

# The Picard group of topological modular forms via descent theory

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This paper starts with an exposition of descent-theoretic techniques in the study of Picard groups of  $E_\infty$ -ring spectra, which naturally lead to the study of Picard spectra. We then develop tools for the efficient and explicit determination of differentials in the associated descent spectral sequences for the Picard spectra thus obtained. As a major application, we calculate the Picard groups of the periodic spectrum of topological modular forms  $TMF$  and the nonperiodic and nonconnective  $Tmf$ . We find that  $\text{Pic}(TMF)$  is cyclic of order 576, generated by the suspension  $\Sigma TMF$  (a result originally due to Hopkins), while  $\text{Pic}(Tmf) = \mathbb{Z} \oplus \mathbb{Z}/24$ . In particular, we show that there exists an invertible  $Tmf$ -module which is not equivalent to a suspension of  $Tmf$ .

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## 1 Introduction

Elliptic curves and modular forms occupy a central role in modern stable homotopy theory in the guise of the variants of *topological modular forms*: the connective  $tmf$ , the periodic  $TMF$ , and  $Tmf$ , which interpolates between them. These are structured ring spectra which have demonstrated surprising connections between the arithmetic of elliptic curves and  $v_2$ -periodicity in stable homotopy. For example,  $tmf$  detects a number of 2-torsion and 3-torsion classes in the stable homotopy groups of spheres through the Hurewicz image. Even more interestingly, the more geometric-natured  $TMF$  can be used to detect and describe, using congruences between modular forms, the 2-line of the Adams–Novikov spectral sequence at primes  $p \geq 5$ , according to Behrens [7].

From a different perspective, the structure of topological modular forms as  $E_\infty$ -ring spectra leads to symmetric monoidal  $\infty$ -categories of modules which give rise to well-behaved invariants of algebraic or algebrogeometric type. For instance, Meier [48] has studied  $TMF$ -modules which become free when certain level structures are introduced; these can be thought of as locally free sheaves with respect to a predetermined cover.

Our goal in this paper is to understand another such invariant, the Picard group. Any symmetric monoidal category has an associated group of isomorphism classes of objects invertible under the tensor product, which is called the *Picard group*. The classical examples are the Picard group  $\text{Pic}(R)$  of a ring  $R$ , ie of the category  $\text{Mod}(R)$  of  $R$ -modules, or the Picard group of a scheme  $X$ , ie of the category  $\text{Mod}(\mathcal{O}_X)$  of quasicohherent modules over its structure sheaf. In homotopy theory, the interest in Picard groups arose when Mike Hopkins made the observation that the homotopy categories of  $E_n$ -local and  $K(n)$ -local spectra have interesting Picard groups, particularly when the prime at hand is small in comparison with  $n$ . Here,  $E_n$  is the Lubin–Tate spectrum and  $K(n)$  is the Morava  $K$ -theory spectrum at height  $n$ . In the few existing computations of such groups, notably those in Hopkins, Mahowald and Sadofsky [26], Hovey and Sadofsky [27], Kamiya and Shimomura [29], Goerss, Henn, Mahowald and Rezk [17] and Heard [21], one often uses that an invertible  $E_n$ -module must be a suspension of  $E_n$  itself.

The  $K(2)$ -localization of any of the three versions of topological modular forms gives a spectrum closely related to the Lubin–Tate spectrum  $E_2$ ; namely, this localization is a finite product of homotopy fixed point spectra of finite group actions on  $E_2$  (or slight variants of  $E_2$  with larger residue fields). More generally, each  $E_n$  is an  $E_\infty$ -ring spectrum with an action, through  $E_\infty$ -ring maps, by a profinite group  $\mathbb{G}_n$  called the Morava stabilizer group (see Rezk [57] for the  $E_1$ -ring case). The  $K(n)$ -local sphere is obtained then as the Devinatz–Hopkins homotopy fixed points. However,  $\mathbb{G}_n$  also has interesting finite subgroups when the prime is relatively small with respect to  $n$ . If  $G$  is such a subgroup, the homotopy fixed points  $E_n^{hG}$  are an  $E_\infty$ -ring spectrum, which is in theory easier to study than the  $K(n)$ -local sphere, but hopefully contains a lot of information about the  $K(n)$ -local sphere. For instance, Hopkins has observed that in all known examples, the Picard group of  $E_n^{hG}$  (unlike that of the  $K(n)$ -local category) is very simple as it only contains suspensions of  $E_n^{hG}$ , and raised the following natural question.

**Question** (Hopkins) Let  $G$  be a finite subgroup of the Morava stabilizer group  $\mathbb{G}_n$  at height  $n$ . Is it true that any invertible  $K(n)$ -local module over  $E_n^{hG}$  is a suspension of  $E_n^{hG}$ ?

The periodic TMF is closer to its  $K(2)$ -localization than  $\text{Tmf}$ , and this is demonstrated by the following result, originally due to Hopkins but unpublished.

**Theorem A** (Hopkins) *The Picard group of TMF is isomorphic to  $\mathbb{Z}/576$ , generated by the suspension  $\Sigma \text{TMF}$ .*

In the paper at hand, we prove Theorem A using a descent-theoretic approach. In particular, our method is different from Hopkins’s. The descent-theoretic approach also enables us to prove that, nonetheless, the nonconnective, nonperiodic flavor of topological modular forms  $\mathrm{Tmf}$  behaves differently and has a more interesting Picard group.

**Theorem B** *The Picard group of  $\mathrm{Tmf}$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}/24$ , generated by the suspension  $\Sigma \mathrm{Tmf}$  and a certain 24–torsion invertible object.*

In addition, we explicitly construct the 24–torsion module in Construction 8.4.2. We note that, after the initial submission of this paper, the preprint of Hill and Meier [23] appeared, in which the authors use techniques from  $C_2$ –equivariant stable homotopy to construct exotic torsion elements in the Picard group of  $\mathrm{Tmf}_1(3)$ . In contrast, our construction is given by an unusual gluing of locally trivial modules.

We hope that our method of proof of Theorems A and B, which is very general, will also be of interest to those not directly concerned with  $\mathrm{TMF}$ . Our method is inspired by and analogous to the forthcoming work of Gepner and Lawson [15] on Galois descent of Brauer as well as Picard groups, though the key ideas are classical.

Take, for example, the periodic variant  $\mathrm{TMF}$ . Its essential property is that it arises as the global sections of the structure sheaf  $\mathcal{O}^{\mathrm{top}}$  of a regular “derived stack”  $(\mathfrak{M}_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}})$  refining the moduli stack of elliptic curves  $M_{\mathrm{ell}}$ . Thus

$$\mathrm{TMF} = \Gamma(\mathfrak{M}_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}}) = \varprojlim_{\mathrm{Spec} R \rightarrow M_{\mathrm{ell}}} \Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}}),$$

where the maps  $\mathrm{Spec} R \rightarrow M_{\mathrm{ell}}$  range over all étale morphisms from affine schemes to  $M_{\mathrm{ell}}$ . Moreover, the  $E_\infty$ –ring spectra  $\Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}})$  are weakly even periodic; thus we have  $\mathrm{TMF}$  as the homotopy limit of a diagram of weakly even periodic  $E_\infty$ –rings. It follows by the main result in Mathew and Meier [42] that the *module category* of  $\mathrm{TMF}$  can also be represented as the inverse limit of the module categories  $\mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R))$ , that is, as quasicohherent sheaves on the derived stack. In any analogous situation, our descent techniques for calculating Picard groups apply.

Over an affine chart  $\mathrm{Spec} R \rightarrow M_{\mathrm{ell}}$ , the Picard group of  $\Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}})$  (ie that of an *elliptic spectrum*) is purely algebraic, by a classical argument in Hopkins, Mahowald and Sadofsky [26] and Baker and Richter [4] with “residue fields”. This results from the fact that the ring  $\pi_* \Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}})$  is *homologically simple*: in particular, it has finite global dimension, which makes the study of  $\Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}})$ –modules much easier. One attempts to use this together with descent theory to compute the Picard group of  $\mathrm{TMF}$  itself; however, doing so necessitates the consideration of higher homotopy

coherences. For this, it is important to work with Picard *spectra* rather than Picard groups, as they have a better formal theory of descent.

The Picard spectrum  $\text{pic}(A)$  of an  $E_\infty$ -ring  $A$  is an important spectrum associated to  $A$  that deloops the space of units  $\text{GL}_1(A)$  of May [46]:<sup>1</sup> it is connective, its  $\pi_0$  is the Picard group of  $A$ , and its 1-connective cover  $\tau_{\geq 1} \text{pic}(A)$  is equivalent to  $\Sigma \text{gl}_1(A)$  for  $\text{gl}_1(A)$  the *spectrum of units* of [46]. We find that the *Picard spectrum* of  $\text{TMF}$  is the connective cover of the homotopy limit of  $\text{pic}(\mathcal{O}^{\text{top}}(\text{Spec } R))$ , taken over étale maps  $\text{Spec } R \rightarrow M_{\text{ell}}$ . This statement is a homotopy-theoretic expression of the descent theory that we need. Thus, we get a *descent spectral sequence* for the homotopy groups of  $\text{pic}(\text{TMF})$ , which is a computational tool for understanding the aforementioned homotopy coherences concretely. We use this technique to compute  $\pi_0(\text{pic}(\text{TMF}))$ , the group we are after.

The descent spectral sequence has many consequences in cases where it degenerates simply for dimensional reasons, or in cases where the information sought is coarse. For instance, in a specific example (Proposition 2.4.9), we show that the Picard group of the  $E_\infty$ -ring  $C^*(S^1; \mathbb{Q}[\epsilon]/\epsilon^2)$  is given by  $\mathbb{Z} \times \mathbb{Q}$ , which yields a counterexample to a general conjecture of Balmer [5, Conjecture 74] on the Picard groups of certain tensor-triangulated categories. We also prove the following general results in Sections 4 and 5.

**Theorem C** *Let  $A$  be a weakly even periodic Landweber exact  $E_\infty$ -ring with  $\pi_0 A$  regular noetherian. Let  $n \geq 1$  be an integer, and let  $L_n$  denote localization with respect to the Lubin–Tate spectrum  $E_n$ . The Picard group of  $L_n A$  is*

$$\text{Pic}(L_n A) = \text{Pic}(\pi_* A) \times \pi_{-1}(L_n A),$$

where  $\text{Pic}(\pi_* A)$  refers to the (algebraic) Picard group of the graded commutative ring  $\pi_* A$ .

Note that  $\text{Pic}(\pi_* A)$  sits in an extension

$$0 \rightarrow \text{Pic}(\pi_0 A) \rightarrow \text{Pic}(\pi_* A) \rightarrow \mathbb{Z}/2 \rightarrow 0,$$

which is split if  $A$  is *strongly even periodic*.

**Theorem D** *Let  $A$  be an  $E_\infty$ -ring such that  $\pi_0 A$  is a field of characteristic zero and such that  $\pi_i A = 0$  for  $i > 0$ . Then  $\text{Pic}(A)$  is infinite cyclic, generated by  $\Sigma A$ .*

**Theorem E** *Let  $G$  be a finite group, and let  $A \rightarrow B$  be a faithful  $G$ -Galois extension of  $E_\infty$ -rings in the sense of Rognes [59]. Then the relative Picard group of  $B/A$ , ie the kernel of  $\text{Pic}(A) \rightarrow \text{Pic}(B)$ , is  $|G|$ -power torsion of finite exponent.*

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<sup>1</sup>See Ando, Blumberg, Gepner, Hopkins and Rezk [2] for a very important application.

For  $\mathrm{TMF}$ , the descent spectral sequence does not degenerate so nicely, and we need to work further to obtain our main results. The homotopy groups of the Picard spectrum of an  $E_\infty$ -ring  $A$ , starting with  $\pi_2$ , are simply those of  $A$ : in fact, we have a natural equivalence of spaces

$$\Omega^{\infty+2} \mathrm{pic}(A) \simeq \Omega^{\infty+1} A.$$

This determines the  $E_2$ -page and many of the differentials in the descent spectral sequence for  $\mathrm{Pic}(\mathrm{TMF})$ , but not all the ones that affect  $\pi_0$ . A key step in our argument is the identification of the differentials of the descent spectral sequence for the Picard spectra, in a certain *range of dimensions*, with that of the (known) descent spectral sequence for  $\pi_*(\mathrm{TMF})$ . We prove this in a general setting in Section 5.

At the prime 2, this technique is not sufficient to determine all the differentials in the descent spectral sequence, and we need to determine in addition the first “unstable” differential in the Picard spectral sequence (in comparison to the usual descent spectral sequence). We give a “universal” formula for this first differential in Theorem 6.1.1, which we hope will have further applications.

**Conventions** Throughout, we will write  $\mathcal{S}$  for the  $\infty$ -category of spaces,  $\mathcal{S}_*$  for the  $\infty$ -category of pointed spaces, and  $\mathrm{Sp}$  for the  $\infty$ -category of spectra. We will frequently identify abelian groups  $A$  with their associated Eilenberg–Mac Lane spectra  $HA$ . Finally, all spectral sequences are displayed with the Adams indexing convention, ie the vertical axis represents the cohomological degree, and the horizontal axis represents the total topological degree.

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## Part I Generalities

### 2 Picard groups

We begin by giving an introduction to Picard groups in stable homotopy theory. General references here include [26; 47].

## 2.1 Generalities

Let  $(\mathcal{C}, \otimes, \mathbf{1})$  be a symmetric monoidal category.

**Definition 2.1.1** The *Picard group* of  $\mathcal{C}$  is the group of isomorphism classes of objects  $x \in \mathcal{C}$  which are *invertible*, ie such that there exists an object  $y \in \mathcal{C}$  such that  $x \otimes y \simeq \mathbf{1}$ . We will denote this group by  $\text{Pic}(\mathcal{C})$ .

**Remark 2.1.2** If  $\mathcal{C}$  is a large category, then it is not necessarily clear that the Picard group is a set. However, in all cases of interest,  $\mathcal{C}$  will be *presentable* so that this will be automatic (see Remark 2.1.4).

When  $\mathcal{C}$  is the category of quasicohherent sheaves on a scheme (or stack)  $X$ , then this recovers the usual Picard group of  $X$ : line bundles are precisely the invertible objects. The principal goal of this paper is to compute a Picard group in a homotopy-theoretic setting.

We will repeatedly use the following simple principle, which follows from the observation that tensoring with an invertible object induces an autoequivalence of categories.

**Proposition 2.1.3** Let  $\mathcal{C}_0 \subset \mathcal{C}$  be a full subcategory that is preserved under any autoequivalence of  $\mathcal{C}$ . Suppose the unit object  $\mathbf{1} \in \mathcal{C}$  belongs to  $\mathcal{C}_0$ . Then any  $x \in \text{Pic}(\mathcal{C})$  belongs to  $\mathcal{C}_0$  as well.

For example, if  $\mathbf{1}$  is a compact object (that is, if  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot)$  commutes with filtered colimits), then so is  $x$ .

Suppose now that, more generally,  $\mathcal{C}$  is a symmetric monoidal  $\infty$ -category in the sense of [39], which is the setting that we will be most interested in. Then we can still define the Picard group  $\text{Pic}(\mathcal{C})$  of  $\mathcal{C}$ , which is the same as  $\text{Pic}(\text{Ho}(\mathcal{C}))$ . Moreover, Proposition 2.1.3 is valid, but where one is allowed to (and often should) use  $\infty$ -categorical properties.

**Remark 2.1.4** The theory of *presentable*  $\infty$ -categories [34, Section 5.5] enables one to address set-theoretic concerns. If  $\mathcal{C}$  is a presentable symmetric monoidal  $\infty$ -category, then the unit of  $\mathcal{C}$  is  $\kappa$ -compact for some regular cardinal  $\kappa$ . Therefore, by Proposition 2.1.3 (strictly speaking, its  $\infty$ -categorical analog), every invertible object of  $\mathcal{C}$  is  $\kappa$ -compact, and the collection of  $\kappa$ -compact objects of  $\mathcal{C}$  is essentially small. In particular, the collection of isomorphism classes forms a set and the Picard group is well defined.

**Example 2.1.5** Suppose that  $\mathcal{C}$  is a symmetric monoidal stable  $\infty$ -category such that the tensor product commutes with finite colimits in each variable. Then one has a natural homomorphism

$$\mathbb{Z} \rightarrow \text{Pic}(\mathcal{C}),$$

sending  $n \mapsto \Sigma^n \mathbf{1}$ .

**Example 2.1.6** Let  $\text{Sp}$  be the  $\infty$ -category of spectra with the smash product. Then it is a classical result [26, page 90] that  $\text{Pic}(\mathcal{C}) \simeq \mathbb{Z}$ , generated by the sphere  $S^1$ . A quick proof based on the above principle (which simplifies the argument in [26] slightly) is as follows. If  $T \in \text{Sp}$  is invertible, so that there exists a spectrum  $T'$  such that  $T \wedge T' \simeq S^0$ , then we need to show that  $T$  is a suspension of  $S^0$ .

Since the unit object  $S^0 \in \text{Sp}$  is compact, it follows that  $T$  is compact: that is, it is a finite spectrum. By suspending or desuspending, we may assume that  $T$  is connective,<sup>2</sup> and that  $\pi_0 T \neq 0$ . By the Künneth formula, it follows easily that  $H_*(T; F)$  is concentrated in one dimension for each field  $F$ . Since  $H_*(T; \mathbb{Z})$  is finitely generated, an argument with the universal coefficient theorem implies that  $H_*(T; \mathbb{Z})$  is torsion-free of rank one and is concentrated in dimension zero: ie  $H_0(T; \mathbb{Z}) \simeq \mathbb{Z}$ . By the Hurewicz theorem,  $T \simeq S^0$ .

**Example 2.1.7** Other variants of the stable homotopy category can have more complicated Picard groups. For instance, if  $E \in \text{Sp}$ , one can consider the  $\infty$ -category  $L_E \text{Sp}$  of  $E$ -local spectra, with the symmetric monoidal structure given by the  $E$ -localized smash product  $(X, Y) \mapsto L_E(X \wedge Y)$ . The Picard group of  $L_E \text{Sp}$  is generally much more complicated than  $\mathbb{Z}$ . When  $E$  is given by the Morava  $E$ -theories  $E_n$  or the Morava  $K$ -theories  $K(n)$ , the resulting Picard groups have been studied in [26; 27], among other references.

Another important example of this construction arises for  $R$  an  $E_\infty$ -ring, when we can consider the symmetric monoidal  $\infty$ -category  $\text{Mod}(R)$  of  $R$ -modules.

**Definition 2.1.8** Given an  $E_\infty$ -ring  $R$ , we write  $\text{Pic}(R)$  to denote the Picard group  $\text{Pic}(\text{Mod}(R))$ .

Using the same argument as in Example 2.1.6, it follows that any invertible  $R$ -module is necessarily compact (ie perfect): in particular, the invertible modules actually form a set rather than a proper class. Note that if  $R$  is simply an  $E_2$ -ring spectrum, then  $\text{Mod}(R)$  is a monoidal  $\infty$ -category, so one can still define a Picard group. This raises the following natural question.

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<sup>2</sup>We always use “connective” to mean “ $(-1)$ -connected”.

**Question 2.1.9** Is there an example of an  $E_2$ -ring whose Picard group is nonabelian?

We will only work with  $E_\infty$ -rings in the future, as it is for these highly commutative multiplications that we will be able to obtain good (from the point of view of descent theory) infinite loop spaces that realize  $\text{Pic}(R)$  on  $\pi_0$ .

## 2.2 Picard $\infty$ -groupoids

If  $(\mathcal{C}, \otimes, \mathbf{1})$  is a symmetric monoidal  $\infty$ -category, we reviewed in the previous section the *Picard group* of  $\mathcal{C}$ . There is, however, a more fundamental invariant of  $\mathcal{C}$ , where we remember all isomorphisms (and higher isomorphisms), and which behaves better with respect to descent processes.

**Definition 2.2.1** Let  $\text{Pic}(\mathcal{C})$  denote the  $\infty$ -groupoid (ie space) of *invertible objects* in  $\mathcal{C}$  and equivalences between them. We will refer to this as the *Picard  $\infty$ -groupoid* of  $\mathcal{C}$ ; it is a group-like  $E_\infty$ -space, and thus [45; 60] the delooping of a connective *Picard spectrum*  $\text{pic}(\mathcal{C})$ .

We have in particular

$$\pi_0 \text{Pic}(\mathcal{C}) \simeq \text{Pic}(\mathcal{C}).$$

However, we can also describe the higher homotopy groups of  $\text{Pic}(\mathcal{C})$ . Recall that since  $\mathcal{C}$  is symmetric monoidal,  $\text{End}(\mathbf{1})$  is canonically an  $E_\infty$ -space and  $\text{Aut}(\mathbf{1})$  consists of the grouplike components. Since

$$\Omega \text{Pic}(\mathcal{C}) \simeq \text{Aut}(\mathbf{1}),$$

we get the relations

$$\pi_1 \text{Pic}(\mathcal{C}) = (\pi_0 \text{End}(\mathbf{1}))^\times \quad \text{and} \quad \pi_i \text{Pic}(\mathcal{C}) = \pi_{i-1} \text{End}(\mathbf{1}) \quad \text{for } i \geq 2.$$

**Example 2.2.2** Let  $R$  be an  $E_\infty$ -ring. We will write

$$\mathcal{P}\text{ic}(R) \stackrel{\text{def}}{=} \text{Pic}(\text{Mod}(R)) \quad \text{and} \quad \text{pic}(R) \stackrel{\text{def}}{=} \text{pic}(\text{Mod}(R)).$$

Then  $\mathcal{P}\text{ic}(R)$  is a delooping of the *space of units*  $\text{GL}_1(R)$  studied in [46] and more recently using  $\infty$ -categorical techniques in [2]. In particular, the homotopy groups of  $\mathcal{P}\text{ic}(R)$  look very much like those of  $R$  (with a shift), starting at  $\pi_2$ . In fact, if we take the connected components at the basepoint, we have a natural equivalence of spaces

$$\tau_{\geq 1}(\text{GL}_1 R) \simeq \tau_{\geq 1}(\Omega \mathcal{P}\text{ic}(R)) \simeq \tau_{\geq 1}(\Omega^\infty R),$$

given by subtracting 1 with respect to the group structure on the infinite loop space  $\Omega^\infty R$ . Nonetheless, the *spectra*  $\text{pic}(R)$  and  $R$  are generally very different: that is, the infinite loop structure on  $\mathcal{P}\text{ic}(R)$  behaves very differently from that of  $\Omega^\infty R$ .



Unlike the group-valued functor  $\text{Pic}$ , both  $\mathcal{P}\text{ic}$  and  $\text{pic}$  have the fundamental property, upon which the calculations in this paper are based, that they commute with homotopy limits.

**Proposition 2.2.3** *The functor*

$$\text{pic}: \text{Cat}^{\otimes} \rightarrow \text{Sp}_{\geq 0},$$

from the  $\infty$ -category  $\text{Cat}^{\otimes}$  of symmetric monoidal  $\infty$ -categories to the  $\infty$ -category  $\text{Sp}_{\geq 0}$  of connective spectra, commutes with limits and filtered colimits, and the functor  $\mathcal{P}\text{ic} = \Omega^{\infty} \circ \text{pic}: \text{Cat}^{\otimes} \rightarrow \mathcal{S}_*$  does as well.

**Proof** We will treat the case of limits; the case of filtered colimits is similar and easier. It suffices to show that  $\mathcal{P}\text{ic}$  commutes with homotopy limits, since  $\Omega^{\infty}: \text{Sp}_{\geq 0} \rightarrow \mathcal{S}_*$  creates limits. Let  $\text{CAlg}(\mathcal{S})$  be the  $\infty$ -category of  $E_{\infty}$ -spaces. Now,  $\mathcal{P}\text{ic}$  is the composite  $\text{inv} \circ \bar{\iota}$  where:

- (1)  $\bar{\iota}: \text{Cat}^{\otimes} \rightarrow \text{CAlg}(\mathcal{S})$  sends a symmetric monoidal  $\infty$ -category to the symmetric monoidal  $\infty$ -groupoid (ie  $E_{\infty}$ -space) obtained by excluding all noninvertible morphisms.
- (2)  $\text{inv}: \text{CAlg}(\mathcal{S}) \rightarrow \mathcal{S}_*$  sends an  $E_{\infty}$ -space  $X$  to the union of those connected components which are invertible in the commutative monoid  $\pi_0 X$ , with basepoint given by the identity.

It thus suffices to show that  $\bar{\iota}$  and  $\text{inv}$  both commute with limits.

(1) The functor  $\iota: \text{Cat} \rightarrow \mathcal{S}$  that sends an  $\infty$ -category  $\mathcal{C}$  to its *core*  $\iota\mathcal{C}$  commutes with limits: in fact, it is right adjoint to the inclusion  $\mathcal{S} \rightarrow \text{Cat}$  that regards a space as an  $\infty$ -groupoid. See for instance [58, Section 17.2]. Now, to see that  $\bar{\iota}$  commutes with limits, we observe that limits either in  $\text{Cat}^{\otimes}$  or in  $\text{CAlg}(\mathcal{S})$  are calculated at the level of the underlying spaces (resp.  $\infty$ -categories), so the fact that  $\iota$  commutes with limits implies that  $\bar{\iota}$  does too.

(2) It is easy to see that  $\text{inv}$  commutes with arbitrary products. Therefore, we need to show that  $\text{inv}$  turns pullbacks in  $\text{CAlg}(\mathcal{S})$  into pullbacks in  $\mathcal{S}_*$ . We recall that if  $\mathcal{A}, \mathcal{B}$  are complete  $\infty$ -categories, then a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  preserves limits if and only if it preserves pullbacks and products [34, Proposition 4.4.2.7]. Suppose given a homotopy pullback

$$(2-1) \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in  $\text{CAlg}(S)$ ; we need to show that

$$\begin{array}{ccc} \text{inv}(A) & \longrightarrow & \text{inv}(B) \\ \downarrow & & \downarrow \\ \text{inv}(C) & \longrightarrow & \text{inv}(D) \end{array}$$

is one too, in  $S_*$ . Given the construction of  $\text{inv}$  as a union of connected components, it suffices to show that if  $x \in \pi_0 A$  has the property that  $x$  maps to invertible elements in the monoids  $\pi_0 B, \pi_0 C$ , then  $x$  itself is invertible.

To see this, consider the homotopy pullback square (2-1). Addition of  $x$  induces an endomorphism of the square. Since it acts via homotopy equivalences on  $B, C, D$ , it follows formally that it must act invertibly on  $A$ , ie that  $x \in \pi_0 A$  has an inverse.  $\square$

### 2.3 Descent

Let  $R \rightarrow R'$  be a morphism of  $E_\infty$ -rings. Recall the *cobar construction*, a cosimplicial  $E_\infty$ - $R$ -algebra

$$R' \rightrightarrows R' \otimes_R R' \rightrightarrows \cdots,$$

important in descent procedures, which receives an augmentation from  $R$ . The cobar construction is the *Čech nerve* (see [34, Section 6.1.2]) of  $R \rightarrow R'$ , in the opposite  $\infty$ -category.

**Definition 2.3.1** [37, Definition 5.2] We say that  $R \rightarrow R'$  is *faithfully flat* if the map  $\pi_0 R \rightarrow \pi_0 R'$  is faithfully flat and the natural map  $\pi_* R \otimes_{\pi_0 R} \pi_0 R' \rightarrow \pi_* R'$  is an isomorphism.

In this case, the theory of faithfully flat descent goes into effect. We have:

**Theorem 2.3.2** [37, Theorem 6.1] *Suppose  $R \rightarrow R'$  is a faithfully flat morphism of  $E_\infty$ -rings. Then the symmetric monoidal  $\infty$ -category  $\text{Mod}(R)$  can be recovered as the limit of the cosimplicial diagram of symmetric monoidal  $\infty$ -categories*

$$\text{Mod}(R') \rightrightarrows \text{Mod}(R' \otimes_R R') \rightrightarrows \cdots.$$

As a result, by Proposition 2.2.3,  $\text{Pic}(R)$  can be recovered as a totalization of spaces,

$$(2-2) \quad \text{Pic}(R) \simeq \text{Tot}(\text{Pic}(R'^{\otimes(\bullet+1)})).$$

Equivalently, one has an equivalence of connective spectra

$$(2-3) \quad \text{pic}(R) \simeq \tau_{\geq 0} \text{Tot}(\text{pic}(R'^{\otimes(\bullet+1)})).$$

In this paper, we will apply a version of this, except that we will work with morphisms of ring spectra that are not faithfully flat on the level of homotopy groups. As we will see, the descent spectral sequences given by (2-2) and (2-3) are not very useful in the faithfully flat case for our purposes.

**Example 2.3.3** A more classical example of this technique (eg [20, Exercise 6.9]) is as follows. Let  $X$  be a nodal cubic curve over the complex numbers  $\mathbb{C}$ . Then  $X$  can be obtained from its normalization  $\mathbb{P}^1$  by gluing together  $0$  and  $\infty$ . There is a pushout diagram of schemes:

$$\begin{array}{ccc} \{0, \infty\} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & X \end{array}$$

Therefore, one would *like* to say that the category  $\text{QCoh}(X)$  of quasicoherent sheaves on  $X$  fits into a homotopy pullback square

$$(2-4) \quad \begin{array}{ccc} \text{QCoh}(X) & \longrightarrow & \text{QCoh}(*) \\ \downarrow & & \downarrow \\ \text{QCoh}(\mathbb{P}^1) & \longrightarrow & \text{QCoh}(* \sqcup *) \end{array}$$

and that therefore the Picard *groupoid* of  $X$  fits into the homotopy cartesian square:

$$(2-5) \quad \begin{array}{ccc} \mathcal{P}ic(X) & \longrightarrow & \mathcal{P}ic(*) \\ \downarrow & & \downarrow \\ \mathcal{P}ic(\mathbb{P}^1) & \longrightarrow & \mathcal{P}ic(*) \times \mathcal{P}ic(*) \end{array}$$

Unfortunately, (2-4) is not a pullback square of categories, because restricting to a closed subscheme is not an exact functor. It is possible to remedy this (up to connectivity issues) by working with derived  $\infty$ -categories [36, Theorem 7.1], or by noting that we are working with locally free sheaves and applying a version of [49, Theorems 2.1–2.3]. In any event, one can argue that (2-5) is homotopy cartesian.

Alternatively, we obtain a homotopy pullback diagram of *connective* spectra. Using the long exact sequence on  $\pi_*$ , it follows that we have a short exact sequence

$$0 \rightarrow \mathbb{C}^\times \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(\mathbb{P}^1) \simeq \mathbb{Z} \rightarrow 0.$$

The approach of this paper is essentially an elaboration of this example.

### 2.4 Picard groups of $E_\infty$ -rings

We now specialize to the case of interest to us in this paper. Let  $R$  be an  $E_\infty$ -ring, and consider the Picard group  $\text{Pic}(R)$ , and better yet, the Picard  $\infty$ -groupoid  $\mathcal{P}\text{ic}(R)$  and the Picard spectrum  $\text{pic}(R)$ . The first of these has been studied by Baker and Richter in the paper [4], and we start by recalling some of their results.

We start with the following useful property.

**Proposition 2.4.1** *The functor  $R \mapsto \text{Pic}(R)$  commutes with filtered colimits in  $R$ .*

**Proof** This is a consequence of a form of “noetherian descent” [19, Section 8]. Given an  $E_\infty$ -ring  $T$ , let  $\text{Mod}^\omega(T)$  denote the  $\infty$ -category of perfect  $T$ -modules. If  $I$  is a filtered  $\infty$ -category and  $\{R_i\}_{i \in I}$  is a filtered system of  $E_\infty$ -rings indexed by  $I$ , then the functor of symmetric monoidal  $\infty$ -categories

$$(2-6) \quad \varinjlim_{i \in I} \text{Mod}^\omega(R_i) \rightarrow \text{Mod}^\omega(\varinjlim_I R_i)$$

is an equivalence. We outline the proof of this below.

Assume without loss of generality that  $I$  is a filtered partially ordered set and write  $R = \varinjlim_I R_i$ . To see that (2-6) is an equivalence, observe that the  $\infty$ -category  $\varinjlim_{i \in I} \text{Mod}^\omega(R_i)$  has objects given by pairs  $(M, i)$  where  $i \in I$  and  $M \in \text{Mod}^\omega(R_i)$ . The space of maps between  $(M, i)$  and  $(N, j)$  is given by

$$\varinjlim_{k \geq i, j} \text{Hom}_{\text{Mod}(R_k)}(R_k \otimes_{R_i} M, R_k \otimes_{R_j} N).$$

For instance, this implies that if  $i' \geq i$ , the pair  $(M, i)$  is (canonically) equivalent to the pair  $(R_{i'} \otimes_{R_i} M, i')$ . Thus, the assertion that (2-6) is fully faithful is equivalent to the assertion that if  $M, N \in \text{Mod}^\omega(R_i)$  for some  $i$ , then the natural map

$$(2-7) \quad \varinjlim_{j \geq i} \text{Hom}_{\text{Mod}^\omega(R_j)}(R_j \otimes_{R_i} M, R_j \otimes_{R_i} N) \rightarrow \text{Hom}_{\text{Mod}^\omega(R)}(R \otimes_{R_i} M, R \otimes_{R_i} N)$$

is an equivalence. But (2-7) is clearly an equivalence if  $M = R_i$  for any  $N$ . The collection of  $M \in \text{Mod}^\omega(R_i)$  such that (2-7) is an equivalence is closed under finite colimits, desuspensions, and retracts, and therefore it is all of  $\text{Mod}^\omega(R_i)$ . It therefore follows that (2-6) is fully faithful.

Moreover, the image of (2-6) contains  $R \in \text{Mod}^\omega(R)$  and is closed under desuspensions and cofibers (thus finite colimits). Let  $\mathcal{C} \subset \text{Mod}^\omega(R)$  be the subcategory generated by  $R$  under finite colimits and desuspensions. We have shown the image of the fully faithful functor (2-6) contains  $\mathcal{C}$ . Any object  $M \in \text{Mod}^\omega(R)$  is a retract of an

object  $X \in \mathcal{C}$ , associated to an idempotent map  $e: X \rightarrow X$ . We can “descend”  $X$  to some  $X_i \in \text{Mod}^\omega(R_i)$  and the map  $e$  to a self-map  $e_i: X_i \rightarrow X_i$  such that  $e_i^2$  is homotopic to  $e_i$ . As is classical, we use the idempotent  $e_i$  to split  $X_i$ ; see [52, Proposition 1.6.8] or the older [12] and [13, Theorem 5.3]. Explicitly, form the filtered colimit  $Y_i$  of  $X_i \xrightarrow{e_i} X_i \xrightarrow{e_i} \dots$ , which splits off  $X_i$ . The tensor product  $R \otimes_{R_i} Y_i$  is the direct summand of  $X$  given by the idempotent  $e$  and is therefore equivalent to  $M$ .

The association  $\mathcal{C} \mapsto \text{Pic}(\mathcal{C})$  commutes with filtered colimits of symmetric monoidal  $\infty$ -categories by Proposition 2.2.3. Taking Picard groups in the equivalence (2-6), the proposition follows.  $\square$

Purely algebraic information can be used to begin approaching  $\text{Pic}(R)$ . Let  $\text{Pic}(R_*)$  be the Picard group of the symmetric monoidal category of graded  $R_*$ -modules. The starting point of [4] is the following.

**Construction 2.4.2** There is a monomorphism

$$\Phi: \text{Pic}(R_*) \rightarrow \text{Pic}(R),$$

constructed as follows. If  $M_*$  is an invertible  $R_*$ -module, it has to be finitely generated and projective of rank one. Consequently, there is a finitely generated free  $R_*$ -module  $F_*$  of which  $M_*$  is a direct summand, ie there is a projection  $p_*$  with a section  $s_*$ :

$$F_* \begin{array}{c} \xleftarrow{s_*} \\ \xrightarrow{p_*} \end{array} M_*$$

Clearly,  $F_*$  can be realized as an  $R$ -module  $F$  which is a finite wedge sum of copies of  $R$  or its suspensions. Let  $e_*$  be the idempotent given by composition  $s_* \circ p_*$ . Since  $F$  is free over  $R$ ,  $e_*$  can be realized as an  $R$ -module map  $e: F \rightarrow F$  which must be idempotent. Define  $M$  to be the colimit of the sequence  $F \xrightarrow{e} F \xrightarrow{e} \dots$ , ie the image of the idempotent  $e$ . Observe that the homotopy groups of  $M$  are given by  $M_*$ , as desired. If  $M'_*$  is the inverse to  $M_*$  in the category of graded  $R_*$ -modules, we can construct an analogous  $R$ -module  $M'$ , and clearly  $M \otimes_R M' \simeq R$  by the degeneration of the Künneth spectral sequence. Thus,  $M \in \text{Pic}(R)$ . The association  $M_* \mapsto M$  defines  $\Phi$ .

Note that any two  $R$ -modules that realize  $M_*$  on homotopy groups are equivalent by the degeneration of the Ext spectral sequence, and that  $\Phi$  is a homomorphism by the degeneration of the Künneth spectral sequence. Observe also that  $\Phi$  is clearly a monomorphism as equivalences of  $R$ -modules are detected on homotopy groups.

**Definition 2.4.3** When  $\Phi$  is an isomorphism, we say that  $\text{Pic}(R)$  is *algebraic*.

Baker and Richter [4] determine certain conditions which imply algebraicity. There are, in particular, two fundamental examples. The first one generalizes Example 2.1.6.

**Theorem 2.4.4** [4] *Suppose  $R$  is a connective  $E_\infty$ -ring. Then the Picard group of  $R$  is algebraic.*

**Proof** Since the formulation in [4, Theorem 21] assumed a coherence hypothesis on  $\pi_*R$ , we explain briefly how this (slightly stronger) version can be deduced from the theory of flatness of [39, Section 8.2.2]. Recall that an  $R$ -module  $M$  is flat if  $\pi_0M$  is a flat  $\pi_0R$ -module and the natural map

$$\pi_*R \otimes_{\pi_0R} \pi_0M \rightarrow \pi_*M$$

is an isomorphism.

Since the Picard group commutes with filtered colimits in  $R$ , we may assume that  $R$  is finitely presented in the  $\infty$ -category of connective  $E_\infty$ -rings: in particular, by [39, Proposition 8.2.5.31],  $\pi_0R$  is a finitely generated  $\mathbb{Z}$ -algebra and in particular noetherian; moreover, each  $\pi_jR$  is a finitely generated  $\pi_0R$ -module. These are the properties that will be critical for us.

Let  $M$  be an invertible  $R$ -module. We will show that  $\pi_*M$  is a flat module over  $\pi_*R$ , which immediately implies the claim of the theorem. Localizing at a prime ideal of  $\pi_0R$ , we may assume that  $\pi_0R$  is a noetherian local ring; in this case we will show the Picard group is  $\mathbb{Z}$  generated by the suspension of the unit. We saw that  $M$  is perfect, so we can assume by shifting that  $M$  is connective and that  $\pi_0M \neq 0$ . Now for every map<sup>3</sup>  $R \rightarrow k$ , for  $k$  a field,  $\pi_*(M \otimes_R k)$  is necessarily concentrated in a single degree: in fact,  $M \otimes_R k$  is an invertible object in  $\text{Mod}(k)$  and one can apply the Künneth formula to see that  $\text{Pic}(\text{Mod}(k)) \simeq \mathbb{Z}$  generated by  $\Sigma k$ . By Nakayama’s lemma, since  $\pi_0M \neq 0$ , the homotopy groups of  $M \otimes_R k$  must be concentrated in degree zero. Thus,  $M \otimes_R k \simeq k$  itself. Using Lemma 2.4.5, it follows that  $M$  is equivalent to  $R$  as an  $R$ -module, so we are done.  $\square$

**Lemma 2.4.5** *Let  $R$  be a connective  $E_\infty$ -ring with  $\pi_0R$  noetherian local with residue field  $k$ . Suppose moreover each  $\pi_iR$  is a finitely generated  $\pi_0R$ -module. Suppose  $M$  is a connective (ie  $(-1)$ -connected) perfect  $R$ -module. Then, for  $n \geq 0$ , the following are equivalent:*

- (1)  $M \simeq R^n$ .
- (2)  $M \otimes_R k \simeq k^n$ .

---

<sup>3</sup>Recall that we are using the same symbol to denote an abelian group and its Eilenberg–Mac Lane spectrum.

**Proof** Suppose  $M \otimes_R k$  is isomorphic to  $k^n$  and concentrated in degree zero. Note that  $\pi_0(M \otimes_R k) \simeq \pi_0 M \otimes_{\pi_0 R} k$ . Choose a basis  $\bar{x}_1, \dots, \bar{x}_n$  of this  $k$ -vector space and lift these elements to  $x_1, \dots, x_n \in \pi_0 M$ . These define a map  $R^n \rightarrow M$  which induces an equivalence after tensoring with  $k$ , since  $M \otimes_R k \simeq k^n$ .

Now consider the cofiber  $C$  of  $R^n \rightarrow M$ . It follows that  $C \otimes_R k$  is contractible. Suppose  $C$  itself is not contractible. The hypotheses on  $\pi_* R$  imply that  $C$  is connective and each  $\pi_j C$  is a finitely generated module over the noetherian local ring  $\pi_0 R$ . If  $j$  is chosen minimal such that  $\pi_j C \neq 0$ , then

$$0 = \pi_j(C \otimes_R k) \simeq \pi_j C \otimes_{\pi_0 R} k,$$

and Nakayama’s lemma implies that  $\pi_j C = 0$ , a contradiction. □

Some of our analyses in the computational sections will rest upon the next result about the Picard groups of *periodic* ring spectra.

**Theorem 2.4.6** (Baker and Richter [4, Theorem 37]) *Suppose  $R$  is a weakly even periodic  $E_\infty$ -ring with  $\pi_0 R$  regular noetherian. Then the Picard group of  $R$  is algebraic.*

The result in [4, Theorem 37] actually assumes that  $\pi_0 R$  is a complete regular local ring. However, one can remove the hypotheses by replacing  $R$  with the localization  $R_{\mathfrak{p}}$  for any  $\mathfrak{p} \in \text{Spec } \pi_0 R$  and then forming the completion at the maximal ideal.

We will need a slight strengthening of Theorem 2.4.6, though.

**Corollary 2.4.7** *Suppose  $R$  is an  $E_\infty$ -ring satisfying the following assumptions:*

- (1)  $\pi_0 R$  is regular noetherian.
- (2) The  $\pi_0 R$ -module  $\pi_{2k} R$  is invertible for some  $k > 0$ .
- (3)  $\pi_i R = 0$  if  $i \not\equiv 0 \pmod{2k}$ .

*Then the Picard group of  $R$  is algebraic.*

**Proof** Using the obstruction theory of [3] (as well as localization), we can construct “residue fields” in  $R$  as  $E_1$ -algebras in  $\text{Mod}(R)$  (which will be  $2k$ -periodic rather than  $2$ -periodic). After this, the same argument as in Theorem 2.4.6 goes through. □

**Remark 2.4.8** If  $R$  is a ring spectrum satisfying the conditions of Corollary 2.4.7, then  $\text{Pic}(R) \cong \text{Pic}(\pi_* R)$  sits in a short exact sequence

$$0 \rightarrow \text{Pic}(\pi_0 R) \rightarrow \text{Pic}(\pi_* R) \rightarrow \mathbb{Z}/(2k) \rightarrow 0.$$

The extension is such that the  $(2k)^{\text{th}}$  power of a set-theoretic lift of a generator of  $\mathbb{Z}/(2k)$  to  $\text{Pic}(\pi_* R)$  is identified with the invertible  $\pi_0 R$ -module  $\pi_{2k} R$ .

An example of a nonalgebraic Picard group, based on [41, Example 7.1], is as follows.

**Proposition 2.4.9** *The Picard group of the rational  $E_\infty$ -ring  $R = \mathbb{Q}[\epsilon_0, \epsilon_{-1}]/\epsilon_0^2$  (free on two generators  $\epsilon_0$ , of degree 0, and  $\epsilon_{-1}$ , of degree  $-1$ , and with the relation  $\epsilon_0^2 = 0$ ) is given by  $\mathbb{Z} \times \mathbb{Q}$ .*

**Proof** The key observation is that  $R$  is equivalent, as an  $E_\infty$ -ring, to cochains over  $S^1$  on the (discrete)  $E_\infty$ -ring  $\mathbb{Q}[\epsilon_0]/\epsilon_0^2$ , because  $C^*(S^1; \mathbb{Q})$  is equivalent to  $\mathbb{Q}[\epsilon_{-1}]$ . By [40, Remark 7.9], we have a fully faithful, symmetric monoidal embedding  $\text{Mod}(R) \subset \text{Loc}_{S^1}(\text{Mod}(\mathbb{Q}[\epsilon_0]/\epsilon_0^2))$  into the  $\infty$ -category of local systems (see Definition 4.2.1 below) of  $\mathbb{Q}[\epsilon_0]/\epsilon_0^2$ -modules over the circle, whose image consists of those local systems of  $\mathbb{Q}[\epsilon_0]/\epsilon_0^2$ -modules such that the monodromy action of  $\pi_1(S^1)$  is ind-unipotent.

In particular, to give an object in  $\text{Pic}(R)$  is equivalent to giving an element in the Picard group  $\text{Pic}(\mathbb{Q}[\epsilon_0]/\epsilon_0^2)$  (of which there are only the suspensions of the unit, by Theorem 2.4.4) and an ind-unipotent (monodromy) automorphism, which is necessarily given by multiplication by  $1 + q\epsilon_0$  for  $q \in \mathbb{Q}$ . We observe that this gives the right group structure to the Picard group because  $(1 + q\epsilon_0)(1 + q'\epsilon_0) = 1 + (q + q')\epsilon_0$ .  $\square$

Proposition 2.4.9 provides a counterexample to [5, Conjecture 74], which states that in a tensor triangulated category generated by the unit with a local spectrum (eg with no nontrivial thick subcategories), any element  $\mathcal{L}$  in the Picard group has the property that  $\mathcal{L}^{\otimes n}$  is a suspension of the unit for suitable  $n > 0$ . In fact, one can take the (homotopy) category of perfect  $R$ -modules for  $R$  as in Proposition 2.4.9, which has no nontrivial thick subcategories by [41, Theorem 1.3].

**Remark 2.4.10** Other Picard groups of interest come from the theory of *stable module  $\infty$ -categories* of a  $p$ -group  $G$  over a field  $k$  of characteristic  $p$ , which from a homotopy-theoretic perspective can be expressed as the module  $\infty$ -categories of the Tate construction  $k^{tG}$ . The Picard groups of stable module  $\infty$ -categories have been studied in the modular representation theory literature (under the name *endotrivial modules*) starting with [10], where it is proved that the Picard group is algebraic (and cyclic) in the case where  $G$  is elementary abelian. The classification for a general  $p$ -group appears in [8].



### 3 The descent spectral sequence

In this section, we describe a descent spectral sequence for calculating Picard groups. The spectral sequence (studied originally by Gepner and Lawson [15] in a closely related setting) is based on the observation (Proposition 2.2.3) that the association  $\mathcal{C} \mapsto \mathcal{P}ic(\mathcal{C})$ , from symmetric monoidal  $\infty$ -categories to  $E_\infty$ -spaces, commutes with homotopy limits. We will describe several examples and applications of this in the present section. Explicit computations will be considered in later parts of this paper.

For example, let  $\{\mathcal{C}_U\}$  be a sheaf of symmetric monoidal  $\infty$ -categories on a site, and let  $\Gamma(\mathcal{C})$  denote the global sections (ie the homotopy limit)  $\infty$ -category. Then we have an equivalence of connective spectra

$$pic(\Gamma(\mathcal{C})) \simeq \tau_{\geq 0}\Gamma(pic(\mathcal{C}_U)),$$

and one can thus use the descent spectral sequence for a sheaf of spectra to approach the computation of  $pic(\Gamma(\mathcal{C}))$ . We will use this approach, together with a bit of descent theory, to calculate  $Pic(TMF)$ . The key idea is that while  $TMF$  itself has sufficiently complicated homotopy groups that results such as Theorem 2.4.6 cannot apply, the  $\infty$ -category of  $TMF$ -modules is built up as an inverse limit of module categories over  $E_\infty$ -rings with better behaved homotopy groups.

#### 3.1 Refinements

Let  $X$  be a Deligne–Mumford stack equipped with a flat map  $X \rightarrow M_{FG}$  to the moduli stack of formal groups. We will use the terminology of [42].

**Definition 3.1.1** An *even periodic refinement* of  $X$  is a sheaf  $\mathcal{O}^{top}$  of  $E_\infty$ -rings on the affine, étale site of  $X$ , such that for any étale map

$$Spec R \rightarrow X,$$

the multiplicative homology theory associated to the  $E_\infty$ -ring  $\mathcal{O}^{top}(Spec R)$  is functorially identified with the (weakly) even-periodic Landweber-exact theory<sup>4</sup> associated to the formal group classified by  $Spec R \rightarrow X \rightarrow M_{FG}$ . We will denote the refinement of the ordinary stack  $X$  by  $\mathfrak{X}$ .

A very useful construction from the refinement  $\mathfrak{X}$  is the  $E_\infty$ -ring of “global sections”  $\Gamma(\mathfrak{X}, \mathcal{O}^{top})$ , which is the homotopy limit of the  $\mathcal{O}^{top}(Spec R)$  as  $Spec R \rightarrow X$  ranges over the affine étale site of  $X$ .

<sup>4</sup>See [35, Lecture 18] for an exposition of the theory of weakly even-periodic theories.

**Example 3.1.2** When  $X$  is the moduli stack  $M_{\text{ell}}$  of elliptic curves, with the natural map  $M_{\text{ell}} \rightarrow M_{\text{FG}}$  that assigns to an elliptic curve its formal group, fundamental work of Goerss, Hopkins, and Miller, and (later) Lurie constructs an even periodic refinement  $\mathfrak{M}_{\text{ell}}$ . The global sections of  $\mathfrak{M}_{\text{ell}}$  are defined to be the  $E_\infty$ -ring TMF of *topological modular forms*; for a survey, see [16]. There is a similar picture for the compactified moduli stack  $\overline{M}_{\text{ell}}$ , whose global sections are denoted Tmf.

**Definition 3.1.3** Given the refinement  $\mathfrak{X}$ , one has a natural symmetric monoidal stable  $\infty$ -category  $\text{QCoh}(\mathfrak{X})$  of *quasicoherent sheaves* on  $\mathfrak{X}$ , given as a homotopy limit of the (stable symmetric monoidal)  $\infty$ -categories  $\text{Mod}(\mathcal{O}^{\text{top}}(\text{Spec } R))$  for each étale map  $\text{Spec } R \rightarrow X$ .

There is an adjunction

$$(3-1) \quad \text{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}})) \rightleftarrows \text{QCoh}(\mathfrak{X}),$$

where the left adjoint “tensors up” and the right adjoint takes global sections.<sup>5</sup>

Our main goal in this paper is to investigate the left hand side; however, the right hand side is sometimes easier to work with, since even periodic, Landweber-exact spectra have convenient properties. Therefore, the following result will be helpful.

**Theorem 3.1.4** [42, Theorem 4.1] *Suppose  $X$  is noetherian and separated, and  $X \rightarrow M_{\text{FG}}$  is quasiaffine. Then the adjunction (3-1) is an equivalence of symmetric monoidal  $\infty$ -categories.*

For example, since the map  $M_{\text{ell}} \rightarrow M_{\text{FG}}$  is affine, it follows that  $\text{Mod}(\text{TMF})$  is equivalent to  $\text{QCoh}(\mathfrak{M}_{\text{ell}})$ . This was originally proved by Meier, away from the prime 2, in [48]. Theorem 3.1.4 implies the analog for Tmf and the derived *compactified* moduli stack, as well [42, Theorem 7.2].

Suppose  $X \rightarrow M_{\text{FG}}$  is quasiaffine. In particular, it follows that there is a *sheaf* of symmetric monoidal  $\infty$ -categories on the affine, étale site of  $X$ , given by

$$(\text{Spec } R \rightarrow X) \rightarrow \text{Mod}(\mathcal{O}^{\text{top}}(\text{Spec } R)),$$

whose global sections are given by  $\text{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}}))$ . This diagram of  $\infty$ -categories is a sheaf in view of the descent theory of [37, Theorem 6.1], but [42, Theorem 4.1]

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<sup>5</sup>One way to extract this from [39] is to consider the thick subcategory  $\mathcal{C}$  of  $\text{QCoh}(\mathfrak{X}, \mathcal{O}^{\text{top}})$  generated by the unit. Then, one obtains by the universal property of Ind an adjunction  $\text{Ind}(\mathcal{C}) \rightleftarrows \text{QCoh}(\mathfrak{X}, \mathcal{O}^{\text{top}})$ . However, the symmetric monoidal  $\infty$ -category  $\text{Ind}(\mathcal{C})$  is generated under colimits by the unit, so it is by Lurie’s symmetric monoidal version [39, Proposition 8.1.2.7] of Schwede–Shpley theory equivalent to modules over  $\Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}})$ , which is the ring of endomorphisms of the unit.

gives the global sections. We are now in the situation of the introduction to this section. In particular, we obtain a descent spectral sequence for  $\text{pic}(\Gamma(X, \mathcal{O}^{\text{top}}))$ , and we turn to studying it in detail.

### 3.2 The Gepner–Lawson spectral sequence

Keep the notation of the previous subsection:  $X$  is a Deligne–Mumford stack equipped with a quasiaffine flat map  $X \rightarrow M_{\text{FG}}$ , and  $(\mathfrak{X}, \mathcal{O}^{\text{top}})$  is an even periodic refinement.

Our goal in this subsection is to prove:

**Theorem 3.2.1** *Suppose that  $X$  is a regular Deligne–Mumford stack with a quasiaffine flat map  $X \rightarrow M_{\text{FG}}$ , and suppose  $\mathfrak{X}$  is an even periodic refinement of  $X$ . There is a spectral sequence with*

$$(3-2) \quad E_2^{s,t} = \begin{cases} H^s(X, \mathbb{Z}/2) & \text{if } t = 0, \\ H^s(X, \mathcal{O}_{\mathfrak{X}}^{\times}) & \text{if } t = 1, \\ H^s(X, \omega^{(t-1)/2}) & \text{if } t \geq 3 \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

whose abutment is  $\pi_{t-s}\Gamma(\mathfrak{X}, \text{pic}(\mathcal{O}^{\text{top}}))$ . The differentials run  $d_r: E_r^{s,t} \rightarrow E^{s+r,t+r-1}$ .

The analogous spectral sequence for a faithful Galois extension has been studied in work of Gepner and Lawson [15], and our approach is closely based on theirs.

**Proof** In this situation, as we saw in the previous subsection, we get an equivalence of symmetric monoidal  $\infty$ -groupoids,

$$\text{Pic}(\Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}})) \simeq \text{holim}_{\text{Spec } R \rightarrow X} \text{Pic}(\mathcal{O}^{\text{top}}(\text{Spec } R)),$$

where  $\text{Spec } R \rightarrow X$  ranges over the affine étale maps. Equivalently, we have an equivalence of connective spectra

$$\text{pic}(\Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}})) \simeq \tau_{\geq 0}(\text{holim}_{\text{Spec } R \rightarrow X} \text{pic}(\mathcal{O}^{\text{top}}(\text{Spec } R))).$$

Let us study the descent spectral sequence associated to this. We need to understand the homotopy group *sheaves* of the sheaf of connective spectra

$$(\text{Spec } R \rightarrow X) \mapsto \text{pic}(\mathcal{O}^{\text{top}}(\text{Spec } R)),$$

ie the sheafification of the homotopy group presheaves

$$(\text{Spec } R \rightarrow X) \mapsto \pi_i \text{pic}(\mathcal{O}^{\text{top}}(\text{Spec } R)).$$

First, we know that

$$\pi_1 \text{pic}(\mathcal{O}^{\text{top}}(\text{Spec } R)) \simeq R^\times,$$

and, for  $i \geq 2$ , we have

$$\pi_i(\text{pic}(\mathcal{O}^{\text{top}}(\text{Spec } R)) \simeq \pi_{i-1} \mathcal{O}^{\text{top}}(\text{Spec } R) = \begin{cases} \omega^{(i-1)/2} & \text{for } i \text{ odd,} \\ 0 & \text{for } i \text{ even.} \end{cases}$$

It remains to determine the homotopy group sheaf  $\pi_0$ . If  $X$  is a regular Deligne–Mumford stack, so that each ring  $R$  that enters is regular, then we can do this using Theorem 2.4.6. In fact, it follows that if  $R$  is a local ring, then  $\pi_0 \text{pic}(\mathcal{O}^{\text{top}}(\text{Spec } R))$  is isomorphic to  $\mathbb{Z}/2$ . Thus, up to suitably suspending once, invertible sheaves are locally trivial. Using the descent spectral sequence for a sheaf of spectra, we get that the above descent spectral sequence for  $\Gamma(\mathfrak{X}, \text{pic}(\mathcal{O}^{\text{top}}))$  is almost entirely the same as the descent spectral sequence for  $\Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}})$  in the sense that the cohomology groups that appear for  $t \geq 3$ , ie  $H^s(X, \omega^{(t-1)/2})$ , are the same as those that appear in the descent spectral sequence for  $\Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}})$ . However, the terms for  $t = 1$  are the étale cohomology of  $\mathbb{G}_m$  on  $X$ . In particular, we obtain the term

$$H^1(X, \mathcal{O}_X^\times) \simeq \text{Pic}(X),$$

which is the Picard group of the underlying ordinary stack. □

**Remark 3.2.2** One may think of the spectral sequence as arising from a totalization, or rather as a filtered colimit of totalizations. Choose an étale hypercover  $\mathfrak{U}$  given by  $U_\bullet \rightarrow X$  by affine schemes  $\{U_n\}$ . For any  $E_\infty$ -ring  $A$ , denote by  $\mathcal{P}\text{ic}^{\mathbb{Z}}(A)$  the symmetric monoidal subcategory of  $\mathcal{P}\text{ic}(A)$  spanned by those  $A$ -modules such that, after restricting to each connected component of  $\text{Spec } \pi_0 A$ , become equivalent to a suspension of  $A$ . Denote by  $\text{pic}^{\mathbb{Z}}(A)$  the associated connective spectrum. Then we form the totalization

$$\text{Tot}(\text{pic}^{\mathbb{Z}}(\mathcal{O}^{\text{top}}(U_\bullet))),$$

whose associated infinite loop space  $\Omega^\infty \text{Tot}(\text{pic}^{\mathbb{Z}}(\mathcal{O}^{\text{top}}(U_\bullet)))$  is, by descent theory, the symmetric monoidal  $\infty$ -subgroupoid of  $\mathcal{P}\text{ic}(\Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}}))$  spanned by those invertible modules which become (up to a suspension) trivial after pullback along  $U_0 \rightarrow X$ . In particular, the filtered colimit of these totalizations is the spectrum we are after. The descent spectral sequence of Theorem 3.2.1 is the filtered colimit of these Tot spectral sequences.

### 3.3 Galois descent

We next describe the setting of the spectral sequence that was originally considered in [15]. Let  $A \rightarrow B$  be a faithful  $G$ -Galois extension of  $E_\infty$ -ring spectra in the

sense of [59]. In particular,  $G$  acts on  $B$  in the  $\infty$ -category of  $E_\infty$ - $A$ -algebras and  $A \rightarrow B^{hG}$  is an equivalence. Then  $A \rightarrow B$  is an analog of a  $G$ -Galois étale cover in the sense of ordinary commutative algebra or algebraic geometry. As in ordinary algebraic geometry, there is a good theory of *Galois descent* along  $A \rightarrow B$ , as has been observed by several authors, for instance in [15; 48].

**Theorem 3.3.1** (Galois descent) *Let  $A \rightarrow B$  be a faithful  $G$ -Galois extension of  $E_\infty$ -rings. Then there is a natural equivalence of symmetric monoidal  $\infty$ -categories  $\text{Mod}(A) \simeq \text{Mod}(B)^{hG}$ .*

The “strength” of the descent is in fact very good. As shown in [40, Theorem 3.36], any faithful Galois extension  $A \rightarrow B$  satisfies a form of descent up to nilpotence: the thick tensor-ideal that  $B$  generates in  $\text{Mod}(A)$  is equal to all of  $\text{Mod}(A)$ . This imposes strong restrictions on the descent spectral sequences that can arise.

Applying the Picard functor, we get an equivalence of spaces

$$(3-3) \quad \mathcal{P}\text{ic}(A) \simeq \mathcal{P}\text{ic}(B)^{hG},$$

or an equivalence of *connective* spectra

$$(3-4) \quad \text{pic}(A) \simeq \tau_{\geq 0} \text{pic}(B)^{hG}.$$

**Remark 3.3.2** The spectrum  $\Sigma \text{gl}_1 B$  is equivalent to  $\tau_{\geq 1} \text{pic}(B)$ ; consider the induced map of  $G$ -homotopy fixed point spectral sequences. All the differentials involving the  $t - s = 0$  line will be the same for  $\text{pic} B$  and  $\Sigma \text{gl}_1 B$ . Hence, we obtain a short exact sequence

$$0 \rightarrow \pi_0(\Sigma \text{gl}_1 B)^{hG} \rightarrow \pi_0(\text{pic}(B))^{hG} \rightarrow E_\infty^{0,0} \rightarrow 0,$$

where  $E_\infty^{0,0}$  is the kernel of all the differentials supported on  $H^0(G, \pi_0 \text{pic} B)$ . This short exact sequence exhibits  $\pi_0(\Sigma \text{gl}_1 B)^{hG}$  as the *relative Picard group* of  $A \rightarrow B$ , which consists of invertible  $A$ -modules which after smashing with  $B$  become isomorphic to  $B$  itself.

Our main interest in Galois theory, for the purpose of this paper, comes from the observation, due to Rognes, that there are numerous examples of  $G$ -Galois extensions of  $E_\infty$ -rings  $A \rightarrow B$  where the homotopy groups of  $B$  are significantly simpler than that of  $A$ . In particular, one hopes to understand the homotopy groups of  $\text{pic}(B)$ , and then use (3-3) and (3-4) together with an analysis of the associated homotopy fixed-point spectral sequence

$$(3-5) \quad H^s(G, \pi_t \text{pic}(B)) \Rightarrow \pi_{t-s}(\text{pic}(B))^{hG},$$

whose abutment for  $t = s$  is the Picard group  $\text{Pic}(A)$ .

**Example 3.3.3** [59, Proposition 5.3.1] The map  $\text{KO} \rightarrow \text{KU}$  and the  $C_2$ -action on  $\text{KU}$  arising from complex conjugation exhibit  $\text{KU}$  as a  $C_2$ -Galois extension of  $\text{KO}$ .

Example 3.3.3 is fundamental and motivational to us: the study of  $\text{KO}$ -modules, which is a priori difficult because of the complicated structure of the ring  $\pi_* \text{KO}$ , can be approached via Galois descent together with the (much easier) study of  $\text{KU}$ -modules. In particular, we obtain

$$\text{pic}(\text{KO}) \simeq \tau_{\geq 0} \text{pic}(\text{KU})^{hC_2},$$

and one can hope to use the homotopy fixed-point spectral sequence (HFPSS) to calculate  $\text{pic}(\text{KO})$ . This approach is due to Gepner and Lawson [15],<sup>6</sup> and we shall give a version of it below in Section 7.1 (albeit using a different method of deducing differentials).

Other examples of Galois extensions come from the theory of topological modular forms with *level structure*.

**Example 3.3.4** Let  $n \in \mathbb{N}$ . Let  $\text{TMF}(n)$  denote the periodic version of  $\text{TMF}$  for elliptic curves over  $\mathbb{Z}[\frac{1}{n}]$ -algebras with a *full level  $n$  structure*. Then, by [42, Theorem 7.6],  $\text{TMF}[\frac{1}{n}] \rightarrow \text{TMF}(n)$  is a faithful  $\text{GL}_2(\mathbb{Z}/n)$ -Galois extension. The advantage is that, if  $n \geq 3$ , the moduli stack of elliptic curves with level  $n$  structure is actually a regular affine scheme (by [30, Corollary 2.7.2], elliptic curves with full level  $n \geq 3$  structure have no nontrivial automorphisms). In particular,  $\text{TMF}(n)$  is even periodic with regular  $\pi_0$ , and one can compute its Picard group purely algebraically by Theorem 2.4.6. One can then hope to use  $\text{GL}_2(\mathbb{Z}/n)$ -descent to get at the Picard group of  $\text{TMF}[\frac{1}{n}]$ . We will take this approach below.

### 3.4 The $E_n$ -local sphere

In addition, descent theory can be used to give a spectral sequence for  $\text{pic}(L_n S^0)$ . This is related to work of Kamiya and Shimomura [29] and the upper bounds that they obtain on  $\text{Pic}(L_n S^0)$ .

Consider the cobar construction on  $L_n S^0 \rightarrow E_n$ , ie the cosimplicial  $E_\infty$ -ring

$$E_n \rightrightarrows E_n \wedge E_n \Rrightarrow \cdots,$$

whose homotopy limit is  $L_n S^0$ . It is a consequence of the Hopkins–Ravenel smash product theorem [56, Chapter 8] that this cosimplicial diagram has “effective descent”.

---

<sup>6</sup>The original calculation of the Picard group of  $\text{KO}$ , by related techniques, is unpublished work of Mike Hopkins.

**Proposition 3.4.1** *The natural functor*

$$\text{Mod}(L_n S^0) \rightarrow \text{Tot}(\text{Mod}(E_n^{\wedge(\bullet+1)})),$$

*is an equivalence of symmetric monoidal  $\infty$ -categories.*

**Proof** According to the Hopkins–Ravenel smash product theorem, the map of  $E_\infty$ -rings  $L_n S^0 \rightarrow E_n$  has the property that the thick tensor-ideal that  $E_n$  generates in  $\text{Mod}(L_n S^0)$  is all of  $\text{Mod}(L_n S^0)$ .<sup>7</sup>

According to [40, Proposition 3.21], this implies the desired descent statement (the condition is there called “admitting descent”). The argument is a straightforward application of the Barr–Beck–Lurie monadicity theorem [39, Section 6.2].  $\square$

In particular, we find that

$$\text{pic}(L_n S^0) \simeq \tau_{\geq 0} \text{Tot pic}(E_n^{\wedge(\bullet+1)}).$$

Let us try to understand the associated spectral sequence.

The higher homotopy groups,  $\pi_i$  for  $i \geq 2$ , of  $\text{pic}(E_n^{\wedge(\bullet+1)})$  are determined in terms of those of  $E_n^{\wedge(\bullet+1)}$ . Once again, it remains to determine  $\pi_0$ . Now  $E_n$  is an even periodic  $E_\infty$ -ring whose  $\pi_0$  is regular local, so  $\text{Pic}(E_n) \simeq \pi_0 \text{pic}(E_n) \simeq \mathbb{Z}/2$  by Theorem 2.4.6. The iterated smash products  $E_n^{\wedge m}$  are also even periodic, so their Picard group contains at least a  $\mathbb{Z}/2$ . We do not need to know their exact Picard groups, however, to run the spectral sequence, as only the  $\mathbb{Z}/2$  component is relevant for the spectral sequence (as it is all that comes from  $\pi_0 \text{pic}(E_n)$ ).

Next, we need to determine the algebraic Picard group. After taking  $\pi_0$ , the simplicial scheme

$$\cdots \rightrightarrows \text{Spec } \pi_0(E_n \wedge E_n) \rightrightarrows \text{Spec } \pi_0 E_n$$

is a presentation of the moduli stack  $M_{\text{FG}}^{\leq n}$  of formal groups (over  $\mathbb{Z}_{(p)}$ -algebras) of height at most  $n$ .

**Proposition 3.4.2**  $\text{Pic}(M_{\text{FG}}^{\leq n}) \simeq \mathbb{Z}$ , generated by  $\omega$ .

**Proof** We use the presentation of  $M_{\text{FG}}$  (localized at  $p$ ) via the simplicial stack

$$(3-6) \quad \cdots \rightrightarrows (\text{Spec}(MU \wedge MU)_*)/\mathbb{G}_m \rightrightarrows (\text{Spec } MU_*)/\mathbb{G}_m.$$

<sup>7</sup>The argument in [56, Chapter 8] is stated for the uncompleted Johnson–Wilson theories, but also can be carried out for the completed ones. We refer in particular to the lecture notes of Lurie [35]; Lecture 30 contains the necessary criterion for constancy of the Tot-tower.

Since the Picard group of a polynomial ring over  $\mathbb{Z}_{(p)}$  is trivial,<sup>8</sup> and each smash power of  $MU$  has a polynomial ring for  $\pi_*$ , the Picard group of each of the terms in the simplicial stack *without* the  $\mathbb{G}_m$ -quotient is trivial, and the group of units is  $\mathbb{Z}_{(p)}^\times$ , constant across the simplicial object. In other words, the *Picard groupoid* of each  $\text{Spec}(MU^{\wedge(s+1)})_*$  is  $B\mathbb{Z}_{(p)}^\times$ . When we add the  $\mathbb{G}_m$ -quotient, we get  $\mathbb{Z} \times B\mathbb{Z}_{(p)}^\times$  for the Picard groupoid of each term in the simplicial stack because of the possibility of twisting by a character of  $\mathbb{G}_m$ : this twisting corresponds to the powers of  $\omega$ . By descent theory, this shows that  $\text{Pic}(M_{\text{FG}}) \simeq \mathbb{Z}$ , generated by  $\omega$ . More precisely, the Picard groupoid of  $M_{\text{FG}}$  is the totalization of the Picard groupoids of  $\text{Spec}(MU^{\wedge(s+1)})_*/\mathbb{G}_m$ , and each of these is  $\mathbb{Z} \times B\mathbb{Z}_{(p)}^\times$ : that is, the cosimplicial diagram of Picard groupoids is constant and the totalization is  $\mathbb{Z} \times B\mathbb{Z}_{(p)}^\times$  again.

When we replace  $M_{\text{FG}}$  by  $M_{\text{FG}}^{\leq n}$ , we can replace the above presentation by excising from each term the closed substack cut out by  $(p, v_1, \dots, v_n)$ . This does not affect the Picard *groupoid* since the codimension of the substack removed is at least 2 (ie neither the Picard group nor the group of units is affected).<sup>9</sup> That is, when we modify each term in (3-6) to form the associated presentation of  $M_{\text{FG}}^{\leq n}$ , the Picard groupoid is unchanged. It follows by faithfully flat descent that the inclusion  $M_{\text{FG}}^{\leq n} \rightarrow M_{\text{FG}}$  induces an isomorphism on Picard groups (or groupoids) and that the Picard group is generated by  $\omega$ . □

We obtain the following result.

**Theorem 3.4.3** *There is a spectral sequence*

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/2 & \text{if } t = 0, \\ H^s(M_{\text{FG}}^{\leq n}, \mathcal{O}_{M_{\text{FG}}}^\times) & \text{if } t = 1, \\ H^s(M_{\text{FG}}^{\leq n}, \omega^{(t-1)/2}) & \text{if } t \geq 3 \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

which converges for  $t - s \geq 0$  to  $\pi_{t-s} \text{pic}(L_n S^0)$ . The relevant occurrences of the second case are  $H^0(M_{\text{FG}}^{\leq n}, \mathcal{O}_{M_{\text{FG}}}^\times) \simeq \mathbb{Z}_{(p)}^\times$  and  $H^1(M_{\text{FG}}^{\leq n}, \mathcal{O}_{M_{\text{FG}}}^\times) \simeq \mathbb{Z}$ .

Note in particular that the  $E_2$ -term is determined entirely in terms of the Adams–Novikov spectral sequence for the  $E_n$ -local sphere. As we will see in Section 5, many of the differentials are also determined by the ANSS.

<sup>8</sup>Since the Picard group commutes with filtered colimits, one reduces to the case of a polynomial ring on a finite number of variables, and here it follows from unique factorization.

<sup>9</sup>Once again, this is a familiar result for regular rings, and here one must pass to filtered colimits since one is working with polynomial rings on infinitely many variables.



## 4 First examples

In this section, we will give several examples where descent theory gives a quick calculation of the Picard group. In these examples, we will not need to analyze differentials in the descent spectral sequence (3-5). The main examples of interest, where there will be a number of differentials to determine, will be treated in the last part of this paper.

### 4.1 The faithfully flat case

We begin with the simplest case. Suppose  $R \rightarrow R'$  is a morphism of  $E_\infty$ -rings which is faithfully flat. In this case, we know from [37, Theorem 6.1] that the tensor-forgetful adjunction  $\text{Mod}(R) \rightleftarrows \text{Mod}(R')$  is comonadic and we get a descent spectral sequence for the Picard group of  $R$ , as

$$\text{pic}(R) \simeq \tau_{\geq 0} \text{Tot pic}(R' \otimes^{\bullet+1}).$$

This spectral sequence, however, gives essentially no new information that is not algebraic in nature. That is, the entire  $E_2$ -term  $E_2^{s,t}$  for  $t > 1$  vanishes, as it can be identified with the  $E_2$ -term for the cobar resolution  $R' \otimes^{\bullet+1}$  of  $R$ , and this cobar resolution has a degenerate spectral sequence with nonzero terms only for  $s = 0$  at  $E_2$ . For example, an element in  $\text{Pic}(R)$  is algebraic if and only if its image in  $\text{Pic}(R')$  is algebraic, by faithful flatness.

Thus, faithfully flat descent will be mostly irrelevant to us as a tool of computing the nonalgebraic parts of Picard groups. In the examples of interest, we want  $\pi_* R'$  to be significantly simpler homologically than  $\pi_* R$ , so that we will be able to conclude (using results such as Theorem 2.4.6) that the Picard group of  $R'$  is entirely algebraic. But if  $\pi_* R'$  is faithfully flat over  $\pi_* R$ , it cannot be much simpler homologically. (Recall for example that *regularity* descends under faithfully flat extensions of noetherian rings.)

### 4.2 Cochain $E_\infty$ -rings and local systems

In this subsection, we give another example of a family of  $E_\infty$ -ring spectra whose Picard groups can be determined, or at least bounded.

Let  $X$  be a space and  $R$  an  $E_\infty$ -ring. Let  $R^X = C^*(X; R)$  be the  $E_\infty$ -ring of  $R$ -valued cochains on  $X$ .

**Definition 4.2.1** Let  $\text{Loc}_X(\text{Mod}(R)) = \text{Fun}(X, \text{Mod}(R))$  denote the  $\infty$ -category of *local systems* of  $R$ -module spectra on  $X$ .

Then we have a fully faithful embedding of symmetric monoidal  $\infty$ -categories

$$\text{Mod}^\omega(R^X) \subset \text{Loc}_X(\text{Mod}(R)),$$

which sends  $R^X$  to the constant local system at  $R$  and is determined by that. As discussed in [40, Section 7], this embedding is often useful for relating invariants of  $R^X$  to those of  $R$ . In particular, since any invertible  $R^X$ -module is perfect, we have a fully faithful functor of  $\infty$ -groupoids

$$\text{Pic}(R^X) \rightarrow \text{Pic}(\text{Loc}_X(\text{Mod}(R))) = \text{Map}(X, \text{Pic}(\text{Mod}(R))),$$

where the last identification follows because  $\text{Pic}$  commutes with homotopy limits (Proposition 2.2.3). Thus, we get the following useful *upper bound* for the Picard group of  $R^X$ .

**Proposition 4.2.2** *If  $R$  is an  $E_\infty$ -ring and  $X$  is any space, then  $\text{Pic}(R^X)$  is a subgroup of  $\pi_0(\text{pic}(R)^X)$ .*

Without loss of generality, we will assume that  $X$  is connected. Note that we have a cofiber sequence

$$\Sigma \text{gl}_1(R) \rightarrow \text{pic}(R) \rightarrow H(\text{Pic}(R)),$$

where  $H(\text{Pic}(R))$  is the Eilenberg–Mac Lane spectrum associated to the group  $\text{Pic}(R)$ . If we take the long exact sequence after taking maps from  $X$ , we get an exact sequence

$$(4-1) \quad 0 \rightarrow \pi_{-1}(\text{gl}_1(R)^X) \rightarrow \pi_0(\text{pic}(R)^X) \rightarrow \text{Pic}(R).$$

Our object of interest,  $\text{Pic}(R^X)$ , is a subobject of the middle term, by the above proposition.

Let us unwind the exact sequence further. First, observe that the composite map  $\text{Pic}(R^X) \rightarrow \pi_0(\text{pic}(R)^X) \rightarrow \text{Pic}(R)$  comes from the map of  $E_\infty$ -rings  $R^X \rightarrow R$  given by choosing a basepoint of  $X$ . In particular, it is *split surjective* as it has a section given by  $R \rightarrow R^X$  (so (4-1) is a split exact sequence). Next, using the truncation map  $\text{gl}_1(R) \rightarrow HR_0^\times$ , we have a map  $\pi_{-1}(\text{gl}_1(R)^X) \rightarrow \pi_{-1}((HR_0^\times)^X) = \text{Hom}(\pi_1(X), R_0^\times)$ . We can understand this map in terms of  $\text{Pic}(R^X)$ . Very explicitly, suppose given an invertible  $R^X$ -module  $M$  with associated local system  $\mathcal{L} \in \text{Loc}_X(\text{Mod}(R))$ . Then if the image of  $M$  in  $\text{Pic}(R)$  is trivial, we conclude that  $\mathcal{L}_x \simeq R$  for any basepoint  $x \in X$ . An element in  $\pi_1(X, x)$  induces a monodromy automorphism of  $\mathcal{L}_x$  and thus defines an element of  $R_0^\times$ . This defines a map in  $\text{Hom}(\pi_1(X, x), R_0^\times)$ . Let  $\text{Pic}^0(R^X)$  denote the kernel of  $\text{Pic}(R^X) \rightarrow \text{Pic}(R)$ . Then we have just described the map

$$(4-2) \quad \text{Pic}^0(R^X) \xrightarrow{\phi} \text{Hom}(\pi_1(X, x), R_0^\times),$$

that comes from the exact sequence (4-1).

The monodromy action cannot be arbitrary, since this local system is not arbitrary: it is in the image of  $\text{Mod}^\omega(R^X)$  and therefore belongs to the thick subcategory generated by the unit. As in [40, Section 8], it follows that the monodromy action of any element of the fundamental group must be *ind-unipotent*. In particular, fix an element  $M$  of  $\text{Pic}^0(R^X)$ . Given any loop  $\gamma \in \pi_1(X, x)$ , the associated element  $u = u_{\gamma, M} \in R_0^\times$  under the homomorphism  $\phi(M): \text{Pic}^0(R^X) \rightarrow \text{Hom}(\pi_1(X, x), R_0^\times)$  of (4-2) must have the property that  $u - 1$  is nilpotent.

Hence if  $R_0$  is a *reduced* ring, we deduce from (4-1) the following conclusion.

**Corollary 4.2.3** *If  $R$  is an  $E_\infty$ -ring with  $\pi_0 R$  reduced, and  $X$  is any connected space, then we have a split short exact sequence*

$$0 \rightarrow A \rightarrow \text{Pic}(R^X) \rightarrow \text{Pic}(R) \rightarrow 0,$$

where  $A \subset \pi_{-1}(\text{gl}_1(R)^X)$  is contained in  $\pi_{-1}((\tau_{\geq 1} \text{gl}_1(R))^X) \subset \pi_{-1}((\text{gl}_1(R))^X)$ . In particular, if  $\pi_{-1}((\tau_{\geq 1} \text{gl}_1(R))^X) = 0$ , then  $\text{Pic}(R) \rightarrow \text{Pic}(R^X)$  is an isomorphism.

Again, we note that the map  $\pi_{-1}((\tau_{\geq 1} \text{gl}_1(R))^X) \rightarrow \pi_{-1}(\text{gl}_1(R)^X)$  is injective, by the long exact sequence and the fact that  $\pi_0(\text{gl}_1(R)^X) \rightarrow \pi_0((HR_0^\times)^X) \simeq R_0^\times$  is surjective.

As an application, we obtain a calculation of the Picard group of a nonconnective  $E_\infty$ -ring in a setting far from regularity.

**Theorem 4.2.4** *Let  $A$  be any finite abelian group and let  $E_n$  be Morava  $E$ -theory. Then the Picard group of  $E_n^{BA}$  is  $\mathbb{Z}/2$ , generated by the suspension  $\Sigma E_n^{BA}$ . The same conclusion holds for any finite group  $G$  whose  $p$ -Sylow subgroup is abelian, where  $p$  is the prime of definition for  $E_n$ .*

**Proof** We induct on the  $p$ -rank of  $A$ . When  $A$  has no  $p$ -torsion, then  $E_n^{BA} \simeq E_n$  and Theorem 2.4.6 implies that the Picard group is  $\mathbb{Z}/2$ .

If the  $p$ -rank of  $A$  is positive, write  $A \simeq \mathbb{Z}/p^m \times A'$  where the  $p$ -rank of  $A'$  has smaller cardinality than that of  $A$ . The inductive hypothesis gives us that the Picard group of  $E_n^{BA'}$  is  $\mathbb{Z}/2$ . Now  $E_n^{BA} \simeq (E_n^{BA'})^{B\mathbb{Z}/p^m}$ . Moreover,  $E_n^{BA'}$  is well known to be even periodic (though its  $\pi_0$  is not regular).<sup>10</sup>

<sup>10</sup>We refer to [25, Section 7] for a general analysis of the question of when  $E_n^{BG}$  is even-periodic for  $G$  a finite group.

We claim now that  $\pi_{-1}((\tau_{\geq 1} \mathfrak{gl}_1(E_n^{BA}))^{B\mathbb{Z}/p^m}) = 0$ . To see this, we note that the homotopy groups of  $\tau_{\geq 1} \mathfrak{gl}_1(E_n^{BA'})$  are concentrated in even degrees and are all given by torsion-free  $p$ -complete abelian groups. Therefore, the cohomology groups  $H^i(\mathbb{Z}/p^m, \pi_j \tau_{\geq 1} \mathfrak{gl}_1(E_n^{BA}))$  vanish if  $i$  is odd, since the  $\mathbb{Z}/p^m$ -action on them is trivial. In the homotopy fixed point spectral sequence for  $(\tau_{\geq 1} \mathfrak{gl}_1(E_n^{BA}))^{B\mathbb{Z}/p^m}$  (ie the Atiyah–Hirzebruch spectral sequence), there is no room for contributions to  $\pi_{-1}$ . In fact, there is no room for differentials at all, which indicates that any  $\text{lim}^1$  terms cannot occur either. Now Corollary 4.2.3 shows that the map  $E_n^{BA'} \rightarrow E_n^{BA}$  induces an equivalence on Picard groups, which completes the inductive step.

For the last claim, fix any finite group  $G$  with an abelian  $p$ -Sylow subgroup  $A \subset G$ . For any connected space  $X$ , denote as before  $\text{Pic}^0(R^X)$  the kernel of  $\text{Pic}(R^X) \rightarrow \text{Pic}(R)$ . We have a commutative square:

$$\begin{CD} \text{Pic}^0(E_n^{BG}) @>>> \text{Pic}^0(E_n^{BA}) \\ @VVV @VVV \\ \pi_{-1}(\tau_{\geq 1} \mathfrak{gl}_1(E_n)^{BG}) @>>> \pi_{-1}(\tau_{\geq 1} \mathfrak{gl}_1(E_n)^{BA}) \end{CD}$$

The bottom horizontal map is injective since  $\tau_{\geq 1} \mathfrak{gl}_1(E_n)$  is  $p$ -local and  $BG$  is  $p$ -locally a wedge summand of  $BA$  in view of the transfer  $\Sigma_+^\infty BG \rightarrow \Sigma_+^\infty BA$ , which has the property that the composite  $\Sigma_+^\infty BG \rightarrow \Sigma_+^\infty BA \rightarrow \Sigma_+^\infty BG$  is a  $p$ -local equivalence by inspection of  $p$ -local homology. It follows that  $\text{Pic}^0(E_n^{BG}) \rightarrow \text{Pic}^0(E_n^{BA})$  is injective, and since the latter is zero, the former must be as well.  $\square$

Recall that the spectrum  $E_1$  is  $p$ -complete complex  $K$ -theory.

**Proposition 4.2.5** *Let  $G$  be any finite group. Then the Picard group of  $E_1^{BG}$  is finite.*

**Proof** In fact,  $\pi_{-1}(\tau_{\geq 1} \mathfrak{gl}_1(E_1)^{BG})$  is finite. We know that  $\tau_{\geq 3} \mathfrak{gl}_1(E_1) \simeq \Sigma^4 ku_p^\wedge$  by a theorem of Adams and Priddy [1]. Moreover,  $(ku_p^\wedge)^*(BG)$  is finite in each odd dimension, by comparing with  $E_1^*(BG)$  which vanishes in odd dimensions. It follows now from Corollary 4.2.3 that the desired Picard group has to be finite.  $\square$

**Question 4.2.6** *Let  $G$  be any finite group. Can the Picard group of  $E_1^{BG}$  be any larger than  $\mathbb{Z}/2$ ? What about the higher Morava  $E$ -theories?*

### 4.3 Coconnective rational $E_\infty$ -rings

We can also determine the Picard groups of coconnective rational  $E_\infty$ -ring spectra. A rational  $E_\infty$ -ring  $R$  is said to be *coconnective* if:

- (1)  $\pi_0 R$  is a field (of characteristic zero).
- (2)  $\pi_i R = 0$  for  $i > 0$ .

**Theorem D** *If  $R$  is a coconnective rational  $E_\infty$ -ring, then the Picard group  $\text{Pic}(R)$  is infinite cyclic, generated by  $\Sigma R$ .*

**Proof** Let  $k = \pi_0 R$ . We use [38, Proposition 4.3.3] to conclude that  $R \simeq \text{Tot}(A^\bullet)$ , where  $A^\bullet$  is a cosimplicial  $E_\infty$ - $k$ -algebra with each  $A^i$  of the form  $k \oplus V[-1]$ , where  $V$  is a discrete  $k$ -vector space; the  $E_\infty$ -structure given is the “square-zero” one.

We thus begin with the case of  $R = k \oplus V[-1]$ : we will show that  $\text{Pic}(R) \simeq \mathbb{Z}$  in this case. Since  $\text{Pic}$  commutes with filtered colimits, we may assume that  $V$  is a finite-dimensional vector space. In this case,

$$R \simeq k^{S^1 \vee \dots \vee S^1},$$

where the number of copies of  $S^1$  in the wedge summand is equal to the dimension  $n = \dim_k V$ ; by [38, Proposition 4.3.1], any rational  $E_\infty$ -ring with these homotopy groups is equivalent to  $k \oplus V[-1]$ . But we can now use Corollary 4.2.3 to see that the Picard group of  $k^{S^1 \vee \dots \vee S^1}$  is  $\mathbb{Z}$ , generated by the suspension, because  $\tau_{\geq 1} \mathfrak{gl}_1(k) = 0$ .

Now suppose that  $R$  is arbitrary. As above, we have an equivalence  $R \simeq \text{Tot}(A^\bullet)$  where each  $A^i$  is a coconnective  $E_\infty$ -ring of the form  $k \oplus V[-1]$  for  $V$  a discrete  $k$ -vector space. We have seen above that  $\text{Pic}(A^i) \simeq \mathbb{Z}$ . We know, moreover, that we have a fully faithful embedding of symmetric monoidal  $\infty$ -categories

$$\text{Mod}^\omega(R) \subset \text{Tot}(\text{Mod}(A^\bullet)),$$

which implies that we have a fully faithful functor of  $\infty$ -groupoids

$$\text{Pic}(R) \rightarrow \text{Tot}(\text{Pic}(A^\bullet)).$$

But each  $\text{Pic}(A^i)$ , as an  $\infty$ -groupoid, has homotopy groups given by

$$\pi_j \text{Pic}(A^i) \simeq \begin{cases} \mathbb{Z} & \text{if } j = 0, \\ k^\times & \text{if } j = 1, \end{cases}$$

and in particular, in the cosimplicial diagram  $\text{Pic}(A^\bullet)$ , all the maps are *equivalences*. This is a helpful consequence of coconnectivity. Therefore we find that  $\text{Tot}(\text{Pic}(A^\bullet))$  maps by equivalences to each  $\text{Pic}(A^i)$ , and we get an upper bound of  $\mathbb{Z}$  for  $\text{Pic}(R)$ . This upper bound is realized by the suspension  $\Sigma R$  (which hits the generator of  $\mathbb{Z} \simeq \pi_0 \text{Tot}(\text{Pic}(A^\bullet))$ ). □

**Remark 4.3.1** If  $k = \mathbb{Q}$ , then a large class of coconnective  $E_\infty$ -rings with  $\pi_0 \simeq \mathbb{Q}$  (eg those with reasonable finiteness hypotheses and vanishing  $\pi_{-1}$ ) arise as cochains on a simply connected space, by Quillen and Sullivan’s rational homotopy theory. The comparison with local systems can be carried out directly here to prove Theorem D for these  $E_\infty$ -rings.

### 4.4 Quasiaffine cases

We now consider a case where the descent spectral sequence enables us to *produce* nontrivial elements in the Picard group. Let  $A$  be a weakly even-periodic  $E_\infty$ -ring with  $\pi_0 A$  regular noetherian, and write  $\omega = \pi_2 A$ . Then  $A$  leads to a sheaf of  $E_\infty$ -rings on the affine étale site of  $\text{Spec } \pi_0 A$ . That is, for every étale  $\pi_0 A$ -algebra  $A'_0$ , there is (functorially) associated [39, Section 8.5] an  $E_\infty$ -ring  $A'$  under  $A$  with  $\pi_0 A' \simeq A'_0$  and  $A'$  flat over  $A$ . We will denote this sheaf by  $\mathcal{O}^{\text{top}}$ .

Let  $a_1, \dots, a_n \in \pi_0 A$  be a regular sequence, for  $n \geq 2$ . We consider the complement  $U$  in  $\text{Spec } \pi_0 A$  of the closed subscheme  $V(a_1, \dots, a_n)$  and the sections  $\bar{A} = \Gamma(U, \mathcal{O}^{\text{top}})$ .  $\bar{A}$  is an  $E_\infty$ - $A$ -algebra and is a type of localization of  $A$ , albeit not (directly) an arithmetic one.<sup>11</sup> Note that  $\text{Pic}(A)$  is algebraic by Theorem 2.4.6, but the situation for  $\bar{A}$  is more complicated.

The homotopy groups  $\pi_*(\bar{A})$  are given by the abutment of a descent spectral sequence

$$(4-3) \quad H^s(U, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(\bar{A}).$$

We can first determine the zero-line. We have

$$H^0(U, \omega^{\otimes t}) = H^0(\text{Spec } \pi_0 A, \omega^{\otimes t}),$$

because  $\text{Spec } \pi_0 A$  is regular and  $U \subset \text{Spec } \pi_0 A$  is obtained by removing a subscheme of codimension at least two.

**Proposition 4.4.1** *The only other nonzero term in the descent spectral sequence (4-3) occurs for  $s = n - 1$ . The descent spectral sequence degenerates.*

**Proof** Cover the scheme  $U$  by the  $n$  open affine subsets  $U_i = \text{Spec } \pi_0(A) \setminus V(a_i)$ , for  $1 \leq i \leq n$ . Given any quasicohherent sheaf  $\mathcal{F}$  on  $U$ , it follows that the coherent cohomology  $H^*(U, \mathcal{F})$  is that of the Čech complex (which starts in degree zero)

$$\bigoplus_{i=1}^n \mathcal{F}(U_i) \rightarrow \bigoplus_{i < j} \mathcal{F}(U_i \cap U_j) \rightarrow \dots \rightarrow \mathcal{F}(U_1 \cap \dots \cap U_n).$$

<sup>11</sup>Forthcoming work of Bhatt and Halpern-Leistner identifies the universal property of  $\bar{A}$ .

Let  $R = \pi_0 A$ , and suppose  $\mathcal{F}$  is the restriction to  $U \subset \text{Spec } R$  of the quasicoherent sheaf  $\tilde{M}$  on  $\text{Spec } R$  for an  $R$ -module  $M$ . Then the final term is the cokernel of the map

$$\bigoplus_{i=1}^n M[(a_1 \cdots \widehat{a}_i \cdots a_n)^{-1}] \rightarrow M[(a_1 \cdots a_n)^{-1}],$$

where the hat denotes omission. If  $M$  is flat, the complex is exact away from degrees 0 and  $n - 1$  as the sequence  $a_1, \dots, a_n$  is regular, using a Koszul complex argument (see [28] for a detailed treatment or [18] for a short exposition with a view towards topological applications), and the zeroth cohomology is given by  $M$  itself.

Now, in view of the map  $A \rightarrow \bar{A}$ , clearly everything in the zero-line of the  $E_2$ -page of the spectral sequence survives, so the spectral sequence must degenerate.  $\square$

We now study the Picard group of  $\bar{A}$ : as above,  $\pi_* \bar{A}$  is not regular but instead has a great deal of square-zero material. Let  $\mathcal{U} = (U, \mathcal{O}^{\text{top}}|_U)$  denote the derived scheme consisting of the topological space  $U \subset \text{Spec } \pi_0 A$ , but equipped with the sheaf  $\mathcal{O}^{\text{top}}$  of  $E_\infty$ -rings restricted to  $U$ .  $\bar{A}$  arises as the global sections of the structure sheaf  $\mathcal{O}^{\text{top}}$  over the derived scheme  $\mathcal{U}$ .

Since  $U$  is quasiaffine as an (ordinary!) scheme, it follows by [42, Corollary 3.24] that the global sections functor is the right adjoint of an inverse equivalence

$$\text{Mod}(\bar{A}) \rightleftarrows \text{QCoh}(\mathcal{U}),$$

of symmetric monoidal  $\infty$ -categories. In particular, the Picard group  $\text{Pic}(\bar{A})$  can be computed as  $\text{Pic}(\text{QCoh}(\mathcal{U}))$ .

As before, we have a descent spectral sequence (3-2) converging to  $\pi_{t-s} \text{pic}(\bar{A})$ . But from (3-2), we know that almost all of the terms at  $E_2$  are identified with the descent spectral sequence for  $\pi_* \bar{A}$ . In addition, we know that  $H^1(U, \mathcal{O}_U^\times) \simeq \text{Pic}(\pi_0 A)$ , as  $\pi_0 A$  is regular and the complement of  $U$  has codimension  $\geq 2$ . These classes must be permanent cycles as they are realized in  $\text{Pic}(\bar{A})$ : in fact, they are realized in  $\text{Pic}(A)$  itself. Thus, the descent spectral sequence for  $\text{pic}$  degenerates as well. We get three contributions to the Picard group:  $\mathbb{Z}/2$  and  $\text{Pic}(\pi_0 A)$ , which together build  $\text{Pic}(\pi_* A)$  (compare Remark 2.4.8), and a group that is identified with  $\pi_{-1} \bar{A}$ . The relevant extension problem is solved because of the map  $\text{Pic}(\pi_* A) \cong \text{Pic}(A) \rightarrow \text{Pic}(\bar{A})$  realizing the algebraic part of the Picard group. We get:

**Theorem 4.4.2** *Let  $\bar{A} = \Gamma(U, \mathcal{O}^{\text{top}})$  as above. Then we have a natural isomorphism*

$$\text{Pic}(\bar{A}) \simeq \text{Pic}(\pi_* A) \times \pi_{-1}(\bar{A}).$$

Moreover, observe that

$$(4-4) \quad \pi_{-1}(\bar{A}) = \begin{cases} \operatorname{coker}\left(\bigoplus_{i=1}^n \omega^{n/2-1}[(a_1 \cdots \widehat{a}_i \cdots a_n)^{-1}]\right) \\ \quad \rightarrow \omega^{n/2-1}[(a_1 \cdots a_n)^{-1}] & \text{for } n \geq 4 \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

**Example 4.4.3** Let  $A$  be a Landweber-exact weakly even periodic  $E_\infty$ -ring with  $\pi_0 A$  regular noetherian; for instance,  $A$  could be Morava  $E$ -theory  $E_n$ . In this case, we take  $a_1, \dots, a_k = p, v_1, \dots, v_{k-1}$ , so that  $\bar{A} \simeq L_k A$ . This gives Theorem C as a special case of Theorem 4.4.2.

## Part II Computational tools

### 5 The comparison tool in the stable range

This is a technical section in which we develop a tool that will enable us to compare many of the differentials in a Picard spectral sequence for Galois or étale descent with the analogous differentials in the corresponding descent spectral sequence before taking the Picard functor (ie for the  $E_\infty$ -rings themselves). For example, in the Galois descent setting, we are given a  $G$ -Galois extension  $A \rightarrow B$ , and we know the descent, ie homotopy fixed point, spectral sequence for  $A \simeq B^{hG}$ . The tool we develop in this section will allow us to deduce many differentials in the homotopy fixed point spectral sequence for  $(\operatorname{pic}(B))^{hG}$ .

For a spectrum or a pointed space  $X$ , and integers  $a, b$ , we denote by  $\tau_{\geq a} X$ ,  $\tau_{\leq b} X$  and  $\tau_{[a,b]} X$  the truncations of  $X$  with homotopy groups in the designated range. Our main observation is that if  $R$  is any  $E_\infty$ -ring, then for any  $n \geq 2$ , there is a natural equivalence of spectra

$$\tau_{[n,2n-1]} R \simeq \tau_{[n,2n-1]} \mathfrak{gl}_1(R).$$

This equivalence is natural at the level of  $\infty$ -categories, and enables us to identify a large number of differentials in descent spectral sequences for  $\mathfrak{gl}_1$  and therefore also for  $\operatorname{pic}$ . This observation, however, fails if we increase the range by 1, and an identification of the relevant discrepancy (as observed in such spectral sequences) will be the subject of the following section and the formula (6-1).

The main result of Section 5.1 is essentially a formulation of the classical concept of the “stable range” in  $\infty$ -categorical terms, as can be seen from the fact that the major ingredients of the proof are Freudenthal’s suspension theorem as well as the existence



of Whitehead products in the unstable setting. Nonetheless, our formulation will be extremely useful in the sequel.

### 5.1 Truncated spaces and spectra

Throughout,  $n \geq 2$ .

**Definition 5.1.1** Let  $\text{Sp}_{[n,2n-1]} \subset \text{Sp}$  denote the  $\infty$ -category of spectra with homotopy groups concentrated in degrees  $[n, 2n - 1]$ . Let  $\mathcal{S}_*$  denote the  $\infty$ -category of pointed spaces, and let  $\mathcal{S}_{*,[a,b]} \subset \mathcal{S}_*$  denote the subcategory spanned by those pointed spaces whose homotopy groups are concentrated in the interval  $[a, b]$ .

The main goal of this subsection is to prove the following result identifying spaces and spectra whose homotopy groups are concentrated in a range of dimensions.

**Theorem 5.1.2** *The functor  $\Omega^\infty: \text{Sp}_{[n,2n-1]} \rightarrow \mathcal{S}_*$  is fully faithful. The functor  $\Omega^\infty: \text{Sp}_{[n,2n-2]} \rightarrow \mathcal{S}_{*,[n,2n-2]}$  is an equivalence of  $\infty$ -categories.*

**Proof** Let  $X, Y \in \text{Sp}_{[n,2n-1]}$ . We want to show that the natural map

$$(5-1) \quad \text{Hom}_{\text{Sp}}(X, Y) \rightarrow \text{Hom}_{\mathcal{S}_*}(\Omega^\infty X, \Omega^\infty Y)$$

is a homotopy equivalence. By adjointness, we can identify this with the map

$$\text{Hom}_{\text{Sp}}(X, Y) \rightarrow \text{Hom}_{\text{Sp}}(\Sigma^\infty \Omega^\infty X, Y)$$

that arises from the counit map  $\Sigma^\infty \Omega^\infty X \rightarrow X$ . Observe that we have a natural equivalence  $\text{Hom}_{\text{Sp}}(\Sigma^\infty \Omega^\infty X, Y) \simeq \text{Hom}_{\text{Sp}}(\tau_{\leq 2n-1} \Sigma^\infty \Omega^\infty X, Y)$  because  $Y$  is  $(2n-1)$ -truncated. In particular, to prove Theorem 5.1.2, it will suffice to show that the natural map of spectra

$$\tau_{\leq 2n-1} \Sigma^\infty \Omega^\infty X \rightarrow X \simeq \tau_{\leq 2n-1} X,$$

is an equivalence, for any  $X \in \text{Sp}_{[n,2n-1]}$ . Equivalently, we need to show that for any such spectrum  $X$ , the map

$$(5-2) \quad \pi_k(\Sigma^\infty \Omega^\infty X) \rightarrow \pi_k(X)$$

is an isomorphism for  $k \leq 2n - 1$ . But we have maps of spaces

$$\Omega^\infty X \rightarrow \Omega^\infty \Sigma^\infty \Omega^\infty X \rightarrow \Omega^\infty X,$$

where the composite is the identity. The first map is the unit  $Y \rightarrow \Omega^\infty \Sigma^\infty Y$  applicable for any  $Y \in \mathcal{S}_*$ , and the second map is  $\Omega^\infty$  applied to the counit. By the Freudenthal

suspension theorem, the first map induces an isomorphism on homotopy groups  $\pi_k$  for  $k \leq 2n - 1$ , and therefore the second map does as well. This proves the claim that (5-2) is an equivalence and the first part of the theorem.

The functor  $\Omega^\infty: \text{Sp}_{[n,2n-1]} \rightarrow \mathcal{S}_{*,[n,2n-1]}$  is not essentially surjective, because spaces with homotopy groups concentrated in degrees  $[n, 2n - 1]$  can still have *Whitehead products*, and spaces with nontrivial Whitehead products can never be in the image of  $\Omega^\infty$ . However, we claimed in the statement of the theorem that the functor  $\Omega^\infty: \text{Sp}_{[n,2n-2]} \rightarrow \mathcal{S}_{*,[n,2n-2]}$  is an equivalence of  $\infty$ -categories. To show this, it suffices to show that the functor is essentially surjective.

Given a pointed space  $X$  with homotopy groups in the desired range, we suppose inductively (on  $k$ ) that  $\tau_{\leq k} X$  is in the image of  $\Omega^\infty$ . If  $k \geq 2n - 2$ , then we are done. Otherwise, we have a pullback square:

$$\begin{array}{ccc} \tau_{\leq k+1} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau_{\leq k} X & \longrightarrow & K(\pi_{k+1} X, k+2) \end{array}$$

Observe that the pointed spaces  $\tau_{\leq k} X$ ,  $K(\pi_{k+1} X, k+2)$  and  $*$  are all in the image of  $\Omega^\infty$  (the first by the inductive hypothesis), and  $K(\pi_{k+1} X, k+2) \in \mathcal{S}_{*,[n,2n-1]}$ . Moreover, the *maps* in the diagram are in the image of  $\Omega^\infty$  by the previous part of the result. Therefore, the object  $\tau_{\leq k+1} X$  is in the image of  $\Omega^\infty$ , as  $\Omega^\infty$  preserves homotopy fiber squares. □

Given an integer  $k$ , we could precompose the functor of Theorem 5.1.2 with the equivalence  $\Omega^k: \text{Sp}_{[n+k,2n+k-1]} \rightarrow \text{Sp}_{[n,2n-1]}$ , and obtain the following:

**Corollary 5.1.3** *For any integer  $k$ , the functor  $\Omega^{\infty+k}: \text{Sp}_{[n+k,2n+k-1]} \rightarrow \mathcal{S}_*$  is fully faithful.*

### 5.2 Comparisons for $E_\infty$ -rings

Our basic example for all this comes from the spectrum  $\text{gl}_1(R)$  associated to an  $E_\infty$ -ring  $R$ , and the comparison between the two. This comparison is the main obstacle in understanding the descent spectral sequence for the Picard group: it is generally easier to understand descent spectral sequences for the  $E_\infty$ -rings themselves (eg for TMF).

We emphasize again that given an  $E_\infty$ -ring  $R$ , the *spectra*  $R$  and  $\text{gl}_1(R)$  are generally very different, and for an illustration we provide the following example.

**Example 5.2.1** (Lawson [33]) Consider the commutative differential graded algebra  $\mathbb{F}_2[x]/x^3$  where  $|x| = 1$  and  $dx = 0$  (so  $d \equiv 0$ ). Let  $R$  be the associated  $E_\infty$ -ring under  $\mathbb{F}_2$ . Then  $\mathfrak{gl}_1(R)$  has homotopy groups in dimensions 1, 2 given by  $\mathbb{F}_2$ ; however, they are connected by multiplication by  $\eta$ . In particular,  $\mathfrak{gl}_1(R)$  is not an  $\mathbb{F}_2$ -module spectrum.

More generally, let  $R$  be the  $E_\infty$ -ring associated to the commutative differential graded algebra  $\mathbb{F}_2[x]/x^3$  where  $|x| = n$  and  $dx = 0$ .  $R$  can also be constructed by applying the Postnikov section  $\tau_{\leq 2n}$  to the free  $E_\infty$ - $\mathbb{F}_2$ -algebra on a class in degree  $n$ . Then  $\pi_n(\mathfrak{gl}_1(R)) \simeq \pi_{2n}(\mathfrak{gl}_1(R)) \simeq \mathbb{F}_2$  and all the other homotopy groups of  $\mathfrak{gl}_1(R)$  vanish. Therefore,  $\mathfrak{gl}_1(R)$  is the fiber of a  $k$ -invariant map  $H\mathbb{F}_2[n] \rightarrow H\mathbb{F}_2[2n + 1]$ . In this case, we can identify the  $k$ -invariant and thus identify  $\mathfrak{gl}_1(R)$ .

**Proposition 5.2.2** Given  $R$  as above, the  $k$ -invariant of  $\mathfrak{gl}_1(R)$  is given by the map

$$\mathrm{Sq}^{n+1}: H\mathbb{F}_2[n] \rightarrow H\mathbb{F}_2[2n + 1].$$

**Proof** We begin by arguing, following Lawson, that  $\mathfrak{gl}_1(R)$  cannot be the spectrum  $H\mathbb{F}_2[n] \vee H\mathbb{F}_2[2n]$ . In fact, in this case, the map of spectra  $H\mathbb{F}_2[n] \rightarrow \mathfrak{gl}_1(R)$  would by adjointness [2] lead to a map of  $E_\infty$ -rings

$$\Sigma_+^\infty K(\mathbb{F}_2, n) \rightarrow R,$$

carrying the class in  $\pi_n K(\mathbb{F}_2, n)$  to the nonzero class in  $\pi_n R$ . Smashing with  $H\mathbb{F}_2$ , we would get a map of  $E_\infty$ - $H\mathbb{F}_2$ -algebras

$$H\mathbb{F}_2 \wedge \Sigma_+^\infty K(\mathbb{F}_2, n) \rightarrow R$$

with the same property. Now  $\pi_n(H\mathbb{F}_2 \wedge \Sigma_+^\infty K(\mathbb{F}_2, n)) \simeq \mathbb{F}_2$ , with the nontrivial class coming from  $\pi_n(K(\mathbb{F}_2, n))$ . However, this class squares to zero by [9, Lemma 6.1, Chapter 1] while the nonzero class in  $\pi_n R$  does not square to zero. This is a contradiction and proves that such a map cannot exist. Consequently, the  $k$ -invariant map for  $\mathfrak{gl}_1(R)$  must be nontrivial.

On the other hand,  $\Omega^\infty \mathfrak{gl}_1(R) \simeq K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, 2n)$  because  $\Omega^\infty \mathfrak{gl}_1(R)$  is the connected component at 1 of  $\Omega^\infty R$ . In particular, the  $k$ -invariant  $H\mathbb{F}_2[n] \rightarrow H\mathbb{F}_2[2n + 1]$  defines, upon applying  $\Omega^\infty$ , the trivial cohomology class in  $H^{2n+1}(K(\mathbb{F}_2, n); \mathbb{F}_2)$ .

So, for the  $k$ -invariant of  $\mathfrak{gl}_1(R)$ , we need a nonzero element  $\phi$  of degree  $n + 1$  in the (mod 2) Steenrod algebra such that, if  $\iota_n \in H^n(K(\mathbb{F}_2, n); \mathbb{F}_2)$  is the tautological class, then  $\phi \iota_n = 0$ . By the calculation of the cohomology of Eilenberg–Mac Lane spaces [61] (see also [50, Chapter 9] for a textbook reference), the only possibility is  $\mathrm{Sq}^{n+1}$ . □

Nonetheless, we will show that right below the range of the previous example, the spectra  $\mathfrak{gl}_1(R)$  and  $R$  can be identified.

**Corollary 5.2.3** *Let  $n \geq 2$  and let  $R$  be any  $E_\infty$ -ring. Then there is an equivalence of spectra, functorial in  $R$ ,*

$$\tau_{[n,2n-1]} \mathfrak{gl}_1(R) \simeq \tau_{[n,2n-1]} R.$$

Similarly, there is an equivalence of spectra, functorial in  $R$ ,

$$\tau_{[n+1,2n]} \mathfrak{pic}(R) \simeq \Sigma \tau_{[n,2n-1]} R.$$

**Proof** For any  $E_\infty$ -ring  $R$ , the space  $\Omega^\infty \mathfrak{gl}_1(R) = GL_1(R)$  is a union of those components of  $\Omega^\infty R$  that correspond to units in  $\pi_0 R$ . In particular,  $\Omega^\infty \tau_{\geq 1} \mathfrak{gl}_1(R)$  is canonically identified with  $\Omega^\infty \tau_{\geq 1} R$  in  $S_*$ . Applying Theorem 5.1.2, we now get a canonical identification as desired in the corollary. The second half of Corollary 5.2.3 follows from the first, as  $\tau_{\geq 0} \Omega \mathfrak{pic}(R) \simeq \mathfrak{gl}_1(R)$  as spectra.  $\square$

Take now a faithful  $G$ -Galois extension  $A \rightarrow B$  of  $E_\infty$ -rings, and consider the HFPSS (3-5) for the  $G$ -action on  $\mathfrak{pic}(B)$ . We want to understand  $\pi_0(\mathfrak{pic}(B)^{hG})$ , or equivalently  $\pi_{-1}(\Omega \mathfrak{pic}(B)^{hG})$ , and we can do this by understanding the HFPSS for the  $G$ -action on  $\Omega \mathfrak{pic}(B)$ . Observe first that  $\pi_t \Omega \mathfrak{pic}(B) \simeq \pi_t B$  functorially for  $t \geq 1$ : in fact,  $\Omega^\infty(\Omega \mathfrak{pic}(B)) \simeq GL_1(B)$ . In other words, the spectrum  $\Omega \mathfrak{pic}(B)$  equipped with the  $G$ -action has the property that, after applying  $\Omega^\infty$ , it is identified with a union of connected components of  $\Omega^\infty B$  (with the  $G$ -action on  $B$ ).

As a result, we have a map of spaces with  $G$ -action

$$\Omega^\infty(\Omega \mathfrak{pic}(B)) \rightarrow \Omega^\infty B,$$

which identifies the former with a union of connected components of the latter. As a result, we can identify the respective HFPSS for the spaces  $\Omega^\infty(\Omega \mathfrak{pic}(B))$ ,  $\Omega^\infty B$  for  $t > 0$ , both at  $E_2$  and differentials (including the “fringed” ones). This identification comes from the map  $\tau_{\geq 1} GL_1(B) \rightarrow \Omega^\infty B$  given by subtracting one.

In particular, shifting by one again, most of the differentials in the HFPSS for  $\mathfrak{pic}(B)$  are determined by the HFPSS for  $B$ . More precisely, any differential out of  $E_r^{s,t}$  for  $t - s > 0$ ,  $s > 0$ , depends only on the  $G$ -space  $\Omega \mathcal{P}ic(B)$ , so the equivalence of  $\Omega \mathcal{P}ic(B)$  with a union of connected components of  $\Omega^\infty B$  implies that the differential can be identified with the analogous differential in the HFPSS for  $B$ .

However, to understand  $\pi_0(\mathfrak{pic}(B)^{hG}) \simeq \pi_0(\mathcal{P}ic(B)^{hG}) \simeq \mathfrak{Pic}(A)$ , we need to determine differentials out of  $E_r^{s,t}$  with  $t = s$ . These differentials cannot be determined

by  $\Omega \text{Pic}(B)$ , as a space with a  $G$ -action. Our strategy to determine these differentials is to use the equivalence of spectra with  $G$ -action

$$\tau_{[n+1,2n]} \text{pic}(B) \simeq \Sigma \tau_{[n,2n-1]} B,$$

which is a special case of Corollary 5.2.3.

Assume that  $r \leq t - 1$ . In this case, any differential  $d_r: E_*^{s,t} \rightarrow E_*^{s+r,t+r-1}$  in the HFPSS for  $\text{pic}(B)$  is determined by the  $G$ -action on  $\tau_{[t,t+r-1]} \text{pic}(B)$ . Since we have an equivalence  $\tau_{[t,t+r-1]} \text{pic}(B) \simeq \Sigma \tau_{[t-1,t+r-2]} B$ , compatible with the  $G$ -actions, we can identify the differentials.

Denote the differentials in the homotopy fixed point spectral sequence

$$H^s(G, \pi_t \text{pic } B) \Rightarrow \pi_{t-s}(\text{pic } B)^{hG}$$

by  $d_r^{s,t}(\text{pic } B)$ , and similarly  $d_r^{s,t}(B)$  for those in the HFPSS for  $B$ . The upshot of this discussion is the following.

**Comparison Tool 5.2.4** *Let  $A \rightarrow B$  be a  $G$ -Galois extension of  $E_\infty$ -rings. Whenever  $2 \leq r \leq t - 1$ , we have an equality of differentials  $d_r^{s,t}(\text{pic } B) = d_r^{s,t-1}(B)$ .*

Of course, we also have an identification of differentials out of  $(s, t)$  if  $t - s > 0, s > 0$ .

**Remark 5.2.5** Our original approach to the Comparison Tool 5.2.4 was somewhat more complicated than the above and has been described in [44]. Namely, our strategy was to identify the HFPSS with a Bousfield–Kan spectral sequence for a certain cosimplicial space  $X^\bullet$  built from  $\text{Pic}(B)$  with its  $G$ -action, and argue that these differentials only depended on the fiber of  $\text{Tot}_{t+r}(X^\bullet) \rightarrow \text{Tot}_{t-1}(X^\bullet)$  (as well as the other fibers in between). In the appropriate range, these fibers depend only on  $\Omega X^\bullet$  as a cosimplicial space. However,  $\Omega X^\bullet$  can be (almost) identified with the analogous cosimplicial space for the  $G$ -action on  $\Omega^{\infty-1}(\tau_{\geq 0} B)$  because  $\Omega \text{Pic}(B)$  is a union of components of  $\Omega^\infty B$ . This forces the differentials to correspond to one another.

For the same reasons, we have analogous comparison results for the spectral sequence as Theorem 3.2.1. Again, any differential in the descent spectral sequence for  $\text{pic}(\Gamma(X, \mathcal{O}^{\text{top}}))$  that only depends on the diagram  $\tau_{[n+1,2n]} \text{pic}(\mathcal{O}^{\text{top}})$  can be identified with the corresponding differential in the descent spectral sequence for  $\Gamma(X, \mathcal{O}^{\text{top}})$ , thanks to the equivalence of diagrams of spectra  $\tau_{[n+1,2n]} \text{pic}(\mathcal{O}^{\text{top}}) \simeq \Sigma \tau_{[n,2n-1]} \mathcal{O}^{\text{top}}$ .

**Remark 5.2.6** The equivalence  $\tau_{[n,2n-1]} R \simeq \tau_{[n,2n-1]} \mathfrak{gl}_1(R)$  resembles the following observation in commutative algebra. Let  $A$  be an ordinary commutative ring and

let  $I \subset A$  be a square-zero ideal. Then  $1 + I \subset A^\times$  and there is an isomorphism of groups

$$I \simeq 1 + I \subset A^\times \quad \text{with } x \mapsto 1 + x.$$

This correspondence is a very degenerate version of the exponential and logarithm.

Suppose  $p$  is a prime number and  $(p - 1)!$  is invertible in  $A$ . Then if  $J \subset A$  is an ideal with  $J^p = 0$ , we have  $1 + J \subset A^\times$  and a natural isomorphism of groups

$$J \simeq 1 + J \quad \text{with } x \mapsto 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{p-1}}{(p - 1)!},$$

given by a  $p$ -truncated exponential.

Similarly, let  $R$  be an  $E_\infty$ -ring with  $(p - 1)!$  invertible. Motivated by the above, for any  $n \geq 1$ , one could surmise a *functorial* equivalence of spectra

$$\tau_{[n, pn-1]}R \simeq \tau_{[n, pn-1]} \mathfrak{gl}_1(R).$$

We expect to construct such an equivalence in ongoing joint work with Clausen and Heuts.

### 5.3 A general result on Galois descent

As a quick application of the preceding ideas, we can prove a general result about Galois descent for Picard groups.

**Theorem E** *Let  $A \rightarrow B$  be a faithful  $G$ -Galois extension of  $E_\infty$ -rings. Then the relative Picard group of  $B/A$  is  $|G|$ -power torsion of finite exponent.*

**Proof** We know that the relative Picard group of  $A \rightarrow B$  is given by  $\pi_{-1}(\mathfrak{gl}_1(B)^{hG})$  (compare Remark 3.3.2). There is a HFPSS that converges to the homotopy groups, which begins with the group cohomology of  $G$  with coefficients in  $\pi_*(\mathfrak{gl}_1(B))$ . Every contributing term is  $|G|$ -power torsion: in fact, every term is a  $H^i(G, \cdot)$  for  $i > 0$  and is thus killed by  $|G|$ . However, in view of the potential infiniteness of the filtration, as well as the possibilities of nontrivial extensions, this alone does not force  $\pi_{-1}(\mathfrak{gl}_1(B)^{hG})$  to be  $|G|$ -power torsion.

Our strategy is to compare the HFPSS for  $\pi_{-1}(\mathfrak{gl}_1(B)^{hG})$  with that of  $\pi_{-1}(B^{hG})$ . The map  $A \rightarrow B$  admits descent in the sense of [40, Definition 3.17]. In particular, by [40, Corollary 4.4], the descent spectral sequence for  $A \rightarrow B$  (equivalently, the HFPSS) has a *horizontal vanishing line* at a finite stage. It follows that, above a certain filtration, everything in the HFPSS for  $\pi_*(A) \simeq \pi_*(B^{hG})$  is killed by a  $d_k$  for  $k$  bounded.

In view of our Comparison Tool 5.2.4, it follows that any class in the relative Picard group has bounded filtration (though possibly the bound is weaker than the analog in  $\pi_{-1}(B)$ ). Since every contributing term in the spectral sequence is killed by  $|G|$ , the theorem follows.  $\square$

## 6 The first unstable differential

### 6.1 Context

Let  $R^\bullet$  be a cosimplicial  $E_\infty$ -ring, and consider the Bousfield–Kan spectral sequences (BKSS)  $\{E_r^{s,t}\}$  and  $\{\bar{E}_r^{s,t}\}$  for the two cosimplicial objects  $R^\bullet$  and  $\mathfrak{gl}_1(R^\bullet)$ , converging to  $\pi_{t-s}$  of the respective totalizations in  $\text{Sp}$ .

For  $t - s \geq 0$ , the spectral sequences and the differentials are mostly identified with one another, as the space  $\Omega^\infty \mathfrak{gl}_1(R)$  is a union of connected components of  $\Omega^\infty R$ . But for  $t - s = -1$ , we get differentials

$$d_r: E_r^{t+1,t} \rightarrow E_r^{t+r+1,t+r-1} \quad \text{and} \quad \bar{d}_r: \bar{E}_r^{t+1,t} \rightarrow \bar{E}_r^{t+r+1,t+r-1}.$$

These depend on more than the spaces  $\Omega^\infty R^\bullet$ ,  $\Omega^\infty \mathfrak{gl}_1(R^\bullet)$ : they require the one-fold deloopings. As we saw in Corollary 5.2.3, for any  $n \geq 2$ , in the range  $[n, 2n - 1]$ , the cosimplicial spectra  $\tau_{[n,2n-1]}R^\bullet$  and  $\tau_{[n,2n-1]}\mathfrak{gl}_1(R^\bullet)$  are identified. As a result, for  $r \leq t$ , the groups in question are (canonically) identified and  $d_r = \bar{d}_r$ .

But in general,  $d_{t+1} \neq \bar{d}_{t+1}$ . Since all the previous differentials entering or leaving this spot between the two spectral sequences were identified, the groups in question are identified. We let the correspondence  $E_{t+1}^{t+1,t} \simeq \bar{E}_{t+1}^{t+1,t}$  be given as

$$x \mapsto \bar{x}.$$

Similarly, we have a correspondence  $E_{t+1}^{2t+2,2t} \simeq \bar{E}_{t+1}^{2t+2,2t}$ .

In this subsection, we will give a universal formula for the first differential out of the stable range. We will need this in Section 8.2 to obtain the 2–primary Picard group of  $\text{TMF}$ .

**Theorem 6.1.1** *We have the formula*

$$(6-1) \quad \bar{d}_{t+1}(\bar{x}) = \overline{d_{t+1}(x) + x^2} \quad \text{for } x \in E_{t+1}^{t+1,t}.$$

**Remark 6.1.2** The above formula actually makes  $\bar{d}_{t+1}$  into a linear operator. This follows from the graded-commutativity of the BKSS for  $R^\bullet$ . Note in particular that the difference between  $\bar{d}_{t+1}$  and  $d_{t+1}$  is annihilated by 2.

### 6.2 The universal example

The proof of (6-1) follows a standard technique in algebraic topology: we reduce to a “universal” case and show that (6-1) is essentially the only possibility. We want to consider the universal case of a cosimplicial  $E_\infty$ -ring  $R^\bullet$  with a class in  $E_{t+1}^{t+1,t}$ . This class represents an element in  $\pi_{-1} \text{Tot}_{2t+1}(R^\bullet)$  trivialized in  $\text{Tot}_t(R^\bullet)$ ; the differential  $d_{t+1}$  represents the obstruction to lifting to  $\text{Tot}_{2t+2}$ . So, we need to make the analysis of differentials in the cosimplicial  $E_\infty$ -ring which corepresents the functor  $R^\bullet \mapsto \mathfrak{A}(R^\bullet) = \Omega^\infty(\Sigma^{-1} \text{fib}(\text{Tot}_{2t+1}(R^\bullet) \rightarrow \text{Tot}_t(R^\bullet)))$ .

The relevant cosimplicial  $E_\infty$ -ring  $\mathcal{X}^\bullet$  can be constructed as follows.

**Definition 6.2.1** Let  $\text{Lan}$  denote the operation of left Kan extension; let  $\text{Lan}_{\Delta \leq t \rightarrow \Delta}(*)$  denote the left Kan extension of the constant functor  $\Delta^{\leq t} \rightarrow \mathcal{S}$  at a point to  $\Delta$ . Similarly, define  $\text{Lan}_{\Delta \leq 2t+1 \rightarrow \Delta}(*)$ . Consider the homotopy pushout

$$(6-2) \quad \begin{array}{ccc} \text{Lan}_{\Delta \leq t \rightarrow \Delta}(*)_+ & \longrightarrow & * \\ \downarrow & & \downarrow \\ \text{Lan}_{\Delta \leq 2t+1 \rightarrow \Delta}(*)_+ & \longrightarrow & \mathcal{F}^\bullet \end{array}$$

where  $\mathcal{F}^\bullet: \Delta \rightarrow \mathcal{S}_*$  is a functor to the  $\infty$ -category  $\mathcal{S}_*$  of *pointed spaces*.

Consider

$$\mathcal{G}^\bullet \stackrel{\text{def}}{=} \Sigma^{\infty-1} \mathcal{F}^\bullet: \Delta \rightarrow \text{Sp}$$

and the functor

$$\mathcal{X}^\bullet = \text{Free}_{\text{CAlg}}(\mathcal{G}^\bullet): \Delta \rightarrow \text{CAlg}$$

into the  $\infty$ -category  $\text{CAlg}$  of  $E_\infty$ -rings, obtained by applying the free algebra functor everywhere to  $\mathcal{G}$ . By construction,  $\mathcal{X}^\bullet$  corepresents the functor  $\mathfrak{A}: \text{Fun}(\Delta, \text{CAlg}) \rightarrow \mathcal{S}$  in which we are interested. In particular, it suffices to prove (6-1) for this particular functor. As we will see in the next paragraph,  $\mathcal{G}^\bullet$  takes values in *connective spectra* and therefore so does  $\mathcal{X}^\bullet$ . Since we are only interested in differentials in a particular range, we may (by naturality) only consider the Postnikov section  $\tau_{\leq 2t} \mathcal{X}^\bullet$ . We get the following basic step.

**Proposition 6.2.2** *In order to prove Theorem 6.1.1, it suffices to prove it for  $\tau_{\leq 2t} \mathcal{X}^\bullet$  (and the tautological class).*



In fact, we have a reasonable handle on what the functor  $\tau_{\leq 2t} \mathcal{K}^\bullet$  looks like and can entirely determine the BKSS. To see this, we recall the construction of  $\mathcal{F}^\bullet$ ; compare also the discussion in [44]. The functor

$$\text{Lan}_{\Delta^{\leq t} \rightarrow \Delta} (*): \Delta \rightarrow \mathcal{S}$$

sends any finite nonempty totally ordered set  $T$  to the nerve of the category  $\Delta_{/T}^{\leq t}$  of all order-preserving morphisms  $\{S \rightarrow T\}$  where

- (1)  $S$  is a finite, nonempty totally ordered set, and
- (2)  $|S| \leq t + 1$ .

**Proposition 6.2.3**  *$\text{Lan}_{\Delta^{\leq t} \rightarrow \Delta} (*)$  is naturally equivalent to the functor which sends  $T$  in  $\Delta$  to the nerve of the poset  $P_{\leq t+1}(T)$  of nonempty subsets of  $T$  of cardinality at most  $t + 1$ .*

**Proof** In fact, for any  $T$ , there is a natural map  $P_{\leq t+1}(T) \rightarrow \Delta_{/T}^{\leq t}$ , which is a homotopy equivalence as it is right adjoint to the functor  $\Delta_{/T}^{\leq t} \rightarrow P_{\leq t+1}(T)$  which sends  $S \rightarrow T$  to  $\text{image}(S \rightarrow T) \subset T$ . □

In view of the last proposition, one can also consider the following approach to the left Kan extension. There is a standard cosimplicial simplicial set sending  $[n] \mapsto \Delta^n$ . The functor of the proposition is equivalent to the barycentric subdivision of the cosimplicial simplicial set  $[n] \mapsto \text{sk}_t \Delta^n$ .

As in [44], the nerve of  $P_{\leq t+1}(T)$ , for any choice of  $T$ , is (pointwise) homotopy equivalent to a wedge of  $t$ -spheres, and contractible if  $|T| \leq t + 1$ . We get from (6-2):

**Proposition 6.2.4** *The functor  $\mathcal{F}^\bullet: \Delta \rightarrow \mathcal{S}_*$  constructed above has the following properties:*

- (1) For any  $T$ ,  $\mathcal{F}(T)$  is always a wedge of copies of  $S^{t+1}$  and  $S^{2t+1}$ .
- (2) Restricted to  $\Delta^{\leq t}$ , the functor  $\mathcal{F}^\bullet$  is contractible. Restricted to  $\Delta^{\leq 2t}$ , the functor  $\mathcal{F}^\bullet$  is pointwise a wedge of copies of  $S^{t+1}$ .

### 6.3 Some technical lemmas

Our first goal is to understand the BKSS for  $\mathcal{G}^\bullet = \Sigma^{\infty-1} \mathcal{F}^\bullet$ . Observe that pointwise, this cosimplicial spectrum is a wedge of copies of  $S^t$  and  $S^{2t}$  by Proposition 6.2.4. In order to do this, we need to understand the cosimplicial abelian group  $\pi_*(\Sigma^{\infty-1} \mathcal{F}^\bullet)$ . We will prove the following:

**Proposition 6.3.1** *The cohomology  $H^s(\pi_*(\mathcal{G}^\bullet))$  is given by*

$$(6-3) \quad H^s(\pi_*(\mathcal{G}^\bullet)) \simeq \begin{cases} \pi_* S^t & \text{if } s = t + 1, \\ \pi_* S^{2t} & \text{if } s = 2(t + 1). \end{cases}$$

*In the spectral sequence, the differential  $d_{t+1}$  is an isomorphism.*

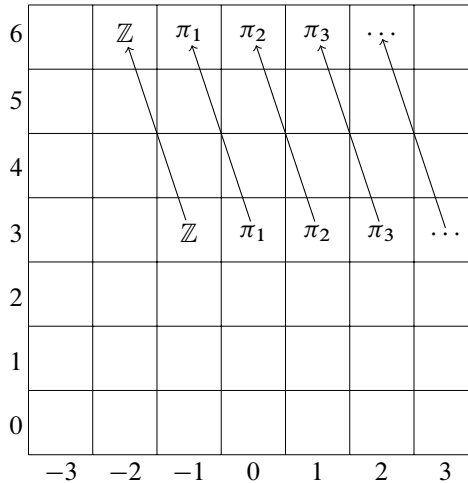


Figure 1: Bousfield–Kan spectral sequence for  $\mathcal{G}^\bullet$ , with  $t = 2$  ( $\pi_k$  denotes  $\pi_k S^0$ )

The spectral sequence is depicted in Figure 1. The proof of Proposition 6.3.1 will take work and will be spread over two subsections. In the present subsection, our main result is that the totalization of  $\mathcal{G}^\bullet$  (and related cosimplicial spectra) is contractible, and we will deduce the differentials from that. The approach to this is not computational and relies instead on ideas involving the  $\infty$ -categorical Dold–Kan correspondence of Lurie.

We recall from [34, Notation 1.2.8.4] the *cone* construction, which associates to a simplicial set  $K$ , the *cone*  $K^\triangleleft$ . If  $K$  is an  $\infty$ -category,  $K^\triangleleft$  is as well, and is obtained by adding a new initial object to  $K$ .

**Lemma 6.3.2** *Let  $K$  be a simplicial set and  $\mathcal{D}$  an  $\infty$ -category with colimits. Let  $F: K^\triangleleft \rightarrow \mathcal{D}$  be a functor with the property that  $F$  carries the cone point to an initial object of  $\mathcal{D}$ . Then the natural map*

$$\lim_{\rightarrow K} F|_K \rightarrow \lim_{\rightarrow K^\triangleleft} F$$

*is an equivalence in  $\mathcal{D}$ .*

**Proof** It suffices to show<sup>12</sup> that the natural map

$$(6-4) \quad \mathcal{D}_{K\triangleleft/} \rightarrow \mathcal{D}_{K/}$$

is an equivalence of  $\infty$ -categories. But we have  $\mathcal{D}_{K\triangleleft/} \simeq \mathcal{D}_{(\Delta^0 \star K)/} \simeq (\mathcal{D}_{\Delta^0/})_{K/}$  in view of the definition of the overcategory [34, Section 1.2.9], where  $\star$  denotes the *join* of simplicial sets [34, Section 1.2.8]. However, we also know that the projection map  $\mathcal{D}_{\Delta^0/} \rightarrow \mathcal{D}$  is an equivalence since  $\Delta^0 \rightarrow \mathcal{D}$  maps to an initial object. Therefore, we obtain that (6-4) is an equivalence, as desired.  $\square$

**Lemma 6.3.3** *Let  $\mathcal{C}, \mathcal{D}$  be  $\infty$ -categories and assume that  $\mathcal{D}$  has colimits. Let  $F: \mathcal{C}^{\triangleleft} \rightarrow \mathcal{D}$  be a functor such that  $F$  carries the cone point to an initial object of  $\mathcal{D}$ . Let  $\mathcal{C}' \subset \mathcal{C}$  be a full subcategory. Then the following are equivalent:*

- (1)  $F|_{\mathcal{C}'}$  is a left Kan extension of its restriction to  $\mathcal{C}'$ .
- (2)  $F$  is a left Kan extension of its restriction to  $\mathcal{C}'^{\triangleleft}$ .

**Proof** Suppose the first condition is satisfied. Then if  $c \in \mathcal{C}$  is arbitrary, the natural map

$$\varinjlim_{c' \rightarrow c \in \mathcal{C}'/c} F(c') \rightarrow F(c)$$

is an equivalence. Now, we have an equivalence of  $\infty$ -categories  $(\mathcal{C}'/c)^{\triangleleft} \simeq (\mathcal{C}'^{\triangleleft})/c$ , because  $\triangleleft$  adds a new initial object. Therefore, for arbitrary  $c \in \mathcal{C}$ , we also get that the natural map

$$\varinjlim_{c' \rightarrow c \in (\mathcal{C}'^{\triangleleft})/c} F(c') \simeq \varinjlim_{c' \rightarrow c \in (\mathcal{C}'/c)^{\triangleleft}} F(c') \rightarrow F(c)$$

is an equivalence, thanks to Lemma 6.3.2. At the cone point, the left Kan extension condition is automatic. Thus, it follows that  $F$  is a left Kan extension of  $F|_{\mathcal{C}'^{\triangleleft}}$ . The converse is proved in the same way.  $\square$

**Proposition 6.3.4** *Let  $\mathcal{C}$  be a stable  $\infty$ -category and let  $F: \Delta^{\leq n} \rightarrow \mathcal{C}$  be any functor. Suppose  $F$  is a left Kan extension of its restriction to  $\Delta^{\leq n-1}$ . Then  $\varprojlim_{\Delta^{\leq n}} F$  is contractible.*

**Proof** Observe that the cone  $(\Delta^{\leq n})^{\triangleleft}$  is given by the category  $\Delta_+^{\leq n}$  of the finite totally ordered sets  $\{[i]\}_{-1 \leq i \leq n}$  since  $[-1]$  is an initial object of this category. Consider the functor  $\tilde{F}: \Delta_+^{\leq n} \simeq (\Delta^{\leq n})^{\triangleleft} \rightarrow \mathcal{C}$  extending  $F$  that sends the cone point to the initial

<sup>12</sup>We are indebted to the referee for substantially simplifying our original argument here.

object (one can always make such an extension). To show that  $\lim_{\Delta \leq n} F$  is contractible, it suffices to show that  $\tilde{F}$  is a right Kan extension of  $F = \tilde{F}|_{\Delta \leq n}$ .

Now, we recall a basic result of Lurie [39, Lemma 1.2.4.19] (which we use for the opposite category), a piece of the  $\infty$ -categorical version of the Dold–Kan correspondence: given any functor  $G: \Delta_{+}^{\leq n} \rightarrow \mathcal{C}$ ,  $G$  is a right Kan extension of  $G|_{\Delta \leq n}$  if and only if  $G$  is a left Kan extension of  $G|_{\Delta_{+}^{\leq n-1}}$ . In our case, it follows that to show that  $\tilde{F}$  is a right Kan extension of  $F$  (as we would like to see), it suffices to show that  $\tilde{F}$  is a left Kan extension of  $\tilde{F}|_{\Delta_{+}^{\leq n-1}}$ . But by Lemma 6.3.3, this follows from the fact that  $\tilde{F}|_{\Delta \leq n} = F$  is a left Kan extension of  $\tilde{F}|_{\Delta \leq n-1} = F|_{\Delta \leq n-1}$ .  $\square$

### 6.4 The BKSS for $\mathcal{F}$

The goal of this subsection is to complete the proof of Proposition 6.3.1. To begin with, we analyze the BKSS for the functor  $\Sigma_{+}^{\infty} \text{Lan}_{\Delta \leq t \rightarrow \Delta}(*): \Delta \rightarrow \text{Sp}$ .

**Proposition 6.4.1** *The BKSS for the cosimplicial spectrum  $\Sigma_{+}^{\infty} \text{Lan}_{\Delta \leq t \rightarrow \Delta}(*)$  satisfies*

$$(6-5) \quad E_2^{s,*} = H^s(\pi_*(\Sigma_{+}^{\infty} \text{Lan}_{\Delta \leq t \rightarrow \Delta}(*))) = \begin{cases} \pi_*(S^0) & \text{if } s = 0, \\ \pi_*(S^t) & \text{if } s = t + 1. \end{cases}$$

The differential  $d_{t+1}$  is an isomorphism. (The result for  $t = 2$  is displayed in Figure 2.)

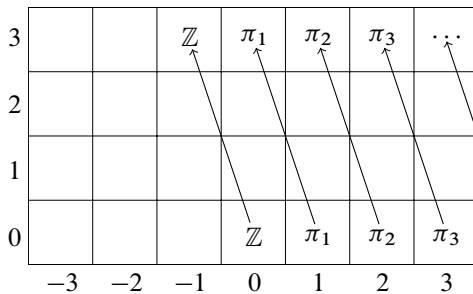


Figure 2: Bousfield–Kan spectral sequence for  $\Sigma_{+}^{\infty} \text{Lan}_{\Delta \leq t \rightarrow \Delta}(*)$ , with  $t = 2$

**Proof** Observe that  $\text{Lan}_{\Delta \leq t \rightarrow \Delta}(*)$  is, pointwise, a wedge of  $t$ -spheres, so to compute the desired cohomology  $H^s(\pi_*(\Sigma_{+}^{\infty} \text{Lan}_{\Delta \leq t \rightarrow \Delta}(*)))$ , it suffices to do this for  $\pi_t$ . (The disjoint basepoint contributes the  $\pi_*(S^0)$  for  $s = 0$  in cohomology.) In other words, we may consider the cosimplicial  $H\mathbb{Z}$ -module  $M^\bullet = H\mathbb{Z} \wedge \Sigma_{+}^{\infty} \text{Lan}_{\Delta \leq t \rightarrow \Delta}(*).$

Now we know, for each  $n$ , that  $\pi_*(M^n)$  is concentrated in degrees 0 and  $t$ , and that  $\pi_0(M^\bullet)$  is the constant cosimplicial abelian group  $\mathbb{Z}$ . Moreover, by Proposition 6.3.4,  $\text{Tot}(M^\bullet)$  is contractible. A look at the spectral sequence for  $\text{Tot}(M^\bullet)$  shows that  $H^s(\pi_t M^\bullet)$  must be concentrated in degree  $s = t + 1$  and must be a  $\mathbb{Z}$  there. The claim about differentials also follows from contractibility of the totalization.  $\square$

**Proof of Proposition 6.3.1** The definition (6-2) of  $\mathcal{F}^\bullet$  and Proposition 6.4.1 together give the  $E_2$ -page of the spectral sequence, when one uses the long exact sequence in homotopy groups. The differentials are forced, again, by Proposition 6.3.4 which implies that  $\text{Tot}(\mathcal{G}^\bullet)$  is contractible.  $\square$

### 6.5 Completion of the proof

Now we need to consider the cosimplicial  $E_\infty$ -ring defined earlier

$$\mathcal{Y}^\bullet \stackrel{\text{def}}{=} \tau_{\leq 2t} \mathcal{X}^\bullet \simeq \tau_{\leq 2t} \text{Free}_{\text{CAlg}}(\mathcal{G}^\bullet).$$

We recall that this is well defined as a cosimplicial  $E_\infty$ -ring because  $\mathcal{G}^\bullet$  is (pointwise) connective.

In this subsection, we will determine the relevant piece of the BKSS for  $\mathcal{Y}$  and then complete the proof of Theorem 6.1.1. We have that

$$\mathcal{Y}^\bullet \simeq \tau_{\leq 2t} S^0 \vee \tau_{\leq 2t} \mathcal{G}^\bullet \vee \tau_{\leq 2t} ((\mathcal{G}^\bullet)_{h\Sigma_2}^{\wedge 2}),$$

because, by a connectivity argument, no other terms contribute. In particular, the cohomology  $H^s(\pi_*(\mathcal{Y}^\bullet))$  picks up a copy of  $\pi_*(S^0)$  for  $s = 0$  (which is mostly irrelevant). In Proposition 6.3.1, we determined the BKSS for  $\mathcal{G}^\bullet$ ; in bidegrees  $(t + 1, t)$  and  $(2t + 2, 2t)$ , this picks up copies of  $\mathbb{Z}$  such that the first one hits the second one with a  $d_{t+1}$ . We will prove:

**Proposition 6.5.1**  $E_2^{2t+2, 2t} \simeq \mathbb{Z} \oplus \mathbb{Z}/2$  in the BKSS for  $\mathcal{Y}^\bullet$ . The  $\mathbb{Z}/2$  is generated by the square of the class in bidegree  $(t + 1, t)$ .

**Proof** We will use the notation and results of Appendix C. Let  $A^\bullet$  be the cosimplicial abelian group  $\pi_t \mathcal{G}^\bullet$ , which is levelwise free and finitely generated. As we have seen (Proposition 6.3.1),  $H^{t+1}(A^\bullet) \simeq \mathbb{Z}$  and the other cohomology of  $A^\bullet$  vanishes. Now, using the notation of Definition C.1,

$$\pi_{2t}((\mathcal{G}^\bullet)_{h\Sigma_2}^{\wedge 2}) = \begin{cases} \text{Sym}_2 A^\bullet & \text{for } t \text{ even,} \\ \widetilde{\text{Sym}}_2 A^\bullet & \text{for } t \text{ odd.} \end{cases}$$

By Proposition C.3, we find that the  $E_2^{2t+2, 2t}$  term of  $(\mathcal{G}^\bullet)_{h\Sigma_2}^{\wedge 2}$  is as claimed.  $\square$

We are now ready to complete the proof and determine the differential in the  $\mathfrak{gl}_1$  spectral sequence. Using the notation of the beginning of this section, it follows that  $E_{t+1}^{t+1,t} \simeq \mathbb{Z}$  and  $E_{t+1}^{2t+1,2t} \simeq \mathbb{Z} \oplus \mathbb{Z}/2$ , and similarly for  $\bar{E}$ . The  $d_{t+1}$  carries the  $\mathbb{Z}$  into the other  $\mathbb{Z}$ . By naturality of the spectral sequence, it follows that there must exist a universal formula

$$(6-6) \quad \bar{d}_{t+1}(\bar{x}) = \overline{ad_{t+1}(x) + \epsilon x^2} \quad \text{for } a \in \mathbb{Z} \text{ and } \epsilon \in \{0, 1\}.$$

The main claim is that  $a = \epsilon = 1$ . Our first goal is to compute  $a$ .

**Lemma 6.5.2** *We have an equivalence of  $\infty$ -categories between the  $\infty$ -category  $\text{Fun}^L(\text{Sp}_{\geq 0}, \text{Sp}_{\geq 0})$  of cocontinuous functors  $\text{Sp}_{\geq 0} \rightarrow \text{Sp}_{\geq 0}$  and  $\text{Sp}_{\geq 0}$  given by evaluating at the sphere. The inverse equivalence sends a connective spectrum  $Y$  to the functor  $X \mapsto X \otimes Y$ .*

**Proof** It suffices to show that evaluation at the sphere induces an equivalence of  $\infty$ -categories  $\text{Fun}^L(\text{Sp}_{\geq 0}, \text{Sp}) \simeq \text{Sp}$  (with inverse given as above). But the  $\infty$ -category  $\text{Sp}$  is the stabilization [39, Section 1.4] of  $\text{Sp}_{\geq 0}$  (as one sees easily from the fact that  $\Sigma$  is fully faithful on  $\text{Sp}_{\geq 0}$  and an equivalence on  $\text{Sp}$ ), so that, by [39, Corollary 1.4.4.5], we have an equivalence  $\text{Fun}^L(\text{Sp}, \text{Sp}) \simeq \text{Fun}^L(\text{Sp}_{\geq 0}, \text{Sp})$  given by restriction. But we know that  $\text{Fun}^L(\text{Sp}, \text{Sp}) \simeq \text{Sp}$  by evaluation at the sphere spectrum, with inverse given by the smash product; see [39, Section 4.8.2].  $\square$

We need the following fact about  $\mathfrak{gl}_1$ .

**Proposition 6.5.3** *Let  $X$  be a connective spectrum, and let  $S^0 \vee X$  be the square-zero  $E_\infty$ -ring. Then there is a natural equivalence of spectra,*

$$\mathfrak{gl}_1(S^0 \vee X) \simeq \mathfrak{gl}_1(S^0) \vee X.$$

On homotopy groups, this equivalence is compatible with the purely algebraic equivalence  $\pi_t \mathfrak{gl}_1(S^0 \vee X) \simeq \pi_t(S^0 \vee X) \simeq \pi_t(S^0) \oplus \pi_t(X) \simeq \pi_t(\mathfrak{gl}_1(S^0)) \oplus \pi_t(X)$ .

**Proof** Given the connective spectrum  $X$ , we can use the composite  $S^0 \rightarrow S^0 \vee X \rightarrow S^0$ , in which the second map sends  $X$  to 0, to get a natural splitting

$$\mathfrak{gl}_1(S^0 \vee X) \simeq \mathfrak{gl}_1(S^0) \vee F(X),$$

where  $F: \text{Sp}_{\geq 0} \rightarrow \text{Sp}_{\geq 0}$  is a certain functor that we want to claim is naturally isomorphic to the identity. First, observe that  $F$  commutes with colimits. Namely,  $F$

commutes with filtered colimits (as one can check on homotopy groups),  $F$  takes  $*$  to  $*$ , and given a pushout square

$$(6-7) \quad \begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_3 & \longrightarrow & X_4 \end{array}$$

in  $\mathrm{Sp}_{\geq 0}$ , the analogous diagram

$$(6-8) \quad \begin{array}{ccc} F(X_1) & \longrightarrow & F(X_2) \\ \downarrow & & \downarrow \\ F(X_3) & \longrightarrow & F(X_4) \end{array}$$

is a pushout square in  $\mathrm{Sp}_{\geq 0}$ . This in turn follows by considering long exact sequences in homotopy groups. More precisely, given the pushout square (6-7), the diagram of  $\mathbf{E}_\infty$ -rings

$$\begin{array}{ccc} S^0 \vee X_1 & \longrightarrow & S^0 \vee X_2 \\ \downarrow & & \downarrow \\ S^0 \vee X_3 & \longrightarrow & S^0 \vee X_3 \end{array}$$

is a homotopy *pullback* in  $\mathbf{E}_\infty$ -rings, so that applying  $\mathrm{gl}_1$  (which is a *right adjoint*) leads to a pullback square

$$\begin{array}{ccc} \mathrm{gl}_1(S^0 \vee X_1) & \longrightarrow & \mathrm{gl}_1(S^0 \vee X_2) \\ \downarrow & & \downarrow \\ \mathrm{gl}_1(S^0 \vee X_3) & \longrightarrow & \mathrm{gl}_1(S^0 \vee X_4) \end{array}$$

and in particular, (6-8) is homotopy cartesian too in  $\mathrm{Sp}_{\geq 0}$ . Therefore, it is homotopy cocartesian as well if we can show that the map

$$\pi_0(\mathrm{gl}_1(S^0 \vee X_3)) \oplus \pi_0(\mathrm{gl}_1(S^0 \vee X_2)) \rightarrow \pi_0(\mathrm{gl}_1(S^0 \vee X_4))$$

is surjective. This follows from the analogous fact that  $\pi_0(X_3) \oplus \pi_0(X_2) \rightarrow \pi_0(X_4)$  is surjective as (6-7) is a pushout.

Therefore, as  $F$  commutes with colimits,  $F$  is necessarily of the form  $X \mapsto X \otimes Y$  for some  $Y \in \mathrm{Sp}_{\geq 0}$ , by Lemma 6.5.2. For  $X = H\mathbb{Z}$ , we find  $F(X) = H\mathbb{Z}$ , so that

$H\mathbb{Z} \otimes Y$  is concentrated in degree zero and is isomorphic to  $H\mathbb{Z}$ . This forces  $Y \simeq S^0$  and proves the claim. □

**Proof of Theorem 6.1.1** Proposition 6.5.3 implies that in the universal formula (6-6), the constant  $a = 1$ . In fact, we know that if  $X^\bullet$  is any cosimplicial spectrum, then the cosimplicial spectra  $\mathrm{gl}_1(S^0 \vee X^\bullet)$  and  $\mathrm{gl}_1(S^0) \vee X^\bullet$  are identified in a manner compatible with the identifications of homotopy groups. In particular, the differentials in the spectral sequence for  $\mathrm{gl}_1(S^0 \vee X^\bullet)$  and in the spectral sequence for  $S^0 \vee X^\bullet$  are identified, forcing  $a = 1$ .

It remains to show that  $\epsilon = 1$ . For this, we need an example where the two differentials do *not* agree. This will be a generalization of Example 5.2.1. Consider the  $E_\infty$ -ring  $R$  of Proposition 5.2.2, with  $n = t$ , so that, in particular,  $\mathrm{gl}_1(R)$  has homotopy groups in dimensions  $t$  and  $2t$  only. Proposition 5.2.2 shows that the  $k$ -invariant is *nontrivial*.

Consider the space  $X = K(\mathbb{F}_2, t + 1)$ , and consider the Atiyah–Hirzebruch spectral sequences for the homotopy groups of  $\mathrm{gl}_1(R)^X$  and  $R^X$  (these can be identified with BKSS’s by choosing simplicial resolutions of  $X$  by points). The latter clearly degenerates because  $R$  is an Eilenberg–Mac Lane spectrum, but we claim that the former does not.

More precisely, we claim that there is no map of spectra

$$\Sigma^{-1} \Sigma^\infty K(\mathbb{F}_2, t + 1) \rightarrow \mathrm{gl}_1(R),$$

inducing an isomorphism on  $\pi_t$ . The degeneration of the AHSS would certainly imply the existence of such a map. To see this, it is equivalent to showing that there is no map of (pointed) spaces

$$K(\mathbb{F}_2, t + 1) \rightarrow BGL_1(R),$$

with the same properties. If there existed such a map, then we could combine it with the map  $\tau_{\geq 2t+1} BGL_1(R) \simeq K(\mathbb{F}_2, 2t + 1) \rightarrow BGL_1(R)$  via the infinite loop structure to obtain a map

$$K(\mathbb{F}_2, t + 1) \times K(\mathbb{F}_2, 2t + 1) \rightarrow BGL_1(R),$$

which would be an equivalence by inspection of homotopy groups. However, this contradicts Proposition 5.2.2, which shows that the *space*  $BGL_1(R)$  has a nontrivial  $k$ -invariant.

This completes the proof of Theorem 6.1.1. □



### Part III Computations

## 7 Picard groups of real $K$ -theory and its variants

Before we embark on the lengthy computations for the Picard groups of the various versions of topological modular forms, let us work out in detail the case of real  $K$ -theory, as well as the Tate  $K$ -theory spectrum  $KO((q))$ . In particular, these examples will illustrate our methodology without being computationally cumbersome.

### 7.1 Real $K$ -theory

In this subsection, we compute the Picard group of  $KO$  using  $C_2$ -Galois descent from the  $C_2$ -Galois extension  $KO \rightarrow KU$  and the Comparison Tool 5.2.4 (but not the universal formula of Theorem 6.1.1).

We begin with the basic case of *complex*  $K$ -theory.

**Example 7.1.1** (complex  $K$ -theory) The complex  $K$ -theory spectrum has a very simple ring of homotopy groups  $KU_* = \mathbb{Z}[u^{\pm 1}]$  with  $u$  in degree 2. In particular,  $KU$  is even periodic with a regular noetherian  $\pi_0$ , so its Picard group is algebraic by Theorem 2.4.6. The inner workings of Theorem 2.4.6 would use that the only (homogeneous) maximal ideals of  $KU_*$  are generated by prime numbers  $p$ ; for each  $p$ , there is a corresponding residue field spectrum, namely  $\text{mod } p$   $K$ -theory, also known as an extension of the Morava  $K$ -theory of height one at the given prime. As the Picard group of  $KU_0 = \mathbb{Z}$  is trivial, and  $\text{Pic}(KU_*) \simeq \mathbb{Z}/2$ , any invertible  $KU$ -module is equivalent to either  $KU$  or  $\Sigma KU$ .

To compute  $\text{Pic}(KO)$ , we start with this knowledge that, thanks to Example 7.1.1,  $\pi_0 \text{pic}(KU) = \text{Pic}(KU)$  is  $\mathbb{Z}/2$ . We have the spectral sequence from (3-5)

$$H^*(C_2, \pi_* \text{pic}(KU)) \Rightarrow \pi_*(\text{pic}(KU))^{hC_2}$$

which will allow us to compute  $\pi_0(\text{pic}(KU))^{hC_2} \simeq \text{Pic}(KO)$ . We note that

$$\pi_1 \text{pic}(KU) \simeq (KU_0)^\times = \mathbb{Z}/2$$

and

$$H^*(C_2, \mathbb{Z}/2) = \mathbb{Z}/2[x],$$

where  $x$  is in cohomological degree 1. The higher homotopy groups of  $\text{pic}(KU)$  coincide (as  $C_2$ -modules) with those of  $KU$ , suitably shifted by one.

Recall, moreover, that the  $E_2$ -page of the HFPSS for  $\pi_* \text{KO}$  is given by the bigraded ring

$$E_2^{*,*} = \mathbb{Z}[u^2, u^{-2}, h_1]/(2h_1) \quad \text{with } |u^2| = (4, 0) \text{ and } |h_1| = (1, 2),$$

where  $u^2$  is the square of the Bott class in  $\pi_* \text{KU} \simeq \mathbb{Z}[u^{\pm 1}]$ , and  $h_1$  detects in homotopy the Hopf map  $\eta$ . The class  $h_1$  is in bidegree  $(s, t) = (1, 2)$ , so it is drawn using Adams indexing in the  $(1, 1)$  place. The differentials are determined by  $d_3(u^2) = h_1^3$  and the spectral sequence collapses at  $E_4$ . For convenience, we reproduce a picture in Figure 3; the interested reader can find the detailed computation of this spectral sequence in [22, Section 5].

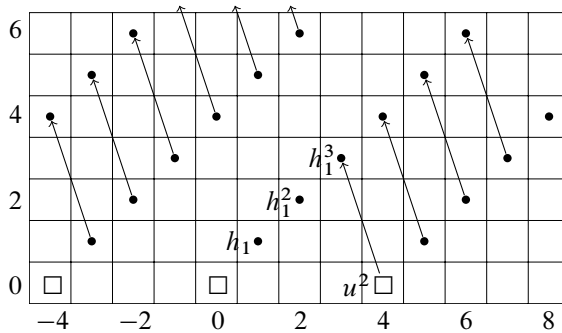


Figure 3: Homotopy fixed point spectral sequence for  $\pi_* \text{KO} \simeq \pi_*(\text{KU}^{hC_2})$  ( $\bullet$  denotes  $\mathbb{Z}/2$  and  $\square$  denotes  $\mathbb{Z}$ )

Therefore, the  $E_2$ -page of the spectral sequence for  $(\text{pic}(\text{KU}))^{hC_2}$  is as in Figure 4. To deduce differentials, we use our Comparison Tool 5.2.4: in the homotopy fixed point spectral sequence for  $\text{KU}$ , there are only (nontrivial)  $d_3$ -differentials. By the Comparison Tool 5.2.4, we conclude that we can “import” those differentials to the HFPSS for  $\text{pic}(\text{KU})$  when they involve terms with  $t \geq 4$ . In particular, we see that the differentials drawn in Figure 4 are nonzero; moreover, everything that is above the drawn range and in the  $s = t$  column either supports or is the target of a nonzero differential. Note that we are not claiming that there are no other nonzero differentials, but these suffice for our purposes.

We deduce from this that  $\pi_0 \text{pic}(\text{KU})^{hC_2} = \text{Pic}(\text{KO})$  has cardinality at most eight. On the other hand, the fact that  $\text{KO}$  is 8-periodic gives us a lower bound  $\mathbb{Z}/8$  on  $\text{Pic}(\text{KO})$ . Thus we get:

**Theorem 7.1.2** (Hopkins; Gepner and Lawson [15])  *$\text{Pic}(\text{KO})$  is precisely  $\mathbb{Z}/8$ , generated by  $\Sigma \text{KO}$ .*

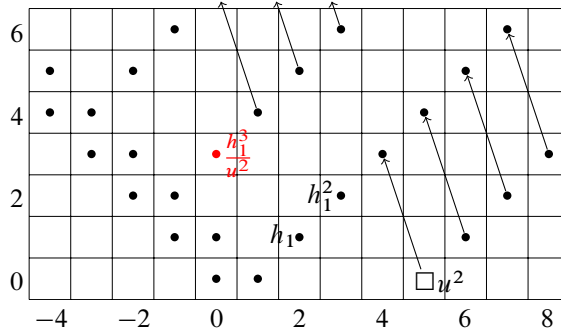


Figure 4: Homotopy fixed point spectral sequence for  $\text{pic}(\text{KU})^{hC_2}$

Theorem 7.1.2 was proved originally by Hopkins (unpublished) using related techniques. The approach via descent theory is due to Gepner and Lawson in [15]. Their identification of the differentials in the spectral sequence is, however, different from ours: they use an explicit knowledge of the structure of  $\mathfrak{gl}_1(\text{KU})$  with its  $C_2$ -action (which one does not have for  $\text{TMF}$ ).

**Remark 7.1.3** In view of Remark 3.3.2, we conclude that the relative Picard group of the  $C_2$ -extension  $\text{KO} \rightarrow \text{KU}$  is  $\pi_{-1}(\mathfrak{gl}_1 \text{KU})^{hC_2} \simeq \mathbb{Z}/4$ .

**Remark 7.1.4** In the usual descent spectral sequence for  $\text{KO}$ , the class  $h_1^3/u^2$  (in red) supports a  $d_3$ . By Theorem 6.1.1 and the multiplicative structure of the usual SS,  $h_1^3/u^2$  does *not* support a  $d_3$  in the descent SS for  $\text{Pic}$ . We saw that above by counting: if  $h_1^3/u^2$  did not survive, the Picard group of  $\text{KO}$  would be too small. For 2-local  $\text{TMF}$ , simple counting arguments will not suffice and we will actually need to use Theorem 6.1.1 as well.

**Remark 7.1.5** We can also deduce from the spectral sequence that the cardinality of the relative Brauer group for  $\text{KO} / \text{KU}$ , which is isomorphic to  $\pi_{-1}(\text{pic}(\text{KU}))^{hC_2}$ , is at most eight. However, we do not know how to construct necessarily nontrivial elements of this Brauer group in order to deduce a lower bound as in the Picard group case.

## 7.2 $\text{KO}[q]$ , $\text{KO}[[q]]$ and $\text{KO}((q))$

We now include a variant of the above example where one adds a polynomial (resp. power series, Laurent series) generator, where we will also be able to confirm the answer using a different argument. This example can be useful for comparison with  $\text{TMF}$  using topological  $q$ -expansion maps. We begin by introducing the relevant  $E_\infty$ -rings. This subsection will not be used in the sequel and may be safely skipped by the reader.

**Definition 7.2.1** We write for  $S^0[x]$  the suspension spectrum  $\Sigma_+^\infty \mathbb{Z}_{\geq 0}$ . Since  $\mathbb{Z}_{\geq 0}$  is an  $E_\infty$ -monoid in spaces (in fact, a commutative topological monoid),  $S^0[x]$  naturally acquires the structure of an  $E_\infty$ -ring. Given an  $E_\infty$ -ring  $R$ , we will write  $R[x] = R \wedge S^0[x]$ .

We can also derive several other variants:

- (1) We will let  $R[[x]]$  denote the  $x$ -adic completion of  $R[x]$ , so its homotopy groups look like a power series ring over  $\pi_* R$ .
- (2) We will let  $R[x^{\pm 1}]$  denote the localization  $R[x][1/x]$ , so its homotopy groups are given by Laurent polynomials in  $\pi_* R$ .
- (3) We will let  $R((x)) = R[[x]][1/x]$ , so that its homotopy groups look like formal Laurent series over  $\pi_* R$ .

On the one hand,  $\pi_*(R[x]) \simeq (\pi_* R)[x]$  is a polynomial ring over  $\pi_* R$  on a generator in degree zero. On the other hand, as an  $E_\infty$ -algebra under  $R$ , the universal property of  $R[x]$  is significantly more complicated than that of the “free”  $E_\infty$ - $R$ -algebra on a generator (often denoted  $R\{x\}$ ). A map  $R[x] \rightarrow R'$ , for an  $E_\infty$ - $R$ -algebra  $R'$ , is equivalent to an  $E_\infty$ -map

$$\mathbb{Z}_{\geq 0} \rightarrow \Omega^\infty R',$$

where  $\Omega^\infty R'$  is regarded as an  $E_\infty$ -space under multiplication. In general, given a class in  $\pi_0 R'$ , there is no reason to expect an  $E_\infty$ -map  $R[x] \rightarrow R'$  carrying  $x$  to it, since  $\mathbb{Z}_{\geq 0}$  as an  $E_\infty$ -monoid is quite complicated. Classes for which this is possible (together with the associated maps  $R[x] \rightarrow R'$ ) have been called “strictly commutative” by Lurie.

**Example 7.2.2** There is a map  $R[x] \rightarrow R$  satisfying  $x \mapsto 1$ . This comes from the map of  $E_\infty$ -spaces  $\mathbb{Z}_{\geq 0} \rightarrow * \rightarrow \Omega^\infty S^0$  where  $*$  maps to the unit in  $\Omega^\infty S^0$ .

**Example 7.2.3** There is a map  $R[x] \rightarrow R$  satisfying  $x \mapsto 0$ .<sup>13</sup>

To obtain this in the universal case  $R = S^0$ , we consider the adjunction

$$(\Sigma^\infty, \Omega^\infty): \mathcal{S}_* \rightleftarrows \text{Sp}.$$

Here  $\mathcal{S}_*$  and  $\text{Sp}$  are symmetric monoidal with the smash product and  $\Sigma^\infty$  is a symmetric monoidal functor. In particular,  $\Sigma^\infty$  carries commutative algebra objects in  $\mathcal{S}_*$  to  $E_\infty$ -ring spectra.

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<sup>13</sup>We are grateful to the referee for suggesting this argument over our previous one.

We start with the commutative monoid  $M$  with a single element  $m$ . Then we have that  $M_+ = \{*, m\} \in \mathcal{S}_*$  is a commutative algebra object of  $\mathcal{S}_*$  with respect to the smash product: in fact, it is the unit  $S^0$  as a pointed space. Similarly,  $(\mathbb{Z}_{\geq 0})_+$  is a commutative algebra object of  $\mathcal{S}_*$ . Now we have equivalences of  $E_\infty$ -ring spectra  $\Sigma^\infty(M_+) \simeq S^0$  and  $\Sigma^\infty(\mathbb{Z}_{\geq 0})_+ \simeq \Sigma_+^\infty \mathbb{Z}_{\geq 0}$ . There is a map of commutative monoids in  $\mathcal{S}_*$

$$(\mathbb{Z}_{\geq 0})_+ \rightarrow M_+,$$

which carries  $0 \in \mathbb{Z}_{\geq 0}$  to  $m$  and everything else to  $*$ . After applying  $\Sigma^\infty$ , we obtain the desired map  $S^0[x] \rightarrow S^0$  of  $E_\infty$ -rings.

The map  $R[x] \rightarrow R$  given in Example 7.2.3 has the property that it exhibits the  $R[x]$ -module  $R$  as the cofiber  $R[x]/x$ . It follows in particular that if  $R'$  is any  $E_\infty$ - $R$ -algebra and  $x' \in \pi_0 R'$  is a strictly commutative element, then we can give the cofiber  $R'/x' \simeq R' \otimes_{R[x]} R$  the structure of an  $E_\infty$ - $R'$ -algebra.

**Remark 7.2.4** Consider the sphere spectrum  $S^0$ . No cofiber  $S^0/n$  for  $n \notin \{\pm 1, 0\}$  can admit the structure of an  $E_\infty$ -ring by, for example, [43, Remark 4.3].<sup>14</sup> It follows that the only element of  $\pi_0 S^0 \simeq \mathbb{Z}$ , besides 0 and 1, that can potentially be strictly commutative is  $-1$ . Now,  $-1$  is not strictly commutative in the  $K(1)$ -local sphere  $L_{K(1)} S^0$  at the prime 2 because of the operator  $\theta$  of [24]: we have  $\theta(-1) = \frac{1}{2}((-1)^2 - (-1)) = 1 \neq 0$ , while power operations such as  $\theta$  annihilate strictly commutative elements. Therefore,  $-1$  cannot be strictly commutative in  $S^0$ . (One could have applied a similar argument with power operations to every other integer, too.) However, we observe that it is strictly commutative in  $S^0[\frac{1}{2}]$ : the obstruction is entirely 2-primary (Proposition 7.2.6 below).

**Example 7.2.5** Let  $a, b \in \pi_0 R$  be strictly commutative elements for  $R$  an  $E_\infty$ -ring. Then  $ab$  is also strictly commutative. If  $a$  is a unit, then  $a^{-1}$  is strictly commutative. This follows because there is a natural addition on  $E_\infty$ -maps  $\mathbb{Z}_{\geq 0} \rightarrow \Omega^\infty R$ .

**Proposition 7.2.6** Let  $R$  be an  $E_\infty$ -ring with  $n$  invertible. Then any  $u \in \pi_0 R$  with  $u^n = 1$  (ie an  $n^{\text{th}}$  root of unity) admits the structure of a strictly commutative element.

**Proof** We consider the map of  $E_\infty$ -monoids  $\mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}/n\mathbb{Z}$  and the induced map of  $E_\infty$ -ring spectra

$$(7-1) \quad R[x] \rightarrow R \wedge \Sigma_+^\infty \mathbb{Z}/n\mathbb{Z}.$$

<sup>14</sup>It is an unpublished result of Hopkins that no Moore spectrum can even admit the structure of an  $E_1$ -algebra.

Since  $1/n \in \pi_0 R$ , we have that  $R \wedge \Sigma_+^\infty \mathbb{Z}/n\mathbb{Z}$  is étale over  $R$  and the homotopy groups are given by  $\pi_* R[x]/(x^n - 1)$ . We can thus produce a map of  $E_\infty$ -rings  $R \wedge \Sigma_+^\infty (\mathbb{Z}/n\mathbb{Z}) \rightarrow R$  sending  $1 \in \mathbb{Z}/n\mathbb{Z}$  to  $u$  by étaleness.<sup>15</sup> Composing with (7-1) gives us the strictly commutative structure on  $u$ . □

Using these ideas, we will be able to give a direct computation of the Picard group of the  $E_\infty$ -ring  $\mathrm{KO}[[q]]$ . (We have renamed the power series variable to “ $q$ ” in accordance with “ $q$ -expansions”.)

**Proposition 7.2.7** *The map  $\mathrm{Pic}(\mathrm{KO}) \rightarrow \mathrm{Pic}(\mathrm{KO}[[q]])$  is an isomorphism, where  $q$  is in degree zero.*

**Proof** Suppose  $M$  is an invertible  $\mathrm{KO}[[q]]$ -module such that  $M/qM \simeq M \otimes_{\mathrm{KO}[[q]]} \mathrm{KO}$  is equivalent to  $\mathrm{KO}$ . We will show that then  $M$  is equivalent to  $\mathrm{KO}[[q]]$  using Bocksteins. Specifically, consider the generating class in  $\pi_0(M/qM) \simeq \mathbb{Z}$ ; we will lift this to a class in  $\pi_0 M$ . It will follow that the induced map  $\mathrm{KO}[[q]] \rightarrow M$  becomes an equivalence after tensoring with  $\mathrm{KO} \simeq \mathrm{KO}[[q]]/q$ . Since  $M$  is  $q$ -adically complete, it will follow that  $\mathrm{KO}[[q]] \simeq M$ .

By induction on  $k$ , suppose that:

- (1)  $\pi_{-1}(M/q^k M) = 0$ .
- (2)  $\pi_0(M/q^k M) \rightarrow \pi_0(M/qM)$  is a surjection.

These conditions are clearly satisfied for  $k = 1$ . If these conditions are satisfied for  $k$ , then the cofiber sequence of  $\mathrm{KO}[[q]]$ -modules

$$M/q^k M \rightarrow M/q^{k+1} M \rightarrow M/qM$$

shows that they are satisfied for  $k + 1$ . In the limit, we find that there is a map  $\mathrm{KO}[[q]] \rightarrow M$  which lifts the generator of  $\pi_0(M/qM)$ , which proves the claim. □

Proposition 7.2.7 can also be proved using Galois descent, but unlike for  $\mathrm{KO}$ , we need to use Theorem 6.1.1.

**Second proof of Proposition 7.2.7** The faithful  $C_2$ -Galois extension  $\mathrm{KO} \rightarrow \mathrm{KU}$  induces upon base-change a faithful  $C_2$ -Galois extension  $\mathrm{KO}[[q]] \rightarrow \mathrm{KU}[[q]]$ . The Picard group of  $\mathrm{KU}[[q]]$ , again by Theorem 2.4.6, is  $\mathbb{Z}/2$  generated by the suspension.

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<sup>15</sup>The étale obstruction theory has been developed by a number of authors; a convenient reference for the result that we need is [39, Theorem 8.5.4.2].

Consider now the descent spectral sequence for  $(\text{pic}(\text{KU}[[q]]))^{hC_2}$ , which is a modification of the descent spectral sequence for  $\text{KU}^{hC_2}$  in Figure 4. One difference is that every term with  $t \geq 2$  is replaced by its tensor product over  $\mathbb{Z}$  with  $\mathbb{Z}[[q]]$ ; the other is that the  $t = 1$  line now contains the  $C_2$ -cohomology of the units in  $\pi_0 \text{KU}[[q]]$ , which is a bigger module than  $(\pi_0 \text{KU})^\times = \mathbb{Z}/2$ . Namely, these units are  $\mathbb{Z}/2 \oplus q\mathbb{Z}[[q]]$ , with trivial  $C_2$ -action. The resulting  $E_2$ -page is displayed in Figure 5.

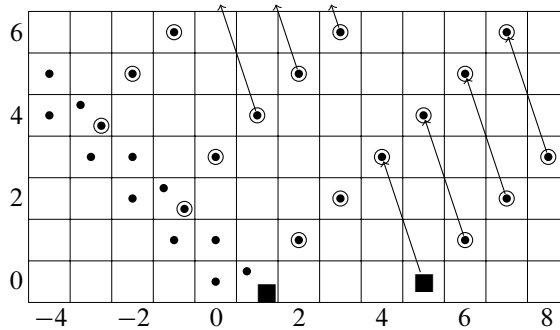


Figure 5: Homotopy fixed point spectral sequence for  $\text{pic}(\text{KU}[[q]])^{hC_2}$  (• denotes  $\mathbb{Z}/2$ , ⊙ denotes  $\mathbb{Z}/2[[q]]$ , and ■ denotes  $\mathbb{Z}[[q]]$ )

Since the  $d_3$  is the only differential in the ordinary HFPSS for  $\pi_* \text{KO}[[q]]$ , as before, it follows that the only contributions to  $\text{Pic}(\text{KO}[[q]])$  can come from the  $\mathbb{Z}/2$  with  $t = s = 0$  (the suspension), the  $\mathbb{Z}/2$  with  $(s, t) = (1, 1)$  (ie the algebraic Picard group), and the  $\mathbb{Z}/2[[q]]$  in bidegree  $(s, t) = (3, 3)$ .

But here,  $E_2^{3,3} = \mathbb{Z}/2[[q]](h_1^3/u^2)$  is infinite, so unlike previously, we do not get the automatic upper bound of eight on  $|\text{Pic}(\text{KO}[[q]])|$ . On the other hand, we can use Theorem 6.1.1 to determine the  $d_3$  supported here. Note that in the HFPSS for  $(\text{KU}[[q]])^{hC_2}$ , we have

$$d_3(f(q)(h_1^3/u^2)) = f(q)(h_1^6/u^4) \quad \text{for } f(q) \in \mathbb{Z}/2[[q]].$$

Therefore, in view of (6-1), in the HFPSS for  $\text{pic}(\text{KU}[[q]])^{hC_2}$ , we have

$$d_3(f(q)(h_1^3/u^2)) = (f(q) + f(q)^2)(h_1^6/u^4).$$

(Note that a crucial point here is that in the HFPSS for  $\text{KO}$ , squaring or applying  $d_3$  to  $h_1^3/u^2$  yields the same result.) It follows from this that in the HFPSS, the kernel of  $d_3$  on  $E_2^{3,3}$  is  $\mathbb{Z}/2$  generated by  $1(h_1^3/u^2)$ : the equation  $f(q) + f(q)^2 = 0$  has only the solutions  $f(q) \equiv 0, 1$ . Therefore, we do get an upper bound of eight on the cardinality of  $\text{Pic}(\text{KO}[[q]])$  after all, as nothing else in  $E_2^{3,3}$  lives to  $E_4$ . □

**Corollary 7.2.8** *The maps  $\mathrm{KO} \rightarrow \mathrm{KO}[q]$  and  $\mathrm{KO} \rightarrow \mathrm{KO}((q))$  induce isomorphisms on Picard groups.*

**Proof** This result is not a corollary of Proposition 7.2.7 but rather of its second proof. In fact, the same argument shows that  $d_3$  has a  $\mathbb{Z}/2$  as kernel on the relevant term  $E_2^{3,3}$ , which gives an upper bound of cardinality eight on the Picard group of  $\mathrm{KO}[q]$  or  $\mathrm{KO}((q))$  as before.  $\square$

**Remark 7.2.9** Corollary 7.2.8 cannot be proved using the Bockstein spectral sequence argument used in the first proof of Proposition 7.2.7. However, a knowledge of the Picard group of  $\mathrm{KO}[[q]]$  can be used to describe enough of the  $C_2$ -descent spectral sequence to make it possible to prove Corollary 7.2.8 without the explicit formula (6-1). We leave this to the reader.

## 8 Picard groups of topological modular forms

In the rest of the paper we proceed to use descent to compute the Picard groups of various versions of topological modular forms. We will analyze the following descent-theoretic situations:

- The Galois extension  $\mathrm{TMF}[\frac{1}{2}] \rightarrow \mathrm{TMF}(2)$ , with structure group  $\mathrm{GL}_2(\mathbb{Z}/2)$ , also known as the symmetric group on three letters.
- The Galois extension  $\mathrm{TMF}[\frac{1}{3}] \rightarrow \mathrm{TMF}(3)$ , with structure group  $\mathrm{GL}_2(\mathbb{Z}/3)$ , a group of order 48 which is a nontrivial extension of the binary tetrahedral group and  $C_2$ .
- Étale descent from the (derived) moduli stack of elliptic curves or its compactification.

In each of these cases, we will start with the knowledge of the original descent spectral sequence, computing the homotopy groups of the global sections or homotopy fixed point spectrum. This information plus some additional computation of the differing cohomology groups will provide the data for the  $E_2$ -page of the descent spectral sequence for the Picard spectrum. The additional computations are somewhat lengthy, hence we are including them separately in the appendices.

### 8.1 The Picard group of $\mathrm{TMF}[\frac{1}{2}]$

When 2 is inverted, the moduli stack of elliptic curves  $M_{\mathrm{ell}}$  has a  $\mathrm{GL}_2(\mathbb{Z}/2)$ -Galois cover by  $M_{\mathrm{ell}}(2)$ , the moduli stack of elliptic curves with full level 2 structure. This



remains the case for the derived versions of these stacks, and on global sections gives a faithful Galois extension  $\mathrm{TMF}[\frac{1}{2}] \rightarrow \mathrm{TMF}(2)$  by [42, Theorem 7.6]. The extension is useful for the purposes of descent as the homotopy groups of  $\mathrm{TMF}(2)$  are cohomologically very simple.

To be precise, we have that

$$\mathrm{TMF}(2)_* = \mathbb{Z}[\frac{1}{2}][\lambda_1^{\pm 1}, \lambda_2^{\pm 1}][(\lambda_1 - \lambda_2)^{-1}],$$

where the (topological) degree of each  $\lambda_i$  is four. To see this, one can use the presentation of the moduli stack  $M_{\mathrm{ell}}(2)$  from [63, Section 7]. There it is computed that  $\overline{M}_{\mathrm{ell}}(2)$  is equivalent to (the stacky)  $\mathrm{Proj} \mathbb{Z}[\frac{1}{2}][\lambda_1, \lambda_2]$ . Moreover, the substack classifying smooth curves, ie  $M_{\mathrm{ell}}(2)$ , is the locus of nonvanishing of  $\lambda_1^2 \lambda_2^2 (\lambda_1 - \lambda_2)^2$ . More precisely,  $M_{\mathrm{ell}}(2)$ , as a stack, is the  $\mathbb{G}_m$ -quotient of the ring

$$\mathbb{Z}[\frac{1}{2}][\lambda_1, \lambda_2, (\lambda_1^2 \lambda_2^2 (\lambda_1 - \lambda_2)^{-1})^{-1}],$$

where the  $\mathbb{G}_m$ -action is as follows: a unit  $u$  acts as  $\lambda_i \mapsto u^2 \lambda_i$  for  $i = 1, 2$ , so that it is an open substack of a *weighted projective stack*.

In particular,  $\mathrm{TMF}(2)_*$  has a unit in degree 4, and is zero in degrees not divisible by 4. It will be helpful to write  $\mathrm{TMF}(2)_*$  differently, so as to reflect this periodicity more explicitly; for example, we have that  $\mathrm{TMF}(2)_* = \mathrm{TMF}(2)_0[\lambda_2^{\pm 1}]$ , and

$$(8-1) \quad \mathrm{TMF}(2)_0 = \mathbb{Z}[\frac{1}{2}][s^{\pm 1}, (s - 1)^{-1}],$$

where  $s = \lambda_1 / \lambda_2$ . Therefore, Corollary 2.4.7 applies to give the following conclusion.

**Lemma 8.1.1** *Pic(TMF(2)) is  $\mathbb{Z}/4$ , generated by the suspension  $\Sigma \mathrm{TMF}(2)$ .*

**Remark 8.1.2** The proof of Corollary 2.4.7 relies on the construction of “residue field” spectra; let us specify what they are in the case at hand. The maximal ideals in  $\mathrm{TMF}(2)_0$  are  $\mathfrak{m} = (p, f(s))$ , where  $p$  is an odd prime and  $f(s)$  a monic polynomial irreducible modulo  $p$  (and not congruent mod  $p$  to  $s, s - 1$ ). For each of these ideals, we have an associative ring spectrum (the “residue field”) with homotopy groups  $\mathrm{TMF}(2)_*/\mathfrak{m}$  by [3]; denote it temporarily by  $\mathrm{TMF}(2)/\mathfrak{m}$ . After extending scalars so that  $f$  splits, we get that  $\mathrm{TMF}(2)/\mathfrak{m}$  is a product of (extensions of) mod  $p$  Morava  $K$ -theory spectra at height one or two, one for each zero of  $f$ . By [62, Chapter V, Theorem 4.1], the factor associated to the zero  $a$  of  $f$  has height two precisely when

$$\sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i} a^i$$

is zero modulo  $p$ .

Next we use descent from  $\mathrm{TMF}(2)$  to  $\mathrm{TMF}[\frac{1}{2}]$  to obtain the following result.

**Theorem 8.1.3**  $\mathrm{Pic}(\mathrm{TMF}[\frac{1}{2}])$  is  $\mathbb{Z}/72$ , generated by the suspension  $\Sigma \mathrm{TMF}[\frac{1}{2}]$ . In particular, this Picard group is algebraic.

**Proof** We use the homotopy fixed point spectral sequence (3-5)

$$(8-2) \quad H^s(\mathrm{GL}_2(\mathbb{Z}/2), \pi_t \mathrm{pic}(\mathrm{TMF}(2))) \Rightarrow \pi_{t-s} \mathrm{pic}(\mathrm{TMF}(2))^{h\mathrm{GL}_2(\mathbb{Z}/2)}.$$

To begin with, note that the homotopy groups  $\pi_t \mathrm{pic}(\mathrm{TMF}(2))$  for  $t \geq 2$  are isomorphic to  $\pi_{t-1} \mathrm{TMF}(2)$  as  $\mathrm{GL}_2(\mathbb{Z}/2)$ -modules. This tells us that the  $t \geq 2$  part of the  $E_2$ -page of the HFPSS (8-2) for  $\mathrm{pic}(\mathrm{TMF}(2))$  is a shifted version of the corresponding part for  $\mathrm{TMF}(2)$ .

The latter is immediately obtained from the analogous computation for  $\mathrm{Tmf}(2)$  depicted in [63, Figure 2], as we now describe. Recall that  $\mathrm{TMF}(2) \simeq \mathrm{Tmf}(2)[\Delta^{-1}]$ ; the nonnegative homotopy groups  $\pi_{\geq 0} \mathrm{Tmf}(2)$  are the graded polynomial ring  $\Lambda = \mathbb{Z}[\frac{1}{2}][\lambda_1, \lambda_2]$  [63, Proposition 8.1], and the class  $\Delta \in \pi_{24} \mathrm{Tmf}(2)$  is

$$\Delta = 16\lambda_1^2\lambda_2^2(\lambda_2 - \lambda_1)^2$$

by [63, Proposition 10.3]. Now, by [63, Proposition 10.8] we have that

$$H^*(\mathrm{GL}_2(\mathbb{Z}/2), \pi_* \mathrm{TMF}(2)) = H^*(\mathrm{GL}_2(\mathbb{Z}/2), \Lambda)[\Delta^{-1}].$$

In particular, the invariants  $H^0(\mathrm{GL}_2(\mathbb{Z}/2), \Lambda)[\Delta^{-1}]$  are the ring of  $\Delta$ -inverted modular forms

$$\mathbb{Z}[\frac{1}{2}][c_4, c_6, \Delta^{\pm 1}]/(12^3\Delta - c_4^3 + c_6^2).$$

The higher cohomology  $H^{>0}(\mathrm{GL}_2(\mathbb{Z}/2), \Lambda)$  is computed in [63, Section 10.1], and in particular is killed by  $c_4$  and  $c_6$ . Consequently,

$$\begin{aligned} H^{>0}(\mathrm{GL}_2(\mathbb{Z}/2), \pi_{\geq 0} \mathrm{TMF}(2)) &= H^{>0}(\mathrm{GL}_2(\mathbb{Z}/2), \Lambda) \\ &= H^{>0}(\mathrm{GL}_2(\mathbb{Z}/2), \pi_{\geq 0} \mathrm{Tmf}(2)). \end{aligned}$$

Let us recall (the names of) certain interesting classes in these cohomology groups:

- (1) There is the class  $a$  in  $H^1(\mathrm{GL}_2(\mathbb{Z}/2), \pi_4 \mathrm{TMF}(2)) = \mathbb{Z}/3$ , hence also in  $H^1(\mathrm{GL}_2(\mathbb{Z}/2), \pi_5 \mathrm{pic}(\mathrm{TMF}(2)))$  (so,  $a$  is in bidegree  $(s, t) = (1, 5)$  in the Picard HFPSS, and depicted in position  $(s, t - s) = (1, 4)$  using the Adams convention). In homotopy, this element detects the Greek letter element  $\alpha_1$  in the Hurewicz image in  $\mathrm{TMF}[\frac{1}{2}]$ .
- (2) There is  $b$  in  $H^2(\mathrm{GL}_2(\mathbb{Z}/2), \pi_{13} \mathrm{pic}(\mathrm{TMF}(2))) = \mathbb{Z}/3$  ( $b$  is in bidegree  $(2, 13)$  or position  $(2, 11)$ ); in homotopy it detects  $\beta_1$ .

Then,  $H^{>0}(\mathrm{GL}_2(\mathbb{Z}/2), \mathrm{TMF}(2)_*)$  is precisely the ideal of  $\mathbb{Z}/3[a, b][[\Delta^{\pm 1}]]/(a^2)$  of positive cohomological degree. For example

$$H^5(\mathrm{GL}_2(\mathbb{Z}/2), \pi_5 \mathrm{pic}(\mathrm{TMF}(2))) = H^5(\mathrm{GL}_2(\mathbb{Z}/2), \pi_4 \mathrm{TMF}(2)) = \mathbb{Z}/3,$$

generated by  $ab^2\Delta^{-1}$ . We see this class depicted in red in Figure 6.

Next, we turn to the information which is new for the Picard HFPSS, ie the group cohomology of  $\pi_0$  and  $\pi_1$  of the spectrum  $\mathrm{pic}(\mathrm{TMF}(2))$ . By Lemma 8.1.1, we know that the zeroth homotopy group is  $\mathbb{Z}/4$ , and since it is generated by the suspension  $\Sigma \mathrm{TMF}(2)$ , the action of  $\mathrm{GL}_2(\mathbb{Z}/2)$  on this  $\mathbb{Z}/4$  is trivial. Even though for our purposes only the invariants  $H^0(\mathrm{GL}_2(\mathbb{Z}/2), \pi_0 \mathrm{pic}(\mathrm{TMF}(2)))$  are necessary, we can in fact compute all the cohomology groups. This is done in Lemma A.1.

The last piece of data needed for the determination of the  $E_2$ -page of the Picard HFPSS is the group cohomology with coefficients in  $\pi_1 \mathrm{pic}(\mathrm{TMF}(2)) = (\pi_0 \mathrm{TMF}(2))^\times$ . This is done in Proposition A.2. The range  $s \leq 15$  and  $-6 \leq t - s \leq 7$  of the spectral sequence is depicted in Figure 6. Note that in this range, the  $t - s = 0$  column has three nonzero entries: there is a  $\mathbb{Z}/4$  for  $s = 0$ , a  $\mathbb{Z}/6$  for  $s = 1$  and a  $\mathbb{Z}/3$  for  $s = 5$ .

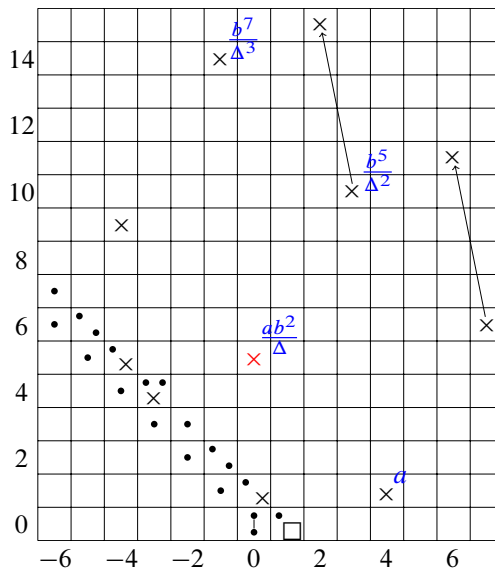


Figure 6: Homotopy fixed point spectral sequence for  $(\mathrm{pic}(\mathrm{TMF}(2)))^{h\mathrm{GL}_2(\mathbb{Z}/2)}$   
 ( $\square$  denotes  $\mathbb{Z}$ ,  $\bullet$  denotes  $\mathbb{Z}/2$ , and  $\times$  denotes  $\mathbb{Z}/3$ )

Now we are ready to study the differentials in the HFPSS for  $\mathrm{pic}(\mathrm{TMF}(2))^{h\mathrm{GL}_2(\mathbb{Z}/2)}$ . Comparison with the HFPSS for the  $\mathrm{GL}_2(\mathbb{Z}/2)$ -action on  $\mathrm{TMF}(2)$  gives a number of

differentials, using our Comparison Tool 5.2.4. To distinguish between the differentials in the two spectral sequences, let us denote by  $d_r^o$  those in the HFPSS of  $\mathrm{TMF}(2)$ . The superscript  $o$  stands for “original”.

Recall that in the HFPSS for  $\mathrm{TMF}(2)$ , there are nonzero  $d_5^o$  and  $d_9^o$  differentials, which are obtained, for example, by a comparison with the HFPSS for  $\mathrm{Tmf}(2)$  which is fully determined in [63]. In particular, in the HFPSS for  $\mathrm{TMF}(2)$ , the first differential is  $d_5^o(\Delta) = ab^2$ , and the rest of the  $d_5^o$ 's are determined by multiplicativity and the fact that  $a$  and  $b$  are permanent cycles. In particular, we have

$$(8-3) \quad d_5^o\left(\frac{b^5}{\Delta^2}\right) = \frac{ab^7}{\Delta^3} \quad \text{and} \quad d_5^o\left(\frac{b^3}{\Delta}\right) = -\frac{ab^5}{\Delta^2}.$$

Next (and last) is  $d_9^o$ ; we have that  $d_9^o(a\Delta^2) = b^5$ . Consequently, we also have

$$(8-4) \quad d_9^o\left(\frac{ab^2}{\Delta}\right) = \frac{b^7}{\Delta^3}.$$

Let us now see which of these differentials also occur in the HFPSS for  $\mathrm{pic}(\mathrm{TMF}(2))$ ; according to Comparison Tool 5.2.4, the  $d_5$ -differentials are imported in the  $t > 9$  range, and the  $d_9$ -differentials in the  $t > 9$  range. In particular, the differentials in (8-3) are the same in the Picard HFPSS; these are the two differentials drawn in Figure 6. Moreover, everything in the zero column and above the depicted region, ie such that  $s = t > 16$ , either supports a differential or is killed by one which originates in the  $t > 9$  range. Hence, everything above the depicted region is killed in the spectral sequence and nothing survives to the  $E_\infty$ -page.

Note, however, that we cannot (and should not attempt to) import the differential (8-4); this would be a  $d_9$ -differential with  $t = 5$ , so it does not satisfy the hypothesis of Comparison Tool 5.2.4.

Let us analyze the potentially remaining contributions to  $\pi_0 \mathrm{pic}(\mathrm{TMF}(2))^{\mathrm{GL}_2(\mathbb{Z}/2)}$ ; regardless of what the rest of the differentials could possibly be, we have

- a group of order at most 4 (and dividing 4) in position  $(0, 0)$ ,
- a group of order at most 6 (and dividing 6) in position  $(0, 1)$ , and
- a group of order at most 3 (and dividing 3) in position  $(0, 5)$ .

Therefore  $\mathrm{Pic}(\mathrm{TMF}[\frac{1}{2}]) = \pi_0 \mathrm{pic}(\mathrm{TMF}(2))^{\mathrm{GL}_2(\mathbb{Z}/2)}$  has order at most  $4 \times 6 \times 3 = 72$ , and dividing 72. This is an upper bound. But we also have a well-known lower bound: the suspension  $\Sigma \mathrm{TMF}[\frac{1}{2}]$  generates a nontrivial element of  $\mathrm{Pic}(\mathrm{TMF}[\frac{1}{2}])$  of order 72 because  $\mathrm{TMF}[\frac{1}{2}]$  is 72-periodic. Thus we have proven the result.  $\square$

**Remark 8.1.4** Our computations give an independent proof of the result of Fulton and Olsson [14] that the Picard group of the classical moduli stack of elliptic curves  $M_{\text{ell}}$  over  $\mathbb{Z}[\frac{1}{2}]$  is  $\mathbb{Z}/12$ . (Fulton and Olsson carry out the analysis over any base, though.) This is a toy analog of the above analysis, as we now see.

The Picard *groupoid* of the moduli stack  $M_{\text{ell}}[\frac{1}{2}]$  is the homotopy fixed points of the  $\text{GL}_2(\mathbb{Z}/2)$ -action on the Picard groupoid of  $M_{\text{ell}}(2)$ . Now the Picard *group* of  $M_{\text{ell}}(2)$  is  $\mathbb{Z}/2$ , as  $M_{\text{ell}}(2)$  is an open subset in a weighted projective stack over a UFD, so that quasicoherent sheaves on  $M_{\text{ell}}(2)$  correspond simply to graded modules over  $\mathbb{Z}[\frac{1}{2}, \lambda_1, \lambda_2, (\lambda_1^2 \lambda_2^2 (\lambda_1 - \lambda_2))^{-1}]$  and the only nontrivial invertible object is the shift by one of the unit. Note that this is the *algebraic* setting: the generator of  $\text{Pic}(M_{\text{ell}}(2))$  would correspond to the *two-fold* suspension of  $\text{TMF}(2)$ .

Next, in the HFPSS for computing  $\text{Pic}(M_{\text{ell}}[\frac{1}{2}])$ , we see by the above computation of

$$H^1(\text{GL}_2(\mathbb{Z}/2), \Gamma(M_{\text{ell}}(2), \mathcal{O}^\times))$$

that one gets a contribution of order 6. Together with  $\text{Pic}(M_{\text{ell}}(2)) = \mathbb{Z}/2$  from the previous paragraph, we get that  $|\text{Pic}(M_{\text{ell}}[\frac{1}{2}])| \leq 12$ , but we know that  $\omega$  has order twelve, so we are done.

## 8.2 The Picard group of $\text{TMF}[\frac{1}{3}]$

This section will be similar to Section 8.1, but with more complicated computations as is to be expected from 2-torsion. In this case we will use the  $\text{GL}_2(\mathbb{Z}/3)$ -Galois extension  $\text{TMF}[\frac{1}{3}] \rightarrow \text{TMF}(3)$ , coming from the Galois cover  $M_{\text{ell}}(3) \rightarrow M_{\text{ell}}[\frac{1}{3}]$  of the moduli stack of elliptic curves with 3 inverted by the moduli stack of elliptic curves equipped with a full level 3-structure.

From [64, Section 4.2], we can immediately compute the homotopy groups of  $\text{TMF}(3)$ : the moduli stack  $M_{\text{ell}}(3)$  is affine, and is given as the locus of nonvanishing of

$$\Delta = 3^{-5} \zeta (1 - \zeta) \gamma_1^3 \gamma_2^3 (\gamma_1 + \zeta \gamma_2)^3 (\gamma_2 - \zeta \gamma_1)^3$$

in the compact moduli stack  $\overline{M}_{\text{ell}}(3) = \text{Proj } \mathbb{Z}[\frac{1}{3}, \zeta][\gamma_1, \gamma_2]$ . Here  $\gamma_i$  are variables in (topological) degree 2, and  $\zeta$  is a primitive third root of unity, whose appearance is due to the fact that the Weil pairing on the 3-torsion points of an elliptic curve equips  $\overline{M}_{\text{ell}}(3)$  with a map to  $\text{Spec } \mathbb{Z}[\frac{1}{3}, \zeta]$ .<sup>16</sup> Hence the descent spectral sequence computing  $\text{TMF}(3)_*$  collapses to give

$$\text{TMF}(3)_* = \mathbb{Z}[\frac{1}{3}, \zeta][\gamma_1^{\pm 1}, \gamma_2^{\pm 1}][(\gamma_1 + \zeta \gamma_2)^{-1}, (\gamma_2 - \zeta \gamma_1)^{-1}].$$

<sup>16</sup>The map is given by the usual Weil pairing on the locus of smooth curves; for what it does at the cusps, see for example [11, IV.3.21].

Written differently, we have that  $\mathrm{TMF}(3)_* = \mathrm{TMF}(3)_0[\gamma_2^{\pm 1}]$ , and

$$(8-5) \quad \mathrm{TMF}(3)_0 = \mathbb{Z}\left[\frac{1}{3}, \zeta\right][t^{\pm 1}, (1 - \zeta t)^{-1}, (1 + \zeta^2 t)^{-1}],$$

for  $t = \gamma_1/\gamma_2$ . In particular  $\mathrm{TMF}(3)_0$  is regular noetherian, and  $\mathrm{TMF}(3)$  is even periodic. Thus, Theorem 2.4.6 (together with the fact that the ring  $\mathbb{Z}[\zeta, t]$  and hence any of its localizations has unique factorization) implies the following conclusion.

**Lemma 8.2.1** *The Picard group  $\mathrm{Pic}(\mathrm{TMF}(3))$  is  $\mathbb{Z}/2$ , generated by  $\Sigma \mathrm{TMF}(3)$ .*

Naturally, we will use this lemma as an input in computing the HFPSS for the associated Picard spectra.

**Theorem 8.2.2**  *$\mathrm{Pic}(\mathrm{TMF}[\frac{1}{3}])$  is  $\mathbb{Z}/192$ , generated by the suspension  $\Sigma \mathrm{TMF}[\frac{1}{3}]$ . In particular, this Picard group is algebraic.*

**Proof** As is to be expected, we use the HFPSS (3-5)

$$(8-6) \quad H^s(\mathrm{GL}_2(\mathbb{Z}/3), \pi_t \mathrm{pic}(\mathrm{TMF}(3))) \Rightarrow \pi_{t-s} \mathrm{pic}(\mathrm{TMF}(3))^{h\mathrm{GL}_2(\mathbb{Z}/3)}.$$

The homotopy groups  $\pi_t(\mathrm{pic}(\mathrm{TMF}(3)))$  for  $t \geq 2$  are isomorphic to  $\pi_{t-1} \mathrm{TMF}(3)$  as  $\mathrm{GL}_2(\mathbb{Z}/3)$ -modules; therefore the  $t \geq 2$  part of the  $E_2$ -page of the HFPSS for  $\mathrm{pic}(\mathrm{TMF}(3))$  is same as the corresponding part in the HFPSS for  $\mathrm{TMF}(3)$ . We will use the fact that  $\mathrm{TMF}(3) \simeq \mathrm{Tmf}(3)[\Delta^{-1}]$  to identify this part of the spectral sequence for  $\mathrm{TMF}(3)$  and therefore for  $\mathrm{pic}(\mathrm{TMF}(3))$ .

Computed in [64], and depicted in Figure 9 of loc. cit., is the  $E_2$ -page of the HFPSS computing the homotopy groups of  $\mathrm{Tmf}\widehat{2}$  as  $(\mathrm{Tmf}(3)\widehat{2})^{h\mathrm{GL}_2(\mathbb{Z}/3)}$ . Since we are working with 3 inverted, and 2 and 3 are the only primes dividing the order of  $\mathrm{GL}_2(\mathbb{Z}/3)$ , we conclude that

$$H^{>0}(\mathrm{GL}_2(\mathbb{Z}/3), \pi_* \mathrm{Tmf}(3)) = H^{>0}(\mathrm{GL}_2(\mathbb{Z}/3), \pi_* \mathrm{Tmf}(3)\widehat{2}).$$

The invariants  $H^0(\mathrm{GL}_2(\mathbb{Z}/3), \pi_{\geq 0} \mathrm{Tmf}(3))$  are the ring of modular forms

$$\mathbb{Z}\left[\frac{1}{3}\right][c_4, c_6, \Delta]/(12^3 \Delta - c_4^3 + c_6^2).$$

Let  $\Gamma$  denote the graded ring  $\mathbb{Z}[\frac{1}{3}, \zeta][\gamma_1, \gamma_2]$ . As in the case of level 2-structures, we have that

$$H^*(\mathrm{GL}_2(\mathbb{Z}/3), \pi_* \mathrm{TMF}(3)) = H^*(\mathrm{GL}_2(\mathbb{Z}/3), \Gamma)[\Delta^{-1}].$$

In the group cohomology of  $\Gamma$ , computed and depicted in [64, Figure 7], there are a number of interesting torsion classes, including

- (1)  $h_1$  in bidegree  $(s, t) = (1, 2)$ , depicted in position  $(s, t - s) = (1, 1)$ , which detects (the Hurewicz image of) the Hopf map  $\eta$  in homotopy,
- (2)  $h_2$  in position  $(1, 3)$ , which detects (the Hurewicz image of) the Hopf map  $\nu$ ,
- (3)  $d$  in position  $(2, 14)$ , which detects in homotopy the class known as  $\kappa$ ,
- (4)  $g$  in position  $(4, 20)$ , which detects in homotopy the class  $\bar{\kappa}$ , and
- (5)  $c$  in position  $(2, 8)$ , which detects in homotopy the class  $\epsilon$ .

The homotopy elements detected by these classes satisfy some relations; for example,

$$\eta^3 = 4\nu \quad \text{and} \quad \kappa\nu^2 = 4\bar{\kappa}.$$

Let us also name one of the less famous elements in the descent spectral sequence for  $\text{tmf}_{(2)}$ , which also appears in the HFPSS for  $\text{TMF}[\frac{1}{3}]$ . Namely, there is a  $\mathbb{Z}/2$  in position  $(1, 5)$ ; we will denote the generating class by the generic name  $x$  (in [6] it bears the name  $a_1^2 h_1$ ).

All torsion classes with the exception of (powers of)  $h_1$  are annihilated by  $c_4$  and  $c_6$ . In the Picard spectral sequence, all of these classes appear shifted by one to the right; we have labeled some such classes in Figure 9. A zoomed in portion of the Picard spectral sequence is depicted in Figure 8. There, and in all of the related spectral sequences, lines of slope 1 denote  $h_1$ -multiplication, and lines of slope  $\frac{1}{3}$  denote  $h_2$ -multiplication.

A zoomed out portion of the Picard HFPSS (8-6) is depicted in Figure 7; the elements that are to the right of the  $t = 2$  line are, of course, a shift of the corresponding elements in the spectral sequence for  $\text{TMF}[\frac{1}{3}]$ . However, to avoid cluttering the picture, a family of classes is not shown. The family consists precisely of the  $h_1$ -power multiples of nontorsion classes. An exception is made for the elements depicted in green, namely  $h_1^3 c_4 c_6 / \Delta$  and  $h_1^6 c_4^2 / \Delta$  (in the  $(0, 3)$  and  $(-1, 6)$  positions, respectively; these classes are also labeled in Figure 8), as well as the tower supported on 1, which do belong to this family, but are nonetheless depicted. In the zoomed in Figure 9 this family is also not shown.

More specifically, the nontorsion subring of the  $E_2$ -page of the  $\text{TMF}[\frac{1}{3}]$  spectral sequence is precisely the part in cohomological degree 0 and consists of the ring of modular forms  $\text{MF}_*[\frac{1}{3}] = \mathbb{Z}[\frac{1}{3}][c_4, c_6, \Delta^{\pm 1}] / (12^3 \Delta - c_4^3 + c_6^2)$ . On the  $E_2$ -page, these support infinite  $h_1$ -multiples, ie  $\text{MF}_*[\frac{1}{3}][h_1] / (2h_1)$  is a subring of the  $E_2$ -page. Note in degree zero,  $\text{MF}_0[\frac{1}{3}] = \mathbb{Z}[\frac{1}{3}, j]$ , where  $j = c_4^3 / \Delta$  is the classical  $j$ -invariant. What we have omitted drawing in Figure 7 and 9 are all of the elements coming from this subring, with the exception of the mentioned classes. For comparison, these elements are drawn in the smaller-range Figure 8.

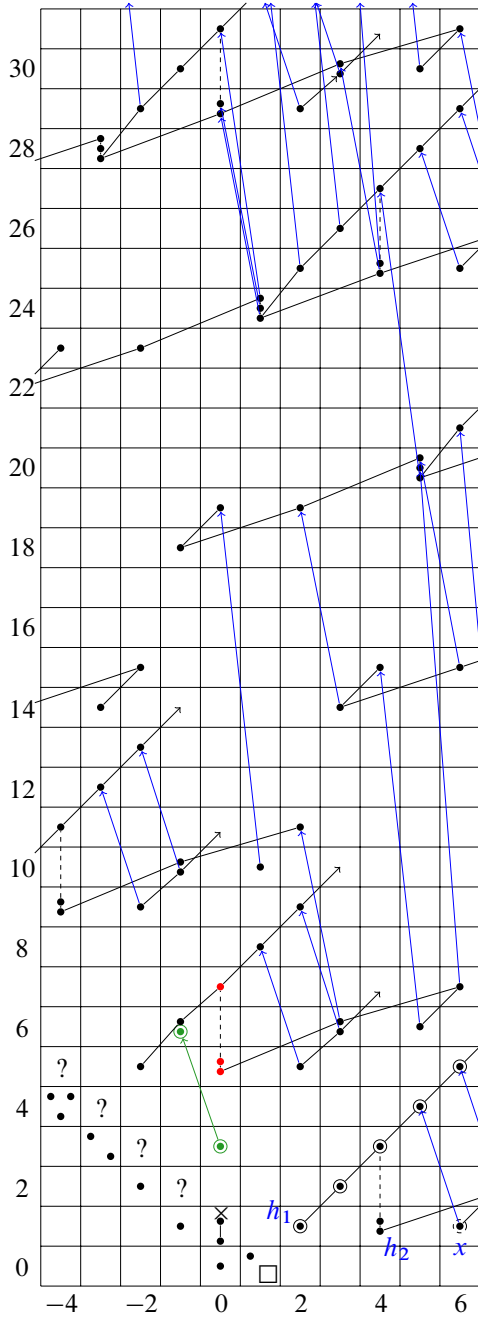


Figure 7: Homotopy fixed point spectral sequence for  $\text{pic}(\text{TMF}(3))^{h_{\text{GL}_2(\mathbb{Z}/3)}}$ : zoomed out version with some  $h_1$ -omissions ( $\square$  denotes  $\mathbb{Z}$ ,  $\bullet$  denotes  $\mathbb{Z}/2$ ,  $\odot$  denotes  $\mathbb{Z}/2[j]$ , and  $\times$  denotes  $\mathbb{Z}/3$ )



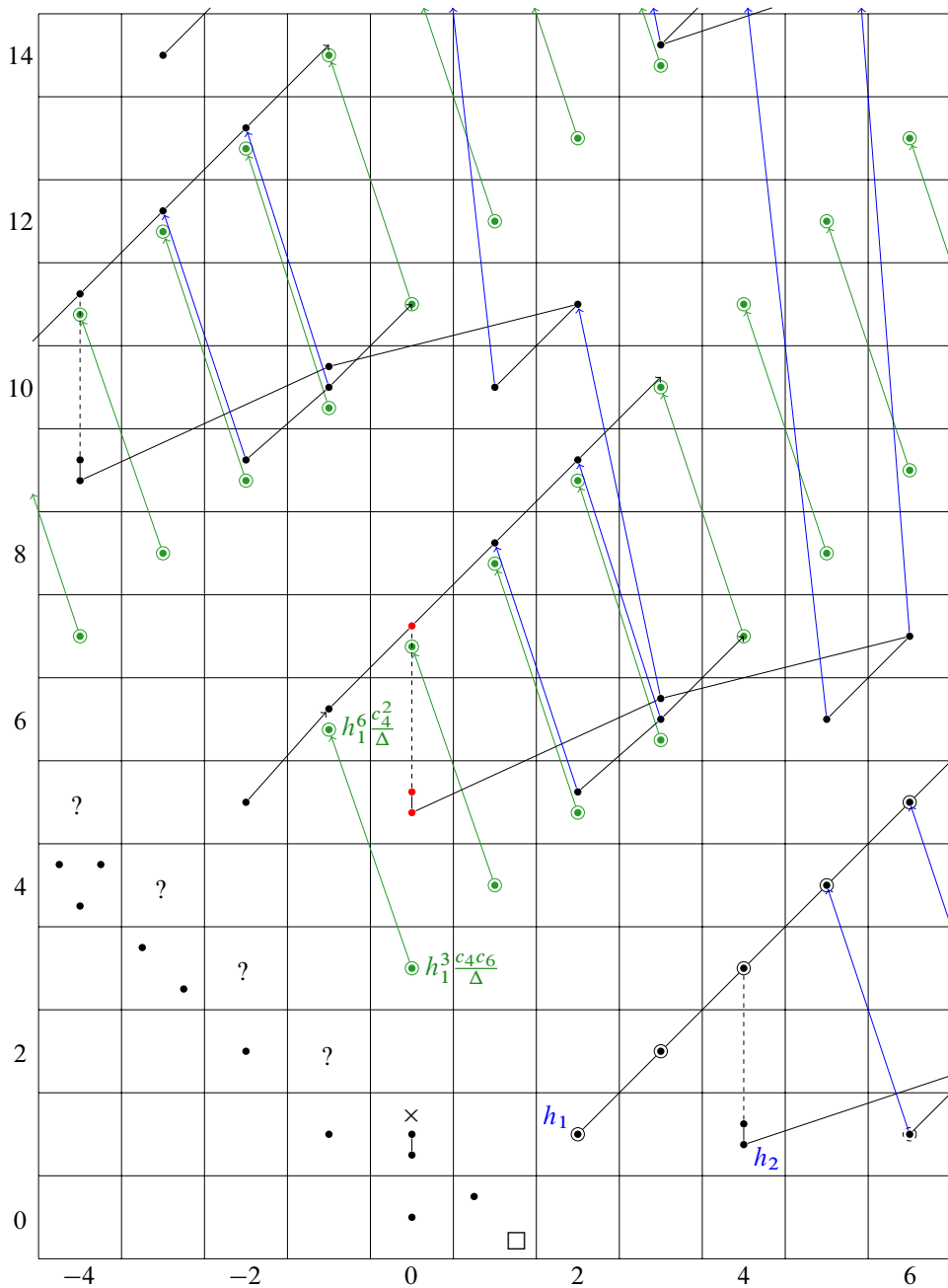


Figure 8: Homotopy fixed point spectral sequence for  $\text{pic}(\text{TMF}(3))^{h_{\text{GL}_2(\mathbb{Z}/3)}}$ : zoomed in version without omissions ( $\square$  denotes  $\mathbb{Z}$ ,  $\bullet$  denotes  $\mathbb{Z}/2$ ,  $\odot$  denotes  $\mathbb{Z}/2[j]$ , and  $\times$  denotes  $\mathbb{Z}/3$ )

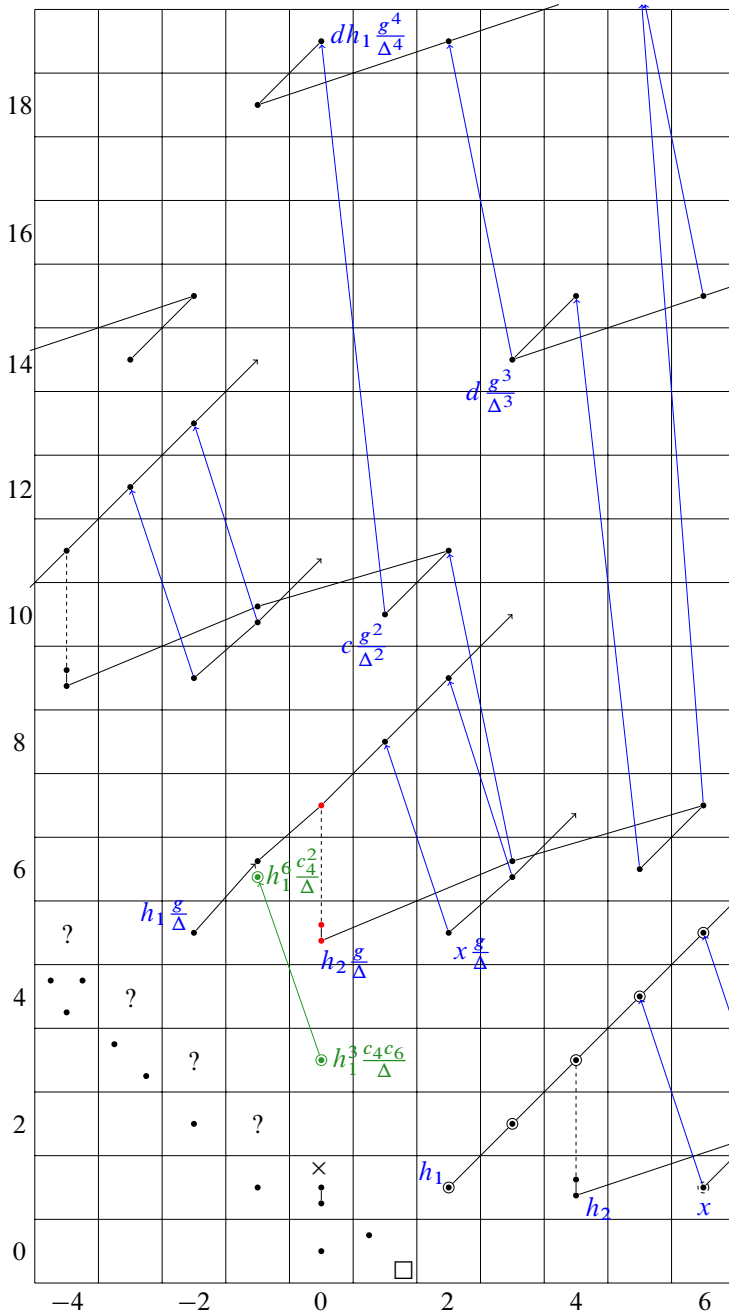


Figure 9: Homotopy fixed point spectral sequence for  $\text{pic}(\text{TMF}(3))^{h_{\text{GL}_2(\mathbb{Z}/3)}}$ : zoomed in version with some  $h_1$ -omissions ( $\square$  denotes  $\mathbb{Z}$ ,  $\bullet$  denotes  $\mathbb{Z}/2$ ,  $\odot$  denotes  $\mathbb{Z}/2[j]$ , and  $\times$  denotes  $\mathbb{Z}/3$ )

**Remark 8.2.3** The two classes  $h_1^3 c_4 c_6 / \Delta$  and  $h_1^6 c_4^2 / \Delta$ , which we have depicted in green (in the  $(0, 3)$  and  $(-1, 6)$  positions, respectively), do not appear in the spectral sequence for  $\text{Tmf}[\frac{1}{3}]$ , as they involve a negative power of  $\Delta$ . Another difference between the  $\text{Tmf}$  and  $\text{TMF}$  situation is that in the  $E_2$ -page of the latter, there are infinite groups, isomorphic to  $\mathbb{Z}/2[j]$  and generated by  $h_1, h_1^2, h_1^3$ , etc, in positions  $(1, 1), (2, 2), (3, 3)$ , etc. Moreover, the element  $x$  in position  $(1, 5)$  also generates an infinite  $\mathbb{Z}/2[j]$ , as do all of its  $h_1$ -multiples.

Note that in the range that we are considering (namely,  $t > 1$ ), the HFPSS for the  $\text{GL}_2(\mathbb{Z}/3)$ -action on  $\text{Tmf}(3)$  coincides with the descent spectral sequence for  $\text{Tmf}[\frac{1}{3}]$  as the sections of  $\mathcal{O}^{\text{top}}$  over  $\overline{\mathcal{M}}_{\text{ell}}[\frac{1}{3}]$ , and the differentials in the latter have been fully determined in Johan Konter’s master thesis [32]. Of course, these differentials really come from the connective  $\text{tmf}$ , whose descent spectral sequence is fully computed in [6]. In these spectral sequences,  $d_3^o$  is the first nontrivial differential, followed by  $d_5^o, d_7^o, d_9^o, \dots, d_{23}^o$ . In particular, we have the following differentials [6, Section 8]:

$$\begin{aligned}
 (8-7) \quad & d_3^o(c_6) = c_4 h_1^3, & d_3^o(x) &= h_1^4, \\
 & d_5^o(\Delta) = g h_2, & d_7^o(4\Delta) &= g h_1^3, \\
 & d_9^o(\Delta^2 h_1) = g^2 c, & d_{11}^o(d\Delta^2) &= g^3 h_1,
 \end{aligned}$$

and a number of others.

Let us see now which of these differentials we can import using our Comparison Tool 5.2.4. In the  $\text{TMF}[\frac{1}{3}]$  spectral sequence, we have that  $d_3^o(h_1^3 c_4 c_6 / \Delta) = h_1^6 c_4^2 / \Delta$ ; in the Picard SS, the element corresponding to  $h_1^3 c_4 c_6 / \Delta$  has  $t = 3$ , thus we *cannot* import this differential. We deal with this class later, ie in the next paragraph. However, all the other classes which are on the  $s = t$  column and are  $h_1$ -power multiples of nontorsion classes, ie members of the family which we have not drawn in Figure 7, are well within the  $t > 3$  range, so that we can indeed conclude by Comparison Tool 5.2.4 that they either support a differential or are killed by one. For example, the  $h_1$ -multiple of the differential just discussed does happen, ie in the Picard SS we have  $d_3(h_1^4 c_4 c_6 / \Delta) = h_1^7 c_4^2 / \Delta$ . In particular, we need not worry about these omitted classes any more.

We turn to the question of whether any differentials are supported on the  $(s, t-s) = (3, 0)$  position in the HFPSS for  $\text{pic}(\text{TMF}(3))^{h_{\text{GL}_2(\mathbb{Z}/3)}}$ . For this purpose we use the universal formula (6-1) of Theorem 6.1.1, just as we did in the second proof of Proposition 7.2.7. We have that  $E_2^{3,3}$  of the Picard spectrum HFPSS is  $\mathbb{Z}/2[j]$  generated by  $h_1^3 c_4 c_6 / \Delta$ ; the corresponding element in the original HFPSS has

$$d_3^o \left( h_1^3 \frac{c_4 c_6}{\Delta} \right) = h_1^6 \frac{c_4^2}{\Delta}.$$

Now we have that

$$\left(h_1^3 \frac{c_4 c_6}{\Delta}\right)^2 = h_1^6 \frac{c_4^2 c_6^2}{\Delta^2} = (j - 12^3) h_1^6 \frac{c_4^2}{\Delta} = j h_1^6 \frac{c_4^2}{\Delta},$$

using the fact that  $12^3 \Delta = c_4^3 - c_6^2$  and that by definition,  $j = c_4^3/\Delta$ . Thus we conclude by (6-1) that in the Picard HFPSS, the differential  $d_3: E_3^{3,3} \rightarrow E_3^{6,5}$  is given by

$$d_3\left(f(j)h_1^3 \frac{c_4 c_6}{\Delta}\right) = (f(j) + jf(j)^2)h_1^6 \frac{c_4^2}{\Delta},$$

where  $(f(j)h_1^3 c_4 c_6/\Delta)$  is an arbitrary element of  $E_3^{3,3}$ . However,  $(f(j) + jf(j)^2)$  in  $\mathbb{Z}/2[j]$  is zero only if  $f(j)$  is zero, hence this  $d_3$  is injective and has trivial kernel. (Note this is an interesting difference between the present situation and the one in Proposition 7.2.7.) Consequently,  $E_4^{3,3}$  is zero.

Further use of Comparison Tool 5.2.4 determines that all the differentials we have drawn in blue in Figures 7–9 are nonzero. Note that of the classes in the  $s = t$  column, ie the one which contributes to the Picard group of  $\text{TMF}[\frac{1}{3}]$ , everything with  $s \geq 8$  is killed. However,  $h_2 g/\Delta$ , generating a  $\mathbb{Z}/4$  in  $s = 5$ , and  $h_1^3 g/\Delta$  generating a  $\mathbb{Z}/2$  in  $s = 7$ , remain. In the original spectral sequence, the first one of these supported a  $d_5^o$  and a  $d_{13}^o$ , and the second supported a  $d_{23}^o$ .

Next we need to determine the rest of the spectral sequence, ie the part which involves  $\pi_0$  and  $\pi_1$  of the Picard spectrum of  $\text{TMF}(3)$ . Detailed computations for this are deferred until Appendix B. The piece in which we are most interested is  $H^1(\text{GL}_2(\mathbb{Z}/3), \pi_1 \text{pic}(\text{TMF}(3)))$ , which is a cyclic group of order 12 according to Proposition B.1; we have also determined  $H^*(\text{GL}_2(\mathbb{Z}/3), \pi_0 \text{pic}(\text{TMF}(3)))$  in Proposition B.2 using a more general result of Quillen.

Now we are ready to make conclusions about the Picard group of  $\text{TMF}[\frac{1}{3}]$ : in the  $t = s$  vertical line of the HFPSS, ie the one that abuts to  $\pi_0 \text{pic}(\text{TMF}[\frac{1}{3}]) = \text{Pic}(\text{TMF}[\frac{1}{3}])$ , nothing above the  $s = 7$  line survives the spectral sequence. The following might survive:

- at most a group of order 2 in position  $(0, 0)$ ,
- at most a group of order 12 in  $(1, 0)$ ,
- at most a group of order 4 in  $(5, 0)$ , and
- at most a group of order 2 in  $(7, 0)$ .

The upshot is that we get an upper bound of  $2 \times 12 \times 4 \times 2 = 192$  on the order of the Picard group. But  $\text{TMF}[\frac{1}{3}]$  is 192–periodic, so this upper bound must also be a lower bound. In conclusion,  $\text{Pic}(\text{TMF}[\frac{1}{3}]) = \mathbb{Z}/192$ , as claimed, generated by  $\Sigma \text{TMF}[\frac{1}{3}]$ . □

**Remark 8.2.4** As in Remark 8.1.4, we can use some of our computations to reprove Fulton and Olsson’s [14] result that the moduli stack of elliptic curves  $M_{\text{ell}}[\frac{1}{3}]$  also has a Picard group  $\mathbb{Z}/12$ . Namely, we start with the knowledge that  $\text{Pic}(M_{\text{ell}}(3))$  is trivial, as  $M_{\text{ell}}(3)$  is the prime spectrum of a UFD. Then, we consider the Picard HFPSS for the algebraic stack  $M_{\text{ell}}[\frac{1}{3}]$ , which must collapse. The only contribution towards the Picard group is

$$H^1(\text{GL}_2(\mathbb{Z}/3), \Gamma(M_{\text{ell}}(3), \mathcal{O}^\times)),$$

which we saw by Proposition B.1 has order 12. But  $\omega$  has order 12, hence  $\text{Pic}(M_{\text{ell}}[\frac{1}{3}])$  is cyclic of order 12.

### 8.3 Calculation of $\text{Pic}(\text{TMF})$

In this section we will compute the Picard group of the integral periodic version of topological modular forms TMF. The result, as stated in the introduction, is:

**Theorem A** *The Picard group of integral TMF is  $\mathbb{Z}/576$ , generated by  $\Sigma \text{TMF}$ .*

**Proof** There is no nontrivial Galois extension of the integral TMF by [40, Theorem 10.1], but we can use étale descent, as TMF is obtained as the global sections of the sheaf  $\mathcal{O}^{\text{top}}$  of even-periodic  $E_\infty$ -rings on the moduli stack of elliptic curves. Namely, we can use Theorem 3.2.1 because the map  $M_{\text{ell}} \rightarrow M_{\text{FG}}$  is known to be affine. The spectral sequence is

$$H^s(M_{\text{ell}}, \pi_t \text{pic } \mathcal{O}^{\text{top}}) \Rightarrow \pi_{t-s} \Gamma(\text{pic } \mathcal{O}^{\text{top}}),$$

and we are interested in  $\pi_0$ . Using Theorem 3.2.1, the  $E_2$ -page of this spectral sequence is given by (for  $t - s \geq 0$ )

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/2 & \text{if } t = s = 0, \\ H^s(M_{\text{ell}}, \mathcal{O}_{M_{\text{ell}}}^\times) & \text{if } t = 1, \\ H^s(M_{\text{ell}}, \omega^{(t-1)/2}) & \text{if } t \geq 3 \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Over a field  $k$  of characteristic  $\neq 2, 3$ , Mumford [51] showed that

$$H^1((M_{\text{ell}})_k, \mathcal{O}_{M_{\text{ell}}}^\times) \simeq \mathbb{Z}/12,$$

ie the Picard group of the moduli stack is  $\mathbb{Z}/12$ , generated by the line bundle  $\omega$  that assigns to an elliptic curve the dual of its Lie algebra. This result is also true over  $\mathbb{Z}$  by the work of Fulton and Olsson [14]. However, using descent we can reprove that result. Namely, in Remarks 8.1.4 and 8.2.4 we saw that the Picard groups of both  $M_{\text{ell}}[\frac{1}{2}]$

and  $M_{\text{ell}}[\frac{1}{3}]$  are  $\mathbb{Z}/12$ , both generated by  $\omega$ . Cover the integral stack  $M_{\text{ell}}$  by these two; their intersection is  $M_{\text{ell}}[\frac{1}{6}]$ , which is the weighted projective stack  $\text{Proj } \mathbb{Z}[\frac{1}{6}][c_4, c_6]$  (with  $c_4$  and  $c_6$  in degrees<sup>17</sup> 4 and 6 respectively), and which therefore has Picard group  $\mathbb{Z}/12$  also generated by  $\omega$ . The descent spectral sequence for  $\text{pic}$  associated to this cover gives the result.

Since  $M_{\text{ell}}[\frac{1}{6}]$  has no higher cohomology, the groups  $H^s(M_{\text{ell}}, \omega^{(t-1)/2})$ , when  $s > 0$ , are given as the direct sum of the corresponding cohomology groups of  $M_{\text{ell}}[\frac{1}{2}]$  and  $M_{\text{ell}}[\frac{1}{3}]$ . These groups, in turn, are isomorphic to

$$H^s(\text{GL}_2(\mathbb{Z}/p), \pi_{t-1} \text{TMF}(p)) = H^s(\text{GL}_2(\mathbb{Z}/p), H^0(M_{\text{ell}}(p), \omega^{(t-1)/2})),$$

where  $p$  is 2 or 3, as the map  $M_{\text{ell}}(p) \rightarrow M_{\text{ell}}[\frac{1}{p}]$  is Galois, and  $M_{\text{ell}}(p)$  has no higher cohomology. We computed these groups in the previous examples.

The machinery of Section 5 now allows us to compare this Picard descent spectral sequence to the one which computes the homotopy groups of  $\text{TMF}$ . From Corollary 5.2.3 and an analogue of Comparison Tool 5.2.4, we conclude that the differentials involving 3-torsion classes wipe out everything above the  $s = 5$  line, and those involving 2-torsion classes wipe out everything above the  $s = 7$  line. These differentials are identical to what happens in the homotopy fixed point spectral sequences in the previous two examples. We conclude that the following are the only groups that can survive:

- at most a group of order 2 in  $(t - s, s) = (0, 0)$ ,
- at most a group of order 12 in  $(0, 1)$ ,
- at most a group of order 12 in  $(0, 5)$ , and
- at most a group of order 2 in  $(0, 7)$ .

This gives us an upper bound  $2^6 3^2 = 576$  on the cardinality of  $\pi_0$ , which is exactly the periodicity of  $\text{TMF}$ . The spectral sequence is depicted in Figure 10. □

### 8.4 Calculation of $\text{Pic}(\text{Tmf})$

We will now prove the following result stated in the introduction.

**Theorem B** *The Picard group of  $\text{Tmf}$  is  $\mathbb{Z} \oplus \mathbb{Z}/24$ , generated by  $\Sigma \text{Tmf}$  and a certain 24-torsion invertible module.*

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<sup>17</sup>These are the algebraic degrees, which get doubled in topology.

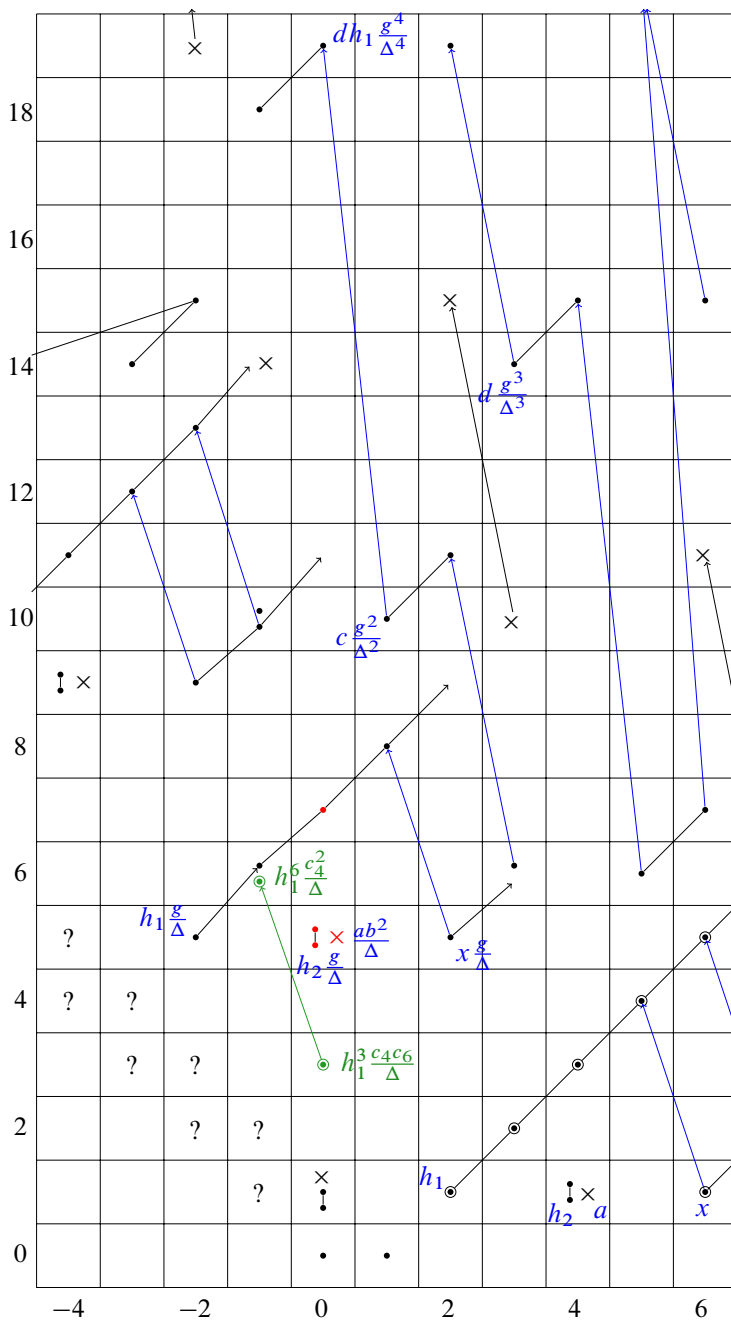


Figure 10: Descent spectral sequence for  $\Gamma(\text{pic } \mathcal{O}^{\text{top}})$  on  $\mathfrak{M}_{\text{ell}}$  with some  $h_1$ -omissions as in Figure 7 ( $\square$  denotes  $\mathbb{Z}$ ,  $\bullet$  denotes  $\mathbb{Z}/2$ ,  $\odot$  denotes  $\mathbb{Z}/2[j]$ , and  $\times$  denotes  $\mathbb{Z}/3$ )

While  $\mathrm{Tmf}[\frac{1}{n}]$  can be described as the homotopy fixed point spectrum  $\mathrm{Tmf}(n)^{h\mathrm{GL}_2(\mathbb{Z}/n)}$  for  $n = 2, 3$  just as in the periodic case, the extension  $\mathrm{Tmf}[\frac{1}{n}] \rightarrow \mathrm{Tmf}(n)$  is *not* Galois, and therefore we cannot use Galois descent to compute the Picard group. However, we can use Theorem 3.2.1 for the compactified moduli stack  $\overline{M}_{\mathrm{ell}}$ .

First, we need a lemma.

**Lemma 8.4.1** *Let  $\mathcal{L}$  be the line bundle on  $\overline{M}_{\mathrm{ell}}$  obtained by gluing the trivial line bundles on  $M_{\mathrm{ell}} = \overline{M}_{\mathrm{ell}}[\Delta^{-1}]$  and  $\overline{M}_{\mathrm{ell}}[c_4^{-1}]$  via the clutching function  $j$ . Then  $\mathcal{L} \simeq \omega^{-12}$ .*

**Proof** To give a section of  $\mathcal{L} \otimes \omega^{12}$  over  $\overline{M}_{\mathrm{ell}}$  is equivalent to giving sections  $s_1 \in \Gamma(M_{\mathrm{ell}}, \omega^{12})$  and  $s_2 \in \Gamma(\overline{M}_{\mathrm{ell}}[c_4^{-1}], \omega^{12})$  such that

$$(js_1)|_{M_{\mathrm{ell}}[c_4^{-1}]} = (s_2)|_{M_{\mathrm{ell}}[c_4^{-1}]}$$

We take  $s_1 = \Delta$  and  $s_2 = c_4^3$ , and we get a nowhere vanishing section of  $\mathcal{L} \otimes \omega^{12}$ .  $\square$

**Proof of Theorem B** The relevant part of the Picard descent spectral sequence is similar to that of  $\mathrm{TMF}$ , with the following exceptions: the algebraic part  $H^1(\overline{M}_{\mathrm{ell}}, \mathcal{O}^\times)$  is now  $\mathbb{Z}$  generated by  $\omega$ , according to Fulton and Olsson [14], and all the torsion groups are now finite, ie there are no  $\mathbb{Z}/2[j]$ 's appearing. In particular,  $E_2^{3,3}$  is zero, and we have

- at most a group of order 2 in  $(t - s, s) = (0, 0)$ ,
- a subquotient of  $\mathbb{Z}$  in  $(0, 1)$ ,
- at most a group of order 12 in  $(0, 5)$ , and
- at most a group of order 2 in  $(0, 7)$

as potential contributions to the  $s = t$  line of the  $E_\infty$ -page. The depiction is in Figure 11.

Note that the  $\mathbb{Z}/2$  in  $(0, 0)$ , which corresponds to a single suspension of the even-periodic spectra that  $\mathrm{Tmf}$  is built from, is represented by  $\Sigma \mathrm{Tmf}$  in the Picard group of  $\mathrm{Tmf}$ . Similarly, the element  $1 \in \mathbb{Z} = E_2^{0,1} = \mathrm{Pic}(\overline{M}_{\mathrm{ell}})$  corresponds to the line bundle  $\omega$ , which topologically is represented by  $\Sigma^2 \mathrm{Tmf}$ . Thus these groups survive to the  $E_\infty$ -page and are related by an extension. The rest of the  $E_\infty$ -filtration now tells us that  $\mathrm{Pic}(\mathrm{Tmf})$  sits in an extension

$$0 \rightarrow A \rightarrow \mathrm{Pic}(\mathrm{Tmf}) \rightarrow \mathbb{Z} \rightarrow 0,$$

where  $A$  is a finite group of order at most 24.



We claim that  $A = \mathbb{Z}/24$  and therefore  $\text{Pic}(\text{Tmf}) = \mathbb{Z} \oplus \mathbb{Z}/24$ . To see this, we will construct a line bundle  $\mathcal{I}$  such that  $\mathcal{I}^{\otimes 24} \simeq \mathcal{O}^{\text{top}}$ , but no lower power of  $\mathcal{I}$  is equivalent to  $\mathcal{O}^{\text{top}}$ .

In order to proceed with the construction, we make the preliminary observation that the modular function  $j = c_4^3/\Delta$  is a homotopy class in  $\pi_0 \text{TMF}[c_4^{-1}]$ , ie it survives the descent spectral sequence

$$H^*(\overline{M}_{\text{ell}}[\Delta^{-1}, c_4^{-1}], \omega^*) \cong H^*(M_{\text{ell}}, \omega^*)[c_4^{-1}] \Rightarrow \pi_* \text{TMF}[c_4^{-1}].$$

In fact, it is an invertible element of  $\pi_0 \text{TMF}[c_4^{-1}]$ . We reason as follows. The torsion in the  $E_2$ -page consists only of  $h_1$ -towers supported on the nontorsion classes, since all other torsion classes in  $H^*(M_{\text{ell}}, \omega^*)$  are annihilated by  $c_4$ . Therefore, when  $c_4$  is inverted only  $d_3$ -differentials can be nonzero, and they wipe out everything above the line  $s = 3$ . As  $\Delta$  and  $c_4$  do not support any of those differentials,  $j$  is a permanent cycle, as is  $j^{-1}$ .

**Construction 8.4.2** Consider the cover of  $\overline{M}_{\text{ell}}$  by  $\overline{M}_{\text{ell}}[\Delta^{-1}] = M_{\text{ell}}$  and  $\overline{M}_{\text{ell}}[c_4^{-1}]$  which fit in the pushout diagram:

$$\begin{array}{ccc} \overline{M}_{\text{ell}}[\Delta^{-1}, c_4^{-1}] & \longrightarrow & \overline{M}_{\text{ell}}[\Delta^{-1}] \\ \downarrow & & \downarrow \\ \overline{M}_{\text{ell}}[c_4^{-1}] & \longrightarrow & \overline{M}_{\text{ell}} \end{array}$$

Let  $\mathcal{J}$  be the line bundle on the *derived* moduli stack  $\overline{\mathfrak{M}}_{\text{ell}} = (\overline{M}_{\text{ell}}, \mathcal{O}^{\text{top}})$  obtained by gluing  $\mathcal{O}^{\text{top}}$  on  $\overline{M}_{\text{ell}}[\Delta^{-1}]$  and  $\mathcal{O}^{\text{top}}$  on  $\overline{M}_{\text{ell}}[c_4^{-1}]$  using the clutching function  $j = c_4^3/\Delta$  on  $\overline{M}_{\text{ell}}[\Delta^{-1}, c_4^{-1}]$ .

We claim that  $\mathcal{J}$  is not a suspension of  $\mathcal{O}^{\text{top}}$ , and that  $\mathcal{I} = \Sigma^{24} \mathcal{J}$  is an element of the Picard group of order 24.

To see the first assertion, note that by Lemma 8.4.1,  $\pi_0 \mathcal{J}$  is  $\omega^{-12}$ , so if  $\mathcal{J}$  is a suspension of  $\mathcal{O}^{\text{top}}$ , it ought to be  $\Sigma^{-24} \mathcal{O}^{\text{top}}$ . However,  $\Sigma^{-24} \mathcal{O}^{\text{top}}$  restricted to  $\overline{M}_{\text{ell}}[\Delta^{-1}]$  is  $\Sigma^{-24} \mathcal{O}^{\text{top}}|_{\overline{M}_{\text{ell}}[\Delta^{-1}]}$ , whereas  $\mathcal{J}$  restricts to  $\mathcal{O}^{\text{top}}|_{\overline{M}_{\text{ell}}[\Delta^{-1}]}$ .

This argument can be repeated with any power  $\mathcal{J}^{\otimes m}$  such that  $m$  is not divisible by 24. In this case,  $\pi_0 \mathcal{J}^{\otimes m}$  is  $\omega^{-12m}$ , so if  $\mathcal{J}^{\otimes m}$  were a suspension of  $\mathcal{O}^{\text{top}}$ , it would be the  $(-24)m^{\text{th}}$  suspension. At the same time,  $\mathcal{J}^{\otimes m}$  restricts to

$$(\mathcal{O}^{\text{top}})^{\otimes m}|_{\overline{M}_{\text{ell}}[\Delta^{-1}]} = \mathcal{O}^{\text{top}}|_{\overline{M}_{\text{ell}}[\Delta^{-1}]}$$

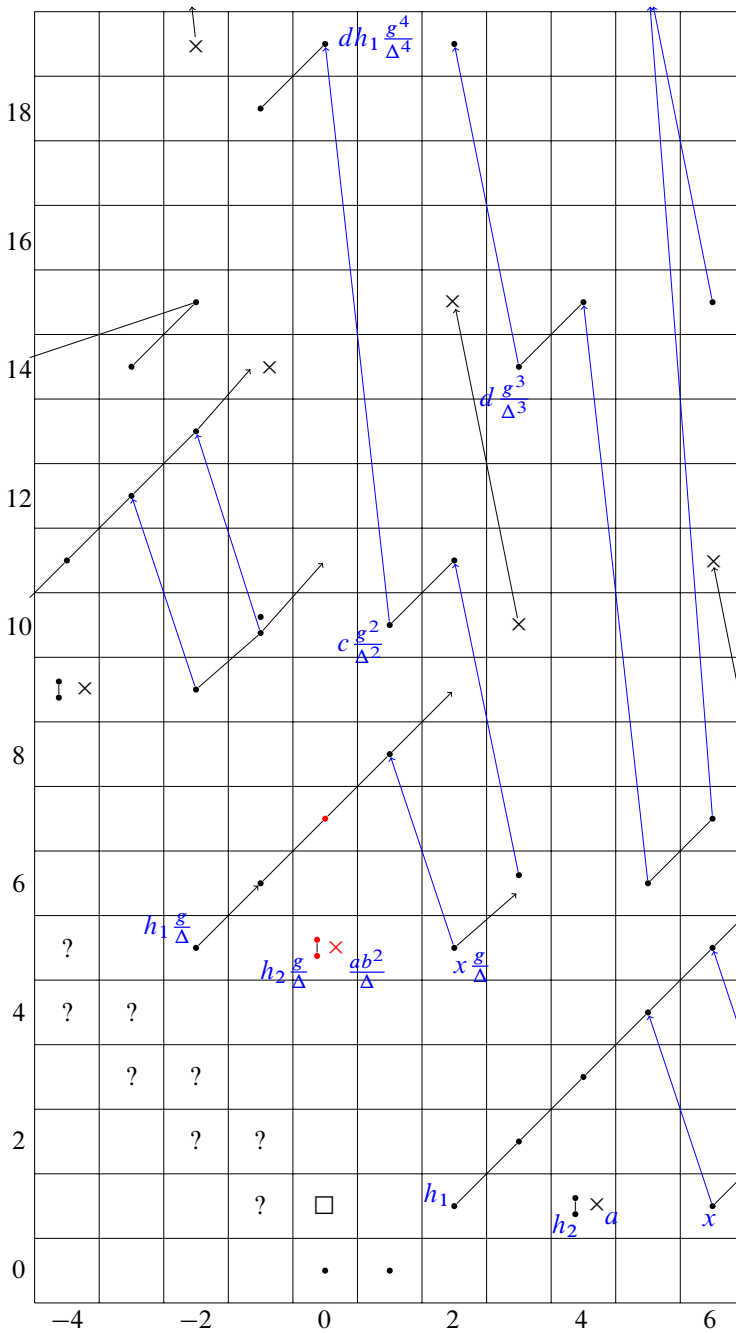


Figure 11: Descent spectral sequence for  $\Gamma(\text{pic } \mathcal{O}^{\text{top}})$  on  $\overline{\mathfrak{M}}_{\text{ell}}$  ( $\square$  denotes  $\mathbb{Z}$ ,  $\bullet$  denotes  $\mathbb{Z}/2$ , and  $\times$  denotes  $\mathbb{Z}/3$ )

upon inverting  $\Delta$ . If  $\mathcal{J}^{\otimes m}$  were a suspension, therefore, one would have that

$$\Sigma^{-24m} \mathcal{O}^{\text{top}} |_{\overline{M}_{\text{ell}}[\Delta^{-1}]} \simeq \mathcal{O}^{\text{top}} |_{\overline{M}_{\text{ell}}[\Delta^{-1}]}.$$

By Theorem A, this holds if and only if  $m$  is divisible by 24.

This shows that the order of  $\mathcal{J}$  in  $\text{Pic}(\mathcal{O}^{\text{top}})/\mathbb{Z}$ , where the  $\mathbb{Z}$  is generated by  $\Sigma \mathcal{O}^{\text{top}}$ , is at least 24. The spectral sequence argument above, however, showed that this quotient has order at most 24.  $\square$

The same analysis shows that  $\text{Pic}(\text{Tmf}_{(2)}) = \mathbb{Z} \oplus \mathbb{Z}/8$  and  $\text{Pic}(\text{Tmf}_{(3)}) = \mathbb{Z} \oplus \mathbb{Z}/3$ , the torsion being generated by the respective localizations of  $\mathcal{I}$ . Moreover, when  $p$  is greater than 3,  $\text{Pic}(\text{Tmf}_{(p)}) = \mathbb{Z}$ .

### 8.5 Relation to the $E_2$ -local Picard group

Notice that  $\mathcal{I}$  is the only “exotic” element in all of our examples involving the various forms of topological modular forms. Let us see how it relates to the exotic piece of the Picard group of the category of  $E_2$ -local spectra, ie modules over the  $E_2$ -local sphere spectrum. The exotic phenomena only occur at  $p = 2$  and  $p = 3$ , but since only the 3-primary  $E_2$ -local Picard group is known, let us concentrate on that case for the remainder of this section.

In [17], the authors compute  $\kappa_2$ , the exotic part of the Picard group of the category of 3-primary  $K(2)$ -local spectra; they show  $\kappa_2 = \mathbb{Z}/3 \times \mathbb{Z}/3$ .

Additionally, they look at the localization map from the  $E_2$ -local category to the  $K(2)$ -local category and show that it induces an isomorphism  $\kappa_{\mathcal{L}_2} \rightarrow \kappa_2$ , where  $\kappa_{\mathcal{L}_2}$  denotes the exotic  $E_2$ -local Picard group.

Consider now the commutative diagram

$$\begin{array}{ccc} \kappa_{\mathcal{L}_2} & \xrightarrow{\quad\quad\quad} & \kappa_2 \\ \downarrow t & & \downarrow t_{K(2)} \\ \text{Pic}(\text{Tmf}_{(3)}) & \xrightarrow{\quad\quad\quad} & \text{Pic}(\text{Tmf}_{K(2)}) \end{array}$$

in which the horizontal maps are given by  $K(2)$ -localization, and the vertical maps are given by smashing with  $\text{Tmf}$  and  $\text{Tmf}_{K(2)}$ , respectively. In [17, Theorem 5.5], the authors show there is an element  $P$  of  $\kappa_2$  such that  $L_{K(2)}(P \wedge \text{Tmf}_{K(2)}) \simeq \Sigma^{48} \text{Tmf}_{K(2)}$ , ie  $t_{K(2)} P = 48 \in \mathbb{Z}/72 \subseteq \text{Pic}(\text{Tmf}_{K(2)})$ . Under the top horizontal isomorphism, this  $P$  lifts to an element  $\tilde{P}$  of  $\kappa_{\mathcal{L}_2}$ , such that  $t(\tilde{P})$  has order three in  $\text{Pic}(\text{Tmf}_{(3)})$  and such that the  $K(2)$ -localization of  $t(\tilde{P})$  is  $L_{K(2)}(\Sigma^{48} \text{Tmf})$ . Thus  $t(\tilde{P})$  must be twice

the class of  $\mathcal{I}$ . In other words, the exotic element  $\tilde{P}$  of  $\kappa_{\mathcal{L}_2}$  is detected as an exotic element of  $\text{Pic}(\text{Tmf}_{(3)})$ .

The other  $\mathbb{Z}/3$  in  $\kappa_2$ , ie  $\kappa_2$  modulo the subgroup generated by  $P$ , is generated by a spectrum  $Q$  such that  $t_{K(2)}Q = 0$ . This  $Q$  lifts to  $\tilde{Q} \in \kappa_{\mathcal{L}_2}$ , still of order 3, which must map under  $t$  to an element of order 3 in  $\text{Pic}(\text{Tmf}_{(3)})$  which is in the kernel of the bottom localization map. But there are no nontrivial elements of finite order in this kernel, hence  $\tilde{Q}$  is not detected in  $\text{Pic}(\text{Tmf}_{(3)})$ .

Perhaps at the prime 2 as well there is an element of the exotic  $E_2$ -local Picard group which is detected in the torsion of  $\text{Pic}(\text{Tmf}_{(2)})$ .

### Appendices

#### Appendix A: Group cohomology computations for TMF(2)

In this appendix, we will compute the group cohomology for the  $\text{GL}_2(\mathbb{Z}/2)$ -action on  $\pi_0 \text{pic}(\text{TMF}(2)) = \mathbb{Z}/4$  (with trivial action), and on  $\pi_1 \text{pic}(\text{TMF}(2)) = \text{TMF}(2)_0^\times$  with the natural action. The group  $\text{GL}_2(\mathbb{Z}/2)$  is the symmetric group on three letters, so it has a (unique) normal subgroup of order 3, which we denote by  $C_3$ , with quotient  $C_2$ . We can therefore use the associated Lyndon–Hochschild–Serre spectral sequence (LHSSS)

$$(A-1) \quad H^p(C_2, H^q(C_3, M)) \Rightarrow H^{p+q}(\text{GL}_2(\mathbb{Z}/2), M)$$

for  $\text{GL}_2(\mathbb{Z}/2)$ -modules  $M$ .

Let us first deal with the easier case.

**Lemma A.1** *The group cohomology for the  $\text{GL}_2(\mathbb{Z}/2)$ -action on the trivial module  $\mathbb{Z}/4$  is*

$$H^*(\text{GL}_2(\mathbb{Z}/2), \pi_0 \text{pic}(\text{TMF}(2))) = \begin{cases} \mathbb{Z}/4 & \text{if } * = 0, \\ \mathbb{Z}/2 & \text{if } * > 0. \end{cases}$$

**Proof** Since 3 is invertible in  $\mathbb{Z}/4$ , we have that  $H^*(C_3, \mathbb{Z}/4) = \mathbb{Z}/4$  concentrated in degree zero, and with trivial action by  $C_2 = \text{GL}_2(\mathbb{Z}/2)/C_3$ . Hence the LHSSS (A-1) collapses, giving

$$H^s(\text{GL}_2(\mathbb{Z}/2), \mathbb{Z}/4) = H^s(C_2, \mathbb{Z}/4),$$

which is  $\mathbb{Z}/4$  for  $s = 0$  and  $\mathbb{Z}/2$  otherwise. □

Next we compute the group cohomology for the  $GL_2(\mathbb{Z}/2)$ -action on  $\pi_1 \text{pic}(\text{TMF}(2))$ , which is the multiplicative group of units in  $\pi_0 \text{TMF}(2)$ . For brevity, we call this module  $M$ , and to begin with, we explicitly describe the action of  $GL_2(\mathbb{Z}/2)$  on  $M$ .

Let  $\sigma$  and  $\tau$  be the generators of  $GL_2(\mathbb{Z}/2)$  of order 3 and 2 respectively as chosen in [63, Lemma 7.3]; of course,  $\sigma$  generates the normal subgroup  $C_3$ . It follows from (8-1) that  $M$  is isomorphic to  $\mathbb{Z}/2 \oplus \mathbb{Z}^{\oplus 3}$ , where  $\mathbb{Z}/2$  is multiplicatively generated by  $-1$ , and the  $\mathbb{Z}$ 's are multiplicatively generated by  $2, s$  and  $(s-1)$ . The action is determined by [63, Lemma 7.3], where it is shown that the chosen generators  $\sigma$  and  $\tau$  act as

$$\sigma: s \mapsto \frac{s-1}{s} \quad \text{and} \quad \tau: s \mapsto \frac{1}{s}.$$

Written additively, so that  $m = (\epsilon, k, a, b) \in M$  represents  $(-1)^\epsilon 2^k s^a (s-1)^b \in \text{TMF}(2)_0^\times$ , the action is given by

$$\begin{aligned} \sigma: m &\mapsto (\epsilon + b, k, -a - b, a), \\ \tau: m &\mapsto (\epsilon + b, k, -a - b, b). \end{aligned}$$

We use this information to compute  $H^*(C_3, M)$  as a  $C_2$ -module. We get that

$$H^s(C_3, M) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z} & \text{if } s = 0, \\ (\mathbb{Z}/3) & \text{if } s \equiv 0, 1(4) \text{ and } s > 0, \\ (\mathbb{Z}/3)_{\text{sgn}} & \text{if } s \equiv 2, 3(4) \text{ and } s > 0. \end{cases}$$

This gives the  $E_2$ -page of the LHSSS (A-1), which must collapse and give that

$$(A-2) \quad H^s(GL_2(\mathbb{Z}/2), M) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z} & \text{if } s = 0, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \text{if } s \equiv 1(4), \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } s \equiv 2(4), \\ \mathbb{Z}/2 & \text{if } s \equiv 3(4), \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \text{if } s \equiv 0(4) \text{ and } s > 0. \end{cases}$$

We have thus proven the following result.

**Proposition A.2** *The group cohomology for the  $GL_2(\mathbb{Z}/2)$ -action on  $\pi_0 \text{pic}(\text{TMF}(2))$  is as in (A-2). In particular, we have that  $H^1(GL_2(\mathbb{Z}/2), \text{TMF}(2)_0^\times) = \mathbb{Z}/6$ .*

### Appendix B: Group cohomology computations for $\text{TMF}(3)$

This appendix is devoted to computing the group cohomology for  $GL_2(\mathbb{Z}/3)$  acting on  $\pi_1 \text{pic}(\text{TMF}(3)_0)^\times$ ; we also determine the cohomology of  $\pi_0 \text{pic}(\text{TMF}(3)) = \mathbb{Z}/2$  as a simple consequence of a result of Quillen [55]. The group  $GL_2(\mathbb{Z}/3)$  has order 48

and has the binary tetrahedral group as a normal subgroup, in the guise of  $SL_2(\mathbb{Z}/3)$ . We have found it difficult to compute the higher cohomology groups of  $(TMF(3)_0)^\times$ , but since we are only using  $H^1(GL_2(\mathbb{Z}/3), (TMF(3)_0)^\times)$  in Section 8.2, we will concentrate on computing this group only.

In this section, we denote  $(TMF(3)_0)^\times$  by  $M$ . From (8-1), we see that  $M \subset TMF(3)_0$  is isomorphic to  $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}^{\oplus 4}$  multiplicatively generated by  $-1, \zeta, (1 - \zeta), t, (1 - \zeta t)$  and  $(1 + \zeta^2 t)$ . (To see the appearance of  $(1 - \zeta)$ , note that  $(1 - \zeta)^2 = -3\zeta$ .) The  $GL_2(\mathbb{Z}/3)$ -module structure is determined in [64, Section 4.3]; to describe it, let  $x, y, z$  be the elements of  $GL_2(\mathbb{Z}/3)$  chosen in loc. cit. Explicitly,

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then  $x$  and  $y$  generate a quaternion group  $Q_8$ , and  $x, y, z$  generate  $SL_2(\mathbb{Z}/3)$ . Let  $\sigma$  be the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . These generate the whole group, and their action on the element  $t = \gamma_1/\gamma_2$  is as determined in loc. cit.<sup>18</sup> to be

$$x(t) = -\frac{1}{t}, \quad y(t) = \zeta^2 \frac{1 - \zeta t}{1 + \zeta^2 t}, \quad z(t) = \zeta \frac{t}{1 + \zeta^2 t}, \quad \sigma(t) = \frac{1}{t}.$$

The rest is determined by the fact that everything fixes  $\mathbb{Z}[\frac{1}{3}] \subset TMF(3)_0$ , a matrix  $A$  in  $GL_2(\mathbb{Z}/3)$  takes  $\zeta$  to  $\zeta^{\det A}$ , and the action respects the ring structure.

To be brutally explicit, let  $m = (\epsilon, \alpha, \beta, a, b, c) \in M$  denote the element

$$(-1)^\epsilon \zeta^\alpha (1 - \zeta)^\beta t^a (1 - \zeta t)^b (1 + \zeta^2 t)^c.$$

Then the generators  $x, y, z, \sigma \in GL_2(\mathbb{Z}/3)$  act as

$$\begin{aligned} x: m &\mapsto (\epsilon + a + c, \alpha + b - c, \beta, -a - b - c, c, b), \\ y: m &\mapsto (\epsilon + b + c, \alpha - a - c, \beta, b, a, -a - b - c), \\ z: m &\mapsto (\epsilon, \alpha + a, \beta, a, c, -a - b - c), \\ \sigma: m &\mapsto (\epsilon + \beta + b, -\alpha - \beta - b + c, \beta, -a - b - c, b, c). \end{aligned} \tag{B-1}$$

Since we know a set of generators and relations for  $GL_2(\mathbb{Z}/3)$ , and the action is given explicitly, we can compute  $H^1$  directly as crossed homomorphisms modulo coboundaries. We found it a little bit simpler, however, to do this for  $SL_2(\mathbb{Z}/3)$ , and then use the Lyndon–Hochschild–Serre spectral sequence for the extension

$$1 \rightarrow SL_2(\mathbb{Z}/3) \rightarrow GL_2(\mathbb{Z}/3) \rightarrow C_2 \rightarrow 1,$$

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<sup>18</sup>Actually, the formulas in loc. cit. determine a *right* action, although the *left* action that we include here is almost the same.

in which  $C_2$  is generated by the image of  $\sigma \in \text{GL}_2(\mathbb{Z}/3)$ . The contributions to  $H^1(\text{GL}_2(\mathbb{Z}/3), M)$  are from  $H^1(\text{SL}_2(\mathbb{Z}/3), M)^{C_2}$  and  $H^1(C_2, M^{\text{SL}_2(\mathbb{Z}/3)})$ , and there is a potential differential

$$(B-2) \quad d_2: H^1(\text{SL}_2(\mathbb{Z}/3), M)^{C_2} \rightarrow H^2(C_2, M^{\text{SL}_2(\mathbb{Z}/3)}).$$

To compute these groups and the differential, we note that the invariants  $M^{\text{SL}_2(\mathbb{Z}/3)}$  are the submodule  $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}$  with  $a = b = c = 0$ . Here,  $\ker(1 + \sigma) = \text{im}(1 - \sigma)$ , so that  $H^1(C_2, M^{\text{SL}_2(\mathbb{Z}/3)}) = 0$ .

Next, suppose  $f: \text{SL}_2(\mathbb{Z}/3) \rightarrow M$  represents a class in  $H^1(\text{SL}_2(\mathbb{Z}/3), M)^{C_2}$ , ie it is a crossed homomorphism which is  $\sigma$ -invariant modulo coboundaries. Since each  $f(g)$  is in the kernel of the norm of  $g$ , we must have that

$$\begin{aligned} f(x) &= (\epsilon_x, c_x, 0, a_x, -c_x, c_x), \\ f(y) &= (\epsilon_y, -a_y - c_y, 0, a_y, -a_y, c_y), \\ f(z) &= (0, \alpha_z, 0, 0, b_z, c_z). \end{aligned}$$

The relations  $x^2 = y^2$ ,  $xyx = y$ ,  $xz = zy^3$  and  $zyx = yz$ , imply that

$$a_x + c_x = a_y + c_y, \quad b_z = -c_x, \quad c_z = c_y, \quad \epsilon_x = c_x + c_y, \quad \epsilon_y = a_x.$$

One directly checks that any crossed homomorphism of this form is  $\sigma$ -invariant modulo coboundaries. Finally, suppose an  $f$  of this form is itself a coboundary, ie there is an  $m = (\epsilon, \alpha, \beta, a, b, c) \in M$ , such that  $f(g) = gm - m$  for all  $g \in \text{SL}_2(\mathbb{Z}/3)$ . Then  $4b = a_x + 3c_x - 2c_y$ ,  $a = b - a_x - c_x + 2c_y$ ,  $c = b - c_x$  and  $\alpha_z = a$ . Consequently,

$$(B-3) \quad H^1(\text{SL}_2(\mathbb{Z}/3), M)^{C_2} = \mathbb{Z}/12.$$

It remains to compute the differential (B-2). This is a transgression, and we have an explicit formula for it, for example in [31, Section 3.7] or [53, Section I.6]. One checks that this formula gives that  $d_2$  is zero in our case. Thus we have proved the following.

**Proposition B.1**  $H^1(\text{GL}_2(\mathbb{Z}/3), \text{TMF}(3)_0^\times)$  is cyclic of order 12.

Although not directly affecting the computation of  $\text{Pic}(\text{TMF}[\frac{1}{3}])$ , we record the following result of Quillen that determines a few more entries in the spectral sequence (8-6).

**Proposition B.2** [55, Lemma 11] *The cohomology ring  $H^*(\text{GL}_2(\mathbb{Z}/3), \mathbb{Z}/2)$  is  $\mathbb{Z}/2[c_1, c_2] \otimes \Lambda(e_1, e_2)$ , where the cohomological degrees are  $|c_i| = 2i$  and  $|e_i| = 2i - 1$ .*

### Appendix C: Derived functors of the symmetric square

The purpose of this appendix is to prove the necessary auxiliary results on symmetric squares of cosimplicial abelian groups.

**Definition C.1** Let  $A$  be an abelian group. We let  $\text{Sym}_2(A) = (A \otimes A)_{C_2}$  be the coinvariants for the  $C_2$ -action on  $A \otimes A$  given by permuting the factors. We also let  $\widetilde{\text{Sym}}_2(A)$  denote the  $C_2$ -coinvariants in  $(A \otimes A) \otimes \mathbb{Z}_\epsilon$  where the first factor is given the permutation action and  $\mathbb{Z}_\epsilon$  is the sign representation. Note that if  $A$  is a free abelian group, then the 2-torsion in  $\widetilde{\text{Sym}}_2(A)$  is canonically isomorphic to  $A \otimes_{\mathbb{Z}} \mathbb{F}_2$  via the “Frobenius” map

$$A/2A \rightarrow \widetilde{\text{Sym}}_2(A), \quad a \mapsto a \otimes a.$$

In [54], Priddy gives a complete description of the actions of the symmetric algebra functor on cosimplicial vector spaces, or equivalently the analog of the Steenrod algebra for cosimplicial algebras. We will only need a small piece of this, which we state next. We note that the generators in question are the Steenrod squares applied to the fundamental class  $\iota$ . For example, the generator in maximal degree is the cup square.

**Proposition C.2** [54, Theorem 4.0.1] *Let  $A^\bullet$  be a cosimplicial  $\mathbb{F}_2$ -vector space. Suppose that  $H^{t+1}(A^\bullet) \simeq \mathbb{F}_2$  and the cohomology of  $A^\bullet$  is concentrated in degree  $t + 1$  by a class  $\iota$ . Then*

$$H^i(\text{Sym}_2 A^\bullet) \simeq \begin{cases} \mathbb{F}_2 & \text{if } t + 1 \leq i \leq 2(t + 1), \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition C.3** *Let  $t \geq 2$  and let  $A^\bullet$  be a levelwise free, finitely generated cosimplicial abelian group with  $H^*(A^\bullet)$  concentrated in degree  $* = t + 1$  and  $H^{t+1}(A^\bullet) = \mathbb{Z}$  generated by  $\iota$ . Then:*

- (1) *If  $t$  is even, then  $H^{2t+2}(\text{Sym}_2 A^\bullet) \simeq \mathbb{Z}/2$ , generated by  $\iota^2$ .*
- (2) *If  $t$  is odd, then  $H^{2t+2}(\widetilde{\text{Sym}}_2 A^\bullet) \simeq \mathbb{Z}/2$ , generated by  $\iota^2$ .*

**Proof** Consider first the case  $t$  even. In this case, we have maps of cosimplicial abelian groups

$$\text{Sym}_2 A^\bullet \rightarrow A^\bullet \otimes A^\bullet \rightarrow \text{Sym}_2 A^\bullet$$

where the first map is the norm map and the second map is projection. The composite is multiplication by two. Note that  $H^{2t+2}(A^\bullet \otimes A^\bullet) \simeq \mathbb{Z}$ , but since  $t$  is even, the  $C_2$ -action is the sign representation, so that the map  $H^*(\text{Sym}_2 A^\bullet) \rightarrow H^*(A^\bullet \otimes A^\bullet)$



must be the zero map as it lands in the  $C_2$ -invariants on cohomology. In particular, the cohomology of  $\mathrm{Sym}_2(A^\bullet)$  is all annihilated by 2. By the universal coefficient theorem, it suffices to show that  $H^{2t+2}(\mathrm{Sym}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{Z}/2) \simeq \mathbb{Z}/2$  and  $H^k(\mathrm{Sym}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{Z}/2) = 0$  for  $k > 2t + 2$ , which is the statement of Proposition C.2. In addition, we see that  $\iota^2$  is a generator, as desired, by working modulo 2.

Now suppose  $t$  is odd. Again, using the norm maps

$$\widetilde{\mathrm{Sym}}_2 A^\bullet \rightarrow A^\bullet \otimes A^\bullet \otimes \epsilon \rightarrow \widetilde{\mathrm{Sym}}_2 A^\bullet,$$

we find that the cohomology of  $\widetilde{\mathrm{Sym}}_2 A^\bullet$  is annihilated by 2. We note that at the level of cosimplicial abelian groups  $\widetilde{\mathrm{Sym}}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2 \simeq \mathrm{Sym}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2$ , but working with the underived tensor product is problematic here because  $\widetilde{\mathrm{Sym}}_2 A^\bullet$  has 2-torsion. If we take the derived tensor product

$$\widetilde{\mathrm{Sym}}_2(A^\bullet) \overset{\mathbb{L}}{\otimes} \mathbb{F}_2,$$

we obtain in addition a copy of  $A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2$  (ie the 2-torsion in  $\widetilde{\mathrm{Sym}}_2 A^\bullet$ ) in  $\pi_1$  that does not contribute in the relevant dimensions, so we may ignore it. Now, by Proposition C.2, we know that

$$H^k(\widetilde{\mathrm{Sym}}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2) \simeq \begin{cases} \mathbb{F}_2 & \text{for } k = 2t + 2, \\ 0 & \text{for } k > 2t + 2. \end{cases}$$

So we can apply the universal coefficient theorem as in the previous case. We conclude that  $\iota^2$  is a generator similarly.  $\square$

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