adapted to ξ. Among many other requirements (see Definition 4.12 below) the confoliation on each polyhedron is close to the horizontal foliation $\{dz = 0\}$ in terms of adapted coordinates (x, y, z). Moreover, the characteristic foliation is homeomorphic to a foliation on the sphere with exactly two singular points such that no closed leaf is attractive or repelling on both sides.

For two contact structures ξ_0, ξ_1 which are sufficiently close to ξ the polyhedra are Darboux domains (Definition 4.4). Therefore ξ_0 and ξ_1 are tight when restricted to polyhedra and all leaves of the characteristic foliation on the boundary of a polyhedron spiral from one repulsive critical point to an attractive one (both singular points will be particular vertices of the polyhedron). Characteristic foliations with these properties will be called decreasing.

We want to interpolate between ξ_0 and ξ_1 by a family of plane fields ζ_s, $s \in [0, 1]$, such that the characteristic foliation of ζ_s on the boundary of each polyhedron is homeomorphic to a foliation on the sphere which is decreasing.

This is not trivial because a change of the characteristic foliation on a single face which leads to a more decreasing characteristic foliation when the face is viewed as part of the boundary of one polyhedron does the opposite for the polyhedron on the other side of the face.

The strategy to overcome this problem is to modify the characteristic foliation simultaneously on several faces. For this we will connect the faces with cylinders in the region $H(\xi)$ where ξ is contact using ribbons (Definition 4.16 in Section 4B1). At this point we are using the transitivity of the confoliation. On each cylinder $C = D^2 \times [-1, 1] \subset H(\xi)$ all confoliations ξ, ξ_0, ξ_1 are required to be transverse to the second factor. Then the holonomy on $S^1 \times [-1,1] \subset \partial C$ is decreasing by a positive amount and this will be used to manipulate characteristic foliations in the desired way.

Once this is achieved, it is relatively straightforward to find a family ξ_s, $s \in [0,1]$, of contact structures on the polyhedra which induces the same characteristic foliation as ζ_s. In order to use the techniques developed here in later sections we will insist that all plane fields appearing in the construction are transverse to a foliation $\mathcal{I}$ of rank 1 which is transverse to ξ. The main tool for this is Lemma 4.14.

The following two sections contain preliminaries for the proof of Theorem 4.2, which can be found in Section 4C. The structure of the proof is similar to Colin's proof of Theorem 4.1. Since this technique will be used later, we present it in a way that makes it amenable to further adaptation. We shall also use the following theorem of Varela [57].

Theorem 4.3 *Let ξ be a positive confoliation on M which is somewhere nonintegrable. Then there is a C^0–neighbourhood of ξ such that every confoliation in that neighbourhood is somewhere nonintegrable and positive.*

Originally, this theorem was stated in [57] only for contact structures (and therefore there is no further reference to nonintegrability). However, the proof uses only properties of characteristic foliations on the boundaries of a family of tubular neighbourhoods of a single knot transverse to ξ. It thus carries over immediately to yield Theorem 4.3 since every open set of a contact domain, like $H(\xi)$, contains a transverse knot.

4A Adapted polyhedral decompositions

Colin's proof of Theorem 4.1 uses polyhedral decompositions which are adapted to ξ. We will make use of similar decompositions which we explain in this section.

4A1 Darboux domains The original Darboux theorem for contact structures states that every positive contact structure is locally diffeomorphic to a domain in $(\mathbb{R}^3, \ker(dz + x\,dy))$. This is of course not true for confoliations but a slightly weakened notion is part of the following definition.

Definition 4.4 A pair (P, V), where P is a compact set in M and V an open neighbourhood of P, is a *Darboux domain* if there is a bounded smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $\partial f/\partial x \geq 0$ and a confoliated embedding

$$\varphi\colon (V, \xi|_V) \to (\mathbb{R}^3, \ker(dz + f(x,y,z)\,dy))$$

so that the intersection of every flow line of the Legendrian vector field ∂_x with the image of V is connected. Furthermore, we require that there is a disc $D \subset V$ contained in a (y,z)–plane such that the lines parallel to the x–axis which intersect D cover P and their intersection with P is connected.

We say that a relatively compact set $P \subset M$ is a Darboux domain if there is a neighbourhood V such that (P,V) is a Darboux domain; (x,y,z) are called *Darboux coordinates*.

When a manifold is equipped with a contact structure ξ and a foliation $\mathcal{I}$ transverse to ξ, we require that ∂_z is tangent to $\mathcal{I}$ on the Darboux domain.

We will consider Darboux domains which are polyhedra or even simplices. New Darboux domains will often arise by subdivision of simplices. The importance of Darboux domains comes from the following stability property. The role of the disc D from the above definition is to simplify later proofs.

Lemma 4.5 *Let (M,ξ) be a confoliated manifold and (P,V) a Darboux domain. Then there is a C^0–neighbourhood U of ξ in the space of smooth plane fields such that P is a Darboux domain for every positive confoliation ξ' in U.*

If a foliation $\mathcal{I}$ of rank 1 is transverse to ξ, then the Darboux coordinates (x',y',z') for ξ' can be chosen so that $\partial_{z'}$ is tangent to $\mathcal{I}$.

Proof Let (P,V) be a Darboux domain. We fix a confoliated embedding φ and a surface D as in Definition 4.4. In order to determine U we fix an open neighbourhood $V' \subset V$ of P such that $\overline{V'}$ is compact. Then U is determined by the following requirements:

(i) Every plane field in U is transverse to $\varphi_*^{-1}(\partial_z)$ on V'.

(ii) For a smooth plane field ξ' on M let X' be the projection of $\varphi_*^{-1}(\partial_x)$ along $\varphi_*^{-1}(\partial_z)$ to ξ'. We require that X' is transverse to $V' \cap D$ and the flow lines of X' starting at D are well defined as long as they stay in V' and the image of $V' \cap D$ under the flow of X' covers P.

The vector field X' is smooth and therefore flow lines are uniquely defined. By standard theorems about the continuous dependence of solutions of ordinary differential equations on parameters (see [30, Chapter V], for example) U is open in the C^0–topology.

By construction X' preserves the foliation $\mathcal{I}$. The new coordinate vector fields $\partial_{y'}$ and $\partial_{z'}$ are obtained by transporting ∂_z and ∂_y using the flow of X'. These coordinates define an embedding φ'.

Since $\partial_{x'}$ is Legendrian and $\partial_{z'}$ is transverse to ξ', this plane field is defined by a defining form $dz' + f'(x',y',z')\,dy'$ (here f' is a new function, not the derivative of f).

We have defined φ' on connected segments of flow lines of X' starting at points of $V' \cap D$. Hence all requirements in Definition 4.4 are satisfied except that f is not yet defined on all of $\mathbb{R}^3$ but only on the image of φ'. But the construction allows one to choose an extension of f' to $\mathbb{R}^3$ so that $dz + f'(x,y,z)\,dy$ defines a positive confoliation and f' is bounded. $\square$

According to Theorem 2.27 every confoliation in U is tight when restricted to V where (P,V) is a Darboux domain: the boundedness of the function f implies that $\ker(dz + f(x,y,z)\,dy)$ is a complete connection of the fibration $\mathbb{R}^3 \to \mathbb{R}^2$ given by $(x,y,z) \mapsto (x,y)$.

Remark 4.6 The notion of a Darboux domain can be extended to general smooth plane fields by omitting the requirement that $f(\cdot\,,y,z)$ is weakly monotone for all (y,z). Then one can formulate the stability property for all smooth plane fields.

4A2 Polyhedral decompositions adapted to ξ In this section we describe polyhedral decompositions adapted to a confoliation. Such decompositions are also used in [6]. Let ξ be a positive cooriented confoliation on M and $\mathcal{I}$ a line field transverse to ξ.

Definition 4.7 Given a polyhedron P in M, we say that $\xi(x)$ is *transverse to* ∂P at $x \in \partial P$ if

- $\xi(x)$ is transverse to a face of P if x is contained in the interior of that face,
- $\xi(x)$ is transverse to all edges of P whose closure contains x, and
- if x is a vertex, then for a germ of a surface Σ_x tangent to $\xi(x)$ the intersection $\Sigma_x \cap P$ is a manifold with piecewise smooth boundary. If $\Sigma_x \cap P = \{x\}$, then x is *elliptic*.

We will never need hyperbolic singularities on boundaries of polyhedra. These could be incorporated by requiring that x is a simple crossing of $\Sigma_x \cap \partial P$.

The following notion (introduced by Thurston [55]) will be applied to line fields and plane fields. We therefore formulate it in complete generality.

Definition 4.8 Let τ be a distribution of codimension k on an n–manifold and $\mathbb{R}^n \supset P \hookrightarrow M$ an embedded polyhedron. Then P is *in general position* with respect to τ if for all $x \in P$ and all k–subsimplices of P the map

$$\mathbb{R}^n \to \mathbb{R}^n/\tau_x = \tau_x^\perp$$

restricted to the k–simplex is a diffeomorphism onto its image.

General position is a C^0–open condition and it implies that ∂P is transverse to τ. Later M will be a 3–manifold, and $\tau = \xi$ a confoliation ($k = 1$) or $\tau = \mathcal{I}$ a foliation of rank 1 transverse to ξ ($k = 2$). In order to obtain simplices in general position, one can use Thurston's jiggling lemma [55]. This lemma uses triangulations (and not just polyhedral decompositions).

Lemma 4.9 (Thurston's jiggling lemma) *Let M be a compact manifold, $\mathcal{T}$ a triangulation and τ^{n-k} a continuous distribution of codimension k. Then there is a subdivision $\mathcal{T}'$ of $\mathcal{T}$ such that after a small perturbation of the vertices one obtains a triangulation $\mathcal{T}''$ in general position with respect to τ.*

For the proof of this lemma, it is convenient to embed M into a Euclidean space. Then M is approximated by a simplicial complex; subdivisions and perturbations of this complex provide the desired triangulation. For more details we refer to [55].

The way simplices are subdivided is essential. A possible subdivision is due to Thurston [55]. Another method was used by Whitney. In both cases the simplices obtained by subdivision depend on the ordering of the vertices of a simplex P at least if $n \geq 3$. Whitney's method in dimension 3 goes as follows [59, page 358]:

Let $P \subset \mathbb{R}^3$ be a simplex and p_0, p_1, p_2, p_3 its vertices. Let p_{ij} be the midpoint between p_i and p_j with $p_{ii} = p_i$. The first Whitney subdivision of P consists of the following simplices (with an ordering of the vertices):

$$\begin{array}{llll} p_0 p_{01} p_{02} p_{03}, & p_1 p_{01} p_{02} p_{03}, & p_1 p_{12} p_{02} p_{03}, & p_2 p_{12} p_{02} p_{03}, \\ p_1 p_{12} p_{13} p_{03}, & p_2 p_{12} p_{13} p_{03}, & p_2 p_{23} p_{13} p_{03}, & p_3 p_{23} p_{13} p_{03}. \end{array}$$

Both subdivision schemes have the property that consecutive subdivisions of a simplex yield only finitely many subsimplices of $\mathbb{R}^n$ up to rescaling and translation. An important consequence is the following: if $\mathcal{T}''$ is obtained as in Lemma 4.9 and is in general position, then all simplices obtained by further Whitney subdivisions of $\mathcal{T}''$ are still in general position with respect to τ. It is therefore possible to apply Lemma 4.9 to a finite collection of distributions (with varying codimensions).

According to Lemma 4.9 there is a triangulation such that every simplex is in general position with respect to a confoliation ξ and a foliation $\mathcal{I}$ of rank 1 transverse to it. If we start with a triangulation such that every simplex is contained in a Darboux domain, then we can ensure in addition that all simplices are contained in a Darboux domain and are in general position with respect to the vector field ∂_x associated to the Darboux domain (see Definition 4.4), ie we consider different vector fields ∂_x for each polyhedron of a given simplicial decomposition. After subdividing a simplex P from such a decomposition we obtain a collection of Darboux domains, and the Darboux coordinates can be chosen to be restrictions of the Darboux coordinates around P. The proof of the jiggling lemma (Lemma 4.9) applies.

Thus we have established the existence of a triangulation satisfying the requirements of the following definition. It refers to polyhedra rather than simplices because there is one condition we will want to impose later on the decomposition of M, which will require the consideration of polyhedra, not only simplices.

Definition 4.10 A decomposition of M into polyhedra is *weakly adapted to* ξ *and* $\mathcal{I}$ if each polyhedron P of the decomposition has the following properties:

(i) P is in general position with respect to ξ and $\mathcal{I}$. In particular, there are exactly two singular vertices x_1^P and x_2^P which are both elliptic and P is homeomorphic to a ball.

(ii) P is a Darboux domain in general position with respect to the vector field ∂_x from Definition 4.4.

(iii) For $i = 1, 2$ a neighbourhood of x_i^P in P is contained in the half-space determined by the plane $\xi(x_i^P)$ in some coordinate chart near x_i^P. (This property is independent of the choice of a chart.)

(iv) The faces where $\mathcal{I}$ enters P form a disc in ∂P. Moreover, the intersection of every leaf of $\mathcal{I}$ with P is connected.

We say that $\xi(x_i^P)$, $i \in \{1, 2\}$, *supports* P and x_i^P is a *supporting vertex* of P; all other vertices of P are *nonsupporting*. If P lies on the side of $\xi(x_i^P)$ determined by the coorientation of ξ, then x_i^P is *negative*, otherwise this supporting vertex is *positive*.

Later we will often say that a polyhedron/polyhedral decomposition is adapted to ξ and implicitly require that it is also adapted to a fixed line field $\mathcal{I}$ transverse to ξ.

Let $P \subset (M, \xi)$ be a polyhedron in a confoliated manifold. If the boundary of a polyhedron P is transverse to ξ, then one can define the characteristic foliation on ∂P as follows. Since faces are smooth they have a characteristic foliation which is oriented by the usual conventions. Where two faces meet along an edge we concatenate the corresponding oriented leaves. The leaves we obtain are piecewise smooth curves. At edges and nonsupporting vertices the characteristic foliation is tangent to a pair of vectors, one of them tangent to one face adjacent to the edge or vertex while the other vector is tangent to the other face.

By requirement (i) of Definition 4.10 the characteristic foliation $\partial P(\xi)$ has exactly two singular points corresponding to the supporting vertices x_1^P, x_2^P, and both are elliptic.

On a neighbourhood of P the confoliation can be viewed as a connection on a fibre bundle $(x,y,z) \mapsto (x,y)$ determined by Darboux coordinates. Since ξ is a positive confoliation $\partial P(\xi)$ is spiralling away from x_i^P (again in the weak sense if ξ is not a contact structure at x_i^P) if this vertex is positive and towards x_i^P if it is negative. Thus positive supporting vertices are sources of $\partial P(\xi)$ while negative supporting vertices are sinks.

By the Poincaré–Bendixson theorem all limit sets of leaves of $\partial P(\xi)$ are singularities, closed cycles (passing through singularities) or closed leaves, and because P is adapted to ξ there are no cycles other than closed leaves. By Theorem 2.27 the restriction of ξ to a neighbourhood of each polyhedron is tight. Hence $\partial P(\xi)$ has no closed orbits if ξ is a contact structure. If ξ is a positive confoliation and $\partial P(\xi)$ has a closed leaf, then this closed leaf bounds a disc D tangent to ξ inside P and the holonomy of $\partial P(\xi)$ near ∂D is weakly attractive on the side of the positive supporting vertex while it is weakly repelling on the other side.

We still have to modify our triangulation further (in the process we will turn it into a polyhedral decomposition). For each simplex P of a triangulation we denote the unique edge connecting the supporting vertices by $\gamma(P)$. We parametrize $\gamma(P)$ so that it points away from the negative vertex. If P is a polyhedron, then $\gamma(P)$ is a (not uniquely determined) simple path consisting of edges of P and connects the supporting vertices of P. Moreover, $\gamma(P)$ is positively transverse to ξ when $\gamma(P)$ is oriented pointing away from the negative supporting vertex. When P is in general position with respect to $\mathcal{I}$ then the space of leaves of $\mathcal{I}|_P$ is homeomorphic to a disc. The leaf space of $\mathcal{I}|_P$ is denoted by $P/\mathcal{I}$.

Definition 4.11 Let M be a 3–manifold carrying a smooth plane field ξ and a foliation $\mathcal{I}$ transverse to ξ. A weakly adapted simplex $P \subset M$ is *graphical* if the projection of $\partial P(\xi)$ to $P/\mathcal{I}$ has the following properties:

(i) The projections of any two segments of $\partial P(\xi)$ intersect transversely.

(ii) Let σ be a segment of a leaf of $\partial P(\xi)$ such that σ connects two consecutive intersection points of the leaf with $\gamma(P)$. Then the projection of σ to $P/\mathcal{I}$ has at most one self-intersection.

A polyhedral decomposition is graphical if all polyhedra are graphical.

The self-intersections of projections of segments like σ in (ii) of Definition 4.11 form a smooth curve with boundary contained in the projection of $\gamma(P)$ to $P/\mathcal{I}$. In Figure 10 at the end of Section 4A2 this curve is dotted.

The purpose of this definition is to ensure that one can easily find a foliation of P by discs such that the boundary of a leaf is either transverse to ξ (when the intersection point of D with $\gamma(P)$ is not a fixed point of the holonomy of $\partial P(\xi)$) or tangent to ξ (when a closed leaf of $\partial P(\xi)$ goes through $D \cap \gamma(P)$): if P is graphical, then such discs are obtained as sections of $P \to P/\mathcal{I}$.

Definition 4.12 A polyhedral decomposition of M which is weakly adapted to ξ and $\mathcal{I}$ is *adapted to* ξ *and* $\mathcal{I}$ if it satisfies the following conditions:

(i) All polyhedra are graphical.

(ii) Exactly three edges of P meet at a supporting vertex.

(iii) For all vertices x of the polyhedral decomposition there is at most one polyhedron supported by $\xi(x)$.

(iv) For each polyhedron P the projection of the characteristic foliations of ξ on the faces of P to the leaf space of the foliation $\mathcal{I}|_P$ form a collection of line fields which are transverse to each other.

So far we have established the existence of a weakly adapted triangulation. Because ξ and $\mathcal{I}$ are continuous every supporting vertex of a polyhedron has a neighbourhood where $\partial P(\xi)$ is graphical. Subdividing the simplices of a weakly adapted triangulation further following the method of Whitney, one obtains a graphical triangulation which remains graphical when subdivided further by the same method.

In order to ensure (iii) of Definition 4.12 we modify the triangulation as in [6]. In this step the triangulation is modified in neighbourhoods of supporting vertices. In an inductive process one adds/removes tetrahedra from polyhedra of the decomposition (here the triangulation is replaced by a polyhedral decomposition). This is described in detail in [6]; we therefore only indicate the main idea. In Figure 9 (which is [6, Figure 1]), x is a supporting vertex of P_0 but not of P_1. If there is yet another polyhedron with x as a negative supporting vertex, then a piece is removed from P_0 and added to P_1 and we obtain P_0' and P_1'. Now x' supports P_0' and no other polyhedron of the modified decomposition.

Note that P_1 is not convex (with respect to the coordinate system supplied by the Darboux domain). However, the intersection of P with leaves of $\mathcal{I}$ respectively planes parallel to a contact plane in P is connected respectively a disc. The characteristic foliation remains graphical.

Considering the various cases (one half-space of $T_x\mathcal{I}$ is contained in P or not) and choosing the segment τ close enough to other edges of P one sees that the properties

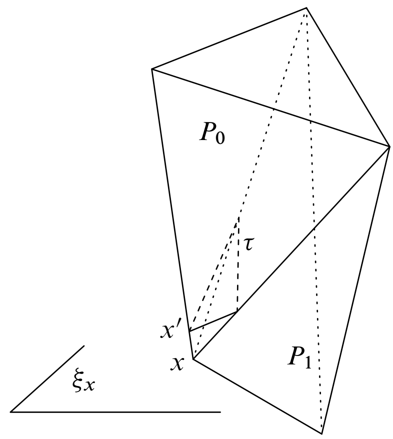

Figure 9: Modification of polyhedra

(i)–(iv) of Definition 4.10 can be preserved by this construction. When a polyhedron is modified then it is sometimes necessary to modify $\gamma(P)$ but it is clear how to do this when P is modified as indicated in Figure 9. It is equally clear that the new polyhedron is still part of a Darboux domain. Finally, requirement (iv) of Definition 4.12 is achieved by a slight perturbation of the polyhedral decomposition.

Thus we have proved the following lemma (essentially due to Colin).

Lemma 4.13 *Let ξ be a confoliation on a 3–manifold and $\mathcal{I}$ a foliation of rank 1 transverse to it. Then there is a polyhedral decomposition of M adapted to ξ and $\mathcal{I}$.*

Now consider plane fields ζ which are transverse to $\mathcal{I}$ and sufficiently close to ξ to ensure that P is still adapted to ζ and $\mathcal{I}$. Moreover, we require that the characteristic foliation on ∂P has decreasing holonomy (this is automatic if ζ is a contact structure).

Under these circumstances, Theorem 2.35 implies that there is a tight contact structure ξ' on P, unique up to isotopy, such that $\partial P(\zeta) = \partial P(\xi')$. For later applications we want to keep ξ' transverse to $\mathcal{I}$ and the purpose of the condition that P is graphical is to ensure that such an extension ξ' can be constructed quite easily.

Lemma 4.14 *Let ζ be a plane field, $\mathcal{I}$ a line field transverse to ζ and P a polyhedron adapted to ζ and $\mathcal{I}$ such that every leaf of $\partial P(\zeta)$ spirals from the positive supporting to the negative supporting vertex.*

Then there is a contact structure ξ' on P transverse to $\mathcal{I}$ so that $\partial P(\zeta) = \partial P(\xi')$. It is unique up to homotopy through contact structures in that class.

If ζ is a contact structure near supporting vertices, than one can arrange that ζ and ξ coincide on small neighbourhoods of supporting vertices.

Proof Given ζ, first construct a foliation on ∂P by circles transverse to $\partial P(\zeta)$ (of course, x_1^P, x_2^P are singular points of this foliation). This is possible since we assume that the characteristic foliation of ζ on ∂P does not have closed leaves. Since P is graphical, these circles can be chosen so that they project to simple closed curves in $P/\mathcal{I}$.

As indicated in Figure 10, this is achieved as follows: Let σ be a segment of a leaf of $\partial P(\zeta)$ connecting two consecutive intersection points of the leaf with $\gamma(P)$. If the projection to $P/\mathcal{I}$ of σ has a self-intersection, then choose an arc transverse to $\partial P(\zeta)$ connecting one endpoint of σ to a point on the segment (close to the intersection point) so that one obtains a piecewise smooth circle in ∂P consisting of σ and one arc transverse to $\partial P(\zeta)$. If the projection has no self-intersection, then the transverse arc can be chosen almost parallel to a piece of $\gamma(P)$.

The piecewise smooth circles obtained in this way can be approximated by circles transverse to $\partial P(\zeta)$ that are projected to simple closed curves in $P/\mathcal{I}$.

In Figure 10 the intersection points of segments like σ lie on the dotted line and the arcs transverse to $\partial P(\zeta)$ are dashed.

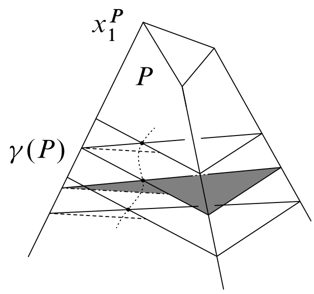

Figure 10: Construction of discs transverse to the characteristic foliation of a graphical polyhedron ($\mathcal{I}$ is orthogonal to the page). The projection of one of the discs is shaded.

This construction can be carried out parametrically. In this way we obtain a foliation of $\partial P \setminus \{x_1^P, x_2^P\}$ by circles transverse to $\partial P(\zeta)$.

Since these circles project to simple closed curves in $P/\mathcal{I}$ we obtain a foliation of P by discs transverse to $\mathcal{I}$. Now pick a curve transverse to the discs connecting x_1^P and x_2^P. The intersection points of this arc with the discs serve as midpoints of the discs. Then a contact structure ξ' which is tangent to a radial line field on the discs is obtained by twisting the tangent plane around the radial line field starting at the centre of each disc. (This can be done in such a way that a given contact structure near the supporting vertices is extended.)

The proof of the last part of the statement should be clear. $\square$

Remark 4.15 This lemma only shows how to fill one polyhedron. However, we will need to fill many polyhedra simultaneously. Then different polyhedra are adjacent to the same face and additional care is needed to ensure that the plane fields we are constructing are smooth.

When we apply Lemma 4.14, ζ will be a contact structure near supporting vertices. Therefore smoothness near supporting vertices will not be a problem.

In order to ensure that the resulting plane field is smooth everywhere we first extend the plane field from the 2–skeleton to a contact structure on a small neighbourhood of the 2–skeleton (as in [6, Lemma 3.3]) by modifying the plane field inductively near vertices (except supporting vertices), edges and faces. Then one can apply Lemma 4.14 to slightly shrunken polyhedra without changing the contact structure on a neighbourhood of the 2–skeleton (for this one chooses the radial foliation in the proof of Lemma 4.14 tangent to the contact structure near the boundary of the discs).

4B Ribbons

The proof of Colin's stability result Theorem 4.1 does not carry over immediately to Theorem 4.2. This is because $\partial P(\xi)$ can have closed leaves for some polyhedron P of the decomposition if ξ is a confoliation, while all leaves of the characteristic foliation pass from the source to the sink if ξ is a contact structure. Characteristic foliations with the latter property are rather stable under C^0–perturbations among contact structures and this is used in Colin's proof of his stability theorem (Theorem 4.1). The goal of the construction presented in this section is to modify (depending on the choice of a contact structure ξ_s close to ξ) a given polyhedral decomposition which is adapted to ξ and $\mathcal{I}$ in order to ensure that the characteristic foliation of ξ_s on the boundary of modified polyhedra does not have closed orbits.

We will see that if the confoliation is transitive we can isotope the polyhedral decomposition so that

- the interior of no isotoped polyhedron contains an integral disc with boundary on the polyhedron, and
- all supporting vertices lie in the interior of the region $H(\xi)$ where ξ is a contact structure.

This is relatively straightforward in the case of a transitive confoliation: One can use Legendrian vector fields whose flow lines connect points of ∂P with $H(\xi)$ (a detailed description of similar isotopies is given below). The characteristic foliation of ξ on the boundary of each isotoped polyhedron has no closed leaves and Colin's proof of Theorem 4.1 yields a proof of Theorem 4.2.

In order to have a proof of Theorem 4.2 which applies to more general — and more interesting — situations, we formalize the isotopies used above in terms of ribbons attached to ∂P in the context of transitive confoliations. This will be adapted to nontransitive confoliations later.

4B1 Definitions Let P be a polyhedron of a polyhedral decomposition adapted to ξ and to a fixed line field $\mathcal{I}$ transverse to ξ.

Definition 4.16 A *ribbon attached to* P is a smooth embedding of a rectangle $\sigma\times[0,1]$ into M with the following properties:

(i) $\sigma\times\{0\}=\sigma$ is transverse to ξ and $\sigma\times\{1\}\subset H(\xi)$ lies in the interior of a polyhedron.

(ii) The projection of σ to $P/\mathcal{I}$ is disjoint from the curve consisting of self-intersection points of projections of segments of leaves of the characteristic foliation on $\partial P\setminus\{x_1^P,x_2^P\}$ which connect consecutive intersection points of leaves of the characteristic foliation with $\gamma(P)$ (see Figure 10). The analogous requirement holds for the intersection of $\sigma\times[0,1]$ with every face of the decomposition through which the ribbon exits.

(iii) The curves $\{z\}\times[0,1]$, $z\in\sigma$, are Legendrian. Close to ∂P they are tangent to $X_\varphi:=\varphi_*^{-1}(\partial_x)$, where φ denotes the embedding associated to the Darboux domain (P,V). In particular, the ribbon is transverse to ∂P. When the ribbon leaves another polyhedron P', then it is tangent to the coordinate vector field $\partial_{x'}$ associated to P'.

(iv) The curve $\sigma\times\{1\}$ is contained in a leaf of $\mathcal{I}$ and $\mathcal{I}$ is tangent to $\sigma\times[0,1]$.

(v) The ribbon is disjoint from the 1–skeleton of the polyhedral decomposition except that $\sigma\times\{0\}$ may contain one supporting vertex of P. The intersection of a ribbon with the faces of polyhedra consists of arcs connecting the two Legendrian curves coming from the endpoints of σ.

It is possible to satisfy (iii) of this definition since the flow of X_φ preserves the foliation $\mathcal{I}$ (see Definition 4.4).

On its way from the polyhedron P to the contact zone $H(\xi)$ the ribbon $\sigma\times[0,1]$ meets other parts of the 2–skeleton of the polyhedral decomposition. Let P^1 be the first polyhedron the ribbon leaves. We view a copy of the remaining part of the ribbon which lies between P^1 and $\sigma\times\{1\}$ as a ribbon $\sigma^1\times[0,1]$ attached to P^1.

For later constructions it is useful to enlarge $\sigma^1\times[0,1]$ in the transverse direction. We replace $\sigma^1\times[0,1]$ by a slightly larger ribbon which is attached to P^1 and whose opposite end is still contained in $H(\xi)$ (see Figure 11). This extension is again denoted by $\sigma^1 \times [0,1]$. We continue until we enter the polyhedron P^C containing $\sigma \times \{1\}$ in its interior. At each step the ribbon that is attached gets a little bit broader. Still, all the induced ribbons $\sigma^1 \times [0,1], \sigma^2 \times [0,1], \ldots, \sigma^k \times [0,1]$ are attached to $P^1, P^2, \ldots, P^k$, ie they satisfy the conditions of Definition 4.16. We will sometimes denote the ribbon $\sigma \times [0,1]$ by $\sigma^0 \times [0,1]$.

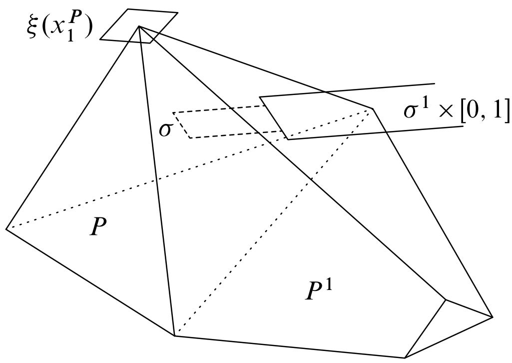

Figure 11: The ribbons $\sigma \times [0,1]$ and $\sigma^1 \times [0,1]$

Lemma 4.17 *Let P be a polyhedron adapted to ξ and $\mathcal{I}$ and let $\sigma_j \times [0,1]$, $j = 1, \ldots, l$, be disjoint ribbons attached to ∂P. Then*

$$P \cup \bigcup_j (\sigma_j \times [0,1])$$

is a Darboux domain in M.

Proof Since only finitely many ribbons are attached it suffices to consider the case $l = 1$. By (iii) of Definition 4.16 we can extend the embedding $\varphi|_P$ to an embedding of $P^\sigma = P \cup_\sigma \sigma \times [0,1]$, and the vector field ∂_x is extended to a Legendrian vector field X which is tangent to the ribbon and whose support is contained in a small neighbourhood of P^σ. We then extend the embedding φ to an open neighbourhood of P^σ. Since $\sigma \times [0,1]$ is transverse to ξ and compact, the function f remains well defined and bounded. $\square$

Let $\sigma \times [0,1]$ be a ribbon attached to P and let $\sigma^1 \times [0,1], \ldots, \sigma^k \times [0,1]$ be the ribbons induced by $\sigma \times [0,1]$. At the end of the broadest ribbon $\sigma^k \times [0,1]$ opposite to P we attach a small cylinder C which can be thought of as a reservoir of nonintegrability. We now specify properties of such cylinders:

(1) $C \cong I \times I \times I_{\sigma^k}$. Here $I = [-1, 1]$ and I_{σ^k} is a compact interval.

(2) C is the total space of a fibration over the base $I \times I$ whose fibres are tangent to $\mathcal{I}$.

(3) $C \subset H(\xi)$ and C is disjoint from the 2–skeleton of the polyhedral decomposition.

(4) The first factor in $C \cong I \times I \times I_{\sigma^k}$ is tangent to the ξ–Legendrian vector field X from the proof of Lemma 4.17 and C is contained in the support of X.

(5) The intersection of the ribbons $\sigma^i \times [0, 1]$ with C is $\sigma^i \times \{1\}$ for all $i = 0, \dots, k$, and $\sigma^k \times \{1\}$ is contained in the interior of the vertical part of ∂C. The projection of $\sigma^k \times \{1\}$ to $I \times I$ is $(-1, -1)$.

(6) The holonomy of $\partial C(\xi)$ and its inverse are defined for all points of $\sigma^k \times \{1\}$.

Because $\sigma^k \times \{1\}$ is compact, there is a number $\delta > 0$ so that the difference between x and both its preimage and image under the holonomy are separated by an interval whose length is at least 2δ.

Lemma 4.18 *Let C be a cylinder with the properties listed above, ξ a contact structure transverse to $\mathcal{I}$ and $h\colon I_{\sigma^k} \to I_{\sigma^k}$ the monodromy of $\partial C(\xi)$. For every diffeomorphism $g\colon I_{\sigma^k} \to I_{\sigma^k}$ with support in the interior of I_{σ^k} such that*

$$h \leq g \leq \mathrm{id}, \tag{4-1}$$

there is a domain $C(g) \subset C$ with piecewise smooth boundary containing I_{σ^k} such that g is the monodromy of $(\partial C(g))(\xi)$.

Proof Let X be the Legendrian vector field tangent to the first factor of $C = I \times I \times \mathbb{R}$. We use the flow of X to identify the front $\partial_+ C = \{1\} \times I \times \mathbb{R}$ of C with the back $\partial_- C = \{-1\} \times I \times \mathbb{R}$. Because ξ is a contact structure the image $\mathcal{L}_+$ of the characteristic foliation on $\partial_+ C$ is transverse to the characteristic foliation $\mathcal{L}_-$ on $\partial_- C$. We use the characteristic foliation on $\partial_- C$ and then the flow of X to identify all fibres with the fibre over $(-1, -1)$.

For x in the fibre over the base point we move along the leaf of $\mathcal{L}_+$ starting at the point above $\{(+1, -1)\}$ which corresponds to x until this leaf intersects the leaf of $\mathcal{L}_-$ coming from the point $g(x)$ above $(-1 - 1)$. Such an intersection point exists because of (4-1). It is unique by transversality of $\mathcal{L}_+$ and $\mathcal{L}_-$.

The domain $C(g)$ is bounded be flow lines of X and the segments of $\mathcal{L}_\pm$ for varying initial points x above $(-1, -1)$. □

4B2 Attaching a full collection of ribbons Recall that for each polyhedron of the decomposition we fixed an edge path $\gamma(P)$ connecting the supporting vertices such that when we use the coorientation of ξ to orient $\gamma(P)$ then $\gamma(P)$ is directed from the negative supporting vertex of P to the positive supporting vertex.

Definition 4.19 Given a polyhedral decomposition of M adapted to ξ we say that a pairwise disjoint collection $(\sigma_i \times [0,1])$, $i = 1, \dots, l$, of ribbons attached to polyhedra of the decomposition is *full* if for every polyhedron P

- each supporting vertex is contained in a ribbon, and
- for each leaf of the characteristic foliation on $\partial P \setminus \gamma(P)$ there is a ribbon $\sigma_i \times [0,1]$ such that the interior of $\sigma_i \times \{0\}$ intersects the leaf.

Here and in the following we stick to the notational convention that upper indices for ribbons indicate ribbons induced by one ribbon $\sigma \times [0,1]$ while lower indices refer to ribbons which are not induced by longer ones.

Lemma 4.20 *Let ξ be a transitive confoliation on a closed manifold M and $\mathcal{I}$ a line field transverse to ξ. For every polyhedral decomposition of M which is adapted to ξ and $\mathcal{I}$ there is a full collection of ribbons.*

Proof For each P we choose a pair of Legendrian curves connecting the supporting vertices x_1^P, x_2^P to $H(\xi)$ so that these curves intersect the 1–skeleton of the polyhedral decomposition exactly in the supporting vertex. The Legendrian curves are chosen tangent to X near the supporting vertices and such that they satisfy the conditions from Definition 4.16 which can be applied to Legendrian curves (ie the second part of (i), (ii), (iii) and (v) of Definition 4.16). For example, they avoid the rest of the 1–skeleton and when they leave a polyhedron they are tangent to the Legendrian vector field ∂_x associated to the Darboux coordinates of that polyhedron.

Then we use the flow of a vector field tangent to $\mathcal{I}$ which commutes with X and extend these curves to obtain ribbons $\sigma_i \times [0,1]$, $i = 1, 2$, so that $\mathcal{I}$ is tangent to the ribbon (it may be necessary to extend the ribbon a little to ensure that it meets P in a segment.)

Similarly, one obtains a collection of ribbons σ_i, $i = 3, \dots, l$, such that $\bigcup_i (\sigma_i \times \{0\})$ intersects every segment of the characteristic foliation on $\partial P \setminus \{x_1^P, x_2^P\}$ and the ribbons satisfy the conditions in Definition 4.16 individually. All polyhedra are treated in the same way.

The ribbons may still intersect each other. We assume that they do so transversely. We remove the intersections inductively starting with one ribbon $\sigma_1 \times [0,1]$. If the ribbon

$\sigma_2 \times [0,1]$ meets $\sigma_1 \times [0,1]$, then we replace $\sigma_2 \times [0,1]$ by narrower ribbons consisting of pieces which are parallel to pieces of the original ribbons. This is indicated in Figure 12 in the case when the first intersection of $\sigma_2 \times [0,1]$ with $\sigma_1 \times [0,1]$ contains exactly one boundary point of $\sigma_1 \times [0,1]$.

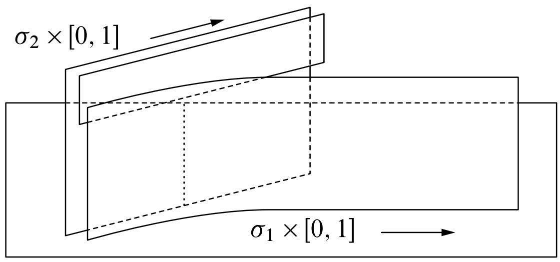

Figure 12: Removing intersections of ribbons

The arrows in Figure 12 indicate the direction away from the polyhedron where the ribbon starts; the original intersection between the ribbons is dotted. After this process is repeated a finite number of times we obtain a full collection of ribbons. □

Definition 4.21 Let $\sigma \times [0,1]$ be a ribbon ending at a cylinder $C \subset H(\xi)$. A collection of diffeomorphisms of $\sigma^{(k)} \times \{1\}$ with compact support is *admissible* if

$$g^{(k)} \leq \cdots \leq g^{(1)} \leq g \leq \mathrm{id} \tag{4-2}$$

and these diffeomorphisms can be realized as the holonomy of the characteristic foliation of a domain in C.

For an admissible collection of diffeomorphisms $g, g^{(1)}, \dots, g^{(k)}$ we obtain domains

$$C(g^{(k)}) \supset \cdots \supset C(g^{(1)}) \supset C(g)$$

by Lemma 4.18.

Let $\sigma \times [0,1]$ be a ribbon attached to a polyhedron P and $g\colon \sigma \to \sigma$ a decreasing map with compact support in the interior which corresponds to the monodromy of the boundary of a domain $C(g)$ in the cylinder containing $\sigma \times \{1\}$. We use the ribbon to identify $\sigma \times \{0\} = \sigma$ with $\sigma \times \{1\}$. Fix a tubular neighbourhood $N(\sigma) \subset \partial P$ whose fibres are segments of leaves of $P(\xi)$ such that $N(\sigma)$ is disjoint from the 1–skeleton of the polyhedral decomposition (for this one has to remove the supporting vertices which σ might contain). The size of $N(\sigma)$ can be chosen arbitrarily small.

The part of $P \cup (\sigma \times [0,1]) \cup C(g)$ which is close to the ribbon or $C(g)$ can be smoothed to a domain with smooth boundary such that

- the resulting domain is homeomorphic to a ball, the boundary contains $\partial P \setminus N(\sigma)$,
- the characteristic foliation on the boundary has no new singular points, and
- its boundary is arbitrarily close to $\partial P \cup (\sigma \times [0,1]) \cup C(g)$.

This is illustrated in Figure 13, where the ribbon, $C(g)$ and $N(\sigma)$ are thickened. We will not introduce notation to distinguish $P \cup (\sigma \times [0,1]) \cup C(g)$ from its partial smoothing.

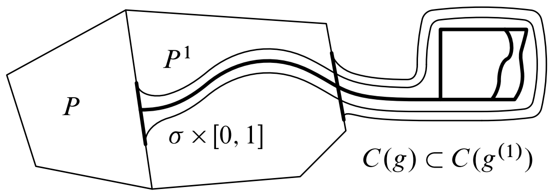

Figure 13: Smoothing parts of polyhedra with ribbons attached, illustrated on a surface transverse to $\mathcal{I}$

In order to identify ∂P with the boundary of the smoothing of $P \cup (\sigma \times [0,1]) \cup C(g)$, we extend the original ribbon $\sigma \times [0,1]$ to a family of ribbons covering the thickening of $(\sigma \times [0,1]) \cup C(g)$ such that all ribbons are tangent to $\mathcal{I}$. We then use a flow tangent to the characteristic foliation on the ribbons to push the new piece of the boundary to $N(\sigma)$.

As indicated in Figure 13, the thickenings of the domains $C(g^{(i)})$ can be chosen such that they satisfy the corresponding strict inclusion relations and their boundaries are disjoint. Moreover, parts of the boundary of thickenings of $(\sigma^i \times [0,1]) \cup C(g^{(i)})$ are identified with parts of ∂P^i, $i = 1, \dots, k$. An isotopy which does this can be constructed similarly to the one we constructed above.

This isotopy is the flow of a vector field tangent to the characteristic foliation of ξ on surfaces which consist of segments of $\mathcal{I}$, and the support of the vector field is contained in the union of such surfaces. Therefore the pullback of ξ under the isotopy remains transverse to $\mathcal{I}$.

Since admissible functions are nowhere increasing, the new monodromy of the characteristic foliation ∂P is decreasing by a larger amount than the monodromy of $\partial P(\xi)$. Also, by construction of the isotopy, P is still a Darboux domain with respect to the pulled-back contact structure.

We attach all the ribbons together with the parts of the cylinders obtained from Lemma 4.18 to the polyhedra. For a polyhedron P we denote the result by P^σ. This is meant to take into account all ribbons meeting P; the functions $g^{(i)}$ defining pieces $C(g^{(i)})$ of a cylinder C are omitted in this notation.

4C Proof of Theorem 4.2

The proof of Theorem 4.2 has two main parts. We first determine a C^0–neighbourhood of ξ and then we show that it has the desired properties. Before we start let us note that in view of future adaptations it is desirable to ensure that all contact structures/plane fields in the construction are transverse to $\mathcal{I}$. The proof without this control would be somewhat simpler; in particular, we could use Theorem 2.35 instead of Lemma 4.14.

4C1 Determining the neighbourhood of ξ in Theorem 4.2 Let ξ be a transitive confoliation, $\mathcal{I}$ a foliation of rank 1 transverse to ξ. We fix a polyhedral decomposition of M which is adapted to ξ and $\mathcal{I}$ together with a full collection of ribbons $\sigma_i \times [0,1]$ and cylinders.

There is a number $\delta_{\mathrm{cyl}} > 0$ such that for each ribbon $\sigma \times [0,1]$ from our collection ending at the cylinder C the monodromy on ∂C is at least $2\delta_{\mathrm{cyl}}$–decreasing when it is defined. This is measured with respect to the parametrization of σ obtained by identifying pieces of $\gamma(P)$ with σ following the leaves of $\partial P(\xi)$ and with $\sigma \times \{1\}$ following the leaves of the characteristic foliation of ξ on $\sigma \times [0,1]$.

We fix a positive number ρ such that the boundary of the 2ρ–ball around a supporting vertex is transverse to the ribbon and meets the interval $\sigma \times \{0\}$ starting at the supporting vertex exactly once.

By (ii) of Definition 4.16 in Section 4B1 the characteristic foliation on ∂P remains graphical if the characteristic foliation on ∂P is changed on an arbitrarily small tubular neighbourhood $N(\sigma) \cong (\sigma \times \{0\}) \times J \subset \partial P$ of σ (where J is a short interval and the second factor is tangent to $\partial P(\xi)$) by a small amount. The purpose of (ii) in Definition 4.16 is that we do not have to worry about the exact shape of the characteristic foliation on $N(\sigma)$ but only about the diffeomorphism g of the two boundary components of $N(\sigma)$ induced by the new characteristic foliation as long as g has compact support in the interior. The new characteristic foliation will later be obtained by pulling back contact structures along the ribbon $\sigma \times [0,1]$ (as explained in Section 4B2).

For sufficiently small positive constants k, k_{supp} the characteristic foliation remains graphical after a modification of the characteristic foliation on P along all ribbons of a full collection (including induced ribbons) which are attached to P if

$$|h_\sigma^{-1} \circ g \circ h_\sigma(x) - x| < 2k \qquad \text{when } \sigma \text{ does not contain a supporting vertex} \tag{4-3}$$

and

$$|h_\sigma^{-1} \circ g \circ h_\sigma(x) - x| < 2k_{\mathrm{supp}} \operatorname{dist}(x, x_i^P) \qquad \text{when } \sigma \text{ does contains the supporting vertex } x_i^P. \tag{4-4}$$

Here h_σ identifies a boundary component of $N(\sigma)$ with a segment of $\gamma(P)$ using the holonomy of the characteristic foliation on ∂P.

The constants $\delta_{\mathrm{cyl}}, \rho, k, k_{\mathrm{supp}}$ have to satisfy the following compatibility relations:

- To ensure that variations of the monodromy by attachments of ribbons which will be allowed later near supporting vertices are not bigger than variations allowed for other ribbons we require

$$\rho \cdot k_{\mathrm{supp}} \leq k. \tag{4-5}$$

- To guarantee that the monodromy of $\partial P(\xi)$ can be modified by the attachment of ribbons and pieces of cylinders so that the monodromy of ∂P is decreasing by at least $2k$ away from arbitrarily small neighbourhoods of supporting vertices we require that

$$2k < \tfrac{1}{2}\delta_{\mathrm{cyl}}. \tag{4-6}$$

These relations can be satisfied by choosing k and k_{supp} appropriately and sufficiently small after ρ and δ_{cyl} are fixed.

When ξ is replaced by a smooth plane field ζ (sufficiently C^0–close to ξ) the Darboux coordinates on a neighbourhood of a polyhedron P from the decomposition and the data associated to a ribbon $\sigma \times [0,1]$ from the full collection vary as follows:

- The vector field defining the ξ–Legendrian curves on the original ribbon $\sigma \times [0,1]$ is replaced by its projection to ζ along $\mathcal{I}$, the initial condition $\sigma \times \{0\}$ is fixed. The curves $(\{p\} \times [0,1]) \subset (\sigma \times [0,1])$ are now ζ–Legendrian.

  The ribbons $\sigma^j \times [0,1]$ which were induced by $\sigma \times [0,1]$ are now treated independently from each other (recall that $\sigma^{j+1} \times [0,1]$ is slightly broader than $\sigma^j \times [0,1]$ for $0 \leq j \leq k-1$).
- The Darboux coordinates and the Legendrian vector field X on the cylinder C where $\sigma \times [0,1]$ ends are deformed in the same way (see the proofs of Lemma 4.5 and Lemma 4.17).
- The identifications h_σ in (4-3)–(4-4) and $N(\sigma) \cong (\sigma \times \{0\}) \times J$ are induced by $\partial P(\zeta)$.

We do not introduce any new notation reflecting these variations.

The following conditions on $\varepsilon > 0$ ensure that the ε-C^0–neighbourhood $U_\varepsilon(\xi)$ of ξ has the stability property in Theorem 4.2; ζ denotes a smooth plane field in $U_\varepsilon(\xi)$.

(1) ζ is transverse to $\mathcal{I}$ and the polyhedral decomposition is adapted to ζ and $\mathcal{I}$ for all plane fields $\zeta \in U_\varepsilon(\xi)$.

(2) If ζ is a confoliation, then it is positive (see Theorem 4.3).

(3) The flows of the deformed vector fields used in the attachments of ribbons and associated constructions are well defined and have the same properties as before for all $\zeta \in U_\varepsilon(\xi)$. In particular, $\sigma^j \times [0,1]$ (induced by $\sigma \times [0,1]$) meets $\sigma^{j+1} \times [0,1]$ in the interior of $\sigma^{j+1} \times \{0\}$ and the end $\sigma^k \times \{1\}$ of the modified ribbon $\sigma^k \times [0,1]$ is contained in the interior of the vertical part of ∂C.

(4) Each polyhedron with ribbons and cylinders attached is contained in a Darboux domain (see Lemma 4.5 and Lemma 4.17). The collection of ribbons is full for $\partial P(\zeta)$ for all polyhedra of the decomposition.

(5) The projection to $P/\mathcal{I}$ of each arc where a ribbon is attached to a polyhedron P remains disjoint from self-intersection points of projections of segments of leaves of $\partial P(\zeta)$ which connect consecutive intersection points of leaves of $\partial P(\zeta)$ with $\gamma(P)$ (see Figure 10).

(6) The monodromy of $\partial P(\zeta)$ is
   - at most $k/2$–increasing on $\gamma(P)$, and
   - at most $(k_{\mathrm{supp}}/2)\operatorname{dist}(\cdot\,, x_i^P)$–increasing on $\gamma(P) \cap B_{2\rho}(x_i^P)$.

(7) The monodromy on the boundary of the cylinders is at least δ_{cyl}–decreasing and there is a collection of admissible diffeomorphisms such that the attachment of the ribbons changes the monodromy of the characteristic foliation on the boundary of polyhedra
   - at least by k on $\gamma(P) \setminus (B_{2\rho}(x_1^P) \cup B_{2\rho}(x_2^P))$, and
   - at least by $k_{\mathrm{supp}} \operatorname{dist}(\cdot\,, x_i^P)$ on $\gamma(P) \cap (B_{2\rho}(x_1^P) \cup B_{2\rho}(x_2^P))$.

(8) The characteristic foliation remains graphical even after a full collection of ribbons with domains in cylinders is attached and the monodromy is changed by an amount bounded by
   - k for each ribbon which does not contain a supporting vertex, and by
   - $k_{\mathrm{supp}} \operatorname{dist}(\cdot\,, x_i^P)$ for each ribbon containing a supporting vertex x_i^P of a polyhedron P.

This is a finite list of requirements restricting the C^0–distance of ζ from ξ. It can be summarized as follows: $\varepsilon > 0$ is chosen so small that ribbons and adapted polyhedral decompositions persist. The conditions on $\rho, \delta_{\mathrm{cyl}}, k, k_{\mathrm{supp}}$ and ε ensure that the characteristic foliation on boundaries of polyhedra can be modified by the attachment of ribbons and pieces of cylinders so that the leaves of the characteristic foliation spiral from the positive supporting vertex towards the negative supporting vertex in a way that allows polyhedra to be filled by contact structures transverse to $\mathcal{I}$.

As indicated by the conditions referring to supporting vertices more care is required near supporting vertices: we will ensure that all plane fields are positive contact structures on a priori unspecified neighbourhoods of the supporting vertices.

4C2 Proof that the neighbourhood of ξ has the desired property We will prove that the space of positive contact structures in the ε–neighbourhood of ξ is weakly contractible inside the space of all contact structures. The proof does *not* show that the neighbourhood itself is weakly contractible.

Proof of Theorem 4.2 We have to show that the C^0–neighbourhood U of ξ described in Section 4C1 has the following property: for every compact polyhedron S and every family of contact structures ξ_s, $s \in S$ in U, there is an extension of this family ξ_s to a family of contact structures $\xi_{\hat{s}}$ with

$$\hat{s} = (s,t) \in \widehat{S} = S \times [0,1]/S \times \{1\}.$$

Here we view S as the subspace $S \times \{0\}$ of the cone $\widehat{S}$ of S.

The construction will be carried out in two main steps, the first consisting of two substeps which are to be carried out at the same time. The first step consists in constructing a family of smooth plane fields $\zeta_{\hat{s}}$ so that for each polyhedron the characteristic foliation $\partial P(\zeta_{\hat{s}})$ has the following properties:

- All leaves spiral from the positive supporting vertex to the negative supporting vertex.
- It is graphical for all $\hat{s} \in \widehat{S}$.
- $\zeta_{\hat{s}}$ is a contact structure on a neighbourhood of every supporting vertex.

First, we discuss the construction near supporting vertices, then we finish the construction of $\zeta_{\hat{s}}$. The second step is an application of Lemma 4.14 and Remark 4.15.

We start with the most delicate part, namely the construction of the family $\xi_{\hat{s}}$ around supporting vertices. This is also the part in which plane fields appear which are *not* contact.

Let x_i^P be one of the supporting vertices of a polyhedron P of the decomposition. Recall that P is a Darboux domain, so we are given a ξ–Legendrian vector field X on P and a surface $D \subset P$ intersecting every flow line of X which meets P exactly once. We denote the Darboux coordinates on (P,ξ) by (x,y,z) so that X is the coordinate vector field of x. By the conditions on ε the polyhedron P is a Darboux domain for all ξ_s, $s \in S$, and the corresponding ξ_s–Legendrian vector fields X_s vary continuously. The same is true for the characteristic foliations $D(\xi_s)$ and the Darboux coordinates (x_s, y_s, z_s). The contact structure ξ_s near x_i^P is defined by $dz_s + f_s(x_s, y_s, z_s)\, dy_s$ with $\partial f_s/\partial x_s > 0$.

We fix a point $s_0 \in S$. Because S is compact there is a small neighbourhood $N(x_i^P) \subset B_\rho(x_i^P)$ such this data can be extended from the parameter space S to $S \times [0,1]$ on $N(x_i^P)$. This means that there are families $X_{\widehat{s}}$, $(x_{\widehat{s}}, y_{\widehat{s}}, z_{\widehat{s}})$, $f_{\widehat{s}}$ and $\xi_{\widehat{s}}$ defined on $N(x_i^P)$ with $\widehat{s} = (s,t)$, $t \in [0,1]$ such that

- $X_{\widehat{s}}$ is tangent to $\partial/\partial x_{\widehat{s}}$,
- $\partial f_{\widehat{s}}/\partial x_{\widehat{s}} > 0$,
- the contact structure
$$\xi_{\widehat{s}} = \ker\big(dz_{\widehat{s}} + f_{\widehat{s}}(x_{\widehat{s}}, y_{\widehat{s}}, z_{\widehat{s}})\big)\, dy_{\widehat{s}}$$
is ε–close to ξ, and
- for $\widehat{s} = (s,1)$ the extended data coincides with the data associated to $s_0 \in S \subset \widehat{S}$ (in particular, it is independent of s).

Next we extend $\xi_{\widehat{s}}$ from $N(x_i^P)$ to a family of *smooth plane fields* $\zeta_{\widehat{s}}$ on M such that

- $\xi_{s_0} = \zeta_{(s,1)}$ for all $s \in S$, and
- $\zeta_{\widehat{s}}$ remains ε–close to ξ.

Because $\zeta_{\widehat{s}}$ is only a plane field it may happen that the characteristic foliation on ∂P has closed orbits in the 2ρ–ball around x_i^P. This problem will be fixed by the attachment of a ribbon with a cylinder as follows.

For the ribbon $\sigma \times [0,1]$ containing x_i^P we pick a family of admissible diffeomorphisms $g_s^{(j)}\colon \sigma \to \sigma$, $j = 0, \dots, k$, such that after the attachment of $(\sigma \times [0,1]) \cup C(g_s^{(0)})$ all closed leaves of $\partial P(\sigma_s)$ close to x_i^P disappear. Moreover, the remaining closed leaves disappear after the other ribbons are attached. That this is possible is due to the conditions (6)–(8) from the numbered list in Section 4C1.

Figure 14 summarizes the situation near x_i^P. The horizontal axis measures the distance of a point in $\gamma(P)$ from x_i^P while the vertical axis corresponds to the displacement of a point by the holonomy of the characteristic foliation. The solid curve represents the holonomy of $\zeta_{\widehat{s}}$ on ∂P (before attachment of ribbons) while one dashed curve represents the effect of the attachment of $C(\widehat{g}_0)$ and the ribbon $\sigma \times [0,1]$. The other dashed curve corresponds to another ribbon and the dotted curves correspond to condition (6) and

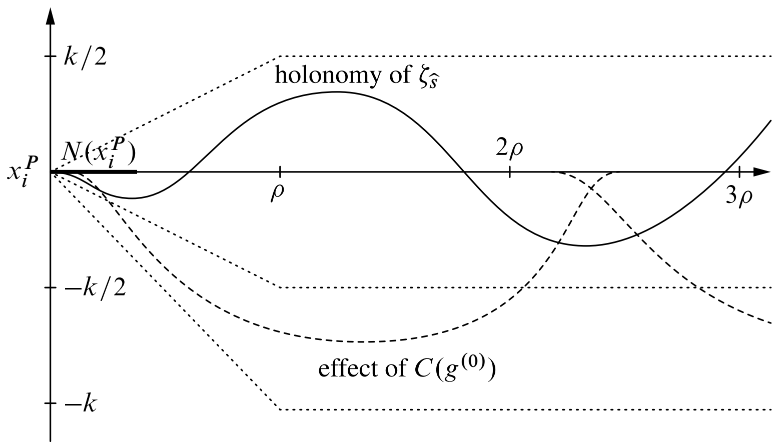

Figure 14: Monodromy of $\zeta_{\hat{s}}$ near supporting vertices

the thickened horizontal arc corresponds to points in $N(x_i^P)$. (We assumed that (4-5) is an equality.)

It is important to note that since $\zeta_{\hat{s}}$ is a positive contact structure near supporting vertices, there is no need to correct the monodromy in a small neighbourhood of the supporting vertex (the size of that neighbourhood does not matter). Hence the condition on admissible diffeomorphisms that they have compact support in the interior of the initial segment $\sigma \times \{0\}$ of the ribbon to which they are associated is not restrictive.

We deal simultaneously with all supporting vertices and attach all ribbons (including ribbons which do not end at supporting vertices). The resulting characteristic foliations $\partial P(\zeta_{\hat{s}})$ have all the properties required in order to apply Lemma 4.14 and the subsequent Remark 4.15. Therefore we can now replace $\zeta_{\hat{s}}$ by a family of contact structures $\xi_{\hat{s}}$ such that $\xi_{(s,1)}$ is independent of s. □

There are several places in this proof where we did not attempt to ensure that $\zeta_{\hat{s}}$ or $\xi_{\hat{s}}$ is contained in the ε–neighbourhood of ξ:

- When the plane fields $\zeta_{\hat{s}}$ are pulled back along ribbons, the resulting plane field is not C^0–close to the original plane fields.
- Lemma 4.14 does not provide such contact structures.

These problems seem to be solvable.

5 Exceptional minimal sets

The purpose of this section is to prove a parametric version of Theorem 1.4 for confoliations (which have holonomy if they are foliations) such that either $\mathcal{F}$ is a minimal foliation or every minimal set of $\mathcal{F}$ is exceptional.

Theorem 5.1 *Let ξ be a C^2–confoliation which has no closed leaf but is not a foliation without holonomy. Then there is a C^0–neighbourhood U of ξ such that the space of positive contact structures in U is weakly contractible.*

Let us outline the main difficulty in adapting the proof from Section 4C. We will use the terminology introduced there and we warn the reader that now we merely sketch the strategy, which we simplify quite a bit. We do this with the hope of clarifying the structure of the proof.

Let ξ be a confoliation which is not transitive. As before we choose an adapted polyhedral decomposition (which is now also compatible with a fixed foliation $\mathcal{I}$ of rank 1 transverse to ξ). Now there are points which can't be connected to $H(\xi)$ by a Legendrian curve and hence we no longer have at our disposal cylinders whose characteristic foliation has decreasing monodromy. This is what we had used in the previous section.

Now assume ξ contains only one minimal set and that this minimal set is exceptional (see Definition 2.5). Then every point can be connected by a Legendrian curve to $H(\xi)$ or to a neighbourhood N of a particular simple closed curve γ (whose existence is guaranteed by a theorem of Sacksteder) contained in a leaf of the minimal set of ξ. The important property of γ is that the holonomy of $\mathcal{F}$ is attractive along γ. Moreover, we can fix annuli containing Legendrian curves parallel to γ such that the characteristic foliation of plane fields which are sufficiently close to ξ has an attractive closed leaf on each of these annuli.

The neighbourhood N is chosen as the union of these annuli, so that N is diffeomorphic to a solid torus and so that every ribbon $\sigma \times [0,1]$ which ends in N can be extended to a semi-infinite ribbon accumulating on a closed leaf of the characteristic foliation of ξ on the annuli. After a modification of the ribbons this remains true when ξ is replaced by a plane field ζ which is sufficiently close to ξ.

Ribbons will only be attached to polyhedra in the complement of N. As before, the attachment of neighbourhoods of pieces of ribbons (now without cylinders) to a polyhedron as in the previous section changes the monodromy of the characteristic foliation on the boundary of the polyhedron. If ζ is a positive contact structure, then the size of this modification increases with the length of the ribbon. A direct computation (the result (3-4) of this computation is explained right before Section 3A1) or an examination of the arguments in [21] shows that if the ribbon can be extended to a semi-infinite ribbon $\sigma \times [0, \infty)$ such that the entire ribbon except for a compact piece is contained in an annulus, then one can change the monodromy of the characteristic foliation again by a definite amount which depends only on the geometry of the ribbon. Thus semi-infinite ribbons will replace the cylinders we used in the previous section.

Now let ξ_0, ξ_2 be two contact structures which are sufficiently close to ξ. First we use ribbons to isotope ξ_0 into a contact structure ξ_1 which coincides with ξ_2 outside of a well-chosen solid torus $\widehat{N}$ containing N.

On $\widehat{N}$ we use the classification of tight contact structures on the solid torus obtained in Theorem 2.36 to isotope ξ_1 into ξ_2. For this we need to show that ξ_1 and ξ_2 are tight. We use the fact that all plane fields appearing in the construction are transverse to $\mathcal{I}$ and we choose $\widehat{N}$ in a particular way.

5A Facts about exceptional minimal sets

We first review the relevant definitions and results concerning Sacksteder curves.

Theorem 5.2 (Sacksteder [52]) *Let $\mathcal{F}$ be a C^2–foliation and $N \subset M$ an exceptional minimal set. Then there is a leaf $L \subset N$ containing an embedded closed curve γ such that the holonomy $h_\gamma\colon \tau \to \tau$ along γ satisfies*

$$h'_\gamma(x) < 1,$$

where $\tau \subset M$ is an embedded interval transverse to $\mathcal{F}$ containing $x \in \gamma$.

A curve with the properties of γ in this theorem will be referred to as *Sacksteder curve*. Going through the proof of Theorem 5.2 one can easily verify that the theorem remains true when one considers C^2–confoliations instead of C^2–foliations, so there is a leaf L containing a curve γ such that the characteristic foliation on an annulus transverse to L containing γ has nontrivial linear holonomy along γ.

Sacksteder's theorem is one of the instances where the C^2–hypothesis is used in an essential way. As it turns out, the other place where C^2–smoothness is used, namely the theory of foliations without holonomy and the following observation about minimal foliations with holonomy, are also based on Theorem 5.2. The proof of the following result from [12] can be found in [49].

Theorem 5.3 (Ghys) *Let $\mathcal{F}$ be a C^2–foliation on M such that M is a minimal set and $\mathcal{F}$ is not a foliation without holonomy. Then there is a Sacksteder curve tangent to $\mathcal{F}$.*

From the fact that Sacksteder curves have nontrivial linear holonomy it follows that a Sacksteder curve cannot bound a compact subsurface of the leaf L it is contained in (this holds for both orientations of γ). In order to see this recall that

$$\pi_1(L) \to \mathbb{R}, \quad \alpha \mapsto \log(h'_\alpha(0)) \tag{5-1}$$

determines a cohomology class in $H^1(L,\mathbb{R})$ (see [4] or [12]). Again, this remains true when L is a leaf of the fully foliated part of a confoliation.

5B Adapting definitions related to ribbons

In the proof of Theorem 5.1 we use the setup from the proof of Theorem 4.2. We describe the required changes below.

Either ξ is a foliation all of whose leaves are dense or all minimal sets of the fully foliated part of ξ are exceptional. Fix a foliation $\mathcal{I}$ of rank 1 transverse to ξ. Because M is compact there are only finitely many exceptional minimal sets $N_1,\dots,N_\kappa$ (this follows immediately from Theorem 5.2; see [4]). If ξ is minimal we pick one Sacksteder curve γ_1, otherwise let $\gamma_1,\dots,\gamma_\kappa$ be a collection of Sacksteder curves such that $\gamma_j \subset N_j$ for $j=1,\dots,\kappa$. The curves γ_j are contained in leaves L_j of the fully foliated part of ξ.

We choose a pairwise disjoint collection of tubular neighbourhoods of the Sacksteder curves $\gamma_1,\dots,\gamma_\kappa$. Each of these tubular neighbourhoods is diffeomorphic to $\gamma_j \times I \times I$, where $I=[-1,1]$, and the fibres of the projection

$$\pi_j\colon \gamma_j \times I \times I \to \gamma_j \times I \times \{0\} \subset L_j$$

along the third factor are tangent to $\mathcal{I}$. Moreover, we require that ξ is transverse to the boundaries $\gamma_j \times \{*\} \times \{\pm 1\}$ of each annulus $\gamma_j \times \{*\} \times I$. We then fix a pair of smaller tubular neighbourhoods of γ_j

$$\gamma_j \subset N_j \subset \widehat{N}_j \subset (\gamma_j \times I \times I)$$

such that $\widehat{N}_j$ is diffeomorphic to the product of γ_j and a disc with two corners such that $\mathcal{I}$ is tangent to the discs and ξ is transverse to all smooth boundary arcs of each disc. Let

$$N := \bigcup_{j=1}^{\kappa} N_j, \quad \widehat{N} := \bigcup_{j=1}^{\kappa} \widehat{N}_j.$$

As in Lemma 4.13, using a fine enough subdivision of a triangulation we obtain polyhedral decomposition of M adapted to ξ and $\mathcal{I}$ such that the following conditions are satisfied:

(i) ∂N_j and $\partial \widehat{N}_j$ are transverse to the 1–skeleton and the 2–skeleton of the polyhedral decomposition. No vertices are contained in $\partial N \cup \partial \widehat{N}$.

(ii) A polyhedron which meets N_j does not meet a polyhedron which intersects $\partial \widehat{N}_j$.

We will use the following modified definition (compare Definition 4.19) of the notion of a full collection of ribbons.

Definition 5.4 A finite collection of ribbons $(\sigma_i \times [0,1])_i$ is *full* if it satisfies the requirements of Definition 4.19 with the following modifications:

- No ribbon begins at a face contained in N. Moreover, no ribbon which enters N leaves N again.
- For all i, the segment $\sigma_i \times \{1\}$ either is contained in $H(\xi)$ or it has the following properties:
  - $\sigma_i \times \{1\}$ is contained in the interior of N_j and at the same time in a fibre of π_j for some $j \in \{1, \dots, \kappa\}$ such that a neighbourhood of the end $\sigma_i \times \{1\}$ is contained in a vertical annulus $\gamma_j \times \{*\} \times I$.
  - The union of semi-infinite segments of the characteristic foliation $\gamma_i \times \{*\} \times I$ which point away from $\sigma_i \times [0,1]$ and start in $\sigma_i \times \{1\}$ is an immersion of $\sigma_i \times [1, \infty)$ which accumulates on $\gamma_i \times \{*\} \times \{0\}$.
  - When one ribbon ends in $\gamma_j \times \{*\} \times I$ then no other ribbon intersects this annulus.

Figure 15 depicts one ribbon $\sigma \times [0,1]$ whose extension accumulates on γ (the horizontal line inside of the rectangle). The two short vertical lines represent $\sigma \times \{0\}$ and $\sigma \times \{1\}$ and the annulus is obtained from the rectangle identifying its vertical sides.

Lemma 5.5 *Every foliation with holonomy and without compact leaves admits a complete collection of ribbons.*

Proof The proof of Lemma 4.20 carries over almost immediately. Since curves with nontrivial holonomy cannot separate the leaf they are contained in (see (5-1) at the end of Section 5A) there is a path tangent to $\mathcal{F}$ from every point in $(\widehat{N} \setminus N) \cap L_j$, $j = 1, \dots, \kappa$, to N without intersecting one of the annuli $\gamma_j \times I$ where another ribbon ends. $\square$

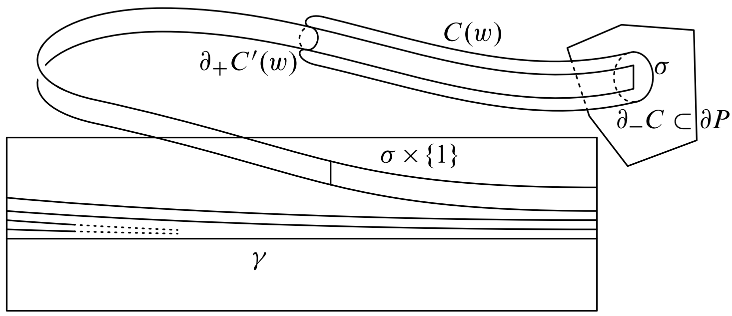

Figure 15: A ribbon accumulating on γ

5C Determining the neighbourhood of $\mathcal{F}$

Before we describe the neighbourhood of ξ in the space of plane fields we first explain how to modify characteristic foliations on the boundary of polyhedra using ribbons (more precisely we will use extensions of ribbons) which do *not* end in $H(\xi)$.

When ξ is not contact near $\sigma_i \times \{1\}$ for some particular i, then the fact that there are integral surfaces in small cylinders near $\sigma_i \times \{1\}$ implies that we cannot realize the holonomy g_i with $g_i < \mathrm{id}$ at given prescribed points in the interior of σ_i.

However, it is possible to realize a given decreasing holonomy with compact support in $\sigma_i = \sigma_i \times \{0\}$ if ξ' is a contact structure which is sufficiently close to ξ. This is done by attaching a thickening of a sufficiently long extension of the ribbon.

Fix an extension of the vector field X (tangent to the ribbon) from the proof of Lemma 4.17 which is ξ–Legendrian, vanishes nowhere and is tangent to the characteristic foliation on the ribbon $\sigma \times [0,1]$ and on nearby ribbons parallel to $\sigma \times [0,1]$, and also on $\gamma_j \times \{*\} \times I$ and nearby annuli parallel to $\gamma_j \times \{*\} \times I$. Using the flow of X we obtain an extension of $\sigma \times [0,1]$ to a semi-infinite ribbon $\sigma_i \times [0,\infty)$ accumulating on $\gamma_j \times \{*\}$.

For each ribbon which does not end inside $H(\xi)$ we pick a neighbourhood $\partial_- C_i \subset \partial P$ which is homeomorphic to a disc and contains σ. The notation $\partial_- C$ is meant to indicate that this surface will play a similar role as the part of the boundary of C which was denoted by $\partial_- C_i$ in Lemma 4.18. We require that $\partial_- C_i$ is disjoint from its images under the flow of X for positive times and that the neighbourhoods $\partial_- C_i \supset \sigma_i$ are so small that their images under the flow never intersect.

If ζ is C^0–close enough to ξ, then we can consider the projection X' of X along $\mathcal{I}$ to ζ. We assume that ζ is so close to ξ that the flow ψ'_τ of X' can be used to modify the ribbon $\sigma_i \times [0,1]$. Moreover, we assume that X' — just like X — is transverse to

$\gamma_j \times I \times \{\pm 1\}$. Then the modified ribbon can still be extended to a semi-infinite ribbon $\sigma_i \times [0,\infty)$ which accumulates on $\gamma_j \times \{*\}$.

For $w > 0$ let

$$C_i(w) := \bigcup_{\tau=0}^{w} \psi'_\tau(\partial_- C_i),$$

where ψ'_τ denotes the flow of X' (the ζ–Legendrian vector field determined by X and the plane field ζ close to ξ). As w increases this is a neighbourhood of longer and longer pieces of the semi-infinite ribbon; for $w > 0$ it is homeomorphic to a ball. This is illustrated in Figure 15. In analogy with the notation employed in Lemma 4.18 we set $\partial_+ C_i(w) := \psi'_w(\partial_- C_i)$.

If $\xi' = \zeta$ is a positive contact structure which is so close to ξ that the above construction can be carried out, then the nonintegrability of ξ' can be used as follows: as $w \to \infty$, the pullback of the characteristic foliation on $\partial_+ C_i(w)$ to $\partial_- C_i$ converges to the line field determined by the intersection of the annuli $\gamma_j \times \{*\} \times [-1,1]$ with $\partial_- C_i$ by (3-4) at the end of Section 3A. (The convergence is monotone and by Dini's theorem the convergence is even uniform.)

As in Lemma 4.18 we can therefore prescribe the holonomy $g_i\colon \sigma_i = \sigma_i \times \{0\} \to \sigma_i \times \{0\}$ corresponding to the domain contained in a long enough tube $C'_i(w)$ containing long pieces of the semi-infinite ribbon as $w \to \infty$. When this domain is added to P the change of the monodromy is determined by g_i (and the characteristic foliation on ∂P, as in (4-3)–(4-4) in Section 4C1).

The size of the shift by which we can change the monodromy when we attach a domain surrounding a long piece of a semi-infinite ribbon is bounded by the length of the attaching arc $\sigma_i \times \{0\}$ of the ribbon.

Again, this works parametrically, ie the domain varies continuously when the contact structures ξ'_s depend continuously (again with the C^0–topology) on a parameter s as long as ξ'_s is sufficiently close to ξ. By the last requirement in Definition 5.4 we can ensure that all these domains are pairwise disjoint for all w by choosing $\partial_- C_i$ sufficiently thin.

As before, the ribbon $\sigma_i \times [0,\infty)$ intersects other polyhedra on its way to $\gamma_j \times \{*\} \times I$ and induces further ribbons but we will ignore induced ribbons which start inside N. As in the previous section we consider neighbourhoods of induced ribbons, which get thicker and thicker as we move towards $\sigma_i \times \{1\}$.

We modify the list of requirements listed in Section 4C1. For all ribbons which end in $H(\xi)$ no further modification is needed; they are treated as in Section 4. There are the following additional restrictions on the neighbourhood of ξ in the space of plane fields. As before, ζ denotes a smooth plane field from that neighbourhood.

(1) The characteristic foliations of ζ on the annuli $\gamma_j \times \{*\} \times I$ described in Definition 5.4 remain transverse to the boundary of the annuli. (The characteristic foliation on these annuli may have more than one closed orbit after a small perturbation of ξ. What matters to us is that all leaves of the characteristic foliation which enter the annulus stay in the annulus even after a C^0–small perturbation.)

(2) ζ is so close to ξ that the construction of semi-infinite ribbons described above works for all ribbons from the full collection which do not end in $H(\xi)$.
This requirement applies to all ribbons (induced or not) which are attached to polyhedra outside of N. Since ribbons either end at cylinders in $H(\xi)$ or in N we have to deal only with a finite number of ribbons.

(3) Recall that each connected component of $\widehat{N}$ is the product of a circle and a disc with two corners such that $\mathcal{I}$ is tangent to the discs. We ask that ζ remains transverse to both smooth boundary segments of the discs with corners. (This will be used to show that certain contact structures on $\widehat{N}$ are tight.)

(4) The constant δ_{cyl} used in Section 4C1 (measuring the possible alterations of the holonomy of characteristic foliations when a ribbon is attached) is smaller than the length of the shortest ribbon σ_i from the full collection.

These conditions together with those from Section 4C1 determine a C^0–neighbourhood of ξ in the space of smooth plane fields.

5D The proof of the stability theorem for confoliations with holonomy and without closed leaves

We shall construct a family of contact structures $\xi_{(s,t)}$ using a modification of the technique developed in Section 4 together with Theorem 2.36. Also, the domains we attach to ends of ribbons depend on s and t. None of these dependencies will be reflected in the notation.

Proof Let ξ_s be a family of contact structures with compact parameter space S that lies in the ε–neighbourhood of ξ. As before, we want to extend ξ_s to a family of contact structures $\xi_{\widehat{s}}$ with parameter space $\widehat{S}$. This will be done in two steps, so it is convenient to put $\widehat{S} := S \times [0,2]/S \times \{2\}$. We denote elements of $\widehat{S}$ by $\widehat{s} = (s,t)$.

We fix a particular perturbation of ξ into a contact structure: by the proof of the approximation theorem of Eliashberg and Thurston (Theorem 1.1) there is a contact structure in the ε–neighbourhood of ξ (which we shall denote by $\widetilde{\xi}$) so that the characteristic foliation on each connected component of $\partial\widehat{N}$ has exactly two closed orbits on each connected component (one of them attractive, the other one repulsive). Like ξ_{s_0} in the proof of Theorem 4.2 in Section 4C2, $\widetilde{\xi}$ will serve as base point.

In the first step we construct a family of contact structures $\xi_{(s,t)}$ with $t \in [0,1]$ so that outside of $\widehat{N}$ we have $\xi_{(s,1)} = \widetilde{\xi}$. This is possible at the expense of losing some control over the contact structure $\xi_{(s,1)}$ inside $\widehat{N}$. However, we will show that $\xi_{(s,1)}$ is tight on $\widehat{N}$. Then we can use Theorem 2.36 to find a homotopy $\xi_{(s,t)}$, $t \in [1,2]$ such that in the end $\xi_{(s,2)} = \widetilde{\xi}$. This will conclude the proof of Theorem 5.1.

The first step is analogous to the first part of the proof of Theorem 4.2 at the end of the previous section: For each supporting vertex of a polyhedron which meets the complement of $\widehat{N}$, we choose a family of contact structures on a neighbourhood of the supporting vertex which interpolates between ξ_s and $\widetilde{\xi}$ so that this family remains ε–close to ξ. Then we extend this family of contact structures to a family of plane fields $\zeta_{(s,t)}$ which coincides with ξ_s

- on all polyhedra meeting N, and
- on all cylinders in $H(\xi)$ where ribbons from the full collection end

such that the plane field is ε–close to ξ. Recall that by the choice of the polyhedral decomposition ((ii) in Section 5B) there is a layer of polyhedra in $\widehat{N}$ separating $\partial\widehat{N}$ from N.

The characteristic foliation of $\zeta_{(s,t)}$ on the boundary of polyhedra not intersecting N may have closed leaves. But since $\zeta_{(s,t)}$ is ε–close to ξ this can be corrected by the attachment

- of ribbons together with pieces of cylinders lying in $H(\xi)$ such that the monodromy on the boundary of the cylinder is sufficiently decreasing, or
- of domains $C_i'(w)$ containing long enough pieces of a ribbon $\sigma_i \times [0,\infty)$ accumulating on a closed leaf of the characteristic foliation on an annulus $\gamma_j \times \{*\} \times I$.

It is irrelevant that a semi-infinite ribbon $\sigma_i \times [0,\infty)$ might meet a region where $\zeta_{(s,t)}$ is not contact since this happens only on the piece $\sigma \times [0,1]$.

Attaching all ribbons we obtain a family of plane fields such that its characteristic foliation on each polyhedra outside of N satisfies the conditions of Lemma 4.14. On the complement of N we therefore obtain a family of contact structures $\xi_{(s,t)}$ which is transverse to $\mathcal{I}$ such that $\xi_{(s,1)} = \widetilde{\xi}$ on the complement of $\widehat{N}$ and also on a neighbourhood of $\partial\widehat{N}$.

On the remaining polyhedra $\xi_{(s,t)}$ is obtained by pulling back the contact structure ξ_s by the isotopy used to identify boundary of polyhedra with ribbons/cylinders or domains surrounding long pieces of semi-infinite ribbons with the original boundary. Recall that these identifications are done so that the resulting plane fields remain transverse to $\mathcal{I}$. This will imply the following claim.

Claim *$\xi_{(s,1)}$ is tight on $\widehat{N}$ for all s.*

We now use the particular shape of tubular neighbourhoods $\widehat{N}_j$ of γ_j which were formed as unions of products of a circle and discs with two corners. By construction $\xi_{(s,1)}$ is transverse to $\mathcal{I}$ and therefore the characteristic foliation of $\xi_{(s,1)}$ on the discs with corners consists of arcs which pass from one smooth piece of the boundary of the disc to the other. Thus $\xi_{(s,1)}$ can be extended to a contact structure on $\mathbb{R}^2\times\mathbb{R}$ such that the second factor corresponds to $\mathcal{I}$ and the extended contact structure is transverse to the second factor and defines a complete connection on $\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$. Therefore $\xi_{(s,1)}$ is tight for all $s\in S$.

On the interior of $\widehat{N}$ the contact structures $\xi_{(s,1)}$ depend on s while the characteristic foliation on the boundary is constant. By the claim above we can use Theorem 2.36 to extend the homotopy of contact structures by a family $\xi_{(s,t)}$, $t\in[1,2]$, so that $\widetilde{\xi}=\xi_{(s,2)}$ is independent from s. The homotopy is constant outside of $\widehat{N}$. □

In this proof we have first modified contact structures outside of a particular set $\widehat{N}$ using semi-infinite ribbons which accumulate inside $\widehat{N}$ in a controlled fashion. Then we have appealed to classification results from contact topology to deal with contact structure inside $\widehat{N}$. The scheme will appear again in the following two sections. The case of foliations without holonomy is somewhat different and will be explained in Section 8.

6 Uniqueness with closed leaves: tori

The proof of Theorem 1.6 is easier when the only closed leaves of $\mathcal{F}$ are stable tori (the terminology is explained below). We give this proof first before proceeding to leaves of higher genus in the next section. The main results discussed in Section 6A apply to all orientable surfaces and will be used later, too.

Recall that for stable torus leaves we only obtain a weakened uniqueness result because every neighbourhood of a confoliation with a stable torus leaf contains contact structures with different Giroux torsion.

The strategy to deal with foliations with stable torus leaves is parallel to the one used in the previous section. The main differences are:

- Neighbourhoods $\widehat{N} \subset N$ of the Sacksteder curves are now replaced by neighbourhoods of torus leaves.
- The classification of tight contact structures on the thickened torus $T^2 \times I$ is more complicated. This leads to the fact that in the presence of torus leaves we only obtain a weakened uniqueness statement.

6A Fixing neighbourhoods of closed leaves

The purpose of this section is to introduce part of the data we shall use to determine the neighbourhood U of the confoliation ξ in Theorem 1.4.

In contrast to exceptional minimal sets, whose number is always finite, a foliation can have uncountably many compact leaves. However, according to a fundamental theorem of A Haefliger [29] the set of compact leaves of a foliation of codimension one is a closed subset of M. This result does not need the C^2–smoothness assumption. Moreover, if $\mathcal{F}$ is coorientable, then the union of leaves of a given diffeomorphism type is compact. In our situation this implies that there is an integer $g_{\max}$ such that the genus of a given closed leaf of $\mathcal{F}$ is at most $g_{\max}$. In order to give a more precise description of the union of closed leaves of $\mathcal{F}$ we recall the following definition from [1].

Definition 6.1 Let Σ_0 and Σ_1 be two closed leaves of ξ. These leaves are *equivalent* if there is an immersion

$$\psi\colon \Sigma \times [0,1] \to M$$

with the following properties:

(i) The restriction of ψ to $\Sigma \times \{t\}$ is an embedding for all $t \in [0,1]$.

(ii) $\psi(\Sigma \times \{0\}) = \Sigma_0$ and $\psi(\Sigma \times \{1\}) = \Sigma_1$.

(iii) For all $p \in \Sigma$ the curve $\psi(p,\cdot)$ is transverse to $\mathcal{F}$.

Clearly, equivalent leaves are diffeomorphic. A diffeomorphism is provided by the holonomy of the image of the foliation by the second factor on $\Sigma \times [0,1]$. Definition 6.1 has an obvious generalization to all foliations of codimension one. The closed leaves of the foliation on $S^1 \times [-1,1]$ shown in Figure 16 (the S^1–factor is horizontal) which lie in the centre of the figure are all equivalent while the other two closed leaves are not equivalent to any other closed leaf in the figure.

Haefliger's compactness theorem implies that there is only a finite number of equivalence classes of closed leaves [1]. Because $\mathcal{F}$ is not a foliation without holonomy we can actually assume that ψ is an embedding and extend it to a tubular neighbourhood for both Σ_0 and Σ_1. Using this terminology, the assumption (i$'$) in Theorem 1.6 can be replaced by the following slightly weaker requirement:

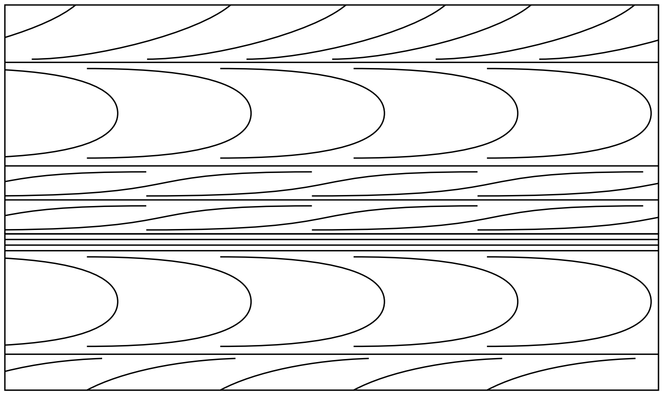

Figure 16: Foliation on $S^1 \times [-1,1]$ with three equivalence classes of closed leaves

(i$''$) The union of all torus leaves is covered by a finite collection of embeddings $\psi\colon T^2 \times [0,1] \to M$ as in Definition 6.1 each of which has attractive holonomy, ie there is a simple nonseparating closed curve $\gamma \subset T^2$ so that the holonomy along $\psi(\gamma \times \{i\})$ in the torus $T^2 \times \{i\}$ is attractive on the side not contained in $\psi(T^2 \times [0,1])$ for $i \in \{0,1\}$.

The two upper equivalence classes of closed leaves in Figure 16 have attractive holonomy; the closed leaf at the bottom does not.

Before we proceed with the proof of Theorem 1.6 let us recall the following result of M Hirsch [33] which explains the terminology *stable/unstable* torus leaf. This result is reproved as [1, Theorem 3.f.1].

Theorem 6.2 *Let $\mathcal{F}$ be a transversely coorientable foliation of codimension 1 on M and L a closed leaf with abelian fundamental group such that $\mathcal{F}$ is not a foliation by fibres of a fibration $M \to S^1$. Fix a tubular neighbourhood $N(L)$ of L.*

(i) *If L has attractive holonomy along some simple closed curve $\gamma \subset L$, then there is a C^0–neighbourhood of $\mathcal{F}$ in the space of plane fields such that every foliation in that neighbourhood has a closed leaf diffeomorphic to L inside $N(L)$.*

(ii) *If there is no curve in L with attractive holonomy, then every C^1–neighbourhood of $\mathcal{F}$ contains a foliation which has no closed leaf inside $N(L)$.*

The case when $\mathcal{F}$ is given by the fibres of a torus fibration over S^1 is also understood: according to a theorem of J Plante [50], a fibration with torus fibres has a C^0–neighbourhood such that every foliation in that neighbourhood has a torus leaf close to an original fibre if and only if the map

$$\phi_M\colon H_1(T^2,\mathbb{Z}) \to H_1(T^2,\mathbb{Z})$$

induced by the monodromy of the torus bundle does not have a positive real eigenvalue. However, it will turn out in Section 8A that not only these fibrations have a neighbourhood with the properties described in Theorem 1.6. Therefore the analogy between stable torus leaves and stability of the approximating contact structures up to isotopy is not perfect.

6B Determining the neighbourhood in the space of plane fields

We now describe the neighbourhood U of ξ in the space of smooth plane fields whose existence is claimed in Theorem 1.6. In order to simplify the presentation we assume that ξ has a unique minimal set which is a closed torus leaf with attractive holonomy. How to treat the case when there are several minimal sets, either exceptional ones or other torus leaves, will then be clear. In Section 5 we have already considered a situation where both $H(\xi)$ and the fully foliated set of ξ are not empty. From now on we will assume that $\xi = \mathcal{F}$ is a foliation. As before $\mathcal{I}$ is a line field transverse to ξ.

Let $T \subset M$ be the unique torus leaf and γ a simple closed curve in T with attractive holonomy.

There is a foliation $\mathcal{G}$ on T by simple closed curves such that γ is a leaf and the holonomy along each leaf of $\mathcal{G}$ is attractive (in the case of confoliations this is true by Lemma 2.7). Now we fix a pair of tubular neighbourhoods

$$T = T \times \{0\} \subset N(T) \cong T \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \subset \widehat{N}(T) \cong T \times [-1, 1] \tag{6-1}$$

of T and a polyhedral decomposition of M adapted to $\mathcal{F}$ and $\mathcal{I}$ such that the following conditions are satisfied:

(i) The foliation $\mathcal{I}$ is tangent to the interval factor in (6-1) and the characteristic foliation on $\gamma \times [-1, 1]$ has one closed orbit, and all other leaves enter this annulus through its boundary and accumulate on γ. The characteristic foliation is transverse to $\partial N(T) \cap (\gamma \times [-1, 1])$. The same requirement applies to the other leaves of $\mathcal{G}$.

(ii) $\partial N(T)$ and $\partial \widehat{N}(T)$ are both transverse to all faces and edges of the polyhedral decomposition. No vertex lies on $\partial \widehat{N} \cup \partial N$.

(iii) The characteristic foliations $\partial \widehat{N}(T)(\mathcal{F})$ and $\partial N(T)(\mathcal{F})$ contain no 2–dimensional Reeb component.

(iv) No polyhedron which intersects $\partial \widehat{N}(T)$ meets a polyhedron intersecting $\partial N(T)$ nontrivially.

We fix a full collection of ribbons for all faces of polyhedra which do not meet $N(T)$, ie we choose a finite collection of ribbons $\sigma_j \times [0,1]$ satisfying the following conditions:

(i) The end of each ribbon which is not contained in the face of a polyhedron is contained in $N(T)$ and the ribbons are pairwise disjoint. Each closed leaf of the characteristic foliation of a polyhedron outside of $N(T)$ meets the attaching arc σ_j of a ribbon.

(ii) For each ribbon the projection of the part of the ribbon lying in $N(T)$ along $\mathcal{I}$ to the torus is contained in a single leaf of $\mathcal{G}$.

(iii) As in Definition 5.4 we extend the ribbon $\sigma_j \times [0,1]$ to a semi-infinite ribbon $\sigma_j \times [0,\infty)$ so that the piece $\sigma_j \times [1,\infty)$ is contained in an annulus to which $\mathcal{I}$ is tangent and which contains a leaf γ of $\mathcal{G}$. The semi-infinite ribbon accumulates on γ (this is shown in Figure 15).

We are now in a position to choose $\varepsilon > 0$ which determines the C^0–neighbourhood of $\mathcal{F}$ in Theorem 1.6 in the present context (ie no closed leaves of higher genus). For every plane field ζ in the ε–neighbourhood of $\mathcal{F}$ we require:

- ζ is transverse to $\mathcal{I}$.
- The characteristic foliation of ζ on $\gamma' \times [-1,1]$ remains transverse to the boundary and inward-pointing for all leaves γ' of $\mathcal{G}$. In particular, all leaves of $\gamma' \times [-1,1]$ entering through the boundary accumulate on a closed leaf. It is irrelevant to our discussion how many closed leaves this characteristic foliation has or whether or not they are nondegenerate.
- All semi-infinite ribbons lift to semi-infinite ribbons adapted to ζ while the ribbons still have the necessary properties (pairwise disjointness, ending in $N(T)$, characteristic foliations remain graphical, etc) explained in Section 4C1 and Section 5C.

6C The proof of Theorem 1.6 in the absence of closed leaves of higher genus

It remains to show that the ε-C^0–neighbourhood of $\mathcal{F}$ from the previous section has the desired properties.

Proof of Theorem 1.6 We start with two contact structures ξ, ξ' which are ε–close to $\mathcal{F}$. By the procedure from Section 4C2 and Section 5D we can connect ξ to a contact structure $\widehat{\xi}$ which coincides with ξ' on all polyhedra which are not contained in the interior of $\widehat{N}(T)$. Using Lemma 4.14 we can ensure that the contact structure remains transverse to $\mathcal{I}$.

Thus we obtain a family of contact structures ξ_t, $t \in [0,1]$, such that ξ_1 coincides with ξ' on all polyhedra which are not contained in $\widehat{N}(T)$, so that ξ_1 is transverse to $\mathcal{I}$ on $\widehat{N}(T)$.

After a C^∞–small perturbation we may assume that $\partial\widehat{N}(T)$ is convex and nonsingular. Then the dividing set on $\partial N(T)$ contains no homotopically trivial component. (By Lemma 3.25, $\widehat{\xi}$ and ξ' are tight on $N(T)$.)

Because the characteristic foliation of ξ_1 on $\partial\widehat{N}(T)$ is the same as the one induced by ξ', there is no Reeb component in $\partial\widehat{N}(T)$. Now we apply Theorem 2.37. For this we have to check that for both contact structures ξ_1, ξ' there is a torus in the interior of $\widehat{N}(T)$ isotopic to T and whose characteristic foliation is a foliation by closed leaves.

But this follows from the fact that for each leaf γ' of $\mathcal{G}$ the characteristic foliation on $\gamma' \times [-1,1]$ has a closed leaf in the interior. By the results from Section 3A the union of these closed Legendrian curves contains an embedded torus with the desired properties. By Theorem 2.37, ξ_1 and ξ' are stably isotopic. □

If T is a torus leaf (stable or unstable) of a (con-)foliation $\mathcal{F}$, then $\mathcal{F}$ can be C^0–approximated by confoliations $\mathcal{F}_n$ containing a domain foliated by tori. It is then easy to approximate $\mathcal{F}_n$ by a contact structure with arbitrarily large Giroux torsion along T.

Finally, let us mention two points where the above proof fails for unstable torus leaves. When T is unstable, then

(1) we are no longer sure that our ribbons can be extended to semi-infinite ribbons in $\widehat{N}(T)$, and

(2) we can no longer guarantee that (iii) of Theorem 2.37 is satisfied.

As indicated in the bottom part of Figure 1 in Section 1A there may be sheets of the contact structure which connect the two boundary components of $\widehat{N}(T)$. If this happens, then according to [24] there are infinitely many contact structures on $\widehat{N}(T)$ with vanishing Giroux torsion which are pairwise nonisotopic and still satisfy assumptions (i)–(ii) of Theorem 2.37. For these contact structures, the sheets connect the two boundary components of $\partial\widehat{N}(T)$. The conditions on ε formulated in Section 7B ensure — among other things — that no sheet of the contact structure ε-C^0–close to $\mathcal{F}$ will connect the boundary components of a tubular neighbourhood of the closed leaf.

In Example 9.12 we show that in this case it may happen that any neighbourhood (it will turn out that we may even take a C^∞–neighbourhood) of a foliation with unstable torus contains two positive contact structures which are not stably isotopic on M.

7 Uniqueness in the presence of closed leaves with higher genus

In this section we shall complete the proof of Theorem 1.4 for foliations with holonomy.

The previous sections have covered the situation when $\mathcal{F}$ is a foliation (or a positive confoliation) which belongs to one of the following classes:

- $\mathcal{F}$ is a foliation such that every leaf is dense and there is holonomy.
- $\mathcal{F}$ is a (positive con-)foliation all of whose minimal sets are either exceptional or stable torus leaves.

Actually, T Tsuboi [56] has shown that every C^1–neighbourhood of a foliation contains a foliation without closed leaves of higher genus.

In this section we deal with closed leaves of genus $g \geq 2$. As in the case of torus leaves the set of closed leaves of a fixed genus is not finite but at least it is compact and the discussion in Section 6A applies. In order to simplify the presentation, we assume throughout this section that there is exactly one minimal set, which is the closed leaf Σ.

The overall strategy to prove the uniqueness result is the same as in Section 5 and Section 6 and the introduction in Section 1A contains an outline which deals with closed leaves of higher genus. Compared to the case of minimal sets which are exceptional or stable torus leaves, there are two main difficulties which we will have to deal with:

(1) We do not know in general whether the characteristic foliation of ξ on Σ has closed leaves on fixed annuli transverse to Σ. In Section 5 and Section 6 we made essential use of such curves since semi-infinite ribbons accumulated on them.

(2) The available classification theorems for tight contact structures on $\Sigma \times [-1, 1]$ are much less satisfactory than those for tight contact structures on the solid torus or $T^2 \times [-1, 1]$: the assumptions of Theorem 2.39 are much more restrictive than those of Theorem 2.36 or Theorem 2.37.

We will use Theorem 2.39 to first modify ξ without changing the isotopy type of ξ to ensure that the new characteristic foliation has closed leaves in a place suitable for extending ribbons from a fixed collection. In order to apply Theorem 2.39, one has to isotope the boundary of $\Sigma \times I$, where I denotes an interval, so the assumptions of Theorem 2.39 are satisfied on the boundary of the isotoped region. The construction of these isotopies is guided by sheets. We will apply Theorem 2.39 twice, once to be able to extend ribbons and then in a way analogous to those in which Theorems 2.36 and 2.37 were applied in Sections 5 and 6, respectively.

7A Geometry of surfaces of higher genus

Let Σ be a closed surface of genus ≥ 2. We fix a hyperbolic metric on Σ and a universal covering $\mathbb{H}^2 \to \Sigma$. On all surfaces covering Σ we use the pulled-back metric, and $\partial\mathbb{H}^2$ denotes the ideal boundary of the hyperbolic plane.

Lemma 7.1 *There is a constant K which depends only on the hyperbolic surface Σ with the following property:*

Let γ_t, $t \in [0, 1]$, be a family of homotopically essential simple closed curves and let $\widetilde{\gamma}_t$ be a lift of the isotopy to $\mathbb{H}^2$. Then there is a pair of points $p_0 \in \widetilde{\gamma}_0$ and $p_1 \in \widetilde{\gamma}_1$ such that the distance between the points is smaller than K.

If $\Sigma' \to \Sigma$ is an abelian covering, then the same constant can be used for Σ'.

The only interesting case is when $\widetilde{\gamma}_1$ lies entirely on one side of $\widetilde{\gamma}_0$ in $\mathbb{H}^2$. This will be the case in our applications of this lemma. Finally, the part of the lemma concerning abelian coverings will be used only in Section 8C3 when we discuss minimal foliations without holonomy.

Proof of Lemma 7.1 We will use some facts from the geometry of hyperbolic surfaces which can be found for example in the first chapters of [16].

Because γ_0 is a simple closed curve it is isotopic to a unique closed geodesic γ which is also simple and nontrivial because γ_0 is not null-homotopic. We fix a lift $\widetilde{\gamma}$ of γ in the universal covering. In order to prove the lemma we will first show that there is a constant K' and a point $\widetilde{x} \in \widetilde{\gamma}$ such that the lift (with the same endpoints on the ideal boundary) of every curve isotopic to γ contains a point whose distance from $\widetilde{x}$ is at most K'.

Let $a_1, \dots, a_l \subset \Sigma$ be a collection of oriented null-homologous homotopically essential simple closed curves such that the complement of the curves is a union of discs (as shown in Figure 17).

We assume that the curves a_i are geodesics. Let B_R be a large disc in $\mathbb{H}^2$ such that the restriction of the universal covering map to B_R is surjective.

For each lift $\widetilde{a}_i$ of a_i intersecting B_R we pick connected neighbourhoods $\widetilde{J}^{\pm}(\widetilde{a}_i) \subset \partial\mathbb{H}^2$ of the endpoints of $\widetilde{a}_i$ that are pairwise disjoint. (There are only finitely many lifts of a_i which intersect B_R.) Given a lift $\widetilde{\alpha}$ of an oriented closed geodesic α in Σ, we denote the corresponding isometry of $\mathbb{H}^2$ by $f_{\widetilde{\alpha}}$.

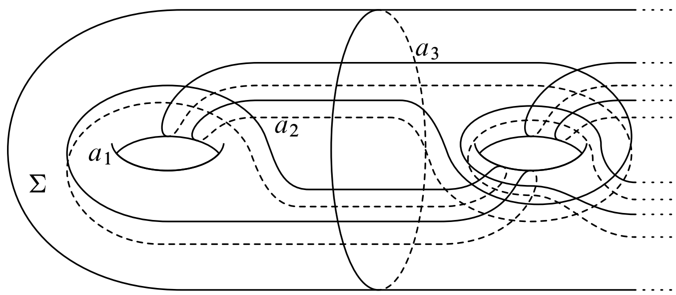

Figure 17: Collection of null-homologous curves in Σ

For each $\alpha > 0$ there is a number $N = N(\alpha)$ with the following property: if $\widetilde{\gamma}$ is a geodesic in $\mathbb{H}^2$ which intersects $\widetilde{a}_i$ in a point $y \in B_R$ such that

$$|\measuredangle_y(\widetilde{\gamma}, \widetilde{a}_i)| > \alpha,$$

then both ideal endpoints of $f_{\widetilde{a}_i}^{\pm N}(\widetilde{\gamma})$ are contained in the same interval $\widetilde{J}^{\pm}(\widetilde{a}_i)$ (the thickened arcs in Figure 18 correspond to two such intervals).

In the following we choose $\varepsilon > 0$ close to zero (and the corresponding integer N) such that if a geodesic γ intersects one curve a_j and the angle at that intersection is smaller than ε, then the absolute value of the angle at the intersection points of γ and $\bigcup_{i \neq j} a_i$ which lie next to y on γ is bigger than

$$\alpha_0 = \tfrac{1}{2} \min\bigl\{|\measuredangle_z(a_i, a_j)| \mid z \in a_i \cap a_j \text{ and } i \neq j\bigr\}. \tag{7-1}$$

We pick the number N corresponding to this angle.

Pick two intersection points x, x' of γ with $a_1, \dots, a_l$ which are either consecutive along γ and the angle of the two geodesics at both intersection points is greater than ε or x, x' are separated by exactly one other intersection of γ with $\bigcup_i a_i$ where the angle between the two curves is smaller than ε (then the angle at the intersections x, x' is again greater than α_0).

Now consider preimages $\widetilde{x}, \widetilde{x}' \in \widetilde{\gamma}$ of x, x' such that there is at most one intersection point of the segment of $\widetilde{\gamma}$ between $\widetilde{x}, \widetilde{x}'$ and lifts of the curves a_i. The distance between $\widetilde{x}$ and $\widetilde{x}'$ is bounded by $2K_1$, where K_1 is the maximal diameter of the discs $\Sigma \setminus \bigcup_i a_i$. This bound is independent of γ.

Let $\widetilde{a}, \widetilde{a}'$ be lifts of two curves from our collection passing through $\widetilde{x}, \widetilde{x}'$. We assume that these curves are oriented to the side of $\widetilde{\gamma}$ containing $\widetilde{\gamma}_0$. Since our collection of curves is finite, we can consider

$$K' := \max\bigl\{\operatorname{dist}(y, f_{\widetilde{a}_i}^{\pm N}(y)) \mid \operatorname{dist}(y, \widetilde{a}_i) \leq 2K_1 \text{ and } \widetilde{a}_i \text{ is a lift of } a_i\bigr\}.$$

This number depends only on the maximal displacement of the isometries associated to our system of curves but not on γ. The point $\widetilde{x}$ and the number K' have the desired properties.

In order to see this, assume that $\widetilde{\gamma}_0$ is the lift of a simple closed curve isotopic to γ with the same endpoints in the ideal boundary of $\mathbb{H}^2$ which does not meet the K'–ball around $\widetilde{x}$. The curves $f_{\widetilde{a}}^{-N}(\widetilde{\gamma}_0)$ and $f_{\widetilde{a}'}^{-N}(\widetilde{\gamma}_0)$ each have endpoints in one of the intervals $\widetilde{J}^{\pm}(\widetilde{a}_i)$. In particular, the endpoints of these curves on $\partial\mathbb{H}^2$ are unlinked. Because the discs bounded by $f_{\widetilde{a}}^{-N}(\widetilde{\gamma}_0)$ and $f_{\widetilde{a}'}^{-N}(\widetilde{\gamma}_0)$ and pieces of the intervals $\widetilde{J}^{\pm}(\widetilde{a}_i)$ contain $\widetilde{x}$, the images of $\widetilde{\gamma}_0$ under deck transformations intersect. This is indicated in Figure 18. But this contradicts the assumption that γ_0 is simple.

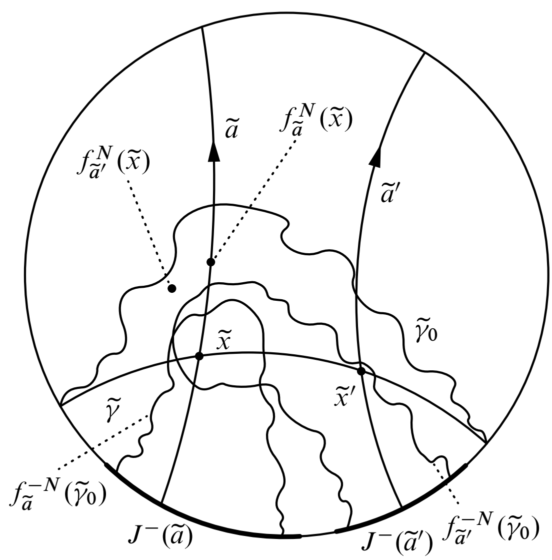

Figure 18: Nondisjoint lifts of γ_0 to $\mathbb{H}^2$

It is therefore impossible that both points $f_{\widetilde{a}}^N(\widetilde{x})$ and $f_{\widetilde{a}'}^N(\widetilde{x})$ lie on the same side of $\widetilde{\gamma}_0$ as γ. Therefore $\widetilde{\gamma}_0$ intersects the K'–ball around $\widetilde{x}$.

What we used to show this is that the angle between the geodesic $\widetilde{\gamma}$ and a lift of one of the curves $a_1, \dots, a_l$ at their intersection point $\widetilde{x}$ is bigger than ε, the distance between two such intersection points is at most twice the diameter of the discs $\Sigma \setminus \left(\bigcup_i a_i\right)$. The constant K' is determined by a collection of disjoint neighbourhoods of endpoints of lifts and the hyperbolic isometries of $\mathbb{H}^2$ which are translations of distance $\text{length}(a_i)$ along a lift of $a_1, \dots, a_l$.

Now if γ_0, γ_1 are isotopic, then the lifts (with the same endpoints on $\partial\mathbb{H}^2$) of $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ could lie on opposite sides of the geodesic connecting the two endpoints in $\mathbb{H}^2$.

The only condition on x, x' that we used above was that the angle between γ and curves from our collection $a_1, \dots, a_l$ at these intersection points is bounded from below by ε. Therefore we can use the same point $\widetilde{x}$ for both γ_0 and γ_1. Hence there are points on $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ whose distance from each other is smaller than $2K'$.

When the lift of γ_0 intersects $\widetilde{\gamma}$ while $\widetilde{\gamma}_1$ does not, there are points on $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ whose distance is smaller than

$$K' + \max\bigl\{\operatorname{diam}\bigl(\text{connected component of } \bigl(\Sigma \setminus \textstyle\bigcup_i a_i\bigr)\bigr)\bigr\}$$

(the second summand bounds the distance between an intersection point of γ and $\widetilde{\gamma}_0$ from a point $\widetilde{x}$ with the properties used above). Finally, if both $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ intersect $\widetilde{\gamma}$, then we can take

$$K = 2K' + 2\max\bigl\{\operatorname{diam}\bigl(\text{connected component of } \bigl(\Sigma \setminus \textstyle\bigcup_i a_i\bigr)\bigr)\bigr\}. \tag{7-2}$$

This constant works in the previous cases, too.

Finally, if $\Sigma' \to \Sigma$ is an abelian covering of Σ, then we lift the collection of separating curves and the hyperbolic metric to the abelian covering. Because the covering is abelian all lifts of curves $a_1, \dots, a_k$ are still closed. The discs obtained by cutting Σ' along all lifts of $a_1, \dots, a_l$ are isometric to the discs obtained by cutting Σ. Note also that although the ball $B_R \subset \mathbb{H}^2$ (used above to determine ε) does not necessarily surject onto the abelian covering we can still use a deck transformation to ensure that the intersection point of $\widetilde{\gamma}$ (a lift of a simple closed geodesic in the abelian cover) with lifts of a_i lies in B_R. Therefore there is no need to change R or any other constant appearing in (7-2) when Σ is replaced by an abelian covering. $\square$

This lemma is not true if Σ is a torus $T^2 \cong S^1 \times S^1$ since one can use the flow along the first circle direction to displace $\{1\} \times S^1$ from itself in such a way that the distance becomes unbounded when everything is lifted to the universal covering. The crucial point in the proof of Lemma 7.1 is that lifts of isotopies of closed curves do not move the endpoints in the ideal boundary.

In order to see that Lemma 7.1 does not hold for homotopies (instead of isotopies) consider a closed geodesic $\gamma \subset \Sigma$ and a lift $\widetilde{\gamma} \subset \mathbb{H}^2$. Now let $\widetilde{\gamma}_t$ be a family of curves in $\mathbb{H}^2$ consisting of points whose distance from $\widetilde{\gamma}$ is t. This family of curves projects to a homotopy γ_t of closed curves in Σ and this homotopy violates the conclusion of Lemma 7.1 since t can be chosen arbitrarily large.

7B Determining the neighbourhood in the space of plane fields

Let Σ be a closed leaf of $\mathcal{F}$ with genus $g \geq 2$. As before we will assume that it is the unique minimal set of $\mathcal{F}$; if Σ is not the only closed leaf then we consider a neighbourhood of an equivalence class of closed leaves.

Fix a foliation $\mathcal{I}$ of rank 1 transverse to $\mathcal{F}$. For a hyperbolic metric on Σ let K be the constant K from Lemma 7.1. We also need to fix a pair of tubular neighbourhoods

$$\Sigma \times [-2, 2] \cong \widehat{N}(\Sigma) \supset N(\Sigma) \supset \Sigma$$

such that the second factor of the product decomposition is tangent to $\mathcal{I}$. Compared to the conditions formulated in Section 5C and Section 6B, the conditions the pair $(\widehat{N}(\Sigma), N(\Sigma))$ of tubular neighbourhoods has to satisfy are now more complicated. To formulate this condition we fix $\widehat{N}(\Sigma) \cong \Sigma \times [-2, 2]$ such that the second factor is tangent to $\mathcal{I}$.

Inside $\widehat{N}$ we will define two sets of $(2g+1)$ levels as follows. For $\delta^- \leq 0$ consider

$$\tau^-_{\mathcal{F}}(\delta^-) = \inf\bigl\{t \in [-2, \delta^-] \mid \text{there is a geodesic of length} \leq K+1 \text{ whose } \mathcal{F}\text{–horizontal lift starting in } \Sigma \times \{\delta\} \text{ meets } \Sigma \times \{t\}\bigr\}. \tag{7-3}$$

This is a negative number and $\tau^-_{\mathcal{F}}(\delta^-) \geq -2$. We choose $\delta^-_{g+1} < 0$ so close to zero that the levels

$$\delta^-_{g+1} > \delta^-_g = \tau^-_{\mathcal{F}}(\delta^-_{g+1}) > \delta^-_{g-1} = \tau^-_{\mathcal{F}}(\delta^-_g) > \cdots > \delta^-_{-(g+1)} > -2 \tag{7-4}$$

are all different and greater than -2. (Note that this iteration is always possible without reaching the boundary of $\widehat{N}(\Sigma)$ when one starts in $\Sigma \times \{0\}$.) Without loss of generality we assume that $\delta^-_0 = -1$.

In the part of $\Sigma \times [-2, 2]$ lying above Σ_0 we fix an analogous sequence of levels. Instead of $\tau^- \mathcal{F}$ we consider $\tau^+_{\mathcal{F}}$ defined by replacing inf by sup and $[-2, \delta]$ by $[\delta, 2]$ in (7-3), and replacing (7-4) by

$$0 < \delta^+_{g+1} < \delta^+_g = \tau^+_{\mathcal{F}}(\delta^+_{g+1}) < \delta^+_{g-1} = \tau^+_{\mathcal{F}}(\delta^+_g) < \cdots < \delta^+_{-(g+1)} < 2.$$

These levels will be used in one of our requirements for ε; namely, we ask that $\varepsilon > 0$ is so small that there is no geodesic in Σ of length $\leq K$ whose ζ–horizontal lift connects $\Sigma \times \{\delta^-_i\}$ to $\Sigma \times \{\delta^-_{i+1}\}$ or $\Sigma \times \{\delta^-_{i-1}\}$ for $i = -g, \dots, g$, where ζ is any smooth plane field whose C^0–distance to $\mathcal{F}$ is smaller than ε. We also require that ε satisfies the analogous requirement with respect to the levels δ^+_i.

In contrast to the case of stable tori and neighbourhoods of Sacksteder curves in exceptional minimal sets, we do not know a priori that there are distinguished annuli in $\Sigma \times [-2, 2]$ such that the characteristic foliation of $\mathcal{F}$ on the annuli contains an attractive closed leaf and $\mathcal{I}$ is tangent to the annuli.

In order to deal with this difficulty we choose yet another collection of levels as follows. Let

$$\delta := \tfrac{1}{2} \min\{|\delta^-_{g+1}|, \delta^+_{g+1}\} \tag{7-5}$$

and $N(\Sigma) = \Sigma \times [-\delta, \delta] \subset \widehat{N}(\Sigma)$.

Inside $N(\Sigma)$ we choose yet another smaller neighbourhood $\widehat{N}'(\Sigma)$ and constants $\widetilde{\delta}_i^{\pm}$ and δ' which are analogous to the constants $\delta_i^{\pm}$ and δ (defined in (7-4)–(7-5)). Finally let $N'(\Sigma) = \Sigma \times [-\delta', \delta']$. Using this data we obtain additional restrictions on ε: namely, starting in $\Sigma \times \{\pm\delta\}$ we do not reach the boundary of $\widehat{N}'(\Sigma)$ when considering $2g + 1$ consecutive ζ–horizontal lifts of geodesics in Σ which are shorter than K.

Since Σ is an isolated closed leaf, there are simple closed nonseparating curves γ_+, γ_- embedded in Σ such that the holonomy along γ_+ (resp. γ_-) is attractive on the side lying above (resp. below) Σ. Note that γ_+, γ_- are not isotopic or disjoint in general. However, we can choose both of them nonseparating because if the holonomy of Σ is trivial on one side for all nonseparating curves, then all leaves of $\mathcal{F}$ in a neighbourhood of Σ which meet the same side of the neighbourhood are compact and equivalent to Σ. Hence there are annuli $A_\pm$ containing $\gamma_\pm$ in their interior such that

- one boundary component (above Σ for A_+, below Σ for A_-) is transverse to $\mathcal{F}$,
- $A_\pm$ are both contained in the interior of $\Sigma \times [\delta, \delta]$, and
- $A_\pm$ is tangent to $\mathcal{I}$.

We pick product neighbourhoods of $A_\pm$ such that every annulus in that family has the same properties as the original annulus $A_\pm$. (The case when Σ is not an isolated closed leaf is only slightly different: the curves on annuli which are attractive on one side then lie in different leaves.)

So far we have been concerned with neighbourhoods of Σ. The number $\varepsilon > 0$ must also satisfy conditions related to polyhedral decompositions and ribbons. We fix the following data on M:

- A polyhedral decomposition of M adapted to $\mathcal{F}$ and $\mathcal{I}$ such that no polyhedron which meets $\Sigma \times [\delta, \delta]$ meets the complement of $\Sigma \times (-2\delta, 2\delta)$.
- No polyhedron which meets the complement of $\Sigma \times (-2, 2)$ meets $\Sigma \times \{\delta^{\pm}_{-(g+1)}\}$.

- A complete collection of ribbons $\sigma_i \times [0,1]$ for all polyhedra in the decomposition which meet the complement of $\Sigma \times [-2\delta, 2\delta]$. As usual, $\mathcal{I}$ is assumed to be tangent to the ribbons. The ends of the ribbons opposite to the faces lie in $\widehat{N}'(\Sigma)$, they are tangent to annuli parallel to $A_\pm$, and they enter annuli near A_+ and A_- through the boundary components above and below Σ, respectively.

We require that ε is such that the properties of polyhedra are also satisfied for plane fields ε–close to $\mathcal{F}$ and ribbons vary continuously when the plane field varies in the ε-C^0–ball around $\mathcal{F}$ in the space of smooth plane fields. Moreover, the boundary components of the annuli $A_\pm$ which are transverse to $\mathcal{F}$ are also transverse to ζ for all plane fields ε–close to $\mathcal{F}$.

Comparing this with the torus case, $\widehat{N}(\Sigma) = \Sigma \times [-2,2]$ will play the role of $\widehat{N}(T)$ and $N(\Sigma) = \Sigma \times [-\delta, \delta]$ will be the analogue of $N(T)$ (as indicated by the notation). Since we made no assumptions on the holonomy of Σ we need one more ingredient, because after the perturbation of $\mathcal{F}$ into a contact structure we can't be sure that the ribbons ending in the annuli $A_\pm$ can still be extended to semi-infinite ribbons. If $\gamma_+ = \gamma_-$ and this curve has attractive holonomy on both sides, then the neighbourhoods $\widehat{N}'(\Sigma) \supset N'(\Sigma)$ are not needed for the proof of Theorem 1.4.

7C The proof of Theorem 1.4 in the presence of closed leaves of higher genus

Let ξ_0, ξ_2 be two positive contact structures ε-C^0–close to $\mathcal{F}$.

(1) We isotope ξ_0, ξ_2 inside $N'(\Sigma)$ such that after the isotopy the annuli $A_\pm$ (and the annuli parallel to $A_\pm$ fixed above) contain a closed leaf in their interior. For this we show that the contact structure on $\widehat{N}'(\Sigma) \setminus N'(\Sigma)$ determines the contact structure on $\widehat{N}'(\Sigma)$ up to isotopy. The resulting contact structures are still transverse to $\mathcal{I}$ and they are still denoted by ξ_0, ξ_2. (This step is not needed if Σ has attractive holonomy.)

(2) We construct a homotopy ξ_s, $s \in [0,1]$, of contact structures on M such that $\xi_1 = \xi_2$ outside of $\Sigma \times [-2\delta, 2\delta]$ and such that the contact structures ξ_1, ξ_2 are tight on $\Sigma \times [\delta^-_{-(g+1)}, \delta^+_{-(g+1)}]$. In this step we use the restrictions on ε related to polyhedra and ribbons.

(3) Using the constraints on ε coming from the holonomy of $\mathcal{F}$ near Σ, we show that the restrictions of ξ_1 and ξ_2 to $\widehat{N}(\Sigma)$ are isotopic relative to the boundary. For this we show that the contact structures are determined up to isotopy by their restriction to the smaller space $\widehat{N}(\Sigma) \setminus N(\Sigma)$.

This is slightly more complicated than in the case of torus leaves. If there is a curve in Σ such that the holonomy of $\mathcal{F}$ along that curve is attractive, then one can omit the first step and proceed as in Section 6.

For steps (1) and (3) we first show the following proposition (formulated for the pair $N(\Sigma) \subset \widehat{N}(\Sigma)$).

Proposition 7.2 *Let ξ be a contact structure ε–close to $\mathcal{F}$ on $\Sigma \times ([-2,2] \setminus (-\delta,\delta))$. Up to isotopy there is a unique tight contact structure on $\Sigma \times [-2,2]$ which coincides with ξ on $\Sigma \times ([-2,2] \setminus (-\delta,\delta))$.*

Since the pair $N'(\Sigma) \subset \widehat{N}'(\Sigma)$ has analogous properties, Proposition 7.2 also holds for this pair: a tight contact structure on $\widehat{N}'(\Sigma)$ which is ε–close to $\mathcal{F}$ is determined up to isotopy by its restriction to the collar $\widehat{N}'(\Sigma) \setminus N'(\Sigma)$.

We postpone the proof of Proposition 7.2 and explain first how it implies Theorem 1.4. In order to finish step (1) it suffices to extend the contact structure on $\widehat{N}'(\Sigma) \setminus N'(\Sigma)$ in such a way that $A_{\pm} \cap N'(\Sigma)$ contain closed leaves. This can be done as in Example 3.24. One can also use [37, Proposition 6.2].

Then step (2) works as in the case of torus leaves or the case of Sacksteder curves. Finally, Proposition 7.2 finishes the proof of Theorem 1.4 for confoliations which are not foliations without holonomy.

Let us summarize the main differences between the case of torus leaves and leaves of higher genus before we prove Proposition 7.2:

(1) The relative Euler class essentially determines the isotopy type of tight contact structures on $\Sigma \times [-1,1]$ if Σ is not a torus.

(2) If $\Sigma = T$ is a torus, then we cannot change the contact structure on $N'(T) \subset \widehat{N}'(T)$ in order to ensure the existence of closed leaves of the characteristic foliation on annuli transverse to T.

(3) The last problem occurs for contact structures which have sheets connecting the two boundary components of $\widehat{N}(T)$. If T does not have attractive holonomy we cannot prevent this by reducing ε since there is no analogue of Lemma 7.1 for tori.

In order to prove Proposition 7.2 we want to apply the classification of tight contact structures outlined in Section 2D2. We need to arrange that the dividing sets on Σ_{-1} and Σ_{+1} have exactly two connected components which are nonseparating. This is done in Section 7C1. In Section 7C2 we determine the relative Euler class and if the relative Euler class vanishes we determine which basic slice embeds into $\Sigma \times [-2,2]$ as stated in Proposition 2.41.

7C1 Correcting the boundary of the neighbourhood of the closed leaf In this section we explain how to find a domain in $\widehat{N}(\Sigma)$ which contains $N(\Sigma)$ and satisfies the assumptions of the classification theorem of Honda, Kazez and Matić (Theorem 2.39).

Lemma 7.3 *The constant $\varepsilon > 0$ and the levels $\delta_i^{\pm}$, $i = -(g+1), \dots, g+1$, have the following property: Let ξ be a contact structure ε–close to $\mathcal{F}$, $i = -g, \dots, g$, and $\beta \subset \Sigma\{\delta_i^{\pm}\}$ a closed attractive leaf of the characteristic foliation. Then the sheet $A(\beta)$ does not meet $\Sigma \times \{\delta_{i+1}^{\pm}\}$ or $\Sigma \times \{\delta_{i-1}^{\pm}\}$.*

Proof We consider the case $\beta \subset \Sigma_{-1}$. As shown in Section 3A4 there is an embedded annulus $A(\beta)$ in $\Sigma \times [-1,1]$ such that $\beta \subset \partial A(\beta)$ and $A(\beta)$ is foliated by closed Legendrian curves parallel to β which are contained in one of the surfaces Σ_t, $t > -1$. The sheet $A(\beta)$ has the following properties:

(i) $A(\beta)$ is transverse to both $\mathcal{I}$ and ξ.

(ii) When a connected component $A(\beta) \cap \Sigma_t$ is an attractive closed leaf, then at each point p of that leaf ξ is steeper than the tangent space of $A(\beta)$ at that point (see (3-5) in Section 3A4).

(iii) The projection of the closed Legendrian curves foliating $A(\beta)$ to Σ along $\mathcal{I}$ provide an isotopy of simple closed curves $\beta_\tau \subset \Sigma$ starting with the curve $\beta = \beta_0$. We lift this isotopy of curves to an isotopy $\widetilde{\beta}_\tau$ with fixed points on the ideal boundary of the universal covering $\mathbb{H}^2 \to \Sigma$.

By Lemma 7.1, $\widetilde{\beta}_\tau$ contains a point which is connected to a point in $\widetilde{\beta}_0$ by a geodesic $\widetilde{\gamma}_\tau$ whose length is less than the constant K from Lemma 7.1. Consider the characteristic foliation of the lifted contact structure $\widetilde{\xi}$ on $\widetilde{\Sigma} \times [-1,1]$ on the surface $\widetilde{\Gamma} = \widetilde{\gamma}_\tau \times [-1,1]$. Let $\widetilde{\xi}, \widetilde{\mathcal{F}}$ denote the lifts of $\xi, \mathcal{F}$ to the universal covering $\mathbb{H}^2 \times [-2,2]$ of $\Sigma \times [-2,2]$.

If $A(\beta)$ meets $\Sigma \times \{\delta_1^-\}$, then by the discussion in Section 3A4 (in particular (3-5)) the above properties of $A(\beta)$ imply that the leaf of the characteristic foliation on $\widetilde{\gamma}_\tau \times [-2,2]$ lies *above* the intersection of $A(\beta)$ with $\widetilde{\gamma}_\tau \times [-2,2]$. Therefore, if $A(\beta)$ meets $\Sigma \times \{\delta_1^-\}$ then so does the leaf of $\widetilde{\Gamma}(\widetilde{\xi})$ which starts at β_0. But this is excluded by the choice of ε. This proves the lemma. □

Lemma 7.4 *There is a convex surface Σ' in $\Sigma \times [\delta_{-g+1}^-, \delta_{g-1}^-]$ transverse to $\mathcal{I}$ such that the dividing set on Σ' has no separating component.*

Proof After a C^∞–small perturbation of ξ we may assume that Σ_{-1} is convex. Assume that the dividing set on Σ_{-1} has separating components. Then there is a separating attractive closed leaf of Σ_{-1}. Let $A(\beta)$ be the sheet of the movie $(\Sigma \times [-1,1], \xi)$

whose boundary contains β. Then $A(\beta)$ is an annulus and we put

$$t_{\max}(\beta) = \max\{t \in [-1,1] \mid A(\beta) \cap \Sigma_t \neq \varnothing\}.$$

This is the highest level the sheet $A(\beta)$ reaches. Since the movie has no negative singularities $A(\beta) \cap \Sigma_{t_{\max}(\beta)}$ is a degenerate closed curve $\widehat{\beta}$.

We assume that $A(\beta)$ has the property that $t_{\max} := t_{\max}(A(\beta))$ is maximal among the levels $t_{\max}(\widetilde{\beta})$ for the finite number of separating attractive closed leaves $\widetilde{\beta}$ of $\Sigma_{-1}(\xi)$ isotopic to β. We can also arrange that the sheet $A(\beta)$ is as simple as possible, ie there is no degenerate closed orbit on $A(\beta)$ between β and $\widehat{\beta}$. By Lemma 7.3,

$$t_{\max}(A(\beta)) < \delta_1^-.$$

We now replace Σ_{-1} by a smooth surface Σ' transverse to $\mathcal{I}$ and close to but never below the surface consisting of

(i) the connected component of $\Sigma_{-1} \setminus \beta$ on the side of β determined by the coorientation of β inside Σ_{-1},

(ii) the part of the sheet $A(\beta)$ which lies between β and $\widehat{\beta}$, and

(iii) the connected component $\widehat{\Sigma}$ of $\Sigma_{t_{\max}} \setminus \widehat{\beta}$ on the repelling side of $\widehat{\beta}$.

If $\widehat{\Sigma}$ contains an attractive closed curve isotopic to $\widehat{\beta}$, then we pick the curve closest to $\widehat{\beta}$ and denote it by β'. There are two possibilities: either β is parallel to β' or not.

In the first case, the sheet $A(\beta')$ is part of a sheet containing $A(\beta)$. This contradicts the definition of $t_{\max}(\beta)$. Hence β' is anti-parallel to $\widehat{\beta}$. Notice that the sheet $A(\beta')$ does not intersect Σ_{-1} since this would contradict the maximality of $t_{\max}(\beta)$. We proceed by replacing Σ' by a surface Σ'' consisting of

(i) the part of Σ' lying on the side of β' which contains $\widehat{\beta}$,

(ii) the part of the sheet of $A(\beta')$ below Σ' and Σ_{-1} and the degenerate closed curve $\widehat{\beta}'$ which is contained in $\Sigma \times \{t_{\min}(\beta')\}$ (the definition of $t_{\min}(\beta') \in (-1, t_{\max})$ should be obvious by now), and

(iii) the part of $\Sigma_{t_{\min}(\beta')}$ on the side of $\widehat{\beta}'$ which is determined by the coorientation of $\widehat{\beta}'$ in $\Sigma_{t_{\min}(\beta')}$.

Now we repeat this process. The next attractive closed curve we encounter in the process is part of a sheet containing $A(\beta)$. Therefore we cannot pass the level $\Sigma_{t_{\max}}$ in the next step and the same applies to all steps that follow it. This procedure terminates since otherwise we would find a degenerate closed curve isotopic to β which is the limit of degenerate closed curves isotopic to β. The resulting surface will be called Σ'.

By construction, Σ' has no attractive closed curves isotopic to β which lie in the part of $\Sigma' \setminus \widehat{\beta}$ not containing β.

The remaining attractive closed curves parallel to β are easy to eliminate since the sheets such curves can be completed to are sheets which are properly embedded, ie both of whose boundary components are contained in Σ_{-1}. The attractive closed curves which are anti-parallel to β are dealt with in the same way explained above. This step does not interfere with what we have achieved already since it all happens in the half of Σ' which is contained in Σ_{-1}.

We have removed all separating attractive closed curves which are parallel or anti-parallel to β and we want to eliminate the remaining separating attractive closed curves.

We proceed with the separating curve $\widetilde{\beta}$ which, together with β, bounds a subsurface containing no other attractive separating leaf. Now $\widetilde{\beta}$ can be treated essentially in the same way as β, except that we leave the part of the surface on the side of $\widetilde{\beta}$ which contains β as it is. Depending on whether the other side of $\widetilde{\beta}$ coincides with the side determined by the coorientation of ξ or not, the surface is shifted towards higher or lower levels.

Figure 19 shows this schematically. The isotoped surface is thickened. Only parts of sheets where the corresponding curve in Σ_t is attractive and neighbourhoods of degenerate closed curves are shown.

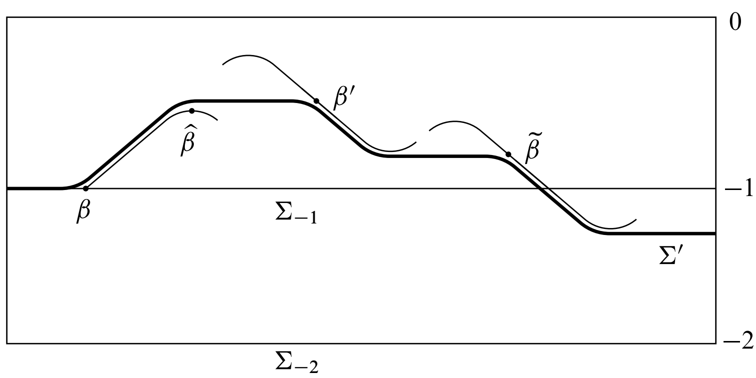

Figure 19: Elimination of separating curves

Since a pairwise disjoint collection of separating homotopically essential simple closed curves contains at most $g-1$ (where g is the genus of Σ) isotopy classes of nonoriented curves, this process stops after a finite number of steps and we have found the desired surface. By our choice of ε the resulting surface is contained in $\Sigma \times [\delta^-_{-g+1}, \delta^-_{g-1}]$. $\square$

After the elimination of all separating attractive curves we obtain a convex surface Σ' in $\Sigma\times[\delta^-_{-g+1},\delta^-_{g-1}]$ such that no dividing curve is separating. The procedure from the proof of Lemma 7.4 allows us prove that the contact structures are tight.

Proposition 7.5 *The restrictions of ξ_1 and ξ_2 to $\Sigma\times[\delta^-_{-g},\delta^+_{-g}]$ are tight.*

Proof By construction, ξ_1,ξ_2 are transverse to $\mathcal{I}$ on $\Sigma\times[\delta^-_{-g},\delta^+_{-g}]$. After a C^∞–small perturbation we may assume that $\Sigma\times\{\delta^\pm_{-g}\}$ is convex. We want to apply Lemma 3.25. For this we have to show that no connected component of the dividing set bounds a disc. Assume that there is a component of the dividing set which bounds a disc. By Remark 3.26 there is a closed attractive orbit β bounding a disc. Let $A(\beta)$ be the corresponding sheet.

After going through the procedure in the proof of Lemma 7.4 we find a disc on a surface Σ_t in $\Sigma\times[-2,2]$ such that this surface contains no closed orbits although the characteristic foliations point inwards along the boundary. But this requires the presence of a negative singularity. However, if ξ is ε–close to $\mathcal{F}$, then there are no such singularities. Hence Lemma 3.25 proves the claim. □

We are now in a position to eliminate all but one pair of dividing curves from the surface Σ' obtained in Lemma 7.4.

Lemma 7.6 *Let $\Sigma\subset\Sigma\times[\delta^-_{-g+1},\delta^-_{g-1}]$ be a convex surface transverse to $\mathcal{I}$ such that all dividing curves are nonseparating. Then Σ is isotopic to a convex surface Σ' in $\Sigma\times[\delta^-_{-g},\delta^-_{g}]$ which is transverse to $\mathcal{I}$ and whose dividing set consists of two nonseparating closed curves bounding an annulus.*

Proof We assume that there are at least four dividing curves, all of which are nonseparating. Fix an attractive closed leaf β in $\Sigma(\xi)$ such that $t_{\max}(\beta)$ is minimal among the finitely many attractive closed leaves in Σ_{-1}. Since β is nonseparating and there is another attractive closed orbit, we can use the theory of convex surfaces to change the characteristic foliation on Σ so that

- there is a repulsive closed leaf β' parallel to β on the side of β opposite to the side determined by the coorientation of ξ, and
- all leaves of the characteristic foliation which do not lie in the annulus bounded by β and β' accumulate on an attractive closed curve different from β.

This can be done without changing any of the sheets containing closed attractive leaves of $\Sigma(\xi)$.

By our assumptions on ε and the choice of neighbourhoods of the closed leaf Σ, the sheet $A(\beta)$ containing β does not enter $\Sigma\times[-\delta,\delta]$. We choose an identification of the region bounded by Σ and $\Sigma_{-\delta}$ with $\Sigma\times[-1,-\delta]$ such that

- Σ corresponds to Σ_{-1},
- the foliation corresponding to the second factor is tangent to $\mathcal{I}$, and
- the Legendrian foliation on the sheets containing closed attractive curves of $\Sigma(\xi)$ is tangent to the level surfaces of the product decomposition $\Sigma\times[-1,-\delta]$.

Now we apply the pre-Lagrangian extension lemma (Lemma 3.29) to β relative to the sheets containing other closed leaves of $\Sigma_{-1}(\xi)$. We obtain a properly embedded sheet $A'(\beta)$ of a contact structure ξ' isotopic to ξ so that $A'(\beta)$ connects β to β'. We now replace Σ_{-1} by a surface Σ' close to the union of

(i) the sheet $A'(\beta)$ and
(ii) Σ_{-1} with the annulus bounded by β' and β removed.

If Σ' is sufficiently close to this union and lies above it, then no new closed curves or negative singular points have appeared on Σ' and we have eliminated one pair of dividing curves. This process can be iterated as long as there are at least two pairs of parallel dividing curves. $\square$

7C2 Identification of the contact structure By the lemmas from the previous section, we find a domain $N(\Sigma_0)$ diffeomorphic to $\Sigma\times[-1,1]$ inside M such that the boundary has the following properties:

- It is convex and contained inside $\Sigma\times([-2,2]\setminus[-\delta,\delta])$.
- One boundary component lies above Σ_0 while the other boundary component lies below Σ_0.
- The dividing set of the characteristic foliation of ξ consists of two nonseparating closed curves on each boundary component.

Since the contact structures ξ_1,ξ_2 are tight on $N(\Sigma_0)$ we can apply Theorem 2.39 to determine the isotopy class of ξ. By our assumptions on ε, both contact structures have the following property: the sheet containing an attractive closed leaf of the characteristic foliation on a boundary component of $N(\Sigma_0)$ does not enter $\Sigma\times[-\delta,\delta]$.

From now on we identify $N(\Sigma_0)$ with $\Sigma\times[-1,1]$ in such a way that sheets containing attractive closed leaves of the characteristic foliation on the boundary are preserved and nothing changes on $\Sigma\times[-\delta,\delta]$.

First, we determine the relative Euler class of a contact structure ξ on $\Sigma\times[-1,1]$.

For this we apply the pre-Lagrangian extension lemma (Lemma 3.29) to $A(\beta_{\pm1})$, where $\beta_\pm$ is the unique closed attractive leaf of $\Sigma_{\pm1}(\xi)$. We obtain a boundary elementary contact structure such that the boundary component β'_{-1} of $A(\beta_{-1})$ which is different from β_{-1} lies on the side of β_{-1} opposite to the coorientation of β_{-1} determined by the coorientation of ξ. For $A(\beta_{+1})$ the situation is opposite.

The map φ in the following lemma is an automorphism of the surface isotopic to a left-handed Dehn twist along β_{-1}.

Recall that an extremal contact structure ξ on $\Sigma \times [-1,1]$ is boundary elementary if all attractive closed of the characteristic foliation on the boundary are part of properly embedded sheets (see Definition 3.27).

Lemma 7.7 *Let ξ be a boundary elementary tight contact structure on $\Sigma \times [-1,1]$ with the properties from the previous paragraphs. Let $\alpha \subset \Sigma$ be a simple closed oriented curve with $\alpha \cdot \beta_{-1} = 1$. Then for sufficiently large $k > 0$ there is an annulus $S(\alpha_k)$ with primped Legendrian boundary isotopic to $\alpha_k := \varphi^k(\alpha)$ connecting Σ_{-1} to Σ_{+1} such that there is a Legendrian curve $\widehat{\alpha}_k$ in the interior of $S(\alpha_k)$ containing no singular point of $S(\alpha_k)(\xi)$.*

Proof We start with Σ_{-1} and arrange that $\partial A(\beta_{-1}) \subset \Sigma_{-1}$ consists of two circles of singularities (negative along β_{-1} and positive at the other end of $A(\beta_{-1})$). Using Lemma 3.3, we can moreover arrange that α is a Legendrian curve in Σ_{-1} which intersects the annulus bounded by $\partial A(\beta_{-1})$ in a single arc so that α is primped (see Section 2D2) and the only negative singularity on α is $\alpha \cap \beta_{-1}$.

Now consider the smooth surface Σ'' obtained from Σ_{-1} after replacing the annulus bounded by $\partial A(\beta_{-1})$ in Σ_{-1} by $A(\beta)$ and then smoothing out the nonsmooth points. Using a vector field X transverse to the surface, we push Σ'' into the interior of $\Sigma \times [-1,1]$. We may assume that X is a contact vector field outside of a small neighbourhood of $\partial A(\beta_{-1})$. Note that the part of Σ'' contained in $A(\beta_{-1})$ is foliated by closed leaves of the characteristic foliation, so Σ'' is certainly not convex. The closed Legendrian curve close to β is attractive while the closed Legendrian curve at the opposite end of $A(\beta_{-1})$ is repelling.

By Lemma 3.7 the collection of closed Legendrian curves forming on Σ'' curves disappears as we push this surface into $N(\Sigma_0)$ using the flow of X and leaves of the characteristic foliation close to β_{-1} get connected to leaves of the characteristic foliation on the opposite side of $A(\beta_{-1})$. If the flow runs for an appropriate time, the characteristic foliation on the pushed-off surface connects the two arcs of α which lie in the part of the surface further away from $A(\beta)$. (The sequence of instances where this happens has 0 as an accumulation point.)

For an appropriate push-off we obtain a surface Σ' containing a closed curve isotopic to a curve obtained from α by applying a sufficiently high power of a left-handed Dehn twist φ along β. The annulus $S(\alpha_k)$ is now obtained by isotoping $\Sigma_{\pm 1}$ such that there is a Legendrian curve $\widehat{\alpha}_k$ isotopic to $\varphi^k(\alpha)$ which intersects β_{-1} exactly once.

The annulus $S(\alpha_k)$ is then obtained by picking an annulus bounding the Legendrian curves isotopic to α_k in $\Sigma_{\pm 1}$ and containing $\widehat{\alpha}_k$. Since the twisting of ξ along $\widehat{\alpha}_k$ is zero with respect to the framing determined by Σ', this twisting vanishes also with respect to the framing determined by $S(\alpha_k)$. Therefore we can eliminate the singular points of $S(\alpha_k)(\xi)$ which lie on $\widehat{\alpha}_k$. □

We now decompose $\Sigma \times [-1, 1]$ as follows. Start with Σ' and modify this surface using the pre-Lagrangian extension lemma in a similar way as in the proof of Lemma 7.6 to reduce the number of dividing curves to 2 without introducing any new ones. For this recall the following facts:

- All dividing curves intersect β_{-1} at least once and always with the same sign, and thus there are no null-homologous dividing curves in Σ'.
- From the way we obtained Σ' it follows that we may assume that the domain between Σ' and Σ_{-1} does not contain negative singularities except those along $\beta_{-1} \subset \Sigma_{-1}$ (these singularities lie on the Legendrian curve $\partial S(\alpha_k)$.)

The resulting surface is called $\widehat{\Sigma}$, and the dividing set on this surface is such that α_k is isotopic to a curve disjoint from the dividing set (which consists of two parallel copies of the curve γ).

Hence if $\widehat{S}$ is an annulus in the domain bounded by $\widehat{\Sigma}$ and Σ_{-1}, whose boundary is isotopic to α_k, then

$$\langle \widetilde{e}(\xi), \widehat{S} \rangle = -1 = (\pm\gamma + \beta_{-1}) \cdot \alpha_k = -\alpha \cdot \beta_{-1} \tag{7-6}$$

if we use the coorientation of β_{-1} in order to orient $\widehat{S}$. In other words, $\widehat{S}$ is oriented so that $-\alpha_k$ is part of the oriented boundary of $\widehat{S}$. This means that we have determined the sign in front of β_{-1} in the expression of the Poincaré dual of $\widetilde{e}(\xi)$ in (2-5). The coefficient in front of β_{+1} is determined in the same way, looking at the other boundary component Σ_{+1}. Since the pre-Lagrangian annulus $A(\beta_{+1})$ now lies on the side of β_{+1} determined by the coorientation of ξ, we get a minus sign. Hence

$$\mathrm{PD}(\widetilde{e}(\xi)) = -\beta_{+1} + \beta_{-1}. \tag{7-7}$$

Thus if β_{-1} and β_{+1} are not homologous, then we have identified the contact structure up to isotopy because by the classification theorem Theorem 2.39 it is determined by the relative Euler class.

If β_{-1} and β_{+1} are homologous, then (7-7) implies $\widetilde{e}(\xi) = 0$ and we have to study which basic slice admits a contact embedding into $\Sigma \times [-1, 1]$ such that one boundary component gets mapped to Σ_{-1} in an orientation-preserving fashion. If $\widehat{\alpha}_k$ is attractive, then we have already found a basic slice. If α_k is repulsive, then after folding we obtain a surface with four dividing curves. After removing the dividing curves which do not come from this folding procedure, we end up again with a basic slice

$$[\![\beta_{-1}, \beta'; -\beta' + \beta_{-1}]\!]$$

with β' isotopic to $\widehat{\alpha}_k$. (The folding procedure provides us with a pre-Lagrangian annulus below Σ' which lies on the side of α_k determined by the coorientation of ξ.)

Therefore the contact structure on $\Sigma \times [-1, 1]$ is completely determined by the contact structure on the complement of $\Sigma \times [-\delta, \delta]$. This completes the proof of Proposition 7.2 and with it the proof of Theorem 1.4 for all confoliations except foliations without holonomy.

8 Foliations without holonomy

The proof of Theorem 1.4 is almost finished. What is left open is the case of foliations without holonomy, which will be discussed in this section.

We will use of the structure theory of foliations without holonomy which was developed in particular by R Sacksteder and S Novikov. The following theorem can be found in [4] (recall that we assume that M is closed).

Theorem 8.1 *Let $\mathcal{F}$ be a C^2–foliation without holonomy on M. Either every leaf of $\mathcal{F}$ is dense or the leaves of $\mathcal{F}$ are the fibres of a fibration $M \to S^1$.*

In particular, a foliation without holonomy is automatically taut since noncompact leaves in closed manifolds always have a closed transversal. In particular, $\mathcal{F}$ has a neighbourhood in which every contact structure is tight (see Theorem 2.31).

There are some differences between the proof presented in this section and the proofs in Sections 4–7. For example, we do not need polyhedral decompositions or ribbons at this point. On the other hand, more care is needed when trying to apply classification theorems for tight contact structures.

As usual, we fix a foliation $\mathcal{I}$ transverse to $\mathcal{F}$. We shall deal first with the case that every leaf is closed. We also determine precisely which torus fibrations satisfy the conclusion of Theorem 1.6.

Then we finally prove Theorem 1.4 for foliations without holonomy all of whose leaves are dense. As before, we will use the classification of tight contact structures on M

after it has been cut open along a surface Σ. The problem is the choice of the surface we cut along.

It is known that foliations without holonomy can be C^0–approximated by fibrations, and we now focus on the case when the genus of the fibre is positive: in that case we fix a fibration $\varphi\colon M \to S^1$ whose fibres are close to $\mathcal{F}$. In the following we consider only fibrations whose fibre is an abelian covering of a fibre of φ.

Given a fibre Σ_0 of a fibration $M \to S^1$ and a positive contact structure close to $\mathcal{F}$, we want to isotope a fibre so that we can apply the classification Theorem 2.39. Whether these manipulations described in Section 7C1 are possible depends not only the constant K appearing in Lemma 7.1 (which used the geometry of hyperbolic surfaces) but also on the distance between intersection points of Σ with leaves of $\mathcal{I}$ (the distance is measured along leaves of $\mathcal{I}$). For a careful choice of the fibration a fibre Σ can then be manipulated as in Section 7C1.

8A Every leaf of $\mathcal{F}$ is a torus

Let pr: $M \to S^1$ be an orientable torus fibration over the circle. The diffeomorphism type of M is determined by the action of the monodromy of the fibration on the homology of a fibre

$$\phi_M\colon H_1(T^2,\mathbb{Z}) \to H_1(T^2,\mathbb{Z}).$$

Thus we may assume that $M = T^2 \times \mathbb{R}/\cong$, with $(x,t) \cong (Ax, t+1)$ for $A \in \mathrm{Sl}(2,\mathbb{Z})$, since M is orientable.

There are infinitely many isotopy classes of positive contact structures on M. On the universal cover $\mathbb{R}^2 \times \mathbb{R}$ of M consider the 1–form

$$\cos\varphi(t)\, dx_1 - \sin\varphi(t)\, dx_2,$$

where φ is a strictly increasing function. According to [23], for each integer $n \geq 0$ one can chose φ such that the corresponding contact structure on $\mathbb{R}^3$ is invariant under the action of $\pi_1(M)$ and

$$2n\pi < \sup_{t\in\mathbb{R}}(\varphi(t+1) - \varphi(t)) \leq 2(n+1)\pi.$$

The resulting contact structures do not depend on the particular choice of φ, but the induced contact structures ξ_n on M are not isotopic for different integers n.

The universally tight contact structures on M have been classified by Honda [35] and Giroux [24; 23]. For our purposes the following statement of their results is sufficient:

Theorem 8.2 *If $|\mathrm{tr}(\phi_M)| \neq 2$, then all positive universally tight contact structures on M are isotopic to one of the contact structures ξ_n defined above.*

This implies that the conclusion of Theorem 1.6 also holds for torus bundles over S^1 whose monodromy is elliptic or hyperbolic. If the monodromy $A \in \mathrm{Sl}(2,\mathbb{Z})$ satisfies $\mathrm{tr}(A) = 2$, then this is not the case, as will be shown in Example 9.7.

Theorem 8.3 *Let $\mathcal{F}$ be the foliation defined by a torus bundle* pr: $M \to S^1$*. Then there is a C^0–neighbourhood of $\mathcal{F}$ in the space of plane fields such that all positive contact structures in that neighbourhood are stably isotopic if and only if $+1$ is not an eigenvalue of A.*

Proof If the monodromy is elliptic or hyperbolic, then $+1$ is not an eigenvalue of A and Theorem 8.2 proves the claim. The case $\mathrm{tr}(A) = +2$ will be treated in Example 9.7 below. Thus we are left with the case when $\mathrm{tr}(A) = -2$, in which the classification of universally tight contact structures is slightly more complicated and we have to prove our claim more directly.

Let U be the neighbourhood of $\mathcal{F}$ determined by the requirement that all plane fields on U are transverse to the line field ∂_t on M. Since such contact structures lift to complete connections of a fibre bundle equivalent to $\mathbb{R}^3 \to \mathbb{R}^2$, all contact structures in U are universally tight by Theorem 2.27.

Consider a positive contact structure ξ in U and consider the characteristic foliations on the fibres of M. After a C^∞–small perturbation of ξ the fibre $T_0 = \mathrm{pr}^{-1}(0)$ is convex. Let β be an attractive closed leaf of $T_0(\xi)$ and $A(\beta)$ the sheet containing β.

Because -1 is the only eigenvalue of A the characteristic foliation $T_0(\xi)$ has another closed attractive leaf β' which is isotopic to β after its orientation is reversed.

If $A(\beta)$ connects the two boundary components of $\overline{M \setminus T_0}$ we consider the cyclic covering $\widehat{M}$ of M given by:

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{\mathrm{pr}}} & \mathbb{R} \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{\mathrm{pr}} & S^1 \end{array}$$

Let $\widehat{T} = \widehat{\mathrm{pr}}^{-1}(0)$ and consider the maximal sheets $\widehat{A}(\beta), \widehat{A}(\beta')$ containing lifts of $\beta, \beta' \subset \widehat{T}$. Let

$$\tau = \sup\{\hat{t} \in \mathbb{R} \mid \widehat{A}(\beta) \cap \widehat{\mathrm{pr}}^{-1}(\hat{t}) \neq \varnothing\},$$
$$\tau' = \sup\{\hat{t} \in \mathbb{R} \mid \widehat{A}(\beta') \cap \widehat{\mathrm{pr}}^{-1}(\hat{t}) \neq \varnothing\}.$$

One of these numbers is finite because otherwise the sheets $\widehat{A}(\beta)$ and $\widehat{A}(\beta')$ would intersect. This is impossible unless they actually coincide. But ξ is transverse to $\mathcal{I}$ and, as explained in the discussion of the behaviour of sheets in Remark 3.15, the sheets $\widehat{A}(\beta), \widehat{A}(\beta')$ cannot coincide. Thus we can reduce the number of dividing curves on T by isotoping the sheet $\widehat{A}(\beta')$. After finitely many such isotopies (as in Section 7C1) we find an embedded torus $T' \subset M$ isotopic to the fibre such that $T'(\xi)$ has no singularities and no closed orbits.

The classification of universally tight contact structures on $T^2\times[0,1]$ (see Theorem 2.37) such that the characteristic foliation on the boundary is of the same type as $T'(\xi)$ implies the claim. □

8B Every leaf of $\mathcal{F}$ is compact and has genus $g \geq 2$

Fix a hyperbolic metric on a fibre Σ_0 and let $K > 0$ be the constant from Lemma 7.1. We identify a foliated tubular neighbourhood of Σ_0 with $\Sigma \times (-\delta, \delta)$ and a foliation $\mathcal{I}$ transverse to the leaves of $\mathcal{F}$ such that $\mathcal{I}$ is tangent to the fibres of $\Sigma \times (-\delta, \delta) \to \Sigma$.

We assume that $\varepsilon > 0$ satisfies the following condition:

For every path γ of length at most $K+1$ and $i = -g+1, \dots, g-1$ and every smooth plane field ζ which is ε-C^0–close to $\mathcal{F}$, the ζ–horizontal lift of γ with starting point in $\Sigma_{i\delta/g}$ does not meet $\Sigma_{(i-1)\delta/g}$ or $\Sigma_{(i+1)\delta/g}$. Moreover, we require that ε is so small that every contact structure ε–close to $\mathcal{F}$ is tight (such an ε exists because $\mathcal{F}$ is taut).

Now we show that all positive contact structures which are ε–close to $\mathcal{F}$ are isotopic.

The first three steps are the same as those used in the proof of Proposition 7.2. We do not go through the details again but here are the steps: Let ξ_0 and ξ_2 be two contact structures ε–close to $\mathcal{F}$.

(1) After a C^∞–small perturbation of ξ_0 and ξ_2 we may assume that Σ_0 is convex for both contact structures.

(2) By Lemma 7.4 we can isotope the surface Σ_0 inside
$$\Sigma \times (-(g-1)\delta/g, (g-1)\delta/g)$$
so that the resulting surfaces $\widehat{\Sigma}^{(0)}$ and $\widehat{\Sigma}^{(2)}$ are convex with respect to ξ_0 and ξ_2, respectively, and whose dividing set has no separating component.

(3) After sufficiently many applications of Lemma 7.6 we end up with surfaces $\Sigma^{(0)}, \Sigma^{(2)} \subset (-\delta, \delta)$ whose dividing set consists of exactly two nonseparating closed curves which bound a single annulus.

When we cut M along $\Sigma^{(0)}$ we obtain $\overline{M \setminus \Sigma^{(0)}}$. This a manifold which is diffeomorphic to $\Sigma \times [0,1]$ and the boundary satisfies the assumptions of the classification theorem of Honda, Kazez and Matić (Theorem 2.39). The contact structure on $\Sigma \times [0,1]$ is determined by the orientations of the attractive closed leaves of $\Sigma^{(0)}(\xi)$ (see Lemma 7.7 and the discussion following it).

To see this, let γ_0 denote the attractive closed leaf of $\Sigma^{(0)}(\xi_0)$. The relative Euler class of ξ_0 on the cut open manifold is Poincaré-dual to $\phi_*(\gamma_0) - \gamma_0$, where $\phi\colon \Sigma^{(0)} \to \Sigma^{(0)}$ denotes the monodromy of the fibration. If the relative Euler class vanishes, then the contact structure is still determined by embedding properties of basic slices, since there are no sheets which pass from one boundary component of $\overline{M \setminus \Sigma^{(0)}}$ to the other.

The analogous statement is true for $\overline{M \setminus \Sigma^{(2)}}$. The remaining problem is that the dividing sets of $\Sigma^{(0)}(\xi_0)$ and $\Sigma^{(2)}(\xi_2)$ are not isotopic in general.

As in the first step of the proof of Theorem 1.4 in Section 7C, we can find a tight contact structure ξ_1 on $\overline{M \setminus \Sigma^{(0)}}$ with the following properties:

- ξ_1 is transverse to $\mathcal{I}$ and hence tight.
- ξ_1 coincides with ξ_0 near the boundary $\overline{M \setminus \Sigma^{(0)}}$.
- Recall that $\Sigma^{(0)}$ and $\Sigma^{(2)}$ are isotopic. We use this isotopy to compare dividing sets on these surfaces. The surface

$$\Sigma_1 = \Sigma^{(0)} \times \left\{\tfrac{1}{2}\right\} \subset \Sigma^{(0)} \times [0,1] \cong \overline{M \setminus \Sigma^{(0)}} \tag{8-1}$$

  is convex with respect to ξ_1 and the dividing set on Σ_1 coincides with the dividing set of ξ_2 on $\Sigma^{(2)}$.

The construction of ξ_1 is similar to the one described in Example 3.24, where we considered two disjoint nonseparating curves. Here we also use that the part of the curve complex of Σ whose vertices are nonseparating is connected (this is [37, Proposition 3.3]) in order to find a path in the curve complex (containing only nonseparating curves) connecting the dividing set on $\Sigma^{(0)}$ with the dividing set of ξ_1 on Σ_1 through nonseparating curves.

Now ξ_1 is isotopic to ξ_0 on $\overline{M \setminus \Sigma^{(0)}}$ relative to the boundary. Thus we can replace ξ_0 by ξ_1 on M. Now, inside of M, the surface Σ_1 is isotopic to $\Sigma^{(2)}$ by shifting Σ_1 along leaves of $\mathcal{I}$. The classification of tight contact structures now implies that ξ_1 and ξ_2 are isotopic.

This concludes the proof of the uniqueness theorem for foliations formed by fibres of a fibration over S^1 such that the genus of the fibres is ≥ 2.

8C Every leaf of $\mathcal{F}$ is dense

First, we review standard facts about foliations without holonomy and explain how a general foliation without holonomy can be understood in terms of a foliated fibre bundle on a manifold of higher dimension. This uses results of Novikov [47] and our presentation follows [44] closely.

This will be used to approximate a foliation without holonomy by the fibres of a fibration over S^1. The main point will be to find an approximation of $\mathcal{F}$ by fibrations which are sufficiently close to $\mathcal{F}$ so that a chosen fibre Σ can be manipulated as in the proofs of Lemma 7.4 and Lemma 7.6. For this one has to exclude the possibility that a sheet intersecting Σ_0 in an attractive closed leaf of the characteristic foliation is part of a pre-Lagrangian torus.

We fix a fibration whose fibres are transverse to a chosen line field $\mathcal{I}$ which is also transverse to $\mathcal{F}$. (It may be helpful to think of $\mathcal{I}$ as defined by a vector field which projects to a fixed nowhere-vanishing vector field on the circle.) We then replace the original fibration by another one such that the new fibres are coverings of the old fibration and the angle between the fibres and $\mathcal{F}$ decreases faster than the distance between intersection points of the new fibres with leaves of $\mathcal{I}$ (measured along the leaves of $\mathcal{I}$). In this process we will use a result concerning the approximation of irrational numbers by rational numbers.

We now start summarizing facts about foliations without holonomy. As always, the underlying manifold is closed and the foliation is at least C^2–smooth. We fix a simple closed curve C transverse to $\mathcal{F}$ (such a curve exists because there are noncompact leaves).

Theorem 8.4 (Sacksteder [52]) *If $\mathcal{F}$ is a C^2–foliation without holonomy, then there is a C^0–flow*

$$\psi\colon M \times \mathbb{R} \to M \tag{8-2}$$

which preserves the leaves of $\mathcal{F}$ and acts transitively on the set of leaves. The flow can be chosen tangent to any previously fixed foliation $\mathcal{L}$ of rank 1 transverse to $\mathcal{F}$. Given a closed curve C transverse to $\mathcal{F}$ we can choose ψ such that C becomes a flow line.

In order to prove this theorem, one establishes the existence of a holonomy-invariant transverse measure μ. The relationship between μ and ψ is $\mu(\psi(x,[0,s])) = s$. The transverse measure determines a group homomorphism

$$\varphi_\mu\colon \pi_1(M) \to \mathbb{R}, \quad [\gamma] \mapsto \int_{S^1} \gamma^*(d\mu).$$

The image $\mathcal{P}(\mu)$ of φ_μ is called the group of periods of $\mathcal{F}$. It consists of those times s where $\psi(s,\cdot)$ maps one (or equivalently every) leaf to itself. The leaves of $\mathcal{F}$ are fibres of a fibration if and only if $\mathcal{P}(\mu)$ is discrete.

Let $\widetilde{M} \to M$ be the universal covering and $\widetilde{\mathcal{F}}$ the induced foliation, L a leaf of $\mathcal{F}$ and $\widetilde{L} \to L$ the universal covering of L. Since M is taut, $\widetilde{L}$ is a leaf of $\widetilde{\mathcal{F}}$, and, lifting ψ to $\widetilde{M}$, we obtain a diffeomorphism

$$\widetilde{\psi}\colon \widetilde{L} \times \mathbb{R} \to \widetilde{M}.$$

We denote the projection onto the second factor of $\widetilde{L} \times \mathbb{R}$ by π. We get a representation

$$q\colon \pi_1(M) = \mathrm{Deck}(\widetilde{M}) \to \mathrm{Diff}^2_+(\mathbb{R}), \quad \alpha \mapsto \big(x \mapsto \pi((p_0, x)\cdot\alpha^{-1})\big). \tag{8-3}$$

Here p_0 is a base point in $\widetilde{L}$ and we abbreviate $\{p_0\} \times \mathbb{R} \subset \widetilde{L} \times \mathbb{R}$ by $\mathbb{R}$. Novikov proves the following facts about q:

(i) If $q(\alpha)$ has a fixed point, then $q(\alpha) = \mathrm{id}$.

(ii) The image G of q is an abelian group (it is obviously free and finitely generated).

The action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations and by q on $\mathbb{R}$ determines a foliated $\mathbb{R}$–bundle

$$E = \widetilde{M} \times \mathbb{R}/\pi_1(M). \tag{8-4}$$

We denote the induced foliation on E by $\mathcal{F}_E$. By the definition of q, the embedding

$$\widetilde{\sigma}\colon \widetilde{M} \to \widetilde{M} \times \mathbb{R}, \quad x \mapsto (x, \pi(x))$$

is $\pi_1(M)$–equivariant, the resulting embedding $\sigma\colon M \to E$ is transverse to $\mathcal{F}_E$ and $\mathcal{F} = \sigma^*\mathcal{F}_E$.

The element $q(C) = f_0$ is nontrivial and $f_0(x) > x$. Then according to [54] we may (smoothly) reparametrize $\mathbb{R}$ so that $f_0(x) = x + 1$. Because the other elements of G commute with f_0 we have that $f(x+1) = f(x) + 1$ for all $f \in G$. Therefore the elements of G have the same properties as lifts of orientation-preserving C^2–diffeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ (we call such diffeomorphisms 1–periodic).

The outline we have given is somewhat misleading since Novikov's result can be used to prove Theorem 8.4. This is explained in [4, Chapter 9], where the representation (8-3) is obtained without using Theorem 8.4 and the holonomy-invariant transverse measure is then obtained by an averaging procedure.

8C1 1–periodic diffeomorphisms of $\mathbb{R}$ Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a 1–periodic homeomorphism of $\mathbb{R}$. We denote the group consisting of such diffeomorphisms by $\mathrm{Diff}^1(\mathbb{R})$. It is well known (and explained in many places, eg [25]) that the sequence $(\phi^n - \mathrm{id})/n$ converges uniformly to a constant $\tau(\phi)$ as $n \to \infty$. This number is called the *translation number* of ϕ, and the fractional part of this number is the rotation number $\rho \in \mathbb{R}/\mathbb{Z}$ of φ when ϕ is a lift of $\varphi\colon S^1 \to S^1$ to the universal covering $\mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$. All the following statements are consequences of well-known properties of the rotation number:

- If $r_\alpha(x) = x + \alpha$ then $\tau(r_\alpha) = \alpha$.
- The rotation number depends continuously on ϕ with respect to the uniform topology.
- If ϕ_1, ϕ_2 commute, then $\tau(\phi_1 \circ \phi_2) = \tau(\phi_1) + \tau(\phi_2)$.

The following theorem is just a translation of a fundamental result in the theory of circle diffeomorphisms to the context of 1–periodic diffeomorphisms of $\mathbb{R}$.

Theorem 8.5 (Denjoy) *If the translation number τ of ϕ is irrational and ϕ is C^2–smooth, then there is a 1–periodic homeomorphism h of $\mathbb{R}$ such that $h \circ \varphi \circ h^{-1} = r_\tau$. The centralizer of r_τ in the group of 1–periodic homeomorphisms of $\mathbb{R}$ consists of all translations of $\mathbb{R}$.*

In particular, the conjugating homeomorphism h is unique up to composition with a translation. Moreover, note that if $\phi_1, \phi_2, \dots, \phi_n$ are pairwise commuting homeomorphisms and one of them is conjugate to a translation, then one can use the same conjugating homeomorphism in order to conjugate $\phi_1, \dots, \phi_n$ to translations simultaneously.

8C2 Diophantine approximations The theory of diffeomorphisms of the circle has strong connections to the theory of Diophantine approximations [32]. However, our use of the following theorem from the theory of Diophantine approximations is much more modest. A reference for the following result is [53, page 27].

Theorem 8.6 (Dirichlet) *Let $\alpha_1, \dots, \alpha_n$ be real numbers such that at least one of them is irrational. Then there are infinitely many $(n+1)$–tuples $(q, p_1, \dots, p_n)$ of integers such that $\gcd(q, p_1, \dots, p_n) = 1$ and*

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+1/n}}. \tag{8-5}$$

The Thue–Siegel–Roth theorem states that if α is a real algebraic number and $\delta > 0$, then

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\delta}}$$

has only finitely many solutions p, q where p, q are coprime integers. Therefore one cannot expect to improve the exponent in (8-5).

It is not known to the author whether one can arrange $\gcd(q, p_i) \in \{1, q\}$ for all i, maybe at the expense of replacing the function $q^{-1-1/n}$ in (8-5) by another function $f(q)$ such that $qf(q) \to 0$ as $q \to \infty$. This would allow us to reduce the case of minimal foliations without holonomy to the previous case. That is, we could find a fibration such that $\mathcal{F}$ lies in a neighbourhood of the fibration in which the uniqueness theorem holds.

8C3 The uniqueness theorem for minimal foliations without holonomy In the proof of the approximation theorem of Eliashberg and Thurston (Theorem 1.1) one uses the fact that all foliations without holonomy can be C^0–approximated by fibrations. This is explained in [12, Section 1.2.2] and later we shall find particular sequences of fibrations converging to $\mathcal{F}$. We fix a fibration transverse to the flow lines of the ψ (see [12, page 10]) and a fibre Σ_0. There are two cases:

Σ_0 has genus ≤ 1 If Σ_0 has genus 0, then $\mathcal{F}$ has to be a foliation by spheres, and there is an ε–neighbourhood of $\mathcal{F}$ which does not contain any contact structure at all. When the fibres have genus 1, then the leaves of $\mathcal{F}$ are either all cylinders or they are all planes since the leaves of $\mathcal{F}$ cover the fibres of the fibration. These are the cases excluded in Theorem 1.4, and as we will see in Section 9B there are infinitely many contact structures with vanishing Giroux torsion in any neighbourhood of a linear foliation by planes or cylinders on T^3.

Σ_0 has genus ≥ 2 Our first goal is to find a very good approximation of the foliation by a fibration using Theorem 8.6. For future reference we fix a hyperbolic metric on Σ_0.

Let $f_0, \dots, f_n$ be generators of G (the image of the map defined in (8-3)). As explained above, $f_i \circ f_j = f_j \circ f_i$ and we may assume $f_0(x) = x + 1$. Because no map f_i has a fixed point and all these maps are 1–periodic they are conjugate to translations (either because they are all lifts of periodic circle diffeomorphisms, or because of Theorem 8.5). Since we assumed that $\mathcal{F}$ is minimal there is one generator, say f_1, with irrational translation number.

By Theorem 8.5 the representation $q\colon \pi_1(M) \to \mathrm{Diff}^2_+(M)$ defined in (8-3) is conjugate to a representation q' with $(q'(\alpha))(x) = x + \tau(q(\alpha))$ via a 1–periodic homeomorphism h of $\mathbb{R}$. We consider the foliated bundle induced by this representation.

Using Theorem 8.6 we approximate the numbers $\tau_i := \tau(f_i)$, $i = 1, \dots, n$, by rational numbers (recall that $\tau(f_1)$ is irrational).

There might be relations among the maps f_i which then translate to relations among the τ_i. Since $\pi_1(M)$ is finitely presented, these relations are automatically satisfied for an approximation $p_1/q, \dots, p_n/q$ of $\tau_1, \dots, \tau_n$ provided that q is big enough and the approximation satisfies (8-5) in Theorem 8.6.

In order to turn the approximations of the translation numbers into foliations approximating $\mathcal{F}_E$ we fix a handle decomposition of M. We can C^0–approximate the foliation $\mathcal{F}_E$ on $E \to M$ on the preimage of 1–handles by a foliation whose monodromy along a curve in M is $h' \circ \tau_{p_i/q} \circ h'^{-1}$ when the original monodromy along the curve was f_i (here h' is a diffeomorphism sufficiently close to the homeomorphism h).

Since the maps $h' \circ \tau_{p_i/q} \circ h'^{-1}$ satisfy the relations in $\pi_1(M)$ coming from the 2–handles we can extend the approximating foliation to the union of preimages of the 2–handles.

Then extension over 3–handles is no problem since a 1–periodic foliation on $S^2 \times \mathbb{R}$ transverse to the $\mathbb{R}$–fibres is a product foliation by the Reeb stability theorem (Theorem 2.8). In this way, we obtain a sequence of foliations $\mathcal{F}_{q,E}$ on E. The extensions can be chosen so that the tangent spaces of the foliations $\mathcal{F}_{q,E}$ converge uniformly to the original foliation $\mathcal{F}_E$ on E as $q \to \infty$.

By construction, the leaves of $\mathcal{F}_{q,E}$ are properly embedded submanifolds. Hence the pullback $\mathcal{F}_q := \sigma^* \mathcal{F}_{q,E}$ is a smooth foliation on M by compact leaves provided that σ is transverse to $\mathcal{F}_{q,E}$. This happens when q is sufficiently large and h' is sufficiently close to h.

So far we have shown that $\mathcal{F}$ can be approximated by foliations $\mathcal{F}_q$ all of whose leaves are compact. Such foliations define fibrations over the circle.

The distance between two distinct points of a leaf of $\mathcal{F}_q$ which lie on the same $\mathbb{R}$–fibre can be chosen in the interval $[1/2q, 2/q]$ (the factor 2 accounts for the additional approximations eg of h). Because $(p_1/q, \dots, p_n/q)$ satisfy (8-5), $\mathcal{F}_q$ can be chosen such that the angle between leaves of $\mathcal{F}_q$ and $\mathcal{F}_E$ is bounded by a constant proportional to $1/q^{1+1/n}$ while the distance between two points on a fibre along leaves of the Sacksteder flow decreases linearly.

If q is sufficiently large, then the $\mathcal{F}$-horizontal lift with respect to the Sacksteder flow of a curve γ in a fiber of $\mathcal{F}_q$ does not intersect *all* fibres of $\mathcal{F}_q$ provided that the length of γ is smaller than $K+1$. The leaves of $\mathcal{F}_q$ are coverings of the fibres of $\mathcal{F}_0$ and the group of deck transformations is abelian, ie the leaves of $\mathcal{F}_q$ are abelian coverings of Σ_0 with index q.

We say that $\mathcal{F}_q$ is *well-approximating*. The fibres of the fibration $\sigma^*\mathcal{F}_q$ of M are abelian coverings of Σ_0.

Let K be the constant from Lemma 7.1 (with respect to a fixed hyperbolic metric on a fibre of the original fibration). According to (8-5) we can choose q so that

$$\frac{K+1}{q^{1+1/n}} < \frac{1}{8q}, \tag{8-6}$$

and consider the corresponding approximation $\mathcal{F}_q$ of $\mathcal{F}$. We now view a tubular neighbourhood of a fibre/leaf Σ of $\mathcal{F}_q$ as a fibration over Σ; the normal fibres of the tubular neighbourhood are segments of $\mathcal{I}$ and we may choose these segments to have length $1/4q$ in both directions away from Σ. Then we view $\mathcal{F}$ as a connection on this interval bundle.

Then (8-6) implies that no $\mathcal{F}$–horizontal lift of a curve of length $\leq K+1$ which starts at Σ can leave the chosen tubular neighbourhood. Recall that Σ is an abelian covering of a fibre of the original fibration.

The following conditions on ε determine the neighbourhood of $\mathcal{F}$ whose existence is claimed in the uniqueness theorem (Theorem 1.4).

Fixing the neighbourhood of $\mathcal{F}$ We choose $\varepsilon > 0$ such that

- every plane field ζ which is ε–close to $\mathcal{F}$ is transverse to the flow lines of the Sacksteder flow, and
- no ζ–horizontal lift of a curve of length $\leq K+1$ which starts at a fibre Σ can leave its tubular neighbourhood chosen above.

We now show that the ε–neighbourhood of $\mathcal{F}$ has the desired property. Let ξ_0, ξ_2 be C^∞–generic positive contact structures ε–close to $\mathcal{F}$. We consider the movie of characteristic foliations on the fibres/leaves of $\mathcal{F}_q$.

From the conditions on ε it follows that no sheet containing an attractive closed curve of the characteristic foliation of a fibre can be a closed torus in M. Therefore, when Σ is a fibre of the well-approximating fibration which is a convex surface with respect to ξ_0, we can isotope Σ along the flow lines of the Sacksteder flow and thereby reduce the dividing set to a single nonseparating pair using Lemma 7.4 and Lemma 7.6. The fact that one sheet which arises in the constructions of these two lemmas may hit the surface we are about to isotope more than once is not a problem since several pieces of the surface can be isotoped at the same time. (From the definition of the sheets it follows that two sheets either coincide or are disjoint.)

We now proceed as in the proof of Theorem 1.4 when $\mathcal{F}$ is a fibration over the circle. Using the construction from Example 3.24 together with the classification of tight contact structures, we conclude that ξ_0 is isotopic to ξ_2. This concludes the proof of Theorem 1.4.

9 Applications and examples

In this section we apply Theorem 1.4 to prove results about the topology of the space of taut foliations. Moreover, we give a few examples of approximations of foliations by contact structures where the foliation violates the assumptions of Theorem 1.4 and Theorem 1.6 and every neighbourhood of the foliation contains contact structures which are not (stably) isotopic.

9A Homotopies through atoral foliations

Definition 9.1 A foliation $\mathcal{F}$ is *atoral* if there is no torus leaf, not every leaf is a plane and not every leaf is a cylinder.

In other words, atoral foliations are just those foliations which satisfy the assumptions of Theorem 1.4. This definition, like the following, makes sense for positive confoliations. However, in the next two sections we shall focus on foliations. According to a result from [27], a foliation on a closed 3–manifold without torus leaves is taut. On the class of atoroidal manifolds, *atoral* foliations coincides with *taut* foliations.

Definition 9.2 A contact structure ξ *approximates* a foliation $\mathcal{F}$ if every C^0–neighbourhood of $\mathcal{F}$ contains a contact structure isotopic to ξ.

Theorem 1.4 then just says that there is a unique positive contact structure approximating $\mathcal{F}$ whenever $\mathcal{F}$ is atoral. We have the following simple consequence:

Theorem 9.3 *Let $\mathcal{F}_t$, $t \in [0,1]$, be a C^0–continuous family of atoral C^2–foliations. Then the positive contact structures ξ_0 and ξ_1 approximating $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively, are isotopic.*

Proof By Theorem 1.4, for each $t \in [0,1]$ there is a C^0–neighbourhood U_t of $\mathcal{F}_t$ in the space of plane fields such that all positive contact structures in U_t are pairwise isotopic. By compactness we can cover the path $\mathcal{F}_t$ by finitely many C^0–open sets $U_0, \dots, U_N$ such that $U_i \cap U_{i+1}$ contains a foliation from the family $\mathcal{F}_t$. According to Theorem 1.1 there is a positive contact structure ξ_i' in $U_i \cap U_{i+1}$ since this is a C^0–open neighbourhood of a foliation. By the choice of U_i, we have

$$\xi_0 \cong \xi_0' \cong \xi_1' \cong \dots \cong \xi_{N-1}' \cong \xi_1. \qquad \square$$

We obtain an obstruction for two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ being homotopic through taut foliations when the underlying manifold is atoroidal: if the positive contact structures approximating the foliations are not isotopic, then there is no homotopy through taut foliations connecting $\mathcal{F}_0$ and $\mathcal{F}_1$. This is of interest because the following h-principle due to Eynard-Bontemps ([14], building on [42]) reduces the question of when two taut foliations are homotopic through foliations to a purely homotopy-theoretic problem.

Theorem 9.4 *Two taut foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ on 3–manifolds are homotopic through foliations if and only if the corresponding plane fields are homotopic.*

The first step in the proof of this theorem is the introduction of Reeb components.

The following example shows that one can indeed use Theorem 9.3 to show that a pair of taut foliations is not homotopic through foliations without Reeb components (also, they are not homotopic through taut foliations) although they are homotopic through foliations. The example therefore shows that the introduction of Reeb components in Eynard-Bontemps' proof of Theorem 9.4 is necessary. To the best of the author's knowledge, this is the first example of this kind.

More information concerning the question of which contact structures appear in neighbourhoods of foliations can be found in Bowden [3].

Example 9.5 We consider the Brieskorn homology sphere

$$M = \Sigma(2,3,11) = \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^{11} = 0\} \cap S^5.$$

This manifold is a Seifert fibred space over S^2 with three singular fibres. In terms of Seifert invariants, this manifold is often denoted by $M\left(\frac{1}{2}, -\frac{1}{3}, -\frac{2}{11}\right)$ (see [19]). Since M is a homology sphere it does not fibre over S^1.

The tight contact structures on M were classified by Ghiggini and Schönenberger [19]. They showed that this manifold carries exactly two positive tight contact structures up to isotopy. From the surgery description in [19, Section 4.1.4] of the two contact structures it follows that if ξ is a tight contact structure on M, then $\overline{\xi}$ (this is ξ with its orientation reversed) represents the other isotopy class of tight contact structures.

In [38] (together with [45]) it is shown that M admits a smooth foliation $\mathcal{F}$ transverse to the fibres. Since M is a homology sphere, no taut foliation on M has a closed leaf. In particular, there are no torus leaves, and since M does not fibre over S^1 there are no smooth foliations without holonomy. Hence $\mathcal{F}$ satisfies the conditions of Theorem 1.4.

Now let ξ be a contact structure in a sufficiently small C^0–neighbourhood of $\mathcal{F}$. Notice that $\overline{\xi}$ approximates $\overline{\mathcal{F}}$. By Theorem 9.3 the foliations $\mathcal{F}$ and $\overline{\mathcal{F}}$ are not homotopic through foliations without torus leaves. Since M is a homology sphere this is equivalent to saying that $\mathcal{F}$ and $\overline{\mathcal{F}}$ are not homotopic through taut foliations.

This is nontrivial since $\mathcal{F}$ and $\overline{\mathcal{F}}$ are homotopic as plane fields, as can be shown using the invariants from [26], which form a complete invariant for oriented plane fields on 3–manifolds up to homotopy. The most obvious of these invariants is the Euler class $e(\xi) \in H^2(M, \mathbb{Z})$. When $e(\xi)$ is torsion and $H^2(M;\mathbb{Z})$ has no 2–torsion (recall that the manifold under consideration is a homology sphere), then the homotopy class of the plane field is determined by $e(\xi)$ and the rational number $\theta(\xi)$ defined by

$$\theta(\xi) = (\mathrm{PD}(c_1(X, J)))^2 - 2\chi(X) - 3\sigma(X) \in \mathbb{Q}. \tag{9-1}$$

Here (X, J) is a 4–manifold oriented by an almost complex structure J with signature $\sigma(X)$ and Euler characteristic $\chi(X)$ such that $M = \partial X$ as oriented manifolds, $J(\xi) = \xi$ and J induces the original orientation of ξ. As explained in [26], these invariants can be computed effectively when the contact structure is given by a Legendrian surgery diagram.

In the case at hand it is clear that the Euler class of $e(\xi) = -e(\overline{\xi})$ vanishes since M is a homology sphere. It follows from (9-1) and the equality $c_1(X, -J) = -c_1(X, J)$ that $\theta(\xi) = \theta(\overline{\xi})$. Hence ξ and $\overline{\xi}$ are homotopic as oriented plane fields and the same is true for $\mathcal{F}$ and $\overline{\mathcal{F}}$. So by Theorem 9.4 the foliations $\mathcal{F}$ and $\overline{\mathcal{F}}$ are homotopic through foliations but by Theorem 9.3 this homotopy has to contain torus leaves. One can easily show that every foliation without Reeb components on M is taut.

Example 9.6 The work of Ghys and Sergiescu [20] provides another class of pairs of foliations which are not homotopic through atoral foliations. These foliations are the stable and unstable foliations $\mathcal{F}^s, \mathcal{F}^u$ of Anosov flows on T^2–bundles over S^1 which are suspensions of $A \in \mathrm{Gl}(2, \mathbb{Z})$ such that $|\mathrm{tr}(A)| > 2$. The reason why these foliations are not homotopic through atoral foliations is that atoral foliations on suspensions of orientation-preserving Anosov diffeomorphisms of T^2 admit a classification up to diffeomorphism: each smooth atoral foliation is smoothly equivalent to $\mathcal{F}^s$ or $\mathcal{F}^u$ [20]. In many instances these two types of foliations are not even diffeomorphic. It is obvious from [12; 43] that both foliations have isotopic approximating contact structures and that the corresponding plane fields are homotopic (through positive confoliations). Note that $\mathcal{F}^s$ and $\mathcal{F}^u$ are both homotopic to the foliation by the torus fibres of the fibration if $\mathrm{tr}(A) > 2$. The homotopy is given by

$$\alpha_s = s\,dt + (1-s)\lambda^{-t}\beta_\lambda,$$

where $s \in [0, 1]$ and β_λ is a 1–form on T^2 with constant coefficients such that $A^*(\beta_\lambda) = \lambda\beta_\lambda$. The coordinate on the base circle is denoted by t. This eigenvalue λ is positive since $\mathrm{tr}(A) > 2$. Thus $\mathcal{F}^u$ and $\mathcal{F}^s$ are even homotopic through taut foliations.

We conclude from the last example that there are foliations which are not homotopic through atoral foliations despite the fact that the approximating contact structures are isotopic.

9B Parabolic torus fibrations and linear foliations on T^3

In this section we discuss foliations given by fibres of particular torus fibrations over S^1 and we show that there are foliations with the property that every isotopy class of contact structures has positive C^0–distance from the foliation. Clearly, these foliations have to belong to the classes where Theorem 1.4 does not apply.

Example 9.7 Let M be the total space of a torus bundle over the circle whose monodromy ϕ_M satisfies $\mathrm{tr}(\phi_M) = +2$ (see Section 8A for the notation). According to [31], these are exactly those orientable manifolds which admit C^2–foliations all of whose leaves are cylinders. We may assume that

$$\phi_M = A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix},$$

with $k \in \mathbb{Z}$. We view M as a fibration over T^2 with the bundle projection

$$M \to T^2, \quad (x_1, x_2, t) \mapsto (x_1, t)$$

and typical fibre S^1. Let $\mathcal{F}_0$ denote the foliation by tori defined by dt on M. Thus the leaves are the fibres of $M \to S^1$. Let $\xi_{\varepsilon,m}$ be the contact structure defined by

$$dt + \varepsilon\big((|k|+1)\cos(2\pi m t)\,dx_1 - \sin(2\pi m t)(dx_2 - kt\,dx_1)\big), \tag{9-2}$$

with m a positive integer and $\varepsilon > 0$. This 1–form is a contact form on M and as $\varepsilon \to 0$ the corresponding plane field converges to $\mathcal{F}_0$. The contact structures $\xi_{\varepsilon,m}$ are distinguished by their Giroux torsion.

However, there are other contact structures in every neighbourhood of $\mathcal{F}_0$: Let $\eta_{\varepsilon,m}$ be the positive contact structure defined by the 1–form

$$\alpha_{\varepsilon,m} = dx_1 + \varepsilon\big(\sin(2\pi m x_1)(dx_2 - kt\,dx_1) + |k+1|\cos(2\pi m x_1)\,dt\big),$$

where m is a positive integer and $\varepsilon > 0$ (these contact forms are taken from [23]). Then as $\varepsilon \to 0$ the contact structures defined by $\alpha_{\varepsilon,m}$ converge to the foliation $\mathcal{F}_1$ defined by dx_1 on M. Now consider the following automorphism of M:

$$\psi_p \colon M \to M, \quad (x_1, x_2, t) \mapsto \big(x_1 + pt, x_2 + \tfrac{1}{2}kpt(t+1), t\big).$$

This map covers the p–fold Dehn twist of T^2 given by $(x_1, t) \mapsto (x_1 + pt, t)$. We consider the foliations defined by

$$\psi_p^*(dx_1) = dx_1 + p\,dt.$$

As $p \to \infty$ these foliations converge to $\mathcal{F}_0$. Hence the contact structures $\eta_{\varepsilon,m,p}$ defined by $\psi_p^*(\alpha_{\varepsilon,m})$ form a sequence of positive contact structures converging to $\mathcal{F}_0$. It is shown in [23] that the contact structures $\eta_{\varepsilon,m,p}$ and $\eta_{\varepsilon',m',p'}$ are isotopic if and only if $m = m'$ and $p = p'$. Moreover, they are not isotopic to any of the contact structures $\xi_{\varepsilon,m}$ defined by (9-2).

A foliation $\mathcal{F}$ on T^3 is *linear* if $\mathcal{F} = \ker(\beta = a\,dx + b\,dy + c\,dz)$ with $a, b, c \in \mathbb{R}$. In the following we establish restrictions on the contact structures lying in a given C^0–neighbourhood of $\mathcal{F}$. For this we first recall the classification of tight contact structures on $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ (with corresponding coordinates x, y, z and oriented by the volume form $dx \wedge dy \wedge dz$). The following result sums up work of Kanda, Giroux and earlier results from [11].[1]

Theorem 9.8 (Kanda, Giroux) *A positive tight contact structure on T^3 is diffeomorphic to*

$$\xi_m = \ker\big(\alpha_m = \cos(2\pi m z)\,dx - \sin(2\pi m z)\,dy\big) \tag{9-3}$$

for a unique $m \in \{1, 2, \dots\}$. Two tight contact structures ξ, ξ' are isotopic if and only if there are contactomorphisms

$$\psi\colon (T^3, \xi) \to (T^3, \xi_m),$$
$$\psi'\colon (T^3, \xi') \to (T^3, \xi_{m'})$$

such that $m = m'$ and the pre-Lagrangian tori $\psi^{-1}(\{z = z_0\})$ and $\psi'^{-1}(\{z = z_0'\})$ are isotopic.

This theorem implies in particular that every oriented tight contact structure ξ on T^3 is isotopic to $\overline{\xi}$ since ξ_m is isotopic to $\overline{\xi}_m$ via the isotopy

$$h_s\colon T^3 \to T^3, \quad (x, y, z) \mapsto (x, y, z + \pi s/m),$$

with $s \in [0, 1]$. Moreover, we can associate to ξ a pre-Lagrangian torus $T(\xi)$ which is well-defined up to isotopy. Using this we will establish the following result:

[1] The author is grateful to Hansjoerg Geiges for pointing out this reference.

Proposition 9.9 *Let $\mathcal{F}$ be a linear foliation on T^3 and ξ a tight contact structure.*

(a) *If $\mathcal{F}$ is not a foliation by closed leaves, then there is a neighbourhood U_ξ of $\mathcal{F}$ which does not contain any contact structure isotopic to ξ.*

(b) *Every C^0–neighbourhood of $\mathcal{F}$ contains a contact structure isotopic to ξ if and only if $T(\xi)$ is isotopic to a leaf of $\mathcal{F}$. In particular, all leaves of $\mathcal{F}$ are closed.*

Proof We fix a fibration pr: $T^3 \to T^2$ by circles which are transverse to $\mathcal{F}$ and a pair $\mathcal{G}_1, \mathcal{G}_2$ of oriented foliations by tori T^2 such that for any two leaves Σ_1 and Σ_2 of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, the intersection $\Sigma_1 \cap \Sigma_2$ is a fibre of pr. We view $\mathcal{G}_1, \mathcal{G}_2$ as an S^1–family of embedded tori and we orient S^1 so that the orientation of the leaves followed by the orientation of S^1 is the orientation of T^3.

We will consider only plane fields which are transverse to the fibres of pr. The characteristic foliation of every such plane field on leaves of $\mathcal{G}_1$ or $\mathcal{G}_2$ is nonsingular. Therefore it has a well-defined asymptotic direction on every leaf of $\mathcal{G}_1$ or $\mathcal{G}_2$. This slope is an element of the space of oriented lines in $H_1(T^2, \mathbb{R})$ (this vector space inherits a natural orientation from the orientation of the leaf).

If $\zeta = \xi$ is a positive contact structure, then the characteristic foliation on leaves of $\mathcal{G}_1$ or $\mathcal{G}_2$ cannot be diffeomorphic to a linear foliation, according to the proof of the classification of tight contact structures on T^3 in [24]. We recall the argument briefly: If the characteristic foliation was diffeomorphic to a linear foliation on a leaf of $\mathcal{G}_i$, $i = 1, 2$, then the slope of the characteristic foliation would have to vary and it can only change in one direction as one moves along S^1. Therefore the slope would have to make at least one full twist. In particular, the characteristic foliation of ξ would be somewhere tangent to the fibres of pr. This contradicts our initial assumptions on $\zeta = \xi$.

Hence the characteristic foliation on the leaves of $\mathcal{G}_1, \mathcal{G}_2$ has constant slope. Moreover, the characteristic foliation of ξ on leaves of $\mathcal{G}_i$, $i = 1, 2$, has closed leaves and the union of these closed leaves forms a collection of sheets which are transverse to the fibres of pr; these sheets are pre-Lagrangian tori of ξ.

Thus we have shown that the characteristic foliation of ξ on a leaf of $\mathcal{G}_1$ or $\mathcal{G}_2$ has a closed leaf whose slope coincides with the slope of a pre-Lagrangian torus of ξ intersected with that leaf. Thus the C^0–distance between ξ and $\mathcal{F}$ is bounded from below by the angle between $\mathcal{F}$ and a linear torus which is isotopic to a pre-Lagrangian torus of ξ.

If $\mathcal{F}$ is a foliation by cylinders or planes, then the slope of the characteristic foliation of $\mathcal{F}$ on a leaf of $\mathcal{G}_1$ or $\mathcal{G}_2$ is irrational while the slope of a linear torus is rational, thus the angle between ξ and $\mathcal{F}$ is bounded from below once the pre-Lagrangian torus of ξ is fixed up to isotopy. This proves (a) and one direction of (b).

Conversely, if every neighbourhood of a linear foliation on T^3 contains a given contact structure, then the slopes of the pre-Lagrangian tori of ξ and the slopes of $\mathcal{F}$ have to coincide. In particular, all leaves of $\mathcal{F}$ are closed. Thus we have shown (b). $\square$

Example 9.10 Let $(a,b,c) \neq (0,0,0)$ and a_n, b_n, c_n, q_n a sequence of integers such that

$$\begin{aligned} q_n &\neq 0, \\ (a_n, b_n, c_n) &\neq (0,0,0), \\ \gcd(a_n, b_n, c_n) &= 1, \\ \lim_n (a_n/q_n, b_n/q_n, c_n/q_n) &= (a,b,c). \end{aligned}$$

We pick 1–forms α, β such that $(\alpha_\infty = a\,dx + b\,dy + c\,dz, \alpha, \beta)$ is a positive framing of T^*T^3. For a positive sequence ε_n with $\lim_n \varepsilon_n = 0$, the plane fields defined by

$$\alpha_n = \alpha_\infty + \varepsilon_n \big(\cos(a_n x + b_n y + c_n z)\alpha - \sin(a_n x + b_n y + c_n z)\beta\big)$$

converge to the foliation $\mathcal{F} = \ker(a\,dx + b\,dy + c\,dz)$ on $T^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$. For large n, α_n is a positive contact form. The Giroux torsion of all these contact structures vanishes.

A similar argument shows that foliations by cylinders on torus bundles over the circle as in Example 9.7 (defined by $dx_1 + a\,dt$ with $a \in \mathbb{R} \setminus \mathbb{Q}$ in the coordinates used in Example 9.7) have properties analogous to those of the foliations by planes considered in Proposition 9.9.

Corollary 9.11 *Let $\mathcal{F}$ be a linear foliation on T^3 with noncompact leaves. Then $\mathcal{F}$ cannot be deformed to a contact structure, ie there is no family of plane fields ξ_s with $\xi_0 = \mathcal{F}$ and ξ_s a contact structure for $s > 0$.*

Proof If ξ_s is a deformation of $\mathcal{F}$ into a contact structure, then Gray's theorem implies that every neighbourhood of $\mathcal{F}$ contains a contact structure isotopic to ξ_1. But according to Proposition 9.9 a sufficiently small neighbourhood of $\mathcal{F}$ does not contain a contact structure isotopic to ξ_1. $\square$

A different type of example for this phenomenon was found much earlier by Etnyre [13]. His example is slightly different since it refers to Giroux torsion rather than pre-Lagrangian tori, but both examples make essential use of pre-Lagrangian tori. It is natural to ask to whether foliations without holonomy on surface bundles over S^1 can be deformed into contact structures when the Euler characteristic of the fibres is negative.

Finally, we give an example of a foliation on T^3 with an unstable torus leaf such that every C^∞–neighbourhood contains positive contact structures which are not stably isotopic.

Example 9.12 Let $Z = f(z)\,\partial_z$ be a smooth vector field on $S^1 = \mathbb{R}/\mathbb{Z}$ with $f(z) > 0$ for $z \neq 0$ such that $f(z) = z^2$ on a neighbourhood of 0. We denote the flow of Z by φ_t. Mapping the two generators of $\pi_1(T^2)$ to the commuting diffeomorphisms φ_t and $\varphi_{t'}$ with $0 < t < t'$, we obtain a foliation on T^3 transverse to the fibres of $T^3 \to T^2$, and the torus $z = 0$ is the only minimal set of $\mathcal{F}_0$. This torus is unstable by construction.

In order to show that this example has the desired properties we proceed as follows. Approximate Z by $\widetilde{Z} = \widetilde{f}(z)\partial_z$ such that $\widetilde{f}(z) > 0$ for all $z \in S^1$. We denote the flow of $\widetilde{Z}$ by $\widetilde{\varphi}$. Replacing $\varphi_t, \varphi_{t'}$ in the representation $\pi_1(T^2) \to \mathrm{Diff}_+(S^1)$ by $\widetilde{\varphi}_t, \widetilde{\varphi}'_t$, we replace $\mathcal{F}$ by $\widetilde{\mathcal{F}}$. If the rotation numbers $\widetilde{\rho}, \widetilde{\rho}'$ of $\widetilde{\varphi}_t, \widetilde{\varphi}'_t$ are rationally independent, the foliation $\widetilde{\mathcal{F}}$ is a foliation by planes. Moreover, if there are integers (c, d) with $c > 0$ and $d > 2$ such that

$$\left|\widetilde{\rho} - \frac{p}{q}\right| > \frac{c}{q^d} \tag{9-4}$$

for all $q \in \mathbb{N}^+$ and $p \in \mathbb{Z}$, then according to Herman's thesis [32] $\widetilde{\mathcal{F}}$ is smoothly conjugate to one of the linear foliations discussed in Proposition 9.9. The numbers satisfying the Diophantine condition (9-4) are dense. Therefore we can approximate $\mathcal{F}$ by a sequence of foliations $\widetilde{\mathcal{F}}_n$ all of whose leaves are planes, and every neighbourhood of $\widetilde{\mathcal{F}}_n$ contains nonisotopic positive contact structures with vanishing Giroux torsion (see Example 9.10).

9C Further applications

Colin showed in [7] that foliations without Reeb components can be approximated by tight contact structures and asked whether or not this is true for *every* contact structure in a sufficiently small neighbourhood of the foliation. Theorem 1.6 together with the gluing results from [5] provide the following partial answer to this question.

Proposition 9.13 *Let $\mathcal{F}$ be a C^2–foliation without Reeb components such that all torus leaves have attractive holonomy. Then $\mathcal{F}$ has a neighbourhood such that all contact structures in that neighbourhood are universally tight.*

Bowden showed in [2] that Proposition 9.13 remains true when the assumption on the holonomy of torus leaves is dropped.

Recall the following theorem from [58]:

Theorem 9.14 *Let (M,ξ) be a confoliation admitting an overtwisted star. Then ξ can be C^0–approximated by overtwisted contact structures.*

Together with the uniqueness result, Theorem 1.4 has the following consequence for confoliations which are not s-tight.

Corollary 9.15 *Let M be a closed manifold and ξ be a confoliation without torus leaves which is not s–tight. Then there is a C^0–neighbourhood of ξ such that every contact structure in that neighbourhood is overtwisted.*

It is not clear whether or not the conclusion of the corollary also holds in the presence of incompressible torus leaves.

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