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Asymptotic formulae for curve operators in TQFT

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The Reshetikhin–Turaev topological quantum field theories with gauge group SU_2 associate to any oriented surface Σ a sequence of vector spaces $V_r(\Sigma)$ and to any simple closed curve γ in Σ a sequence of Hermitian operators T_r^{γ} on the spaces $V_r(\Sigma)$. These operators are called curve operators and play a very important role in TQFT.

We show that the matrix elements of the operators T_r^{γ} have an asymptotic expansion in orders of $1/r$, and give a formula to compute the first two terms from trace functions, generalizing results of Marché and Paul for the punctured torus and the 4–holed sphere to general surfaces.

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1 Introduction

Witten [28] proposed in 1989, by a method using Feynman path integrals, a family of new invariants of 3–manifolds derived from the Jones polynomial, together with the structure of a full topological quantum field theory. Reshetikhin and Turaev [24] formalized the ideas of Witten to construct a family $(Z_{2r}(M))_{r\in\mathbb{N}^*}$ of 3–manifolds invariants. Also they defined a TQFT-structure for these invariants in [24] and Turaev [27]. An alternative method to define these 3–manifold invariants and TQFTs using skein theory of 3–manifolds was later developed by Blanchet, Habegger, Masbaum and Vogel [11].

Let Σ be a closed oriented surface maybe with marked points p_i colored by elements $\widehat{c}_i$ of $\mathcal{C}_r=\{1,\dots,r-1\}$. Neglecting the so-called framing anomaly, the construction of [11] associates a vector space $V_r(\Sigma,\widehat{c})$ to $(\Sigma,\widehat{c})$ and, for any cobordism (M,Σ_0,Σ_1) containing a link L, there is a morphism

$$V_r(M,L)\colon V_r(\Sigma_0)\to V_r(\Sigma_1)$$

such that for every closed orientable 3–manifold M we have $V_r(M)=Z_{2r}(M)$.

Let us recall that a multicurve on Σ is a disjoint union of simple closed curves on Σ. In particular, the construction associates to any multicurve γ on Σ a curve operator

$$T_r^{\gamma}=V_r\big(\Sigma\times[0,1],\gamma\times\{\tfrac{1}{2}\}\big)\in\mathrm{End}(V_r(\Sigma,\widehat{c})).$$

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Curve operators often play a central role in TQFT; they were used to derive the asymptotic faithfulness of quantum representations, or to relate the combinatorial and the geometric framework of TQFT; see Andersen [1; 2] or Andersen and Ueno [7; 8; 9; 10].

From the construction of [11] it follows also that each vector space $V_r(\Sigma,\hat{c})$ comes with a natural Hermitian form.

Recall that a pants decomposition of a surface Σ with marked points is a finite family of simple closed curves on Σ which cut Σ into either pair of pants containing no marked point or disks containing exactly one marked point.

We will say that a trivalent banded graph Γ inside Σ is *compatible* with a pair of pants decomposition $\mathcal{C}=(C_e)_{e\in E}$ if the following conditions are satisfied:

- Γ has a trivalent vertex v_P lying in each pair of pants P of the decomposition, and these are the only trivalent vertices of Γ.
- For every $e\in E$, Γ has exactly one edge (labeled also by e) that intersects the curve C_e. This edge is disjoint from the other curves C_f for $f\in E\setminus\{e\}$, and intersects C_e exactly once.
- The graph Γ has n univalent vertices labeled by $p_1,\dots,p_n$ corresponding to the marked points of Σ. These are the only univalent vertices of Γ.

See Figure 1 for an example of such a graph.

The construction of [11] provides the space $V_r(\Sigma,\hat{c})$ with a Hermitian basis $(\varphi_c)_{c\in U_r}$ for any choice of a pair of pants decomposition $\mathcal{C}$ of Σ and trivalent graph Γ compatible with $\mathcal{C}$. The index set U_r of this basis is the set of r–admissible colorings of the edges of Γ, defined as follows:

Let $\mathcal{C}_r=\{1,\dots,r-1\}$ be the set of colors.

An r–admissible coloring of Γ is a map $c\colon E\to\mathcal{C}_r$ such that the following conditions are met:

(1) For any $i\in\{1,\dots,n\}$, the edge adjacent to p_i is colored by $c_i=\hat{c}_i$.

(2) Let S be the set of all triples (e,f,g) such that the curves C_e, C_f and C_g bound a pair of pants (possibly two of these curves are the same). Then for any $(e,f,g)\in S$ we have
   (i) $c_e+c_f+c_g<2r$ and $c_e+c_f+c_g\equiv 1 \pmod 2$;
   (ii) $c_e<c_f+c_g$.

If we have a sequence of coloring of the marked points $\hat{c}_i=rt_i$ with $t\in\mathbb{Q}^n$, then for $c_r\in U_r$ the E–tuple c_r/r is in the set $U\subset\mathbb{R}^E$ defined by $x\in U$ if and only if

(1) $x_i = t_i$ if i is the edge adjacent to the marked point p_i; and

(2) for any $(e, f, g) \in S$, we have

  (i) $x_e + x_f + x_g < 2$,

  (ii) $x_e < x_f + x_g$.

Let γ_i be small simple closed curves encircling the marked points p_i. We introduce the SU_2–moduli space of Σ with marked points (p_i, t_i), $t_i \in [0,1]$,

$$\mathcal{M}(\Sigma, t_1, \dots, t_n) = \big\{\rho\colon \pi_1(\Sigma) \to \mathrm{SU}_2 \mid \mathrm{Tr}(\rho(\gamma_i)) = 2\cos(\pi t_i)\big\}/\mathrm{SU}_2.$$

The quotient here corresponds to the conjugation of representations by an element of SU_2.

We recall that the subset of irreducible representations in $\mathcal{M}(\Sigma)$ has a natural Atiyah–Bott–Goldman–Seshadri symplectic form, which we call ω.

Any curve γ on Σ induces a natural *trace function* f_γ on $\mathcal{M}(\Sigma)$ by the formula

$$f_\gamma\colon \rho \to -\mathrm{Tr}(\rho(\gamma)).$$

Moreover for any pants decomposition $\mathcal{C}$ of Σ, Jeffrey and Weitsman [20] introduced a momentum map $h_{\mathcal{C}}$ on $\mathcal{M}(\Sigma)$ whose image is the closure of the set U introduced above. This momentum mapping is given by the formula

$$h_{\mathcal{C}}\colon \rho \to (h_{C_e}(\rho))_{e\in E} = \Big(\frac{1}{\pi}\,\mathrm{Acos}\Big(\frac{\mathrm{Tr}(\rho(C_e))}{2}\Big)\Big)_{e\in E}.$$

Here U is exactly the set of regular values of the momentum map $h_{\mathcal{C}}$. Jeffrey and Weitsman showed that the h_{C_e} are independent Poisson-commuting functions, and that these Hamiltonians induce an action of a torus T on each level set. Thus the momentum map induces action-angle coordinates on the subset $h_{\mathcal{C}}^{-1}(U)$ of $\mathcal{M}(\Sigma)$: there is a map

$$R\colon U \times T \to h_{\mathcal{C}}^{-1}(U), \quad (\tau, \theta) \mapsto R(\tau_e, \theta_e).$$

The map R satisfies that $h_{\mathcal{C}}(R(\tau,\theta)) = \tau$ and $R_*(\omega) = \sum_{e\in E} d\tau_e \wedge d\theta_e$. These action-angle coordinates are unique up to a shift in angle coordinates.

Marché and Paul [21] proved from skein calculus that in the case of the once-punctured torus and the case of the four-punctured sphere, the matrix coefficients of curve operators $\langle T_r^\gamma \varphi_c, \varphi_{c+k}\rangle$ converge to the k^{th} Fourier coefficient of the trace functions

$$\theta \mapsto f_\gamma\Big(R\Big(\frac{c}{r}, \theta\Big)\Big), \quad \theta \in T.$$

They also gave an expression for the $O(1/r)$ term in the expansion of $\langle T_r^\gamma \varphi_c, \varphi_{c+k}\rangle$.

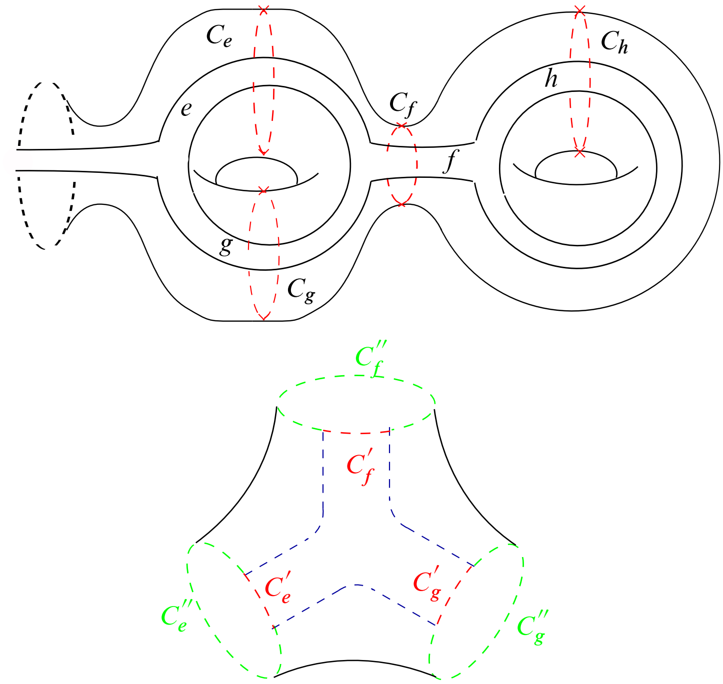

Figure 1: A banded graph compatible with a pants decomposition of Σ by curves $\{C_e\}$ and the associated cell decomposition of a pants into hexagons

Our paper aims to give a generalization of the asymptotic expansion in [21] for any marked surface Σ. We observed a new phenomenon when studying general surfaces: the asymptotic coefficients are again related to Fourier coefficients of trace functions, but they are twisted by rapidly oscillating signs.

To give an expression for these signs, we introduce some cocycles on Σ.

Equip Σ with a pants decomposition $\mathcal{C}$ and a compatible graph Γ. As we can see in the example in Figure 1, $\Sigma \setminus \Gamma$ is a trivalent banded graph diffeomorphic to Γ, so we get a continuous folding map $p\colon \Sigma \to \Gamma$ that pastes the two copies of Γ.

For any r–admissible color c we can define a multicurve L_c inside Γ: take $c_e - 1$ parallel strands at any edge e and connect at vertices in the unique way avoiding crossings.

We define a cocycle $\overline{c}$ in $H^1(\Sigma, \mathbb{Z}/2)$ by the formula

$$\overline{c}(\gamma) = L_c \cap p(\gamma).$$

Here $\cap$ is the $\cap$–product map $H_1(\Gamma, \mathbb{Z}/2) \times H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2) \to \mathbb{Z}/2$, and we view $p(\gamma)$ as an element of $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$.

Theorem 1.1 *Let γ be a multicurve in $\Sigma \setminus \{p_1, \dots, p_n\}$.*

For $e \in E$ we write I_e^γ for the geometric intersection number of γ with C_e.

We introduce an open set $V_\gamma \subset U \times [0,1]$ by the formula

$$V_\gamma = \big\{(\tau, \hbar) \mid (\tau_e + \varepsilon_e \hbar I_e^\gamma)_{e \in E} \in U \ \text{ for all } \varepsilon \in \{\pm 1\}^E \big\}.$$

Then

(1) *Whenever $k_e > I_e^\gamma$ or $k_e \neq I_e^\gamma \pmod 2$, the matrix coefficient $\langle T_r^\gamma \varphi_c, \varphi_{c+k}\rangle$ vanishes.*

(2) *If $k_e \le I_e^\gamma$ and $k_e = I_e^\gamma \pmod 2$, there exists a smooth function $(F_k^\gamma)_{k\colon E \to \mathbb{Z}}$ defined on V_γ such that, for any $c \in U_r$, the matrix coefficient $\langle T_r^\gamma \varphi_c, \varphi_{c+k}\rangle$ is $\overline{c}(\gamma) F_k^\gamma(c/r, 1/r)$.*

If we set $F_k = 0$ for any other $k\colon E \to \mathbb{Z}$, we can write

$$T_r^\gamma \varphi_c = \overline{c}(\gamma) \sum_{k\colon E \to \mathbb{Z}} F_k^\gamma\Big(\frac{c}{r}, \frac{1}{r}\Big)\varphi_{c+k}.$$

As $\overline{c}$ is an element of $H^1(\Sigma, \mathbb{Z}/2)$, $\overline{c}(\gamma)$ is just a sign. This sign factor, which did not appear in [21], will be shown to be trivial when the banded trivalent graph Γ is planar (which was the case for the punctured torus and the four-holed sphere).

The coefficients F_k^γ can be computed by hand for any multicurve γ on Σ, but to give an explicit formula for a general γ is out of reach. However, we will provide a formula for the first two terms of the Taylor expansion of F_k^γ in the second variable.

In [21], to make sense of the coefficients of T_r^γ Marché and Paul introduce a complex-valued function σ^γ, which they called the ψ–symbol of T_r^γ. We follow their approach, but the signs in our formulae lead us to define the ψ–symbol as a function with values in some algebra A_Γ, which we call the intersection algebra. We define A_Γ as follows:

Let π be the map $H^1(\Gamma, \mathbb{Z}/2) \to H^1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$ and B be its image. The folding map p and the map π induce a map $p_*\colon H^1(\Sigma, \mathbb{Z}/2) \to H^1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$. We define

$$A_\Gamma = \bigoplus_{[\gamma] \in B} \mathbb{C}[\gamma]$$

with the product $[\gamma][\delta] = (-1)^{\gamma \cap \widetilde{\delta}}[\gamma + \delta]$, where $\pi(\widetilde{\delta}) = [\delta]$ and $\cap$ is the intersection form $H^1(\Gamma, \partial\Gamma, \mathbb{Z}/2) \times H^1(\Gamma, \mathbb{Z}/2) \to \mathbb{Z}/2$.

Definition 1.2 Let γ be a multicurve on Σ. We define the ψ–symbol of T_r^γ as the map

$$\sigma^\gamma\colon V_\gamma \times (\mathbb{R}/2\pi\mathbb{Z}) \to A_\Gamma$$

such that

$$\sigma^\gamma(\tau,\hbar,\theta) = \sum_{k\colon E\to\mathbb{Z}} F_k(\tau,\hbar)e^{ik\cdot\theta}[p_*(\gamma)].$$

If $\chi\colon A_\Gamma \to \mathbb{C}$ is a morphism of algebras, we also introduce $\sigma_\chi^\gamma(\tau,\theta) = \chi(\sigma^\gamma)(\tau,0,\theta)$.

Let us add a few remarks on this definition:

(1) $k\cdot\theta$ stands for $\sum_{e\in E} k_e\theta_e$.

(2) The sum over $k\colon E\to\mathbb{Z}$ is actually a finite sum, as only a finite number of coefficients F_k^γ does not vanish.

(3) We will often omit the p_* and just write $[\gamma]$ for the element $[p_*(\gamma)]$, when γ is a multicurve.

(4) We will often refer to the zeroth order in $\hbar$ of the ψ–symbol, that is, $\sigma^\gamma(\tau,0,\theta)$, as the principal symbol of T_r^γ.

We use this definition to state our main result:

Theorem 1.3 *Let γ be a multicurve on Σ. The ψ–symbol $\sigma^\gamma(\tau,\hbar,\theta)$ of the curve operator T_r^γ has the following asymptotic expansion:*

$$\sigma^\gamma(\tau,\hbar,\theta) = \sigma^\gamma(\tau,0,\theta) + \frac{\hbar}{2i}\sum_{e\in E}\frac{\partial^2}{\partial\tau_e\,\partial\theta_e}\sigma^\gamma(\tau,0,\theta) + o(\hbar)$$

and, for $\chi\colon A_\Gamma\to\mathbb{C}$ a morphism of algebras, we have $\sigma_\chi^\gamma(\tau,\theta) = f_\gamma(R_\chi(\tau,\theta)) = -\mathrm{Tr}(R_\chi(\tau,\theta)(\gamma))$, where the R_χ are action-angle parametrizations on

$$\mathcal{M}(\Sigma) = \mathrm{Hom}\big(\pi_1(\Sigma\setminus\{p_1,\dots,p_n\}),\mathrm{SU}_2\big)/\mathrm{SU}_2$$

defined up to a choice of origin of the angles.

The above theorem is quite similar to results obtained by Andersen and Gammelgaard [6] in the geometric framework of the Witten–Reshetikhin–Turaev TQFT.

Recall that, for any complex structure σ on Σ representing a point in the Teichmüller space $\mathcal{T}$ of Σ, the smooth part of the moduli space of Σ has the structure of a Kähler manifold M_σ. It is then possible to identify the TQFT vector spaces $V_r(\Sigma)$ with the space of holomorphic sections $H^0(M_\sigma, L^r)$, where L is the Chern–Simons vector bundle; see Andersen and Ueno [7; 8; 9; 10].

Theorem 7 of [6] shows that curve operators T_r^γ are approximated at order 1 by Toeplitz operators of principal symbol f_γ and subprincipal symbols

$$\tfrac{1}{4}\Delta_\sigma f_\gamma + i\nabla_{X_F''} f_\gamma,$$

where X_F'' is the $(0,1)$–part of the Hamiltonian vector field for the Ricci potential.

An alternative proof of Theorem 1.3 could be to combine the results of [6] with results explaining how these Laplace operators degenerate when the complex structure on Σ converges to the pair of pants decomposition. See Andersen [5] for an outline of such techniques.

The methods in [6] rely on the geometric framework of TQFT or the Hitchin connection so they are quite different from ours, which is based on skein theory and is the continuation of the work of Marché and Paul [21].

The proof of [21] in the case where Σ is the punctured torus and the four-holed sphere relied on explicit computations for some simple set of curves that generates the Kauffman algebra of Σ, then extending the result to general curves. This approach failed in higher genus as no simple set of generators is known. Instead, we developed a more conceptual and systematic method, which relies on the study of algebraic properties of the ψ–symbol and the Kauffman algebra of Σ.

Marché and Paul [21] used the asymptotic estimation to construct a framework for curve operators on the punctured torus and the four-holed sphere as Toeplitz operators on the sphere. This allowed the application of the WKB-approximation for eigenvectors. From this they deduced asymptotic expansions of quantum invariants (such as a new proof of the asymptotic expansion of $6j$–symbols, and an expression for the punctured S–matrix). Therefore, we hope to use our asymptotic expansions for general marked surface to make a connection to the framework of curve operators as Toeplitz operators on toric varieties, or at least apply the tools of microlocal analysis. Such a Toeplitz framework for curve operators may be a useful tool to study combinatorial TQFT. Indeed, in a different approach, Andersen [1] introduced some geometrical curve operators that are Toeplitz operators to prove the asymptotic fidelity of the quantum representations of the mapping class group. We think that the idea, initiated by Andersen, of viewing the standard curve operators as Toeplitz operators is a powerful idea, as has been demonstrated in various work of his [2; 3; 4]. We believe that our result and methods, based on the BHMV approach to TQFT, could provide interesting applications in other directions.

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2 A quick overview of TQFT and curve operators

In this section we will outline the BHMV approach to TQFT. Their construction relies on the notion of Kauffman bracket skein modules of 3–manifolds and Kauffman algebras of marked surfaces.

For M a compact oriented 3–manifold (which can have a boundary), we define $K(M,A)$ as the quotient of the free $\mathbb{C}[A^{\pm 1}]$–module generated by links modulo isotopy and the Kauffman relations (see Figure 2).

For $t \in \mathbb{C}^*$, we can define a Kauffman module evaluated at t: we write $K(M,t) = K(M,A) \otimes_{A=t} \mathbb{C}$.

Now, if Σ is a surface with marked points $p_1, \dots, p_n$, we denote by $K(\Sigma, A)$ the Kauffman module $K\big((\Sigma \setminus \{p_1, \dots, p_n\}) \times [0,1], A\big)$.

We call a disjoint union of simple curves on Σ which is disjoint from the marked points of Σ a *multicurve* on Σ. It is easy to see that $K(\Sigma, A)$ is spanned by multicurves on Σ, and actually multicurves give a basis of this vector space, as shown in [14].

The module $K(\Sigma, A)$ has an algebra structure: the product $\gamma \cdot \delta$ of two elements of $K(\Sigma, A)$ is obtained by isotoping γ and δ so they are included in $\Sigma \times \big(\frac{1}{2}; 1\big]$ and $\Sigma \times \big[0; \frac{1}{2}\big)$, respectively, then gluing the two parts into $\Sigma \times [0,1]$.

For $t \in \mathbb{C}^*$, we define $K(\Sigma, t) = K(\Sigma, A) \otimes_{A=t} \mathbb{C}$, which is also an algebra, and admits the set of multicurves as a basis. Using this basis, we get a linear isomorphism between $K(\Sigma, t)$ and $K(\Sigma, -1)$ and we embed $K(\Sigma, -e^{i\pi\hbar/2}) = K(\Sigma, A) \otimes_{A=-e^{i\pi\hbar/2}} \mathbb{C}[\![\hbar]\!]$ into $K(\Sigma, -1)[\![\hbar]\!]$.

The vector spaces $V_r(\Sigma, \hat{c})$ are quotients of Kauffman modules at roots of unity, as explained below:

Definition [11] Let H be a handlebody with $\partial H = \Sigma$, where Σ is a surface with marked points $p_1, \dots, p_n$.

Given a coloration $\hat{c}$ of the marked points, we choose $c_i - 1$ points in a small neighborhood of p_i for each i, and write P for the set of all resulting points for i from 1 to n.

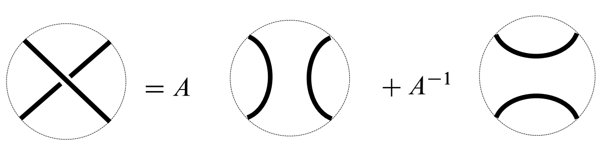


Figure 2: The first Kauffman relation. The other relation states that any trivial component is identified with $-A^2 - A^{-2}$.

We define the relative Kauffman module $K(H,\hat{c},\zeta_r)$ as the $\mathbb{C}[A^{\pm 1}]$–module generated by banded tangles in H whose intersection with Σ is the set P.

For r a positive integer, we write $\zeta_r = -e^{i\pi/(2r)}$. For any embedding j of H in $\boldsymbol{S}^3$, we define the following submodule of $K(H,\hat{c},\zeta_r)$:

$$N_r^j = \left\{ x \in K(H,\hat{c},\zeta_r) \;\middle|\; \left\langle x \middle| \bigotimes_{i=1}^{r} f_{c_i-1} \middle| y \right\rangle = 0 \text{ for all } y \in K(\boldsymbol{S}^3 \setminus \mathrm{Im}(j),\hat{c},\zeta_r) \right\},$$

where we write f_k for the k^{th} Jones–Wenzl idempotent, and $\langle x| \bigotimes_{i=1}^r f_{c_i-1}|y\rangle$ stands for the element of $K(\boldsymbol{S}^3,\zeta_r)$ obtained from x and y by pasting H with $\boldsymbol{S}^3 \setminus \mathrm{Im}(j)$, inserting the Jones–Wenzl idempotent at each marked point.

Theorem 2.1 [11] *N_r^j is in fact independent of j and of finite codimension, and we may define*

$$V_r(\Sigma,\hat{c}) = K(H,\hat{c},\zeta_r)/N_r^j.$$

With this setting, there is a simple description of the curve operator T_r^γ associated to a multicurve γ on Σ disjoint from the marked points $p_1,\dots,p_n$, or more generally to an element of $K(\Sigma,\zeta_r)$.

Indeed, we can take an element z of $K(H,\hat{c},\zeta_r)$ and stack a multicurve γ over it to obtain another element $\gamma\cdot z$ of $K(H,\hat{c},\zeta_r)$. The induced map factors through N_r^j, since for any $n \in N_r^j$ and any $z \in K(\boldsymbol{S}^3 \setminus \mathrm{Im}(j),\hat{c},\zeta_r)$, we have that $\langle \gamma\cdot n| \bigotimes_{i=1}^r f_{c_i-1}|z\rangle = \langle n| \bigotimes_{i=1}^r f_{c_i-1}|\gamma\cdot z\rangle$. Thus we have defined an endomorphism T_r^γ of $V_r(\Sigma,\hat{c})$ associated to $\gamma \in K(\Sigma,\zeta_r)$.

Furthermore, the map

$$T_r^{\cdot}\colon K(\Sigma,\zeta_r) \to \mathrm{End}(V_r(\Sigma,\hat{c})), \quad \gamma \mapsto T_r^\gamma,$$

is a morphism of algebras.

In [11] it is shown that the bracket $\langle\cdot,\cdot\rangle$ that we introduced above induces a Hermitian structure on $V_r(\Sigma,\hat{c})$.

The construction of [11] provides for each admissible coloring c a vector $\varphi_c \in V_r(\Sigma,\hat{c})$. This vector is obtained by cabling the graph Γ by a specific combination of multicurves (we will detail this construction in Section 4). Moreover, the family (φ_c) when c runs over all admissible colorings is a Hermitian basis of $V_r(\Sigma,\hat{c})$.

For a multicurve γ, the operators T_r^γ are Hermitian operators for the Hermitian structure on $V_r(\Sigma,\hat{c})$ given by [11]. The spectrum and the eigenvectors of T_r^γ are known:

First, as all components of γ are disjoint, there exists a pants decomposition of Σ by a family of curves $\mathcal{C}=\{C_e\}_{e\in E}$ such that γ can be isotoped to the union of n_e parallel copies of C_e, for some integers $n_e\in\mathbb{N}$. Then the Hermitian basis (φ_c) coming from the pants decomposition $\mathcal{C}$ is an eigenbasis of T_r^{γ}, and we have

$$T_r^{\gamma}\varphi_c=\left(\prod_{e\in E}\left(-2\cos\frac{\pi c_e}{r}\right)^{n_e}\right)\varphi_c.$$

We should take note that the spectral radius $\|T_r^{\gamma}\|$ is thus always less than $2^{n(\gamma)}$, where we write $n(\gamma)$ for the number of components of the multicurve γ.

Let

$$\mathcal{M}'(\Sigma)=\mathrm{Hom}(\pi_1(\Sigma),\mathrm{SL}_2(\mathbb{C}))/\!/\mathrm{SL}_2(\mathbb{C})$$

be the space of characters of the fundamental group of $\Sigma\setminus\{p_1,\dots,p_n\}$ in $\mathrm{SL}_2(\mathbb{C})$. This space is actually an affine algebraic variety.

Also let $\mathrm{Reg}(\mathcal{M}'(\Sigma))$ be the algebra of regular functions from $\mathcal{M}'(\Sigma)$ to $\mathbb{C}$.

The following theorem, which describes the Kauffman algebra $K(\Sigma,-1)$, will have a central role in the proof of Theorem 1.3:

Theorem 2.2 *The map*

$$\sigma\colon K(\Sigma,-1)\to\mathrm{Reg}(\mathcal{M}'(\Sigma)),\qquad \gamma\mapsto f_{\gamma}\quad \textit{such that }\ f_{\gamma}(\rho)=-\mathrm{Tr}(\rho(\gamma)),$$

is an isomorphism of algebras.

This theorem follows from the work of various authors. Bullock [13] and Brumfiel and Hilden [12] first independently proved that the map from $K(\Sigma,-1)$ to $\mathcal{M}'(\Sigma)$ is surjective and has the nilradical of $K(\Sigma,-1)$ as kernel. It was proved later by Przytycki and Sikora [23] and independently by Charles and Marché [14] that the algebras $K(\Sigma,-1)$ are indeed reduced, which concluded the proof of Theorem 2.2.

Finally, we end this preliminary section with a formula for products of elements of the Kauffman algebra at $-e^{i\pi\hbar/2}$ to first order in $\hbar$. We recall that $\mathcal{M}'(\Sigma)$ is a Poisson manifold for the Poisson structure given in [16]. This Poisson structure depends on a choice of normalization of the symplectic structure on $\mathcal{M}(\Sigma)$. We normalize the symplectic form ω as the symplectic reduction of the form $\omega(\alpha,\beta)=(1/2\pi)\int_{\Sigma}\mathrm{Tr}(\alpha\wedge\beta)$ for $\alpha,\ \beta\in\Omega^1(\Sigma,\mathrm{su}_2)$. Since, by the previous theorem, it is possible to link the product of elements of $K(\Sigma,-1)$ with products of trace functions on $\mathcal{M}'(\Sigma)$, the work of Goldman [18] and Turaev [26] gives a way to think of the first order in $\hbar$ of a product of elements in $K(\Sigma,-e^{i\pi\hbar/2})$ as a Poisson bracket of trace functions.

Notice that from the fact that Kauffman algebras have the set of multicurves as a basis, as linear spaces $K(\Sigma, -e^{i\pi\hbar/2})$ is isomorphic to $K(\Sigma, -1)[\![\hbar]\!]$. This last space is isomorphic to a subspace of $\mathrm{Reg}(\mathcal{M}'(\Sigma))[\![\hbar]\!]$ via the map σ of Theorem 2.2.

Theorem 2.3 [26] *Let γ and δ be multicurves, viewed as elements of $K(\Sigma, -e^{i\pi\hbar/2})$. We have that*

$$\gamma \cdot \delta = f_\gamma f_\delta + \frac{\hbar}{i}\{f_\gamma, f_\delta\} + o(\hbar).$$

This result is due to the work of Goldman and Turaev. First Goldman [18] was able to compute the Poisson bracket of the trace functions of two simple closed curves as the sum of other trace functions. Then Turaev [26] was able to identify the terms in Goldman formula for the Poisson bracket with the order 1 terms of the product in the Kauffman algebra.

3 Algebraic properties of ψ–symbols

3.1 Some remarks on the intersection algebra

In this section, we fix a surface Σ with marked points $p_1, \dots, p_n$, with a pants decomposition $\mathcal{C} = \{C_e\}_{e \in E}$ of Σ and a compatible trivalent banded graph Γ drawn on Σ.

We see from Figure 1 that $\mathcal{C}$ and Γ give us a cell decomposition of Σ into a bunch of hexagons, their sides being the boundary components of Γ and segments of the curves C_e. For each $e \in E$, we name by C_e' (resp. C_e'') the segment $\Gamma \cap C_e$ (resp. $C_e \setminus \mathrm{Int}(C_e \cap \Gamma)$); see Figure 1.

We remark that the cocycle $\overline{c}$ of $H^1(\Sigma, \mathbb{Z}/2)$ can then be computed as

$$\overline{c}(\gamma) = \prod_{e \in E} (-1)^{(c_e - 1)(C_e'^*(\gamma) + C_e''^*(\gamma))}.$$

In this formula, $C_e'^*$ (resp. $C_e''^*$) is the cellular cochain dual to ${C_e}'$ (resp. ${C_e}''$). We can directly check from the formula that $\overline{c}$ is a cocycle, as its value on the boundary of each hexagon is of the form $(-1)^{c_e + c_f + c_g - 1}$ for e, f and g three adjacent edges, which equals 1 as c is an admissible color. Also it is easy to see that the formula gives exactly the intersection number $L_c \cap p_*(\gamma)$.

Now, for α and β in B, the image of $\pi\colon H_1(\Gamma, \mathbb{Z}/2) \to H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$, we write $\langle \alpha, \beta \rangle = \widetilde{\alpha} \cap \beta$, where $\pi(\widetilde{\alpha}) = \alpha$. Recall that we defined the *intersection algebra* A_Γ as

$$A_\Gamma = \bigoplus_{\alpha \in B} \mathbb{C} \cdot [\alpha],$$

with the product structure given by $[\gamma]\cdot[\delta]=(-1)^{\langle\gamma,\delta\rangle}[\gamma+\delta]$. It is not clear at this point that A_Γ is an algebra, and not even that it is well defined. This comes from the following lemma:

Lemma 3.1 *The form*

$$\langle\,,\rangle\colon B\times B\to\mathbb{Z}/2$$

given by $\langle\alpha,\beta\rangle=\tilde{\alpha}\cap\beta$ does not depend on the choice of a lift $\pi(\tilde{\alpha})=\alpha$ and is symmetric and bilinear.

Proof Indeed, two lifts of α differ by an element of $H_1(\partial\Gamma,\mathbb{Z}/2)$. Furthermore, any element γ of $H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)$ can be seen as a linear combination of closed curves and curves with extremities in $\partial\Gamma$, and $\gamma\in B$ if and only if its number of extremities in each component of $\partial\Gamma$ is even. Thus the intersection of an element of $H_1(\partial\Gamma,\mathbb{Z}/2)$ with any element of B vanishes, and the form $\langle\,,\rangle$ is independent of the choice of lift.

Actually, this shows that we can think of B as the quotient of $H_1(\Gamma,\mathbb{Z}/2)$ by the kernel of the intersection form on $H_1(\Gamma,\mathbb{Z}/2)$ and $\langle\,,\rangle$ as the corresponding quotient form.

The bilinearity of the form $\langle\,,\rangle$ is then evident.

Finally we show that the form is symmetric. Given lifts $\tilde{\alpha}$ and $\tilde{\beta}$ to $H_1(\Gamma,\mathbb{Z}/2)$ of two elements α and β in B, $\langle\alpha,\beta\rangle=\tilde{\alpha}\cap\beta$ is also the intersection number mod 2 of $\tilde{\alpha}$ and $\tilde{\beta}$, so it is symmetric. □

From the lemma we get that the product on A_Γ is well-defined, associative and commutative, so A_Γ is a commutative $\mathbb{C}$–algebra of dimension 2^d, where d is the dimension of B. This dimension can be computed using the exact sequence

$$H_1(\Gamma,\mathbb{Z}/2)\to H_1(\Gamma,\partial\Gamma\mathbb{Z}/2)\xrightarrow{\delta}H_0(\partial\Gamma,\mathbb{Z}/2)\to H_0(\Gamma,\mathbb{Z}/2)\to 0.$$

We have $B=\operatorname{Ker}\delta$ and $\dim(\operatorname{Ker}\delta)+\operatorname{rk}(\delta)=g$, where g is the genus of Γ, and $\operatorname{rk}(\delta)=b-1$, where b is the number of boundary components of Γ. Thus the dimension of B is $g-b+1$.

Note that when Γ can be embedded in the plane this dimension is 0 and $A_\Gamma=\mathbb{C}$.

As a finite-dimensional commutative $\mathbb{C}$–algebra, A_Γ is isomorphic to the algebra $\mathbb{C}^l$, where $l=\dim(A_\Gamma)=\operatorname{Card}(\widehat{A}_\Gamma)$ and we recall that $\widehat{A}_\Gamma$ is the (finite) set of algebra morphisms from A_Γ to $\mathbb{C}$. The isomorphism is given by

$$\alpha\mapsto(\chi(\alpha))_{\chi\in\widehat{A}_\Gamma}\quad\text{for }\alpha\in A_\Gamma.$$

An element χ of $\widehat{A}_\gamma$ must send each $[\alpha]$ with $\alpha \in B$ to some $(-1)^{q(\alpha)}$, with the conditions that $q(\alpha+\beta)-q(\alpha)-q(\beta) = \langle \alpha, \beta \rangle \pmod 2$. Thus $\widehat{A}_\Gamma$ is in bijective correspondence with the set of "relative spin-structures" on $(\Gamma, \partial\Gamma)$.

We end this section with the following lemma, providing a computation of products in A_Γ based on the cellular decomposition on Σ into hexagons:

Lemma 3.2 *Let γ and δ be two simple closed curves on Σ, and set*

$$i(\gamma,\delta) = \prod_{e \in E} (-1)^{I_e^\delta (C_e'^*(\gamma) + C_e''^*(\gamma))}.$$

Then $i(\gamma,\delta) = \langle p_*(\gamma), p_*(\delta)\rangle$.

Proof Let γ and δ be two curves on Σ. After an isotopy of $p(\gamma)$ and $p(\delta)$ in Γ we can arrange that $p(\delta)$ lies in the interior of Γ, and $p(\gamma)$ follows the edges of the cell decomposition of Γ. Then the intersection points lie only in the curves $p(C_e) = L_e$. The number of intersection points of $p(\gamma)$ and $p(\delta)$ in L_e is congruent modulo 2 to $\sharp(p(\delta) \cap L_e) L_e^*(p_*(\gamma))$, where L_e^* is the dual to the cell L_e.

But $L_e^*(p_*(\gamma)) = C_e'^*(\gamma) + C_e''^*(\gamma)$ and $\sharp(p(\delta) \cap L_e) = \sharp(\delta \cap C_e) \pmod 2$, hence the formula for $i(\widetilde{\gamma}, \widetilde{\delta})$ computes the number of intersection points of γ and δ modulo 2, that is, $\langle p_*(\gamma), p_*(\delta)\rangle$. □

3.2 The multiplicativity property

In this section, we will temporarily assume that Theorem 1.1 holds. We can then define ψ–symbols, and we will show here that these ψ–symbol have a property of compatibility with the product in Kauffman modules. From this algebraic property alone and the theorem of Bullock, the ψ–symbols are almost constrained to have the form predicted by Theorem 1.3. Theorem 1.1 will be proved in Section 4.1 without using any of the results in this section.

For a fixed $(\tau, \hbar, \theta)$, the definition of the ψ–symbol only introduces $\gamma \mapsto \sigma^\gamma(\tau,\hbar,\theta)$ as a map from multicurves to A_Γ. We extend it by multilinearity to obtain a map

$$\sigma(\tau,\hbar,\theta)\colon K(\Sigma, -e^{i\pi\hbar/2}) \to A_\Gamma[\![\hbar]\!],$$

as $K(\Sigma, -e^{i\pi\hbar/2})$ is spanned by multicurves.

The proof of Theorem 1.3, giving an asymptotic formula for the ψ–symbol, will be the goal of Sections 5 and 6. It will rely heavily on the following property of the ψ–symbol, which explains its compatibility with the product in $K(\Sigma, -e^{i\pi\hbar/2})$:

Proposition 3.3 *Let γ and δ be two multicurves on Σ. Then we have the asymptotic expression*

$$\sigma^{\gamma\cdot\delta}(\tau,\hbar,\theta)=\left(\sigma^{\gamma}(\tau,\hbar,\theta)\sigma^{\delta}(\tau,\hbar,\theta)+\frac{\hbar}{i}\sum_{e}\partial_{\tau_e}\sigma^{\gamma}(\tau,\hbar\theta)\,\partial_{\theta_e}\sigma^{\delta}(\tau,\hbar,\theta)\right)+o(\hbar).$$

This expression is similar to the composition of symbols of Toeplitz operators. This is not a surprise, as curve operators can be approximated at order 1 by Toeplitz operators, by [6]. Theorem 8 of [6] gives the order 1 of the symbols of the composition of two such operators. It could again be possible to derive this result by degenerating the complex structure to a pair of pants decomposition.

A version of this proposition appeared already in [21] for the four-holed sphere and the pointed torus, but they worked with another definition of the ψ–symbol, which took values in $\mathbb{C}$, whereas in our definition, the ψ–symbol takes values in A_Γ.

We can however extract $\mathbb{C}$–valued functions from the ψ–symbol. As A_Γ is isomorphic to $\mathbb{C}^l$, we denote the components of the principal symbol $\sigma^\gamma(\tau,0,\theta)$ by $\sigma^\gamma_\chi(\tau,\theta)=\chi(\sigma^\gamma(\tau,0,\theta))$ for every $\chi\in\widehat{A}_\Gamma$.

Proof of Proposition 3.3 We fix $r>0$ and we take two multicurves γ and δ on Σ. The two functions appearing in the equality are smooth functions on a neighborhood of $U\times\{0\}$ in $U\times[0,1]$. We remark that any point of U can be approximated by a sequence c_r/r with $c_r\in U_r$. Hence it suffice to show that they have the same asymptotic expansion at order 1 on sequences $(c_r/r,\theta,1/r)$ where $c_r/r\to x\in U$. According to Theorem 1.1, writing $\tau=c_r/r$ and $\hbar=1/r$, the matrix coefficients of the operator T^γ_r can be written as

$$T^\gamma_r\varphi_c=\overline{c}(\gamma)\sum_{k\colon E\to\mathbb{Z}}F^\gamma_k(\tau,\hbar)\varphi_{c+k},$$

with the F^γ_k being smooth functions on V_γ such that $F^\gamma_k=0$ as soon as there is some $e\in E$ such that $|k_e|>I^\gamma_e$ or $k_e\not\equiv I^\gamma_e\pmod 2$.

As $\gamma\in K(\Sigma,-e^{i\pi/(2r)})\to T^\gamma_r\in\operatorname{End}(V_r(\Sigma))$ is an morphism of algebras, we have

$$T^{\gamma\cdot\delta}_r\varphi_c=T^\gamma_r(T^\delta_r\varphi_c)$$

and, from the above expression of the matrix coefficients, we get

$$\begin{aligned}T^{\gamma\cdot\delta}_r\varphi_c&=\sum_{m\colon E\to\mathbb{Z}}\left(\sum_{k+l=m}F^\gamma_l(\tau+k\hbar,\hbar)F^\delta_k(\tau,\hbar)\overline{c}(\delta)\overline{c+k}(\gamma)\right)\varphi_{c+m}\\&=\overline{c}(\gamma)\overline{c}(\delta)i(\gamma,\delta)\sum_{m\colon E\to\mathbb{Z}}\left(\sum_{k+l=m}F^\gamma_l(\tau+k\hbar,\hbar)F^\delta_k(\tau,\hbar)\right)\varphi_{c+m}.\end{aligned}$$

To obtain the second equality, note that $\overline{c+k}(\gamma) = \overline{c}(\gamma)\overline{k}(\gamma)$ and observe that if there exists e such that $k_e \neq I_e^\delta \pmod 2$ then, by Theorem 1.1, F_k^δ is 0.

However, if $k_e = I_e^\delta \pmod 2$ for all $e \in E$ then $\overline{k}(\gamma) = \prod_{e\in E}(-1)^{I_e^\delta(C_e'^*(\gamma)+C_e''^*(\gamma))} = i(\gamma,\delta)$ is independent of k. Hence we can factor $\overline{k}(\gamma)$ out of the sum.

Now, as $K(\Sigma, -e^{i\pi\hbar/2})$ is generated by multicurves, we can write $\gamma\cdot\delta = \sum_\lambda f_\lambda(\hbar)\lambda$, and, in this sum, $f_\lambda \neq 0$ only when $[\lambda] = [\gamma]+[\delta] \in H_1(\Sigma, \mathbb{Z}/2)$, according to the Kauffman relations. Thus we have $\overline{c}(\lambda) = \overline{c}(\gamma)\overline{c}(\delta)$. We can write another formula for the curve operator of the product:

$$T_r^{\gamma\cdot\delta}\varphi_c = \sum_m \biggl(\sum_\lambda \overline{c}(\lambda) f_\lambda(\hbar) F_m^\lambda(\tau,\hbar)\biggr)\varphi_{c+m}.$$

So, identifying coefficients in the two formulae, we get

$$\sum_\lambda f_\lambda(\hbar)F_m^\lambda(\tau,\hbar) = \biggl(\sum_{k+l=m} F_l^\gamma(\tau+k\hbar,\hbar)F_k^\delta(\tau,\hbar)\biggr) i(\gamma,\delta).$$

Now, recall that we defined the ψ–symbol of an arbitrary element of $K(\Sigma,-e^{i\pi\hbar/2})$ by extending linearly the formula for multicurves. Thus, we have

$$\sigma^{\gamma\cdot\delta}(\tau,\hbar,\theta) = \sum_m\sum_\lambda f_\lambda(\hbar)F_m^\lambda(\tau,\hbar)e^{im\theta}[\lambda],$$

recalling that $[\lambda]=[\gamma]+[\delta]$ and using the previous identity of coefficients

$$\sigma^{\gamma\cdot\delta}(\tau,\hbar,\theta) = i(\gamma,\delta)\sum_m\biggl(\sum_{k+l=m} F_l^\gamma(\tau+k\hbar,\hbar)F_k^\delta(\tau,\hbar)\biggr)e^{im\theta}[\gamma+\delta].$$

Now the Taylor expansion at order 1 in $\hbar$ of F_l^γ near $(\tau,\hbar)$ in the first variable gives

$$\begin{aligned} F_l^\gamma(\tau+k\hbar,\hbar) &= F_l^\gamma(\tau,\hbar) + \hbar\sum_{e\in E} k_e\frac{\partial}{\partial\tau_e}F_l^\gamma(\tau,\hbar) + o(\hbar) \\ &= F_l^\gamma(\tau,\hbar) + \hbar\sum_{e\in E} k_e\frac{\partial}{\partial\tau_e}F_l^\gamma(\tau,0) + o(\hbar). \end{aligned}$$

Substituting into the previous equation gives us that

$$\begin{aligned} \sigma^{\gamma\cdot\delta}(\tau,\hbar,\theta) = i(\gamma,\delta)\sum_m\biggl(\sum_{k+l=m}\biggl(F_l^\gamma(\tau,\hbar)+\hbar\sum_{e\in E}k_e\frac{\partial}{\partial\tau_e}F_l^\gamma(\tau,\hbar)\biggr) \\ \times e^{il\theta}F_k^\delta(\tau,\hbar)e^{ik\theta}\biggr)[\gamma+\delta]+o(\hbar) \end{aligned}$$

$$= i(\gamma,\delta)\langle p_*(\gamma), p_*(\delta)\rangle \biggl(\sigma^\gamma(\tau,\hbar,\theta)\sigma^\delta(\tau,\hbar,\theta) + \frac{\hbar}{i}\sum_{e\in E} \partial_{\tau_e}\sigma^\gamma(\tau,\hbar,\theta)\,\partial_{\theta_e}\sigma^\delta(\tau,\hbar,\theta)\biggr) + o(\hbar).$$

To obtain the second equality recall that $[\gamma][\delta] = \langle p_*(\gamma), p_*(\delta)\rangle[\gamma+\delta]$ in A_Γ. From Lemma 3.2 we have that $i(\gamma,\delta) = \langle p_*(\gamma), p_*(\delta)\rangle$, which completes the proof. □

According to this proposition, the principal symbol $\sigma^{\cdot}(\tau,0,\theta)\colon K(\Sigma,-1)\to A_\Gamma$ is a morphism of algebras. Furthermore, the components $\sigma_\chi(\tau,\theta) = \chi(\sigma(\tau,0,\theta)$ are algebra morphisms from $K(\Sigma,-1)$ to $\mathbb{C}$.

Using the theorem of Bullock, we will show in Section 5.1 that these morphisms have the form $f\mapsto f(R_\chi)$, $f\in \mathrm{Reg}(\mathcal{M}'(\Sigma))$, for some representations R_χ of $\pi_1(\Sigma\setminus\{p_1,\dots,p_n\})$.

Identifying precisely the representations R_χ will come from checking the special values of the ψ–symbol on the curves C_e.

As for the computation of the first-order term, we will proceed in Section 6 in a similar fashion: first we will show, using only Proposition 3.3, that this term is related to derivations of algebras $K(\Sigma,-1)\to A$, then, by studying the values of the ψ–symbol on the curves C_e and on another family of curves D_e, we will show the first-order term is indeed given by the formula in Theorem 1.3.

4 Computations of curve operators using fusion rules

This section is devoted to the skein theory computations that will be needed in order to prove Theorem 1.1. We describe the general form of the matrix coefficients of the curve operators, and give examples of explicit computations of the coefficients F_k^γ and the ψ–symbol σ^γ for some curves γ.

4.1 Fusion rules in a pants decomposition

In this subsection, we will work with a fixed closed oriented surface Σ, along with a pants decomposition by a family of curves $\mathcal{C} = \{C_e\}_{e\in E}$. We can consider $n_e\ge 1$ parallel copies $(C_e^k)_{1\le k\le n_e}$ of the curves C_e such that the curves C_e^k cut the surface Σ into a collection of pants $\{P_s\}_{s\in S}$ and annuli $\{A_e^k \mid e\in E,\ 1\le k\le n_e-1\}$.

We recall that to this pants decomposition is associated a Hermitian basis φ_c of $V_r(\Sigma)$, of which we will recall the construction:

Let Γ be a banded trivalent graph compatible with the pants decomposition $\mathcal{C}$ of Σ as in Section 2. We recall that Γ is viewed as drawn on Σ. Given an admissible coloring $c\colon E \to C_r$, we define $\psi_c \in K(\Sigma; \hat{c}; \zeta_r)$ as follows:

- Replace each edge e of Γ by $c_e - 1$ parallel copies of e lying on Σ.
- Insert in the middle of each edge the idempotent f_{c_e-1}, where we recall that f_k is the k^{th} Jones–Wenzl idempotent.
- In the neighborhood of each trivalent vertex, join the three sets of lines in Σ in the unique possible way avoiding crossings.

This family of vectors is actually an orthogonal basis of $V_r(\Sigma, c)$ for a natural Hermitian structure defined in [11], which we do not recall here. We refer to [11, Theorem 4.11] for the proof and the formula

$$\|\psi_c\|^2 = \left(\frac{2}{r}\right)^{\chi(\Gamma)/2} \frac{\prod_P \langle c_P^1, c_P^2, c_P^3 \rangle}{\prod_e \langle c_e \rangle}. \tag{1}$$

Here the first product is over all vertices P corresponding to pants of the pants decomposition, the second over the edges e of the graph Γ. We write $\langle n \rangle$ for $\sin(\pi n/r)$; $\langle n \rangle!$ for $\prod_{i=1}^n \langle i \rangle$; c_P^1, c_P^2, and c_P^3 for the colors of the 3 edges adjacent to P; and we also set

$$\langle a, b, c \rangle = \frac{\left\langle \frac{a+b+c-1}{2} \right\rangle! \left\langle \frac{a+b-c-1}{2} \right\rangle! \left\langle \frac{a-b+c-1}{2} \right\rangle! \left\langle \frac{b+c-a-1}{2} \right\rangle!}{\langle a-1 \rangle! \langle b-1 \rangle! \langle c-1 \rangle!}.$$

As we will work with TQFT vectors locally, inside a pants of the pants decomposition for example, we will need to give a local version of this norm. Notice that if we forget the global factor $(2/r)^{\chi(\Gamma)/2}$ in the norm, we will not change the matrix coefficients of the curve operators T_r^γ.

Also, after applying fusion rules, we may get trivalent graphs with vertices other than those in the graph associated to the decomposition. We say then that a vertex is internal if it is trivalent or univalent and associated to a marked point, and that it is external otherwise. Then, we will define the square of the norm of a trivalent graph as

$$\frac{\prod_P \langle c_P^1, c_P^2, c_P^3 \rangle}{\prod_{e \in E_2} \langle c_e \rangle \prod_{e \in E_1} \langle c_e \rangle^{1/2}},$$

where the products in the denominator are over E_2, the set of edges adjacent to 2 internal vertices, and E_1 the set of edges adjacent to 1 internal vertex and 1 external vertex. The other edges bear no contribution to the norm. With this definition, if we paste pieces of colored graph to get the graph Γ, we obtain the previous norm as the product of the norm of the pieces.

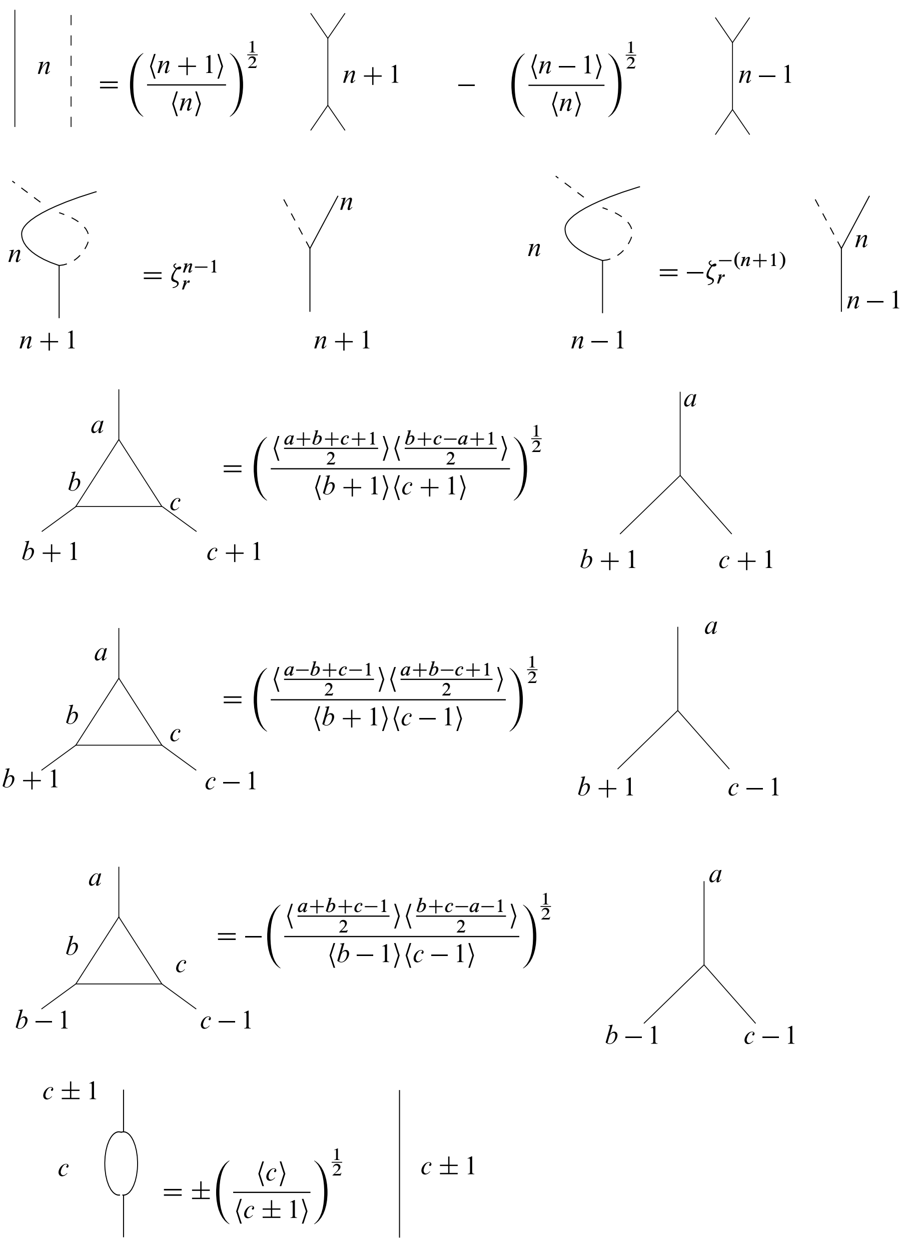


Figure 3: Fusion rules. These "normalized" fusion rules allow us to simplify the union of a colored banded graph and a curve colored by 2. The dotted edges are colored by 2. The first rule allows to merge an edge colored by 2 with another one. The second line consists of the "half-twist formulae" of [22]. When all curves have been merged with the graph, the 3^{rd}, 4^{th} and 5^{th} lines can be used to remove trigons, and the last rule to remove bigons.

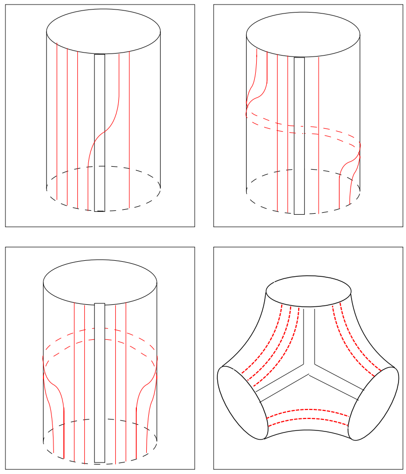


Figure 4: Dehn presentation of multicurves

With this setting, we give a normalized version of the fusion rules in TQFT. The fusion rules derived in [22], give a way to compute the image of the vector φ_c under the curves operators. We list the fusion rules that we will need in Figure 3; our version differs from the rules in [22], as we express them with the normalized vectors φ_c instead of the vectors ψ_c from [22].

We will perform the computations by using the fusion rules only locally, that is only inside of a pair of pants of the pants decomposition, or inside an annulus in the neighborhood of one of the curves C_e.

Indeed, for γ a multicurve, by a classification provided by Dehn, we can isotope γ so that the intersection of γ with each pants P_s of the decomposition looks like the 4^{th} picture of Figure 4, and the intersection with each of the annuli A_e^k looks like one of the first three pictures of Figure 4.

Furthermore, in this isotopy class, the intersection of γ with each C_e is the smallest in the isotopy class of γ. We refer to [15, Section 4.3] for this classification.

Now, we do the computations in two steps:

First, we use fusion rules to reduce each type of piece to elements corresponding to the intersection of the graph Γ in a pants or annulus with a certain coloring, glued with "candlesticks".

A *candlestick* is an element of the TQFT vector space of an annulus that is the normalized vector associated to a banded trivalent graph in an annulus, consisting of a central edge joining the boundary components (with no twist), colored by $n \in \mathcal{C}_r$ on the bottom

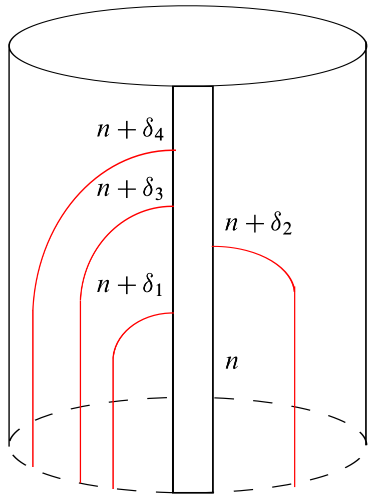
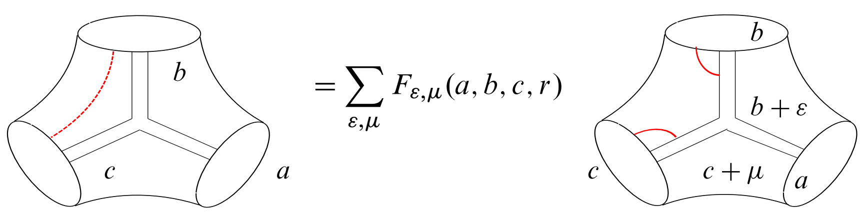


Figure 5: A candlestick $C(n,\varepsilon,\theta)$ with 4 legs. We denote by $\delta_i=\sum_{j=1}^{i}\varepsilon_j$ the partial sums of the color shifts ε_j. Notice that the legs can go alternatively to the left or to the right of the central edge.

component, a collection of legs colored by 2, joining the central edge and the bottom component, as in Figure 5.

The data that defines a candlestick with k legs $C(n,\varepsilon,\Theta)$ is the color $n\in\mathcal{C}_r$ of the central edge at the bottom, the order Θ in which the legs join the central edge, and the shifts of the color of the central edge $(\varepsilon_i)_{i=1\dots k}$ when we pass each vertex corresponding to a leg.

Reduction of the different pieces Simple computations using fusion rules give us the following formulae when the pants or the annuli contain only one curve:



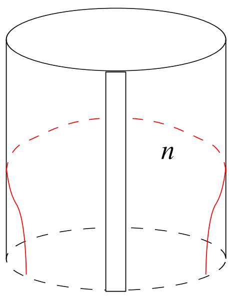
where we set

$$F_{+,+}(a,b,c,r)=\left(\frac{\left\langle\frac{a+b+c+1}{2}\right\rangle\left\langle\frac{b+c-a+1}{2}\right\rangle}{\langle b\rangle\langle c\rangle}\right)^{\frac{1}{2}},$$

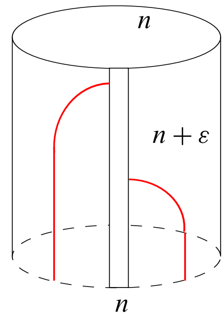
$$F_{+,-}(a,b,c,r)=F_{-,+}(a,c,b,r)=-\left(\frac{\left\langle\frac{a-b+c-1}{2}\right\rangle\left\langle\frac{a+b-c-1}{2}\right\rangle}{\langle b\rangle\langle c\rangle}\right)^{\frac{1}{2}},$$

$$F_{-,-}(a,b,c,r) = -\left(\frac{\left\langle\frac{a+b+c-1}{2}\right\rangle\left\langle\frac{b+c-a-1}{2}\right\rangle}{\langle b\rangle\langle c\rangle}\right)^{\frac{1}{2}};$$

next,



$$= \sum_{\varepsilon} G_{\varepsilon}(n,r)$$

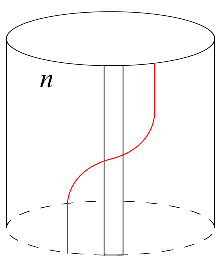


where

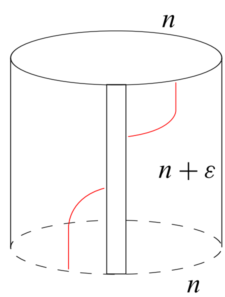
$$G_{+}(n,r) = (-1)^{n+1} e^{-i\pi(n-1)/(2r)} \left(\frac{\langle n+1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}},$$

$$G_{-}(n,r) = (-1)^{n+1} e^{i\pi(n+1)/(2r)} \left(\frac{\langle n-1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}};$$

third,



$$= \sum_{\varepsilon} H_{\varepsilon}(n,r)$$

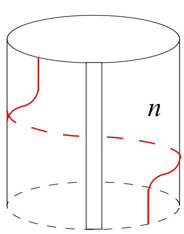


where

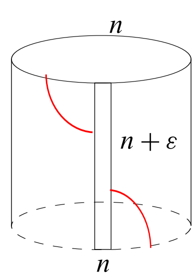
$$H_{+}(n,r) = (-1)^{n+1} e^{i\pi(n-1)/(2r)} \left(\frac{\langle n+1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}},$$

$$H_{-}(n,r) = (-1)^{n+1} e^{-i\pi(n+1)/(2r)} \left(\frac{\langle n-1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}};$$

and lastly



$$= \sum_{\varepsilon} L_{\varepsilon}(n,r)$$



where

$$L_+(n,r) = (-1)^{n+1} e^{i\pi(n+2)/(2r)} \left(\frac{\langle n+1 \rangle}{\langle n \rangle} \right)^{\frac{1}{2}},$$

$$L_-(n,r) = (-1)^{n+1} e^{-i\pi(n-2)/(2r)} \left(\frac{\langle n-1 \rangle}{\langle n \rangle} \right)^{\frac{1}{2}}.$$

All these coefficients are of the required form $\overline{c}(\gamma)F(c/r, 1/r)$ for some smooth function F defined on V_γ.

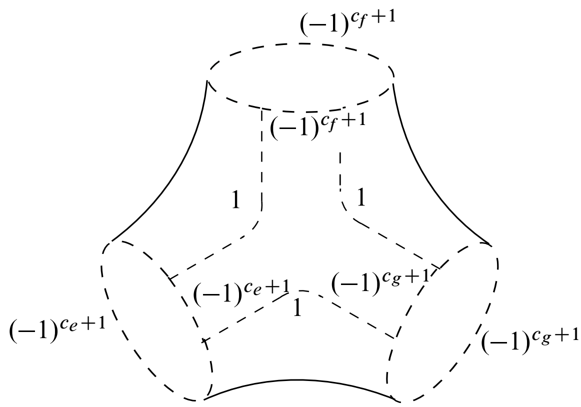
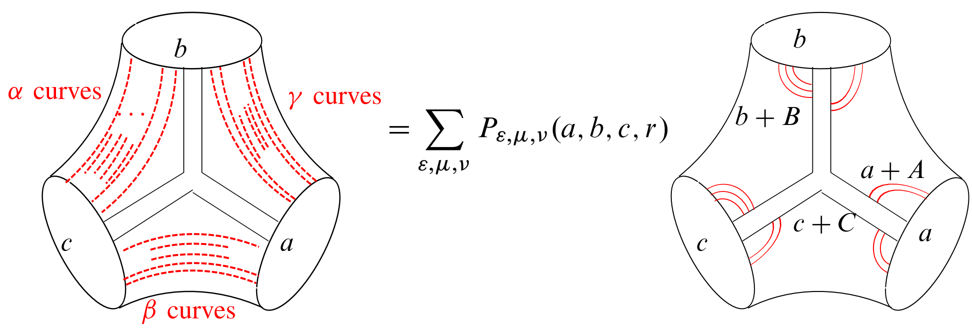


Figure 6: The cocycle $\overline{c}$ on the pants bounded by the curves C_e, C_f and C_g

If we have many curves in a pants or annulus, we only need to choose an order to make the fusions, and apply the latter formulae. For example, in the case of the pants, we obtain:



where we use the notation $A = \sum_{i=1}^{\beta+\gamma} \varepsilon_i$, $B = \sum_{j=1}^{\alpha+\gamma} \mu_j$ and $C = \sum_{k=1}^{\alpha+\beta} \nu_k$.

Here we have first used fusion on the α curves that go from C_b to C_c, then the β curves that run from C_a to C_c, and finally the γ curves from C_a to C_c. With this order for the fusions, the coefficients $P_{\varepsilon,\mu,\nu}(a,b,c,r)$ are products of three factors corresponding to each series of fusions:

$$F_{\mu_1,\nu_1}(a,b,c,r)F_{\mu_2,\nu_2}(a,b+\mu_1,c+\nu_1,r)\cdots F_{\mu_\alpha,\nu_\alpha}\left(a,b+\sum_{i=1}^{\alpha-1}\mu_i,c+\sum_{i=1}^{\alpha-1}\nu_i,r\right),$$

$$F_{\nu_{\alpha+1},\varepsilon_1}\left(b+\sum_{i=1}^{\alpha}\mu_i,a,c+\sum_{i=1}^{\alpha}\nu_i,r\right) \cdots F_{\nu_{\alpha+\beta},\varepsilon_\beta}\left(b+\sum_{i=1}^{\alpha}\mu_i,a+\sum_{i=1}^{\beta-1}\varepsilon_i,c+\sum_{i=1}^{\alpha+\beta-1}\nu_i,r\right),$$

$$F_{\mu_{\alpha+1},\varepsilon_{\beta+1}}\left(c+\sum\nu,b+\sum_{i=1}^{\alpha}\mu_i,a+\sum_{i=1}^{\beta}\varepsilon_i,r\right) \cdots F_{\mu_{\alpha+\gamma},\varepsilon_{\beta+\gamma}}\left(c+\sum\nu,b+\sum_{i=1}^{\alpha+\gamma-1}\mu_i,a+\sum_{i=1}^{\beta+\gamma-1}\varepsilon_i,r\right).$$

Notice that, at every step of the fusion, the shifts in the color c_e are sums of ± 1 terms, one term for each arc intersecting C_e that has been merged with Γ. Thus the coefficients $P_{\varepsilon,\mu,\nu}$ are defined and smooth on the required domain $V_\gamma = \{(\tau,\hbar) \mid \tau_e \pm I_e^\gamma \hbar \in U\}$. Furthermore, in the end the shift of c_e is no greater than the number of curves that intersect C_e and of the same parity as this number.

We now only need to explain what happens when we glue together two candlesticks.

First, note that we can only paste candlesticks with the same number of legs, and the same bottom color n. Moreover, if we paste two candlesticks $C(n,\varepsilon,\Theta)$ and $C(n,\mu,\Theta')$ with $\sum_j \mu_j \neq \sum_i \varepsilon_i$, then we always obtain 0 (as the vector space $V_r(\Sigma)$ of a sphere Σ with two points marked by different colors is 0).

Proposition 4.1 *The gluing of candlesticks $C(n,\varepsilon,\Theta)$ and $C(n,\mu,\Theta')$ with k legs with $\sum_{i=1}^k \varepsilon_i = \sum_{j=1}^k \mu_j$ is proportional to a band colored by $n+\sum \varepsilon_i$ joining the two boundary components of the annulus with no twist, the proportionality constant being $G(n/r, 1/r)$, where G is a smooth function on $\{(\tau,\hbar) \mid \tau \pm k\hbar \in (0,1)\}$.*

We should point out that, in this proposition, the function G depends on Θ, Θ', ε and μ.

Proof We prove this proposition by induction on the number of legs of the candlestick. If we paste two candlesticks with only one leg, this is direct from the fusion rule eliminating bigons (see Figure 3), as it only produces a factor $(\langle c\pm 1\rangle/\langle c\rangle)^{1/2}$. Now,

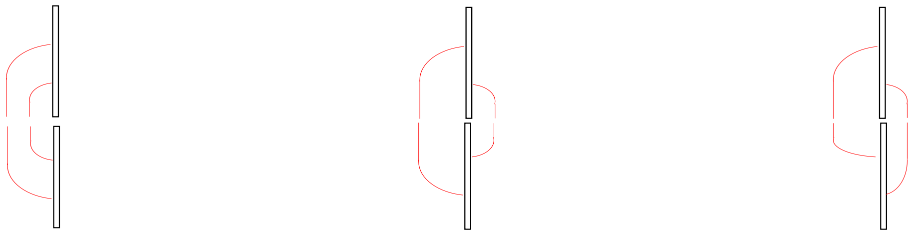
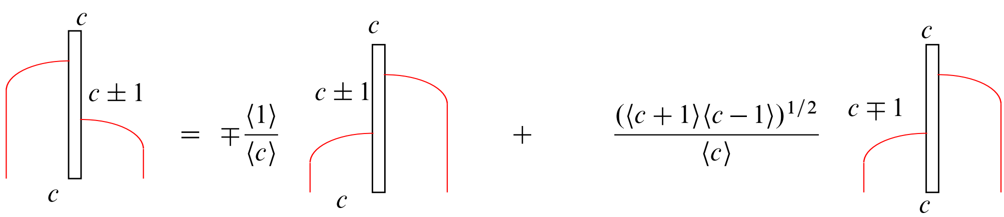
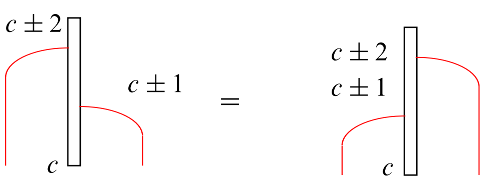


Figure 7

if $n = 2$, the only delicate case is when the legs of the two parts are positioned as in Figure 7(c).

Indeed, in cases (a) and (b), we can simply eliminate two bigons. For (c), we use the following switching legs formulae:





To get such formulae, we have to verify that gluing the left-hand side or the right-hand side with a two-legs candlestick on the bottom, with any color shifts, we get the same result after using the fusion rules for bigons and triangles elimination. This is a straightforward computation, so we will omit it here.

This shows Proposition 4.1 for $k \leq 2$.

Now, suppose we glue two candlesticks with $k + 1$ legs. We have two cases:

In Figure 8 (left), the upper leg of the upper candlestick and the bottom leg of the bottom candlestick both go to the right (or both to the left); the gluing is obtained by gluing two candlesticks with k legs, then suppressing a bigon. The factor we get is of

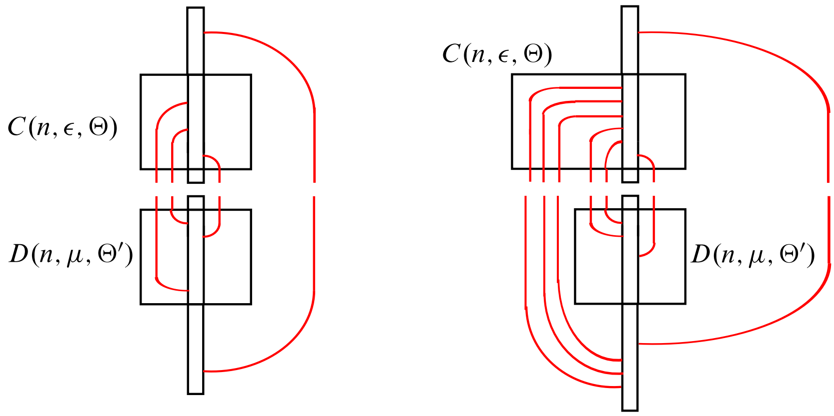


Figure 8: The two cases of pasting candlesticks with k legs

the form

$$G\Big(\frac{n}{r},\frac{1}{r}\Big)\Bigg(\frac{\big\langle n+\sum_{i=1}^{k+1}\varepsilon_i\big\rangle}{\big\langle n+\sum_{i=1}^{k}\varepsilon_i\big\rangle}\Bigg)^{\frac{1}{2}},$$

the factor $G(n/r,1/r)$ coming from k–leg candlestick elimination, and the other factor from the bigon elimination rule. It is indeed a function of $(n/r,1/r)$ that is smooth on the domain we claimed.

In Figure 8 (right), the upper leg of the upper part and the bottom leg of the bottom part go to different sides. We apply a sequence of switching legs formulae until the leg connected to the upper leg of the candlestick is the bottom leg of the bottom candlestick. Each of these operations yields a smooth function on V_γ as a factor; this comes from the switching legs formulae and the fact that all intermediate colors on the central edge are of the form $n+\sum_{i=1}^{j}\varepsilon_i$, with $j\le I_e^\gamma$. Then we are back to the former case. □

4.2 Examples of the ψ–symbol

We derive expressions of the ψ–symbol for two families of curves on Σ: the first family consists of the curves C_e of the pants decomposition itself, and the other of curves D_e, $e\in E$, that are in some sense dual to the curves C_e. The D_e are defined this way: if e is an internal edge that joins a vertex to itself, then D_e is a loop parallel to e. If e joins two different vertices, then D_e consists of two arcs parallel to e that we close into a loop as in Figure 9.

Note that C_e and D_f intersect each other if and only if $e=f$, and in this case they intersect once or twice. Finally, the classes in $H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)$ represented by $p_*(C_e)$ and by $p_*(D_e)$ are all zero. Note that in the case where D_e and C_e have one

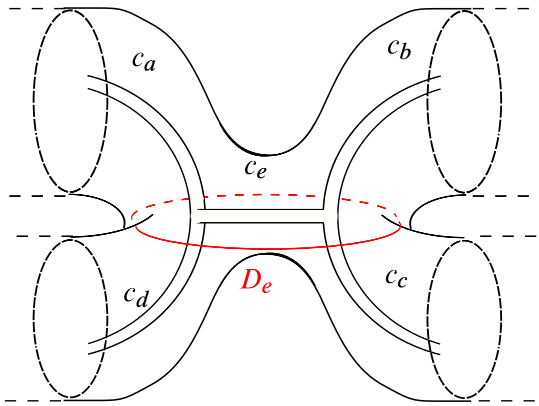


Figure 9: The curve D_e when e joins two distinct trivalent vertices of Γ

point of intersection, $p_*(D_e)$ is not zero as a class in $H_1(\Gamma, \mathbb{Z}/2)$, however it is in $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$ as $p(D_e)$ is homotopic to a boundary curve in the surface Γ.

Proposition 4.2 *We have, for any $e \in E$ and $c \in U_r$:*

(1) $T_r^{C_e}\varphi_c = -2\cos(\pi c_e/r)\varphi_c$ and $\sigma^{C_e}(\tau, \hbar, \theta) = -2\cos(\pi\tau_e)[0]$.

(2) *In the case where e is an edge joining a trivalent vertex to itself as in Figure 10 we have*

$$\sigma^{D_e}(\tau, \hbar, \theta) = \big(W(\pi\tau_e, \pi\tau_f, \hbar)e^{i\theta_e} + W(\pi\tau_e, \pi\tau_f, -\hbar)e^{-i\theta_e}\big)[0],$$

where

$$W(\tau, \alpha, \hbar) = \left(\frac{\sin(\tau + \alpha/2 + \hbar/2)\sin(\tau - \alpha/2 + \hbar/2)}{\sin\tau\,\sin(\tau + \hbar)}\right)^{\frac{1}{2}}.$$

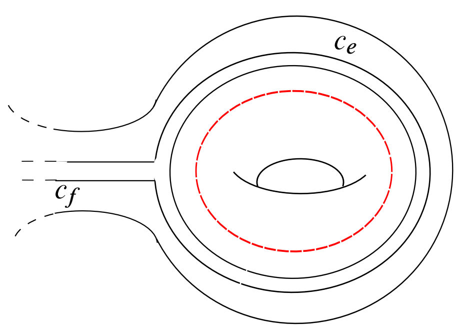


Figure 10: The curve D_e when e joins a trivalent vertex of Γ to itself

(3) *In the case where e is an edge between two distinct trivalent vertices as in Figure 9 we have*

$$\sigma^{D_e}(\tau,\hbar,\theta) = -\big(I(\pi\tau,\pi\hbar) + J(\pi\tau,\pi\hbar)e^{2i\theta_e} + J(\pi(\tau-2\hbar\delta_e),\pi\hbar)e^{-2i\theta_e}\big)[0].$$

Here, we have set δ_e for the element in $\mathbb{R}^E$ such that $\delta_{e,f}=1$ if and only if $e=f$,

$$\begin{aligned} I(\tau,\hbar) = {}& 2\cos(\tau_c+\tau_d-\hbar) \\ &+ 4\frac{\sin\frac{\tau_a+\tau_d-\tau_e-\hbar}{2}\sin\frac{\tau_a-\tau_d+\tau_e+\hbar}{2}\sin\frac{\tau_b+\tau_c-\tau_e-\hbar}{2}\sin\frac{\tau_b-\tau_c+\tau_e+\hbar}{2}}{\sin\tau_e\sin(\tau_e+\hbar)} \\ &+ 4\frac{\sin\frac{\tau_a+\tau_d+\tau_e-\hbar}{2}\sin\frac{-\tau_a+\tau_d+\tau_e-\hbar}{2}\sin\frac{\tau_b+\tau_c+\tau_e-\hbar}{2}\sin\frac{-\tau_b+\tau_c+\tau_e-\hbar}{2}}{\sin\tau_e\sin(\tau_e-\hbar)} \end{aligned}$$

and

$$\begin{aligned} J(\tau,\hbar) = 4\Bigg(&\frac{\sin\frac{\tau_a+\tau_d-\tau_e-\hbar}{2}\sin\frac{\tau_a-\tau_d+\tau_e+\hbar}{2}\sin\frac{\tau_b+\tau_c-\tau_e-\hbar}{2}\sin\frac{\tau_b-\tau_c+\tau_e+\hbar}{2}}{\sin\tau_e\sin(\tau_e+\hbar)} \\ &\times\frac{\sin\frac{\tau_a+\tau_d+\tau_e+\hbar}{2}\sin\frac{-\tau_a+\tau_d+\tau_e+\hbar}{2}\sin\frac{\tau_b+\tau_c+\tau_e+\hbar}{2}\sin\frac{-\tau_b+\tau_c+\tau_e+\hbar}{2}}{\sin(\tau_e+\hbar)\sin(\tau_e+2\hbar)}\Bigg)^{\frac{1}{2}}. \end{aligned}$$

The expressions of $T_r^{C_e}$ and $T_r^{D_e}$ can be derived by using the fusion rules. The computations are rather long in the last case, but straightforward.

These expressions, as well as the expressions of the ψ–symbol of the curves C_e and D_e were already given in [21]. They also checked by hand that the formulae of Theorem 1.3 were satisfied by these curves. We will only derive from the formulae that the zeroth- and first-order term for these curves are related as in Theorem 1.3, a fact that we will use later:

Proposition 4.3 *Let γ be any of the curves C_e or D_e. Then*

$$\sigma^\gamma(\tau,\hbar,\theta) = \sigma^\gamma(\tau,0,\theta) + \frac{\hbar}{2i}\sum_{e\in E}\frac{\partial^2}{\partial\tau_e\,\partial\theta_e}\sigma^\gamma(\tau,0,\theta) + o(\hbar).$$

Proof For C_e, there is not much to prove: as σ^{C_e} does not depend on $\hbar$, the first-order term vanishes, and $\partial^2\sigma^\gamma(\tau,0,\theta)/\partial\tau_e\,\partial\theta_e$ also vanishes as σ^{C_e} does not depend on θ_e.

For the curves D_e, we need to separate the case where e joins a vertex to itself, and the case where it joins two distinct vertices.

In the first case, depicted by Figure 10, we have

$$\sigma^{D_e}(\tau,\hbar,\theta) = (W(\pi\tau_e,\pi\tau_f,\pi\hbar)e^{i\theta_e} + W(\pi\tau_e,\pi\tau_f,-\pi\hbar)e^{-i\theta_e})[0].$$

Notice that we get $W(\pi\tau_e,\pi\tau_f,\pi\hbar) = W\big(\pi\big(\tau_e+\frac{1}{2}\hbar\big),\pi\tau_f,0\big)+o(\hbar)$ from the formula for W given above. Thus

$$\begin{aligned}\sigma^{D_e}&(\tau,\hbar,\theta)\\ &=\sigma^{D_e}(\tau,0,\theta)+\frac{\hbar}{2}\Big(\frac{\partial}{\partial\tau_e}[W(\pi\tau_e,\pi\tau_f,0)e^{i\theta_e}]-\frac{\partial}{\partial\tau_e}[W(\pi\tau_e,\pi\tau_f,0)e^{-i\theta_e}]\Big)[0]+o(\hbar)\\ &=\sigma^{D_e}(\tau,0,\theta)+\frac{\hbar}{2i}\sum_{e\in E}\frac{\partial^2}{\partial\tau_e\,\partial\theta_e}\sigma^{D_e}(\tau,0,\theta)+o(\hbar),\end{aligned}$$

as expected.

Finally, in the second case above, we have

$$\sigma^{D_e}(\tau,\hbar,\theta) = -\big(I(\pi\tau,\pi\hbar)+J(\pi\tau,\pi\hbar)e^{2i\theta_e}+J(\pi(\tau-2\hbar\delta_e),\pi\hbar)e^{-2i\theta_e}\big)[0].$$

It is easily seen that $J(\tau,\hbar)=J(\tau+\hbar\delta_e,0)$. Thus we only need to prove that $I(\tau,\hbar)=I(\tau,0)+o(\hbar)$. This is a bit more tricky:

First, notice that we can write

$$I(\tau,\hbar)=2\cos(\tau_c+\tau_d-\hbar)+\frac{1}{\sin\tau_e}(F(\tau_e+\hbar)-F(-\tau_e+\hbar))+o(\hbar),$$

where

$$\begin{aligned}F(\tau_e)&=4\frac{\sin\frac{\tau_a+\tau_d-\tau_e}{2}\sin\frac{\tau_a-\tau_d+\tau_e}{2}\sin\frac{\tau_b+\tau_c-\tau_e}{2}\sin\frac{\tau_b-\tau_c+\tau_e}{2}}{\sin\tau_e}\\ &=\frac{(\cos(\tau_d-\tau_e)-\cos\tau_a)(\cos(\tau_c-\tau_e)-\cos\tau_b)}{\sin\tau_e}.\end{aligned}$$

Therefore, the first-order term for $I(\tau,\hbar)$ is

$$\hbar\bigg(2\sin(\tau_c+\tau_d)+\frac{2}{\sin\tau_e}\frac{d}{d\tau_e}\mathcal{P}(F)(\tau_e)\bigg),$$

where $\mathcal{P}(F)$ is the even part of the function F. From the formula above, we have

$$\mathcal{P}(F)(\tau_e)=\sin(\tau_c+\tau_d)\cos\tau_e-\cos\tau_a\sin\tau_c-\cos\tau_b\sin\tau_d,$$

so that $(1/\sin\tau_e)\,d\mathcal{P}(F)(\tau_e)/d\tau_e = -\sin(\tau_c+\tau_d)$, and the first order of $I(\tau,\hbar)$ vanishes. □

The computations of σ^{C_e} and σ^{D_e} were previously used in [21] to prove a version of Theorem 1.3 for the punctured torus and the 4–holed sphere. Their approach was to derive from the above formulae that the asymptotic estimate of Theorem 1.3 is valid for the curves C_e, D_e and $\tau_{C_e}(D_e)$, where τ_{C_e} denotes the Dehn twist along C_e. Then they used the compatibility of the ψ–symbol with the product in $K(\Sigma,-e^{i\pi\hbar/2})$ to prove that if Theorem 1.3 is verified for γ and δ two multicurves, then it is also true for their product $\gamma \cdot \delta$. This yielded Theorem 1.3 for all multicurves in the punctured torus and the 4–holed sphere, as the curves C_e, D_e and $\tau_{C_e}(D_e)$ were sufficient to generate the Kauffman algebra.

However, this approach fails in higher genus, as this set of curves no longer generate the Kauffman algebra. Therefore, we developed another approach to tackle the higher-genus cases, which was also more conceptual and required less computations. Our fundamental idea is to use the multiplicativity of the ψ–symbol together with the theorem of Bullock (recalled in Section 2) to view the zeroth- and first-order term of the ψ–symbol in terms of algebra morphism and derivation of algebras on $\mathrm{Reg}(\mathcal{M}'(\Sigma))$. We then only need to compare this general shape with the values of the ψ–symbol on a few curves to get the formula of Theorem 1.3. (In fact, for the zeroth-order term we will only need the values on the C_e, while the first-order term also requires the values on the D_e).

5 Principal symbol and representation spaces

This section will be centered on the study of the principal symbol $\sigma^\gamma(\tau, 0, \theta)$, that is the zeroth order of the ψ–symbol $\sigma^\gamma(\tau, \hbar, \theta)$. The goal of the first subsection is to establish the formula for the principal symbol, which is stated in our main theorem: $\sigma_\chi^\gamma(\tau, 0, \theta) = f_\gamma(R_\chi(\tau, \theta))$, where f_γ is the function on $\mathcal{M}(\Sigma)$ such that $f_\gamma(\rho) = -\mathrm{Tr}(\rho(\gamma))$ and R_χ are action-angles parametrization on $\mathcal{M}(\Sigma)$.

5.1 Principal symbol and the SL_2–character variety

This section aims to establish a link between the components of the principal symbol σ_χ and functions on the space of representations $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$.

We will start our study of the principal symbol by the following proposition, which describes which values $\sigma_\chi^\gamma(\tau, \theta)$ can take:

Proposition 5.1 *For any multicurve γ and $\chi \in \widehat{A}_\Gamma$, we have:*

(1) $\sigma_\chi^\gamma(\tau, \theta) \in \mathbb{R}$.

(2) $|\sigma_\chi^\gamma(\tau, \theta)| \leq 2^{n(\gamma)}$, *where $n(\gamma)$ is the number of components of γ.*

Proof (1) We recall that the components of the ψ–symbol σ_χ^γ are complex-valued. The stated property comes from the fact that curve operators are Hermitian: for any multicurve γ, and every r, the operator T_r^γ is a Hermitian endomorphism of $V_r(\Sigma)$.

By definition, we have $T_r^\gamma \varphi_c = \sum_k F_k^\gamma(c/r, 1/r)\varphi_{c+k}$. As the basis $(\varphi_c)_{c\in U_r}$ is a Hermitian basis, we get

$$F_{-k}^\gamma\Big(\frac{c+k}{r}, \frac{1}{r}\Big) = \overline{F_k^\gamma\Big(\frac{c}{r}, \frac{1}{r}\Big)}$$

for all $c \in U_r$. Then for $r \to +\infty$ we have $F_{-k}^\gamma(\tau, 0) = \overline{F_k^\gamma(\tau, 0)}$.

Hence $\sigma_\chi^\gamma(\tau,\theta) = \chi(\gamma)\sum_k F_k^\gamma(\tau,0)e^{ik\cdot\theta} \in \mathbb{R}$ for all $(\tau,\theta) \in U \times (\mathbb{R}/2\pi\mathbb{Z})^E$.

(2) We want to find a bound for $|\sigma_\chi^\gamma(\tau,\theta)|$, where γ is a multicurve. By definition, we have $\sigma_\chi^\gamma(\tau,\theta) = \chi(\gamma)\sum_k F_k^\gamma(\tau,0)e^{ik\cdot\theta}$. On the one hand, we know that the coefficients F_k^γ are zero as soon as there is an e such that $|k_e| > I_e^\gamma = \sharp(\gamma \cap C_e)$. The number of nonzero coefficients is then lower than $M_\gamma = \prod_{e\in E}(2I_e^\gamma + 1)$. On the other hand, for any $r \geq 2$ and $c \in U_r$,

$$F_k^\gamma\Big(\frac{c}{r}, \frac{1}{r}\Big) = \langle T_r^\gamma \varphi_c, \varphi_{c+k}\rangle \leq \|T_r^\gamma\|.$$

We recalled in Section 2 that the spectral radius of T_r^γ is always $\leq 2^{n(\gamma)}$. Thus we have $|F_k^\gamma(c/r, 1/r)| \leq 2^{n(\gamma)}$ for every r > 0 and every $c \in U_r$. Taking the limit, we get $|F_k^\gamma(\tau,0)| \leq 2^{n(\gamma)}$.

These two estimations only allow us to write $|\sigma_\chi^\gamma(\tau,\theta)| \leq M_\gamma 2^{n(\gamma)}$. To obtain the promised inequality, we use the multiplicativity of $\sigma_\chi^\cdot(\tau,\theta)$:

We have $|\sigma_\chi^{\gamma^p}(\tau,\theta)| = |\sigma_\chi^\gamma(\tau,\theta)|^p$ for any integer p. But γ^p is also a multicurve, obtained by taking p parallel copies of each component of γ.

So we have that $|\sigma_\chi^{\gamma^p}(\tau,\theta)| \leq M_{\gamma^p} 2^{n(\gamma^p)}$.

But the number of components $n(\gamma^p)$ is just $pn(\gamma)$, and the geometric intersection numbers

$$I_e^\gamma = \sharp(\gamma \cap C_e)$$

verify $I_e^{\gamma^p} \leq pI_e^\gamma$.

From the product formula defining M_γ, we get that $M_{\gamma^p} \leq p^{|E|}M_\gamma$.

We conclude that $|\sigma_\chi^{\gamma^p}(\tau,\theta)| \leq p^{|E|}M_\gamma 2^{pn(\gamma)}$.

Then, taking the limit $p \to +\infty$, we get that $|\sigma_\chi^\gamma(\tau,\theta)| \leq 2^{n(\gamma)}$ for all (τ,θ) in $U \times (\mathbb{R}/2\pi\mathbb{Z})^E$. □

Now, recall that the components of the ψ–symbol

$$\sigma_\chi(\tau,\theta)\colon K(\Sigma,-1) \to \mathbb{C}$$

are morphisms of algebras. There is a simple description of all such morphism of algebras: indeed, by Theorem 2.2, we have an isomorphism

$$K(\Sigma,-1) \simeq \mathrm{Reg}(\mathcal{M}'(\Sigma)),$$

where $\mathcal{M}'(\Sigma)$ stands for $\mathrm{Hom}(\pi_1\Sigma, \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C})$, the space of characters of the fundamental group of Σ in $\mathrm{SL}_2(\mathbb{C})$. This space is an affine algebraic variety, and we are writing $\mathrm{Reg}(\mathcal{M}'(\Sigma))$ for the set of regular functions from $\mathcal{M}'(\Sigma)$ to $\mathbb{C}$. A morphism of algebras ϕ from $\mathrm{Reg}(\mathcal{M}'(\Sigma))$ to $\mathbb{C}$ is always of the form

$$\phi\colon f \mapsto f(\rho)$$

for some $\rho \in \mathcal{M}'(\Sigma)$. We deduce the existence of maps

$$R_\chi\colon U \times (\mathbb{R}/2\pi\mathbb{Z})^E \to \mathcal{M}'(\Sigma)$$

such that $\sigma_\chi^\gamma(\tau,\theta) = f_\gamma(R_\chi(\tau,\theta))$.

5.2 A system of action-angle coordinates on the SU_2–character variety

This subsection will be devoted to the study of the maps R_χ more closely, the aim being to prove that it actually gives action-angle coordinates on the character variety $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}_2)/\mathrm{SU}_2$, which we will denote by $\mathcal{M}(\Sigma)$.

In $\mathcal{M}(\Sigma)$ there is an open dense subset $\mathcal{M}_{\mathrm{irr}}(\Sigma)$ consisting of all conjugacy of irreducible representations. It is a well-known fact that $\mathcal{M}_{\mathrm{irr}}(\Sigma)$ consists only of smooth points of $\mathcal{M}(\Sigma)$ and it has a symplectic structure.

The maps R_χ have at first sight their image in $\mathcal{M}'(\Sigma)$. Again, we have a subset $\mathcal{M}'_{\mathrm{irr}}(\Sigma) \subset \mathcal{M}'(\Sigma)$ consisting of conjugacy classes of irreducible representations, and there is a structure of complex symplectic variety on this subspace. Moreover, $\mathcal{M}_{\mathrm{irr}}(\Sigma) \subset \mathcal{M}'_{\mathrm{irr}}(\Sigma)$.

We have two remarks:

First, we point out that $R_\chi(\tau,\theta)$ is always a noncommutative representation. Indeed, for a commutative representation, we would have, for three adjacent edges e, f and g,

$$h_{C_e}(\rho) + h_{C_f}(\rho) = h_{C_g}(\rho)$$

for one of the three orderings of e, f and g, or have $h_{C_e}(\rho) + h_{C_f}(\rho) + h_{C_g}(\rho) = 2$. This can not happen for $R_\chi(\tau,\theta)$ as $(h_{C_e})_{e\in E}$ maps it to $\tau \in U$, and we have strict inequalities $\tau_g < \tau_e + \tau_f$ and $\tau_e + \tau_f + \tau_g < 2$.

Our second point is that the map R_χ is smooth. By our first remark its image is indeed in the smooth part of $\mathcal{M}'(\Sigma)$. Note that for any $\gamma \in K(\Sigma,-1)$ the map

$(\tau,\theta)\to\sigma^{\gamma}(\tau,0,\theta)$ is smooth on $U\times(\mathbb{R}/2\pi\mathbb{Z})^E$, so $(\tau,\theta)\to\mathrm{Tr}(R_\chi(\tau,\theta)(\gamma))$ is smooth for every $\gamma\in\pi_1(\Sigma)$. As the space $\mathcal{M}'(\Sigma)$ can be parametrized by a finite collection of coordinates $\rho\to\mathrm{Tr}(\rho(\gamma_j))$, where $\gamma_j\in\pi_1(\Sigma)$, the map

$$R_\chi\colon U\times(\mathbb{R}/2\pi\mathbb{Z})^E\to\mathcal{M}'(\Sigma)$$

is smooth.

Proposition 5.2 *The maps R_χ take values in $\mathcal{M}_{\mathrm{irr}}(\Sigma)=\mathrm{Hom}(\pi_1\Sigma,\mathrm{SU}_2)/\mathrm{SU}_2$.*

Proof Indeed, we have seen with Proposition 5.1 that $\sigma_\chi^{\gamma}(\tau,\theta)$ is real-valued. We can use a well-known lemma:

Lemma *Any irreducible subgroup $G\subset\mathrm{SL}_2(\mathbb{C})$ such that the trace of all elements of G are real is conjugated to either a subgroup of $\mathrm{SL}_2(\mathbb{R})$ or a subgroup of SU_2.*

The proof of this lemma is based only on elementary algebra, manipulating trace of products of elements of G. A detailed proof can be found for example in [19, pages 3040–3041].

As we have $\sigma^{\gamma}(\tau,0,\theta)=-\mathrm{Tr}(R(\tau,\theta)(\gamma))\in\mathbb{R}$, we get that $R(\tau,\theta)$ is conjugated to either a representation in $\mathrm{SL}_2(\mathbb{R})$ or a representation in SU_2.

To prove Proposition 5.2, we still need to dismiss the case where the image of $R_\chi(\tau,\theta)$ would be conjugated to a subgroup of $\mathrm{SL}_2(\mathbb{R})$. To this end, we use Proposition 5.1(2), which states that $|\mathrm{Tr}(R_\chi(\tau,\theta)\gamma)|\le 2$ for every $\gamma\in\pi_1(\Sigma)$ representing a simple closed curve on Σ. We use the following lemma, proved in [17, Lemma 3.1.1]:

Lemma *Let $\rho\colon\pi_1(\Sigma)\to\mathrm{PSL}_2(\mathbb{C})$ be a nonelementary representation, then there exist two simple loops a and b intersecting once such that $\rho(a)$ and $\rho(b)$ are loxodromic* (meaning $|\mathrm{Tr}(\rho(a))|>2$ and $|\mathrm{Tr}(\rho(b))|>2$) *and noncommuting.*

This lemma follows from elementary considerations in hyperbolic geometry. From the lemma, we get that, since $R(\tau,\theta)(a)$ is never loxodromic, it must be an elementary representation into $\mathrm{PSL}_2(\mathbb{C})$. But if $R(\tau,\theta)$ was conjugated to a representation in $\mathrm{SL}_2(\mathbb{R})$, it would be a commutative representation, and we saw that $R(\tau,\theta)$ is not. □

Proposition 5.3 *For any $\chi\in\widehat{A}_\Gamma$, the map*

$$R_\chi\colon U\times(\mathbb{R}/2\pi\mathbb{Z})^E\to\mathcal{M}(\Sigma),\quad(\tau,\theta)\mapsto R_\chi(\tau,\theta),$$

gives action-angle coordinates on the symplectic variety $\mathcal{M}_{\mathrm{irr}}(\Sigma)$.

Proposition 5.2 of [20] shows that when a pants decomposition $\mathcal{C} = \{C_e\}_{e\in E}$ of Σ is given, the family of functions $h_{C_e} = \frac{1}{\pi}\operatorname{Acos}\bigl(-\frac{1}{2} f_{C_e}\bigr)$ constitutes a moment mapping h: $h^{-1}(U) \to U$ and $h^{-1}(U)$ is an open dense subset of $\mathcal{M}(\Sigma)$. The variables τ_e are the action coordinates associated to this moment mapping:

$$h_{C_e}(R_\chi(\tau,\theta)) = \frac{1}{\pi}\operatorname{Acos}\bigl(-\tfrac{1}{2} f_{C_e}(R_\chi(\tau,\theta))\bigr) = \frac{1}{\pi}\operatorname{Acos}\bigl(-\tfrac{1}{2}\sigma_\chi^{C_e}(\tau,\theta)\bigr) = \tau_e,$$

where the third equality comes from the computation of the operator $T_r^{C_e}$ given in Section 4: for any coloration c of E, we have $T_r^{C_e}\varphi_c = -2\cos(\pi c/r)\varphi_c$, so that $\sigma_\chi^{C_e}(\tau,\theta,\hbar) = F_0^{C_e}(\tau,\hbar)\chi([0]) = -2\cos(\pi\tau_e)$.

The only missing condition for (τ,θ) to be a system of action-angle coordinates on $\mathcal{M}(\Sigma)$ is that

$$R_\chi^*(\omega) = \sum_{e\in E} d\tau_e \wedge d\theta_e,$$

where ω refers to the symplectic form on the variety $\mathcal{M}(\Sigma)$.

It also amounts to the fact that the vector fields ∂_{θ_e} and $X_{h_{C_e}}$ (the symplectic gradient associated to the function h_{C_e}) on $\mathcal{M}(\Sigma)$ are equal. This equality of vector fields can be rewritten in terms of Poisson brackets:

$$\{h_{C_e}, f\} = \frac{\partial}{\partial\theta_e} f(R_\chi(\tau,\theta)) \quad \text{for all } f \in C^\infty(\mathcal{M}(\Sigma),\mathbb{C}) \text{ and all } \tau,\ \theta.$$

As the map $f \to \{h_{C_e}, f\}$ is a first-order differential operator, and any function f on $\mathcal{M}(\Sigma)$ can be approximated at order 1 near any point $\rho \in \mathcal{M}(\Sigma)$ by a linear combination of trace functions f_γ associated to multicurves, we only need to verify the equality when $f = f_\gamma$, the trace function of a multicurve γ.

To compute such Poisson brackets, we can apply Theorem 2.3:

We denote by ε the linear map

$$\varepsilon\colon K(\Sigma,-e^{i\pi\hbar/2}) \to K(\Sigma,-1) \simeq \operatorname{Reg}(\mathcal{M}'(\Sigma)),$$
$$\sum_{\gamma \text{ multicurve}} c_\gamma(\hbar)\gamma \mapsto \sum_{\gamma \text{ multicurve}} c_\gamma(0)\gamma.$$

For γ and $\delta \in K(\Sigma,-e^{i\pi\hbar/2})$ we have

$$\{f_{\varepsilon(\gamma)}, f_{\varepsilon(\delta)}\} = f_{\varepsilon((i/\hbar)[\gamma,\delta])}$$

with $[\gamma,\delta] = \gamma\cdot\delta - \delta\cdot\gamma \in K(\Sigma,-e^{i\pi\hbar/2})$.

We apply the above formula to compute $\{h_{C_e}, f_\gamma\}$ for any $\gamma \in K(\Sigma,-e^{i\pi\hbar/2})$: We recall that $h_{C_e} = \frac{1}{\pi}\operatorname{Acos}\bigl(-\frac{1}{2} f_{C_e}\bigr)$. Our strategy to compute the Poisson bracket is to approximate h_{C_e} with polynomials in f_{C_e}.

Since $\tau \in U$ we have $-2\cos(\pi\tau_e) \in (-2,2)$ and we can choose a polynomial P such that $P\big(-2\cos(\pi(\tau_e+x))\big) = x + o(x^2)$.

Now, the maps

$$\{\,\cdot\,, f_\gamma\}\colon C^\infty(\mathcal{M}(\Sigma)) \to C^\infty(\mathcal{M}(\Sigma)) \quad \text{and} \quad (i/\hbar)[\,\cdot\,,\gamma]\colon K(\Sigma,-1) \to K(\Sigma,-1)$$

being derivations of algebras, we have, by Goldman's formula,

$$\{P(f_{C_e}), f_\gamma\}(R_\chi(\tau,\theta)) = f_{\varepsilon((i/\hbar)[P(C_e),\gamma])}(R_\chi(\tau,\theta)) = \sigma_\chi^{\varepsilon((i/\hbar)[P(C_e),\gamma])}(\tau,\theta,0).$$

We compute this last quantity: we recall that we wrote $T_r^\gamma \varphi_c = \sum_k F_k^\gamma(\tau,\hbar)\varphi_{c+k}$ and we gave in Section 4.2 the expression $T_r^{C_e}\varphi_c = -2\cos(\pi\tau_e)\varphi_c$. Hence $T_r^{P(C_e)}\varphi_c = P(-2\cos(\pi\tau_e))\varphi_c$. We deduce that, for $c \in U_r$,

$$T_r^{[P(C_e),\gamma]}\varphi_c = \sum_k P\big(-2\cos(\pi(\tau_e+k_e\hbar))\big)F_k^\gamma(\tau,\hbar)\varphi_{c+k} - \sum_k P(-2\cos(\pi\tau_e))F_k^\gamma(\tau,\hbar)\varphi_{c+k}.$$

But, since $[C_e^k] = [0]$ in A_Γ,

$$\sigma_\chi^{\varepsilon((i/\hbar)[P(C_e),\gamma])}(\tau,\theta,0) = i\sum_k \frac{P\big(-2\cos(\pi(\tau_e+k_e\hbar))\big) - P(-2\cos(\pi\tau_e))}{\hbar}\bigg|_{\hbar=0} F_k^\gamma(\tau,0)e^{ik\cdot\theta}\chi(\gamma).$$

By our choice of P this reduces to

$$\sum_k ik_eF_k^\gamma(\tau,\hbar)e^{ik\cdot\theta}\chi(\gamma) = \frac{\partial}{\partial\theta_e}\sigma_\chi^\gamma(\tau,0,\theta) = \frac{\partial}{\partial\theta_e}f_\gamma(R_\chi(\tau,\theta)).$$

The last equality ends the proof: we have $\{h_{C_e}, f_\gamma\}(R_\chi(\tau,\theta)) = \partial f_\gamma(R_\chi(\tau,\theta))/\partial\theta_e$ for every multicurve γ, and R_χ gives an action-angle parametrization of $\mathcal{M}_{\mathrm{irr}}(\Sigma)$. $\square$

5.3 Origin of angle coordinates

We want to investigate how exactly R_χ varies with $\chi \in \widehat{A}_\Gamma$. We recall that according to Section 3.2, the values of two different morphisms χ and χ' on $[\gamma]$ differ by a representation $\rho\colon H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2) \to \{\pm1\}$.

Let us also get more precise information about angle coordinates. We recall that we have a hamiltonian $h\colon \mathcal{M}_{\mathrm{irr}}(\Sigma) \to U$, given by $(h(\rho))_e = \frac{1}{\pi}\,\mathrm{Acos}\big(\frac{1}{2}\mathrm{Tr}(\rho(C_e))\big)$. The hamiltonian flow gives an action of $\mathbb{R}^E$ on $\mathcal{M}_{\mathrm{irr}}(\Sigma)$. This action has a kernel

$$\Lambda = \mathrm{Vect}_{\mathbb{Z}}\{(2\pi u_e)_{e\in E},\ \pi(u_e+u_f+u_g)_{(e,f,g)\in S}\},$$

where $(u_e)_{e\in E}$ is the canonical basis of $\mathbb{R}^E$, E is the set of edges of Γ and S is the set of triples of edges adjacent to the same vertex in Γ. We also define $\Lambda' = \mathrm{Vect}_{\mathbb{Z}}(\pi u_e) \supset \Lambda$. The quotient Λ'/Λ then acts on $\mathcal{M}^{\mathrm{irr}}(\Sigma)$ by $\pi u_e \cdot \rho(\gamma) = (-1)^{(C_e,\gamma)}\rho(\gamma)$, where $(\cdot,\cdot)$ is the intersection form in Σ.

Now that we know that the maps R_χ give action-angle coordinates on $\mathcal{M}_{\mathrm{irr}}(\Sigma)$, the only ambiguity is the choice of the origin of the angle part. That is, we must have, for any $\chi,\ \chi' \in \widehat{A}_\Gamma$, that $R_{\chi'}(\tau,\theta) = R_\chi(\tau,\theta + v_{\chi,\chi'}(\tau))$, where $v_{\chi,\chi'}$ is a continuous function from U to $\mathbb{R}/\Lambda$.

We use the values of R_χ on the curves D_e to get the origin of the angle coordinates. We have $\mathrm{Tr}(R_\chi(\tau,\theta)(D_e)) = -\sigma_\chi^{D_e}(\tau,0,\theta) = -2W(\pi\tau,0)\cos\theta_e$ if e joins a vertex to itself, and $\mathrm{Tr}(R_\chi(\tau,\theta)(D_e)) = I(\pi\tau,0) + 2J(\pi\tau,0)\cos(2\theta_e)$ otherwise. We see that, in the first case, $\theta_e = 0$ is the unique minimum of $\mathrm{Tr}(R_\chi(\tau,\theta)(D_e))$, so that the origin of this coordinate is the same for all $\chi \in \widehat{A}_\Gamma$. In the second case, $\theta_e \mapsto \mathrm{Tr}(R_\chi(\tau,\theta)(D_e))$ has exactly two maxima, one for $\theta_e = 0$ and one for $\theta_e = \pi$. So θ is fixed modulo πu_e. Thus, for $\chi,\ \chi' \in \widehat{A}_\Gamma$, we have $v_{\chi,\chi'}(\tau) \in \Lambda'/\Lambda$. Furthermore, $v_{\chi,\chi'}$ is continuous, hence it has to be constant.

Taking two elements χ and χ' in $\widehat{A}_\Gamma$, we know that they differ by a morphism

$$\rho\colon H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2) \to \{\pm 1\}.$$

It is possible to recover the vector $v_{\chi,\chi'} \in \Lambda'/\Lambda$ from the representation ρ: by Poincaré duality, one can write $\rho(p_*(\gamma)) = (-1)^{\langle C,\gamma\rangle}$, where $C \in H_1(\Sigma,\mathbb{Z}/2)$, p_* is the projection $H_1(\Sigma,\mathbb{Z}/2) \to H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)$ and $\langle\cdot,\cdot\rangle$ is the intersection form in $H_1(\Sigma,\mathbb{Z}/2)$. Remember that p_* maps each C_e to zero, so that the intersection of C with each C_e must vanish. As the C_e generate a Lagrangian of $H_1(\Sigma,\mathbb{Z}/2)$, C is a linear combination of the C_e and this yields a vector $v_\rho \in \Lambda'/\Lambda$ such that $R_{\rho\chi}(\tau,\theta) = R_\chi(\tau,\theta + v_\rho)$.

We need to note that when Γ is a planar graph we can drop this complicated consideration of angle origins and we could have taken the ψ–symbol to be just $\mathbb{C}$–valued. Indeed, in this case the intersection form in $H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)$ is trivial, and the image of $H_1(\Sigma,\mathbb{Z}/2) \to H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)$ is $\{0\}$, so the ψ–symbol is $\mathbb{C}$–valued.

6 First order of the ψ–symbol

In this section, we investigate the first-order term in $\hbar$ of the asymptotic expansion of the ψ–symbol. We identify this term by linking it with the principal symbol, for which we already know a formula.

We recall that for γ a multicurve, the map $(\tau,\hbar,\theta)\mapsto\sigma^\gamma(\tau,\hbar,\theta)$ is defined as a finite sum of smooth functions on V_γ, and V_γ is a neighborhood of $U\times\{0\}$ in $U\times[0,1]$. We may write, for any multicurve γ,

$$\sigma^\gamma(\tau,\hbar,\theta)=\sigma^\gamma(\tau,0,\theta)+\hbar(\Delta_\gamma(\tau,\theta)+D_\gamma(\tau,\theta))+o(\hbar).$$

Here, $\Delta_\gamma(\tau,\theta)$ refers to the expected first order as in Theorem 1.3:

$$\Delta_\gamma(\tau,\theta)=\frac{1}{2i}\sum_e\frac{\partial^2}{\partial\tau_e\,\partial\theta_e}\sigma^\gamma(\tau,0,\theta).$$

Hence, what we want to prove in this section is that the remainder $D_\gamma(\tau,\theta)$ is zero for all γ and $(\tau,\theta)\in U\times(\mathbb{R}/2\pi\mathbb{Z})^E$.

We remark that the previous expressions define $\Delta(\tau,\theta)$ and $D(\tau,\theta)$ as maps from the set of multicurves to A_Γ, which we can extend by linearity to linear maps $K(\Sigma,-e^{i\pi\hbar/2})\to A_\Gamma[\![\hbar]\!]$.

Furthermore, Δ_γ and D_γ are some linear combinations of partial derivatives of the smooth functions F_k on V_γ, so they are both smooth on $U\times(\mathbb{Z}/2\pi\mathbb{Z})^E$.

Proposition 6.1 *For any multicurve γ and for all (τ,θ), the remainder term $D_\gamma(\tau,\theta)$ vanishes, so that the first-order term of $\sigma^\gamma(\tau,\hbar,\theta)$ is*

$$\Delta_\gamma(\tau,\theta)=\frac{1}{2i}\sum_e\frac{\partial^2}{\partial\tau_e\,\partial\theta_e}\sigma^\gamma(\tau,0,\theta).$$

The proof relies on the following two lemmas:

Lemma 6.2 *Let (τ,θ) be in $U\times(\mathbb{R}/2\pi\mathbb{Z})^E$. We will provide $\mathbb{C}$ with the structure of a $K(\Sigma,-1)$–module (or equivalently of $\mathrm{Reg}(\mathcal{M}'(\Sigma))$–module): for $x\in\mathbb{C}$ and $f\in\mathrm{Reg}(\mathcal{M}'(\Sigma))$, we define $f\cdot x=f(R_\chi(\tau,\theta))x$. Then the corresponding component of the remainder term $\gamma\mapsto\chi(D_\gamma(\tau,\theta))$ is a derivation of $K(\Sigma,-1)$–modules from $K(\Sigma,-1)$ to $\mathbb{C}$.*

Lemma 6.3 *With respect to the above-discussed $\mathrm{Reg}(\mathcal{M}'(\Sigma))$–module structure on $\mathbb{C}$ as above, we have an isomorphism $\mathrm{Der}(\mathrm{Reg}(\mathcal{M}'(\Sigma)),\mathbb{C})\simeq T_{R_\chi(\tau,\theta)}\mathcal{M}(\Sigma)$ sending a vector $X\in T_{R_\chi(\tau,\theta)}\mathcal{M}(\Sigma)$ to the derivation $f\to\mathcal{L}_X f(R_\chi(\tau,\theta))$, and the vector fields $(R_\chi^*\,\partial/\partial\tau_e,R_\chi^*\,\partial/\partial\theta_e)$ give a basis of the tangent spaces $T_{R_\chi(\tau,\theta)}\mathcal{M}(\Sigma)$.*

Proof of Lemma 6.2 We use Proposition 3.3 to determine how the remainder term $D(\tau,\theta)$ behaves with the product of elements in $K(\Sigma,-e^{i\pi\hbar/2})$. We work with one component σ_χ of the ψ–symbol at a time. For $\gamma \in K(\Sigma,-1)$, we will use the notation $E_\gamma = \chi(\Delta_\gamma + D_\gamma)$, so that we can write $\sigma_\chi^\gamma(\tau,\hbar,\theta) = \sigma_\chi^\gamma(\tau,0,\theta) + \hbar E_\gamma(\tau,\theta) + o(\hbar)$.

Then, applying $\chi \in \widehat{A}_\Gamma$ to Proposition 3.3 we have

$$\sigma_\chi^{\gamma\cdot\delta}(\tau,\hbar,\theta) = \sigma_\chi^\gamma(\tau,\hbar,\theta)\sigma_\chi^\delta(\tau,\hbar,\theta) + \frac{\hbar}{i}\sum_e \partial_{\tau_e}\sigma_\chi^\gamma(\tau,\hbar,\theta)\,\partial_{\theta_e}\sigma_\chi^\delta(\tau,\hbar,\theta) + o(\hbar).$$

We have $\sigma_\chi^\gamma(\tau,0,\theta) = f_\gamma(R_\chi(\tau,\theta))$. Recall that, by Theorem 2.3,

$$f_{\gamma\cdot\delta} = f_\gamma f_\delta + \hbar\frac{\pi}{i}\{f_\gamma, f_\delta\} + o(\hbar).$$

So, isolating terms of order 1 in $\hbar$, we get

$$\begin{aligned}\frac{\pi}{i}\{f_\gamma, f_\delta\}(R_\chi(\tau,\theta)) + E_{\gamma\cdot\delta}(\tau,\theta)\\ = E_\gamma(\tau,\theta) f_\delta(R_\chi(\tau,\theta)) + E_\delta(\tau,\theta) f_\gamma(R_\chi(\tau,\theta))\\ + \frac{1}{i}\sum_e \partial_{\tau_e} f_\gamma(R_\chi(\tau,\theta))\,\partial_{\theta_e} f_\delta(R_\chi(\tau,\theta)),\end{aligned}$$

but $\{f_\gamma, f_\delta\} = (1/2\pi)\sum_e \partial_{\tau_e} f_\gamma\,\partial_{\theta_e} f_\delta - \partial_{\tau_e} f_\delta\,\partial_{\theta_e} f_\gamma$. We deduce that

$$E_{\gamma\cdot\delta} = E_\gamma\sigma_\chi^\delta + E_\delta\sigma_\chi^\gamma + \frac{1}{2i}\sum_e \partial_{\tau_e}\sigma_\chi^\gamma\,\partial_{\theta_e}\sigma_\chi^\delta + \partial_{\theta_e}\sigma_\chi^\gamma\,\partial_{\tau_e}\sigma_\chi^\delta.$$

However, as for $\gamma,\ \delta \in K(\Sigma,-1)$ we have, by Theorem 2.2, that $f_{\gamma\cdot\delta} = f_\gamma f_\delta$, and

$$\chi(\Delta_\gamma) = \frac{1}{2i}\sum_e \frac{\partial^2 f_\gamma}{\partial\tau_e\,\partial\theta_e}\circ R_\chi,$$

the Leibniz rule implies that $\chi(\Delta_\gamma)$ satisfies the same law of composition:

$$\chi(\Delta_{\gamma\cdot\delta}) = \chi(\Delta_\gamma) f_\delta + \chi(\Delta_\delta) f_\gamma + \frac{1}{2i}\sum_e \partial_{\tau_e} f_\gamma\,\partial_{\theta_e} f_\delta + \partial_{\theta_e} f_\gamma\,\partial_{\tau_e} f_\delta.$$

This concludes the proof of Lemma 6.2: $\chi\circ D$ is a derivation. □

Proof of Lemma 6.3 It is well known that $\mathcal{M}'(\Sigma)$ is an affine algebraic variety whose smooth points is the open dense subset $\mathcal{M}'_{\mathrm{irr}}(\Sigma)$ (see [25], for example). The point $R_\chi(\tau,\theta)$ is thus a smooth point of $\mathcal{M}'(\Sigma)$ for any $(\tau,\theta)\in U\times\mathbb{R}/2\pi\mathbb{Z}$.

Then the proof comes from elementary considerations of algebraic geometry: when V is an affine algebraic variety and x a point of V, we put a structure of $\mathrm{Reg}(V)$–module on $\mathbb{C}$ by defining $f\cdot\lambda = f(x)\lambda$. Then $\mathrm{Der}_x(V,\mathbb{C})$ identifies with $T_xV = m_x/(m_x)^2$, the algebraic tangent space to V at x (where $m_x = \{f \mid f(x)=0\}$), and the algebraic tangent space at a smooth point is the same as the tangent space of V at x in the sense of differential manifolds. As the affine variety $\mathcal{M}'(\Sigma)$ is smooth on the image of R_χ, by this general property, derivations of $\mathrm{Reg}(\mathcal{M}(\Sigma))$ can be viewed as vectors of the tangent space. As $(\tau,\theta)\mapsto R_\chi(\tau,\theta)$ is a parametrization of $\mathcal{M}(\Sigma)$, the vector fields $((R_\chi)_*\,\partial/\partial\tau_e,(R_\chi)_*\,\partial/\partial\theta_e)$ give a basis of the tangent space $T_{R_\chi(\theta,\tau)}\mathcal{M}(\Sigma)$ for each (τ,θ). □

Proof of Proposition 6.1 Combining Lemmas 6.2 and 6.3 allows us to assert that $\chi(D(\tau,\theta))$, viewed as a map $\mathrm{Reg}(\mathcal{M}'(\Sigma))\to\mathbb{C}$, is of the form $f\mapsto\mathcal{L}_X f(R_\chi(\tau,\theta))$ for some $X\in T_{R_\chi(\tau,\theta)}\mathcal{M}'(\Sigma)$ and we may write $X=\sum_e a_e\,\partial/\partial_{\tau_e}+b_e\,\partial/\partial_{\theta_e}$ for some coefficients a_e, b_e: $\mathcal{M}(\Sigma)\to\mathbb{C}$. As D_γ is smooth, so are the coefficients a_e and b_e.

We want to prove that these coefficients all vanish. To this end, we recall that we proved in Section 4.2 that the remainder term vanishes for the curves C_e and D_e. Furthermore, we have the formula of Section 4:

We have $\sigma^{C_e}(\tau,\hbar,\theta)=-2\cos\pi\tau_e[0]$, so that $\chi(D_{C_e})(\tau,\theta)=2a_e\pi\sin(\pi\tau_e)$. Since the remainder term vanishes on C_e, we must have $a_e=0$.

To show the vanishing of the b_e, we use the formulae for D_e:

For the first kind of curve D_e, described in Section 4.2, we have $f_{D_e}(R_\chi(\tau,\theta))=\sigma_\chi^{D_e}(\tau,0,\theta)=2W(\pi\tau,0)\cos\theta_e$, where W does not vanish for $\tau\in U$.

We know that the remainder term D_{D_e} vanishes, so we have

$$\chi(D_{D_e}(\tau,\theta))=b_e\frac{\partial}{\partial\theta_e}f_{D_e}(R_\chi(\tau,\theta))=-2b_e\pi\sin(\theta_e)W(\pi\tau,0)=0.$$

This yields $b_e=0$.

In the second case, $f_{D_e}(R_\chi(\tau,\theta))=\sigma_\chi^{D_e}(\tau,0,\theta)=-2J(\pi\tau,0)\cos 2\theta_e-I(\pi\tau,0)$ for the functions I and J defined in Section 4.2, which are nonvanishing for $\tau\in U$.

Again since $\chi(D_{D_e}(\tau,\theta))=b_e\,\partial f_{D_e}(R_\chi(\tau,\theta))/\partial\theta_e=4\pi b_e\sin(2\theta_e)J(\pi\tau,0)$ vanishes, we must have $b_e=0$. It follows that the remainder term $\gamma\mapsto D_\gamma$ is the zero derivation on $K(\Sigma,-1)\mapsto A_\Gamma$, which is the last ingredient we needed to complete the proof of Proposition 6.1. □

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