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Cylindrical contact homology and topological entropy

MARCELO R R ALVES

We establish a relation between the growth of the cylindrical contact homology of a contact manifold and the topological entropy of Reeb flows on this manifold. We show that if a contact manifold (M,ξ) admits a hypertight contact form λ_0 for which the cylindrical contact homology has exponential homotopical growth rate, then the Reeb flow of every contact form on (M,ξ) has positive topological entropy. Using this result, we provide numerous new examples of contact 3–manifolds on which every Reeb flow has positive topological entropy.

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1 Introduction

The aim of this paper is to establish a relation between the behaviour of cylindrical contact homology and the topological entropy of Reeb flows. The topological entropy is a nonnegative number associated to a dynamical system which measures the complexity of the orbit structure of the system. Positivity of the topological entropy means that the system possesses some type of exponential instability. We show that if the cylindrical contact homology of a contact 3–manifold is "complicated enough" from a homotopical viewpoint, then every Reeb flow on this contact manifold has positive topological entropy.

1.1 Basic definitions and history of the problem

We first recall some basic definitions from contact geometry. A 1–form λ on a $(2n+1)$–dimensional manifold Y is called a *contact form* if $\lambda \wedge (d\lambda)^n$ is a volume form on Y. The hyperplane $\xi = \ker\lambda$ is called the *contact structure*. For us a *contact manifold* will be a pair (Y,ξ) such that ξ is the kernel of some contact form λ on Y (these are usually called co-oriented contact manifolds in the literature). When λ satisfies $\xi = \ker\lambda$, we will say that λ is a contact form on (Y,ξ). On any contact manifold there always exist infinitely many different contact forms. Given a contact form λ, its *Reeb vector field* is the unique vector field X_λ satisfying $\lambda(X_\lambda) = 1$ and $i_{X_\lambda} d\lambda = 0$. The *Reeb flow* ϕ_{X_λ} of λ is the flow generated by the vector field X_λ. We will refer

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to the periodic orbits of ϕ_{X_λ} as *Reeb orbits* of λ. The action $A(\gamma)$ of a Reeb orbit is defined by $A(\gamma) := \int_\gamma \lambda$.

We study the topological entropy of Reeb flows from the point of view of contact topology. More precisely, we search for conditions on the topology of a contact manifold (M,ξ) that force *all* Reeb flows on (M,ξ) to have positive topological entropy. The condition we impose is on the behaviour of a contact topological invariant called cylindrical contact homology. We show that if a contact manifold (M,ξ) admits a contact form λ_0 for which the cylindrical contact homology has *exponential homotopical growth*, then all Reeb flows on (M,ξ) have positive topological entropy.

The notion of exponential homotopical growth of cylindrical contact homology, which is introduced in this paper, differs from the notion of growth of contact homology studied by Colin and Honda [12] and by Vaugon [40]. For reasons explained in Section 2, the growth of contact homology is not well adapted to study the topological entropy of Reeb flows, while the notion of homotopical growth rate is (as we show) well suited for this purpose. We begin by explaining the results which were previously known relating the behaviour of contact topological invariants to the topological entropy of Reeb flow.

The study of contact manifolds all of whose Reeb flows have positive topological entropy was initiated by Macarini and Schlenk [36]. They showed that if Q is an energy hyperbolic manifold and ξ_{geo} is the contact structure on the unit tangent bundle T_1Q associated to the geodesic flows, then every Reeb flow on (T_1Q, ξ_{geo}) has positive topological entropy. Their work was based on previous ideas of Frauenfelder and Schlenk [20; 21] which related the growth rate of Lagrangian Floer homology to entropy invariants of symplectomorphisms. The strategy to estimate the topological entropy used in [36] can be briefly sketched as follows:

$$\begin{gathered}\text{Exponential growth of Lagrangian Floer homology of the tangent fibre } (TQ)|_p \\ \Downarrow \\ \text{Exponential volume growth of the unit tangent fibre } (T_1Q)|_p \\ \text{for all Reeb flows in } (T_1Q, \xi_{\text{geo}}) \\ \Downarrow \\ \text{Positivity of the topological entropy for all Reeb flows in } (T_1Q, \xi_{\text{geo}}).\end{gathered}$$

To obtain the first implication, Macarini and Schlenk use the fact that (T_1Q, ξ_{geo}) has the structure of a Legendrian fibration, and apply the geometric idea of [20; 21] to show that the number of trajectories connecting a Legendrian fibre to another Legendrian fibre can be used to obtain a volume growth estimate. The second implication in this scheme follows from Yomdin's theorem, which states that exponential volume growth of a submanifold implies positivity of topological entropy.[1]

In the author's Ph D thesis [2; 3], this approach was extended to deal with 3–dimensional contact manifolds which are not unit tangent bundles. This was done by designing a localized version of the geometric idea of [20; 21]. Globally most contact 3–manifolds are not Legendrian fibrations, but a sufficiently small neighbourhood of a given Legendrian knot in a contact 3–manifold can always be given the structure of a Legendrian fibration. It turns out that this is enough to conclude that if the linearized Legendrian contact homology of a pair of Legendrian knots in a contact 3–manifold (M^3, ξ) grows exponentially, then the length of these Legendrian knots grows exponentially for any Reeb flow on (M^3, ξ). We then apply Yomdin's theorem to obtain that all Reeb flows on (M^3, ξ) have positive topological entropy.

One drawback of these approaches is that they only give lower entropy bounds for C^∞–smooth Reeb flows. The reason is that Yomdin's theorem holds only for C^∞–smooth flows. The approach presented in the present paper *does not* use Yomdin's theorem and gives lower bounds for the topological entropy of C^1–smooth Reeb flows.

Another advantage is that the cylindrical contact homology is usually easier to compute than the linearized Legendrian contact homology. In fact, to apply the strategy of [2; 3] to a contact 3–manifold (M^3, ξ), one must first find a pair of Legendrian curves which "should" have exponential growth of linearized Legendrian contact homology. This is highly nontrivial since on any contact 3–manifolds there exist many Legendrian links for which the linearized Legendrian contact homology does not even exist. On the other hand, the definition of cylindrical contact homology involves only the contact manifold (M^3, ξ), and no Legendrian submanifolds.

1.2 Main results

Our results are inspired by the philosophy that a "complicated" topological structure should force chaotic behaviour for dynamical systems associated to this structure. Two important examples of this phenomenon are: the fact that on manifolds with complicated loop space the geodesic flow always has positive topological entropy (see Paternain [38]), and the fact that every diffeomorphism of a surface which is isotopic to a pseudo-Anosov diffeomorphism has positive topological entropy (see Fel'shtyn [16]).

To state our results we introduce some notation. Let M be a manifold and X be a C^k $(k \geq 1)$ vector field. Our first result relates the topological entropy of the flow ϕ_X to

[1] The same scheme was used by Frauenfelder and Schlenk [22] and by Frauenfelder, Labrousse and Schlenk [19] to obtain positive lower bounds for the intermediate and slow entropies of Reeb flows on unit tangent bundles.

the growth (relative to T) of the number of distinct homotopy classes which contain periodic orbits of ϕ_X with period at most T. More precisely, let Λ_X^T be the set of free homotopy classes of M which contain a periodic orbit of ϕ_X with period at most T. We denote by $N_X(T)$ the cardinality of Λ_X^T.

Theorem 1 *If for real numbers $a > 0$ and b there is a sequence $T_n \to +\infty$ such that*

$$N_X(T_n) \geq e^{aT_n+b}$$

for all T_n, then $h_{\text{top}}(\phi_X) \geq a$.

Theorem 1 might be a folklore result in the theory of dynamical systems. However, as we have not found it in the literature, we provide a complete proof in Section 2. It contains as a special case Ivanov's inequality for surface diffeomorphisms; see Jiang [31]. Our motivation for proving this result is to apply it to Reeb flows. Contact homology allows one to carry over information about the dynamical behaviour of one special Reeb flow on a contact manifold to all other Reeb flows on the same contact manifold. In Section 4, we introduce the notion of exponential homotopical growth of cylindrical contact homology. As we already mentioned, this growth rate differs from the ones previously considered in the literature and is specially designed to allow one to use Theorem 1 to obtain results about the topological entropy of Reeb flows. Recall that a contact form is called hypertight if its Reeb flow has no contractible closed orbits. We prove the following result:

Theorem 8 *Let λ_0 be a hypertight contact form on a contact manifold (M, ξ), and assume that the cylindrical contact homology of λ_0 has exponential homotopical growth with exponential weight $a > 0$. Then for every C^k $(k \geq 2)$ contact form λ on (M, ξ), the Reeb flow of X_λ has positive topological entropy. More precisely, if f_λ is the function such that $\lambda = f_\lambda \lambda_0$, then*

$$h_{\text{top}}(\phi_{X_\lambda}) \geq \frac{a}{\max f_\lambda}. \tag{1-1}$$

Notice that Theorem 8 allows us to conclude the positivity of the topological entropy for *all* Reeb flows on a given contact manifold (M, ξ), once we show that (M, ξ) admits one special hypertight contact form for which the cylindrical contact homology has exponential homotopical growth. It is worth remarking that our proof of Theorem 8 is carried out in full rigour, and does *not* make use of the polyfold technology which is being developed by Hofer, Wysocki and Zehnder. The reason is that we do not use the linearized contact homology considered by Bourgeois, Ekholm and Eliashberg [7] and Vaugon [40], but resort to a topological idea used by Hryniewicz, Momin and Salomão [30] to prove existence of Reeb orbits in prescribed homotopy classes.

Theorem 8 allows one to obtain estimates for the topological entropy for C^1–smooth Reeb flows. As previously observed, the strategy used in [36; 2; 3] produces estimates for the topological entropy only for C^∞–smooth contact forms as they depend on Yomdin's theorem, which fails for finite regularity.

Our other results are concerned with the existence of examples of contact manifolds which have a contact form with exponential homotopical growth rate of cylindrical contact homology. We show that in dimension 3 they exist in abundance, and it follows from Theorem 8 that every Reeb flow on these contact manifolds has positive topological entropy. In Section 5, we use a construction of Colin and Honda [12] to obtain many such examples of contact 3–manifolds. In these examples, the underlying differentiable 3–manifold has nontrivial JSJ decomposition and a hyperbolic component that fibres over the circle.

Theorem 9 *Let M be a closed connected oriented 3–manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i,\ 0 \le i \le q\}$ of irreducible manifolds with boundary such that*

- *M_0 is the mapping torus of a diffeomorphism $h\colon S \to S$ with pseudo-Anosov monodromy on a surface S with nonempty boundary.*

Then M can be given infinitely many nondiffeomorphic contact structures ξ_k such that for each ξ_k, there exists a hypertight contact form λ_k on (M, ξ_k) which has exponential homotopical growth of cylindrical contact homology. It follows that on each (M, ξ_k), all Reeb flows have positive topological entropy.

The contact structures studied in Theorem 9 are among the tight contact structures constructed by Colin and Honda [12] in closed connected irreducible toroidal 3–manifolds.

In Section 6, we study the cylindrical contact homology of contact 3–manifolds $(M, \xi_{(q,\mathfrak{r})})$ obtained via a special integral Dehn surgery on the unit tangent bundle $(T_1 S, \xi_{\mathrm{geo}})$ of a hyperbolic surface (S, g). This Dehn surgery is performed on a neighbourhood of a Legendrian curve $L_{\mathfrak{r}}$ which is the Legendrian lift of a simple closed separating geodesic $\mathfrak{r}$. The surgery we consider is the contact version of Handel–Thurston surgery, which was introduced by Foulon and Hasselblatt in [18] to produce nonalgebraic Anosov Reeb flows on 3–manifolds. We call this contact surgery the Foulon–Hasselblatt surgery. This surgery produces not only a contact 3–manifold $(M, \xi_{(q,\mathfrak{r})})$, but also a special contact form, which we denote by λ_{FH}, on $(M, \xi_{(q,\mathfrak{r})})$. In [18], the authors restrict their attention to integer surgeries with positive surgery coefficient q and prove that, in this case, the Reeb flow of λ_{FH} is Anosov. Our methods also work for negative coefficients as the Anosov condition on λ_{FH} does not play a role in our results. We obtain:

Theorem 16 *Let $(M, \xi_{(q,\mathfrak{r})})$ be the contact manifold obtained from performing the Foulon–Hasselblat q–surgery on the Legendrian curve $L_{\mathfrak{r}} \subset (T_1 S, \xi_{\text{geo}})$, and denote by λ_{FH} the contact form on $(M, \xi_{(q,\mathfrak{r})})$ obtained from this surgery. Then λ_{FH} is hypertight, and its cylindrical contact homology has exponential homotopical growth. It follows that every Reeb flow on $(M, \xi_{(q,\mathfrak{r})})$ has positive topological entropy.*

Organization of the paper In Section 2, we recall one of the definitions of the topological entropy and present the proof of Theorem 1. In Section 3, we recall the definition of cylindrical contact homology and its basic properties. In Section 4, we introduce the notion of exponential homotopical growth of cylindrical contact homology and prove Theorem 8. Section 5 is devoted to the proof of Theorem 9. In Section 6, we present the definition of the integral Foulon–Hasselblatt surgery and prove Theorem 16. In Section 7, we discuss the results obtained in this paper and propose some questions for future research.

Remark We again would like to point out that all the results above *do not* depend on the polyfolds technology which is being developed Hofer, Wysocki and Zehnder. This is the case because the versions of contact homology used for proving the results above involve only somewhere injective pseudoholomorphic curves. In this situation, transversality can be achieved by "classical" perturbation methods as in Dragnev [13].

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2 Homotopic growth of periodic orbits and topological entropy

Throughout this section, M will denote a compact manifold. We endow M with an auxiliary Riemannian metric g, which induces a distance function d_g on M, whose injectivity radius we denote by ϵ_g. Let $\widetilde{M}$ be the universal cover of M, $\tilde{g}$ be the Riemannian metric that makes the covering map $\pi\colon \widetilde{M} \to M$ an isometry, and $d_{\tilde{g}}$ be the distance induced by the metric $\tilde{g}$.

Let X be a vector field on M with no singularities and ϕ_X^t the flow generated by X. We call $P^X(T)$ the number of periodic orbits of ϕ^t with period in $[0,T]$. For us, a periodic orbit of X is a pair $([\gamma]_c, T)$, where $[\gamma]_c$ is the set of parametrizations of a given *immersed* curve $c\colon S^1 \to M$, and T is a positive real number (called the period of the orbit), such that

- $\gamma \in [\gamma]_c$ if and only if $\gamma\colon \mathbb{R} \to M$ parametrizes c and $\dot{\gamma}(t) = X(\gamma(t))$,
- for all $\gamma \in [\gamma]_c$, we have $\gamma(T+t) = \gamma(t)$ and $\gamma([0,T]) = c$.

We say that a periodic orbit $([\gamma]_c, T)$ is in a free homotopy class l of M if $c \in l$.

By a parametrized periodic orbit (γ, T) we mean a periodic orbit $([\gamma]_c, T)$ with a fixed choice of parametrization $\gamma \in [\gamma]_c$. A parametrized periodic orbit (γ, T) is said to be in a free homotopy class l when the underlying periodic orbit $([\gamma]_c, T)$ is in l.

We now recall a definition of topological entropy due to Bowen [10] which will be very useful for us. Let T and δ be positive real numbers. A set S is said to be T,δ–separated if for all $q_1 \neq q_2 \in S$, we have

$$\max_{t\in[0,T]} d_g\big(\phi_X^t(q_1), \phi_X^t(q_2)\big) > \delta. \tag{2-1}$$

We denote by $n^{T,\delta}$ the maximal cardinality of a T,δ–separated set for the flow ϕ_X. Then we define the δ–entropy $h_\delta(\phi_X)$ as

$$h_\delta(\phi_X) = \limsup_{T\to+\infty} \frac{\log(n^{T,\delta})}{T}. \tag{2-2}$$

The topological entropy h_{top} is then defined by

$$h_{\mathrm{top}}(\phi_X) = \lim_{\delta\to 0} h_\delta(\phi_X).$$

One can prove that the topological entropy does not depend on the metric d_g but only on the topology determined by the metric. For these and other structural results about topological entropy, we refer the reader to any standard textbook in dynamics such as [34] and [39].

From the work of Kaloshin and others it is well known that the exponential growth rate of periodic orbits,

$$\limsup_{T\to+\infty} \frac{\log(P^X(T))}{T}, \tag{2-3}$$

can be much bigger than the topological entropy. This implies that the growth rate (2-3) does not give a lower bound for the topological entropy of an arbitrary flow. There is, however, a different growth rate that measures how quickly periodic orbits appear in different free homotopy classes, which can be used to give such a lower bound of the topological entropy of a flow.

Let Λ denote the set of free homotopy classes of loops in M, and $\Lambda_0 \subset \Lambda$ the subset of primitive free homotopy classes. We define the set $\Lambda_X^T \subset \Lambda$ in the following way: $\varrho \in \Lambda_X^T$ if and only if there exists a periodic orbit of ϕ_X^t with period at most T that belongs to ϱ. We denote by $N_X(T)$ the cardinality of Λ_X^T.

Let $\{(\gamma_i, T_i) : 1 \leq i \leq n\}$ be a finite set of parametrized periodic orbits of X. For a number T satisfying $T \geq T_i$ for all $i \in \{1, \dots, n\}$ and a constant $\delta > 0$, we denote by $\Lambda_X^{T,\delta}((\gamma_1, T_1), \dots, (\gamma_n, T_n))$ the subset of Λ such that

- $l \in \Lambda_X^{T,\delta}((\gamma_1, T_1), \dots, (\gamma_n, T_n))$ if and only if there exist a parametrized periodic orbit $(\hat{\gamma}, \hat{T})$ with period $\hat{T} \leq T$ in the free homotopy class l and a number $i_l \in \{1, \dots, n\}$ for which $\max_{t \in [0,T]}\big(d_g(\gamma_{i_l}(t), \hat{\gamma}(t))\big) \leq \delta$.

Notice that

$$\Lambda_X^{T,\delta}((\gamma_1, T_1), \dots, (\gamma_n, T_n)) = \bigcup_{i \in \{1,\dots,n\}} \Lambda_X^{T,\delta}((\gamma_i, T_i)). \tag{2-4}$$

We are ready to prove the main result in this section. Theorem 1 below is well known to be true in the particular cases when ϕ_X is a geodesic flow, where it follows from Manning's inequality (see [33] and [38]), and when ϕ_X is the suspension of a surface diffeomorphism with pseudo-Anosov monodromy, where it follows from Ivanov's theorem (see [31]). It can be seen as a generalization of these results in the sense that it includes them as particular cases and that it applies to many other situations. Our argument is inspired by the remarkable proof of Ivanov's inequality given by Jiang in [31].

Theorem 1 *If for real numbers $a > 0$ and b there is a sequence $T_n \to +\infty$ such that*

$$N_X(T_n) \geq e^{aT_n + b}$$

for all T_n, then $h_{\mathrm{top}}(\phi_X) \geq a$.

Proof The theorem will follow if we prove that for all $0 < \delta < \epsilon_g/32$, we have $h_\delta(\phi_X) \geq a$. From now on, fix $0 < \delta < \epsilon_g/32$.

Step 1 For any point $p \in M$, let $V_{4\delta}(p)$ be the 4δ–neighbourhood of $\pi^{-1}(p)$. Because $\delta < \epsilon_g/32$, it is clear that $V_{4\delta}(p)$ is the disjoint union

$$V_{4\delta}(p) = \bigcup_{\tilde{p} \in \pi^{-1}(p)} B_{4\delta}(\tilde{p}), \tag{2-5}$$

where the ball $B_{4\delta}(\tilde{p})$ is taken with respect to the metric $\tilde{g}$.

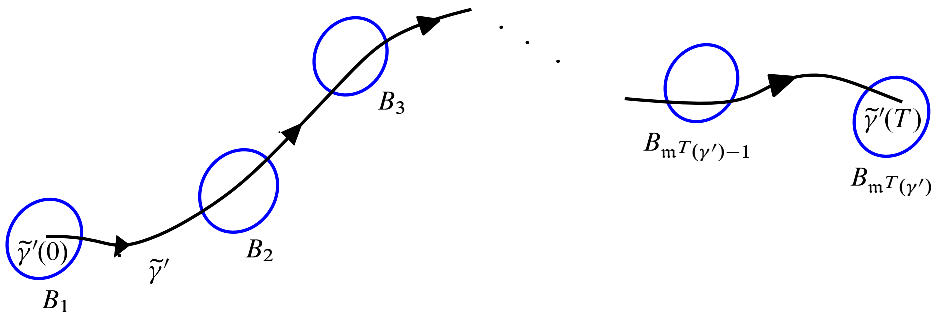

Figure 1: The set $\{B_j : 1 \leq j \leq \mathfrak{m}^T(\gamma', T')\}$

Because of our choice of $\delta < \epsilon_g/32$, it is clear that there exists a constant $k_1 > 0$, which does not depend on p, such that if B and B' are two distinct connected components of $V_{4\delta}(p)$, we have $d_{\widetilde{g}}(B, B') > k_1$.

Because of compactness of M, we know that the vector field $\widetilde{X} := \pi^* X$ is bounded in the norm given by the metric $\widetilde{g}$. Combining this with the inequality in the last paragraph, one obtains the existence of a constant $k_2 > 0$, which again does not depend on p, such that if $\widetilde{v}$: $[0, R] \to \widetilde{M}$ is a parametrized trajectory of $\phi_{\widetilde{X}}$ with $\widetilde{v}(0) \in B$ and $\widetilde{v}(R) \in B'$, then $R > k_2$.

From the last assertion, we deduce the existence of a constant $\widetilde{K}$, depending only g and X, such that for every $p \in M$ and every parametrized trajectory $\widetilde{v}$: $[0, T] \to \widetilde{M}$ of $\phi_{\widetilde{X}}$, the number $L^T(p, \widetilde{v})$ of distinct connected components of $V_{4\delta}(p)$ intersected by the curve $\widetilde{v}([0, T])$ satisfies

$$L^T(p, \widetilde{v}) < \widetilde{K}T + 1. \tag{2-6}$$

Step 2 We claim that for every parametrized periodic orbit (γ', T') of X, we have

$$\#\big(\Lambda_X^{T,\delta}((\gamma', T'))\big) < \widetilde{K}T + 1 \tag{2-7}$$

for all $T > T'$.

To see this, take a lift $\widetilde{\gamma}'$ of γ' to $\widetilde{M}$, and let $p' = \gamma'(0)$ and $\widetilde{p}' = \widetilde{\gamma}'(0)$. We consider (see Figure 1) the set $\{B_j : 1 \leq j \leq \mathfrak{m}^T(\gamma', T')\}$ of connected components of $V_{4\delta}(p')$ satisfying:

- $B_j \neq B_k$ if $j \neq k$,
- if B is a connected component of $V_{4\delta}(p')$ which intersects $\widetilde{\gamma}'([0, T])$, then $B = B_j$ for some $j \in \{1, \dots, \mathfrak{m}^T(\gamma', T')\}$,
- if $j < i$, then B_j is visited by the trajectory $\widetilde{\gamma}'$: $[0, T] \to \widetilde{M}$ before B_i.

From step 1, we know that $\mathfrak{m}^T(\gamma', T') < \widetilde{K}T + 1$.

For each $l \in \Lambda_X^{T,\delta}((\gamma', T'))$, pick a parametrized periodic orbit (χ_l, T_l) in l which satisfies $d_g(\chi_l(t), \gamma'(t)) < \delta$ for all $t \in [0, T]$. There exists a lift $\widetilde{\chi}_l$ of χ_l satisfying $d_{\widetilde{g}}(\widetilde{\chi}_l(t), \widetilde{\gamma}'(t)) < \delta$ for all $t \in [0, T]$.

From the triangle inequality, it is clear that the point $q_l = \widetilde{\chi}_l(0)$ is in the connected component B_1 which contains $\widetilde{p}'$. We will show that $\widetilde{\chi}_l(T_l)$ is contained in B_j for some $j \in \{1, \dots, \mathfrak{m}^T(\gamma')\}$. Because $\pi(\widetilde{\chi}_l(0)) = \pi(\widetilde{\chi}_l(T_l))$, we have

$$d_{\widetilde{g}}\big(\widetilde{\chi}_l(T_l), \pi^{-1}(p')\big) = d_{\widetilde{g}}\big(\widetilde{\chi}_l(0), \pi^{-1}(p')\big) < \delta, \tag{2-8}$$

which already implies that $\widetilde{\chi}_l(T_l) \in V_{4\delta}(p')$. We denote by $\widetilde{p}_l{}'$ the unique element in $\pi^{-1}(p')$ for which we have $d_{\widetilde{g}}(\widetilde{\chi}_l(T_l), \widetilde{p}_l{}') < \delta$. Using the triangle inequality we now obtain

$$d_{\widetilde{g}}\big(\widetilde{\gamma}'(T_l), \widetilde{p}_l{}'\big) \le d_{\widetilde{g}}\big(\widetilde{\gamma}'(T_l), \widetilde{\chi}_l(T_l)\big) + d_{\widetilde{g}}\big(\widetilde{\chi}_l(T_l), \widetilde{p}_l{}'\big) < \delta + \delta.$$

From the inequalities above we conclude that $\widetilde{\gamma}'(T_l)$ and $\widetilde{\chi}_l(T_l)$ are in the connected component of $V_{4\delta}(p')$ that contains $\widetilde{p}_l{}'$. Because this connected component contains $\widetilde{\gamma}'(T_l)$, it is therefore one of the B_j for $j \in \{1, \dots, \mathfrak{m}^T(\gamma', T')\}$ as we wanted to show. We can thus define a map

$$\Upsilon^{T,\delta}_{(\gamma',T')} \colon \Lambda_X^{T,\delta}((\gamma', T')) \to \{1, \dots, \mathfrak{m}^T((\gamma', T'))\} \tag{2-9}$$

which associates to each $l \in \Lambda_X^{T,\delta}(\gamma')$ the unique $j \in \{1, \dots, \mathfrak{m}^T(\gamma', T')\}$ for which $\widetilde{\chi}_l(T_l) \in B_j$.

We now claim that if $l \neq l'$, then $\widetilde{\chi}_l(T_l)$ and $\widetilde{\chi}_{l'}(T_{l'})$ are in different connected components of $V_{4\delta}(p')$. To see this, notice that both $\widetilde{\chi}_l(0)$ and $\widetilde{\chi}_{l'}(0)$ are in the component B_1. Therefore, it is clear, because $\delta < \epsilon_g/32$, that if $\widetilde{\chi}_l(T_l)$ and $\widetilde{\chi}_{l'}(T_{l'})$ are in the same component of $V_{4\delta}(p')$, then the closed curves $\chi_l([0, T_l])$ and $\chi_{l'}([0, T_{l'}])$ are freely homotopic. This contradicts our choice of (χ_l, T_l) and $(\chi_{l'}, T_{l'})$ and the fact that $l \neq l'$.

We thus conclude that the map (2-9) is injective, which implies that $\#(\Lambda_X^{T,\delta}((\gamma', T'))) \le \mathfrak{m}^T(\gamma', T') < \widetilde{K}T + 1$.

Step 3 (inductive step) As an immediate consequence of step 2, we have that if $\{(\gamma_i, T_i) : 1 \le i \le m\}$ is a set of parametrized periodic orbits of X, we have

$$\#(\Lambda_X^{T,\delta}((\gamma_1, T_1), \dots, (\gamma_m, T_m))) \le m(\widetilde{K}T + 1).$$

Inductive claim *Fix $T > 0$, and suppose that $S_m^T = \{(\gamma_i, T_i) : 1 \leq i \leq m\}$ is a set of parametrized periodic orbits such that $T \geq T_i$ for every $i \in \{1, \dots, m\}$, and that satisfies:*

(a) *The free homotopy classes l_i of (γ_i, T_i) and l_j of (γ_j, T_j) are distinct if $i \neq j$.*

(b) *For every $i \neq j$ we have $\max_{t \in [0,T]} d_g(\gamma_i(t), \gamma_j(t)) > \delta$.*

Then, if

$$m < \frac{N_X(T)}{\widetilde{K}T + 1},$$

there exists a parametrized periodic orbit $(\gamma_{m+1}, T_{m+1} \leq T)$ such that its homotopy class l_{m+1} does not belong to the set $\{l_i : 1 \leq i \leq m\}$ and such that

$$\max_{t \in [0,T]} d_g(\gamma_{m+1}(t), \gamma_i(t)) > \delta \tag{2-10}$$

for all $i \in 1, \dots, m$.

Proof of claim First, recall that $\#(\Lambda_X^{T,\delta}((\gamma_1, T_1), \dots, (\gamma_m, T_m))) \leq m(\widetilde{K}T + 1)$. Therefore, because $m < N_X(T)/(\widetilde{K}T + 1)$, there exists a free homotopy class $l_{m+1} \in \Lambda_X^T \backslash \Lambda_X^{T,\delta}((\gamma_1, T_1), \dots, (\gamma_m, T_m))$. Choose a parametrized periodic orbit (γ_{m+1}, T_{m+1}) with $T_{m+1} \leq T$ in the homotopy class l_{m+1}.

As $l_{m+1} \notin \Lambda_X^{T,\delta}((\gamma_1, T_1), \dots, (\gamma_m, T_m))$, we must have (2-10) for all $i \in 1, \dots, m$, thus completing the proof of the claim. □

Step 4 Obtaining a T, δ–separated set.

As usual, we denote by $\lfloor N_X(T)/(\widetilde{K}T + 1) \rfloor$ the largest integer which is at most $N_X(T)/(\widetilde{K}T + 1)$. The strategy is now to use the inductive step to obtain a set $S_X^T = \{(\gamma_i, T_i) : 1 \leq i \leq \lfloor N_X(T)/(\widetilde{K}T + 1) \rfloor\}$, satisfying conditions (a) and (b) above, with the maximum possible cardinality. We start with a set $S_1^T = \{(\gamma_1, T_1)\}$, which clearly satisfies conditions (a) and (b), and if $1 < \lfloor N_X(T)/(\widetilde{K}T + 1) \rfloor$ we apply the inductive step to obtain a parametrized periodic orbit $(\gamma_2, T_2 \leq T)$ such that $S_2^T = \{(\gamma_1, T_1), (\gamma_2, T_2 \leq T)\}$ satisfies (a) and (b). We can go on applying the inductive step to produce sets $S_m^T = \{(\gamma_i, T_i) : 1 \leq i \leq m\}$ satisfying the desired conditions (a) and (b) as long as $m - 1$ is smaller than $\lfloor N_X(T)/(\widetilde{K}T + 1) \rfloor$. By this process, we can construct a set $S_X^T = \{(\gamma_i, T_i) : 1 \leq i \leq \lfloor N_X(T)/(\widetilde{K}T + 1) \rfloor\}$ such that for all $i, j \in \{1, \dots, \lfloor N_X(T)/(\widetilde{K}T + 1) \rfloor\}$, (a) and (b) above hold true.

For each $i \in \{1, \dots, \lfloor N_X(T)/(\widetilde{K}T + 1) \rfloor\}$, let $q_i = \gamma_i(0)$. We define the set $P_X^T := \{q_i : 1 \leq i \leq \lfloor N_X(T)/(\widetilde{K}T) \rfloor + 1\}$. The condition (b) satisfied by S_X^T implies that P_X^T is a T, δ–separated set. It then follows from the definition of the δ–entropy h_δ that

$$h_\delta(\phi_X) \geq \limsup_{T \to +\infty} \frac{\log\bigl(\lfloor N_X(T)/(\widetilde{K}T + 1) \rfloor\bigr)}{T}. \tag{2-11}$$

Step 5 From the hypothesis of the theorem, we know that for the real numbers $a > 0$ and b, there exists a sequence $T_n \to +\infty$ such that $N_X(T_n) \geq e^{aT_n+b}$ for all T_n.

For every $\epsilon > 0$, we know that for T_n big enough we have $e^{\epsilon T_n} > \widetilde{K}T_n + 1$. This implies that

$$\limsup_{T_n \to +\infty} \frac{\log\bigl(\lfloor N_X(T_n)/(\widetilde{K}T_n+1)\rfloor\bigr)}{T_n} \geq \limsup_{T_n \to +\infty} \frac{\log(\lfloor e^{aT_n+b}/e^{\epsilon T_n}\rfloor)}{T_n} = \limsup_{T_n \to +\infty} \frac{\log(\lfloor e^{(a-\epsilon)T_n+b}\rfloor)}{T_n}. \tag{2-12}$$

It is clear that $\limsup_{T_n \to +\infty} \log(\lfloor e^{(a-\epsilon)T_n+b}\rfloor)/T_n = a - \epsilon$. Combining this with (2-11), we conclude that under the hypothesis of the theorem, $h_\delta(\phi_X) \geq a - \epsilon$. Because ϵ can be taken arbitrarily small, we obtain

$$h_\delta(\phi_X) \geq a. \tag{2-13}$$

Step 6 So far, we have shown that for all $\delta < \epsilon_g/32$, we have $h_\delta(\phi_X) \geq a$. It then follows that

$$h_{\mathrm{top}}(\phi_X) = \lim_{\delta \to 0} h_\delta(\phi_X) \geq a, \tag{2-14}$$

finishing the proof of the theorem. □

Remark One could naively believe that there exists a constant $\delta_g > 0$ depending only on the metric g such that if two parametrized closed curves $\sigma_1\colon \mathbb{R} \to M$ of period T_1 and $\sigma_2\colon \mathbb{R} \to M$ of period T_2 satisfy $\sup_{t \in [0,\max\{T_1,T_2\}]}\{d_g(\sigma_1(t), \sigma_2(t))\} < \delta_g$, then (γ_1, T_1) and (γ_2, T_2) are freely homotopic to each other. This would make the proof of Theorem 1 much shorter. However such a constant does not exist. One can easily find for any $\delta > 0$ two parametrized curves in the 3–torus which are in different primitive free homotopy classes and satisfy $\sup_{t \in [0,\max\{T_1,T_2\}]}\{d_g(\sigma_1(t), \sigma_2(t))\} < \delta$. We sketch the construction below.

Consider coordinates $(x, y, z) \in (\mathbb{R}/\mathbb{Z})^3$ on the three-dimensional torus $\mathbb{T}^3$. Figure 2 above represents the universal cover of the two-dimensional torus $\mathbb{T}^2 \subset \mathbb{T}^3$ obtained by fixing the coordinate $z = 0$ in $\mathbb{T}^3$. The dotted points p_0, $\widehat{p}$, p_1 and p_2 in Figure 2 represent lifts of a point $p \in \mathbb{T}^2$. It is then clear that the curve c represented in Figure 2 projects to a smooth immersed curve in $\mathbb{T}^2 \subset \mathbb{T}^3$.

We consider a parametrization by arc length $\varsigma_1\colon [0, T_1] \to \mathbb{R}^2$ of the piece of c connecting p_0 and p_1. We can extend ς_1 periodically to $\mathbb{R}$ by demanding that $\varsigma_1(t + T_1) = \varsigma_1(t) + (1, 2)$ for all $t \in \mathbb{R}$. This extension is a lift to $\mathbb{R}^2$ of the closed immersed curve

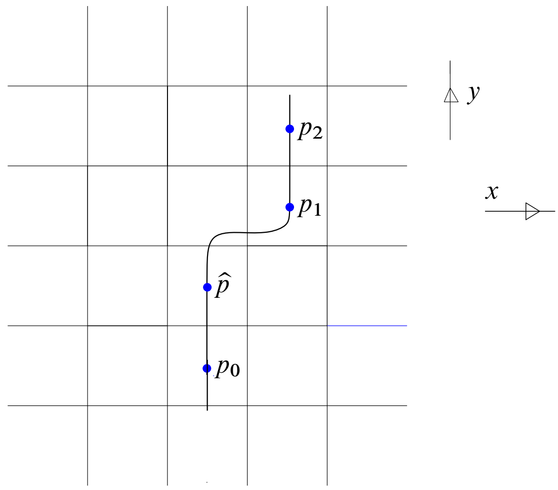

Figure 2: The universal cover of $\mathbb{T}^2 \subset \mathbb{T}^3$

obtained by projecting $\varsigma_1([0,T_1])$ to $\mathbb{T}^2$. By a very small perturbation of the projection of $\varsigma_1([0,T_1])$, we can produce a closed smooth embedded curve $\sigma_1\colon [0,T_1] \to \mathbb{T}^3$ which closes at the point $(p,0) = \sigma_1(0) = \sigma_1(T_1)$. We consider the natural extension of σ_1 to $\mathbb{R}$ obtained by demanding that $\sigma_1(t) = \sigma_1(t-T_1)$ for all $t \in \mathbb{R}$.

Analogously, we consider a parametrization by arc length $\varsigma_2\colon [0,T_1+1] \to \mathbb{R}^2$ of the piece of c connecting p_0 and p_2. We can also extend ς_2 periodically to $\mathbb{R}$, this time demanding that $\varsigma_2(t+T_1+1) = \varsigma_2(t) + (1,3)$. By making a very small perturbation of ς_2, we can produce a closed smooth embedded curve $\sigma_2\colon [0,T_1+1] \to \mathbb{T}^3$ which closes at the point $(p,\delta/K) = \sigma_2(0) = \sigma_2(T_1+1)$ and which is disjoint from the image of σ_1. We consider the natural extension of σ_2 to $\mathbb{R}$ obtained by demanding that $\sigma_2(t) = \sigma_2(t-(T_1+1))$ for all $t \in \mathbb{R}$.

We point out that the extensions $\varsigma_1\colon \mathbb{R} \to \mathbb{R}^2$ and $\varsigma_2\colon \mathbb{R} \to \mathbb{R}^2$ coincide on the interval $[0,T_1+1]$. To see this just notice that the piece of c connecting p_0 and $\widehat{p}$ and the piece of c connecting p_1 and p_2 project to the same circle in $\mathbb{T}^2$.

Now let $\sigma_0\colon [0,T_1+1] \to \mathbb{T}^2$ be the parametrized curve obtained by projecting $\varsigma_1\colon [0,T_1+1] \to \mathbb{R}^2$, which equals $\varsigma_2\colon [0,T_1+1] \to \mathbb{R}^2$, to the torus $\mathbb{T}^2$. The curves $\sigma_1|_{[0,T_1+1]}$ and $\sigma_2|_{[0,T_1+1]}$ are both perturbations of the parametrized curve σ_0. By making the perturbations sufficiently small we can guarantee that $\sigma_1|_{[0,T_1+1]}$ and $\sigma_2|_{[0,T_1+1]}$ are arbitrarily close. It is immediate to see that $\sigma_1|_{[0,T_1+1]}$ and $\sigma_2|_{[0,T_1+1]}$ are in distinct homotopy classes.

3 Contact homology

3.1 Pseudoholomorphic curves in symplectic cobordisms

To define the contact homologies used in this paper, we use pseudoholomorphic curves in symplectizations of contact manifolds and symplectic cobordisms. Pseudoholomorphic curves in symplectic manifolds were introduced by Gromov in [24] and adapted to symplectizations and symplectic cobordisms by Hofer [26]; see also [8] as a general reference for pseudoholomorphic curves in symplectic cobordisms.

3.1.1 Cylindrical almost complex structures Let (Y,ξ) be a contact manifold and λ a contact form on (Y,ξ). The symplectization of (Y,ξ) is the product $\mathbb{R}\times Y$ with the symplectic form $d(e^s\lambda)$ (where s denotes the $\mathbb{R}$ coordinate in $\mathbb{R}\times Y$). The 2–form $d\lambda$ restricts to a symplectic form on the vector bundle ξ, and it is well known that the set $\mathrm{j}(\lambda)$ of $d\lambda$–compatible almost complex structures on the symplectic vector bundle ξ is nonempty and contractible. Notice that if Y is 3–dimensional, the set $\mathrm{j}(\lambda)$ does not depend on the contact form λ on (Y,ξ).

For $j\in \mathrm{j}(\lambda)$, we can define an $\mathbb{R}$–invariant almost complex structure J on $\mathbb{R}\times Y$ by demanding that

$$J\partial_s = X_\lambda \quad\text{and}\quad J|_\xi = j. \tag{3-1}$$

We will denote by $\mathcal{J}(\lambda)$ the set of almost complex structures in $\mathbb{R}\times Y$ that are $\mathbb{R}$–invariant, $d(e^s\lambda)$–compatible and satisfy (3-1) for some $j\in\mathrm{j}(\lambda)$.

3.1.2 Exact symplectic cobordisms with cylindrical ends An exact symplectic cobordism is, roughly, an exact symplectic manifold (W,ϖ) that, outside a compact subset, is like the union of cylindrical ends of symplectizations. We restrict our attention to exact symplectic cobordisms having only one positive end and one negative end.

Let $(W,\varpi=d\kappa)$ be an exact symplectic manifold without boundary, and let (Y^+,ξ^+) and (Y^-,ξ^-) be contact manifolds with contact forms λ^+ and λ^-. We say that $(W,\varpi=d\kappa)$ is an exact symplectic cobordism from λ^+ to λ^- if there exist subsets W^-, W^+ and $\widehat{W}$ of W and diffeomorphisms $\Psi^+\colon W^+\to[0,+\infty)\times Y^+$ and $\Psi^-\colon W^-\to(-\infty,0]\times Y^-$, such that

$$\begin{gathered}\widehat{W}\text{ is compact},\quad W=W^+\cup\widehat{W}\cup W^-,\quad W^+\cap W^-=\varnothing,\\ (\Psi^+)^*(e^s\lambda^+)=\kappa\quad\text{and}\quad(\Psi^-)^*(e^s\lambda^-)=\kappa.\end{gathered}\tag{3-2}$$

In such a cobordism, we say that an almost complex structure $\bar{J}$ is cylindrical if

$$\bar{J} \text{ coincides with } J^+ \in \mathcal{J}(C^+\lambda^+) \text{ in the region } W^+, \tag{3-3}$$

$$\bar{J} \text{ coincides with } J^- \in \mathcal{J}(C^-\lambda^-) \text{ in the region } W^-, \tag{3-4}$$

$$\bar{J} \text{ is compatible with } \varpi \text{ in } \widehat{W}, \tag{3-5}$$

where $C^+ > 0$ and $C^- > 0$ are constants.

For fixed $J^+ \in \mathcal{J}(C^+\lambda^+)$ and $J^- \in \mathcal{J}(C^-\lambda^-)$, we denote by $\mathcal{J}(J^-, J^+)$ the set of cylindrical almost complex structures in $(\mathbb{R} \times Y, \varpi)$ coinciding with J^+ on W^+ and J^- on W^-. It is well known that $\mathcal{J}(J^-, J^+)$ is nonempty and contractible. We will write $\lambda^+ \succ_{\text{ex}} \lambda^-$ if there exists an exact symplectic cobordism from λ^+ to λ^- as above. We remind the reader that $\lambda^+ \succ_{\text{ex}} \lambda$ and $\lambda \succ_{\text{ex}} \lambda^-$ implies $\lambda^+ \succ_{\text{ex}} \lambda^-$, or in other words that the exact symplectic cobordism relation is transitive; see [8] for a detailed discussion on symplectic cobordisms with cylindrical ends. Notice that a symplectization is a particular case of an exact symplectic cobordism.

Remark We point out to the reader that in many references in the literature, a slightly different definition of cylindrical almost complex structures is used: instead of demanding that $\bar{J}$ satisfies conditions (3-3) and (3-4), the stronger condition that $\bar{J}$ coincides with $J^\pm \in \mathcal{J}(\lambda^\pm)$ in the region $W^\pm$ is demanded. We need to consider this more relaxed definition of cylindrical almost complex structures when we study the cobordism maps of cylindrical contact homologies in Section 3.2.3.

3.1.3 Splitting symplectic cobordisms Let λ^+, λ and λ^- be contact forms on (Y, ξ) such that $\lambda^+ \succ_{\text{ex}} \lambda$ and $\lambda \succ_{\text{ex}} \lambda^-$. For $\epsilon > 0$ sufficiently small, it is easy to see that one also has $\lambda^+ \succ_{\text{ex}} (1+\epsilon)\lambda$ and $(1-\epsilon)\lambda \succ_{\text{ex}} \lambda^-$. Then for each $R > 0$, we can construct an exact symplectic form $\varpi_R = d\kappa_R$ on $W = \mathbb{R} \times Y$, where

$$\kappa_R = e^{s-R-2}\lambda^+ \quad \text{in } [R+2, +\infty) \times Y, \tag{3-6}$$

$$\kappa_R = f(s)\lambda \quad \text{in } [-R, R] \times Y, \tag{3-7}$$

$$\kappa_R = e^{s+R+2}\lambda^- \quad \text{in } (-\infty, -R-2] \times Y, \tag{3-8}$$

and $f\colon [-R, R] \to [1-\epsilon, 1+\epsilon]$ satisfies $f(-R) = 1-\epsilon$, $f(R) = 1+\epsilon$ and $f' > 0$. In $(\mathbb{R} \times Y, \varpi_R)$, we consider a compatible cylindrical almost complex structure $\widetilde{J}_R$, but we demand an extra condition on $\widetilde{J}_R$:

$$\widetilde{J}_R \text{ coincides with } J \in \mathcal{J}(\lambda) \text{ in } [-R, R] \times Y. \tag{3-9}$$

Again we divide W into regions: $W^+ = [R+2, +\infty) \times Y$, $W(\lambda^+, \lambda) = [R, R+2] \times Y$, $W(\lambda) = [-R, R] \times Y$, $W(\lambda, \lambda^-) = [-R-2, -R] \times Y$ and $W^- = (-\infty, -R-2] \times Y$.

The family of exact symplectic cobordisms with cylindrical almost complex structures $(\mathbb{R}\times Y, \varpi_R, \widetilde{J}_R)$ is called a splitting family from λ^+ to λ^- along λ.

3.1.4 Pseudoholomorphic curves Let (S,i) be a closed Riemann surface without boundary and $\Gamma\subset S$ a finite set. Let λ be a contact form on (Y,ξ), and let $J\in\mathcal{J}(\lambda)$. A finite energy pseudoholomorphic curve in the symplectization $(\mathbb{R}\times Y, J)$ is a map $\widetilde{w}=(s,w)\colon S\setminus\Gamma\to\mathbb{R}\times Y$ that satisfies

$$\bar\partial_J(\widetilde{w}) = d\widetilde{w}\circ i - J\circ d\widetilde{w} = 0 \tag{3-10}$$

and

$$0 < E(\widetilde{w}) = \sup_{q\in\mathcal{E}} \int_{S\setminus\Gamma} \widetilde{w}^* d(q\lambda), \tag{3-11}$$

where $\mathcal{E}=\{q\colon \mathbb{R}\to[0,1] : q'\geq 0\}$. The quantity $E(\widetilde{w})$ is called the Hofer energy and was introduced in [26]. The operator $\bar\partial_J$ above is called the Cauchy–Riemann operator for the almost complex structure J.

For an exact symplectic cobordism (W,ϖ) from λ^+ to λ^- as considered above, and for $\bar{J}\in\mathcal{J}(J^-,J^+)$, a finite energy pseudoholomorphic curve is again a map $\widetilde{w}\colon S\setminus\Gamma\to W$ satisfying

$$d\widetilde{w}\circ i = \bar{J}\circ d\widetilde{w} \tag{3-12}$$

and

$$0 < E_{\lambda^-}(\widetilde{w}) + E_c(\widetilde{w}) + E_{\lambda^+}(\widetilde{w}) < +\infty, \tag{3-13}$$

where

$$E_{\lambda^-}(\widetilde{w}) = \sup_{q\in\mathcal{E}} \int_{\widetilde{w}^{-1}(W^-))} \widetilde{w}^* d(q\lambda^-),$$
$$E_{\lambda^+}(\widetilde{w}) = \sup_{q\in\mathcal{E}} \int_{\widetilde{w}^{-1}(W^+)} \widetilde{w}^* d(q\lambda^+),$$
$$E_c(\widetilde{w}) = \int_{\widetilde{w}^{-1}(W(\lambda^+,\lambda^-))} \widetilde{w}^*\varpi.$$

These energies were also introduced in [26].

In splitting symplectic cobordisms, we use a slightly modified version of energy. Instead of demanding $0 < E_-(\widetilde{w}) + E_c(\widetilde{w}) + E_+(\widetilde{w}) < +\infty$, we demand that

$$0 < E_{\lambda^-}(\widetilde{w}) + E_{\lambda^-,\lambda}(\widetilde{w}) + E_\lambda(\widetilde{w}) + E_{\lambda,\lambda^+}(\widetilde{w}) + E_{\lambda^+}(\widetilde{w}) < +\infty, \tag{3-14}$$

where

$$E_\lambda(\widetilde{w}) = \sup_{q\in\mathcal{E}} \int_{\widetilde{w}^{-1}W(\lambda)} \widetilde{w}^* d(q\lambda),$$
$$E_{\lambda^-,\lambda}(\widetilde{w}) = \int_{\widetilde{w}^{-1}(W(\lambda,\lambda^-))} \widetilde{w}^*\varpi,$$
$$E_{\lambda,\lambda^+}(\widetilde{w}) = \int_{\widetilde{w}^{-1}(W(\lambda^+,\lambda))} \widetilde{w}^*\varpi,$$

and where $E_{\lambda^-}(\widetilde{w})$ and $E_{\lambda^+}(\widetilde{w})$ are as above.

The elements of the set $\Gamma \subset S$ are called punctures of the pseudoholomorphic curve $\widetilde{w}$. According to [26; 27], punctures fall into two classes, positive and negative, according to the behaviour of $\widetilde{w}$ in the neighbourhood of the puncture. Before presenting this classification, we introduce some notation. Let $B_\delta(z)$ be the ball of radius δ centred at the puncture z, and denote by $\partial(B_\delta(z))$ its boundary. We can describe the types of punctures as follows:

- $z \in \Gamma$ is called a positive interior puncture if $\lim_{z'\to z} s(z') = +\infty$ and there exist a sequence $\delta_n \to 0$ and a Reeb orbit γ^+ of X_{λ^+} such that $w(\partial(B_{\delta_n}(z)))$ converges in C^∞ to γ^+ as $n \to +\infty$,
- $z \in \Gamma$ is called a negative interior puncture if $\lim_{z'\to z} s(z') = -\infty$, and there exist a sequence $\delta_n \to 0$ and a Reeb orbit γ^- of X_{λ^-} such that $w(\partial(B_{\delta_n}(z)))$ converges in C^∞ to γ^- as $n \to +\infty$.

The results in [26] and [27] imply that these are indeed the only possibilities we need to consider for the behaviour of $\widetilde{w}$ near punctures. Intuitively, we have that at the punctures, the pseudoholomorphic curve $\widetilde{w}$ detects Reeb orbits. When for a puncture z, there is a subsequence δ_n such that $w(\partial(B_{\delta_n}(z)))$ converges to a Reeb orbit γ, we will say that $\widetilde{w}$ is asymptotic to this Reeb orbit γ at the puncture z.

If a pseudoholomorphic curve $\widetilde{w}$ is asymptotic to a nondegenerate Reeb orbit at a puncture z, more can be said about its asymptotic behaviour in neighbourhoods of this puncture. Take a neighbourhood $U \subset S$ of z that admits a holomorphic chart $\psi_U\colon (U,z) \to (\mathbb{D},0)$. Using polar coordinates $(r,t) \in (0,+\infty) \times S^1$, we can write $x \in (\mathbb{D} \setminus 0)$ as $x = e^{-r}t$. With this notation, it is shown in [26; 27], that if z is a positive interior puncture at which $\widetilde{w}$ is asymptotic to a nondegenerate Reeb orbit γ^+ of X_{λ^+}, then $\widetilde{w} \circ \psi_U^{-1}(r,t) = (s(r,t), w(r,t))$ satisfies

- $w^r(t) = w(r,t)$ converges in C^∞ to a Reeb orbit γ^+ of X_{λ^+}, exponentially in r and uniformly in t.

Similarly, if z is a negative interior puncture at which $\widetilde{w}$ is asymptotic to a nondegenerate Reeb orbit γ^- of X_{λ^-}, then $\widetilde{w} \circ \psi_U^{-1}(r,t) = (s(r,t), w(r,t))$ satisfies

- $w^r(t) = w(r,t)$ converges in C^∞ to a Reeb orbit γ^- of $-X_{\lambda^-}$ as $r \to +\infty$, exponentially in r and uniformly in t.

Remark The fact that the convergence of pseudoholomorphic curves near punctures to Reeb orbits is of exponential nature is a consequence of the asymptotic formula obtained in [27]. Such formulas are necessary for the Fredholm theory developed in [28] that gives the dimension of the space of pseudoholomorphic curves with fixed asymptotic data.

The discussion above can be summarized by saying that, near punctures, the finite energy pseudoholomorphic curves detect Reeb orbits. It is exactly this behaviour that makes these objects useful for the study of dynamics of Reeb vector fields.

For us, it will be important to consider the moduli spaces $\mathcal{M}(\gamma, \gamma_1', \dots, \gamma_m'; J)$ of genus-0 pseudoholomorphic curves, modulo biholomorphic reparametrization, with one positive puncture asymptotic to a nondegenerate Reeb orbit γ, and negative punctures asymptotic to nondegenerate Reeb orbits $\gamma_1', \dots, \gamma_m'$. It is well known that the linearization $D\bar\partial_J$ of $\bar\partial_J$ at any element of $\mathcal{M}(\gamma, \gamma_1', \dots, \gamma_m'; J)$ is a Fredholm map (we remark that this property is valid for more general moduli spaces of curves with prescribed asymptotic behaviour). One would like to conclude that the dimension of a connected component of $\mathcal{M}(\gamma, \gamma_1', \dots, \gamma_m'; J)$ is given by the Fredholm index of an element of $\mathcal{M}(\gamma, \gamma_1', \dots, \gamma_m'; J)$. However, this is not always the case if the moduli space contains multiply covered pseudoholomorphic curves.

Fact As a consequence of the exactness of the symplectic cobordisms considered above, we obtain that the energy $E(\widetilde{w})$ of $\widetilde{w}$ satisfies $E(\widetilde{w}) \leq 5A(\widetilde{w})$, where $A(\widetilde{w})$ is the sum of the action of the Reeb orbits detected by the punctures of $\widetilde{w}$ counted with multiplicity; see [8; 29].

3.2 Contact homologies

Contact homologies were introduced in [14] as homology theories which are topological invariants of contact manifolds. In Sections 3.2.1 and 3.2.2, we give an introduction to the more basic and well-known versions of contact homologies. This serves mainly as a motivation to Section 3.2.3, where we present the version of contact homology that will be used in this paper.[2]

[2] We stress that while the versions of contact homology presented in Sections 3.2.1 and 3.2.2 do depend on the polyfold technology currently being developed by Hofer, Wysocki and Zehnder, the version of contact homology which we use in this paper and present in Section 3.2.3 *does not* depend on polyfold and can be constructed in complete rigour with technology that is available in the literature. See the detailed discussion in Section 3.2.3 below.

3.2.1 Full contact homology Full contact homology was introduced in [14] as an important invariant of contact structures. We refer the reader to [14] and [6] for detailed presentations of the material contained in this subsection.

Let $(Y^{2\mathrm{n}+1}, \xi)$ be a contact manifold with λ a nondegenerate contact form. We denote by $\mathcal{P}(\lambda)$ the set of good periodic orbits of the Reeb vector field X_λ. To each orbit $\gamma \in \mathcal{P}(\lambda)$, we define a $\mathbb{Z}_2$–degree $|\gamma| = (\mu_{CZ}(\gamma) + (\mathrm{n}-2)) \bmod 2$. An orbit γ is called good if it is either simple, or if $\gamma = (\gamma')^i$ for a simple orbit γ' with $|\gamma| = |\gamma'|$.

$\mathfrak{A}(Y, \lambda)$ is defined to be the supercommutative, $\mathbb{Z}_2$–graded $\mathbb{Q}$–algebra with unit generated by $\mathcal{P}(\lambda)$ (an algebra with these properties is called a commutative superalgebra or a super-ring). The $\mathbb{Z}_2$–grading on the elements of the algebra is obtained by considering (on the generators) the grading mentioned above and extending it to $\mathfrak{A}(Y, \lambda)$.

$\mathfrak{A}(Y, \lambda)$ can be equipped with a differential d_J. Denote by $\mathcal{M}^k(\gamma, \gamma'_1, \dots, \gamma'_m; J)$ the moduli space of finite energy pseudoholomorphic curves of genus 0 and Fredholm index k, modulo reparametrization, with one positive puncture asymptotic to γ and negative punctures asymptotic to $\gamma'_1, \dots, \gamma'_m$ in the symplectization $(\mathbb{R} \times Y, J)$. As the almost complex structure J is $\mathbb{R}$–invariant in $\mathbb{R} \times Y$, we have an $\mathbb{R}$–action on $\mathcal{M}^k(\gamma, \gamma'_1, \dots, \gamma'_m; J)$, and we write

$$\widehat{\mathcal{M}}^k(\gamma, \gamma'_1, \dots, \gamma'_m; J) = \mathcal{M}^k(\gamma, \gamma'_1, \dots, \gamma'_m; J)/\mathbb{R}.$$

Lastly, we denote by $\overline{\mathcal{M}}^k(\gamma, \gamma'_1, \dots, \gamma'_m; J)$, as presented in [8], the compactification of $\widehat{\mathcal{M}}^k(\gamma, \gamma'_1, \dots, \gamma'_m; J)$. The compactified moduli space $\overline{\mathcal{M}}^k(\gamma, \gamma'_1, \dots, \gamma'_m; J)$ also involves pseudoholomorphic buildings that appear as limits of a sequence of curves in $\widehat{\mathcal{M}}^k(\gamma, \gamma'_1, \dots, \gamma'_m; J)$ that "breaks"; we refer the reader to [8] for a more detailed description of these moduli spaces. To define our differential, we need the following hypothesis:

Hypothesis H *There exists an abstract perturbation of the Cauchy–Riemann operator ∂_J such that the compactified moduli spaces $\overline{\mathcal{M}}(\gamma, \gamma'_1, \dots, \gamma'_m; J)$ of solutions of the perturbed equation are unions of branched manifolds with corners and rational weights whose dimension is given by the Conley–Zehnder index of the asymptotic orbits and the relative homology class of the solution.*

The proof that Hypothesis H is true is still not written. Establishing its validity is one of the main reasons for the development of the polyfold technology by Hofer, Wysocki and Zehnder. We define

$$d_J \gamma = m(\gamma) \sum_{\gamma'_1, \dots, \gamma'_m} \frac{C(\gamma, \gamma'_1, \dots, \gamma'_m)}{m!} \gamma'_1 \gamma'_2 \dots \gamma'_m, \tag{3-15}$$

where $C(\gamma, \gamma'_1, \dots, \gamma'_m)$ is the algebraic count of points in the 0–dimensional manifold

$$\widehat{\mathcal{M}}^1(\gamma, \gamma'_1, \dots, \gamma'_m; J), \tag{3-16}$$

and $m(\gamma)$ is the multiplicity of γ. The map d_J is extended to the whole algebra by the Leibnitz rule. Under Hypothesis H, it was proved in [14] that $(d_J)^2 = 0$. We therefore have that $(\mathfrak{A}(Y, \lambda), d_J)$ is a differential $\mathbb{Z}_2$–graded supercommutative algebra.

Definition 2 The *full contact homology* $\mathbb{CH}(\lambda, J)$ of λ is the homology of the complex $(\mathfrak{A}, d_J)$.

Under Hypothesis H, it was also proved in [14] that the full contact homology does not depend on the contact form λ on (Y, ξ), nor on the choice of the cylindrical almost complex structure $J \in \mathcal{J}(\lambda)$.

3.2.2 Cylindrical contact homology Suppose now that (Y, ξ) is a contact manifold, and λ is a nondegenerate hypertight contact form on (Y, ξ). Fix a cylindrical almost complex structure $J \in \mathcal{J}(\lambda)$. For hypertight contact manifolds, we can define a simpler version of contact homology called cylindrical contact homology. We denote by $\mathrm{CH}_{\mathrm{cyl}}(\lambda)$ the $\mathbb{Z}_2$–graded $\mathbb{Q}$–vector space generated by the elements of $\mathcal{P}(\lambda)$. The differential $d_J^{\mathrm{cyl}}\colon \mathrm{CH}_{\mathrm{cyl}}(\lambda) \to \mathrm{CH}_{\mathrm{cyl}}(\lambda)$ will count elements in the moduli space $\widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$. For the generators $\gamma \in \mathcal{P}(\lambda)$, we define

$$d_J^{\mathrm{cyl}}(\gamma) = \mathrm{cov}(\gamma) \sum_{\gamma' \in \mathcal{P}(\lambda)} C(\gamma, \gamma'; J)\gamma', \tag{3-17}$$

where $C(\gamma, \gamma'; J)$ is the algebraic count of elements in $\widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$ and $\mathrm{cov}(\gamma)$ is the covering number of γ. For λ hypertight, and assuming Hypothesis H is true, Eliashberg, Givental and Hofer proved in [14] that $(d_J^{\mathrm{cyl}})^2 = 0$.

Definition 3 The *cylindrical contact homology* $\mathbb{CH}_{\mathrm{cyl}}(\lambda)$ of λ is the homology of the complex $(\mathrm{CH}_{\mathrm{cyl}}(\lambda), d_J^{\mathrm{cyl}})$.

Under Hypothesis H, the cylindrical contact homology does not depend on the hypertight contact form λ on (Y, ξ), nor on the cylindrical almost complex structure $J \in \mathcal{J}(\lambda)$.

Denote by Λ the set of free homotopy classes of Y. It is easy to see that for each $\rho \in \Lambda$, the subspace $\mathrm{CH}^{\rho}_{\mathrm{cyl}}(\lambda) \subset \mathrm{CH}_{\mathrm{cyl}}(\lambda)$ generated by the set $\mathcal{P}_\rho(\lambda)$ of good periodic orbits in ρ is a subcomplex of $(\mathrm{CH}_{\mathrm{cyl}}(\lambda), d_J^{\mathrm{cyl}})$. This follows from the fact that the number $C(\gamma, \gamma'; J)$ can only be nonzero for Reeb orbits γ' that are freely homotopic to γ, which implies that the restriction $d_J^{\mathrm{cyl}}|_{\mathrm{CH}^{\rho}_{\mathrm{cyl}}}$ has image in $\mathrm{CH}^{\rho}_{\mathrm{cyl}}(\lambda)$. From now on, we will denote the restriction $d_J^{\mathrm{cyl}}|_{\mathrm{CH}^{\rho}_{\mathrm{cyl}}}\colon \mathrm{CH}^{\rho}_{\mathrm{cyl}}(\lambda) \to \mathrm{CH}^{\rho}_{\mathrm{cyl}}(\lambda)$ by d_J^{ρ}. Denoting by $\mathrm{C}\mathbb{H}^{\rho}_{\mathrm{cyl}}$ the homology of $(\mathrm{CH}^{\rho}_{\mathrm{cyl}}(\lambda), d_J^{\rho})$, we thus have

$$\mathrm{C}\mathbb{H}_{\mathrm{cyl}}(\lambda) = \bigoplus_{\rho\in\Lambda} \mathrm{C}\mathbb{H}^{\rho}_{\mathrm{cyl}}(\lambda). \tag{3-18}$$

The fact that we can define partial versions of cylindrical contact homology restricted to certain free homotopy classes will be of crucial importance for us. It will allow us to obtain our results without resorting to Hypothesis H. This is explained in the next subsection.

3.2.3 Cylindrical contact homology in special homotopy classes Maintaining the notation of the previous sections, we denote by (Y, ξ) a contact manifold endowed with a hypertight contact form λ.

Let Λ_0 denote the set of primitive free homotopy classes of Y. Let $\rho \in \Lambda$ be either an element of Λ_0, or a free homotopy class which contains only simple Reeb orbits of λ. Assume that all Reeb orbits in $\mathcal{P}_\rho(\lambda)$ are nondegenerate. By the work of Dragnev [13], we know that there exists a generic subset $\mathcal{J}^{\rho}_{\mathrm{reg}}(\lambda)$ of $\mathcal{J}(\lambda)$ such that for all $J \in \mathcal{J}^{\rho}_{\mathrm{reg}}(\lambda)$ we have:

- For all Reeb orbits $\gamma_1, \gamma_2 \in \rho$, the moduli space of pseudoholomorphic cylinders $\mathcal{M}(\gamma_1, \gamma_2; J)$ is transverse, ie the linearized Cauchy–Riemann operator $D\bar{\partial}_J(\widetilde{w})$ is surjective for all $\widetilde{w} \in \mathcal{M}(\gamma_1, \gamma_2; J)$.
- For all Reeb orbits $\gamma_1, \gamma_2 \in \rho$, each connected component $\mathcal{L}$ of the moduli space $\mathcal{M}(\gamma_1, \gamma_2; J)$ is a manifold whose dimension is given by the Fredholm index of any element $\widetilde{w} \in \mathcal{L}$.

In this case, for $J \in \mathcal{J}^{\rho}_{\mathrm{reg}}(\lambda)$, we define

$$d_J^{\rho}(\gamma) = \mathrm{cov}(\gamma) \sum_{\gamma' \in \mathcal{P}_\rho(\lambda)} C^{\rho}(\gamma, \gamma'; J)\gamma' = \sum_{\gamma' \in \mathcal{P}_\rho(\lambda)} C^{\rho}(\gamma, \gamma'; J)\gamma', \tag{3-19}$$

where $C^{\rho}(\gamma, \gamma'; J)$ is the number of points of the moduli space $\widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$. The second equality follows from the fact that all Reeb orbits in ρ are simple, which implies $\mathrm{cov}(\gamma) = 1$.

For λ and ρ as above and $J \in \mathcal{J}^{\rho}_{\mathrm{reg}}(\lambda)$, the differential $d_J^{\rho}\colon \mathrm{CH}^{\rho}_{\mathrm{cyl}}(\lambda) \to \mathrm{CH}^{\rho}_{\mathrm{cyl}}(\lambda)$ is well-defined and satisfies $(d_J^{\rho})^2 = 0$. Thus, in this situation, we can define the cylindrical contact homology $\mathrm{C}\mathbb{H}^{\rho,J}_{\mathrm{cyl}}(\lambda)$ without imposing Hypothesis H. Once the transversality for J has been achieved, and using coherent orientations constructed in [9], the proof that d_J^{ρ} is well-defined and that $(d_J^{\rho})^2 = 0$ is a combination of compactness and gluing, similar to the proof of the analogous result for Floer homology. For the convenience of the reader, we sketch these arguments below:

Claim *For ρ as above, $d^\rho_J\colon \mathrm{CH}^\rho_{\mathrm{cyl}}(\lambda) \to \mathrm{CH}^\rho_{\mathrm{cyl}}(\lambda)$ is well-defined, and for every $\gamma \in \mathcal{P}_\rho(\lambda)$, the differential $d^\rho_J(\gamma)$ is a finite sum.*

Proof The moduli space $\widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$ can be nonempty only if $A(\gamma') \le A(\gamma)$. It then follows from the nondegeneracy of λ that, for a fixed γ, the numbers $C^{\mathrm{cyl}}(\gamma, \gamma'; J)$ can be nonzero for only finitely many γ'. To see that $C^{\mathrm{cyl}}(\gamma, \gamma'; J)$ is finite for every $\gamma' \in \rho$, suppose by contradiction that there is a sequence $\widetilde{w}_i$ of distinct elements of $\widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$. By the SFT compactness theorem [8], such a sequence has a convergent subsequence that converges to a pseudoholomorphic building $\widetilde{w}$ which has Fredholm index 1. Because of the hypertightness of λ, no bubbling can occur and all the levels $\widetilde{w}^1, \dots, \widetilde{w}^k$ of the building $\widetilde{w}$ are pseudoholomorphic cylinders. As all Reeb orbits of λ in ρ are simple, it follows that all these cylinders are somewhere injective pseudoholomorphic curves, and the regularity of J implies that they must all have Fredholm index at least 1. As a result, we have $1 = I_F(\widetilde{w}) = \sum (I_F(\widetilde{w}^l)) \ge k$, which implies $k = 1$. Thus $\widetilde{w} \in \widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$, and it is the limit of a sequence of distinct elements of $\widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$. This is absurd, because $\widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$ is a 0–dimensional manifold. We thus conclude that the numbers $C^{\mathrm{cyl}}(\gamma, \gamma'; J)$ are all finite. □

Claim *For ρ as above, $(d^\rho_J)^2 = 0$.*

Proof If we write

$$d^\rho_J \circ d^\rho_J(\gamma) = \sum_{\gamma'' \in \mathcal{P}_\rho(\lambda)} m_{\gamma,\gamma''} \gamma'', \tag{3-20}$$

we know that $m_{\gamma,\gamma''}$ is the number of two-level pseudoholomorphic buildings $\widetilde{w} = (\widetilde{w}^1, \widetilde{w}^2)$ such that $\widetilde{w}^1 \in \widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$ and $\widetilde{w}^2 \in \widehat{\mathcal{M}}^1(\gamma', \gamma''; J)$ for some $\gamma' \in \mathcal{P}_\rho(\lambda)$. Because of transversality of $\widetilde{w}^1$ and $\widetilde{w}^2$, we can perform gluing. This implies that $\widetilde{w}$ is in the boundary of the moduli space $\overline{\mathcal{M}}^2(\gamma, \gamma''; J)$. Taking a sequence $\widetilde{w}_i$ of elements in $\widehat{\mathcal{M}}^2(\gamma, \gamma''; J)$ converging to the boundary of $\overline{\mathcal{M}}^2(\gamma, \gamma''; J)$ and arguing similarly as above, we have that this sequence converges to a pseudoholomorphic building $\widetilde{w}_\infty$ whose levels are somewhere injective pseudoholomorphic cylinders. Using that $I_F(\widetilde{w}_\infty) = 2$, we obtain that $\widetilde{w}_\infty$ can have at most 2 levels. As $\widetilde{w}_\infty$ is in the boundary of $\overline{\mathcal{M}}^2(\gamma, \gamma''; J)$, it cannot have only one level, and is therefore a two-level pseudoholomorphic building whose levels have Fredholm index 1. Summing up, $\widetilde{w}_\infty = (\widetilde{w}^1_\infty, \widetilde{w}^2_\infty)$, where $\widetilde{w}^1_\infty \in \widehat{\mathcal{M}}^1(\gamma, \gamma'; J)$ and $\widetilde{w}^2_\infty \in \widehat{\mathcal{M}}^1(\gamma', \gamma''; J)$, for some $\gamma' \in \mathcal{P}_\rho(\lambda)$.

The discussion above implies that $m_{\gamma,\gamma''}$ is the count with signs of boundary components of the compactified moduli space $\overline{\mathcal{M}}^2(\gamma,\gamma'';J)$ which is homeomorphic to a one-dimensional manifold with boundary. Because the signs of this count are determined by coherent orientations of $\overline{\mathcal{M}}^2(\gamma,\gamma'';J)$, it follows that $m_{\gamma,\gamma''}=0$. □

These claims give us the following:

Proposition 4 *Let (Y,ξ) be a contact manifold with a hypertight contact form λ. Let $\rho\in\Lambda$ be either an element of Λ_0 or a free homotopy class which contains only simple Reeb orbits of λ. Assume that all Reeb orbits in $\mathcal{P}_\rho(\lambda)$ are nondegenerate and pick $J\in\mathcal{J}^\rho_{\mathrm{reg}}(\lambda)$. Then d^ρ_J is well defined and $(d^\rho_J)^2=0$. Under these conditions we define $\mathrm{C}\mathbb{H}^\rho_{\mathrm{cyl}}(\lambda)$ as the homology of the pair $(\mathrm{CH}^\rho_{\mathrm{cyl}}(\lambda),d^\rho_J)$.*

Exact symplectic cobordisms induce homology maps for the SFT-invariants. We describe how this is done for the version of cylindrical contact homology considered in this section. Let (Y^+,ξ^+) and (Y^-,ξ^-) be contact manifolds, with hypertight contact forms λ^+ and λ^-. Let (W,ω) be an exact symplectic cobordism from λ^+ to λ^-. Assume that ρ is either a primitive free homotopy class or that all the closed Reeb orbits of both λ^+ and λ^- which belong to ρ are simple. Assume moreover that all Reeb orbits of both $\mathcal{P}_\rho(\lambda^+)$ and $\mathcal{P}_\rho(\lambda^-)$ are nondegenerate. Choose almost complex structures $J^+\in\mathcal{J}^\rho_{\mathrm{reg}}(\lambda^+)$ and $J^-\in\mathcal{J}^\rho_{\mathrm{reg}}(\lambda^-)$. From the work of Dragnev [13] (see also Section 2.3 in [37]) we know that there is a generic subset $\mathcal{J}^\rho_{\mathrm{reg}}(J^-,J^+)\in\mathcal{J}(J^-,J^+)$ such that for $\widehat{J}\in\mathcal{J}^\rho_{\mathrm{reg}}(J^-,J^+)$, $\gamma^+\in\mathcal{P}_\rho(\lambda^+)$ and $\gamma^-\in\mathcal{P}_\rho(\lambda^-)$, we have that

- all the curves $\widetilde{w}$ in the moduli spaces $\mathcal{M}(\gamma^+,\gamma^-;\widehat{J})$ are Fredholm regular,
- the connected components $\mathcal{V}$ of $\mathcal{M}(\gamma^+,\gamma^-;\widehat{J})$ have dimension equal to the Fredholm index of any pseudoholomorphic curve in $\mathcal{V}$.

In this case, we can define a map $\Phi^{\widehat{J}}\colon \mathrm{CH}^\rho_{\mathrm{cyl}}(\lambda^+)\to\mathrm{CH}^\rho_{\mathrm{cyl}}(\lambda^-)$, given on elements of $\mathcal{P}_\rho(\lambda^+)$, by

$$\Phi^{\widehat{J}}(\gamma^+)=\sum_{\gamma^-\in\mathcal{P}_\rho(\lambda^-)} n_{\gamma^+,\gamma^-}\gamma^-, \tag{3-21}$$

where n_{γ^+,γ^-} is the number of pseudoholomorphic cylinders with Fredholm index 0, positively asymptotic to γ^+ and negatively asymptotic to γ^-. Using a combination of compactness and gluing (see [6]) one proves that $\Phi^{\widehat{J}}\circ d^\rho_{J^+}=d^\rho_{J^-}\circ\Phi^{\widehat{J}}$. As a result we obtain a map $\Phi^{\widehat{J}}\colon \mathrm{C}\mathbb{H}^{\rho,J^+}_{\mathrm{cyl}}(\lambda^+)\to\mathrm{C}\mathbb{H}^{\rho,J^-}_{\mathrm{cyl}}(\lambda^-)$ on the homology level.

We study the cobordism map in the following situation: take $(V=\mathbb{R}\times Y,\varpi)$ to be an exact symplectic cobordism from $C\lambda$ to $c\lambda$, where $C>c>0$ and λ is a hypertight contact form. Suppose that one can make an isotopy of exact symplectic cobordisms $(\mathbb{R}\times Y, \varpi_t)$ from $C\lambda$ to $c\lambda$, with ϖ_t satisfying $\varpi_0 = \varpi$ and $\varpi_1 = d(e^s\lambda_0)$. We consider the space $\widetilde{\mathcal{J}}(J,J)$ of smooth homotopies

$$J_t \in \mathcal{J}(J,J), \quad t\in[0,1], \tag{3-22}$$

such that $J_0 = J_V$, $J_1 \in \mathcal{J}_{\mathrm{reg}}(\lambda)$, and J_t is compatible with ϖ_t for every $t\in[0,1]$. Here J_t is a deformation of J_0 to J_1 through asymptotically cylindrical almost complex structures in the cobordisms $(\mathbb{R}\times Y, \varpi_t)$. For Reeb orbits $\gamma,\gamma'\in\mathcal{P}_\rho(\lambda)$, we consider the moduli space

$$\widetilde{\mathcal{M}}^1(\gamma,\gamma';J_t) = \big\{(t,\widetilde{w}) \mid t\in[0,1] \text{ and } \widetilde{w}\in\widehat{\mathcal{M}}^1(\gamma,\gamma';J_t)\big\}. \tag{3-23}$$

By using the techniques of [13], we know that there is a generic subset $\widetilde{\mathcal{J}}_{\mathrm{reg}}(J,J)\subset\widetilde{\mathcal{J}}(J,J)$ such that $\widetilde{\mathcal{M}}^1(\gamma,\gamma';J_t)$ is a 1–dimensional smooth manifold with boundary. The crucial condition that makes this valid is again the fact that all the pseudoholomorphic curves that make part of this moduli space are somewhere injective.

The following proposition follows from combining the work of Eliashberg, Givental and Hofer [14] and Dragnev [13].

Proposition 5 *Let (Y,ξ) be a contact manifold with a hypertight contact form λ. Let $\lambda^+ = C\lambda$ and $\lambda^- = c\lambda$, where $C > c > 0$ are constants, and let ρ be either a primitive free homotopy class or a free homotopy class in which all Reeb orbits of λ are simple. Assume that all Reeb orbits in $\mathcal{P}_\rho(\lambda)$ are nondegenerate. Choose an almost complex structure $J\in\mathcal{J}^\rho_{\mathrm{reg}}(\lambda)$, and set $J^+ = J^- = J$. Let $(W=\mathbb{R}\times Y,\varpi)$ be an exact symplectic cobordism from $C\lambda$ to $c\lambda$, and choose a regular almost complex structure $\widehat{J}\in\mathcal{J}^\rho_{\mathrm{reg}}(J^-,J^+)$. Then, if there is a homotopy $(\mathbb{R}\times Y,\varpi_t)$ of exact symplectic cobordisms from $C\lambda$ to $c\lambda$ with $\varpi_0=\varpi$ and $\varpi_1 = d(e^s\lambda)$, it follows that the map*

$$\Phi^{\widehat{J}}\colon \mathrm{CH}^{\rho,J}_{\mathrm{cyl}}(\lambda)\to\mathrm{CH}^{\rho,J}_{\mathrm{cyl}}(\lambda)$$

is chain homotopic to the identity.

The proof is again a combination of compactness and gluing, and we sketch it below. We refer the reader to [6] and [14] for the details.

Sketch of the proof We initially define the map

$$K\colon \mathrm{CH}^\rho_{\mathrm{cyl}}(\lambda)\to\mathrm{CH}^\rho_{\mathrm{cyl}}(\lambda) \tag{3-24}$$

that counts finite energy Fredholm index-(-1) pseudoholomorphic cylinders in the cobordisms $(\mathbb{R}\times Y,\varpi_t)$ for $t\in[0,1]$. Because of the regularity of our homotopy, the moduli space of index -1 cylinders whose positive puncture detects a fixed Reeb orbit γ is finite, and therefore the map K is well defined.

Notice that for $t=1$, the cobordism map $\Phi^{\hat{J}_1}$ is the identity, and the pseudoholomorphic curves that define it are just trivial cylinders over Reeb orbits. For $t=0$, the map $\Phi^{\hat{J}_0}=\Phi^{\hat{J}}$ counts index-0 cylinders in the cobordism $(\mathbb{R}\times Y,\varpi)$. From the regularity of J_0, J_1 and the homotopy J_t, we have that the pseudoholomorphic cylinders involved in these two maps belong to the 1–dimensional moduli spaces $\widetilde{\mathcal{M}}^1(\gamma,\gamma';J_t)$.

By using a combination of compactness and gluing we can show that the boundary of the moduli space $\widetilde{\mathcal{M}}^1(\gamma,\gamma';J_t)$ is exactly the set of pseudoholomorphic buildings $\widetilde{w}$ with two levels, $\widetilde{w}_{\mathrm{cob}}$ and $\widetilde{w}_{\mathrm{symp}}$, such that $\widetilde{w}_{\mathrm{cob}}$ is an index-(-1) cylinder in a cobordism $(\mathbb{R}\times Y,\varpi_t)$, and $\widetilde{w}_{\mathrm{symp}}$ is an index-1 pseudoholomorphic cylinder in the symplectization of λ above or below $\widetilde{w}_{\mathrm{cob}}$. Such two-level buildings are exactly the ones counted in the map $K\circ d_J^{\mathrm{cyl}}+d_J^{\mathrm{cyl}}\circ K$. As a consequence, one has that the difference between the maps $\Phi^{\hat{J}_1}=\mathrm{Id}$ and $\Phi^{\hat{J}}$ is equal to $K\circ d_J^{\mathrm{cyl}}+d_J^{\mathrm{cyl}}\circ K$. This implies that $\Phi^{\hat{J}}$ is chain homotopic to the identity. □

The result above can be used to show that $\mathrm{C}\mathbb{H}^{\rho}_{\mathrm{cyl}}(\lambda)$ does not depend on the regular almost complex structure J used to define the differential d_J.

4 Exponential homotopical growth rate of cylindrical contact homology and estimates for h_{top}

In this section, we define the exponential homotopical growth of contact homology and relate it to the topological entropy of Reeb vector fields. The basic idea is to use the fact that the cylindrical contact homology of (M,ξ) in a free homotopy class is nonvanishing to obtain existence of Reeb orbits in such a homotopy class for any contact form on (M,ξ); this idea is present in [30; 37]. It is straightforward to see that the period and action of a Reeb orbit are equal, and in the sequel, we will use the same notation to refer to period and action of Reeb orbits.

Definition 6 Let (M,ξ) be a contact manifold and λ_0 a hypertight contact form on (M,ξ). We denote by $\Lambda(M)$ the set of free homotopy classes of loops in M. For $T>0$, we define the subset $\widetilde{\Lambda}_T(\lambda_0)\subset\Lambda(M)$ by the following condition:

- $\rho\in\widetilde{\Lambda}_T(\lambda_0)$ if and only if all Reeb orbits of X_{λ_0} in ρ are simply covered, nondegenerate, have action/period at most T, and $\mathrm{C}\mathbb{H}^{\rho}_{\mathrm{cyl}}(\lambda_0)\neq 0$.

We define $N_T^{\mathrm{cyl}}(\lambda_0):=\#\widetilde{\Lambda}_T(\lambda_0)$.

Definition 7 We say that the cylindrical contact homology of λ_0 has exponential homotopical growth with exponential weight $a > 0$ if there exist a number b and a sequence $T_n \to +\infty$ such that $N^{\mathrm{cyl}}_{T_n}(\lambda_0) \geq e^{aT_n+b}$ for all T_n.

Remark Notice that in Definition 7, we do not demand that λ_0 is nondegenerate. We only demand the weaker condition that the Reeb orbits of λ_0 belonging to some free homotopy classes are nondegenerate.

The main result of this section is the following:

Theorem 8 *Let λ_0 be a hypertight contact form on a contact manifold (M,ξ), and assume that the cylindrical contact homology of λ_0 has exponential homotopical growth with exponential weight $a > 0$. Then for every C^k $(k \geq 2)$ contact form λ on (M,ξ), the Reeb flow of X_λ has positive topological entropy. More precisely, if f_λ is the function such that $\lambda = f_\lambda \lambda_0$, then*

$$h_{\mathrm{top}}(\phi_{X_\lambda}) \geq \frac{a}{\max f_\lambda}. \tag{4-1}$$

Proof We write $E = \max f_\lambda$.

Step 1 We assume first that λ is nondegenerate and C^∞. For every $\epsilon > 0$ we can construct an exact symplectic cobordism from $(E+\epsilon)\lambda_0$ to λ. Analogously, for $\epsilon' > 0$ small enough, it is possible to construct an exact symplectic cobordism from λ to $\epsilon'\lambda_0$.

Using these cobordisms, we can construct a splitting family $(\mathbb{R}\times M, \varpi_R, J_R)$ from $(E+\epsilon)\lambda_0$ to $\epsilon'\lambda_0$, along λ, such that for every $R>0$, we have that $(\mathbb{R}\times M, \varpi_R, J_R)$ is homotopic to the symplectization of λ_0. For a fixed $\rho \in \widetilde{\Lambda}_T(\lambda_0)$, we pick a regular almost complex structure $J_0 \in \mathcal{J}^\rho_{\mathrm{reg}}(\lambda_0)$ and $J \in \mathcal{J}(\lambda)$, and demand that J_R coincides with J_0 in the positive and negative ends of the cobordism, and with J on $[-R,R]\times M$.

We claim that for every R, there exists a finite energy pseudoholomorphic cylinder $\widetilde{w}$ in $(\mathbb{R}\times M, J_R)$ that is positively asymptotic to a Reeb orbit in $\mathcal{P}_\rho(\lambda_0)$ and negatively asymptotic to an orbit in $\mathcal{P}_\rho(\lambda_0)$.

If this was not true for a certain $R>0$, then because of the absence of pseudo-holomorphic cylinders asymptotic to Reeb orbits in $\mathcal{P}_\rho(\lambda_0)$, we would have that $J_R \in \mathcal{J}^\rho_{\mathrm{reg}}(J_0, J_0)$. Therefore, the map $\Phi^{J_R}\colon \mathrm{C}\mathbb{H}^\rho_{\mathrm{cyl}}(\lambda_0) \to \mathrm{C}\mathbb{H}^\rho_{\mathrm{cyl}}(\lambda_0)$ induced by $(\mathbb{R}\times M, \varpi_R, J_R)$ is well-defined. But because there are no pseudoholomorphic cylinders asymptotic to Reeb orbits in $\mathcal{P}_\rho(\lambda_0)$, we have that Φ^{J_R} vanishes. On the other hand, from Proposition 5 in Section 3.2.3, we know that Φ^{J_R} is the identity. As Φ^{J_R} vanishes and is the identity, we conclude that $\mathrm{C}\mathbb{H}^\rho_{\mathrm{cyl}}(\lambda_0) = 0$, contradicting that $\rho \in \widetilde{\Lambda}_T(\lambda_0)$.

Step 2 Let $\rho \in \widetilde{\Lambda}_T(\lambda_0)$, let $R_n \to +\infty$ be a strictly increasing sequence, and let $\widetilde{w}_n\colon (S^1 \times \mathbb{R}, i) \to (\mathbb{R} \times M, J_{R_n})$ be a sequence of pseudoholomorphic cylinders with one positive puncture asymptotic to an orbit in $\mathcal{P}_\rho(\lambda_0)$ and one negative puncture asymptotic to an orbit in $\mathcal{P}_\rho(\lambda_0)$. Notice that because of the properties of ρ, the energy of $\widetilde{w}_n$ is uniformly bounded.

Therefore, we can apply the SFT compactness theorem to obtain a subsequence of $\widetilde{w}_n$ which converges to a pseudoholomorphic building $\widetilde{w}$. Notice that in order to apply the SFT compactness theorem, we need to use the nondegeneracy of λ. Moreover, we can give a very precise description of the building.

Let $\widetilde{w}^k$ for $k \in \{1, \dots, m\}$ be the levels of the pseudoholomorphic building $\widetilde{w}$. Because the topology of our curve $\widetilde{w}$ does not change after breaking, we have the following picture:

- The upper level $\widetilde{w}^1$ is composed of one connected pseudoholomorphic curve, which has one positive puncture asymptotic to an orbit $\gamma_0 \in \mathcal{P}_\rho(\lambda_0)$, and several negative punctures. All of the negative punctures detect contractible orbits, except one that detects a Reeb orbit γ_1 which is also in ρ.
- On every other level $\widetilde{w}^k$, there is a special pseudoholomorphic curve which has one positive puncture asymptotic to a Reeb orbit γ_{k-1} in ρ, and at least one, but possibly several, negative punctures. Of the negative punctures, there is one that is asymptotic to an orbit γ_k in ρ, while all the others detect contractible Reeb orbits.

Because of the splitting behaviour of the cobordisms $(\mathbb{R} \times M, J_{R_n})$, it is clear that there exists a k_0 such that the level $\widetilde{w}^{k_0}$ is in an exact symplectic cobordism from $(E+\epsilon)\lambda_0$ to λ. This implies that the special orbit γ_{k_0} is a Reeb orbit of X_λ in the homotopy class ρ.

Notice that $A(\gamma_0) \leq (E+\epsilon)T$. This implies that all the other orbits appearing as punctures of the building $\widetilde{w}$ have action smaller than $(E+\epsilon)T$ and, in particular, that γ_{k_0} has action smaller than $(E+\epsilon)T$.

As we can do the construction above for any $\epsilon > 0$, we can obtain a sequence of Reeb orbits γ_j^ρ which are all in ρ such that $A(\gamma_j^\rho) \leq (E + 1/j)T$. Using the Arzela–Ascoli theorem, one can extract a convergent subsequence of γ_j^ρ. Its limit γ_ρ is clearly a Reeb orbit of λ in the free homotopy class ρ, with action at most ET.

Step 3 (estimating $N_{X_\lambda}(T)$)[3] From step 2, we know that if $\rho \in \widetilde{\Lambda}_T(\lambda_0)$, then there is a Reeb orbit γ_ρ of the Reeb flow of X_λ with $A(\gamma_\rho) \leq ET$. Recalling that the

[3]Recall from Section 2 that $N_{X_\lambda}(T)$ is the number of free homotopy classes of M that contain periodic orbits of X_λ with period at most T.

period and the action of a Reeb orbit coincide, we obtain that $N_{X_\lambda}(T) \geq \#\widetilde{\Lambda}_{T/E}(\lambda_0)$. Under the hypothesis of the theorem, there exists a sequence $T_n \to +\infty$ such that $\#\widetilde{\Lambda}_{T/E}(\lambda_0) \geq e^{aT_n/E+b}$ for all T_n. We then conclude that

$$N_{X_\lambda}(T_n) \geq e^{aT_n/E+b} \tag{4-2}$$

for all elements of the sequence T_n. Applying Theorem 1, we obtain $h_{\text{top}}(\phi_{X_\lambda}) \geq a/E$. This proves the theorem in the case that λ is C^∞ and nondegenerate.

Step 4 Here we pass to the case of a general $C^{k\geq 2}$ contact form λ (the case where λ is degenerate is included here).

Let λ_i be a sequence of nondegenerate smooth contact forms converging in the C^k–topology to a contact form λ which is C^k $(k \geq 2)$ and possibly degenerate. For every $\epsilon > 0$, there is i_0 such that for $i > i_0$, there exists an exact symplectic cobordism from $(E+\epsilon)\lambda_0$ to λ_i.

Fixing a homotopy class $\rho \in \widetilde{\Lambda}_T(\lambda_0)$, we know, by the previous steps, that there exists a Reeb orbit $\gamma_\rho(i)$ of λ_i in the homotopy class ρ with action smaller than $(E+\epsilon)T$. By applying the Arzela–Ascoli theorem to $\gamma_\rho(i)$, we obtain a subsequence which converges to a Reeb orbit $\gamma_{\epsilon,\rho}$ of X_λ with $A(\gamma_{\epsilon,\rho}) \leq (E+\epsilon)T$. Notice that here we use that λ is at least C^2 (so that X_λ is at least C^1) in order to be able to use the Arzela–Ascoli theorem.

Because $\epsilon > 0$ above can be taken arbitrarily close to 0, we can actually obtain a sequence $\gamma_{j,\rho}$ of Reeb orbits of X_λ, whose homotopy class is ρ, such that the actions $A(\gamma_{j,\rho})$ converges to a number at most ET. Again applying the Arzela–Ascoli theorem, we obtain that the sequence $\gamma_{j,\rho}$ has a convergent subsequence which converges to an orbit γ_ρ satisfying $A(\gamma_\rho) \leq ET$.

Reasoning as in step 3 above, we conclude that $N_{X_\lambda}(T_n) \geq e^{aT_n/E+b}$ for all elements of the sequence $T_n \to +\infty$. Applying Theorem 1, we obtain the desired estimate for the topological entropy. This finishes the proof of the theorem. □

5 Contact 3–manifolds with a hyperbolic component

In this section, we prove the following theorem:

Theorem 9 *Let M be a closed connected oriented 3–manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0 \leq i \leq q\}$ of irreducible manifolds with boundary, such that*

- *M_0 is the mapping torus of a diffeomorphism $h\colon S \to S$ with pseudo-Anosov monodromy on a surface S with nonempty boundary.*

Then M can be given infinitely many nondiffeomorphic contact structures ξ_k such that for each ξ_k, there exists a hypertight contact form λ_k on (M, ξ_k) which has exponential homotopical growth of cylindrical contact homology. It follows that on each (M, ξ_k), all Reeb flows have positive topological entropy.

We denote by S a compact surface with nonempty boundary and by ω a symplectic form on S. Let h be a symplectomorphism of (S, ω) to itself, with pseudo-Anosov monodromy and which is the identity on a neighbourhood of ∂S. We follow a well-known recipe to construct a suitable contact form on the mapping torus $\Sigma(S, h)$.

We choose a primitive β for ω such that, for coordinates $(r, \theta) \in [-\epsilon, 0] \times S^1$ in a neighbourhood V of ∂S, we have $\beta = f(r)d\theta$, where $f > 0$ and $f' > 0$. We pick a smooth nondecreasing function $F_0\colon \mathbb{R} \to [0, 1]$ which satisfies $F_0(t) = 0$ for $t \in \left(-\infty, \frac{1}{100}\right)$ and $F_0(t) = 1$ for $t \in \left(\frac{1}{100}, +\infty\right)$. For $i \in \mathbb{Z}$, define $F_i(t) = F_0(t - i)$. Fixing $\epsilon > 0$, we define a 1–form $\widetilde{\alpha}$ on $\mathbb{R} \times S$ by

$$\widetilde{\alpha} = dt + \epsilon(1 - F_i(t))(h^i)^*\beta + \epsilon F_i(t)(h^{i+1})^*\beta \quad \text{for } t \in [i, i+1). \tag{5-1}$$

This defines a smooth 1–form on $\mathbb{R} \times S$, and a simple computation shows that if ϵ is small enough, the 1–form $\widetilde{\alpha}$ is a contact form. For $t \in [0, 1]$, the Reeb vector field $X_{\widetilde{\alpha}}$ is equal to $\partial_t + v(p, t)$, where $v(p, t)$ is the unique vector tangent to S that satisfies $\omega(v(p, t), \cdot) = F_0'(t)(\beta - h^*\beta)$.

Consider the diffeomorphism $H\colon \mathbb{R} \times S \to \mathbb{R} \times S$ defined by $H(t, p) = (t - 1, h(p))$. The mapping torus $\Sigma(S, h)$ is defined by

$$\Sigma(S, h) := (\mathbb{R} \times S)/(t, p) \sim H(t, p), \tag{5-2}$$

and we denote by $\pi\colon \mathbb{R} \times S \to \Sigma(S, h)$ the associated covering map.

Because $H^*\widetilde{\alpha} = \widetilde{\alpha}$, there exists a unique contact form α on $\Sigma(S, h)$ such that $\pi^*\alpha = \widetilde{\alpha}$. Notice that in the neighbourhood $S^1 \times V$ of $\partial\Sigma(S, h)$, we have $\alpha = dt + \epsilon f(r)d\theta$, which implies that X_α is tangent to $\partial\Sigma(S, h)$.

The Reeb vector field X_α on $\Sigma(S, h)$ is transverse to the surfaces $\{t\} \times S$ for $t \in \mathbb{R}/\mathbb{Z}$. This implies that $\{0\} \times S$ is a global surface of section for the Reeb flow of α, and by our expression of $X_{\widetilde{\alpha}}$, the first return map of the Reeb flow of α is isotopic to h.

It follows from [1, Theorem 13] that we can make a perturbation of α supported in the interior $\breve{\Sigma}(S, h)$ of $\Sigma(S, h)$ to obtain a contact form $\widehat{\alpha}$ on $\Sigma(S, h)$ (whose kernel coincides with that of α) satisfying that all Reeb orbits of $\widehat{\alpha}$ which are contained in $\breve{\Sigma}(S, h)$ are nondegenerate. By doing the perturbation small enough, we can also guarantee that $\{0\} \times S$ is still a global surface of section for the flow of $X_{\widehat{\alpha}}$. Since the perturbation is supported in the interior of $\Sigma(S, h)$, the Reeb flow of $\widehat{\alpha}$ is also tangent to the boundary of $\Sigma(S,h)$. It is clear that the first return map $\widehat{h}\colon \{0\}\times S \to \{0\}\times S$ of $\phi_{X_{\widehat{\alpha}}}$ is a diffeomorphism isotopic to h.

5.1 Contact 3–manifolds containing $(\Sigma(S,h),\widehat{\alpha})$ as a component

Let M be a closed connected oriented 3–manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0\le i\le q\}$ of irreducible manifolds with boundary, such that the component M_0 is diffeomorphic to $\Sigma(S,h)$. Then it is possible to construct hypertight contact forms on M which match with $\widehat{\alpha}$ in the component M_0. More precisely, we have the following result due to Colin and Honda, and Vaugon:

Proposition 10 [12; 40] *Let M be a closed connected oriented 3–manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0\le i\le q\}$ of irreducible manifolds with boundary, such that the component M_0 is diffeomorphic to $\Sigma(S,h)$. Then, there exists an infinite family $\{\xi_k, k\in\mathbb{Z}\}$ of nondiffeomorphic contact structures on M such that*

- *for each $k\in\mathbb{Z}$, there exists a hypertight contact form λ_k on (M,ξ_k) which coincides with $\widehat{\alpha}$ on the component M_0.*

We briefly recall the construction of the contact forms λ_k and refer the reader to [12; 40] for the details. For $i\ge 1$, we apply [12, Theorem 1.3] to obtain a hypertight contact form α_i on M_i which is compatible with the orientation of M_i, and whose Reeb vector field X_{α_i} is tangent to the boundary of M_i. On the special piece M_0, we consider the contact form α_0 equal to $\widehat{\alpha}$ constructed above.

Let $\{\mathfrak{T}_j \mid 1\le j\le m\}$ be the family of incompressible tori along which we cut M to obtain the pieces M_i. Then the contact forms α_i give a hypertight contact form on each component of $M\setminus\bigcup_{j\ge 1}^m \mathbb{V}(\mathfrak{T}_j)$, where $\mathbb{V}(\mathfrak{T}_j)$ is a small open neighbourhood of $\mathfrak{T}_j$. This gives a contact form $\widehat{\lambda}$ on $M\setminus\bigcup_{j\ge 1}^m \mathbb{V}(\mathfrak{T}_j)$. Using an interpolation process (see [40, Section 7]), one can construct contact forms on the neighbourhoods $\overline{\mathbb{V}(\mathfrak{T}_j)}$ which coincide with $\widehat{\lambda}$ on $\partial\overline{\mathbb{V}(\mathfrak{T}_j)}$. The interpolation process is not unique and can be done in ways so as to produce an infinite family of distinct contact forms $\{\lambda_k \mid k\in\mathbb{Z}\}$ on M that extend $\widehat{\lambda}$, and which are associated to contact structures $\xi_k := \ker\lambda_k$ that are all nondiffeomorphic. The contact topological invariant used to distinguish the contact structures ξ_k is the *Giroux torsion*; see [40, Section 7].

5.2 Proof of Theorem 9

It is clear that Theorem 9 will follow if we establish that the cylindrical contact homology of λ_k has exponential homotopical growth. This is the content of the following:

Proposition 11 *The cylindrical contact homology of λ_k has exponential homotopical growth.*

Before proving the proposition, we introduce some necessary ideas and notation. The first return map of $X_{\widehat{\alpha}}$ is a diffeomorphism $\widehat{h}\colon S \to S$ which is homotopic to h and, therefore, to a pseudo-Anosov map $\psi\colon S \to S$. The Reeb orbits of $X_{\widehat{\alpha}}$ are in one-to-one correspondence with periodic orbits of $\widehat{h}$. Moreover, we have that two Reeb orbits γ_1 and γ_2 of $X_{\widehat{\alpha}}$ are freely homotopic if and only if their associated periodic orbits are in the same Nielsen class. Thus there is an injective map Ξ from the set $\mathcal{N}$ of Nielsen classes to the set $\Lambda(\Sigma(S,h))$ of free homotopy classes of Reeb orbits in $\Sigma(S,h)$.

We now recall some facts about Nielsen theory for pseudo-Anosov maps in surfaces with boundary, which the reader can find in [11; 15; 16]. Let P_n be the set of periodic orbits of ψ with period n which are contained in the interior of S. Because pseudo-Anosov maps have Markov partitions [15; 16], we know that there exist numbers $a > 0$ and b such that

$$\#P_n > e^{an+b}$$

for every $n \in \mathbb{N}$. It follows from [11, Lemma 1.1] that all periodic orbits in P_n belong to distinct Nielsen classes, and that these Nielsen classes are unrelated to the boundary of S. By this, we mean that for every periodic orbit in P_n, its suspension is a curve in $\Sigma(S,h)$ which cannot be homotoped to a curve completely contained in the boundary of $\Sigma(S,h)$.

We denote by $\mathcal{N}_n$ the set of Nielsen classes associated to the periodic orbits P_n of ψ. Notice that $\mathcal{N}$ equals the disjoint union $\bigcup_{n\in\mathbb{N}} \mathcal{N}_n$. It follows from the discussion above that

$$\#\mathcal{N}_n > e^{an+b}$$

for all $n \in \mathbb{N}$. It is immediate to see that the fixed points of $\widehat{h}$ belong to a finite number of Nielsen classes, and we denote by c the number of elements in $\mathcal{N}_1$. We write $\mathcal{N}_1 = \{\nu_1, \dots, \nu_c\}$. For each $\nu_i \in \mathcal{N}_1$ we will denote by ν_i^n the Nielsen class in $\mathcal{N}_n$ which n–covers ν_i in the following sense: if x_i is a fixed point in ν_i, then ν_i^n is the Nielsen class that contains the periodic orbit of period n that "covers" x_i.

As observed previously, there exists an injective map $\Xi\colon \mathcal{N} \to \Lambda(\Sigma(S,h))$. Let p be a prime number, and let $\rho \in \Xi(\mathcal{N}_p)$. Then there are two possibilities:

(a) ρ contains only simple Reeb orbits,

(b) ρ contains a Reeb orbit γ which is a p–cover of a simple orbit γ_0 that intersects $\{0\} \times S$ once.

The reason why these are the only two possibilities is that every Reeb orbit $\gamma \in \rho$ intersects $\{0\} \times S$ exactly p times. If γ is a multiple cover of a simple orbit γ_0, then the number of intersections of γ_0 with $\{0\} \times S$ must be a divisor of p. As p is prime, this number is either p, which implies that γ is simple, or 1. It is clear that if $\rho \in \Xi(\mathcal{N}_p)$ satisfies (b), then $\rho = \Xi(\nu_i^p)$ for some $\nu_i \in \mathcal{N}_1$. We denote by $\mathcal{N}_p^{\text{simp}}$ the set $\mathcal{N}_p \setminus \{\nu_1^p, \dots, \nu_i^p, \dots, \nu_q^p\}$. As a consequence we conclude that if $\Lambda_{\text{simp}}^p := \Xi(\mathcal{N}_p^{\text{simp}})$ is the set of elements in $\Xi(\mathcal{N}_p)$ satisfying (a), then

$$\#\Lambda_{\text{simp}}^p = \#\mathcal{N}_p - c$$

for every prime number p. Since $\#\mathcal{N}_p > e^{ap+b}$ for every prime p, we conclude that there exists a prime number p_0 such that for every prime $p \geq p_0$,

$$\#\Lambda_{\text{simp}}^p \geq e^{ap+q}.$$

Let $\mathfrak{x}$ be a periodic orbit of $\widehat{h}$ of period n. Viewing $\widehat{h}$ as the first return map for a global surface of section of the Reeb flow $\phi_{X_{\widehat{\alpha}}}$ we know that there is a Reeb orbit $\gamma_{\mathfrak{x}}$ of $\widehat{\alpha}$ (and also of λ_k) which is the suspension of $\mathfrak{x}$. Because of the compactness of S, we know that there exists a number $\eta > 0$, depending only on $\widehat{h}$ and $\widehat{\alpha}$, such that $A(\gamma_{\mathfrak{x}}) \leq \eta n$.

We are now ready for the proof of Proposition 11. The main ideas of the argument are due to Vaugon, who estimated in [40] a different growth rate of the cylindrical contact homology λ_k.

Proof of Proposition 11 **Step 1** Let $i\colon \Sigma(S,h) \to M$ be the injection we obtain from viewing $\Sigma(S,h)$ as a component of M. Because of the incompressibility of $\partial\Sigma(S,h)$ in M, the associated map $i_*\colon \Lambda(\Sigma(S,h)) \to \Lambda(M)$ is injective (here $\Lambda(M)$ denotes the free loop space of M).

For each prime number p, we define $T_p := \eta p$. Recall that if $\rho \in \Lambda_{\text{simp}}^p$, then ρ does not contain curves completely contained in the boundary of $\Sigma(S,h)$. It follows from this and from the incompressibility of $\partial\Sigma(S,h)$ in M, that if $\varrho \in i_*(\Lambda_{\text{simp}}^p)$, then every loop in ϱ must intersect the interior $\Sigma(S,h)$.

Using that the Reeb flow of λ_k is tangent to $\partial\Sigma(S,h)$, it follows that if $\varrho \in i_*(\Lambda_{\text{simp}}^p)$, then all Reeb orbits of $\phi_{X_{\lambda_k}}$ that belong to ϱ are contained in the interior of $\Sigma(S,h)$. This implies that ϱ contains only nondegenerate[4] Reeb orbits of $\phi_{X_{\lambda_k}}$. Combining this with the injectivity of i_* and Ξ, we conclude that every Reeb orbit λ_k in ϱ is the suspension of a periodic orbit of $\widehat{h}$ in the Nielsen class $\nu := (i_* \circ \Xi)^{-1}\varrho \in \mathcal{N}_p^{\text{simp}}$. This implies that

[4]Recall that because of our choice of $\widehat{\alpha}$, Reeb orbits contained in $\operatorname{int}(\Sigma(S,h))$ are nondegenerate.

(c) all Reeb orbits of λ_k in the free homotopy class ϱ are nondegenerate and simple,

(d) all Reeb orbits of λ_k in the free homotopy class ϱ have action $\leq T_p$.

Hypertightness of λ_k and (c) imply that if $\varrho \in i_*(\Lambda^p_{\text{simp}})$, then $\mathrm{C}\mathbb{H}^{\varrho}_{\text{cyl}}(\lambda_k)$ is well defined.

Step 2 For every $\varrho \in i_*(\Lambda^p_{\text{simp}})$, we have $\mathrm{C}\mathbb{H}^{\varrho}_{\text{cyl}}(\lambda_k) \neq 0$. Indeed, Vaugon showed (see the proofs of Lemma 7.11 and Theorems 1.3 and 1.2 in [40]) that the number of Reeb orbits in ϱ of even and odd degree differ. For Euler characteristic reasons, this implies that $\mathrm{C}\mathbb{H}^{\varrho}_{\text{cyl}}(\lambda_k) \neq 0$. Combining this with (d) from step 1, we conclude that every $\varrho \in i_*(\Lambda^p_{\text{simp}})$ belongs to the set $\widetilde{\Lambda}_{T_p}(\lambda_k)$ as defined in Definition 6.

Step 3 Recall that in Definition 6 of Section 4, we defined $N^{\text{cyl}}_T(\lambda_k)$ as the cardinality of $\#\widetilde{\Lambda}_T(\lambda_k)$. That is, $N^{\text{cyl}}_T(\lambda_k)$ is the number of free homotopy classes ϱ in $\Lambda(M)$ which contain only nondegenerate simple Reeb orbits with action smaller than T and that satisfy $\mathrm{C}\mathbb{H}^{\varrho}_{\text{cyl}}(\lambda_k) \neq 0$.

Because of the injectivity of i_*, we know that $\#i_*(\Lambda^p_{\text{simp}}) = \#\Lambda^p_{\text{simp}}$. Combining this with steps 1 and 2, it follows that for every element of the sequence $T_p \to +\infty$,

$$N^{\text{cyl}}_{T_p}(\lambda_k) \geq \#i_*(\Lambda^p_{\text{simp}}) = \#\Lambda^p_{\text{simp}} \geq e^{aT_p/\eta+b}, \tag{5-3}$$

which establishes the proposition. □

Proof of Theorem 9 As mentioned previously, Theorem 9 follows directly from combining Proposition 11, Proposition 10 and Theorem 8. □

It would be interesting to obtain an upper bound on the constant η above. This could provide a more precise estimate for the homotopical growth rate of the cylindrical contact homology of λ_k.

6 Graph manifolds and Handel–Thurston surgery

In [25], Handel and Thurston used Dehn surgery to obtain nonalgebraic Anosov flows in 3–manifolds. Their surgery was adapted to the contact setting by Foulon and Hasselblatt in [18], who interpreted it as a Legendrian surgery and used it to produce nonalgebraic Anosov Reeb flows on 3–manifolds. In this section, we apply the Foulon–Hasselblatt Legendrian surgery to obtain more examples of contact 3–manifolds which are distinct from unit tangent bundles, and on which every Reeb flow has positive topological entropy.

Some clarifications regarding the surgeries we consider are in order. On one hand, we restrict our attention to the Foulon–Hasselblatt surgery on Legendrian lifts of embedded separating geodesics on hyperbolic surfaces. This is an important restriction, since Foulon and Hasselblatt perform their surgery on the Legendrian lift of any immersed closed geodesic on a hyperbolic surface. On the other hand, for this restricted class of Legendrian knots, the surgery we consider is a bit more general than the one in [18]. They restrict their attention to Dehn surgeries with positive integer coefficients, while we consider the case of any integer coefficient, as is explained in Section 6.1.

6.1 The surgery

We start by fixing some notation. Let (S,g) be an oriented hyperbolic surface and $\mathfrak{r}\colon S^1\to S$ an embedded oriented separating geodesic of g. We let $\pi\colon(\mathbb{D},g)\to(S,g)$ denote a locally isometric covering of (S,g) by the hyperbolic disc $(\mathbb{D},g)$ with the property that $(-1,1)\times\{0\}\subset\pi^{-1}(\mathfrak{r}(S^1))$. Such a covering always exists since the segment $(-1,1)\times\{0\}$ of the real axis is a geodesic in $(\mathbb{D},g)$. We denote by $v(\theta)$ the unique unitary vector field along $\mathfrak{r}(\theta)$ satisfying $\angle(\mathfrak{r}'(\theta),v(\theta))=-\pi/2$. Our orientation convention is chosen so that for coordinates $z=x+iy\in\mathbb{D}$, the lift of $v(\theta)$ to $(-1,1)\times\{0\}$ is a positive multiple of the vector field $-\partial_y$ along $(-1,1)\times\{0\}$. Also, let $\Pi\colon T_1S\to S$ denote the base point projection.

Because $\mathfrak{r}$ is a separating geodesic, we can cut S along $\mathfrak{r}$ to obtain two oriented hyperbolic surfaces with boundary which we denote by S_1 and S_2. Our labelling is chosen so that the vector field $v(\theta)$ points into S_2 and out of S_1. This decomposition of S induces a decomposition of T_1S into T_1S_1 and T_1S_2. Both T_1S_1 and T_1S_2 are 3–manifolds whose boundary is the torus formed by the unit fibres over $\mathfrak{r}$.

Denote by $V_{\mathfrak{r},\delta}$ the closed $\delta-$neighbourhood of the geodesic $\mathfrak{r}$ for the hyperbolic metric g. For $\delta>0$ sufficiently small, we have that $V_{\mathfrak{r},\delta}$ is an annulus such that the only closed geodesics contained in $V_{\mathfrak{r},\delta}$ are the covers of $\mathfrak{r}$, and such that $V_{\mathfrak{r},\delta}$ satisfies the following convexity property: if $\check{V}$ is the connected component of $\pi^{-1}(V_{\mathfrak{r},\delta})$ containing $(-1,1)\times\{0\}$, then every segment of a hyperbolic geodesic starting and ending in $\check{V}$ is completely contained in $\check{V}$. It also follows from the conventions adopted above that, if we denote by U^+ the upper hemisphere of $\mathbb{D}$ composed of points with positive imaginary part and by U^- the lower hemisphere of the $\mathbb{D}$ composed of points with negative imaginary part, we have

$$\check{V}\cap U^+\subset\pi^{-1}(S_1)\quad\text{and}\quad\check{V}\cap U^-\subset\pi^{-1}(S_2).\tag{6-1}$$

This fact has the following important consequence: if $\nu([0,K])$ is a hyperbolic geodesic segment starting and ending at $V_{\mathfrak{r},\delta}$ and contained in one of the S_i, then $[\nu]$ is a nontrivial homotopy class in the relative fundamental group $\pi_1(S_i,V_{\mathfrak{r},\delta})$.

On the unit tangent bundle T_1S, we consider the contact form λ_g whose Reeb vector field is the geodesic vector field for the hyperbolic metric g. It is well known that the lifted curve $L_{\mathfrak{r}}(\theta) = (\mathfrak{r}(\theta), v(\theta))$ in T_1S is Legendrian on the contact manifold $(T_1S, \ker \lambda_g)$. The geodesic vector field X_{λ_g} along $L_{\mathfrak{r}}$ coincides with the horizontal lift of v (see [38, Section 1.3]), points into T_1S_2 and out of T_1S_1, and is normal to $\partial T_1S_2 = \partial T_1S_1$ for the Sasaki metric on T_1S.

Moreover, if $\delta > 0$ is small enough, we know that for every $\vartheta \in L_{\mathfrak{r}}$, there exist numbers $t_1 < 0$ and $t_2 > 0$ such that

$$\phi^{t_1}_{\lambda_g}(\vartheta) \in T_1S_1 \setminus \Pi^{-1}(V_{\mathfrak{r},\delta}), \tag{6-2}$$

$$\phi^{t_2}_{\lambda_g}(\vartheta) \in T_1S_2 \setminus \Pi^{-1}(V_{\mathfrak{r},\delta}). \tag{6-3}$$

Following [18], we know that there exists a neighbourhood $B^{3\eta}_{2\epsilon}$ of $L_{\mathfrak{r}}$ on which we can find coordinates $(t, s, w) \in (-3\eta, 3\eta) \times S^1 \times (-2\epsilon, 2\epsilon)$ such that

$$\lambda_g = dt + w\,ds, \tag{6-4}$$

$$L_{\mathfrak{r}} = \{0\} \times S^1 \times \{0\}, \tag{6-5}$$

where $\{0\} \times \{\vartheta\} \times (-2\epsilon, 2\epsilon)$ is a local parametrization of the unitary fibre over $\vartheta \in L_{\mathfrak{r}}$, and $\epsilon < \eta/(4|q|\pi)$, with q being a fixed integer. Let $\mathcal{W}^- = \{-3\eta\} \times S^1 \times (-2\epsilon, 2\epsilon)$ and $\mathcal{W}^+ = \{+3\eta\} \times S^1 \times (-2\epsilon, 2\epsilon)$. It is clear that $\Pi(\mathcal{W}^-) \subset S_1$ and $\Pi(\mathcal{W}^+) \subset S_2$. Because on $\bar{B}^{3\eta}_{2\epsilon}$, the Reeb vector field X_{λ_g} is given by ∂_t, it is clear that for every point $p \in B^{3\eta}_{2\epsilon}$, there are $p^- \in \mathcal{W}^-$, $p^+ \in \mathcal{W}^+$, $t^- \in (-6\eta, 0)$ and $t^+ \in (0, 6\eta)$ for which

$$\phi^{t^-}_{X_{\lambda_g}}(p) = p^- \quad \text{and} \quad \phi^{t^+}_{X_{\lambda_g}}(p) = p^+. \tag{6-6}$$

This means that trajectories of the flow of X_{λ_g} that enter the box $B^{3\eta}_{2\epsilon}$ enter through $\mathcal{W}^-$ and exit through $\mathcal{W}^+$. They cannot stay inside $B^{3\eta}_{2\epsilon}$ for a very long positive or negative interval of time. We can say even more about these trajectories.

For $\sigma = (\mathfrak{p}, \dot{\mathfrak{p}}) \in S \times T_pS$ in $\mathcal{W}^+ \cup \mathcal{W}^-$ let $\tilde{\sigma} = (\tilde{\mathfrak{p}}, \dot{\tilde{\mathfrak{p}}})$ be a lift of σ to the unit tangent bundle $T_1\mathbb{D}$ such that $\tilde{\mathfrak{p}} \in \breve{V}$. The geodesic vector field X_{λ_g} at $\tilde{\sigma}$ coincides with the horizontal lift of $\dot{\mathfrak{p}}$ [38, Section 1.3]. For δ, $\eta > 0$ and $\epsilon < \eta/(4|q|\pi)$ sufficiently small, we can guarantee that

- $\Pi(B^{3\eta}_{2\epsilon})$ is contained in $V_{\mathfrak{r},\delta}$,
- for the lifts $\tilde{\sigma} = (\tilde{\mathfrak{p}}, \dot{\tilde{\mathfrak{p}}})$ of points in $\mathcal{W}^+ \cup \mathcal{W}^-$ as above, the vector $\dot{\tilde{\mathfrak{p}}}$ (which is the projection of the geodesic vector field $X_{\lambda_g}(\tilde{\sigma})$) satisfies $\angle(\dot{\tilde{\mathfrak{p}}}, -\partial_y) < \delta$.

With such a choice of $\delta>0$, $\eta>0$ and $0<\epsilon<\eta/(4|q|\pi)$, we obtain that for every $\sigma^+\in\mathcal{W}^+$ there exists $t_{\sigma^+}>0$, and for every $\sigma^-\in\mathcal{W}^-$ there exists $t_{\sigma^-}<0$, such that

$$\phi^{t_{\sigma^+}}_{X_{\lambda_g}}(\sigma^+)\in(T_1S_2)\setminus\Pi^{-1}(V_{\mathfrak{r},\delta})\quad\text{and}\quad\forall t\in[0,t_{\sigma^+}],\ \phi^{t}_{X_{\lambda_g}}(\sigma^+)\notin B^{3\eta}_{2\epsilon},\tag{6-7}$$

$$\phi^{t_{\sigma^-}}_{X_{\lambda_g}}(\sigma^-)\in(T_1S_1)\setminus\Pi^{-1}(V_{\mathfrak{r},\delta})\quad\text{and}\quad\forall t\in[t_{\sigma^-},0],\ \phi^{t}_{X_{\lambda_g}}(\sigma^+)\notin B^{3\eta}_{2\epsilon}.\tag{6-8}$$

To prove the last condition above one uses the fact that $\angle(\dot{\tilde{\mathfrak{p}}},-\partial_y)<\delta$ is small and studies the behaviour of geodesics in $(\mathbb{D},g)$ starting at points close to the real axis and with initial velocity close to $-\partial_y$. It is easy to see that such geodesics have to intersect the region $V_{\mathfrak{r},\delta}$ and visit the interior of both $S_1\setminus V_{\mathfrak{r},\delta}$ and $S_2\setminus V_{\mathfrak{r},\delta}$. From now on we will assume that $\delta>0$, $\eta>0$ and $0<\epsilon<\eta/(8|q|\pi)$ are such that all the properties described above hold simultaneously.

Consider the map $F\colon B^{2\eta}_{2\epsilon}\setminus\bar{B}^{\eta}_{\epsilon}\to B^{2\eta}_{2\epsilon}\setminus\bar{B}^{\eta}_{\epsilon}$ defined by

$$F(t,s,w)=(t,s+f(w),w)\quad\text{for}\quad(t,s,w)\in(\eta,2\eta)\times S^1\times(-2\epsilon,2\epsilon),\tag{6-9}$$

where $f(w)=-q\mathcal{R}(w/\epsilon)$ (for our previously chosen integer q) and $\mathcal{R}\colon[-1,1]\to[0,2\pi]$ satisfies $\mathcal{R}=0$ on a neighbourhood of -1, $\mathcal{R}=2\pi$ on a neighbourhood of 1, $0\le\mathcal{R}'\le4$ and $\mathcal{R}'$ is an even function.

Our new 3–manifold M is obtained by gluing $T_1S\setminus\bar{B}^{\eta}_{\epsilon}$ and $B^{2\eta}_{2\epsilon}$ using the map F:

$$M=(T_1S\setminus\bar{B}^{\eta}_{\epsilon})\cup B^{2\eta}_{2\epsilon}\big/(x\in B^{2\eta}_{2\epsilon}\setminus\bar{B}^{\eta}_{\epsilon})\sim(F(x)\in T_1S\setminus\bar{B}^{\eta}_{\epsilon}).\tag{6-10}$$

Notice that

$$T_1S=(T_1S\setminus\bar{B}^{\eta}_{\epsilon})\cup B^{2\eta}_{2\epsilon}\big/(x\in B^{2\eta}_{2\epsilon}\setminus\bar{B}^{\eta}_{\epsilon})\sim(x\in T_1S\setminus\bar{B}^{\eta}_{\epsilon}).$$

This clarifies our construction of M and shows that M is obtained from T_1S via a Dehn surgery on $L_{\mathfrak{r}}$. We follow [18] to endow M with a contact form which coincides with λ_g outside $B^{2\eta}_{2\epsilon}$. As a preparation, we define the function $\beta\colon(-3\eta,3\eta)\to\mathbb{R}$:

- β is equal to 1 on an open neighbourhood of $[-2\eta,2\eta]$,
- $|\beta'|\le\pi/\eta$ and $\operatorname{supp}\beta$ is contained in $[-3\eta,3\eta]$.

Using β we define

$$r(t,w)=\frac{\beta(t)}{2}\int_{-2\epsilon}^{w}xf'(x)\,dx.\tag{6-11}$$

We point out that $\operatorname{supp}(r)$ is contained in $B^{3\eta}_{\epsilon}$, and therefore, so is $\operatorname{supp}(dr)$. Notice also that in $B^{2\eta}_{2\epsilon}\setminus\bar{B}^{\eta}_{\epsilon}$, one has $dr=\frac{1}{2}wf'(w)dw$.

Again following [18], we define in $T_1S \setminus \overline{B}^{\eta}_{\epsilon}$ the 1–form A_r:

$$A_r = dt + w\,ds + dr \quad \text{for } t \in (-3\eta, -\eta), \tag{6-12}$$

$$A_r = dt + w\,ds - dr \quad \text{for } t \in (\eta, 3\eta), \tag{6-13}$$

$$A_r = \lambda_g \quad \text{otherwise.} \tag{6-14}$$

Notice that because $\operatorname{supp}(dr)$ is contained in $B^{3\eta}_{\epsilon}$, the 1–form A_r is well defined.

On the box $B^{2\eta}_{2\epsilon}$, we define

$$\widetilde{A} = dt + w\,ds + dr. \tag{6-15}$$

A direct computation shows that $F^*(A_r) = \widetilde{A}$, which means that the gluing map F allows us to glue the 1–forms A_r and $\widetilde{A}$. We denote by λ_{FH} the 1–form on M obtained by gluing $\widetilde{A}$ and A_r. We will denote by $\widetilde{B}$ the following region:

$$\widetilde{B} = ((B^{3\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \subset M) \cup B^{2\eta}_{2\epsilon} \big/ (x \in B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \sim (F(x) \in (B^{3\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon})). \tag{6-16}$$

The importance of this region lies in the fact that in $M \setminus \widetilde{B} = T_1S \setminus B^{3\eta}_{2\epsilon}$, the contact form λ_{FH} coincides with λ_g.

Following [18], one shows by a direct computation that $(dt + w\,ds \pm dr) \wedge (dw \wedge ds) = (1 \pm \partial r/\partial t) dt \wedge dw \wedge ds$. Using the fact that $\epsilon < \eta/(8\pi|q|)$, one gets that $|\partial r/\partial t| < 1$, thus obtaining that $(dt + w\,ds \pm dr)$ is a contact form. It follows from this that A_r and $\widetilde{A}$ are contact forms in their respective domains, and therefore, λ_{FH} is a contact form on M. More strongly, Foulon and Hasselblatt proceed to show that if q is nonnegative, the Reeb flow of λ_{FH} is Anosov.

6.2 Hypertightness and exponential homotopical growth of contact homology of λ_{FH}

For $q \in \mathbb{N}$, the hypertightness of λ_{FH} follows from the fact that its Reeb flow is Anosov [17]. In this subsection, we give an independent and completely geometrical proof of the hypertightness of λ_{FH}, which is valid for every $q \in \mathbb{Z}$.

To understand the topology of Reeb orbits of λ_{FH}, we will study trajectories that enter the surgery region $\widetilde{B}$. We start by studying trajectories in $B^{2\eta}_{2\epsilon}$. In this region, we have

$$X_{\lambda_{\mathrm{FH}}} = \frac{\partial_t}{1 + \partial_t r}. \tag{6-17}$$

This implies, similarly to what happens for λ_g, that for points $p \in B^{2\eta}_{2\epsilon}$, the trajectory $\phi^t_{X_{\lambda_{\mathrm{FH}}}}(p)$ leaves the box $B^{2\eta}_{2\epsilon}$ in forward and backward times. More precisely, there exists a constant $\widetilde{a} > 0$, depending only on λ_{FH}, such that for $p \in B^{2\eta}_{2\epsilon}$, there are $\breve{p}^- \in \breve{\mathcal{W}}^- = \{-2\eta\} \times S^1 \times [-2\epsilon, 2\epsilon]$, $\breve{p}^+ \in \breve{\mathcal{W}}^+ = \{+2\eta\} \times S^1 \times [-2\epsilon, 2\epsilon]$, $\breve{t}^- \in (-\widetilde{a}, 0]$ and $\breve{t}^+ \in [0, \widetilde{a})$ such that

$$\begin{gathered} \phi^t_{X_{\lambda_{\mathrm{FH}}}}(\breve{p}) \text{ is in the interior of } B^{2\eta}_{2\epsilon} \text{ for every } t \in (t^-, t^+), \\ \phi^{t^-}_{X_{\lambda_{\mathrm{FH}}}}(\breve{p}) = \breve{p}^- \quad \text{and} \quad \phi^{t^+}_{X_{\lambda_{\mathrm{FH}}}}(\breve{p}) = \breve{p}^+. \end{gathered} \tag{6-18}$$

We now analyse the trajectories of points $\breve{p}^- \in \breve{\mathcal{W}}^-$ and $\breve{p}^+ \in \breve{\mathcal{W}}^+$. For this, we first notice that on $\widetilde{B} \setminus B^{\eta}_{\epsilon}$, the contact form λ_{FH} is given by $dt + w\,ds \pm dr$, and therefore, we have in this region

$$X_{\lambda_{\mathrm{FH}}} = \frac{\partial_t}{1 \pm \partial_t r}, \tag{6-19}$$

which is still a positive multiple of ∂_t.

This implies that for every $\breve{p}^- \in \breve{\mathcal{W}}^-$ and $\breve{p}^+ \in \breve{\mathcal{W}}^+$, there exist $t^{\breve{p}^-} < 0$ and $t^{\breve{p}^+} > 0$ such that

$$\phi^{t^{\breve{p}^-}}_{X_{\lambda_{\mathrm{FH}}}}(\breve{p}^-) \in \mathcal{W}^- \quad \text{and} \quad \phi^{t^{\breve{p}^+}}_{X_{\lambda_{\mathrm{FH}}}}(\breve{p}^+) \in \mathcal{W}^+. \tag{6-20}$$

Again using that $X_{\lambda_{\mathrm{FH}}}$ is a positive multiple of ∂_t on $\widetilde{B} \setminus B^{2\eta}_{2\epsilon}$, we have that for every point p in $\widetilde{B} \setminus B^{2\eta}_{2\epsilon}$ whose t coordinate is in $[2\eta, 3\eta]$, the trajectory of the flow $\phi^t_{X_{\lambda_{\mathrm{FH}}}}$ going through p is a straight line, with fixed coordinates s and w, that goes from $\breve{\mathcal{W}}^+$ to $\mathcal{W}^+$. Analogously, for every point p in $\widetilde{B} \setminus B^{2\eta}_{2\epsilon}$ whose t coordinate is in $[-3\eta, -2\eta]$, the trajectory of the backward flow of $\phi^t_{X_{\lambda_{\mathrm{FH}}}}$ going through p is a straight line from $\breve{\mathcal{W}}^-$ to $\mathcal{W}^-$.

Summing up, with all the cases considered above, we have showed that for every point $p \in \widetilde{B}$, the trajectory of the flow $\phi^t_{X_{\lambda_{\mathrm{FH}}}}$ going through p for $t = 0$ intersects $\mathcal{W}^-$ for nonpositive time and $\mathcal{W}^+$ for nonnegative time. In other words, all trajectories that intersect $\widetilde{B}$ enter through $\mathcal{W}^-$ and leave through $\mathcal{W}^+$, which means that for all $\breve{p} \in \widetilde{B}$, there exist times $\breve{t}^- \leq 0$ and $\breve{t}^+ \geq 0$ such that

$$\phi^{\breve{t}^+}_{X_{\lambda_{\mathrm{FH}}}}(\breve{p}) \in \mathcal{W}^+, \tag{6-21}$$

$$\phi^{\breve{t}^-}_{X_{\lambda_{\mathrm{FH}}}}(\breve{p}) \in \mathcal{W}^-, \tag{6-22}$$

$$\phi^{t}_{X_{\lambda_{\mathrm{FH}}}}(\breve{p}) \in \widetilde{B} \quad \text{for all } t \in [\breve{t}^-, \breve{t}^+]. \tag{6-23}$$

Now, because on $M \setminus \widetilde{B} = T_1 S \setminus B^{3\eta}_{2\epsilon}$, the contact form λ_{FH} coincides with λ_g, we have that trajectories of $X_{\lambda_{\mathrm{FH}}}$ starting at $\mathcal{W}^-$ at time $t = 0$ have to leave $M \setminus N$ (with N defined as in (6-26) below) as time diminishes before reentering on $\widetilde{B}$. Similarly, the trajectories starting at $\mathcal{W}^+$ have to leave $M \setminus N$ for positive time before reentering to $\widetilde{B}$. More precisely, one can use (6-7) and (6-8) to show that for $p^- \in \mathcal{W}^-$ and $p^+ \in \mathcal{W}^+$, there exist $t_{p^-} < 0$ and $t_{p^+} > 0$ such that

$$\phi^{t_{p^+}}_{X_{\lambda_{\mathrm{FH}}}}(p^+) \in M_2 \setminus N \quad \text{and} \quad \forall t \in [0, t_{p^+}],\ \phi^t_{X_{\lambda_{\mathrm{FH}}}}(p^+) \notin \widetilde{B}, \tag{6-24}$$

$$\phi^{t_{p^-}}_{X_{\lambda_{\mathrm{FH}}}}(p^-) \in M_1 \setminus N \quad \text{and} \quad \forall t \in [t_{p^-}, 0],\ \phi^t_{X_{\lambda_{\mathrm{FH}}}}(p^-) \notin \widetilde{B}, \tag{6-25}$$

where, denoting

$$B^{2\eta}_{2\epsilon}(-) = [-2\eta, 0] \times S^1 \times (-2\epsilon, 2\epsilon) \quad \text{and} \quad B^{2\eta}_{2\epsilon}(+) = [0, 2\eta] \times S^1 \times (-2\epsilon, 2\epsilon),$$

the submanifolds M_1, M_2 and N of M are defined as follows:

$$M_1 = {(T_1S_1 \setminus B^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon}(-)} \big/ {\big(x \in B^{2\eta}_{2\epsilon}(-) \setminus \overline{B}^{\eta}_{\epsilon}\big) \sim \big(F(x) \in ((B^{2\eta}_{2\epsilon} \cap T_1S_1) \setminus \overline{B}^{\eta}_{\epsilon})\big)},$$

$$M_2 = {(T_1S_2 \setminus B^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon}(+)} \big/ {\big(x \in B^{2\eta}_{2\epsilon}(+) \setminus \overline{B}^{\eta}_{\epsilon}\big) \sim \big(F(x) \in ((B^{2\eta}_{2\epsilon} \cap T_1S_2) \setminus \overline{B}^{\eta}_{\epsilon})\big)},$$

and

$$N = {(\Pi^{-1}(V_{\mathfrak{r},\delta}) \setminus B^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon}(-)} \big/ {x \sim F(x)}, \tag{6-26}$$

with $x \in B^{2\eta}_{2\epsilon}(-) \setminus \overline{B}^{\eta}_{\epsilon}$ and $F(x) \in ((B^{2\eta}_{2\epsilon} \cap T_1S_1) \setminus \overline{B}^{\eta}_{\epsilon})$.

Remark It is not hard to see that

$$M = {M_1 \cup M_2} \big/ {(x \in \partial M_1) \sim (\widetilde{F}(x) \in \partial M_2)}.$$

Here $\widetilde{F}$ is a Dehn twist which coincides with $(s + f(w), w)$ for $w \in [-2\epsilon, 2\epsilon]$ and is the identity elsewhere. This picture of M is closer to the one in the paper [25] and shows that M is a graph manifold (a graph manifold is one whose JSJ decomposition consists of Seifert S^1 bundles). By using this description of M and applying van Kampen's theorem to analyse the fundamental group of M, Handel and Thurston show that, for q not belonging to a finite subset of $\mathbb{Z}$, no finite cover of M is a Seifert manifold, thus obtaining that M is an "exotic" graph manifold.

From their definition, one sees that as manifolds, $M_1 \cong T_1S_1$ and $M_2 \cong T_1S_2$. This implies that ∂M_1 and ∂M_2 are incompressible tori in M_1 and M_2, respectively. If we look at M_1 and M_2 as submanifolds of M, their boundary $\mathbb{T}$ coincides and is also incompressible in M. We remark that $M_i \setminus N$ is diffeomorphic to $T_1S_i \setminus \Pi^{-1}(V_{c,\delta})$, which is diffeomorphic to T_1S_i for $i = 1, 2$.

In a similar way, we can describe the topology of N. Let $N_i = M_i \cap N$. Reasoning identically as one does to show that M_i is diffeomorphic to T_1S_i, one shows that N_i

is diffeomorphic to a thickened two torus $\mathcal{T}^2 \times [-1,1]$. As N is obtained from N_1 and N_2 by gluing them along $\mathbb{T}$ (which is a boundary component of both of them), we have that N is also diffeomorphic to the product $\mathcal{T}^2 \times [-1,1]$.

The discussion above proves the following:

Lemma 12 *For all $\check{p} \in \widetilde{B}$, the trajectory $\{\phi^t_{X_{\lambda_{\mathrm{FH}}}}(\check{p}) \mid t \in \mathbb{R}\}$ intersects $M_1 \setminus N$ and $M_2 \setminus N$.*

Proof We have already established that for $\check{p} \in \widetilde{B}$, its trajectory intersects $\mathcal{W}^+$ for some nonnegative time and $\mathcal{W}^-$ for some nonpositive time, as shown in (6-21) and (6-22). One now applies (6-24) and (6-25) to finish the proof of the lemma. □

Notice that trajectories can only enter in $\widetilde{B}$ through the wall $\mathcal{W}^-$, which is contained in M_1, and can only exit $\widetilde{B}$ through the wall $\mathcal{W}^+$, which is contained in M_2. We also point out that all trajectories of the flow $\phi^t_{X_{\lambda_{\mathrm{FH}}}}$ are transversal to $\mathbb{T}$, with the exception of the two Reeb orbits which correspond to parametrizations of the hyperbolic geodesic $\mathfrak{r}$ (they continue to exist as periodic orbits after the surgery because they are out of the surgery region).

We will deduce, from the previous discussion, the following important lemma.

Lemma 13 *Let $\gamma([0,T'])$ be a trajectory of $X_{\lambda_{\mathrm{FH}}}$ such that $\gamma(0), \gamma(T') \in \mathbb{T}$ and for all $t \in (0,T')$ we have $\gamma(t) \notin \mathbb{T}$ (notice that in such a situation, $\gamma([0,T']) \subset M_i$ for $i=1$ or $i=2$). Then $\gamma([0,T']) \cap (M_i \setminus N)$ is nonempty.*

Proof We divide the proof in 3 possible scenarios.

Case 1 Suppose that $\gamma([0,T']) \cap \widetilde{B}$ is empty. In this case, $\gamma([0,T'])$ is a hyperbolic geodesic with endpoints on the closed geodesic $\mathfrak{r}$. It follows from the convexity of the hyperbolic metric that $[\gamma([0,T'])] \in \pi_1(T_1S_i, \mathbb{T})$ is nontrivial. This implies that $[\gamma([0,T'])] \in \pi_1(M_i, \mathbb{T})$ is nontrivial, which can be true only if $\gamma([0,T']) \cap (M_i \setminus N)$ is nonempty since N is a tubular neighbourhood of $\mathbb{T}$.

Case 2 Suppose that $\gamma([0,T']) \cap \widetilde{B}$ is nonempty and $\gamma([0,T']) \subset M_2$. Take $\hat{t} \in [0,T']$ such that $\gamma(\hat{t}) \in \widetilde{B}$. We know from our previous discussion that there are $\hat{t}_1 \le \hat{t} \le \hat{t}_2$ such that $\gamma([\hat{t}_1,\hat{t}_2]) \subset \widetilde{B}$, $\gamma(\hat{t}_1) \in (\mathbb{T} \cap \widetilde{B})$ and $\gamma(\hat{t}_2) \in \mathcal{W}^+$; notice that in the coordinates (t,s,w) for $\widetilde{B}$ considered previously, $\mathbb{T} \cap \widetilde{B}$ is the annulus $\{0\} \times S^1 \times (-2\epsilon, 2\epsilon)$. From this picture, it is clear that for t smaller that $\hat{t}_1$, the trajectory enters M_1. Therefore, we must have $\hat{t}_1 = 0$ and $\gamma([0,\hat{t}_2]) \subset \widetilde{B}$. Notice also that for all t slightly bigger than $\hat{t}_2$, the trajectory is outside $\widetilde{B}$. Because trajectories of $X_{\lambda_{\mathrm{FH}}}$ can only enter $\widetilde{B}$

in M_1, we obtain that $\gamma([\hat{t}_2, T'])$ does not intersect the interior of $\widetilde{B}$ and, therefore, is a hyperbolic geodesic in $T_1 S_2$. Now, using (6-7) and (6-8), we obtain that, because $\gamma(\hat{t}_2) \in \mathcal{W}^+$, the trajectory $\gamma\colon [\hat{t}_2, T'] \to M_2$ has to intersect $M_2 \setminus N$ before hitting $\mathbb{T}$ at $t = T'$. Thus there is some $t \in (\hat{t}_2, T')$ for which $\gamma(t) \in M_2 \setminus N$.

Case 3 The proof in the case where $\gamma([0, T']) \cap \widetilde{B}$ is nonempty and $\gamma([0, T']) \subset M_1$ is analogous to the one of case 2.

These three cases exhaust all possibilities and, therefore, prove the lemma. □

Our reason for introducing the above decomposition of M into M_1 and M_2, and for proving the lemmas above, is to introduce the following representation of Reeb orbits of λ_{FH}. Let (γ, T) be a Reeb orbit of λ_{FH} which intersects both $M_1 \setminus N$ and $M_2 \setminus N$. We can assume that the chosen parametrization of γ is such that $\gamma(0) \in \partial N$, and that there are $t_+ > 0$ and $t_- < 0$ such that

$$\gamma(t_+) \in M_1 \setminus N \quad \text{and} \quad \gamma([0, t_+]) \in M_1 \cup N, \tag{6-27}$$

$$\gamma(t_-) \in M_2 \setminus N \quad \text{and} \quad \gamma([t_-, 0]) \in M_2 \cup N. \tag{6-28}$$

This means that in an interval of the origin, γ is coming from $M_2 \setminus N$ and going to $M_1 \setminus N$. It follows from Lemma 13 that there exists a unique sequence $0 = t_0 < t_{1/2} < t_1 < t_{3/2} < \cdots < t_n = T$ such that for all $k \in \{0, \dots, n-1\}$,

- $\gamma([t_k, t_{k+(1/2)}]) \subset M_i$ for $i = 1$ or $i = 2$,
- $\gamma([t_{k+(1/2)}, t_{k+1}]) \in N$ and there is a unique $\widetilde{t}_k \in [t_{k+(1/2)}, t_{k+1}]$ such that $\gamma(\widetilde{t}_k) \in \mathbb{T}$,
- if $\gamma([t_k, t_{k+(1/2)}]) \subset M_i$, then $\gamma([t_{k+1}, t_{k+(3/2)}]) \subset M_j$ for $j \neq i$.

Notice that $\gamma([t_0, t_{1/2}]) \subset M_1$ and $\gamma([t_{n-1}, t_{n-(1/2)}]) \subset M_2$. This implies that n is even, so we can write $n = 2n'$, and that $\gamma([t_k, t_{k+(1/2)}]) \subset M_1$ for k even and $\gamma([t_k, t_{k+(1/2)}]) \subset M_2$ for k odd. For each $k \in \{0, \dots, 2n'-1\}$, the existence of the unique $\widetilde{t}_k$ in the interval $[t_{k+(1/2)}, t_{k+1}]$ for which $\gamma(\widetilde{t}_k) \in \mathbb{T}$ is guaranteed by Lemma 13 and the fact that $\mathbb{T}$ is the hypersurface that separates M_1 and M_2.

In order to obtain information on the free homotopy class of (γ, T), we observe that $\gamma([t_k, t_{k+(1/2)}])$ coincides with a hyperbolic geodesic segment in $T_1 S_i$ starting and ending in $V_{\tau,\delta}$. Therefore, as we have previously seen, the homotopy class $[\gamma([t_k, t_{k+(1/2)}])]$ in $\pi_1(T_1 S_i, V_{\tau,\delta})$ is nontrivial, which implies that $\gamma([t_k, t_{k+(1/2)}])$ is a nontrivial relative homotopy class in $\pi_1(M_i, N)$. We consider now the curve $\gamma([\widetilde{t}_k, \widetilde{t}_{k+1}])$: it is the concatenation of 3 curves, the first and the third ones being completely contained in N and the middle one being $\gamma([t_k, t_{k+(1/2)}])$. From this description and the fact that $\gamma([t_k, t_{k+(1/2)}])$ is a nontrivial relative homotopy class in $\pi_1(M_i, N)$ it is clear that $\gamma([\tilde{t}_k, \tilde{t}_{k+1}])$ is also nontrivial in $\pi_1(M_i, N)$ (and also nontrivial in $\pi_1(M_i, \mathbb{T})$).

We now denote by $\widetilde{M}$ the universal cover of M and $\hat{\pi}\colon \widetilde{M} \to M$ the covering map. From the incompressibility of $\mathbb{T}$, it follows that every lift of $\mathbb{T}$ is an embedded plane in $\widetilde{M}$. We denote by $\widetilde{N}^0$ a lift of N. Because N is a thickened neighbourhood of an incompressible torus, it follows that $\widetilde{N}^0$ is diffeomorphic to $\mathbb{R}^2 \times [-1, 1]$, ie it is a thickened neighbourhood of an embedded plane in $\widetilde{M}$. Because N separates M into two components, it follows that $\widetilde{N}^0$ separates $\widetilde{M}$ into two connected components. Now, $\partial \widetilde{N}^0$ is the union of two embedded planes, P^0_- and P^0_+, which are characterized by the fact that there are neighbourhoods V_- and V_+ of P^0_- and P^0_+, respectively, such that $\hat{\pi}(V_-) \subset M_1$ and $\hat{\pi}(V_+) \subset M_2$. We will denote by C^0_- the connected component of $\widetilde{M} \setminus \widetilde{N}^0$ which intersects V_-, and by C^0_+ the connected component of $\widetilde{M} \setminus \widetilde{N}^0$ which intersects V_+.

As seen earlier, $[\gamma([t_k, t_{k+(1/2)}])]$ is a nontrivial relative homotopy class in $\pi_1(M_i, N)$. We show that this class remains nontrivial when seen in $\pi_1(M, N)$. Let $\mathbb{T}_i = \partial N \cap M_i$. Because N is obtained by attaching over each point of $\mathbb{T}_i$ a small compact interval (ie it is a bundle over $\mathbb{T}_i$ whose fibres are intervals), it follows that $[\gamma([t_k, t_{k+(1/2)}])]$ is trivial in $\pi_1(M_i, \mathbb{T}_i)$ if and only if it is trivial in $\pi_1(M_i, N)$, which is not the case. As $\mathbb{T}_i$ is isotopic to $\mathbb{T}$, it is also an incompressible torus that divides M into two components. Now, $[\gamma([t_k, t_{k+(1/2)}])]$ is trivial in $\pi_1((M_i \setminus \operatorname{int}(N)), \mathbb{T}_i)$ if and only if there exists a curve $\mathfrak{c}$ in $\mathbb{T}_i$, with endpoints $\gamma(t_k)$ and $\gamma(t_{k+(1/2)})$, such that the concatenation $\gamma * \mathfrak{c}$ is contractible in $M_i \setminus \operatorname{int}(N)$. Because of the incompressibility of $\mathbb{T}_i$, such a curve $\gamma * \mathfrak{c}$ is contractible in $M_i \setminus \operatorname{int}(N)$ if and only if it is contractible in M. This implies that $[\gamma([t_k, t_{k+(1/2)}])]$ is trivial in $\pi_1(M, \mathbb{T}_i)$ if and only if it is trivial in $\pi_1((M_i \setminus \operatorname{int}(N)), \mathbb{T}_i)$, which we know not to be the case. Lastly, again because N is an interval bundle over $\mathbb{T}_i$, it is clear that as $[\gamma([t_k, t_{k+(1/2)}])]$ is not trivial in $\pi_1(M, \mathbb{T}_i)$, it cannot be trivial in $\pi_1(M, N)$, as we wished to show.

Let $\tilde{\gamma}$ be a lift of γ such that $\tilde{\gamma}(0) \in \widetilde{N}^0$. We know that $\tilde{\gamma}([t_{2n'-(1/2)} - T, t_{1/2}]) \subset \widetilde{N}^0$. It will be useful to define the sequence

$$\tilde{t}_i = q_i T + t_{r_i}, \tag{6-29}$$

where q_i and $r_i < 2n'$ are the unique integers such that $i = q_i(2n') + r_i$. To $\tilde{t}_i$ we associate the lift $\widetilde{N}^i$ of N which is determined by the property that $\tilde{\gamma}(\tilde{t}_i) \in \widetilde{N}^i$. It is clear that the sequence $\widetilde{N}^i$ contains all lifts of N which are intersected by the curve $\tilde{\gamma}(\mathbb{R})$. For the lifts $\widetilde{N}^i$, we define the connected components C^i_- and C^i_+ of $\widetilde{M} \setminus \widetilde{N}^i$, and the planes P^i_- and P^i_+ in the same way as for $\widetilde{N}^0$. A priori it could be that, for $i \neq j$, we have $\widetilde{N}^i = \widetilde{N}^j$. We will show that this cannot happen.

Firstly, $\widetilde{N}^0 \neq \widetilde{N}^1$ because $\gamma([\widetilde{t}_0, \widetilde{t}_1])$ is nontrivial in $\pi_1(M, N)$. Also, we have that $\widetilde{N}^1 \subset C^0_-$ because $\gamma([t_0, t_{1/2}]) \subset M_1$. The same reasoning shows that $\widetilde{N}^2 \neq \widetilde{N}^1$ and

$$\widetilde{N}^2 \subset C^1_+. \tag{6-30}$$

On the other hand, we have that $\widetilde{N}^0 \subset C^1_-$, because $\widetilde{\gamma}([\widetilde{t}_0, t_{1/2}])$ is a path totally contained in $\widetilde{M} \setminus \widetilde{N}^1$ connecting $\widetilde{N}^0$ and P^1_-. As $\widetilde{N}^2 \subset C^1_+$ and $\widetilde{N}^0 \subset C^1_-$, we must have $\widetilde{N}^2 \neq \widetilde{N}^0$. In the same way, one shows that $\widetilde{N}^3 \neq \widetilde{N}^1$ and, more generally, that $\widetilde{N}^{i+2} \neq \widetilde{N}^i$ and $\widetilde{N}^{i+1} \neq \widetilde{N}^i$. Now for $\widetilde{N}^3$, we have that $\widetilde{N}^3 \subset C^2_-$. As $\widetilde{\gamma}([\widetilde{t}_0, t_{3/2}])$ is a path completely contained in $\widetilde{M} \setminus \widetilde{N}^2$ connecting $\widetilde{N}^0$ and P^2_+, we obtain that $\widetilde{N}^0 \subset C^2_+$ and, therefore, $\widetilde{N}^3 \neq \widetilde{N}^0$.

Proceeding inductively along this line, one obtains that $\widetilde{N}^i \neq \widetilde{N}^0$ for all $i \neq 0$ and, more generally, $\widetilde{N}^i \neq \widetilde{N}^j$ for all $i \neq j$. As a consequence, we obtain that the curve $\widetilde{\gamma}(\mathbb{R})$ cannot be homeomorphic to a circle, and therefore, $\gamma(\mathbb{R})$ cannot be contractible. We are ready for the main result of this subsection.

Proposition 14 λ_{FH} *is hypertight.*

Proof There are two possibilities for Reeb orbits.

Possibility 1 The Reeb orbit γ visits both $M_1 \setminus N$ and $M_2 \setminus N$. In this case, we have just showed that γ is not contractible.

Possibility 2 The Reeb orbit γ is completely contained in M_i for $i = 1$ or $i = 2$. In this case, γ does not visit the surgery region $\widetilde{B}$. Therefore, it also existed before the surgery as a closed hyperbolic geodesic in $M_i \setminus \widetilde{B} = T_1 S_i \setminus B^{3\eta}_{2\epsilon}$. Such a closed geodesic is noncontractible in $T_1 S_i$, which is diffeomorphic to M_i. Thus $\gamma \subset M_i$ is noncontractible in M_i.

Now looking at M_i as a submanifold with boundary of M, we recall that ∂M_i is an incompressible torus in M. This implies that every noncontractible closed curve in M_i remains noncontractible in M. Therefore, γ is also a noncontractible Reeb orbit in this case. □

6.2.1 Exponential homotopical growth of cylindrical contact homology for λ_{FH}

We now obtain more information on the properties of periodic orbits of $X_{\lambda_{\mathrm{FH}}}$.

Lemma 15 *If a Reeb orbit (γ, T) of λ_f visits both $M_1 \setminus N$ and $M_2 \setminus N$, then any curve freely homotopic to (γ, T) must intersect $\mathbb{T}$.*

Proof As we saw earlier, the lift $\widetilde{\gamma}$ intersects all the elements of the sequence $\widetilde{N}_i$ (of lifts of N), which satisfy $\widetilde{N}_i \neq \widetilde{N}_j$ for all $i \neq j$.

Introducing an auxiliary distance d on the compact manifold M (coming from a Riemannian metric), we obtain an auxiliary distance $\widetilde{d}$ on $\widetilde{M}$ by pulling d back by the covering map. It is clear that for i sufficiently large, the $\widetilde{d}$–distance between $\widetilde{N}_{\pm i}$ and $\widetilde{N}_0$ becomes arbitrarily large. As a consequence, one obtains that for each $K > 0$, there exists $t_K > 0$ such that $\widetilde{d}(\widetilde{\gamma}(\pm t_K), \widetilde{N}_0) > K$.

Now let $\zeta\colon [0,T] \to M$ be a closed curve freely homotopic to $\gamma([0,T])$. A homotopy $H\colon [0,T] \times [0,1] \to M$ generates a homotopy $\widetilde{H}\colon \mathbb{R} \times [0,1] \to \widetilde{M}$ from a lift $\widetilde{\gamma}$ to a lift $\widetilde{\zeta}$. Using the fact that H is uniformly continuous, one sees that there exists a constant $\mathfrak{C} > 0$ such that $\widetilde{d}(\widetilde{H}(\{t\} \times [0,1]), \widetilde{\gamma}(t)) < \mathfrak{C}$ for all $t \in \mathbb{R}$.

Now take $K > 2\mathfrak{C}$. Using the inequalities

$$\widetilde{d}((\widetilde{H}(\{t\} \times [0,1])), \widetilde{\gamma}(t)) < \mathfrak{C}, \quad \widetilde{d}(\widetilde{\gamma}(\pm t_K), \widetilde{N}_0) > K,$$

and the triangle inequality, we obtain that $H(\{t_K\} \times [0,1])$ is always in the connected component of $\widetilde{\gamma}(t_K)$. This implies that $\widetilde{\zeta}(\mathbb{R})$ visits both connected components of $\widetilde{M} \setminus \widetilde{N}_0$ and must thus intersect $\widetilde{N}_0$. Even more, because $\widetilde{\zeta}(\mathbb{R})$ intersects both components of $\partial \widetilde{N}_0$, we have that ζ visits both components of $M \setminus N$ and, therefore, has to intersect $\mathbb{T}$. This completes the proof of the lemma. □

We are now ready for the main result of this section:

Theorem 16 *Let $(M, \xi_{(q,\mathfrak{r})})$ be the contact manifold obtained from performing the Foulon–Hasselblat q–surgery on the Legendrian curve $L_{\mathfrak{r}} \subset (T_1 S, \xi_{\text{geo}})$, and denote by λ_{FH} the contact form on $(M, \xi_{(q,\mathfrak{r})})$ obtained from this surgery. Then λ_{FH} is hypertight, and its cylindrical contact homology has exponential homotopical growth. It follows that every Reeb flow on $(M, \xi_{(q,\mathfrak{r})})$ has positive topological entropy.*

Proof It suffices to show that the cylindrical contact homology of λ_{FH} has exponential homotopical growth, since this combined with Theorem 1 establishes the last assertion of the theorem.

Step 1 (a special class of Reeb orbits) We will obtain our estimate by looking at Reeb orbits which are completely contained in the component M_1. As we saw previously, such orbits never cross the surgery region $\widetilde{B}$. Thus they are in a region where λ_{FH} coincides with λ_g, and such Reeb orbits are closed geodesics in (S_1, g). Conversely, every closed geodesic in (S_1, g) does not cross the region $B^{3\eta}_{2\epsilon}$ and thus is a Reeb orbit of λ_{FH}. This gives a bijective correspondence between closed geodesics of (S_1, g) which are not homotopic to a multiple of ∂S_1 and Reeb orbits of λ_{FH} which are contained in M_1.

Let $\Lambda(S_1)$ denote the set of free homotopy classes of curves in S_1 which are not covers of $[\partial S_1]$. We know that each $\rho \in \Lambda(S_1)$ contains exactly one closed geodesic $\mathfrak{c}_\rho$. The canonical lift γ_ρ of $\mathfrak{c}_\rho$ to $T_1 S_1$ is a Reeb orbit of λ_g. As we saw above, each γ_ρ can also be seen as a Reeb orbit of λ_{FH}. Because of the negative curvature of g we know that the geodesic $\mathfrak{c}_\rho$ is hyperbolic. This implies that γ_ρ is a nondegenerate Reeb orbit of λ_g, and as λ_{FH} coincides with λ_g on a neighbourhood of γ_ρ, we conclude that γ_ρ is also nondegenerate when viewed as a Reeb orbit of λ_{FH}.

We will denote by $\Lambda(S_1)^{\leq T}$ the set of primitive of free homotopy classes in $\Lambda(S_1)$ whose unique closed geodesic has period smaller or equal to T. Because g is hyperbolic, it is a well known fact that there exist constants $a > 0$ and b such that $\#(\Lambda(S_1)^{\leq T}) \geq e^{aT+b}$. The map $\Theta\colon \Lambda(S_1) \to \Lambda(T_1 S_1)$ (where $\Lambda(T_1 S_1)$ is the free loop space of $T_1 S_1$) associating with $\mathfrak{c}_\rho$ the Reeb orbit γ_ρ in $T_1 S_1$ is easily seen to be injective. Because $T_1 S_1$ is diffeomorphic to M_1, we can also view $\Theta(\Lambda(S_1))$ as a subset of the free loop space $\Lambda(M_1)$ of M_1.

Step 2 Let $i\colon M_1 \to M$ be the injection. As seen before, the boundary $\partial(i(M_1)) = \mathbb{T}$ is an incompressible torus in M. We consider the induced map of free loop spaces $i_*\colon \Lambda(M_1) \to \Lambda(M)$. As a consequence of the incompressibility of $\partial(i(M_1))$, the restriction of i_* to $\Theta(\Lambda(S_1))$ is injective.

To see this, it suffices to show the following claim: if ζ and ζ' are curves in M_1 which cannot be isotoped to a curve in ∂M_1 and which are in the same free homotopy class in M, then ζ and ζ' are freely homotopic in M_1. For ζ and ζ' satisfying the hypothesis of our claim, there is a cylinder cyl in M, whose boundary components are ζ and ζ', which intersects ∂M_1 transversely. Then cyl intersects ∂M_1 in a finite collection of curves $\{w_n\}$ which are all contractible in M; the contractibility of these curves is due to the fact that both ζ and ζ' cannot be isotoped to a curve contained in ∂M_1. The incompressibility of ∂M_1 implies that these $\{w_n\}$ are all contractible in ∂M_1. Now we cut the discs in cyl whose boundary are the curves w_n and substitute them by discs contained in ∂M_1. This produces a cylinder cyl$'$ completely contained in M_1 whose boundaries are ζ and ζ'. This implies that ζ and ζ' are already in the same free homotopy class in M_1, as we wished to show.

From step 1, we know that for each $\rho \in i_*(\Theta(\Lambda(S_1)))$, there is a Reeb orbit γ_ρ in ρ.

Step 3 We will show that for each $\rho \in i_*(\Theta(\Lambda(S_1)))$, the Reeb orbit γ_ρ considered in step 1 is the unique Reeb orbit of λ_{FH} in ρ.

Let γ be a Reeb orbit in ρ. If it is contained in M_1, we know that γ is a closed geodesic in (S_1, g). Using an argument as in step 2, it is easy to show that γ and γ_ρ are freely homotopic in M_1 and, therefore, also in $T_1 S_1$. Projecting to S_1, we obtain that γ and γ_ρ are lifts of geodesics of (S_1, g) in the same free homotopy class of S_1. But for each free homotopy class of S_1, there is a unique closed geodesic of (S_1, g); this implies that $\gamma = \gamma_\rho$.

Step 3 will now follow if we prove the following:

Claim *Every Reeb orbit of λ_{FH} in ρ is completely contained in M_1.*

Proof of the claim If γ was contained in M_2, then it would be possible to isotope γ_ρ to a curve contained in ∂M_1. This is impossible by the definition of $\Lambda(S_1)$.

The only remaining possibility is that γ visits both M_1 and M_2. In this case, it has to visit both $M_1 \setminus N$ and $M_2 \setminus N$ (indeed, if γ is completely contained in $M_i \cup N$, convexity of the hyperbolic metric implies that γ is in M_i). We then know from Lemma 15 that every curve which is freely homotopic to γ has to intersect the torus $\mathbb{T}$. But γ_ρ, which is freely homotopic to γ, does not intersect $\mathbb{T}$. This contradiction rules out the possibility that γ visits both M_1 and M_2, and establishes the claim. □

Step 4 From the previous steps, we know that for each $\rho \in i_*(\Theta(\Lambda(S_1)))$, there exists a unique nondegenerate[5] Reeb orbit $\gamma_\rho \in \rho$. Hence for such ρ, the cylindrical contact homology $\mathbb{CH}^{\rho}_{\mathrm{cyl}}(\lambda_{\mathrm{FH}})$ is well-defined, and for Euler characteristic reasons, $\mathbb{CH}^{\rho}_{\mathrm{cyl}}(\lambda_{\mathrm{FH}}) \neq 0$.

Let $\rho \in i_*(\Theta(\Lambda(S_1)^{\leq T}))$. Then as we showed in the previous steps, the unique Reeb orbit of λ_{FH} in ρ has action at most T, and $\mathbb{CH}^{\rho}_{\mathrm{cyl}}(\lambda_{\mathrm{FH}}) \neq 0$. This implies that

$$N^{\mathrm{cyl}}_T(\lambda_{\mathrm{FH}}) \geq \#(i_*(\Theta(\Lambda(S_1)^{\leq T}))). \tag{6-31}$$

As i_* restricted to $\Theta(\Lambda(S_1)^{\leq T}))$ is injective, and Θ is injective, we conclude that

$$\#(i_*(\Theta(\Lambda(S_1)^{\leq T}))) = \#(\Lambda(S_1)^{\leq T}) \geq e^{aT+b}. \tag{6-32}$$

Combining formulas (6-31) and (6-32), we obtain

$$N^{\mathrm{cyl}}_T(\lambda_{\mathrm{FH}}) \geq e^{aT+b}. \tag{6-33}$$

□

7 Conclusion

The works of Katok [32; 33] and of Lima and Sarig [35] imply that if ϕ is a smooth flow on a 3–manifold, generated by a nonvanishing vector field, then ϕ has positive topological entropy if and only if there exists a Smale "horseshoe" as a subsystem of

[5] Recall that we established in step 1 that γ_ρ is nondegenerate.

the flow. For a flow, a "horseshoe" is a compact invariant set where the dynamics is conjugate to that of the suspension of a shift map. In particular, the number of hyperbolic periodic orbits on a "horseshoe" of a 3–dimensional flow ϕ grows exponentially with respect to the period. We remark that the result obtained in the recent work of Lima and Sarig [35] is stronger: they show that there exists a compact invariant set $\mathcal{K}$ of ϕ where the dynamics is nonuniformly hyperbolic and such that $h_{\mathrm{top}}(\phi_{\mathcal{K}}) = h_{\mathrm{top}}(\phi)$.[6]

As a consequence, for the contact 3–manifolds (M,ξ) considered in Theorems 9 and 16, we have that for *every* Reeb flow on (M,ξ), the number of hyperbolic Reeb orbits grows exponentially with the action. This can be summarized by saying that all Reeb flows on these contact manifolds posses a "complicated" orbit structure which is forced to exist by the "complicated" contact topology of these contact manifolds.

An interesting property of the entropy estimate used in this paper, and also in [3] and [36], is that it gives estimates on the growth of the number of hyperbolic Reeb orbits for degenerate contact forms as well. This kind of information is not obtainable just by studying the growth rate of contact homology.

It is known that the consequences of positivity of topological entropy in higher dimensions are not as strong as in the low-dimensional case. In particular, positive topological entropy for a flow in dimension greater than 3 does not imply the existence of a "horseshoe" in the flow. It is, however, natural to ask the following question.

Question 1 In dimension greater than or equal to 5, does exponential homotopical growth of periodic orbits for a Reeb flow imply the existence of a compact invariant set where the dynamics is conjugate to a shift?

In another direction, one would like to know if it is possible to obtain more dynamical information about the Reeb flows on the contact manifolds covered by Theorems 9 and 16.

Question 2 Let (M,ξ) be a manifold satisfying the hypothesis of Theorem 9 or 16, and let λ be a contact form on (M,ξ). Is it true that for the Reeb flow ϕ_{X_λ}, there exists an invariant region of positive measure (with respect to the measure $\lambda \wedge d\lambda$) on which the dynamics of the Reeb flow is ergodic?

One important property of many of the contact 3–manifolds covered in Theorem 9 is that they have positive Giroux torsion. By a theorem of Gay [23] (see also [41]),

[6] We remark that in [32], Katok proves analogous results for diffeomorphisms on surfaces and only states the results for flows on 3–manifolds in [33]. To the best of our knowledge, the complete proofs of all the results mentioned above for 3–dimensional flows with positive topological entropy only appeared in [35], which builds on the ideas of [32; 33].

manifolds with positive Giroux torsion are not strongly fillable. This implies that many of the contact manifolds satisfying the claims of Theorem 9 are not strongly fillable and therefore different from the unit tangent bundles studied in [36], which are fillable. It would be interesting to know if such examples also exist in higher dimensions.

Question 3 Are there examples of nonsymplectically fillable contact manifolds, with dimension at least 5, on which every Reeb flow has positive topological entropy? Are there examples, in dimension at least 5, of manifolds which admit infinitely many different contact structures such that, on all of them, every Reeb flow has positive topological entropy?

We remark also that in Theorem 9, we showed the existence of 3–manifolds with hyperbolic components which can be given infinitely many different contact structures whose Reeb flows always have positive topological entropy. From the perspective of 3–dimensional topology, it would be interesting to have examples of contact structures on hyperbolic 3–manifolds on which every Reeb flow has positive topological entropy.

Question 4 Are there examples of contact structures on closed hyperbolic 3–manifolds on which every Reeb flow has positive topological entropy?[7] Are there hyperbolic 3–manifolds which admit multiple nondiffeomorphic contact structures, on which every Reeb flow has positive topological entropy?

Lastly we mention that the techniques used in this paper, and in [3], can also be used in combination with the ideas of Momin [37] to establish chaotic behaviour of Reeb flows on $(S^3, \xi_{\mathrm{tight}})$ when these Reeb flows have a special link as a Reeb orbit. This and similar results will appear in [5].

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Institut de Mathématiques, Université de Neuchâtel,
Rue Emile-Argand 11, CH-2000 Neuchâtel, Switzerland

marcelorralves@gmail.com

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