Geometry & Topology 21 (2017) 253–314

Universal polynomials for tautological integrals on Hilbert schemes

JØRGEN VOLD RENNEMO

We show that tautological integrals on Hilbert schemes of points can be written in terms of universal polynomials in Chern numbers. The results hold in all dimensions, though they strengthen known results even for surfaces by allowing integrals over arbitrary "geometric" subsets (and their Chern–Schwartz–MacPherson classes).

We apply this to enumerative questions, proving a generalised Göttsche conjecture for all isolated singularity types and in all dimensions. So if L is a sufficiently ample line bundle on a smooth variety X, in a general subsystem $\mathbb{P}^d \subset |L|$ of appropriate dimension the number of hypersurfaces with given isolated singularity types is a polynomial in the Chern numbers of (X, L).

When X is a surface, we get similar results for the locus of curves with fixed "BPS spectrum" in the sense of stable pairs theory.

14C05, 14N10, 14N35

1 Results

Let X be a projective, nonsingular, connected, complex variety of dimension d, and let E be an algebraic vector bundle on X. Denote by $X^{[n]}$ the Hilbert scheme of length n subschemes of X, and let $E^{[n]}$ be the tautological bundle on $X^{[n]}$ with fibre $H^0(Z, E|_Z)$ at $Z \in X^{[n]}$.

We study integrals of products of Chern classes of $E^{[n]}$ over what we call *geometric subsets* of $X^{[n]}$. Geometric subsets form a natural class of subsets definable without reference to the global geometry of $X^{[n]}$. We define the geometric subsets as follows.

Denote by $\mathrm{Hilb}^n_0(\mathbb{C}^d)$ the punctual Hilbert scheme, that is, the closed subvariety of $\mathrm{Hilb}^n(\mathbb{C}^d)$ parametrising subschemes supported at $0 \in \mathbb{C}^d$. Let $Q_1, \dots, Q_k$ be constructible subsets of $\mathrm{Hilb}^{n_i}_0(\mathbb{C}^d)$ such that if $Z \in Q_i$ and $Z' \cong Z$ as $\mathbb{C}$–schemes, then $Z' \in Q_i$. We can then define a subset $P \subseteq \mathrm{Hilb}^n(X)$ by

$$P = \{Z \in X^{[n]} \mid Z = Z_1 \sqcup \cdots \sqcup Z,\ Z_i \text{ is of type } Q_i\},$$

Published: 10 February 2017 DOI: 10.2140/gt.2017.21.253

where $\sqcup$ denotes disjoint union, $n = \sum n_i$, and "Z_i is of type Q_i" means that there exists a $Z \in Q_i$ such that $Z_i \cong Z$ as a $\mathbb{C}$–scheme. We declare P to be a geometric subset, and in general define a geometric subset of $X^{[n]}$ to be any subset obtained by taking finite unions, intersections and complements of sets of this form.

The statement that $P \subseteq X^{[n]}$ is geometric implies two properties of P: that P is constructible and that for any $Z, Z' \in X^{[n]}$ such that $Z \cong Z'$, either $Z, Z' \in P$ or $Z, Z' \notin P$. Being geometric is a stronger requirement than satisfying these two properties; see Example 2.9.

A k–variable Chern polynomial is a polynomial in the formal variables $c_i^{(j)}$, where $i \geq 1$ and $1 \leq j \leq k$. We treat such a Chern polynomial as a function from k–tuples of vector bundles to cohomology by the rule

$$c_i^{(j)}(E_1, \dots, E_k) = c_i(E_j),$$

extended linearly and multiplicatively to all Chern polynomials.

A Chern monomial is a monomial in the variables $c_i^{(j)}$. The *weight* of a Chern monomial $c_{i_1}^{(j_1)} \cdots c_{i_k}^{(j_k)}$ is defined to be $\sum_{m=1}^{k} i_m$, so that treating a Chern monomial of weight l as a function, its image will be in $H^{2l}(X)$. Denote by $\mathrm{CM}(k,l)$ the set of k–variable Chern monomials of weight l.

Let Y be a complex, proper scheme. If $P \subseteq Y$ is a closed, pure-dimensional subset, we let $c_{\mathrm{M}}(P) \in H_*(Y)$ denote the Chern–Mather class of P. If $P \subseteq Y$ is a constructible subset, we let $c_{\mathrm{SM}}(P) \in H_*(Y)$ denote the Chern–Schwartz–MacPherson class of P. The constructions and basic properties of these classes are reviewed in Section 2.

Theorem 1.1 *Let X be a smooth, projective, connected variety of dimension d, E an algebraic vector bundle on X, and F a (1–variable) Chern polynomial. Let N be given by either*

(i) $N = \deg\big(F(E^{[n]}) \cap [P]\big)$, *for $P \subseteq X^{[n]}$ closed, pure-dimensional and geometric,*

(ii) $N = \deg\big(F(E^{[n]}) \cap c_{\mathrm{M}}(P)\big)$, *for $P \subseteq X^{[n]}$ closed, pure-dimensional and geometric, or*

(iii) $N = \deg\big(F(E^{[n]}) \cap c_{\mathrm{SM}}(P)\big)$, *for $P \subseteq X^{[n]}$ geometric.*

Then there exists a polynomial G in the variables $\{x_M\}_{M \in \mathrm{CM}(2,d)}$, depending only on F, the rank of E and the type of P, such that if we assign to x_M the Chern number $\deg M(T_X, E) \cap [X]$, we have $N = G((x_M))$.

Moreover, if every point $Z \in P$ represents a subscheme with support in at most m points, the degree of G is at most m.

Part (i) of Theorem 1.1 follows easily from either (ii) or (iii). We state (i) separately because it has applications to counting singular curves and hypersurfaces; see Sections 1.1.1 and 1.1.2. Part (iii) yields a different application to the problem of counting singular curves; see Section 1.1.3. While we give no separate applications of part (ii), the initial step of our proof is a reduction to a statement close to (ii) — see Lemma 4.4 — and so in the main part of the proof we work with Chern–Mather classes.

If $\dim X \le 2$ or $n \le 3$, then $X^{[n]}$ is nonsingular (Fogarty [7, Theorem 2.4]; Cheah [4, Theorem 3.2.2]), and so the tangent sheaf $T_{X^{[n]}}$ is locally free. We can then extend Theorem 1.1 by including Chern classes of $T_{X^{[n]}}$ as follows.

Theorem 1.2 *Assume that either* $\dim X \le 2$ *or* $n \le 3$, *so that* $X^{[n]}$ *is smooth. Let* F *be a* 2*–variable Chern polynomial. Theorem 1.1 then holds with* $F(E)$ *replaced by* $F(T_{X^{[n]}}, E)$ *everywhere.*

Remark Assuming that X is connected is not a big restriction, as the computation of tautological integrals for a general X are easily reduced to the connected case.

An outline of the proof of Theorems 1.1 and 1.2 is given in Section 3, and the formal proof occupies Sections 4 and 5.

In Section 6, we show that a certain generating function for some Chern integrals of part (i) of Theorems 1.1 and 1.2 can be given a particular product form.

The strategy of the proof of the main theorem is motivated by J Li's paper [23], where he shows that the degree of the virtual fundamental class on the Hilbert scheme of points on a threefold X is given by a universal polynomial in the Chern numbers of X. We adopt an overall strategy similar to that in [23], ie to transfer the problem to the Hilbert scheme of ordered points $X^{[\![n]\!]}$ (Definition 2.2) and then approximate by classes defined on the schemes $X^{[\![\alpha]\!]}$ (Section 3.2). Dealing with geometric subsets, the tautological bundles $E^{[n]}$ and Chern–Mather and Chern–Schwartz–MacPherson classes requires new ingredients.

A special case of Theorem 1.2 has been proved by Ellingsrud, Göttsche and Lehn [6] using a completely different method. In our terminology, they treat the case where X is a surface and the geometric subset P is the whole of $X^{[n]}$.

We note that the method of [6] yields a recursion which computes the universal polynomial explicitly. In contrast, our method is nonconstructive and relies at a crucial point on the fact that an element in the cohomology ring of a Grassmannian is a polynomial in the Chern classes of the universal bundle. Lacking a method of obtaining information about this polynomial, there is no apparent way of turning our proof into an algorithm.

When X is a surface, the cohomology groups $H^*(X^{[n]})$ have a well-understood description by the work of Grojnowski [12] and Nakajima [29]. Using this description, one can ask for a computation of tautological Chern classes lying in $H^k(X^{[n]})$ for any k, instead of just the degree of a class in $H^{2n}(X^{[n]})$; see eg Lehn [22], Boissière and Nieper-Wisskirchen [2] and Nieper-Wisskirchen [30] for examples of such computations. Our method is tailored to computing degrees rather than full cohomology classes and does not apparently apply to these more general questions.

For smooth X of any dimension, Cappell, Maxim, Ohmoto, Schürmann and Yokura [3] give an explicit formula for $c_{\mathrm{SM}}(X^{[n]})$, considered as an element in $H_*(\mathrm{Sym}^n(X), \mathbb{Q})$ by pushing forward along the Hilbert–Chow morphism. Our proof relies on the fact that the Chern–Mather class is defined by taking Chern classes of the Nash bundle, and it is not obviously applicable to computations involving characteristic classes not defined via a bundle in this way, such as the Todd and Hirzebruch classes treated in [3].

1.1 Enumerative applications

1.1.1 Counting singular curves in surfaces The main motivation for our result is to generalise the result known as the Göttsche conjecture, which by now has several proofs; see Kazaryan [16], Kool, Shende and Thomas [21], Liu [25] and Tzeng [35]. We recall the statement of the conjecture. Fix a surface S with a line bundle L which is "sufficiently ample", eg L is a sufficiently large power of a very ample line bundle. The precise definition of sufficiently ample uses the concept of N–very ampleness; see Section 7.

Let δ be a positive integer, and call a curve δ–nodal if it has δ nodes and no other singularities. If L is sufficiently ample, the locus of δ–nodal curves in $|L|$ has the expected codimension δ, so that in a general linear subsystem $\mathbb{P}^\delta \subset |L|$ there is a finite number of δ–nodal curves. The simplest form of the conjecture is then that there exists a degree-δ polynomial G_δ in four variables, independent of S and L, such that the number of δ–nodal curves equals

$$G_\delta\big(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)\big).$$

Our main application is the generalisation of this result to the case of curves with more general specified singularity types. Our approach follows the idea of Göttsche used in [10, Section 5] to reduce the problem of counting nodal curves to an integral on the Hilbert scheme. He defines a closed subset $W \subseteq S^{[3\delta]}$ and shows that the number of δ–nodal curves in the linear system $\mathbb{P}^\delta$ equals the degree of

$$c_{2\delta}(L^{[3\delta]}) \cap [W],$$

assuming L is $(5\delta-1)$–very ample. This idea was used by Tzeng [35] in her proof of the Göttsche conjecture, which uses degenerations of S to show that the degree of the above class is a polynomial in the Chern numbers of (S,L).

The set W appearing above is geometric, hence our theorem yields a different proof of Tzeng's result. Since our main theorem deals with more general loci in the Hilbert scheme of points, we may generalise the statement of Tzeng's theorem, replacing δ–nodal curves with curves having other specified singularity types.

Proposition 7.2 *Let S be a smooth, projective, connected surface, let L be a line bundle on S, and let $T_1,\dots,T_k$ be analytic isolated singularity types. There are expected codimensions d_i associated with each T_i, and we let $d=\sum d_i$.*

There is an integer N and a rational polynomial $G_{(T_i)}$ of degree k in four variables, depending only on the T_i, such that if L is N–very ample, then in a general $\mathbb{P}^d\subseteq |L|$ the number of curves having precisely k singularities of types T_i is

$$G_{(T_i)}\big(c_1^2(L),c_1(L)c_1(S),c_2(S),c_1^2(S)\big).$$

The same statement holds when the T_i are topological rather than analytic singularity types.

For the original problem of counting nodal curves, the numbers of curves having k nodes form a generating function

$$G_{\mathrm{nodal}}(S,L)=\sum_{\delta\geq 0}G_\delta\big(c_1(L)^2,c_1(L)c_1(S),c_1(S)^2,c_2(S)\big)q^\delta,$$

which was conjectured by Göttsche [10, Proposition 2.3] and shown by Tzeng [35, Theorem 1.3] to have a specific product form

$$G_{\mathrm{nodal}}(S,L)=B_1^{c_1^2(L)}B_2^{c_1(L)c_1(S)}B_3^{c_1^2(S)}B_4^{c_2(S)}\quad\text{where } B_i\in\mathbb{Q}[\![q]\!].$$

We generalise this statement as Corollary 7.3: Fixing distinct types T_i, collect the universal polynomials for the number of curves having m_i singularities of type T_i in a generating function; this then admits a product expansion similar to the above.

Both Proposition 7.2 and Corollary 7.3 have recently been obtained independently by Li and Tzeng [24] via a generalisation of Tzeng's degeneration approach.

1.1.2 Counting singular hypersurfaces By the same method we are able to count hypersurfaces with isolated singularities in arbitrary dimensions.

Proposition 7.8 *Let X be a smooth, projective, connected variety, let L be a line bundle on X, and let $T_1, \dots, T_k$ be analytic isolated singularity types. There are expected codimensions d_i associated with each T_i, and we let $d = \sum d_i$.*

There is an integer N and a rational polynomial $G_{(T_i)}$ in the Chern numbers of (X, L), depending only on the T_i, such that if L is N–very ample, then in a general $\mathbb{P}^d \subseteq |L|$ the number of divisors having precisely k isolated singularities of types T_i is given by $G_{(T_i)}$.

As in the curve case, a generating function for these universal polynomials can be written in a product form similar to the one of Corollary 7.3.

1.1.3 Counting curves with given BPS spectra A different application of the main result concerns the locus of curves in a $\mathbb{P}^k \subset |L|$ having given "BPS spectrum". For a reduced, complete, locally planar curve C with arithmetic genus $g(C)$ and geometric genus $\overline{g}(C)$, we consider the generating function

$$H_C(q) := \sum_{k=0}^{\infty} \chi(C^{[k]}) q^k.$$

Pandharipande and Thomas [32] show that there are $n_{i,C} \in \mathbb{Z}$ for $i = \overline{g}(C), \dots, g(C)$ such that

$$H_C(q) = \sum_{i=\overline{g}(C)}^{g(C)} n_{i,C}\, q^{g(C)-i} (1-q)^{2i-2}.$$

If C is smooth, we have $H_C(q) = (1-q)^{2g(C)-2}$, so this result can be interpreted as saying that in general $H_C(q)$ decomposes as a sum of $n_{i,C}$ copies of $q^{g(C)-i} H_{C_i}(q)$ where C_i is smooth of genus i. We define $m_{i,C} = n_{g(C)-i,C}$, and it is then easy to check that the sequence of integers $(m_{i,C})_{i=0}^{\infty}$ depends only on the analytic types of the singularities of C. We refer to the sequence $(m_{i,C})$ as the *BPS spectrum* of C.

Recent work of Maulik [28], settling a conjecture of Oblomkov and Shende [31], shows that the BPS spectrum of C is explicitly determined by the Milnor numbers and HOMFLY polynomials of the links of the singularities of C. As a consequence, the BPS spectrum depends only on the topological types of the singularities of C.

We show the following proposition.

Proposition 7.9 *Let S be a smooth, projective, connected surface, let L be a line bundle on S, and let $k \in \mathbb{Z}_{\geq 0}$. Let $m = (m_i)_{i=0}^{\infty}$ be a BPS spectrum, and denote by $|L|_m \subseteq |L|$ the locus of curves with BPS spectrum m.*

There is an integer N and a rational polynomial G_m in four variables, depending only on k and m, such that if L is N–very ample, then for a general $\mathbb{P}^k \subset |L|$ we have

$$\chi(\mathbb{P}^k \cap |L|_m) = G_m\big(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)\big).$$

This generalises the approach of the proof of Kool, Shende and Thomas [21] of the Göttsche conjecture, which implicitly proves the special case of the above proposition where m is the spectrum of a δ–nodal curve; that is $m = (m_i)$ with $m_i = \binom{\delta}{i}$.

Let us sketch part of the proof in order to illustrate how Theorem 1.1(iii) is applied. Let $V = \mathbb{C}^{k+1} \subset H^0(S, L)$ be the linear subspace corresponding to the general $\mathbb{P}^k \subset |L|$. For any n, there is a canonical map $\phi\colon V \otimes \mathcal{O}_{S^{[n]}} \to L^{[n]}$, and we let $D_r(\phi) \subseteq S^{[n]}$ be the r^{th} degeneracy locus, ie where the map ϕ has rank $\le r$. Applying basic properties of the Euler characteristic and an elementary argument allows us to express $\chi(\mathbb{P}^k \cap |L|_m)$ as some sum of terms $\chi(D_r(\phi) \cap P)$, where P is a geometric subset in $S^{[n]}$.

Taking L sufficiently ample and V general, we may assume that ϕ satisfies a certain genericity condition. Then by Parusiński and Pragacz [33, Theorem 2.10] we may express $\chi(D_r(\phi) \cap P)$ as a polynomial in $c_{\mathrm{SM}}(P)$ and the Chern classes of $L^{[n]}$, which we can further express as a universal polynomial in the Chern numbers of (S, L) by Theorem 1.1(iii).

We note that in the proof of Proposition 7.9 it is essential to be able to take integrals over general geometric subsets of $S^{[n]}$. This is in contrast to the argument of [21], where the integrals needed were taken over the whole of $S^{[n]}$, and so were already computed in Ellingsrud, Göttsche and Lehn [6].

1.2 Conventions

We work over the base field $\mathbb{C}$ throughout, and it is essential to our proof that we can consider the underlying complex analytic spaces of the varieties involved.

In the proof of Lemma 5.9, our argument is based on singular (co)homology. Apart from at this point the reader is free to use their favourite (co)homology theory (eg sheaf cohomology).

We always take (co)homology with coefficients in $\mathbb{Q}$. Note, however, that if the polynomial F in the main theorems has integral coefficients, then the numbers N computed will all be integers.

By the degree of a class in $H_*(X)$ we mean its pushforward to $H_*(\mathrm{pt}) \cong \mathbb{Q}$. In dealing with algebraic subsets of Hilbert schemes we always give these the reduced scheme structure; in particular, this applies to the Hilbert schemes themselves.

If m is some number defined in terms of the data X, E, P, F of the theorem, we will use the shorthand "m is universal" to mean that there exists a polynomial in the variables x_M computing m, depending only on F and the type of P, as in the main theorem.

1.2.1 Douady spaces We will need the analogues of Hilbert schemes in the category of complex analytic spaces. These are called *Douady spaces* and were first constructed by Douady in [5]. If U is an analytic variety, we write $U^{[n]}$ for the Douady space parametrising closed 0–dimensional length-n subspaces.

If U is an analytic open subset of the projective algebraic variety X, then $U^{[n]} \subset X^{[n]}$ is an analytic open subset, and the structure of analytic space on $U^{[n]}$ is inherited from that on $X^{[n]}$. In particular the Douady space and Hilbert scheme of X are isomorphic as analytic spaces. The two key properties we will need are:

- There is an analytic map $U^{[n]} \to \mathrm{Sym}^n(U)$ known as the Douady–Barlet morphism, analogous to the Hilbert–Chow morphism in the algebraic setting. Analyticity of the map follows from the fact that the Hilbert–Chow morphism is algebraic; see also Magnússon [27].
- The structure of $U^{[n]}$ is determined by the complex analytic structure of U, so that an isomorphism of analytic varieties $f\colon U \to V$ induces an isomorphism $f^{[n]}\colon U^{[n]} \to V^{[n]}$.

Acknowledgements I thank Martijn Kool, Ragni Piene, my supervisor Richard Thomas and Yu-jong Tzeng for valuable discussions and comments on this paper. In particular, Piene pointed out to me the results of [18] used in Section 7.1.2. Many thanks also to the referees for very useful comments and corrections.

2 Preliminaries

Let X be a smooth, projective, connected variety of dimension d, and let E be an algebraic vector bundle on X. We give the definition of the tautological bundle $E^{[n]}$ and recall the construction of the Chern–Mather and Chern–Schwartz–MacPherson (CSM) classes.

In Section 2.3 we introduce the scheme $X^{[\![n]\!]}$ and in Section 2.4 we discuss the notion of geometric subsets of $X^{[n]}$ and $X^{[\![n]\!]}$.

2.1 The tautological bundle

Denote by $\mathcal{Z} \subset X^{[n]} \times X$ the universal subscheme over $X^{[n]}$, and let $p\colon \mathcal{Z} \to X$ and $q\colon \mathcal{Z} \to X^{[n]}$ be the projections. The *tautological bundle* $E^{[n]}$ on $X^{[n]}$ is defined as

$$E^{[n]} = q_*(p^*(E)).$$

The flatness of q implies that $E^{[n]}$ is locally free, and we see that the fibre of $E^{[n]}$ at a point $Z \in X^{[n]}$ is the vector space $H^0(Z, E|_Z)$.

2.2 Chern classes

We next review the Chern–Mather and Chern–Schwartz–MacPherson classes. These classes are generalisations to singular varieties of the Poincaré dual of $c_\bullet(T_Y)$ for a smooth, proper Y, so for such Y we have

$$c_{\mathrm{SM}}(Y) = c_{\mathrm{M}}(Y) = c_\bullet(T_Y) \cap [Y].$$

2.2.1 Chern–Mather class Let Y be a reduced and pure-dimensional projective scheme. The first step is to construct the Nash blow-up $\widetilde{Y} \to Y$. Suppose for a moment that Y is *affine*, reduced and irreducible of dimension d. Fix an embedding $f\colon Y \to \mathbb{A}^N$, and let Y_{ns} be the nonsingular part of Y. The tangent map $T_{Y_{\mathrm{ns}}} \to f^*(T_{\mathbb{A}^N})$ induces a morphism $g\colon Y_{\mathrm{ns}} \to \mathrm{Gr}(d, N)$, and we take $\widetilde{Y}$ to be the closure of the graph $\Gamma_g \subset Y \times \mathrm{Gr}(d, N)$. The morphism $\widetilde{Y} \to Y$ is defined by the projection $Y \times \mathrm{Gr}(d, N) \to Y$, and we define the rank-d vector bundle $T_{\widetilde{Y}}$ on $\widetilde{Y}$ by restricting the universal bundle on $\mathrm{Gr}(d, N)$.

It can be shown that this construction is independent of the choice of affine embedding and globalises so that for any reduced, equidimensional scheme Y we get a well-defined Y–scheme $\widetilde{Y}$ with a bundle $T_{\widetilde{Y}}$. The morphism $\widetilde{Y} \to Y$ is the *Nash blow-up* of Y and the bundle $T_{\widetilde{Y}}$ is the *Nash bundle*.

Definition 2.1 The Chern–Mather class $c_{\mathrm{M}}(Y) \in H_*(Y)$ is defined as the pushforward of $c_\bullet(T_{\widetilde{Y}}) \cap [\widetilde{Y}]$ along $\widetilde{Y} \to Y$.

An observation which is important for our proof is that this construction of the Nash bundle is valid when Y is a complex analytic variety, in a way which is compatible with restriction to a complex analytic open subset of an algebraic variety.

2.2.2 Chern–Schwartz–MacPherson class We recall the definition and basic properties of the Chern–Schwartz–MacPherson class. For details, see [8, Example 19.1.7] and [26].

Let Y be a projective scheme, let $Z_*(Y)$ be the group of all cycles on Y, and let $F_*(Y)$ denote the group of constructible functions, where a function $f\colon Y \to \mathbb{Z}$ is called constructible if there exists a finite partition of Y into constructible sets such that f is constant on each set of the partition. Given a reduced pure-dimensional scheme V, the local Euler obstruction $\mathrm{Eu}_V\colon V \to \mathbb{Z}$ is a canonical constructible function determined at a point $x \in V$ by the local analytic structure of V at x.

Let $p\colon \widetilde{V} \to V$ be the Nash blow-up. By work of González-Sprinberg and Verdier, we have $\mathrm{Eu}_V(x) = \deg\bigl(c_\bullet(T_{\widetilde{V}}|_{p^{-1}(x)}) \cap s(p^{-1}(x), \widetilde{V})\bigr)$, where $p^{-1}(x)$ is the scheme-theoretic inverse image and $s(-,-)$ denotes the Segre class [9].

There is a group homomorphism $\Omega\colon Z_*(Y) \to F_*(Y)$ defined on a primitive cycle V by

$$\Omega(V)(x) = \begin{cases} \mathrm{Eu}_V(x) & \text{if } x \in V, \\ 0 & \text{if } x \notin V. \end{cases}$$

The map Ω is an isomorphism, as can be shown using the fact that $\mathrm{Eu}_V(x) = 1$ if x is a nonsingular point of V. The Chern–Mather class defines a homomorphism $c\colon Z_*(Y) \to H_*(Y)$ by letting

$$c(V) = i_*(c_{\mathrm{M}}(V)),$$

where $i\colon V \to Y$ is the inclusion of a primitive cycle.

The Chern–Schwartz–MacPherson class $c_{\mathrm{SM}}(f)$ is now defined for any constructible function f by

$$c_{\mathrm{SM}}(f) = c(\Omega^{-1}(f)).$$

It is clear that c_{SM} is a homomorphism. If $S \subset Y$ is a constructible subset, we write $c_{\mathrm{SM}}(S) = c_{\mathrm{SM}}(1_S)$, where 1_S is the characteristic function of S. Additivity of c_{SM} translates to

$$c_{\mathrm{SM}}(S_1 \cup S_2) = c_{\mathrm{SM}}(S_1) + c_{\mathrm{SM}}(S_2) - c_{\mathrm{SM}}(S_1 \cap S_2).$$

Given a morphism of proper schemes $g\colon Y_1 \to Y_2$, one can define a homomorphism $g_*\colon F(Y_1) \to F(Y_2)$ by letting

$$g_*(1_V)(x) = \chi(g^{-1}(x) \cap [V]) \quad \text{for } x \in Y_2,$$

where $V \subset Y_1$ is a primitive cycle, and χ is the topological Euler characteristic.[1] The main property of CSM classes, shown in [26], is that $g_*(c_{\mathrm{SM}}(f)) = c_{\mathrm{SM}}(g_*(f))$.

[1] The fact that this is well defined is shown in [26].

Applying this to the case where Y is proper and g is the map $Y \to \mathrm{pt}$, we find that $\deg c_{\mathrm{SM}}(Y) = \chi(Y)$.

2.3 The Hilbert scheme of ordered points

Following [23], we introduce the scheme $X^{[\![n]\!]}$, which will play an essential role in the proof of Theorems 1.1/1.2.

Definition 2.2 The *Hilbert scheme of ordered points*, denoted $X^{[\![n]\!]}$, is the scheme defined by the Cartesian diagram

$$\begin{array}{ccc} X^{[\![n]\!]} & \longrightarrow & X^{[n]} \\ \downarrow & & \downarrow \\ X^n & \longrightarrow & \mathrm{Sym}^n(X) \end{array}$$

where the right-hand arrow is the Hilbert–Chow morphism taking a subscheme Z to its support cycle. We denote by $(Z,(x_i))$ the point in $X^{[\![n]\!]}$ mapping to $Z \in X^{[n]}$ and $(x_i) \in X^n$.

While this definition endows $X^{[\![n]\!]}$ with a natural scheme structure, possibly nonreduced, it will suffice for our purposes to consider $X^{[\![n]\!]}$ as a reduced scheme throughout. Since $X^{[\![n]\!]} \to X^{[n]}$ is finite, we can reduce questions about the degree of homology classes on $X^{[n]}$ to questions about similar classes on $X^{[\![n]\!]}$. Roughly speaking, the advantage of introducing $X^{[\![n]\!]}$ is that it naturally maps to X^n. This makes it easier to handle than $X^{[n]}$, which maps to the more complicated scheme $\mathrm{Sym}^n(X)$.

2.3.1 A stratification of $\mathrm{Sym}^n(X)$ We may stratify $\mathrm{Sym}^n(X)$ into disjoint locally closed subsets $\mathrm{Sym}^n_{(n_i)}(X)$, where (n_i) is a partition of n, that is, where (n_i) is a sequence $0 < n_1 \le \cdots \le n_k$ such that $\sum n_i = n$. The subset $\mathrm{Sym}^n_{(n_i)}(X)$ consists of 0–cycles of the form $\sum n_i x_i$, where $x_i \in X$ are distinct points. Restricting the map $X^{[n]} \to \mathrm{Sym}^n(X)$ to $\mathrm{Sym}^n_{(n_i)}$ gives an analytic (or étale) locally trivial fibration, the fibres of which are isomorphic to $\prod_i \mathrm{Hilb}^{n_i}_0(\mathbb{C}^d)$.

Similarly, we may define locally closed subsets $X^n_{(A_i)} \subseteq X^n$, where $(A_i)_{i=1}^k$ is an ordered partition of $\{1,\dots,n\}$, that is, a sequence of disjoint nonempty subsets of $\{1,\dots,n\}$ such that $\cup A_i = \{1,\dots,n\}$. The subset $X^n_{(A_i)}$ consists of n–tuples $(x_i) \in X^n$ such that $x_i = x_j$ if and only if i and j are contained in the same A_l for some l. Restricting the map $X^{[\![n]\!]} \to X^n$ to $X^n_{(A_i)}$ gives a Zariski locally trivial fibration with fibres isomorphic to $\prod_i \mathrm{Hilb}^{|A_i|}_0(\mathbb{C}^d)$.

Reordering the A_i does not change $X^n_{(A_i)}$, so letting $\alpha = \{A_1,\dots,A_k\}$ be the unordered partition of $\{1,\dots,n\}$ underlying A, we may define $X^n_\alpha = X^n_{(A_i)}$. Then the sets X^n_α form a stratification of X^n when varying over all partitions α.

2.4 Geometric subsets

We now give the definition of geometric subsets of $X^{[n]}$ and of $X^{[\![n]\!]}$, along with some results on these which will be needed later.

Let $\mathrm{Hilb}_0^n(\mathbb{C}^d)$ be the punctual Hilbert scheme, defined to be the closed subset of $\mathrm{Hilb}^n(\mathbb{C}^d)$ parametrising subschemes supported at the origin. We define punctual geometric subsets to be the constructible subsets of the punctual Hilbert scheme containing all 0–dimensional schemes of given isomorphism types.

Definition 2.3 A punctual geometric set is a subset $Q \subseteq \mathrm{Hilb}_0^n(\mathbb{C}^d)$ which is constructible and satisfies: if $Z \in Q$ and $Z' \in \mathrm{Hilb}_0^n(\mathbb{C}^d)$ are such that $Z \cong Z'$ (as abstract $\mathbb{C}$–schemes), then $Z' \in Q$.

A collection of punctual geometric subsets will naturally defines a subset of $X^{[n]}$:

Definition 2.4 Let $Q_1, \dots, Q_k$ be punctual geometric sets such that $Q_i \subseteq \mathrm{Hilb}_0^{n_i}(\mathbb{C}^d)$, and let $n = \sum n_i$.

We define $P(Q_1, \dots, Q_k) \subseteq X^{[n]}$ to be the set of all $Z = Z_1 \sqcup \dots \sqcup Z_k$, where every Z_i is isomorphic to a $Z_i' \in Q_i$.

If we additionally specify how to label these Z, we obtain a subset of $X^{[\![n]\!]}$:

Definition 2.5 Let Q_i and n_i be as above. Let $A = (A_1, \dots, A_k)$ be a k–tuple of subsets of $\{1, \dots, n\}$ such that $|A_i| = n_i$ and such that the A_i define a partition of $\{1, \dots, n\}$.

We define $R(Q_1, \dots, Q_k; A) \subseteq X^{[\![n]\!]}$ to be the set of all $(Z, (x_i))$ such that Z is equal to $Z_1 \sqcup \dots \sqcup Z_k$, where every Z_i is isomorphic to a $Z_i' \in Q_i$, and such that $x_i = \mathrm{Supp}\, Z_j$ if $i \in A_j$.

Remark The subset $R((Q_i); A)$ is compatible with the locally closed subsets of X^n described in Section 2.3.1. In particular, the image of $R((Q_i); A)$ under the morphism $X^{[\![n]\!]} \to X^n$ is X_A^n, and over any point $x \in X_A^n$ the fibre of $R((Q_i); A)$ is

$$R((Q_i); A)|_x = \prod_{i=1}^{k} Q_i \subset \prod_{i=1}^{k} \mathrm{Hilb}^{|A_i|}(\mathbb{C}^d) = X^{[\![n]\!]}|_x.$$

Similar remarks hold for $P((Q_i))$ and the stratification of $\mathrm{Sym}^n(X)$.

We can now give the definition of geometric subsets of $X^{[n]}$ and $X^{[\![n]\!]}$:

Definition 2.6
- A subset $P \subseteq X^{[n]}$ is geometric if it can be expressed using sets of the form $P(Q_1, \dots, Q_k)$ and a finite composition of the operations union, intersection and complement.
- A subset $R \subseteq X^{[\![n]\!]}$ is geometric if it can be expressed using sets of the form $R(Q_1, \dots, Q_k; A)$ and a finite composition of the operations union, intersection and complement.

An equivalent definition which will be convenient is the following.

Definition 2.7
- A subset $P \subseteq X^{[n]}$ is geometric if it can be expressed using sets of the form $\overline{P(Q_1, \dots, Q_k)}$, where the Q_i are closed and irreducible, together with a finite composition of the operations union, intersection and complement.
- A subset $R \subseteq X^{[\![n]\!]}$ is geometric if it can be expressed using sets of the form $\overline{R(Q_1, \dots, Q_k; A)}$, where the Q_i are closed and irreducible, together with a finite composition of the operations union, intersection and complement.

The equivalence of Definitions 2.6 and 2.7 is shown in Lemma 2.11(vii).

Example 2.8 The only geometric subsets of $X^{[1]} = X$ are $\varnothing$ and X. In $X^{[2]}$ there are four geometric subsets: the sets $\varnothing$, $X^{[2]}$, the set parametrising pairs of disjoint points and the set parametrising length-2 subschemes with support in one point.

When X is a surface, a naturally occurring example of a geometric subset is the subset of $X^{[3\delta]}$ defined as the closure of

$$\{Z \in X^{[3\delta]} \mid Z = Z_1 \sqcup \dots \sqcup Z_\delta,\ Z_i = \operatorname{Spec} \mathcal{O}_{X,x_i}/\mathfrak{m}_{x_i}^2\}.$$

This set appears in Tzeng's proof of the Göttsche conjecture [35].

The statement that $P \subseteq X^{[n]}$ is geometric implies two properties of P: that P is constructible and that for any $Z, Z' \in X^{[n]}$ such that $Z \cong Z'$ as $\mathbb{C}$–schemes, either $Z, Z' \in P$ or $Z, Z' \notin P$. In other words, a geometric subset is a constructible union of isomorphism classes of subschemes $Z \in X^{[n]}$.

Being geometric is a stronger requirement than having the two properties mentioned above, as the following example shows.

Example 2.9 Let X be a surface, and let $P \subset X^{[28]}$ be the set containing all $Z = Z_1 \sqcup Z_2$ such that (1) each Z_i is defined by an ideal $(\mathfrak{m}_{x_i}^5, f_i)$ where f_i is a product of four distinct linear factors in $\mathfrak{m}_{x_i}/\mathfrak{m}_{x_i}^2$, and (2) the cross ratio of the factors of f_1 equals that of the factors of f_2. Then P is constructible and a union of isomorphism classes of subschemes $Z \in X^{[28]}$, but is not geometric.

We have a notion of isomorphism between points of $X^{[\![n]\!]}$, defined by saying that $(Z,(x_i)) \cong (Z',(x_i'))$ if there exists an isomorphism $Z \cong Z'$ which takes $Z|_{x_i}$ to $Z'|_{x_i'}$ for every i. Then similarly a geometric subset of $X^{[\![n]\!]}$ is a union of isomorphism classes of pairs.

Definition 2.10 Let X_1 and X_2 be smooth varieties of equal dimension, and for $i=1,2$, let $P_i \subset X_i^{[n]}$ be a geometric subset. We say that P_1 and P_2 are of the same *type* if the isomorphism classes of the points in P_1 are the same as the isomorphism classes of points in P_2. It is clear that for any geometric subset of $X_1^{[n]}$, there is a unique geometric subset of $X_2^{[n]}$ of the same type.

The type of a geometric subset $R \subseteq X^{[\![n]\!]}$ is defined in the same way, using the notion of isomorphism of pairs $(Z,(x_i)) \in X^{[\![n]\!]}$ defined above.

The following lemma contains some elementary facts about geometric subsets. Note that there is a natural action of the symmetric group S_n on X^n, and hence on $X^{[\![n]\!]}$.

Lemma 2.11 *Let $P \subseteq X^{[n]}$ and $R \subseteq X^{[\![n]\!]}$ be sets, and let $p\colon X^{[\![n]\!]} \to X^{[n]}$ be the natural forgetful morphism.*

(i) *P is geometric $\iff$ $p^{-1}(P)$ is geometric.*

(ii) *R is geometric $\implies$ $p(R)$ is geometric. If R is S_n–invariant and $p(R)$ is geometric, then R is geometric.*

(iii) *P is geometric $\iff$ P is a finite union of sets of the form $P((Q_i))$.*

(iv) *R is geometric $\iff$ R is a finite union of sets of the form $R((Q_i);A)$.*

(v) *P is geometric, closed and irreducible $\iff$ P is of the form $\overline{P((Q_i))}$ for closed, irreducible Q_i.*

(vi) *R is geometric, closed and irreducible $\iff$ R is of the form $\overline{R((Q_i);A)}$ for closed, irreducible Q_i.*

(vii) *Definitions 2.6 and 2.7 are equivalent.*

Proof In this proof, "geometric subset" means a set satisfying Definition 2.6.

(iv) It is sufficient to show that intersections and complements of sets having the form $R((Q_i);A)$ are expressible as unions of such sets. Let $R((Q_i);A)$ be a geometric set with $A=(A_1,\dots,A_k)$ and $R((Q_i');A'))$ a geometric set with $A'=(A_1',\dots,A_l')$. Then, if $R((Q_i);A) \cap R((Q_i');A') \neq \varnothing$, we have $k=l$ and the k–tuple A is a permutation of the k–tuple A'. In this case, we may relabel the indices of the A_i' to get $A=A'$, and then $R((Q_i);A) \cap R((Q_i');A)) = R((Q_i \cap Q_i');A)$.

Next we see that for any $R((Q_i);A)$ the set $X^{[\![n]\!]} \setminus R((Q_i);A)$ is the union of all sets $R((\mathrm{Hilb}_0^{n_i}(\mathbb{C}^d));A')$ where A' is not a permutation of A and the sets

$$R(\mathrm{Hilb}_0^{n_1}(\mathbb{C}^d),\dots,\mathrm{Hilb}_0^{n_i}(\mathbb{C}^d)\setminus Q_i,\dots,\mathrm{Hilb}_0^{n_k}(\mathbb{C}^d);A)$$

for $i=1,\dots,k$.

(i) $\Rightarrow$ This follows from the fact that $p^{-1}(P(Q_i))$ is the union of $R((Q_i);A)$ for all admissible A.

(ii) The first claim follows from (iv) and $p(R((Q_i);A)) = P((Q_i))$. For the second claim: if R is S_n–invariant, then $R = p^{-1}(p(R))$, which is geometric by (i) $\Rightarrow$.

(i) $\Leftarrow$ This follows from (ii) and the surjectivity of p.

(iii) This follows from (i), (ii), (iv) and the surjectivity of p.

(v) By (iii), we may write $P = \cup_j P((Q_{i,j})_i)$, and since P is closed, we have $P = \cup_j \overline{P((Q_{i,j})_i)} = \cup_j P((\overline{Q_{i,j}})_i)$. Irreducibility of P implies $P = P((\overline{Q_{i,j}})_i)$ for some j, so we may take $Q_i = \overline{Q_{i,j}}$. It remains to show that the Q_i can be chosen to be irreducible. Suppose not, then we have for instance Q_1 reducible. Let $Q_1 = \cup_j Q_{1,j}$ be the decomposition of Q_1 into closed, irreducible subsets. Each $Q_{1,j}$ must be equal to the closure of its orbit under the natural action of $\mathrm{Aut}(\mathcal{O}_{\mathbb{A}^d,0}/\mathfrak{m}_0^{n_1})$ on $\mathrm{Hilb}_0^{n_1}(\mathbb{C}^d)$; hence we see that the $Q_{1,j}$ are geometric.

We then have $P = \cup_j \overline{P(Q_{1,j},Q_2,\dots,Q_k)}$, and since P is irreducible, we may replace Q_1 with some $Q_{1,j}$. Repeat to get all Q_i irreducible, proving the $\Rightarrow$ implication. The $\Leftarrow$ implication is easy and omitted, but note that it depends on the hypothesis that X is connected.

(vi) This is similar to (v).

(vii) It is obvious that a P satisfying Definition 2.7 satisfies Definition 2.6. For the converse, note that the closed, geometric P generate all geometric subsets by unions, intersections and complements. The proof of (v) shows that a closed, geometric P is the union of sets of the form $\overline{P((Q_i))}$ with Q_i closed and irreducible. Hence closed, geometric P satisfy Definition 2.7, and the claim follows. The case of R is similar. □

2.4.1 Geometric functions The definitions and results of this section will only be used in the proof of Lemma 4.3. We say that a constructible function on $X^{[n]}$ (resp. $X^{[\![n]\!]}$) is geometric if its level sets are geometric subsets. The geometric functions on $X^{[n]}$ (resp. $X^{[\![n]\!]}$) form a subring of the ring of constructible functions $F(X^{[n]})$ (resp. $F(X^{[\![n]\!]})$).

The morphism $p\colon X^{[\![n]\!]} \to X^{[n]}$ gives a homomorphism $p_*\colon F(X^{[\![n]\!]}) \to F(X^{[n]})$ as described in Section 2.2.2, and a homomorphism $p^*\colon F(X^{[n]}) \to F(X^{[\![n]\!]})$ given by $p^*(f) = f \circ p$.

Lemma 2.12 *Let $f \in F(X^{[n]})$. Then f is geometric if and only if $p^*(f)$ is geometric.*

Proof This follows from Lemma 2.11(i). □

The symmetric group S_n acts on $X^{[\![n]\!]}$ and hence on $X^{[\![n]\!]}$ and $F(X^{[\![n]\!]})$.

Lemma 2.13 *Let $f \in F(X^{[\![n]\!]})$.*

(i) *If f is geometric, then $p_*(f)$ is geometric.*

(ii) *If f is S_n–invariant and $p_*(f)$ is geometric, then f is geometric.*

Proof (i) Consider first the function $1_{X^{[\![n]\!]}} \in F(X^{[\![n]\!]})$. Let $Z \in X^{[n]}$ have the form $Z = Z_1 \sqcup \cdots \sqcup Z_k$, where each Z_i is supported at a point and has length n_i. Then we get

$$p_*(1_{X^{[\![n]\!]}})(Z) = |p^{-1}(Z)| = \frac{n!}{\prod_i n_i!}.$$

It follows from this that $p_*(1_{X^{[\![n]\!]}})$ is geometric, and hence $p^* p_*(1_{X^{[\![n]\!]}})$ is geometric. For any $f \in F(X^{[\![n]\!]})$, we have

$$p_*(f) = \frac{1}{n!} \sum_{\sigma \in S_n} p_*(\sigma f) = \frac{1}{n!} p_*\Big(\sum_{\sigma \in S_n} \sigma f \Big).$$

Let $g = \sum_{\sigma \in S_n} \sigma f$. Since g is S_n–invariant, we get

$$p^* p_*(g) = p^* p_*(1_{X^{[\![n]\!]}}) \cdot g.$$

Hence $p^* p_*(g)$ is geometric. It follows that $p_*(g)$ and hence $p_*(f) = p_*(g)/(n!)$ are both geometric.

(ii) We have $p^* p_*(f) = p^* p_*(1_{X^{[\![n]\!]}}) \cdot f$, and so, since $p^* p_*(1_{X^{[\![n]\!]}})$ is nonvanishing at all points, we may write $f = p^* p_*(f) \cdot (p^* p_*(1_{X^{[\![n]\!]}}))^{-1}$. Since $p^* p_*(f)$ and $p^* p_*(1_{X^{[\![n]\!]}})$ are geometric, it then follows that f is. □

3 Outline of proof

We give an outline of the proof of the main theorems. We restrict our attention in the outline to Theorem 1.1(i), ignoring the extra complications of (ii), (iii) and Theorem 1.2. We assume that P is irreducible; the general case follows from this by Lemma 4.1.

The set-up is then that we are given a closed, irreducible, geometric subset P of $X^{[n]}$, a Chern polynomial F and a vector bundle E, and we want to show that

$$\deg F(E^{[n]}) \cap [P]$$

is given by a universal polynomial.

3.1 Reduction to $X^{[\![n]\!]}$

The first step is to replace $X^{[n]}$ with the Hilbert scheme of ordered points $X^{[\![n]\!]}$.

Define the bundle $E^{[\![n]\!]}$ on $X^{[\![n]\!]}$ as the pullback of $E^{[n]}$ along $X^{[\![n]\!]} \to X^{[n]}$. In Lemma 4.3 we construct a closed, irreducible $R \subseteq X^{[\![n]\!]}$ which is geometric, maps properly and finitely onto P, and which is such that $\deg(R/P)$ and the type of R is determined by the type of P. The projection formula then gives

$$\deg(R/P)\big(\deg F(E^{[n]}) \cap [P]\big) = \deg F(E^{[\![n]\!]}) \cap [R].$$

Thus, it suffices to show that $\deg F(E^{[\![n]\!]}) \cap [R]$ is given by a universal polynomial.

3.2 Approximating spaces

This section corresponds to Section 4.4. Let α be a partition of $\{1,\dots,n\}$. Following [23], we then define the scheme $X^{[\![\alpha]\!]}$ as follows. Considering α as a set of subsets of $\{1,\dots,n\}$, we let

$$X^{[\![\alpha]\!]} = \prod_{S\in\alpha} X^{[\![|S|]\!]}.$$

So, for example, if α is the partition of $\{1,\dots,n\}$ into n one-element sets, we have $X^{[\![\alpha]\!]} = X^n$. At the other extreme, for the trivial partition $\Lambda = \{\{1,\dots,n\}\}$, we have $X^{[\![\Lambda]\!]} = X^{[\![n]\!]}$. In general, the scheme $X^{[\![\alpha]\!]}$ parametrises ordered collections of n points in X, with the additional data that when k points with labels in the same set in the partition α come together at x, one must specify a length-k subscheme supported at x.

Consider the open subset of $X^{[\![n]\!]}$ where no two points with labels in different sets of α come together. This subset is naturally isomorphic to an open subset of $X^{[\![\alpha]\!]}$, so we get a rational map $g_\alpha\colon X^{[\![n]\!]} \dashrightarrow X^{[\![\alpha]\!]}$. In Definition 4.7, we define a bundle $E^{[\![\alpha]\!]}$ on $X^{[\![\alpha]\!]}$ such that $g_\alpha^*(E^{[\![\alpha]\!]}) = E^{[\![n]\!]}$ on the locus where g_α is defined.

A closed, irreducible geometric $R \subseteq X^{[\![n]\!]}$ as in Section 3.1 is, by Lemma 2.11, of the form $R = R((Q_i); A))$. Let $\mu = \{A_i\}$ be the partition of $\{1,\dots,n\}$ induced by A. Then one checks that R intersects the domain of definition for g_α if and only if $\mu \le \alpha$, where $\le$ means refinement of partitions, ie every element of μ is contained in an element of α. If $\mu \leq \alpha$, we may thus define a closed, irreducible subset $R_\alpha := \overline{g_\alpha(R)} \subseteq X^{[\![\alpha]\!]}$. This in particular holds for the maximal partition Λ, for which we get $E^{[\![\Lambda]\!]} = E^{[\![n]\!]}$ and $R_\Lambda = R$.

All the R_α are birational to R. We define Y as the closure of the image of $R \dashrightarrow \prod_\alpha R_\alpha$, where the product is over all α such that R_α is defined. The projections from $\prod R_\alpha$ induce proper, birational morphisms from Y to every R_α.

3.3 Approximating cohomology classes

This section corresponds to Section 4.5. In what follows we restrict attention to partitions α such that R_α is defined. We define the class $C_\alpha \in H^*(Y)$ by

$$C_\alpha = F(E^{[\![\alpha]\!]}),$$

suppressing the pullback of $E^{[\![\alpha]\!]}$ along $Y \to R_\alpha$. Let $C = C_\Lambda$. By the projection formula, $\deg(C \cap [Y]) = \deg\big(F(E^{[\![n]\!]}) \cap [R]\big)$, so the proof of the main theorem is reduced to showing that $\deg C \cap [Y]$ is universal.

In Definition 4.11, we introduce the class

$$D = C + \sum_{\alpha \neq \Lambda} k_\alpha C_\alpha,$$

where the k_α are integers defined combinatorially via the Möbius inversion formula for the partially ordered set of partitions of $\{1, \ldots, n\}$; see Section 4.5. There is a natural morphism $Y \to X^n$, and one should think of the class D as being supported on (a neighbourhood of) the set $Y|_\Delta$, where $\Delta \subset X^n$ is the small diagonal. The choice of the integers k_α is motivated by the fact (shown in Lemma 4.20; see also Remark 4.13) that they make D vanish on the complement of this locus.

For any $\alpha \neq \Lambda$, the scheme $X^{[\![\alpha]\!]}$ is by definition a product of schemes $X^{[\![m]\!]}$ with $m < n$. This induces product decompositions of $E^{[\![\alpha]\!]}$ and R_α, which allow us to express $\deg(C_\alpha \cap [Y])$ in terms of integrals of Chern classes of $E^{[\![m]\!]}$ over geometric subsets of $X^{[\![m]\!]}$ with $m < n$. By induction on n we can thus show that $\deg(C_\alpha \cap [R_\alpha])$ is universal for $\alpha \neq \Lambda$. This argument gives Lemma 4.14, by which it suffices to show that

$$\deg(D \cap [Y])$$

is universal.

3.4 Relative constructions

Consider the tangent bundle $TX \to X$, and let $\overline{TX} := \mathbb{P}(\mathcal{O}_X \oplus TX)$ be the natural compactification. Let $\mathrm{Hilb}^n(\overline{TX}/X)$ be the relative Hilbert scheme, which parametrises length-n subschemes of fibres of $\overline{TX} \to X$. Note that the formation of the relative Hilbert scheme is compatible with base change; see [13, Remark 3.9] or [20, Exercise 1.4.1.5]. Hence if $U \subset X$ is an open subset such that $TX|_U \cong U \times TX$, then $\mathrm{Hilb}^n(\overline{TX}/X) = U \times (\mathbb{P}^d)^{[n]}$. In particular $\mathrm{Hilb}^n(\overline{TX}/X) \to X$ is a Zariski locally trivial fibration with fibres $(\mathbb{P}^d)^{[n]}$.

Emulating the definition of Y and $E^{[\![\alpha]\!]}$ with $\mathrm{Hilb}^n(\overline{TX}/X)$ replacing $X^{[n]}$, we define the scheme $\mathcal{Y}$ and the bundles $\mathcal{E}^{[\![\alpha]\!]}$ on $\mathcal{Y}$. The classes $\mathcal{C}_\alpha, \mathcal{D} \in H^*(\mathcal{Y})$ are defined similarly to C_α and D. The precise definitions of these objects are given in Section 4.7.

The point of introducing these relative objects is that we can show directly (see Sections 3.5 and 4.8) that the numbers $\deg(\mathcal{C}_\alpha \cap [\mathcal{Y}])$ are universal. The numbers $\deg(\mathcal{C}_\alpha \cap [\mathcal{Y}])$ and $\deg(C_\alpha \cap Y)$ are in general unrelated. On the other hand, the key technical result Lemma 4.20 implies that $\deg(D \cap [Y]) = \deg(\mathcal{D} \cap [\mathcal{Y}])$, and the number $\deg(\mathcal{D} \cap [\mathcal{Y}])$ is universal since it is a linear combination of the $\deg(\mathcal{C}_\alpha \cap [\mathcal{Y}])$.

Denote by $\overline{TX}^n$ the n–fold fibre product of $\overline{TX}$ over X. There are natural morphisms $\pi_Y\colon Y \to X^n$ and $\pi_{\mathcal{Y}}\colon \mathcal{Y} \to \overline{TX}^n$, where π_Y is given by composing $Y \to X^{[\![n]\!]}$ and $X^{[\![n]\!]} \to X^n$, and $\pi_{\mathcal{Y}}$ is defined similarly. Let $\Delta \subset X^n$ be the small diagonal, and consider X as a subset of $TX^n \subset \overline{TX}^n$ using n copies of the 0–section.

Let $U \subset Y$ and $\mathcal{U} \subset \mathcal{Y}$ be Euclidean open neighbourhoods of $\pi_Y^{-1}(\Delta)$ and $\pi_{\mathcal{Y}}^{-1}(X)$, respectively. Choosing U and $\mathcal{U}$ small enough, we show in Lemma 5.1 that we can find a topological isomorphism $f_1\colon U \to \mathcal{U}$ together with topological isomorphisms of the bundles $f_1^*(\mathcal{E}^{[\![\alpha]\!]}) \cong E^{[\![\alpha]\!]}$. The map f_1 moreover restricts to an orientation-preserving homeomorphism on the nonsingular parts of U and $\mathcal{U}$, where these are oriented by their complex structure.

To define the map f_1, we follow [23] and begin with an exponential map $\exp\colon TX \to X^2$. This map is defined near the 0–section $X \subset TX$, and maps a neighbourhood of the 0–section homeomorphically onto a neighbourhood of the diagonal $\Delta \subset X^2$. The map exp is analytic when restricted to a fibre of TX.

In Lemma 5.2, we show that f_1 further induces a local homeomorphism of Hilbert schemes $f_2\colon X^{[n]} \to \mathrm{Hilb}^n(\overline{TX}/X)$, defined in a neighbourhood of the locus of subschemes whose support is one point. Finally, tracing through the parallel steps in the definitions of Y and $\mathcal{Y}$ we get Lemma 5.3, which gives the homeomorphism $f\colon U \to \mathcal{U}$. Crucially, we also show that $f^*(\mathcal{E}^{[\![\alpha]\!]}) \cong E^{[\![\alpha]\!]}$ as topological vector bundles.

Recall the class $D = C + \sum_{\alpha \neq \Lambda} k_\alpha C_\alpha \in H^*(Y)$. Lemma 4.20 says that the restriction of D to $Y \setminus U$ vanishes. The proof uses the fact that the bundles $E^{[\![\alpha]\!]}$ are canonically isomorphic over various open subsets of Y, together with the (nontrivial) fact that the contributions from different C_α cancel out appropriately. This cancellation is the property motivating the definition of the integers k_α. Similarly Lemma 4.20 also shows that $\mathcal{D}$ vanishes upon restriction to $\mathcal{Y}\setminus\mathcal{U}$.

Hence the classes D and $\mathcal{D}$ are concentrated over U and $\mathcal{U}$, respectively. In particular, there are relative cohomology classes $D_{\mathrm{rel}} \in H^*(Y, Y \setminus U)$ and $\mathcal{D}_{\mathrm{rel}} \in H^*(\mathcal{Y}, \mathcal{Y}\setminus\mathcal{U})$ lifting D and $\mathcal{D}$.

There is a map $f^*\colon H^*(\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}) \to H^*(Y, Y \setminus U)$, defined by excision, possibly after shrinking U and $\mathcal{U}$. Lemma 4.20 furthermore shows that we can choose D_{rel} and $\mathcal{D}_{\mathrm{rel}}$ in such a way that $f^*(\mathcal{D}_{\mathrm{rel}}) = D_{\mathrm{rel}}$.

We may define the "relative fundamental class" $[Y, Y \setminus U] \in H_*(Y, Y \setminus U)$ as the image of $[Y]$ under the map $H_*(Y) \to H_*(Y, Y \setminus U)$, and we may similarly define $[\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}] \in H_*(\mathcal{Y},\mathcal{Y}\setminus\mathcal{U})$. Since f is locally an orientation-preserving homeomorphism on the nonsingular parts of Y and $\mathcal{Y}$, we get $f_*([Y, Y\setminus U]) = [\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}]$. Hence we get

$$\deg D \cap [Y] = \deg D_{\mathrm{rel}} \cap [Y, Y \setminus U] = \deg \mathcal{D}_{\mathrm{rel}} \cap [\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}] = \deg \mathcal{D} \cap [\mathcal{Y}].$$

The proof of Lemma 4.20 is the most technical part of the paper and occupies Section 5.

See [14, Theorem 2.20] and [14, Section 3.3] for a description of excision and the cap product in singular homology.

3.5 Pullback from the Grassmannian

Let $H_1 \to \mathrm{Gr}(d, N_1)$ be the universal rank-d subbundle over a Grassmannian. Here N_1 is any integer large enough that TX embeds as a topological subbundle of $\mathcal{O}_X^{N_1}$, so that there is a continuous classifying map $\psi_{TX}\colon X \to \mathrm{Gr}(d, N_1)$ with $TX = \psi_{TX}^*(H_1)$ as topological bundles. We define the scheme $\mathcal{Y}_{\mathrm{Gr}}$ by the same construction as $\mathcal{Y}$, replacing $\mathrm{Hilb}^n(\overline{TX}/X)$ with $\mathrm{Hilb}^n(\overline{H}_1/\mathrm{Gr}(d, N_1))$ throughout. There is a natural morphism $\mathcal{Y}_{\mathrm{Gr}} \to \mathrm{Gr}(d, N_1)$ and the Cartesian diagram of topological spaces:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}_{\mathrm{Gr}} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{\psi_{TX}} & \mathrm{Gr}(d, N_1) \end{array}$$

Let $\psi_E\colon X \to \mathrm{Gr}(r, N_2)$ be a continuous classifying map for E, and consider the Cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad g \quad} & \mathcal{Y}_{\mathrm{Gr}} \times \mathrm{Gr}(r, N_2) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{(\psi_{TX},\psi_E)} & \mathrm{Gr}(d, N_1) \times \mathrm{Gr}(r, N_2) \end{array}$$

We emphasise that the horizontal arrows in these diagrams are not required to be analytic. We define bundles $\mathcal{E}_{\mathrm{Gr}}^{[\![\alpha]\!]}$ on $\mathcal{Y}_{\mathrm{Gr}} \times \mathrm{Gr}(r, N_2)$ such that $g^*(\mathcal{E}_{\mathrm{Gr}}^{[\![\alpha]\!]}) = \mathcal{E}^{[\![\alpha]\!]}$.

From the above it follows that there is a class $D_{\mathrm{Gr}} \in H^*(\mathrm{Gr}(d, N_1) \times \mathrm{Gr}(r, N_2))$, depending only on F and the type of R, such that

$$\pi_*(\mathcal{D} \cap [\mathcal{Y}]) = (\psi_{TX}, \psi_E)^*(D_{\mathrm{Gr}}) \cap [X].$$

The rational cohomology ring of a Grassmannian is generated by the Chern classes of its universal bundle. Combining this with the Künneth formula for $\mathrm{Gr}(d, N_1) \times \mathrm{Gr}(r, N_2)$ we find that $(\psi_{TX}, \psi_E)^*(D_{\mathrm{Gr}})$ is a polynomial in the Chern classes of T_X and E. We show that this polynomial is independent of the choice of N_1 and N_2. Hence $\deg \mathcal{D} \cap [\mathcal{Y}]$ is a universal linear combination of the Chern numbers of (X, E), which concludes the proof of the main theorem.

4 Proof of main theorem

We now begin the formal proof of the main theorem. To avoid dealing with Theorems 1.1 and 1.2 separately, we adopt the following convention: when a formula includes $T_{X^{[n]}}$, terms involving $T_{X^{[n]}}$ should be ignored unless $X^{[n]}$ is nonsingular, and so should all other statements involving $T_{X^{[n]}}$. In this section and the next we have X, P, E, d and F as in the main theorem.

4.1 Reduction to irreducible sets

We first show that we may assume P to be irreducible.

Lemma 4.1 *In order to prove Theorems 1.1/1.2, it suffices to prove the same theorems with the extra assumption that P is closed and irreducible.*

Proof We first treat parts (i) and (ii) of the theorems. For a closed and pure-dimensional P, we let $P_1, \dots, P_j$ be its irreducible components. Arguing as in the proof of Lemma 2.11(v), we see that the P_i are geometric of type determined by the type of P. The statement of the lemma now follows from

$$[P] = \sum [P_i] \quad \text{and} \quad c_{\mathrm{M}}(P) = \sum c_{\mathrm{M}}(P_i).$$

For part (iii), let P be any constructible geometric subset. We may write the characteristic function 1_P as $1_P = \sum_i m_i 1_{P_i}$, where the m_i are integers and the P_i are closed, irreducible and geometric subsets. The m_i and the types of the P_i are determined by the type of P. The claim of the lemma follows. □

4.2 Reduction to $X^{[\![n]\!]}$

Recall the Hilbert scheme of ordered points $X^{[\![n]\!]}$, defined by the Cartesian diagram

$$\begin{array}{ccc} X^{[\![n]\!]} & \longrightarrow & X^{[n]} \\ \downarrow & & \downarrow \\ X^n & \longrightarrow & \mathrm{Sym}^n(X) \end{array}$$

where the right-hand arrow is the Hilbert–Chow morphism.

Definition 4.2 Denote the pullbacks of $E^{[n]}$ and $T_{X^{[n]}}$ along $X^{[\![n]\!]} \to X^{[n]}$ by $E^{[\![n]\!]}$ and $T^{[\![n]\!]}_{X^{[n]}}$, respectively.

We use the projection formula to relate the degree of $F(E^{[n]}, T_{X^{[n]}}) \cap [P]$ to a similar class involving $E^{[\![n]\!]}$ and $T^{[\![n]\!]}_{X^{[n]}}$ on $X^{[\![n]\!]}$. The first step is to produce a closed, irreducible geometric subset $R \subseteq X^{[\![n]\!]}$ mapping finitely onto P with universal degree.

Lemma 4.3 *For any closed, irreducible, geometric $P \subseteq X^{[n]}$, there exists a closed, irreducible, geometric $R \subseteq X^{[\![n]\!]}$ such that the map $X^{[\![n]\!]} \to X^{[n]}$ maps R finitely onto P. Up to a permutation of $\{1, \dots, n\}$, the type of this R is uniquely determined by the type of P.*

Proof As P is closed and irreducible, by Lemma 2.11(v) we have $P = \overline{P((Q_i))}$ for closed and irreducible punctual geometric subsets $Q_i \subset \mathrm{Hilb}_0^{n_i}(\mathbb{C}^d)$. We then take $A = (A_1, \dots, A_k)$ to be a k–element partition of $\{1, \dots, n\}$ such that $|A_k| = n_k$, and let $R = \overline{R((Q_i); A)}$. By Lemma 2.11(vi), we see that R must have this form, hence the second claim follows. □

Lemma 4.4 *In order to prove the main theorem, it suffices to prove the following statement: if R is a closed, irreducible, geometric subset of $X^{[\![n]\!]}$, then*

$$\deg F(E^{[\![n]\!]}, T^{[\![n]\!]}_{X^{[n]}}) \cap c_{\mathrm{M}}(R)$$

is given by a universal polynomial depending only on F and the type of R, such that the degree of the polynomial is at most l, where l is the maximum number of components of Z for $(Z, (x_i)) \in R$.

Proof We first show that part (iii) of the theorems follows from the hypothesis of the lemma. Let $p\colon X^{[\![n]\!]} \to X^{[n]}$ be the natural morphism. Let $P \subseteq X^{[n]}$ be a closed and irreducible geometric subset; by Lemma 4.1 it suffices to prove the main theorems for such P. Let $R \subseteq X^{[\![n]\!]}$ be a closed, irreducible geometric set mapping finitely onto P, as provided by Lemma 4.3.

We claim that the level sets of the local Euler obstruction Eu_R are geometric. To see this, begin by writing R as a union of sets of the form $R((Q_i); A)$, as can be done by Lemma 2.11(iv). Next consider a point $(Z, (x_j)) \in R$, and suppose that $(Z, (x_j)) \in R((Q_i); A) \subseteq R$ with $A = (A_1, \dots, A_k)$. We have $Z = \bigsqcup_i Z_i$ with each Z_i isomorphic to an element of Q_i and $x_j = \operatorname{Supp} Z_i$ for $j \in A_i$. The local analytic structure around $(Z, (x_j))$ in R is determined by the isomorphism types of the Z_i. Furthermore, a sufficiently small neighbourhood of $(Z, (x_j))$ in R is analytically isomorphic to a product $\prod_{i=1}^k U_i$, where the analytic structure of U_i is determined by the isomorphism type of Z_i.

Let π_i be the locally defined projection $R \to U_i$. Using the product formula for the local Euler obstruction [26, page 426], we get

$$\mathrm{Eu}_R\big((Z, (x_j))\big) = \mathrm{Eu}_{\prod U_i}\big((Z, (x_j))\big) = \prod_i \mathrm{Eu}_{U_i}\big(\pi_i((Z, (x_j)))\big).$$

Since $\mathrm{Eu}_{U_i}\big(\pi_i(Z, (x_j))\big)$ depends only on the isomorphism type of Z_i, this implies that the level sets of Eu_R intersected with $R((Q_i); A)$ are geometric. It follows that the complete level sets of Eu_R are geometric.

As we have $c_{\mathrm{SM}}(\mathrm{Eu}_R) = c_{\mathrm{M}}(R)$ by definition, we get

$$c_{\mathrm{SM}}(R) = c_{\mathrm{SM}}(1_R) = c_{\mathrm{SM}}(\mathrm{Eu}_R) + c_{\mathrm{SM}}(1_R - \mathrm{Eu}_R) = c_{\mathrm{M}}(R) + \sum i\, c_{\mathrm{SM}}(R_i),$$

where the sum is finite and the R_i are geometric subsets of $X^{[\![n]\!]}$ of lower dimension than R, with type depending only on the type of R. By induction on $\dim R$, we may assume the terms $c_{\mathrm{SM}}(R_i)$ are universal, and the hypothesis of the lemma is that $c_{\mathrm{M}}(R)$ is as well. Hence $\deg F(E^{[\![n]\!]}, T^{[\![n]\!]}_{X^{[n]}}) \cap c_{\mathrm{SM}}(R)$ is universal.

We may write

$$p_*(1_R) = \deg(R/P) \cdot 1_P + \sum_{i \in \mathbb{Z}} i \cdot 1_{P_i},$$

where the P_i are constructible subsets of lower dimension than P. By Lemma 2.13 the P_i are geometric, and it is easy to see that their type is determined by the type of P. The functorial property of CSM classes then gives

$$p_*(c_{\mathrm{SM}}(R)) = \deg(R/P) c_{\mathrm{SM}}(P) + \sum i\, c_{\mathrm{SM}}(P_i).$$

By induction on $\dim P$, we may assume that the integers $\deg F(E^{[n]}, T_{X^{[n]}}) \cap c_{\mathrm{SM}}(P_i)$ are universal. Since $\deg F(E^{[n]}, T_{X^{[n]}}) \cap p_*(c_{\mathrm{SM}}(R))$ is universal by the above, it follows that $\deg F(E^{[n]}, T_{X^{[n]}}) \cap c_{\mathrm{SM}}(P)$ is universal. This proves Theorems 1.1/1.2(iii).

For a closed, irreducible P, we have $[P] = c_{\mathrm{SM}}(P) + C$, where C is a class of lower homological dimension. Hence part (i) follows from (iii).

For part (ii), note that $c_{\mathrm{M}}(P) = c_{\mathrm{SM}}(\mathrm{Eu}_P)$. A similar argument to the one above shows that the level sets of Eu_P are geometric, and hence part (ii) follows. □

4.3 Partitions

By a partition of $\{1,\dots,n\}$, we mean a set α of disjoint, nonempty subsets of $\{1,\dots,n\}$, such that $\bigcup_{A\in\alpha} A = \{1,\dots,n\}$. Following [23], we will define schemes $X^{[\![\alpha]\!]}$ approximating $X^{[\![n]\!]}$ for each such α. We fix some notation and conventions with respect to partitions.

Definition 4.5 We let $\sim_\alpha$ be the equivalence relation on $\{1,\dots,n\}$ given by letting the elements of α form equivalence classes.

We define a partial ordering on the set of partitions of $\{1,\dots,n\}$ by letting $\alpha \le \beta$ if every element of α is contained in an element of β, as in [34, Example 3.1.1.d]. Equivalently, $\alpha \le \beta$ if $\sim_\alpha$ is a finer relation than $\sim_\beta$.

We denote by Λ the maximal partition under this ordering, that is, $\Lambda = \{\{1,\dots,n\}\}$. Given two partitions α and β, we denote by $[\alpha,\beta]$ the set of partitions γ such that $\alpha \le \gamma \le \beta$, and define $[\alpha,\beta)$ etc similarly.

4.4 Approximating constructions

From this point on we fix a closed, irreducible, geometric subscheme $R \subseteq X^{[\![n]\!]}$. In this section, we define the schemes $X^{[\![\alpha]\!]}$, the bundles $E^{[\![\alpha]\!]}$ and $T^{[\![\alpha]\!]}_{X^{[n]}}$, the subsets $R_\alpha \subseteq X^{[\![\alpha]\!]}$ and the cohomology classes C_α and D.

Definition 4.6 If α is a partition of $\{1,\dots,n\}$, define the scheme $X^{[\![\alpha]\!]}$ by

$$X^{[\![\alpha]\!]} = \prod_{A\in\alpha} X^{[\![A]\!]},$$

where $X^{[\![A]\!]} \cong X^{[\![|A|]\!]}$ and parametrises pairs $(Z,(x_i)_{i\in A})$ such that $\sum_{i\in A} x_i$ is the fundamental cycle of Z.

There is a natural morphism $X^{[\![\alpha]\!]} \to X^n$ defined by the decomposition $X^n = \prod_{A\in\alpha} X^A$ and the natural morphisms $X^{[\![A]\!]} \to X^A$.

Definition 4.7 Define the vector bundles $E^{[\![\alpha]\!]}$ and $T^{[\![\alpha]\!]}_{X^{[n]}}$ on $X^{[\![\alpha]\!]}$ by

$$E^{[\![\alpha]\!]} = \bigoplus_{A\in\alpha} E^{[\![A]\!]} \quad\text{and}\quad T^{[\![\alpha]\!]}_{X^{[n]}} = \bigoplus_{A\in\alpha} T^{[\![A]\!]}_{X^{[A]}},$$

where we suppress pullback along the projection $X^{[\![\alpha]\!]} \to X^{[\![A]\!]}$.

Definition 4.8 If α is a partition of $\{1,\dots,n\}$, denote by Δ_α the subset of X^n given by

$$\Delta_\alpha = \{(x_1,\dots,x_n) \in X^n \mid x_i = x_j \text{ if } i \sim_\alpha j\}.$$

We refer to the sets Δ_α as *diagonals*.

Write $R = \overline{R((Q_i);A)}$, as is possible by Lemma 2.11(vi). We define the partition μ of $\{1,\dots,n\}$ by $\mu = \{A_1,\dots,A_k\}$. The image of R under the map $X^{[\![n]\!]} \to X^n$ is Δ_μ.

Our next task is to define schemes $R_\alpha \subseteq X^{[\![\alpha]\!]}$ birational to R, for those α such that this is possible. For any α, let $f_\alpha\colon X^{[\![n]\!]} \dashrightarrow X^{[\![\alpha]\!]}$ be the natural isomorphism defined on the open set where the moduli problems $X^{[\![n]\!]}$ and $X^{[\![\alpha]\!]}$ solve are the same. Specifically, f_α is defined on the set of points $(Z,(x_i))$ where $x_i \neq x_j$ if $i \not\sim_\alpha j$.

The locus where f_α is defined intersects R if and only if $\alpha \geq \mu$. For such α we let R_α be the closure of $f_\alpha(R)$ in $X^{[\![\alpha]\!]}$.

Recall that $\widetilde{R}_\alpha \to R_\alpha$ denotes the Nash blow-up. As $\widetilde{R}_\alpha \to R_\alpha$ is birational, the map f_α induces a natural rational map $g_\alpha\colon R \dashrightarrow \widetilde{R}_\alpha$.

Definition 4.9 Let

$$g := (g_\alpha)_{\alpha\geq\mu}\colon R \dashrightarrow \prod_{\alpha\geq\mu} \widetilde{R}_\alpha,$$

and define Y to be the closure of $g(R)$ in $\prod \widetilde{R}_\alpha$.

For every α there are birational proper morphisms $Y \to \widetilde{R}_\alpha \to R_\alpha$. Any cohomology class on R_α and $\widetilde{R}_\alpha$ may be pulled back along these morphisms without changing the degree, and we will suppress such pullbacks in the notation.

4.5 Approximations of the cohomology classes

The schemes and bundles defined in the previous section give rise to cohomology classes approximate to the one we want to compute; see [23, Section 5.4]. Recall that Λ denotes the maximal partition of $\{1,\dots,n\}$, and that $T_{\widetilde{R}_\alpha}$ is the Nash bundle on $\widetilde{R}_\alpha$.

Definition 4.10 Let α be a partition $\geq \mu$. Define the class $C_\alpha \in H^*(Y)$ by

$$C_\alpha = F(E^{[\![\alpha]\!]}, T_{X^{[n]}}^{[\![\alpha]\!]}) \cup c_\bullet(T_{\widetilde{R}_\alpha}).$$

We let $C = C_\Lambda$.

Note that the main theorem is reduced to the claim that $\deg(C \cap [Y])$ is universal.

Definition 4.11 Let α be a partition $\geq \mu$. Define the classes $D_\alpha \in H^*(Y)$ by putting $D_\mu = C_\mu$ and for $\alpha > \mu$ letting D_α be defined inductively by

$$D_\alpha = C_\alpha - \sum_{\gamma \in [\mu, \alpha)} D_\gamma.$$

We let $D = D_\Lambda$.

Remark 4.12 The definition of the D_α is an instance of Möbius inversion. Namely, consider the map from the partially ordered set $([\mu, \Lambda], \leq)$ to $H^*(Y)$ given by $\alpha \mapsto C_\alpha$. The Möbius inversion formula then gives

$$D_\alpha = \sum_{\gamma \in [\mu, \alpha]} k(\gamma, \alpha) C_\gamma,$$

where $k(\gamma, \alpha)$ is the Möbius function of $([\mu, \Lambda], \leq)$; see eg [34, Proposition 3.7.1].

The function k is easily calculated; see [34, Example 3.10.4]. In particular, we have $k_\alpha := k(\alpha, \Lambda) = (-1)^{|\alpha|-1}(|\alpha|-1)!$ and so

$$D = \sum_{\alpha \geq \mu} (-1)^{|\alpha|-1}(|\alpha|-1)!\, C_\alpha.$$

Except in the proof of Proposition 6.4, we will not need this, and we work instead directly with the inductive definition of D_α and D.

The motivation behind the definition of D_α is as follows. First, it follows directly from the definition that if $\deg(D \cap [Y])$ and $\deg(C_\alpha \cap [Y])$ are universal for $\alpha \neq \Lambda$, then $\deg(C \cap [Y])$ is universal as well. Using induction on n, we will show in Lemma 4.14 that the degree of $C_\alpha \cap [Y]$ is universal for $\alpha \neq \Lambda$, reducing the problem to that of computing D.

Second, as we show in Lemma 4.20, D is such that the restriction of D to $Y \setminus (Y|_{\Delta_\Lambda})$ vanishes, which allows us to reduce the computation of its degree to studying a small neighbourhood of $Y|_{\Delta_\Lambda}$.

Remark 4.13 We can give a formal analogy explaining the combinatorics underlying the definition of the D_α. For any partition α, let c_α be a (not necessarily continuous) function on X^n, such that $c_\alpha = c_\beta$ on the locus in X^n over which $X^{[\![\alpha]\!]}$ and $X^{[\![\beta]\!]}$ agree. Specifically $c_\alpha = c_\beta$ on $X^n \setminus \Delta_{\alpha,\beta}$, where

$$\Delta_{\alpha,\beta} := \bigcup_{\substack{i \sim_\alpha j \\ i \not\sim_\beta j}} \Delta_{ij} \cup \bigcup_{\substack{i \sim_\alpha j \\ i \not\sim_\beta j}} \Delta_{ij}.$$

Here $\Delta_{ij} \subset X^n$ is the set of points $(x_k) \in X^n$ satisfying $x_i = x_j$. The functions c_α are analogous to the classes C_α, and the equality $c_\alpha = c_\beta$ over $X^n \setminus \Delta_{\alpha,\beta}$ corresponds to the fact that C_α and C_β are canonically equal over this set.

We may now define d_α by $d_\alpha = c_\alpha - \sum_{\gamma<\alpha} d_\gamma$; these functions are analogous to D_α. The functions d_α then vanish on $X^n \setminus \Delta_\alpha$; this is a combinatorial fact which is easy to show by ascending induction on the partially ordered set of partitions. In particular d_Λ vanishes on $X^n \setminus \Delta_\Lambda$, in analogy with the vanishing of $D = D_\Lambda$ away from Δ_Λ shown in Lemma 4.20 and Lemma 5.9.

4.6 Reduction to $\deg(D \cap [Y])$

If α is a nonmaximal partition of $\{1,\dots,n\}$, the scheme $X^{[\![\alpha]\!]}$ is by definition a product of schemes $X^{[\![m]\!]}$ with $m<n$. Hence we can reduce the computation of C_α to computations of cohomology classes on such $X^{[\![m]\!]}$, and if these are universal, then C_α will be as well. This argument leads to the following induction result.

Lemma 4.14 *Let m be a positive integer. Suppose that Theorem 1.1/1.2 holds for every $n<m$, and suppose that for $n=m$ the degree of $D \cap [Y]$ is given by a universal linear polynomial in the Chern numbers of (X,E). Then Theorem 1.1/1.2 holds for $n=m$.*

Proof Assume that the theorem holds for every $n<m$. We shall then show that for every partition $\alpha \in [\mu,\Lambda)$, the degree of $C_\alpha \cap [Y]$ is expressed by a universal polynomial. Since we have $C = D_\Lambda - \sum_{\alpha\in[\mu,\Lambda)} k_\alpha C_\alpha$, the statement of the lemma follows.

Let $\alpha \in [\mu,\Lambda)$. Recall first that by definition there is a product decomposition

$$X^{[\![\alpha]\!]} = \prod_{A\in\alpha} X^{[\![A]\!]}.$$

This gives rise to a product decomposition $R_\alpha = \prod_{A\in\alpha} R_A$, where the $R_A \subseteq X^{[\![A]\!]}$ are closed, irreducible, geometric subsets. The Nash blow-up preserves products, by the

$n=1$ case of [36, Theorem 1.1], using the observation that in the notation of loc. cit. the Nash blow-up of X is $\mathrm{fNash}_1(X)$; see [36, page 1001]. We therefore have

$$\widetilde{R}_\alpha = \prod \widetilde{R}_A,$$

as well as bundle decompositions

$$E^{[\![\alpha]\!]} = \bigoplus E^{[\![A]\!]}, \quad T_{\widetilde{R}_\alpha} = \bigoplus T_{\widetilde{R}_A} \quad \text{and} \quad T^{[\![\alpha]\!]}_{X^{[n]}} = \bigoplus T^{[\![A]\!]}_{X^{[n]}}.$$

Now, using the Whitney sum formula we can find an expression for

$$C_\alpha = F(E^{[\![\alpha]\!]}, T^{[\![\alpha]\!]}_{X^{[n]}}) \cdot c_\bullet(T_{\widetilde{R}_\alpha})$$

as a polynomial in the Chern classes of $E^{[\![A]\!]}$, $T^{[\![A]\!]}_{X^{[n]}}$ and $T_{\widetilde{R}_A}$ for different $A \in \alpha$. Since $\alpha < \Lambda$, we have $|A| < m$ for every $A \in \alpha$. By the induction hypothesis, we thus get a universal polynomial for

$$\deg C_\alpha \cap [Y] = \deg C_\alpha \cap [R_\alpha],$$

as required.

The claim about the degree of the universal polynomial G in the main theorem also follows by induction, using the assumption that $\deg(D \cap [Y])$ is linear as a polynomial in the Chern numbers of (X, E). □

Since the theorem is clear for $n=0$, it now suffices to show that the degree of $D \cap [Y]$ is given by a linear polynomial in the Chern numbers of (X, E).

4.7 Relative constructions

We will show in Lemma 4.20 that the class D vanishes when restricted to the part of Y lying over the complement of the small diagonal $\Delta_\Lambda \subset X^n$. It may thus essentially be computed by looking at a neighbourhood of $Y|_{\Delta_\Lambda}$. The next step is now to use this to show that the degree of D equals that of a class $\mathcal{D} \in H^*(\mathcal{Y})$, where $\mathcal{Y}$ is a scheme defined similarly to Y, but with $X^{[n]}$ replaced with the relative Hilbert scheme $\mathrm{Hilb}^n(\overline{TX}/X)$; see [13] or [20, Chapter 1] for background on relative Hilbert schemes.

We therefore repeat the constructions of approximating schemes and classes in this relative setting. These are for the most part straightforward adaptations of the constructions in Sections 4.4 and 4.5. The exception to this is the scheme $\mathcal{R}$ that corresponds to R (and so the schemes $\mathcal{R}_\alpha$ and $\mathcal{Y}$ which are derived from $\mathcal{R}$), where we impose the condition that the first marked point must lie in the 0–section $X \subset \overline{TX}$.

In order to integrate cohomology classes it will be convenient to work with proper schemes. Hence we let $\overline{TX}$ denote the $\mathbb{P}^d$–bundle $\mathbb{P}(\mathcal{O}_X \oplus TX)$, with the convention that $\mathbb{P}(V)$ is the set of lines through the origin in V. Let $\pi\colon \overline{TX} \to X$ be the projection, and let $\overline{TX}^n$, $\mathrm{Sym}^n(\overline{TX}/X)$ and $\mathrm{Hilb}^n(\overline{TX}/X)$ denote, respectively, the fibre product, relative symmetric product and relative Hilbert scheme of $\overline{TX}$ over X.

Definition 4.15 Define the scheme $\overline{TX}^{[\![n]\!]}$ by the cartesian diagram:

$$\begin{array}{ccc} \overline{TX}^{[\![n]\!]} & \longrightarrow & \mathrm{Hilb}^n(\overline{TX}/X) \\ \downarrow & & \downarrow \\ \overline{TX}^n & \longrightarrow & \mathrm{Sym}^n(\overline{TX}/X) \end{array}$$

Note that as usual we will only consider the reduced scheme structures on $\mathrm{Hilb}^n(\overline{TX}/X)$ and $\overline{TX}^{[\![n]\!]}$.

Definition 4.16 Let $\mathcal{E}^{[n]}$ be the tautological bundle on $\mathrm{Hilb}^n(\overline{TX}/X)$ corresponding to the vector bundle $\mathcal{E} = \pi^*(E)$ on $\overline{TX}$, and let $\mathcal{E}^{[\![n]\!]}$ be the pullback of $\mathcal{E}^{[n]}$ to $\overline{TX}^{[\![n]\!]}$. If $X^{[n]}$ is nonsingular, let $T_{\overline{TX}^{[n]}}$ denote the relative tangent bundle of $\mathrm{Hilb}^n(\overline{TX}/X) \to X$, and denote its pullback to $\overline{TX}^{[\![n]\!]}$ by $T^{[\![n]\!]}_{\overline{TX}^{[n]}}$.

Definition 4.17 Let α be a partition of $\{1,\dots,n\}$. Define the scheme $\overline{TX}^{[\![\alpha]\!]}$ by

$$\overline{TX}^{[\![\alpha]\!]} = \prod_{A\in\alpha} \overline{TX}^{[\![A]\!]},$$

where the product is the fibre product over X. Define the bundles $\mathcal{E}^{[\![\alpha]\!]}$ and $T^{[\![\alpha]\!]}_{\overline{TX}^{[n]}}$ on $\overline{TX}^{[\![\alpha]\!]}$ by

$$\mathcal{E}^{[\![\alpha]\!]} = \bigoplus_{A\in\alpha} \mathcal{E}^{[\![A]\!]} \quad \text{and} \quad T^{[\![\alpha]\!]}_{\overline{TX}^{[n]}} = \bigoplus_{A\in\alpha} T^{[\![A]\!]}_{\overline{TX}^{[A]}},$$

suppressing notation for the natural pullbacks.

For $x \in X$, let $\overline{TX}_x = \pi^{-1}(x)$. Then the set of points in $\overline{TX}^{[\![n]\!]}$ (resp. $\overline{TX}^{[\![\alpha]\!]}$) is the union of $(\overline{TX}_x)^{[\![n]\!]}$ (resp. $(\overline{TX}_x)^{[\![\alpha]\!]}$) for all $x \in X$. A point of $\overline{TX}^{[\![n]\!]}$ can thus be described by a pair $(Z, (v_i)_{i=1}^n)$, where $Z \in (\overline{TX}_x)^{[n]}$ and $v_i \in \overline{TX}_x$ for some $x \in X$, subject to the requirement that the support 0–cycle of Z equals $\sum v_i$.

We can now define the subset $\mathcal{R} \subset \overline{TX}^{[\![n]\!]}$ which plays the role of R in the relative setting. First let $\mathcal{R}'$ be the subset of $\overline{TX}^{[\![n]\!]}$ such that for each $x \in X$, the set $\mathcal{R}' \cap (\overline{TX}_x)^{[\![n]\!]} \subset (\overline{TX}_x)^{[\![n]\!]}$ is the geometric subset of the same type as R, where the type of a geometric subset is as in Definition 2.10.

Let $p\colon \overline{TX}^{[\![n]\!]} \to \overline{TX}$ be the morphism defined by $p((Z,(v_i))) = v_1$, and consider X as a subset of $\overline{TX}$ by embedding along the 0–section. We then let $\mathcal{R} = p^{-1}(X) \cap \mathcal{R}'$.

For every partition α, there is a rational map $f_\alpha\colon \overline{TX}^{[\![n]\!]} \dashrightarrow \overline{TX}^{[\![\alpha]\!]}$ defined where the moduli problems the two schemes solve are the same. Using these maps, we may replace R by $\mathcal{R}$ in Definition 4.9 and the preceding paragraphs, thus defining the schemes $\mathcal{R}_\alpha$ (for $\alpha \geq \mu$), $\widetilde{\mathcal{R}}_\alpha$ and $\mathcal{Y}$. We omit the details.

Finally, we define the relative analogues of the classes C_α and D_α.

Definition 4.18 Let α be a partition $\geq \mu$. Define the class $\mathcal{C}_\alpha \in H^*(\mathcal{Y})$ by

$$\mathcal{C}_\alpha = F(\mathcal{E}^{[\![\alpha]\!]}, T^{[\![\alpha]\!]}_{\overline{TX}^{[n]}}) \cup c_\bullet(T_{\widetilde{\mathcal{R}}_\alpha}).$$

We let $\mathcal{C} = \mathcal{C}_\Lambda$.

Definition 4.19 Let α be a partition $\geq \mu$. Define the classes $\mathcal{D}_\alpha \in H^*(\mathcal{Y})$ by putting $\mathcal{D}_\mu = \mathcal{C}_\mu$ and for $\alpha > \mu$ letting $\mathcal{D}_\alpha$ be defined inductively by

$$\mathcal{D}_\alpha = \mathcal{C}_\alpha - \sum_{\gamma \in [\mu,\alpha)} \mathcal{D}_\gamma.$$

We let $\mathcal{D} = \mathcal{D}_\Lambda$.

4.8 Relating D_Λ to $\mathcal{D}_\Lambda$

Let $Y_0 \subseteq Y$ be the restriction of Y to the small diagonal $X = \Delta_\Lambda \subset X^n$. Similarly let $\mathcal{Y}_0 \subseteq \mathcal{Y}$ be the restriction of $\mathcal{Y}$ to the set $X \subset \overline{TX}^n$ where the inclusion of X is given by n copies of the 0–section. The classes D and $\mathcal{D}$ are related by the following lemma and its corollary.

Lemma 4.20 *There exists a relatively compact pair of open neighbourhoods $U' \Subset U$ of Y_0 in Y, a relatively compact pair of open neighbourhoods $\mathcal{U}' \Subset \mathcal{U}$ of $\mathcal{Y}_0$ in $\mathcal{Y}$, and a homeomorphism $f\colon (U, U') \to (\mathcal{U}, \mathcal{U}')$. Furthermore, there exists a class $\mathcal{D}_{\mathrm{rel}} \in H^*(\mathcal{Y}, \mathcal{Y} \setminus \mathcal{U}')$ lifting $\mathcal{D} \in H^*(\mathcal{Y})$ such that the composition*

(4-1) $$H^*(\mathcal{Y}, \mathcal{Y}\setminus\mathcal{U}') \to H^*(\mathcal{U}, \mathcal{U}\setminus\mathcal{U}') \xrightarrow{f^*} H^*(U, U\setminus U') \to H^*(Y, Y\setminus U') \to H^*(Y)$$

sends $\mathcal{D}_{\mathrm{rel}}$ to D.

By construction, the map f will be orientation-preserving when restricted to the nonsingular open subsets U_{ns} and $\mathcal{U}_{\mathrm{ns}}$, where these are oriented by the complex structure.

Corollary 4.21 *The degree of $D \cap [Y]$ equals the degree of $\mathcal{D} \cap [\mathcal{Y}]$.*

Proof of Corollary 4.21 There are relative fundamental classes

$$[(\mathcal{Y}, \mathcal{Y} \setminus \mathcal{U}')], \quad [(\mathcal{U}, \mathcal{U} \setminus \mathcal{U}')], \quad [(U, U \setminus U')] \quad \text{and} \quad [(Y, Y \setminus U')]$$

in the appropriate homology groups. Replacing H^* with H_* in the above sequence (and reversing the arrows) each fundamental class is sent to the next, so in the composition the class $[Y]$ is sent to $[(\mathcal{Y},\mathcal{Y}\setminus\mathcal{U})]$. This implies

$$\deg(D\cap[Y])=\deg\bigl(\mathcal{D}_{\mathrm{rel}}\cap[(\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}')]\bigr).$$

Now, $[(\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}')]$ is the pushforward of $[\mathcal{Y}]$, which shows that

$$\deg\bigl(\mathcal{D}_{\mathrm{rel}}\cap[(\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}')]\bigr)=\deg(\mathcal{D}\cap[\mathcal{Y}]),$$

completing the proof. □

The proof of Lemma 4.20 is quite technical and is postponed to Section 5. We now show how the main theorem follows from Corollary 4.21.

Proof of Theorems 1.1/1.2 By Corollary 4.21, if $\deg\mathcal{D}\cap[\mathcal{Y}]$ is given by a universal linear polynomial, the same is true for $\deg D\cap[Y]$, which by Lemma 4.14 would imply the main theorem.

Every construction made in Section 4.7 starting from $\overline{TX}\to X$ can be carried out with the bundle $TX\to X$ replaced by an arbitrary algebraic rank-d vector bundle. In particular, we may construct the analogue of $\mathcal{Y}$ starting from the universal rank-d subbundle $H_1\to\mathrm{Gr}(d,N_1)$, where N_1 is any integer large enough that TX is the pullback of H_1 along a continuous classifying map $\psi_{TX}\colon X\to\mathrm{Gr}(d,N_1)$. Call this scheme $\mathcal{Y}_{\mathrm{Gr}}$, and denote the analogues of $\mathcal{R}_\alpha$ by $\mathcal{R}_{\alpha,\mathrm{Gr}}$.

Note that as the construction of the relative Hilbert scheme is local on the base ([13, Remark 3.9] or [20, Exercise 1.4.1.5]) and the projection $H_1\to\mathrm{Gr}(d,N_1)$ is a Zariski locally trivial fibration, it follows that $\mathrm{Hilb}^n(\overline{H_1}/\mathrm{Gr}(d,N_1)\to\mathrm{Gr}(d,N_1)$ is a fibration. This further implies that the projections $\mathcal{Y}_{\mathrm{Gr}}\to\mathrm{Gr}(d,N_1)$ and $\mathcal{R}_\alpha\to\mathrm{Gr}(d,N_1)$ are Zariski locally trivial fibrations. The same argument shows that $\mathcal{Y}\to X$ and $\mathcal{Y}_{\mathrm{Gr}}\to X$ are Zariski locally trivial fibrations.

Let r be the rank of E, and let N_2 be a sufficiently large integer. There is then a classifying map $\psi_E\colon X\to\mathrm{Gr}(r,N_2)$ with $\psi_E^*(H_2)\cong E$, where H_2 is the universal subbundle on $\mathrm{Gr}(r,N_2)$. There is then a Cartesian diagram

$$\begin{array}{ccc}\mathcal{Y} & \xrightarrow{\quad g\quad} & \mathcal{Y}_{\mathrm{Gr}}\times\mathrm{Gr}(r,N_2)\\ {\scriptstyle\pi_X}\downarrow & & \downarrow{\scriptstyle\pi_{\mathrm{Gr}}\times\mathrm{id}}\\ X & \xrightarrow{(\psi_{TX},\psi_E)} & \mathrm{Gr}(d,N_1)\times\mathrm{Gr}(r,N_2)\end{array}\tag{4-2}$$

in the category of topological spaces, where π_{Gr} is the product of the natural projection $\mathcal{Y}_{\mathrm{Gr}}\to\mathrm{Gr}(d,N_1)$. Note that the horizontal maps are in general not analytic.

If $\alpha \geq \mu$, let the bundle $H_2^{[\![\alpha]\!]}$ on $\mathcal{Y}_{\mathrm{Gr}} \times \mathrm{Gr}(r, N_2)$ be defined by

$$H_2^{[\![\alpha]\!]} = H_2 \otimes \mathcal{O}_{\overline{H}_1^n}^{[\![\alpha]\!]},$$

where H_2 and $\mathcal{O}_{\overline{H}_1^n}^{[\![\alpha]\!]}$ are pulled back from $\mathrm{Gr}(r, N_2)$ and $\mathcal{Y}_{\mathrm{Gr}}$, respectively. We then have

$$g^*(H_2^{[\![\alpha]\!]}) = \mathcal{E}^{[\![\alpha]\!]}.$$

The scheme $\mathcal{Y}_{\mathrm{Gr}}$ also carries a bundle $T_{\overline{H}_1^{[n]}}^{[\![\alpha]\!]}$, defined in the same way as $T_{\overline{TX}^{[n]}}^{[\![\alpha]\!]}$, and we have

$$g^*(T_{\overline{H}_1^{[n]}}^{[\![\alpha]\!]}) = T_{\overline{TX}^{[n]}}^{[\![\alpha]\!]}.$$

For any $\alpha \geq \mu$, define the relative Nash bundle $T_{\widetilde{\mathcal{R}}_{\alpha,\mathrm{Gr}}/\mathrm{Gr}}$ on $\mathcal{Y}_{\mathrm{Gr}}$ as the kernel of the natural map $T_{\widetilde{\mathcal{R}}_{\alpha,\mathrm{Gr}}} \to \pi_{\mathrm{Gr}}^*\big(T_{\mathrm{Gr}(d,N_1)}\big)$.

This map is surjective since $\mathcal{R}_{\alpha,\mathrm{Gr}} \to \mathrm{Gr}(d, N_1)$ is a locally trivial fibration. Hence there is a short exact sequence

$$0 \to T_{\widetilde{\mathcal{R}}_{\alpha,\mathrm{Gr}}/\mathrm{Gr}} \to T_{\widetilde{\mathcal{R}}_{\alpha,\mathrm{Gr}}} \to \pi_{\mathrm{Gr}}^*\big(T_{\mathrm{Gr}(d,N_1)}\big) \to 0.$$

Similarly define $T_{\widetilde{\mathcal{R}}_\alpha/X}$ by the short exact sequence

$$0 \to T_{\widetilde{\mathcal{R}}_\alpha/X} \to T_{\widetilde{\mathcal{R}}_\alpha} \to \pi_X^*(T_X) \to 0.$$

The restriction of $T_{\widetilde{\mathcal{R}}_\alpha/X}$ to a fibre $\mathcal{Y}|_x$ of $\mathcal{Y} \to X$ is canonically identified with the Nash bundle on the Nash blow-up of $\mathcal{R}_\alpha|_x$. A similar statement holds for $T_{\widetilde{\mathcal{R}}_{\alpha,\mathrm{Gr}}/\mathrm{Gr}}$ when restricted to a fibre of $\mathcal{Y}_{\mathrm{Gr}} \to \mathrm{Gr}(d, N_1)$.

We then have $g^*(T_{\widetilde{\mathcal{R}}_{\alpha,\mathrm{Gr}}/\mathrm{Gr}}) = T_{\widetilde{\mathcal{R}}_\alpha/X}$. Define a bundle G_α on $\mathcal{Y}_{\mathrm{Gr}}$ by

$$G_\alpha = T_{\widetilde{\mathcal{R}}_{\alpha,\mathrm{Gr}}/\mathrm{Gr}} \oplus \pi_{\mathrm{Gr}}^*(H_1).$$

In the Grothendieck K-group of topological vector bundles we then have $g^*(G_\alpha) = T_{\widetilde{\mathcal{R}}_\alpha}$. Define the class $\mathcal{C}_{\alpha,\mathrm{Gr}} \in H^*(\mathcal{Y}_{\mathrm{Gr}} \times \mathrm{Gr}(r, N_2))$ by

$$\mathcal{C}_{\alpha,\mathrm{Gr}} = F(\mathcal{E}_{\mathrm{Gr}}^{[\![\alpha]\!]}, T_{\overline{H}_1^{[n]}}^{[\![\alpha]\!]}) \cdot c_\bullet(G_\alpha).$$

The above discussion shows that $g^*(\mathcal{C}_{\alpha,\mathrm{Gr}}) = \mathcal{C}_\alpha$.

There are Gysin maps $(\pi_X)_!$ and $(\pi_{\mathrm{Gr}} \times \mathrm{id})_!$ in cohomology, defined by

$$(\pi_X)_!(\alpha) = \mathrm{PD}\big((\pi_X)_*(\alpha \cap [\mathcal{Y}])\big)$$

and

$$(\pi_{\mathrm{Gr}} \times \mathrm{id})_!(\alpha) = \mathrm{PD}\big((\pi_{\mathrm{Gr}} \times \mathrm{id})_*(\alpha \cap [\mathcal{Y}_{\mathrm{Gr}} \times \mathrm{Gr}(r, N_2)])\big),$$

where PD denotes the Poincaré dual.

There is a similar Gysin map $(\psi_{TX},\psi_E)^!\colon H_*(\mathrm{Gr}(d,N_1)\times\mathrm{Gr}(r,N_2))\to H_*(X)$, which by base change along the Cartesian diagram (4-2) induces a map $g^!\colon H_*(\mathcal{Y}_{\mathrm{Gr}})\to H_*(\mathcal{Y})$. Since the vertical arrows of (4-2) are fibre bundles, we get $g^!([\mathcal{Y}_{\mathrm{Gr}}])=[\mathcal{Y}]$.

This implies the relation

$$(\pi_X)_! g^* = (\psi_{TX},\psi_E)^*(\pi_{\mathrm{Gr}}\times \mathrm{id})_!.$$

As a consequence, we get $(\pi_X)_!(\mathcal{C}_\alpha)=g^*((\pi_{\mathrm{Gr}}\times\mathrm{id})_!(\mathcal{C}_{\alpha,\mathrm{Gr}}))$, which implies

$$\deg(\mathcal{C}_\alpha\cap[\mathcal{Y}])=\deg\big(g^*((\pi_{\mathrm{Gr}}\times\mathrm{id})_!(\mathcal{C}_{\alpha,\mathrm{Gr}}))\cap[X]\big). \tag{4-3}$$

Any rational cohomology class on $\mathrm{Gr}(d,N_1)\times\mathrm{Gr}(r,N_2)$ can be expressed as a polynomial in Chern classes of the universal bundles. If moreover the N_i are sufficiently large, then there are no relations between these Chern classes in degree $2d$, so this polynomial is unique. In particular, the degree-$2d$ part of $(\pi_{\mathrm{Gr}}\times\mathrm{id})_!(\mathcal{C}_{\alpha,\mathrm{Gr}})$ is equal to such a polynomial. This polynomial is independent of the N_i, because the class $\mathcal{C}_{\alpha,\mathrm{Gr}}$ is preserved by the pullbacks induced by the natural morphisms $\mathrm{Gr}(d,N_1)\to\mathrm{Gr}(d,N_1+1)$ and $\mathrm{Gr}(r,N_2)\to\mathrm{Gr}(r,N_2+1)$.

It follows then that the right-hand side of (4-3) is equal to a linear combination of the Chern numbers of (X,E). Consequently, $\deg\mathcal{C}_\alpha\cap[\mathcal{Y}]$ is a universal linear combination of the Chern numbers of (X,E). As $\mathcal{D}$ is a linear combination of the $\mathcal{C}_\alpha$, the same is true for $\deg\mathcal{D}\cap[\mathcal{Y}]$, which is what we needed to show. □

5 Proof of Lemma 4.20

As the proof is somewhat complicated, let us first give a brief outline. In Section 5.1, specifically in Lemma 5.3, we show that there exist $U\subset Y$ and $\mathcal{U}\subset\mathcal{Y}$, and a homeomorphism $f\colon U\to\mathcal{U}$ where U (resp. $\mathcal{U}$) is neighbourhood of $Y_0\subset Y$ (resp. $\mathcal{Y}_0\subset\mathcal{Y}$). The first claim of Lemma 4.20 follows from this. We also show that this homeomorphism is compatible with the relevant vector bundles on Y and $\mathcal{Y}$, ie we find isomorphisms of topological vector bundles

$$f^*(\mathcal{E}^{[\![\alpha]\!]})\cong E^{[\![\alpha]\!]},\quad f^*(T^{[\![\alpha]\!]}_{TX^{[n]}})\cong T^{[\![\alpha]\!]}_{X^{[n]}}\quad\text{and}\quad f^*(T_{\widetilde{\mathcal{R}}_\alpha/X})\cong T_{\widetilde{R}_\alpha/X}. \tag{5-1}$$

The second claim of Lemma 4.20 is that $\mathcal{D}$ can be lifted to a relative cohomology class $\mathcal{D}_{\mathrm{rel}}\in H^*(\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}')$ which is sent to $D\in H^*(Y)$ by the composed map

$$H^*(\mathcal{Y},\mathcal{Y}\setminus\mathcal{U}')\to H^*(\mathcal{U},\mathcal{U}\setminus\mathcal{U}')\xrightarrow{f^*}H^*(U,U\setminus U')\to H^*(Y,Y\setminus U')\to H^*(Y).$$

Our approach is to construct explicitly singular cocycles $\overline{D}$ and $\overline{\mathcal{D}}$ representing D and $\mathcal{D}$, such that $\overline{D}$ (resp. $\overline{\mathcal{D}}$) vanishes on $Y\setminus U'$ (resp. $\mathcal{Y}\setminus\mathcal{U}'$). Then $\overline{D}$ (resp. $\overline{\mathcal{D}}$)

defines a class $D_{\mathrm{rel}} \in H^*(Y, Y \setminus U')$ (resp. $\mathcal{D}_{\mathrm{rel}} \in H^*(\mathcal{Y}, \mathcal{Y} \setminus \mathcal{U}')$). We will further show that there is an equality of cocycles $f^*(\overline{\mathcal{D}}|_{\mathcal{U}}) = \overline{D}|_U$, and the second claim of Lemma 4.20 then follows.

In Section 5.2 we first introduce certain subsets of X^n and $\overline{TX}^n$, which are needed in the proof of the later Lemma 5.9. For any partition α, we let U_α be a small open neighbourhood of Δ_α in X^n. If $\alpha < \beta$, we also define the open subset $V_{\alpha,\beta} \subset X^n \setminus U_\beta$, which is a certain large subset of the locus over which $X^{[\![\alpha]\!]}$ and $X^{[\![\beta]\!]}$ are canonically isomorphic. We similarly define $\mathcal{U}_\alpha, \mathcal{V}_{\alpha,\beta} \subseteq \overline{TX}^n$.

Let now $F^{[\![\alpha]\!]}$ denote either $E^{[\![\alpha]\!]}$, $T_{\widetilde{R}_\alpha/X}$ or $T^{[\![\alpha]\!]}_{X^{[n]}}$, and denote the trivial line bundle on Y by $\mathcal{O}_Y$. If $\alpha < \beta$, then over $V_{\alpha,\beta}$ the bundles $F^{[\![\alpha]\!]}$ and $F^{[\![\beta]\!]}$ are canonically isomorphic. We use this observation to get Lemma 5.8, which says that we can find an N and for each α an inclusion of topological vector bundles $F^{[\![\alpha]\!]} \hookrightarrow \mathcal{O}_Y^N$, where these inclusions have the following property: if $\alpha < \beta$, then over $V_{\alpha,\beta}$, the bundles $F^{[\![\alpha]\!]}$ and $F^{[\![\beta]\!]}$ are equal as subbundles of $\mathcal{O}_Y^N$.

Lemma 5.8 similarly gives inclusions of $\mathcal{E}^{[\![\alpha]\!]}$, $T_{\widetilde{\mathcal{R}}_\alpha/X}$ and $T^{[\![\alpha]\!]}_{\overline{TX}^{[n]}}$ into $\mathcal{O}_{\mathcal{Y}}^N$, with the same compatibility property over $\mathcal{V}_{\alpha,\beta}$. Moreover it shows that with these inclusions the isomorphisms of (5-1) are strengthened to equalities of subbundles of $\mathcal{O}_Y^N$:

$$f^*(\mathcal{E}^{[\![\alpha]\!]}) = E^{[\![\alpha]\!]}, \quad f^*(T^{[\![\alpha]\!]}_{\overline{TX}^{[n]}}) = T^{[\![\alpha]\!]}_{X^{[n]}} \quad \text{and} \quad f^*(T_{\widetilde{\mathcal{R}}_\alpha/X}) = T_{\widetilde{R}_\alpha/X}. \tag{5-2}$$

Using the above inclusions into $\mathcal{O}_Y^N$, we can find, for $i = 1, 2, 3$, a Grassmannian $\mathrm{Gr}_i = \mathrm{Gr}(r_i, N)$ with canonical subbundle H_i and maps

$$\phi_{\alpha,i} : Y \to \mathrm{Gr}_i \quad \text{and} \quad \psi_{\alpha,i} : \mathcal{Y} \to \mathrm{Gr}_i$$

such that

$$\phi^*_{\alpha,1}(H_1) = E^{[\![\alpha]\!]}, \quad \phi^*_{\alpha,2}(H_2) = T^{[\![\alpha]\!]}_{X^{[n]}}, \quad \phi^*_{\alpha,3}(H_3) = T_{\widetilde{R}_\alpha/X},$$

and

$$\psi^*_{\alpha,1}(H_1) = \mathcal{E}^{[\![\alpha]\!]}, \quad \psi^*_{\alpha,2}(H_2) = T^{[\![\alpha]\!]}_{\overline{TX}^{[n]}}, \quad \psi^*_{\alpha,3}(H_3) = T_{\widetilde{\mathcal{R}}_\alpha/X}.$$

Let $\mathrm{Gr} = \mathrm{Gr}_1 \times \mathrm{Gr}_2 \times \mathrm{Gr}_3$, let $\phi_\alpha = \prod_i \phi_{\alpha,i}$ and let $\psi_\alpha = \prod_i \psi_{\alpha,i}$. We get a diagram, commutative by (5-2):

$$\begin{array}{ccccc} U & \hookrightarrow & Y & & \\ \downarrow f & & & \searrow^{\phi_\alpha} & \\ \mathcal{U} & \hookrightarrow & \mathcal{Y} & \xrightarrow{\psi_\alpha} & \mathrm{Gr} \end{array} \tag{5-3}$$

As a consequence of Lemma 5.8, we have $\phi_\alpha|_{V_{\alpha,\beta}} = \phi_\beta|_{V_{\alpha,\beta}}$ and $\psi_\alpha|_{\mathcal{V}_{\alpha,\beta}} = \psi_\beta|_{\mathcal{V}_{\alpha,\beta}}$ for any $\alpha < \beta$.

Since C_α is a fixed polynomial in the Chern classes of $E^{[\![\alpha]\!]}$, $T_{X^{[n]}}^{[\![\alpha]\!]}$ and $T_{\widetilde{R}_\alpha/X}$, we can find a class C_{Gr} on Gr such that $\phi_\alpha^*(C_{\mathrm{Gr}}) = C_\alpha$ for all α. We choose a singular cocycle $\overline{C}_{\mathrm{Gr}}$ representing C_{Gr}, which in turn leads to singular cocycles $\phi_\alpha^*(\overline{C}_{\mathrm{Gr}}) = \overline{C}_\alpha$ representing C_α. We then get the singular cocycle $\overline{D}$ representing D by the same formula as the one defining D in terms of C_α in Definition 4.11. Similarly we get cocycles $\overline{\mathcal{C}}_\alpha = \psi_\alpha^*(\overline{C}_{\mathrm{Gr}})$ and $\overline{\mathcal{D}}$ on $\mathcal{Y}$.

Now, using the fact that $\phi_\alpha = \phi_\beta$ over $V_{\alpha,\beta}$, in Lemma 5.9 we show that $\overline{D}$ vanishes as a cocycle when restricted to $Y|_{X^n \setminus U_\Lambda}$. Shrinking U_Λ, we can ensure that $Y|_{U_\Lambda} \subset U'$, and so it follows from this that $\overline{D}$ vanishes on $Y \setminus U'$. The argument for this is given in Section 5.4. Similarly $\overline{\mathcal{D}}$ vanishes on $\mathcal{Y} \setminus \mathcal{U}'$.

Finally, the commutativity of (5-3) implies that $f^*(\overline{\mathcal{D}}|_{\mathcal{U}}) = \overline{D}|_U$, and this completes the proof.

Note that our proof depends on the fact that pairs of bundles, eg $E^{[\![\alpha]\!]}$ and $E^{[\![\beta]\!]}$, are isomorphic over the open subset $V_{\alpha,\beta}$, in the strong sense that the bundles can all compatibly be made equal as subbundles of $\mathcal{O}^N$. In particular, the fact that C_α and C_β are equal cohomology classes over $V_{\alpha,\beta}$ is not strong enough to give our claim (in the same way that a cohomology class in general is not determined by its restriction to an open covering).

5.1 Defining the map from Y to $\mathcal{Y}$

Let $p_1, p_2\colon X \times X \to X$ be the projections to the first and second factors, and let $\pi\colon TX \to X$ be the tangent bundle.

Lemma 5.1 *There is an open neighbourhood U_1 of the diagonal $\Delta \subset X \times X$, an open neighbourhood $\mathcal{U}_1$ of the 0–section $X \subset TX$ and a homeomorphism $f_1\colon U_1 \to \mathcal{U}_1$, such that*

$$\pi \circ f_1 = p_1$$

and such that $f_1|_\Delta$ is the identification between Δ and the 0–section of TX. Furthermore, the restriction of f_1 to each fibre $p_1^{-1}(x)$ is holomorphic.

There is an isomorphism of topological vector bundles $p_1^*(E)|_U \to p_2^*(E)|_U$, which is an isomorphism of holomorphic bundles on the restriction to each fibre $p_1^{-1}(x)$.

Proof See [23, Lemma 2.4] for the first two statements. Holomorphic exponential maps can be constructed on small open sets, and these can be globalised using a partition of unity. This globalisation preserves holomorphicity on fibres of p_1 as required.

For the statement about E, we argue similarly. Cover X with open balls B_i, and choose holomorphic trivialisations $\phi_i\colon E|_{B_i} \xrightarrow{\cong} \mathcal{O}^n_{B_i}$. Using these, define local holomorphic isomorphisms

$$\psi_i\colon p_1^*(E)|_{B_i\times B_i} \xrightarrow{p_1^*(\phi_i)} \mathcal{O}^n_{B_i\times B_i} \xrightarrow{p_2^*(\phi_i^{-1})} p_2^*(E)|_{B_i\times B_i}.$$

Let $\{t_i\colon X\to\mathbb{R}\}$ be a smooth partition of unity subordinate to the covering $\{B_i\}$, and let $U_1=\bigcup_i B_i\times B_i$. Define $\psi\colon p_1^*(E)\to p_2^*(E)$ at $x\in U_1$ by $\psi(x)=\sum t_i(p_1(x))\cdot\psi_i$.

This map ψ is holomorphic on fibres of p_1. Restricted to Δ, the map ψ is the identity, and so after shrinking U_1 if necessary, ψ is an isomorphism. □

Let $X_0^{[\![n]\!]}\subset X^{[\![n]\!]}$ be the set of pairs $(Z,(x_i))$ such that Z is supported at a single point, and let $\overline{TX}_0^{[\![n]\!]}\subset\overline{TX}^{[\![n]\!]}$ denote the set of pairs $(Z,(v_i))$ such that Z is supported at the 0–section of $\overline{TX}$.

Let $q_{X^{[\![n]\!]}}\colon X^{[\![n]\!]}\to X$ be defined by $q_{X^{[\![n]\!]}}(Z,(x_i))=x_1$ and $q_{\overline{TX}^{[\![n]\!]}}\colon \overline{TX}^{[\![n]\!]}\to X$ be defined by $q_{\overline{TX}^{[\![n]\!]}}\big((Z,(v_i))\big)=\pi(v_1)$. Let W be the set of pairs $(Z,(v_i))\in\overline{TX}^{[\![n]\!]}$ such that v_1 lies in the 0–section of $\overline{TX}$.

Lemma 5.2 *There is an open neighbourhood U_2 of $X_0^{[\![n]\!]}$ in $X^{[\![n]\!]}$, an open neighbourhood $\mathcal{U}_2$ of $\overline{TX}_0^{[\![n]\!]}$ in W, and a homeomorphism $f_2\colon U_2\to\mathcal{U}_2$, such that $q_{X^{[\![n]\!]}}=q_{\overline{TX}^{[\![n]\!]}}\circ f_2$. Moreover, f_2 restricted to $q^{-1}_{X^{[\![n]\!]}}(x)$ is a biholomorphic map between the fibres $q^{-1}_{X^{[\![n]\!]}}(x)$ and $q^{-1}_{\overline{TX}^{[\![n]\!]}}(x)$ for all $x\in X$. In particular, f_2 is orientation-preserving on the nonsingular subsets of U_2 and $\mathcal{U}_2$. There are isomorphisms of topological vector bundles $f_2^*(T^{[\![n]\!]}_{\overline{TX}^{[n]}})\to T^{[\![n]\!]}_{X^{[n]}}$ and $f_2^*(\mathcal{E}^{[\![n]\!]})\to E^{[\![n]\!]}$.*

Proof Let U_1, $\mathcal{U}_1$ and f_1 be as provided by Lemma 5.1. Define f_2 by

$$f_2\big((Z,(x_i))\big)=\big((f_1)_*(\{q_{X^{[\![n]\!]}}(x)\}\times Z),(f_1(q_{X^{[\![n]\!]}}(x),x_i))\big).$$

The right-hand side is well-defined if $\{q_{X^{[\![n]\!]}}(x)\}\times Z$ is contained in U_1. Let $U_2\subset X^{[\![n]\!]}$ be the open set where this is the case, so that f_2 is defined on U_2. Then f_2 is a local homeomorphism such that $q_{X^{[\![n]\!]}}=q_{\overline{TX}^{[\![n]\!]}}\circ f_2$. We claim that for all $x\in X$, the restriction $f_2|_{q^{-1}_{X^{[\![n]\!]}}(x)}$ is a local analytic isomorphism between $q^{-1}_{X^{[\![n]\!]}}(x)$ and $q^{-1}_{\overline{TX}^{[\![n]\!]}}(x)$.

The analytic structure of $q^{-1}_{X^{[\![n]\!]}}(x)$ (resp. of $q^{-1}_{\overline{TX}^{[\![n]\!]}}(x)$) is determined by the analytic structure on $p_1^{-1}(x)$ (resp. on $\pi^{-1}(x)$), as can be seen using Douady spaces (see Section 1.2.1) and the fact that each step in the construction of $X^{[\![n]\!]}$ (resp. W) could have been carried out in the analytic category. The claim that f_2 is a local isomorphism on fibres follows from the fact that f_1 gives a local isomorphism $p_1^{-1}(x)\cong\pi^{-1}(x)$, and the local isomorphism property mentioned in Section 1.2.1.

We now describe the isomorphism $T^{[\![n]\!]}_{X^{[n]}} \cong f_2^*(T^{[\![n]\!]}_{\overline{TX}^{[n]}})$. Over a point $(Z,(x_i)) \in X^{[\![n]\!]}$, this is the composition

$$(T^{[\![n]\!]}_{X^{[n]}})_{(Z,(x_i))} = (T_{X^{[n]}})_Z \xrightarrow{\cong} (T_{\overline{TX}^{[n]}})_{(f_1)_*(\{x_1\}\times Z)} = (T^{[\![n]\!]}_{\overline{TX}^{[n]}})_{f_2(Z,(x_i))},$$

where the middle map is the differential of the map $Z \mapsto (f_1)_*(\{x_1\}\times Z)$.

Finally, we describe the isomorphism $f_2^*(\mathcal{E}^{[\![n]\!]}) \cong E^{[\![n]\!]}$. Over a point $(Z,(x_i)) \in X^{[\![n]\!]}$, we have

$$\begin{aligned} f_2^*(\mathcal{E}^{[\![n]\!]})_{(Z,(x_i))} &= E_{x_1} \otimes H^0(\mathcal{O}_{f_1(\{x_1\}\times Z)}) \cong E_{x_1} \otimes H^0(\mathcal{O}_{\{x_1\}\times Z}) \\ &\cong H^0(\{x_1\}\times Z,\, p_1^*(E)|_{\{x_1\}\times Z}). \end{aligned}$$

The isomorphism $p_1^*(E) \to p_2^*(E)$ of Lemma 5.1 now gives an isomorphism

$$H^0(\{x_1\}\times Z,\, p_1^*(E)|_{\{x_1\}\times Z}) \xrightarrow{\cong} H^0(\{x_1\}\times Z,\, p_2^*(E)|_{\{x_1\}\times Z}) = E^{[\![n]\!]}_{(Z,(x_i))}. \qquad \square$$

Recall that $Y_0 \subset Y$ and $\mathcal{Y}_0 \subset \mathcal{Y}$ are the subsets of points having image in $\Delta \subset X^n$ and $X \subset \overline{TX}^n$ under the natural morphisms $Y \to X^n$ and $\mathcal{Y} \to \overline{TX}^n$, respectively.

Let the relative Nash bundles $T_{\widetilde{R}_\alpha/X}$ and $T_{\widetilde{\mathcal{R}}_\alpha/X}$ respectively be the kernels of the surjective homomorphisms $T_{\widetilde{R}_\alpha} \to q^*_{X^{[\![n]\!]}}(T_X)$ and $T_{\widetilde{\mathcal{R}}_\alpha} \to q^*_{\overline{TX}^{[\![n]\!]}}(T_X)$.

Lemma 5.3 *There is an open neighbourhood U of Y_0 in Y, an open neighbourhood $\mathcal{U}$ of $\mathcal{Y}_0$ in $\mathcal{Y}$, and a homeomorphism $f\colon U \to \mathcal{U}$, as well as isomorphisms of topological vector bundles*

$$f^*(\mathcal{E}^{[\![\alpha]\!]}) \to E^{[\![\alpha]\!]}, \quad f^*(T^{[\![\alpha]\!]}_{\overline{TX}^{[n]}}) \to T^{[\![\alpha]\!]}_{X^{[n]}} \quad \text{and} \quad f^*(T_{\widetilde{\mathcal{R}}_\alpha/X}) \to T_{\widetilde{R}_\alpha/X}.$$

Proof The map f_2 constructed in Lemma 5.2 gives rise to local isomorphisms $R_\alpha \to \mathcal{R}_\alpha$. The Nash blow-up is determined analytically locally, and $q_{X^{[\![n]\!]}}\colon R_\alpha \to X$ (resp. $q_{\overline{TX}^{[\![n]\!]}}\colon \mathcal{R}_\alpha \to X$) is a locally trivial fibrations in a neighbourhood of Y_0 (resp. $\mathcal{Y}_0$).

In general, if V_1 and V_2 are complex analytic spaces with V_2 smooth, then the Nash blow-up $\widetilde{V_1 \times V_2}$ is canonically isomorphic to $\widetilde{V}_1 \times V_2$; see [36, Corollary 4.1]. It follows that the Nash blow-ups $\widetilde{R}_\alpha$ and $\widetilde{\mathcal{R}}_\alpha$ are locally trivial fibrations over X, and that the local isomorphisms $R_\alpha \to \mathcal{R}_\alpha$ extend uniquely to isomorphisms of the Nash blow-ups. This in turn induces a local isomorphism $Y \to \mathcal{Y}$.

The first two bundle isomorphisms are induced by the ones produced in Lemma 5.2, and the third follows similarly, taking into account the fact that $R_\alpha \to \mathcal{R}_\alpha$ is holomorphic on the fibres of $q_{X^{[\![n]\!]}}$ and $q_{\overline{TX}^{[\![n]\!]}}$. $\square$

5.2 The subsets U_α, $V_{\alpha,\beta}$, $\mathcal{U}_\alpha$ and $\mathcal{V}_{\alpha,\beta}$

In order to prove the claim of Lemma 4.20, we will construct the relative cohomology classes $\overline{D} \in H^*(Y, Y \setminus U)$ and $\overline{\mathcal{D}} \in H^*(\mathcal{Y}, \mathcal{Y} \setminus V)$ explicitly as singular cochains. Adapting the argument in Section 5.3 of [23], we first define certain open subsets U_α and $V_{\alpha,\beta}$ of X^n which we will later use to compare the bundles $E^{[\![\alpha]\!]}$ (or $T_{X^{[n]}}^{[\![\alpha]\!]}$ or $T_{\widetilde{R}_\alpha}$) for various α.

Let d_X be a metric on X inducing the Euclidean topology. Define the metric d_{X^n} on X^n by

$$d_{X^n}\big((x_i)_{i=1}^n, (y_i)_{i=1}^n\big) = \max_{1 \le i \le n} d_X(x_i, y_i).$$

Let $d_{\overline{TX}}$ be a metric on $\overline{TX}$ inducing the Euclidean topology, and let $d_{\overline{TX}^n}$ be the metric on $\overline{TX}^n$ defined in the same way as d_{X^n}.

Let $x = (x_1, \dots, x_n) \in X^n$ and let α be a partition. Recall that $\Delta_\alpha \subseteq X^n$ is the diagonal set

$$\{(x_1, \dots, x_n) \mid x_i = x_j \text{ if } i \sim_\alpha j\},$$

and let $\Delta_{\alpha,\overline{TX}} \subset \overline{TX}^n$ be defined in the same way. In the following, we will use the inequalities

$$\frac{1}{2} \sup_{\{i,j \mid i \sim_\alpha j\}} d_X(x_i, x_j) \le d_{X^n}(x, \Delta_\alpha) \le \sup_{\{i,j \mid i \sim_\alpha j\}} d_X(x_i, x_j) \tag{5-4}$$

and their variants for $d_{\overline{TX}}$ and $d_{\overline{TX}^n}$, all of which follow easily from the definitions and the triangle inequality.

Definition 5.4 Let $P(n)$ be the set of partitions of $\{1, \dots, n\}$, and let $\epsilon\colon P(n) \to \mathbb{R}^{>0}$ be a function. At various points in the proof, the quantities

$$\max_{\alpha \in P(n)} \epsilon(\alpha) \qquad \text{and} \qquad \max_{\alpha < \beta \in P(n)} \frac{\epsilon(\alpha)}{\epsilon(\beta)}$$

will be assumed to be sufficiently small.

Definition 5.5 Let $U_\alpha \subset X^n$ be the open $\epsilon(\alpha)$–neighbourhood of $\Delta_\alpha \subset X^n$, and let $\mathcal{U}_\alpha$ be the open $\epsilon(\alpha)$–neighbourhood of $\Delta_{\alpha,\overline{TX}} \subset \overline{TX}^n$.

Definition 5.6 Let α and β be partitions such that $\alpha < \beta$. Define the set $V_{\alpha,\beta} \subset X^n$ as

$$V_{\alpha,\beta} = (X^n \setminus U_\beta) \setminus \Big(\bigcup_{\substack{\gamma < \beta \\ \gamma \nleq \alpha}} \overline{U}_\gamma \Big),$$

and define the set $\mathcal{V}_{\alpha,\beta} \subset \overline{TX}^n$ as

$$\mathcal{V}_{\alpha,\beta} = (\overline{TX}^n \setminus \mathcal{U}_\beta) \setminus \Big(\bigcup_{\substack{\gamma<\beta \\ \gamma\not\leq\alpha}} \overline{\mathcal{U}}_\gamma \Big).$$

Let $i, j \in \{1, \dots, n\}$. Define an equivalence relation $\sim_{(i,j)}$ on the set of partitions by saying $\alpha \sim_{(i,j)} \beta$ if $\sim_\alpha$ and $\sim_\beta$ agree when evaluated on the pair (i, j). For any pair of partitions α, β, define the set $\Delta_{\alpha\beta} \subseteq X^n$ to be the set of points over which $X^{[\![\alpha]\!]}$ and $X^{[\![\beta]\!]}$ are not canonically equal. Explicitly, we have

$$\Delta_{\alpha\beta} = \bigcup_{\{i,j \mid \alpha \not\sim_{(i,j)} \beta\}} \Delta_{ij},$$

where Δ_{ij} denotes the set of points $x \in X^n$ for which $x_i = x_j$. Define $\Delta_{\alpha\beta,\overline{TX}} \subset \overline{TX}^n$ similarly.

The following lemma summarises the important properties of $V_{\alpha,\beta}$ and $\mathcal{V}_{\alpha,\beta}$.

Lemma 5.7 (i) *Let β be a partition. The sets $\{V_{\alpha,\beta}\}_{\alpha<\beta}$ form an open covering of $X^n \setminus U_\beta$. The sets $\{\mathcal{V}_{\alpha,\beta}\}_{\alpha<\beta}$ form an open covering of $(\overline{TX}^n) \setminus \mathcal{U}_\beta$.*

(ii) *Let α, β and γ be partitions such that $\alpha < \beta$, $\gamma \leq \beta$ and $\gamma \not\leq \alpha$. Then*

$$U_\gamma \cap V_{\alpha,\beta} = \varnothing \quad \text{and} \quad \mathcal{U}_\gamma \cap \mathcal{V}_{\alpha,\beta} = \varnothing.$$

(iii) *Let $\tau = \min_\gamma \epsilon(\gamma)$, and let $\alpha < \beta$. If $x \in V_{\alpha,\beta}$, we have $d(x, \Delta_{\alpha\beta}) \geq \tau$. If $x \in \mathcal{V}_{\alpha,\beta}$, we have $d(x, \Delta_{\alpha\beta,\overline{TX}}) \geq \tau$.*

Proof We prove the statements for $V_{\alpha,\beta}$; the case of $\mathcal{V}_{\alpha,\beta}$ is exactly the same.

(i) Assume $x = (x_i) \in X^n \setminus U_\beta$, and let α be maximal among partitions $< \beta$ such that $x \in \overline{U}_\alpha$. Such a partition exists since for the smallest partition $\omega = \{\{1\}, \dots, \{n\}\}$, we have $U_\omega = X^n$. We claim that $x \in V_{\alpha,\beta}$.

Assume $x \notin V_{\alpha,\beta}$, there is then a partition γ such that $\gamma \not\leq \alpha$, $\gamma < \beta$ and $x \in \overline{U}_\gamma$. By the maximality property of α, we cannot have $\alpha < \gamma$. It follows that $\alpha, \gamma < (\alpha \vee \gamma) \leq \beta$, where $\alpha \vee \gamma$ is the smallest partition majorising α and γ.

Let i, j be two indices such that $i \sim_{\alpha\vee\gamma} j$ and such that $d(x_i, x_j)$ is maximal for pairs with this property. There is a sequence of integers $i_1, i_2, \dots, i_l$ such that $i_1 = i$, $i_l = j$ and such that for every k with $1 \leq k < l$, either $i_k \sim_\alpha i_{k+1}$ or $i_k \sim_\gamma i_{k+1}$ is true. By (5-4), we now have

$$d(x, \Delta_{\alpha\vee\gamma}) \leq d(x_i, x_j) \leq d(x_{i_1}, x_{i_2}) + \dots + d(x_{i_{l-1}}, x_{i_l}).$$

Since $x \in \overline{U}_\alpha$, we have $d(x, \Delta_\alpha) \leq \epsilon(\alpha)$, and similarly for γ. By (5-4), each term in the above sum is therefore $\leq 2\max(\epsilon(\alpha), \epsilon(\gamma))$. The sum is therefore smaller than

$$2(l-1)\max(\epsilon(\alpha), \epsilon(\gamma)) < \epsilon(\alpha \vee \gamma),$$

where the last inequality uses the second smallness assumption in the definition of ϵ. Hence we have $d(x, \Delta_{\alpha\vee\gamma}) < \epsilon(\alpha\vee\gamma)$, so that $x \in U_{\alpha\vee\gamma}$. If $\alpha\vee\gamma \neq \beta$, this contradicts the maximality of α, and if $\alpha \vee \gamma = \beta$, it contradicts the assumption that $x \notin U_\beta$.

(ii) This is obvious from the definition.

(iii) Let $x = (x_i) \in V_{\alpha,\beta}$. For every $i, j \in \{1, \dots, n\}$, let $\gamma_{i,j}$ be the partition defined by the equivalence relation such that $i \sim_{\gamma_{i,j}} j$ and no other nontrivial relations hold. If $\alpha < \beta$, we have

$$\Delta_{\alpha\beta} = \bigcup_{\substack{i\sim_\beta j \\ i\not\sim_\alpha j}} \Delta_{\gamma_{i,j}}.$$

For every pair i, j occurring in the union, we have $\gamma_{i,j} \not\leq \alpha$, hence by part (ii) of the lemma we have $x \notin U_{\gamma_{i,j}}$. This gives

$$d(x, \Delta_{\alpha\beta}) = \min_{\substack{i\sim_\beta j \\ i\not\sim_\alpha j}} d(x, \Delta_{\gamma_{i,j}}) > \min_{\substack{i\sim_\beta j \\ i\not\sim_\alpha j}} \epsilon(\gamma_{i,j}) \geq \tau. \qquad \square$$

5.3 Constructing maps to Grassmannians

Recall that $T_{\widetilde{R}_\alpha/X}$ and $T_{\widetilde{\mathcal{R}}_\alpha/X}$ are the relative Nash bundles defined above Lemma 5.3. Denote the trivial bundle of rank N by $\mathcal{O}^N$.

Lemma 5.8 (i) *For each $\alpha \geq \mu$, let F_α denote one of $E^{[\![\alpha]\!]}$, $T_{\widetilde{R}_\alpha/X}$ or $T^{[\![\alpha]\!]}_{X^{[n]}}$, considered as a topological bundle on Y. There is an integer M and for every α an injection $i_\alpha\colon F_\alpha \to \mathcal{O}^M$ such that if $\alpha \leq \beta$, then over $V_{\alpha,\beta}$ the bundles $i_\alpha(E^{[\![\alpha]\!]})$ and $i_\beta(E^{[\![\alpha]\!]})$ are equal as subbundles of $\mathcal{O}^M$.*

(ii) *For each $\alpha \geq \mu$, let $\mathcal{F}_\alpha$ denote one of $\mathcal{E}^{[\![\alpha]\!]}$, $T_{\widetilde{\mathcal{R}}_\alpha/X}$ or $T^{[\![\alpha]\!]}_{TX^{[n]}}$, considered as a topological bundle on $\mathcal{Y}$. There is an integer N and for every α an injection $j_\alpha\colon \mathcal{F}_\alpha \to \mathcal{O}^N$ such that if $\alpha \leq \beta$, then over $\mathcal{V}_{\alpha,\beta}$ the bundles $j_\alpha(\mathcal{F}^{[\![\alpha]\!]})$ and $i_\beta(\mathcal{F}^{[\![\alpha]\!]})$ are equal as subbundles of $\mathcal{O}^N$.*

(iii) *Let $f\colon U \to \mathcal{U}$ be the local homeomorphism constructed in Lemma 5.3. We may choose $M = N$ and the injections i_α and j_α in such a way that the diagram*

$$\begin{array}{ccc} f^*(\mathcal{F}_\alpha) & \xleftarrow{\cong} & F_\alpha \\ {\scriptstyle f^*(j_\alpha)}\downarrow & & \downarrow{\scriptstyle i_\alpha} \\ f^*(\mathcal{O}^N) & = & \mathcal{O}^N \end{array}$$

of bundles on U commutes, where the upper isomorphism is the one constructed in Lemma 5.3.

Proof In this proof, all partitions are assumed to be $\geq \mu$.

(i) For each α begin by choosing an injective homomorphism $i'_\alpha \colon F_\alpha \to \mathcal{O}^{M_\alpha}$. Let $k_{\alpha\beta}\colon F_\alpha \to F_\beta$ be the natural isomorphisms, defined over $X^n \setminus \Delta_{\alpha\beta}$. Recall that $\tau = \min_\alpha \epsilon(\alpha)$. Let $t\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a continuous function such that when $x \geq \tau$ we have $t(x) = 1$ and such that when $x \leq \tau/2$ we have $t(x) = 0$. For $x \in Y$, put $t_{ij}(x) = t\big(d(\pi_Y(x), \Delta_{ij})\big)$, where $\pi_Y\colon Y \to X^n$ is the natural morphism. Let

$$t_{\alpha\beta} = \prod_{\{i<j \mid \alpha \not\sim_{(i,j)} \beta\}} t_{ij}.$$

As $t_{\alpha\beta}$ is supported away from $\Delta_{\alpha\beta}$, the homomorphism $t_{\alpha\beta} \cdot k_{\alpha\beta}$ is defined on the whole of Y.

For any two partitions α and γ, let $i_{\alpha\gamma} = i'_\gamma \circ (t_{\alpha\gamma} \cdot k_{\alpha\gamma})$. We take $M = \sum M_\gamma$, and set

$$i_\alpha = \bigoplus_\gamma (i_{\alpha\gamma})_\gamma \colon E_\alpha \to \bigoplus \mathbb{C}^{M_\gamma} = \mathbb{C}^M.$$

It remains to show that i_α has the properties stated. As $i_{\alpha\alpha} = i'_\alpha$ is injective, it is clear that i_α is an injection.

Let $\alpha < \beta$, and let γ be arbitrary. First we show that if $x \in Y$ lies over $V_{\alpha,\beta}$, then

$$(t_{\alpha\gamma} \cdot k_{\alpha\gamma})(x) = (t_{\beta\gamma} \cdot k_{\beta\gamma} \circ k_{\alpha\beta})(x). \tag{5-5}$$

To this end, observe that we may write

$$\frac{t_{\alpha\gamma}}{t_{\beta\gamma}} = \prod_{\{i<j \mid \alpha \not\sim_{(i,j)} \beta\}} t_{ij}^{s(i,j)},$$

where each $s(i,j) \in \{-1,0,1\}$. Now, since $p(x) \in V_{\alpha,\beta}$, we have $t_{ij}(x) = 1$ for every factor on the right-hand side, using Lemma 5.7(iii). Hence $t_{\alpha\gamma}(x) = t_{\beta\gamma}(x)$ holds. If $t_{\alpha\gamma} = 0$, this shows (5-5). If not, then all the morphisms $k_{\alpha\beta}$, $k_{\beta\gamma}$, $k_{\alpha\gamma}$ are defined at x, and by the naturality of these the cocycle condition $k_{\alpha\gamma} = k_{\beta\gamma} \circ k_{\alpha\beta}$ holds.

The above paragraph shows that $i_{\alpha\gamma} = i_{\beta\gamma} \circ k_{\alpha\beta}$ over $V_{\alpha,\beta}$ and hence $i_\alpha = i_\beta \circ k_{\alpha\beta}$. Consequently the two subbundles $i_\alpha(E_\alpha)$ and $i_\beta(E_\beta)$ of $\mathbb{C}^M$ are equal as claimed.

(ii) This is similar to (i).

(iii) Let $k'_{\alpha\beta}\colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ be the homomorphisms defined like the $k_{\alpha\beta}$ in the proof of (i). Let $g_\alpha\colon F_\alpha \to f^*(\mathcal{F}_\alpha)$ be the isomorphism of Lemma 5.3. We then have $g_\beta \circ k_{\alpha\beta} = f^*(k'_{\alpha\beta}) \circ g_\alpha$.

Let $U' \subset U$ and $\mathcal{U}' \subset \mathcal{U}$ be smaller open neighbourhoods of Y_0 and $\mathcal{Y}_0$ such that $f(U') = \mathcal{U}'$. Let $s\colon Y \to \mathbb{R}_{>0}$ be a function which is 1 on U' and 0 on $Y \setminus U$. Replace i_α with

$$i'_\alpha := \big((1-t)i_\alpha,\, s \cdot f^*(j_\alpha) \circ g_\alpha\big)\colon F_\alpha \to \mathcal{O}^{M+N},$$

where $j_\alpha\colon \mathcal{F}_\alpha \to \mathcal{O}^N$ is the homomorphism of part (ii). If we also replace j_α with $j'_\alpha = (0, j_\alpha)\colon \mathcal{F}_\alpha \to \mathcal{O}^M \oplus \mathcal{O}^N$, then obviously $i'_\alpha = f^*(j'_\alpha) \circ g_\alpha$ over U'. After replacing U with U', it now remains to be shown that the statement in part (i) of the lemma still holds for i'_α.

After enlarging the metric on $\overline{TX}$ and shrinking U and $\mathcal{U}$, we may assume that if $x \in U$ lies over $V_{\alpha,\beta}$, then $f(x)$ lies over $\mathcal{V}_{\alpha,\beta}$. Over $V_{\alpha,\beta}$ we thus have

$$\begin{aligned} i'_\alpha &= \big((1-s)i_\alpha,\, s\cdot f^*(j_\alpha)\circ g_\alpha\big) = \big((1-s)i_\beta \circ k_{\alpha\beta},\, s \cdot f^*(j_\beta \circ k'_{\alpha\beta}) \circ g_\alpha\big) \\ &= \big((1-s)i_\beta,\, s\cdot f^*(j_\beta)\circ g_\beta\big)\circ k_{\alpha\beta} = i'_\beta \circ k_{\alpha\beta}. \end{aligned}$$ □

5.4 Conclusion of proof of Lemma 4.20

Let α be a partition $\geq \mu$. The inclusions produced in Lemma 5.8 (taking there $M = N$) define continuous maps $\phi_{\alpha,i}\colon Y \to \mathrm{Gr}_i$ and $\psi_{\alpha,i}\colon \mathcal{Y} \to \mathrm{Gr}_i$ for $i = 1,2,3$, where the Gr_i are Grassmannians with universal subbundles H_i, such that

$$\phi^*_{\alpha,1}(H_1) = E^{[\![\alpha]\!]}, \quad \phi^*_{\alpha,2}(H_2) = T^{[\![\alpha]\!]}_{X^{[n]}}, \qquad \phi^*_{\alpha,3}(H_3) = T_{\widetilde{R}_\alpha/X},$$

and

$$\psi^*_{\alpha,1}(H_1) = \mathcal{E}^{[\![\alpha]\!]}, \quad \psi^*_{\alpha,2}(H_2) = T^{[\![\alpha]\!]}_{\overline{TX}^{[n]}}, \quad \psi^*_{\alpha,3}(H_3) = T_{\widetilde{\mathcal{R}}_\alpha/X}.$$

We let $\mathrm{Gr} = \prod_i \mathrm{Gr}_i$, let $\phi_\alpha = \prod_i \phi_{\alpha,i}$ and let $\psi_\alpha = \prod_i \psi_{\alpha,i}$.

Choose a singular cocycle $\overline{C}_{\mathrm{Gr}}$ on Gr representing the class

$$C_{\mathrm{Gr}} = F(H_1, H_2)\cdot c_\bullet(H_3) \in H^*(\mathrm{Gr}).$$

Define singular cocycles $\overline{C}_\alpha$ and $\overline{D}_\alpha$ by $\overline{C}_\alpha = \phi^*_\alpha(A)$ and

$$\overline{D}_\alpha = \overline{C}_\alpha - \sum_{\gamma \in [\mu,\alpha)} \overline{D}_\gamma.$$

Clearly, the classes of $\overline{C}_\alpha$ and $\overline{D}_\alpha$ are C_α and D_α, respectively. Let $\overline{C} = \overline{C}_\Lambda$ and $\overline{D} = \overline{D}_\Lambda$. We similarly define singular cocycles $\overline{\mathcal{C}}_\alpha = \psi^*_\alpha(A)$, $\overline{\mathcal{D}}_\alpha$, $\overline{\mathcal{C}} = \overline{\mathcal{C}}_\Lambda$ and $\overline{\mathcal{D}} = \overline{\mathcal{D}}_\Lambda$.

Lemma 5.9 *The singular cocycle $\overline{D}$ (resp. $\overline{\mathcal{D}}$) vanishes when restricted to $Y|_{X^n \setminus U_\Lambda}$ (resp. $\mathcal{Y}|_{\overline{TX}^n \setminus \mathcal{U}_\Lambda}$).*

Proof We treat the case of $\overline{D}$, the other case follows similarly.

We will show that $\overline{D}_\beta|_{X^n\setminus U_\beta} = 0$ for any partition $\beta \geq \mu$, by ascending induction on the ordering of partitions. The base case is clear, as $Y|_{X^n\setminus U_\mu} = \varnothing$.

Assume now that $\overline{D}_\alpha|_{X^n\setminus U_\alpha} = 0$ for every $\alpha < \beta$. If $\overline{D}$ has cohomological degree k, we must show that for every singular k–simplex $a\colon \Delta^k \to Y|_{X^n\setminus U_\beta}$ we have $\overline{D}_\beta(a) = 0$.

Since $\overline{D}_\beta$ is a cocycle, we may replace a by any subdivision of a and prove the vanishing for each simplex in the subdivision. By Lemma 5.7(i), $\{V_{\alpha,\beta}\}_{\alpha<\beta}$ is an open covering of $X^n \setminus U_\beta$, so we may assume there is an $\alpha < \beta$ such that a maps into $Y|_{V_{\alpha,\beta}}$.

If $\gamma < \beta$ is such that $\gamma \not\leq \alpha$, then by Lemma 5.7(ii) we have $U_\gamma \cap V_{\alpha,\beta} = \varnothing$, and so by the induction hypothesis $\overline{D}_\gamma(a) = 0$. This implies

$$\overline{D}_\beta(a) = \overline{C}_\beta(a) - \sum_{\gamma\leq\alpha} \overline{D}_\alpha(a) = \overline{C}_\beta(a) - \overline{C}_\alpha(a),$$

where the last equality follows directly from the definition of $\overline{D}_\alpha$. By Lemma 5.8(ii) we have $\phi_\alpha = \phi_\beta$ over $V_{\alpha,\beta}$, hence $\overline{D}_\beta(a) = 0$ and the claim follows. □

Taking ϵ small enough we may assume that $Y \setminus U'$ lies over $X^n \setminus U_\Lambda$, and then Lemma 5.9 shows that $\overline{D}$ is a relative cocycle for the pair $(Y, Y \setminus U')$. Similarly $\overline{\mathcal{D}}$ is a relative cocycle for $(\mathcal{Y}, \mathcal{Y} \setminus \mathcal{U}')$.

By Lemma 5.8(iii), the diagram

(5-6)
$$\begin{array}{ccccc} U & \hookrightarrow & Y & & \\ \big\downarrow f & & & \searrow^{\phi_\alpha} & \\ \mathcal{U} & \hookrightarrow & \mathcal{Y} & \xrightarrow{\psi_\alpha} & \mathrm{Gr} \end{array}$$

is commutative. It follows that $f^*(\overline{\mathcal{D}}|_{\mathcal{U}}) = \overline{D}|_U$, and hence the homomorphism

$$H^*(\mathcal{Y}, \mathcal{Y} \setminus \mathcal{U}') \to H^*(\mathcal{U}, \mathcal{U} \setminus \mathcal{U}') \xrightarrow{f^*} H^*(U, U \setminus U') \to H^*(Y, Y \setminus U')$$

sends $\overline{\mathcal{D}}$ to $\overline{D}$. This concludes the proof of Lemma 4.20.

6 A generating function

As was noted in [6] and elsewhere, the existence of universal polynomials can be strengthened to a statement about the form of the generating function of Chern integrals.

Throughout this section, we let X be as in the main theorem and let $Q_1,\dots,Q_k$ be distinct closed, irreducible punctual geometric subsets with $Q_i \subseteq \mathrm{Hilb}_0^{n_i}(\mathbb{C}^d)$. For any sequence of integers $m_1,\dots,m_k \geq 0$, let $n_{(m_i)} = \sum m_i n_i$. We let

$$\mathcal{S}_{(m_i)} = \{(j,l) \mid 1 \leq j \leq k,\ 1 \leq l \leq m_j\} \subset \mathbb{Z}^2.$$

Definition 6.1 We define the geometric set $P((Q_i^{m_i})) \subset X^{[n_{(m_i)}]}$ as the set of Z of the form $Z = \bigsqcup_{(j,l)\in\mathcal{S}_{(m_i)}} Z_{(j,l)}$ such that every $Z_{(j,l)}$ is isomorphic to an element of Q_j.

In other words, $P((Q_i^{m_i}))$ is the set of Z that are the disjoint union of m_1 subschemes from Q_1 and m_2 subschemes from Q_2 and so on. Specifying appropriate additional data, we can define similar geometric subsets of $X^{[\![n_{(m_i)}]\!]}$:

Definition 6.2 Let $A = (A_{(j,l)})_{(j,l)\in\mathcal{S}_{(m_i)}}$ be a collection of subsets of $\{1,\dots,n_{(m_i)}\}$, such that $|A_{(j,l)}| = n_j$ and such that the $A_{(j,l)}$ define a partition of $\{1,\dots,n_{(m_i)}\}$. We define the geometric set $R((Q_i^{m_i}); A) \subseteq X^{[\![n_{(m_i)}]\!]}$ to be the set of all $(Z, (x_i)_{i=1}^{n_{(m_i)}})$ such that $Z = \bigsqcup_{(j,l)\in\mathcal{S}_{(m_i)}} Z_{(j,l)}$, where every $Z_{(j,l)}$ is isomorphic to an element of Q_j, and such that x_i equals $\operatorname{Supp} Z_{(j,l)}$ if $i \in A_{(j,l)}$.

Lemma 6.3 *Choosing an A as above, the generic fibre of $R((Q_i^{m_i}); A) \to P((Q_i^{m_i}))$ has cardinality $\prod_i m_i!$.*

Proof We may assume the Q_i are ordered in such a way that if $i < j$, then $Q_i \not\subset Q_j$. Fix a generic point $Z \in P((Q_i^{m_i}))$. The number of points in $R((Q_i^{m_i}); A)$ lying above Z equals the number of ways of labelling the components of Z by $\mathcal{S}_{(m_i)}$ in such a way that $Z_{(j,l)}$ is isomorphic to an element of $\mathcal{S}_j$.

Fix one such labelling $Z = \bigsqcup_{(j,l)\in\mathcal{S}_j} Z_{(j,l)}$. We claim that if $Z = \bigsqcup_{(j,l)\in\mathcal{S}_j} Z'_{(j,l)}$ is a different labelling, then if $Z_{(j,l)} = Z'_{(j',l')}$, we must have $j = j'$.

Assume for a contradiction that this is not the case. Then there must be an equality $Z_{(j,l)} = Z'_{(j',l')}$ such that $j < j'$. We know that $Z_{(j,l)}$ is isomorphic to an element of Q_j, and by the ordering of the Q_j that $Q_j \not\subset Q_{j'}$. Since Q_j and $Q_{j'}$ are closed and irreducible and Z is generic in $P((Q_i^{m_i}))$, it follows that $Z_{j,l}$ is not isomorphic to an element of $Q_{j'}$, which is a contradiction, since $Z_{(j,l)} = Z'_{(j',l')} \in Q_{j'}$.

Thus the permissible labellings of the components of Z are given by permutations of $\mathcal{S}_{(m_i)}$ such that each (j,l) is sent to some (j,l'), of which there are $\prod_i m_i!$. □

Recall that $\mathrm{CM}(2,d)$ denotes the set of 2–variable Chern monomials of weight d. For a d–dimensional X with a bundle E on it, we denote by $M(X,E)$ the Chern number corresponding to an $M \in \mathrm{CM}(2,d)$, ie

$$M(X,E) = \deg M(T_X, E).$$

Proposition 6.4 *Let X be a smooth, connected projective variety of dimension d, let E be an algebraic vector bundle and let $Q_1, \dots, Q_k$ be distinct closed, irreducible punctual geometric subsets.*

We then have

$$\sum_{(m_i)\in\mathbb{Z}^k_{\geq 0}} \deg c_\bullet(E^{[n_{(m_i)}]}) \cap [\overline{P((Q_i^{m_i}))}] x_1^{m_1}\cdots x_k^{m_k} = \prod_{M\in \mathrm{CM}(2,d)} B_M^{M(X,E)},$$

where the B_M are elements of $\mathbb{Q}[\![x_1,\dots,x_k]\!]$ with $B_M(0)=1$, which depend only on the Q_i and the rank of E.

Note that the term of the left-hand side sum obtained by setting $m_i = 0$ for all i is 1. For degree reasons, the only nonzero term from $c_\bullet(E^{[n_{(m_i)}]})$ appearing in the formula is $c_{\dim P((Q_i^{m_i}))}\big(E^{[n_{(m_i)}]}\big)$.

Proof Let F be the generating function in the proposition. By the main theorem, the coefficients of F, and therefore those of $\log F$, are polynomials in the Chern numbers of (X,E). We will show that the coefficients of $\log F$ are in fact *linear* polynomials in the Chern numbers. This means that $\log F = \sum M(X,E) b_M$ for $b_M \in \mathbb{Q}[\![y_1,\dots y_k]\!]$, and taking $B_M = \exp(b_M)$ will then give the proposition.

Given a sequence $(m_i)_{i=1}^k$, let $R_{(m_i)} = R((Q_i^{m_i}); A_{(m_i)}) \subset X^{[\![n_{(m_i)}]\!]}$, where $A_{(m_i)}$ is some appropriate $\mathcal{S}_{(m_i)}$–indexed partition of $\{1,\dots,n_{(m_i)}\}$. Let

$$G = \sum_{(m_i)\in\mathbb{Z}^k_{\geq 0}} \deg c_\bullet(E^{[\![n_{(m_i)}]\!]}) \cap [\overline{R_{(m_i)}}] x_1^{m_1}\cdots x_k^{m_k}.$$

We use the notation $(m_i)! = \prod_i m_i!$ and denote the coefficient of the $x_1^{m_1}\cdots x_k^{m_k}$–term of a series by a lower index (m_i). By Lemma 6.3 and the projection formula, we have

$$G_{(m_i)} = (m_i)!\, F_{(m_i)}. \tag{6-1}$$

Fix now a sequence (m_i), and let $R = R_{(m_i)}$, $n = n_{(m_i)}$ and $A = A_{(m_i)}$. Let μ be the partition of $\{1,\dots,n\}$ induced by A, and let α be a partition of $\{1,\dots,n\}$ such that $\alpha \geq \mu$. As explained in Section 4.4, for such an α there is an associated $R_\alpha \subset X^{[\![\alpha]\!]}$ birational to R, and we let

$$G_\alpha = \deg c_\bullet(E^{[\![\alpha]\!]}) \cap [\overline{R_\alpha}].$$

For $1 \leq j \leq k$, let $\mathcal{S}_j = \{(j,l) \mid 1 \leq l \leq m_j\} \subset \mathcal{S}$. Giving a partition $\alpha \geq \mu$ is equivalent to giving a partition of $\mathcal{S}$, and we will denote this partition of $\mathcal{S}$ by $\overline{\alpha}$. Thus, given a partition $\alpha \geq \mu$, for every $B \in \overline{\alpha}$ we get a sequence $(|B\cap \mathcal{S}_j|)_{j=1}^k$.

As explained in the proof of Lemma 4.14, we can decompose R_α and $E^{[\![\alpha]\!]}$ as products, and in this case the decompositions can be written as $R_\alpha = \prod_{B\in\overline{\alpha}} R_{(|B\cap\mathcal{S}_j|)}$ and $E^{[\![\alpha]\!]} = \bigoplus_{B\in\overline{\alpha}} E^{[\![n_{(|B\cap\mathcal{S}_j|)}]\!]}$. As in the proof of Lemma 4.14, we can now use the Whitney sum formula and the Künneth decomposition to get

$$\deg c_\bullet(E^{[\![\alpha]\!]}) \cap [\overline{R_\alpha}] = \prod_{B\in\overline{\alpha}} \deg c_\bullet(E^{[\![n_{(|B\cap\mathcal{S}_j|)}]\!]}) \cap [R_{(|B\cap\mathcal{S}_j|)}],$$

which implies that

$$G_\alpha = \prod_{B\in\alpha} G_{(|B\cap\mathcal{S}_i|)} = \prod_{B\in\alpha} (|B\cap\mathcal{S}_i|)!\, F_{(|B\cap\mathcal{S}_i|)}. \tag{6-2}$$

Let $C_\alpha = c_\bullet(E^{[\![\alpha]\!]})$, and let D be defined (on the space Y as in the proof of the main theorem) inductively in terms of the C_α as in Definition 4.11. As explained in Remark 4.12, we have $D = \sum_{\alpha\geq\mu}(-1)^{|\alpha|-1}(|\alpha|-1)!\,C_\alpha$. By the argument in Section 4.8, $\deg D\cap[Y]$ is a linear polynomial in Chern numbers of (X,E). We have

$$\deg D\cap[Y] = \sum_{\alpha\geq\mu}(-1)^{|\alpha|-1}(|\alpha|-1)!\,C_\alpha\cap[Y] = \sum_{\alpha\geq\mu}(-1)^{|\alpha|-1}(|\alpha|-1)!\,G_\alpha,$$

and therefore

$$H_{(m_i)} := \sum_{\alpha\geq\mu}(-1)^{|\alpha|-1}(|\alpha|-1)!\,G_\alpha \tag{6-3}$$

is a linear combination of Chern numbers.

Using (6-1), (6-2) and (6-3), we see that the terms of $H_{(m_i)}$ admit a combinatorial expression in the terms of F, and we claim that

$$H_{(m_i)} = (m_i)!\,(\log F)_{(m_i)}. \tag{6-4}$$

It follows from this that the terms of $\log F$ are linear polynomials in Chern numbers, and so the proposition follows.

We give the proof of (6-4) in the case when $k=1$; the general case can be treated with similar combinatorics. Fix an $m\geq 0$, and consider

$$H_m = \sum_{\alpha}(-1)^{|\alpha|-1}(|\alpha|-1)!\prod_{B\in\alpha}|B|!\,F_{|B|},$$

where the sum is over all partitions α of $\{1,\dots,m\}$ (so that $H_0 := 0$). Given a partition α of $\{1,\dots,m\}$, let $\underline{\alpha}$ be the underlying partition of m, ie the sum $\sum_{B\in\alpha}|B| = m$. For a partition $\mathcal{P} = \sum_{i=1}^{l} k_i$ of m, we use the notation

$$|\mathcal{P}| = l, \quad \mathcal{P}! = \prod_i k_i!, \quad \operatorname{Aut}(\mathcal{P}) = \prod_{j\geq 1}|\{i \mid k_i = j\}|! \quad \text{and} \quad F_{\mathcal{P}} = \prod_i F_{k_i}.$$

Let $\mathcal{P}$ be a partition of m, and let

$$H_{\mathcal{P}} = \sum_{\{\alpha \mid \underline{\alpha} = \mathcal{P}\}} (-1)^{|\alpha|-1}(|\alpha|-1)! \prod_{B \in \alpha} |B|!\, F_{|B|}.$$

All terms in this sum are equal, so we get

$$H_{\mathcal{P}} = |\{\alpha \mid \underline{\alpha} = \mathcal{P}\}| \cdot (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)!\mathcal{P}!\, F_{\mathcal{P}}.$$

The first term of this product equals

$$\frac{m!}{\mathcal{P}! \cdot \operatorname{Aut}(\mathcal{P})}.$$

We thus get

$$H_{\mathcal{P}} = \frac{m!}{\operatorname{Aut}(\mathcal{P})} \cdot (-1)^{|\mathcal{P}|-1} \cdot \frac{|\mathcal{P}|!}{|\mathcal{P}|} F_{\mathcal{P}}.$$

On the other hand, since $F(0) = 1$, we can write

$$\log F = (F-1) - \tfrac{1}{2}(F-1)^2 + \cdots = \sum_{m \geq 0} \sum_{\mathcal{P} \vdash m} r_{\mathcal{P}} F_{\mathcal{P}} x^m$$

for some $r_{\mathcal{P}} \in \mathbb{Q}$. All contributions to $r_{\mathcal{P}}$ come from the term $(-1)^{|\mathcal{P}|-1}(F-1)^{|\mathcal{P}|}/|\mathcal{P}|$. Expanding out we see that $r_{\mathcal{P}}$ equals $(-1)^{|\mathcal{P}|-1}/|\mathcal{P}|$ times the number of distinct ways of ordering the terms in $\mathcal{P}$, which is $|\mathcal{P}|!/\operatorname{Aut}(\mathcal{P})$.

Comparing the terms we see that $H_{\mathcal{P}} = m! r_{\mathcal{P}} F_{\mathcal{P}}$. Since $H_m = \sum_{\mathcal{P} \vdash m} H_{\mathcal{P}}$ and $(\log F)_m = \sum_{\mathcal{P} \vdash m} r_{\mathcal{P}} F_{\mathcal{P}}$, Equation (6-4) follows. □

7 Enumerative applications

We present some applications of the main theorem to the problem of counting geometric objects with prescribed singularities. We treat three different problems. In Section 7.1 we study curves on a surface having prescribed singularity type, where by singularity type we mean either analytic or topological (equisingular) type. If L is a sufficiently ample line bundle on a surface S, we show that the number of such curves in a general linear system $\mathbb{P}^d \subset |L|$ of appropriate dimension is given by a universal polynomial in the Chern numbers of (S, L). Similar and in some ways more general results to this effect have recently been obtained independently by Li and Tzeng in [24].

In Section 7.2, we consider divisors having fixed isolated analytic singularity types on a smooth variety X of arbitrary dimension. We show that the number of such in a general linear system $\mathbb{P}^d \subset |L|$ is universal. The proofs carry over from the analytic curve singularity case and are omitted.

Finally, in Section 7.3, we consider again the case of curves on a surface. We study the locus $|L|_m \subset |L|$ of curves having prescribed BPS spectrum m and show that if L is sufficiently ample, the Euler characteristic of $|L|_m \cap \mathbb{P}^k$ is universal.

All of our results use the assumption that L is sufficiently ample. This is required to ensure that the objects we consider occur in the expected codimension in $|L|$, as well as in other places in the argument. A natural way of measuring the ampleness of a line bundle in this setting is N–very ampleness, defined as follows.

Definition 7.1 Let X be a nonsingular, projective variety, and let L be a line bundle on X. We say that L is N–very ample if for every length-$(N+1)$ subscheme $Z \subset X$, the map $H^0(X, L) \to H^0(Z, L|_Z)$ is surjective.

Equivalently, we say that the line bundle L is N–very ample if the sheaf homomorphism $H^0(X, L) \otimes \mathcal{O}_{X^{[N+1]}} \to L^{[N+1]}$ is surjective. Being 0–very ample is the same as being globally generated, and being 1–very ample is the same as being very ample. If L_1 and L_2 are N_1– and N_2–very ample, then $L_1 \otimes L_2$ is (N_1+N_2)–very ample [15].

7.1 Curves with specified singularities

We begin by fixing some terms. By a curve singularity we mean a pair (C, p) where C is a reduced, locally planar algebraic curve and p is a singular point of C.

Let (C, p) be a curve singularity. By the analytic type of the singularity (C, p) we mean the isomorphism type of the complete local $\mathbb{C}$–algebra $\widehat{\mathcal{O}}_{C,p}$. By the topological type (equivalently, equisingularity type) of (C, p) with C embedded in a smooth surface S, we mean the homeomorphism type of the pair $\big(B_\epsilon(p), C \cap B_\epsilon(p)\big)$, where $B_\epsilon(p)$ is a sufficiently small open ball in S centred at p.

Proposition 7.2 *Let S be a smooth, projective, connected surface, let L be a line bundle on S, and let $T_1, \dots, T_k$ be analytic isolated singularity types. There are expected codimensions d_i associated with each T_i, and we let $d = \sum d_i$.*

There is an integer N and a rational polynomial $G_{(T_i)}$ of degree k in four variables, depending only on the T_i, such that if L is N–very ample, then in a general $\mathbb{P}^d \subseteq |L|$ the number of curves having precisely k singularities of types T_i is

$$G_{(T_i)}\big(c_1^2(L), c_1(L)c_1(S), c_2(S), c_1^2(S)\big).$$

The same statement holds when the T_i are topological rather than analytic singularity types.

The same proposition has recently been obtained by Li and Tzeng in [24], using essentially the same strategy. One could also choose a sequence of singularity types T_i

where some are analytic and some are topological, and count the curves having these singularity types. In this case, the same proposition can be proved by the same approach, ie by constructing a geometric set $W = P\big(W((T_i))\big)$, which is a straightforward mixture of the corresponding sets in the analytic and topological cases. Proving the bijection between $\{(Z,C) \mid Z \subset C, Z \in W\}$ and $\{C \mid C$ has singularities of types $T_i\}$ as in Lemmas 7.6 and 7.7 then becomes somewhat subtle in the case where there are i, j such that T_i is an analytic singularity type and T_j the corresponding topological type. As [24] in fact proves this more general version of Proposition 7.2, we omit the details.

Remark We will not be concerned with the precise ampleness condition required on L for the universal polynomial to give the correct answer, and refrain from making the N in the statement of the proposition explicit. Instead, we will take N large enough whenever the N–very ampleness of L is required; it will be clear that N depends only on the types T_i. A value for N may be recovered from the proof, but already in the case of nodal singularities, the N provided by this method is known to be larger than required by a factor of 5; compare the bounds obtained by our model [10] with those obtained by [21].

The main idea of the proof, taken from [10], is to set up a correspondence between curves having given singularities and curves containing 0–dimensional subschemes of given isomorphism type.

Choosing analytic or topological singularity types T_i, we find an n and a geometric set $W = W((T_i)) \subset S^{[n]}$, such that a generic curve containing a $Z \in W$ has the specified singularities. Then, using a proposition from [10] we get that in a general $\mathbb{P}^d \subseteq |L|$, the number of curves containing a subscheme $Z \in W$ equals $\deg\big(c_{\dim W}(L^{[n]}) \cap [\overline{W}]\big)$, which is universal by Theorem 1.1. We then show that there is a bijection between such pairs (Z,C) and curves in $\mathbb{P}^d$ with singularities of types T_i, completing the proof.

Corollary 7.3 *Let $T_1, \dots, T_k$ be distinct analytic singularity types, and let $m_1, \dots, m_k$ be nonnegative integers. Denote by $G_{(m_i)}$ the universal polynomial computing the number of curves having precisely m_i singularities of type T_i and no other singularities. There are then power series $B_1, B_2, B_3, B_4 \in \mathbb{Q}[\![x_1, \dots, x_k]\!]$ such that*

$$\sum_{(m_i)\in\mathbb{Z}_{\geq 0}^k} G_{(m_i)} x_1^{m_1} \cdots x_k^{m_k} = B_1^{c_1^2(L)} B_2^{c_1(L)c_1(S)} B_3^{c_1^2(S)} B_4^{c_2(S)}.$$

The same statement holds when the T_i are topological types.

Proof This follows from Proposition 6.4 and the proof of Proposition 7.2, using the fact that W is irreducible in both the analytic and the topological case. □

7.1.1 Analytic types We treat first the case of analytic singularity types. Fix a smooth, projective, connected surface S, a line bundle L on S, and analytic singularity types $T_1, \dots, T_k$. We assume that L is N–very ample, where N will be taken to be sufficiently large at various points in the proof.

In order to associate a 0–dimensional subscheme to an analytic singularity type, we need the following lemma, which states that for an analytic singularity type T, there exists an integer $I(T)$ such that a singularity (C, p) is of analytic type T if it looks like a singularity of type T to $I(T)^{\text{th}}$ order at the singular point.

Lemma 7.4 *Let (C, p) be a curve singularity of analytic type T. There is a positive integer $I(T)$ such that if (C', p') is a curve singularity, the analytic type of (C', p') is T if and only if $\mathcal{O}_{C,p}/\mathfrak{m}^{I(T)} \cong \mathcal{O}_{C',p'}/\mathfrak{m}^{I(T)}$.*

Proof This follows from [11, Corollary 2.24]; in fact we can take $I(T) = \tau + 2$, where τ is the Tjurina number of T. □

Given a singularity type T we define a punctual geometric subscheme $W(T)$ as follows. Let (C, p) be a germ of type T, and let $I(T)$ be the integer that is given by Lemma 7.4. Suppose that the length of $\mathcal{O}_{C,p}/\mathfrak{m}_p^{I(T)}$ is $n(T)$. Let $W(T) \subset \operatorname{Hilb}_0^{n(T)}(\mathbb{C}^2)$ be the set of subschemes $Z \in \operatorname{Hilb}_0^{n(T)}(\mathbb{C}^2)$ with $Z \cong \operatorname{Spec} \mathcal{O}_{C,p}/\mathfrak{m}^{I(T)}$.

Let now $n_i = n(T_i)$, let $n = \sum_{i=1}^k n_i$, and define $W \subseteq S^{[n]}$ to be the set of subschemes of the form $Z_1 \sqcup \dots \sqcup Z_k$, where Z_i is isomorphic to a point in $W(T_i)$ for every i.

It is clear that W is a geometric subset, and in the notation of Section 2.4 we have $W = P\big((W(T_i))\big)$. We define the expected codimension of the singularity T_i to be $d_i = n_i - \dim W(T_i)$. We let $d = n - \dim W = \sum d_i$.

Note that $W(T_i)$ is irreducible and locally closed, as it is the orbit of a given point in $\operatorname{Hilb}_0^{n_i}(\mathbb{C}^2)$ under the action of the connected algebraic group $\operatorname{Aut}(\mathcal{O}_{\mathbb{C}^2,0}/\mathfrak{m}^{I(T_i)})$. It follows that W is irreducible and locally closed.

Lemma 7.5 *Let $Y \subset S^{[n]}$ be a locally closed subset, and assume L is $(n-1)$–very ample.*

(i) *Let $\mathcal{Z} \subset S^{[n]} \times |L|$ denote the incidence locus of pairs (Z, C) with $Z \in Y$ and $Z \subset C$. We have $\dim \mathcal{Z} = \dim|L| + \dim Y - n$.*

(ii) *Let $e = n - \dim Y$, and let $\mathbb{P}^e \subset |L|$ be a general linear subspace. The number of pairs (Z, C) such that $Z \in Y$, $C \in \mathbb{P}^e$ and $Z \subset C$ is equal to*

$$\deg c_{\dim Y}(L^{[n]}) \cap [\overline{Y}].$$

Proof (i) For any $Z \in Y$, the fibre of $\mathcal{Z} \to Y$ over Z is the projectivisation of the kernel of $H^0(S,L) \to H^0(Z, L|_Z)$. By the $(n-1)$–very ampleness of L, this homomorphism is surjective, so $\mathcal{Z} \to Y$ is a projective space bundle with fibres of dimension $|L|-n$. The claim follows.

(ii) See the proof of [10, Proposition 5.2]. □

Applying Lemma 7.5(ii) with $Y = W$, the following lemma now concludes the proof of Proposition 7.2 in the analytic case.

Lemma 7.6 *Let $\mathbb{P}^d \subset |L|$ be a general subsystem, and assume L is N–very ample. Suppose (Z, C) is a pair such that $Z \in W$, $C \in \mathbb{P}^d$ and $Z \subset C$. Then C has k singularities of analytic types T_i, and C contains no other point of W.*

Proof Let $\mathbb{P}^d$, C and Z be as in the statement of the lemma, and suppose $Z = \bigsqcup Z_i$, where Z_i is supported at $x_i \in C$ and where Z_i is isomorphic to a point in $W(T_i)$. We show the following claims: (1) that C has precisely k singularities, (2) that C has a singularity of type T_i at x_i, and (3) that C contains precisely one $Z \in W$.

(1) Clearly, C has at least k singularities. Assume for a contradiction that C has more than k singularities. It must then contain a subscheme of the form $Z \sqcup Z'$, where Z' is defined by an ideal $\mathfrak{m}_x^2$ for some $x \in S$ where C is singular. The geometric set

$$W' := \{Z \sqcup Z' \mid Z \in W \text{ and } Z' = \operatorname{Spec} \mathcal{O}_{S,x}/\mathfrak{m}_x^2\} \subset S^{[n+3]}$$

has dimension 2 greater than W. By Lemma 7.5(i), we see that the set of $C \in |L|$ containing an element of W' has codimension $> d+1$ in $|L|$ if L is $(n+2)$–very ample. The intersection of this set with a general $\mathbb{P}^d \subset |L|$ is empty, contradicting the original assumption.

(2) Suppose for a contradiction that the singularity type of C at x_1 is $T_1' \neq T_1$. Let $R = \mathcal{O}_{S,x_1}/\mathfrak{m}^{I(T_1)}$. As $T_1' \neq T_1$, we have $Z_1 \subsetneq C \cap \operatorname{Spec} R$. Let $f, g \in R$ be defining equations of Z_1 and $C \cap \operatorname{Spec} R$ in R, we then have $(g) \subsetneq (f)$. This implies that $(g) \subseteq \mathfrak{m} \cdot (f) \subseteq \mathfrak{m}^{\operatorname{ord}(f)+1} \cap (f)$, where $\operatorname{ord}(f)$ is the maximal integer such that $f \in \mathfrak{m}^{\operatorname{ord}(f)}$.

Hence C contains a subscheme of the form

$$Z_1' \sqcup Z_2 \sqcup \cdots \sqcup Z_k,$$

where $Z_1' = \operatorname{Spec} R/\mathfrak{m}^{\operatorname{ord}(f)+1} \cup Z_1$. Let W' be the set of subschemes which can be written in this way. Then W' is geometric and has dimension $\leq \dim W$.

Let n' be the length of the points of W'. Clearly, we have $n' > n$, so we have

$$n' - \dim W' > n - \dim W = d.$$

As L is N–very ample, applying Lemma 7.5(i) shows that the codimension of the locus of points containing a point from W' is $> d$, if $N \geq n+1$. As $C \in \mathbb{P}^d$ for a general $\mathbb{P}^d$, it is not contained in this locus.

(3) By (1) and (2), we know that C has a singularity of type T_i at p_i, and suppose for a contradiction that there is a $Z' \in W$ with $Z' \subset C$ such that $Z' \neq Z$. Let $Z' = \bigsqcup Z'_i$ with Z'_i supported at p_i. Assume that $Z_i \neq Z'_i$ as subschemes. But by part (2), the singularity type associated to Z'_i must be T_i, and hence we have

$$Z_i = \operatorname{Spec} \mathcal{O}_{C,p_i}/\mathfrak{m}^{I(T_i)} = Z'_i. \qquad \square$$

7.1.2 Topological singularities We now turn to the case of topological singularities. Let S and L be as before, and fix topological singularity types $T_1, \dots, T_k$. For any planar curve singularity (C, p) the infinitely near points in C over p define a combinatorial structure called the Enriques diagram, which determines the equisingularity type of (C, p); see [18]. Let D be the Enriques diagram of the T_i, that is the union of the Enriques diagrams for each T_i.

The degree of an Enriques diagram is defined in [18], and we let $n = \deg(D)$. Let $W(D) \subset S^{[n]}$ be the subscheme denoted $H(D)$ in [18]. It has the property that if $Z \in W(D)$ and C is a generic curve containing Z, then the singularities of C correspond to the Enriques diagram D. The subset $W(D)$ is geometric and irreducible [19, Corollary 5.8].

Let $d = n - \dim W(D)$. The following lemma is a reformulation of [18, Lemma 3.7].

Lemma 7.7 *Let $\mathbb{P}^d \subset |L|$ be a general subsystem, and assume L is $(n-1)$–very ample. Suppose (Z, C) is a pair such that $Z \in W(D)$, $C \in \mathbb{P}^d$ and $Z \subset C$. Then C has k singularities of topological types T_i, and C contains no other point of $W(D)$.*

Proof Let $\mathcal{Z}$ be the incidence locus in $|L| \times W(D)$, let $|L|_T \subset |L|$ be the set of curves having prescribed singularity types, let $\pi\colon |L| \times W \to |L|$ be the projection and let $\mathcal{Z}_T = \pi^{-1}(|L|_T) \subset \mathcal{Z}$. By [18, Lemma 3.7], $\mathcal{Z}_T$ is dense in $\mathcal{Z}$ when L is $(n-1)$–very ample.[2] We therefore have $\dim \mathcal{Z} \setminus \mathcal{Z}_T < \dim \mathcal{Z}$, and applying Lemma 7.5 we find $\dim \mathcal{Z} = \dim|L| - d$. So if $\mathbb{P}^d \subset |L|$ is general, then $\pi^{-1}(\mathbb{P}^d) \cap \mathcal{Z} \subset \mathcal{Z}_T$, proving the first claim of the lemma. By [18, Lemma 3.7], the map $\mathcal{Z}_T \xrightarrow{\pi} |L|_T$ is bijective, which proves the second claim. $\square$

[2]The reference assumes $L = L_1 \otimes L_2^{\otimes n}$ for L_1 globally generated and L_2 very ample. However, the proof given shows that L is then $(n-1)$–very ample, and the stronger assumption on L is not needed.

If we apply Lemma 7.5(ii) with $Y = W(D)$, then Lemma 7.7 concludes the proof of Proposition 7.2 in the topological case.

If D is a connected Enriques diagram, then every element $Z \in W(D)$ will have support in one point. Since $W(D)$ is geometric, it is defined by a punctual geometric subset $W(D)_0 \subset \mathrm{Hilb}_0^{\deg D}(\mathbb{C}^2)$.

For topological singularities $T_1, \dots, T_k$, the associated Enriques diagram D is given by $D = D_1 \sqcup \cdots \sqcup D_k$, where D_i is the connected Enriques diagram associated with T_i. In the notation of Section 2.4, we have $W(D) = P\big((W(D_i)_0)\big)$, so we can apply Proposition 6.4 to prove Corollary 7.3 for the case of topological singularity types.

7.2 General hypersurface singularities

Without any extra work, the above extends to counts of analytic types of isolated singularities of hypersurfaces. Let (D, p) be the pair of a divisor D in a nonsingular variety X and an isolated singular point of D. The analytic type of the singularity (D, p) is the isomorphism type of the complete local $\mathbb{C}$–algebra $\widehat{\mathcal{O}}_{D,p}$.

Lemma 7.4 is valid for hypersurface singularities of all dimensions, and the following proposition can be shown by the proof given in Section 7.1.1, mutatis mutandis. (Note in particular that we do not use the nonsingularity of $X^{[n]}$ anywhere in the argument.)

Proposition 7.8 *Let X be a smooth, projective, connected variety, let L be a line bundle on X, and let $T_1, \dots, T_k$ be analytic isolated singularity types. There are expected codimensions d_i associated with each T_i, and we let $d = \sum d_i$.*

There is an integer N and a rational polynomial $G_{(T_i)}$ in the Chern numbers of (X, L), depending only on the T_i, such that if L is N–very ample, then in a general $\mathbb{P}^d \subseteq |L|$ the number of divisors having precisely k isolated singularities of types T_i is given by $G_{(T_i)}$.

There is a similar corollary for the generating function of these universal polynomials as in the curve case; we leave the statement of this to the reader.

7.3 BPS spectrum loci

Let C be a reduced, complete, locally planar algebraic curve, and consider the generating function

$$H_C(q) = \sum_{k=0}^{\infty} \chi(C^{[k]}) q^k.$$

Let the arithmetic and geometric genus of C be $g(C)$ and $\overline{g}(C)$, respectively. In [32] it is shown that there are integers $n_{i,C}$, with $n_{i,C} = 0$ unless $\overline{g} \leq i \leq g$, such that

$$H_C(q) = \sum_{i=\overline{g}(C)}^{g(C)} n_{i,C}\, q^{g-i}(1-q)^{2i-2}. \tag{7-1}$$

For our purposes, it will be convenient to work with the index-shifted integers $m_{i,C}$ given by $m_{i,C} := n_{g-i,C}$. We define the BPS spectrum of C to be the sequence of integers $(m_{i,C})_{i=0}^{\infty}$. By the above, we have $m_{i,C} = 0$ if $i \geq g - \overline{g}$. If C has k singularities of analytic types $T_1, \dots, T_k$, then by stratifying $C^{[k]}$ one can see that the BPS spectrum of C depends only on the T_i.

By this observation, one may define the BPS spectrum of an analytic singularity type T as the BPS spectrum of a complete, reduced curve having one singularity of type T. The BPS spectrum of a singularity T is shown by Maulik [28] to be determined explicitly by the Milnor number and the HOMFLY polynomial of the link of T. In particular, the BPS spectrum of a locally planar curve depends only on the topological types of the singularities of the curve.

Proposition 7.9 *Let S be a smooth, projective, connected surface, let L be a line bundle on S, and let $k \in \mathbb{Z}_{\geq 0}$. Let $m = (m_i)_{i=0}^{\infty}$ be a BPS spectrum, and denote by $|L|_m \subseteq |L|$ the locus of curves with BPS spectrum m.*

There is an integer N and a rational polynomial G_m in four variables, depending only on k and m, such that if L is N–very ample, then for a general $\mathbb{P}^k \subset |L|$ we have

$$\chi(\mathbb{P}^k \cap |L|_m) = G_m\big(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)\big).$$

If in addition it is known that $\mathbb{P}^k \cap |L|_m$ is 0–dimensional, this implies an enumerative result of the kind found in the previous subsection. This is essentially the argument used in [21] to compute the number of δ–nodal curves and prove the Göttsche conjecture.

Remark Let $d = \dim|L|$, and let us write $c_{\mathrm{SM}}(|L|_m) = \sum a_i[\mathbb{P}^{d-i}] \in H_*(|L|)$. Let $\chi_k = \chi(\mathbb{P}^k \cap |L|_m)$ for a general hyperplane $\mathbb{P}^k$. Aluffi shows in [1] that the sequence (a_i) determines the sequence (χ_i) and vice versa.[3] Concretely, we get

$$a_d = \chi_d \qquad \text{and} \qquad a_i = \sum_{k \leq i} \binom{d-k-1}{i-k} \chi_k.$$

Thus, Proposition 7.9 implies that there is a polynomial in d and the Chern numbers of (S, L) which computes a_i, assuming L is N–very ample for some N depending only on i.

[3] We thank the referee for pointing out this result.

The remainder of this section contains the proof of Proposition 7.9.

Lemma 7.10 *Let $l \geq 0$ be an integer. Then $\chi(\mathrm{Hilb}^l_p(C))$ can take only finitely many values for (C,p) a locally planar curve singularity.*

Proof Let $B = \operatorname{Spec} \mathbb{C}[z_{i,j}]_{0 \leq i+j<l}$, which corresponds to the affine space of all degree $<l$ polynomials in $\mathbb{C}[x,y]$ by the map

$$(a_{i,j}) \in B \mapsto \sum a_{i,j} x^i y^j \in \mathbb{C}[x,y].$$

Let $\mathcal{C} \subset \mathbb{A}^2 \times B$ be the divisor defined by the equation $\sum_{0 \leq i+j<l} z_{i,j} x^i y^j$. Let $\mathrm{Hilb}^j(\mathcal{C}/B)$ denote the relative Hilbert scheme, let $\mathrm{Hilb}^j_0(\mathcal{C}/B) \subset \mathrm{Hilb}^j(\mathcal{C}/B)$ be the subset of those $Z \subset \mathcal{C}$ with support in $\mathcal{C} \cap (\{0\} \times B)$, and let $\pi\colon \mathrm{Hilb}^l_0(\mathcal{C}/B) \to B$ be the projection.

For any given planar curve singularity (C,p), the scheme $\mathrm{Hilb}^l_p(C)$ depends only on the structure of $\operatorname{Spec}(\mathcal{O}_{C,p}/\mathfrak{m}_p^l)$. We can find a $g \in \mathbb{C}[x,y]$ of degree $<l$ such that there is an isomorphism $\phi\colon \operatorname{Spec} \mathcal{O}_{C,p}/\mathfrak{m}_p^l \to \operatorname{Spec} \mathbb{C}[x,y]/(\mathfrak{m}_0^l, g)$.

Denote by g also the corresponding point of B. We then get a bijective morphism $\phi^{[l]}\colon \mathrm{Hilb}^l_p(C) \to \pi^{-1}(g)$. It follows that $\chi(\mathrm{Hilb}^l_p(C)) = \chi(\pi^{-1}(g)) = \pi_*(1)(g)$, where π_* denotes the pushforward of constructible functions. Since $\pi_*(1)$ is a constructible function, it takes only finitely many values, and this completes the proof. □

Lemma 7.11 *If L is k–very ample, then for a general $\mathbb{P}^k \subseteq |L|$ every curve $C \in \mathbb{P}^k$ is reduced and satisfies $\overline{g}(C) \geq g(C) - k$.*

Proof See [21, Proposition 2.1]. □

Let Sp_k be the set of BPS spectra m satisfying the following condition: there exists a k–very ample L such that for a general $\mathbb{P}^k \subset |L|$, there is a $C \in \mathbb{P}^k$ whose BPS spectrum is m.

Lemma 7.12 *The set Sp_k is finite.*

Proof Let L be k–very ample, and let $C \in \mathbb{P}^k \subset |L|$ for a general $\mathbb{P}^k$. By Lemma 7.11, we have $\overline{g}(C) \geq g(C) - k$. By (7-1), the BPS spectrum of C is then determined by $\chi(C^{[i]})$ for $1 \leq i \leq k$.

Denote by $\mathrm{Hilb}^j_p(C) \subset C^{[j]}$ the set of subschemes supported at $p \in C$. Stratifying $C^{[i]}$, we see that $\chi(C^{[i]})$ is determined by $\chi(C)$ and the integers $\chi(\mathrm{Hilb}^l_p(C))$, where p ranges over the singular points of C and $l \leq i$. Applying Lemma 7.10, the claim follows. □

Lemma 7.13 *For each $m \in \mathrm{Sp}_k$, there is an $F_m \in \mathbb{Q}(g)[x_1,\dots,x_k]$ such that*

$$F_m(g(C),\chi(C),\dots,\chi(C^{[k]}))$$

equals 1 *if C has BPS spectrum m and equals* 0 *if C has BPS spectrum in $\mathrm{Sp}_k \setminus \{m\}$.*

Proof Let C be a curve with BPS spectrum m. Using (7-1), we find that $\chi(C^{[i]})$ is a polynomial in $g(C)$:

$$\chi(C^{[i]}) = \sum_{j=0}^{i} (-1)^{i-j} m_j \binom{2(g(C)-j)-2}{i-j}.$$

For each $m \in \mathrm{Sp}_k$, let C_m be a curve of BPS spectrum m. Let

$$p_m = (\chi(C_m^{[1]}),\dots,\chi(C_m^{[k]})) \in \mathbb{Q}(g)^k.$$

As in the proof of Lemma 7.12, we see that m is determined by $\chi(C_m^{[1]}),\dots,\chi(C_m^{[k]})$, and hence the p_m are all distinct. Therefore for each m we can find an element $G_m \in \mathbb{Q}(g)[x_1,\dots,x_k]$ such that $G_m(p_{m'}) = 0$ if and only if $m = m'$. Putting $F_i = \prod_{j\neq i}(G_j/G_j(p_i))$ gives the result. □

Let now $\mathbb{P}^k \subset |L|$ be general, let $\mathcal{C} \to \mathbb{P}^k$ be the family of curves, and let $\mathcal{C}^{[i]}/\mathbb{P}^k$ denote the relative Hilbert scheme. Every monomial $M \in \mathbb{Q}[x_1,\dots,x_k]$ determines a scheme $\mathcal{C}(M)$ by taking

$$\mathcal{C}(x_i) = \mathcal{C}^{[i]}/\mathbb{P}^k$$

and extending this by the rule

$$\mathcal{C}(M_1 \cdot M_2) = \mathcal{C}(M_1) \times_{\mathbb{P}^k} \mathcal{C}(M_2).$$

It is clear that

$$\chi(\mathcal{C}(M)) = \sum_{m\in \mathrm{Sp}_k} \chi(|L|_m \cap \mathbb{P}^k) M(\chi(C_m^{[1]}),\dots,\chi(C_m^{[k]})),$$

where C_m denotes a curve with BPS spectrum m.

We may write the polynomial F_m from Lemma 7.13 in the form

$$F_m = \sum_M f_M(g) M,$$

where the sum is over all monomials M and where $f_M \in \mathbb{Q}(g)$ for each M. Since $g(C) = \frac{1}{2}(c_1(L)^2 - c_1(L)c_1(S)) + 1$, we get

$$\chi(|L|_m \cap \mathbb{P}^k) = \sum_M f_M\bigl(\tfrac{1}{2}(c_1(L)^2 - c_1(L)c_1(S)) + 1\bigr)\chi(\mathcal{C}(M)).$$

Lemma 7.14 below shows that $\chi(\mathcal{C}(M))$ is universal. This implies that $\chi(|L|_m \cap \mathbb{P}^k)$ admits a universal expression as G/H, where G is a polynomial in the Chern numbers of (S, L) and H is a polynomial in $g = \frac{1}{2}(c_1(L)^2 - c_1(L)c_1(S)) + 1$. In Section 7.4 we strengthen this to show that G/H is a polynomial, concluding the proof of Proposition 7.9.

Lemma 7.14 *Let L be a line bundle, and let $\mathbb{P}^k \subseteq |L|$ be a general linear subsystem. Let $\mathcal{C} \to \mathbb{P}^k$ be the universal family of curves, and denote by $\mathcal{C}^{[i]} \to \mathbb{P}^k$ the relative Hilbert scheme of i points for this morphism. Then the Euler characteristic*

$$\chi(\mathcal{C}^{[i_1]} \times_{\mathbb{P}^k} \cdots \times_{\mathbb{P}^k} \mathcal{C}^{[i_l]})$$

is computed by a universal polynomial, provided that L is $((\sum_j i_j)-1)$–very ample.

Proof For notational simplicity, we treat the case where $l = 2$; the general case is essentially the same. The case $l = 1$ is simpler; see [21].

Let $f_n\colon \mathcal{C}^{[n]} \to S^{[n]}$ be the natural morphism. We claim that there exists a finite stratification of $S^{[n]}$ by geometric sets $P_{n,i}$ of universal type, such that

$$\chi(\mathcal{C}^{[k_1]} \times_{\mathbb{P}^k} \mathcal{C}^{[k_2]}) = \sum_{n=1}^{k_1+k_2} \sum_i i \cdot \chi(\mathcal{C}^{[n]} \cap f_n^{-1}(P_{n,i})). \tag{7-2}$$

Consider the function $g\colon S^{[k_1]} \times S^{[k_2]} \to \bigsqcup_{n=1}^{k_1+k_2} S^{[n]}$ defined pointwise by

$$g(Z_1, Z_2) = Z_1 \cup Z_2,$$

where the union is in the scheme-theoretic sense.

Define

$$P_{n,i} = \{Z \in S^{[n]} \mid \chi(g^{-1}(Z)) = i\}.$$

One can check that $P_{n,i}$ is geometric. Observing that $Z_1, Z_2 \subset C \iff Z_1 \cup Z_2 \subset C$, we also see that $P_{n,i}$ satisfies (7-2). Lemma 7.15 now completes the proof. □

Lemma 7.15 *Let P be a geometric subset of $S^{[n]}$, and let $\mathbb{P}^k \subset |L|$ be a general linear subsystem, with L an $(n-1)$–very ample line bundle. Let $\mathcal{C} \to \mathbb{P}^k$ be the family of curves, let $\mathcal{C}^{[n]}$ be the relative Hilbert scheme of the family, and let $f\colon \mathcal{C}^{[n]} \to S^{[n]}$ be the natural morphism.*

Then there exists a universal polynomial in the Chern numbers of (S, L) which computes $\chi(f^{-1}(P) \cap \mathcal{C}^{[n]})$.

Proof The inclusion-exclusion principle for χ lets us reduce to the case where P is closed and irreducible. Consider the diagram:

$$\begin{array}{ccc} f^{-1}(P) \cap \mathcal{C}^{[n]} & \xrightarrow{f} & P \\ \downarrow & & \\ \mathbb{P}^k & & \end{array}$$

The fibres of f are all projective spaces, since for a point $Z \in P$, the fibre over Z is the linear system of curves containing Z. Hence we have $\chi(f^{-1}(Z)) = \dim f^{-1}(Z) + 1$. Let $P_m = \{Z \in P \mid \chi(f^{-1}(Z)) = m\}$. On P, consider the surjective homomorphism

$$H^0(S, L) \otimes \mathcal{O}_P \to L^{[n]},$$

let $V \subseteq H^0(S, L)$ be the $(k+1)$–dimensional subspace defining $\mathbb{P}^k$, and let

$$\phi\colon V \otimes \mathcal{O}_P \to L^{[n]}$$

be the induced homomorphism. Then

$$P_m = \{Z \in P \mid \dim \ker \phi = m\}.$$

Letting $D_r(\phi)$ denote the locus over which ϕ has rank $\leq r$, we then have that $P_m = D_{k+1-m}(\phi) \setminus D_{k+1-m-1}(\phi)$. It thus suffices to compute $\chi(D_r(\phi))$ for all r.

By [33, Theorem 2.10], there exists a formula for the Euler characteristic of $D_r(\phi)$ as a polynomial in the Chern classes of $L^{[n]}$ capped with $c_{\mathrm{SM}}(P)$, assuming that ϕ is r–general in the sense of [33].

Choose a Whitney stratification of P, and let Y be any stratum. We consider the bundle $\mathcal{H}om(V \otimes \mathcal{O}_Y, L^{[n]})$ as a scheme, and let $D_r \subseteq \mathcal{H}om(V \otimes \mathcal{O}_Y, L^{[n]})$ denote the tautological rank-r degeneracy locus. To say that ϕ is r–general means that for each stratum Y, the graph $\Gamma(\phi) \subseteq \mathcal{H}om(V \otimes \mathcal{O}_Y, L^{[n]})$ intersects $D_r \setminus D_{r-1}$ transversely.

The $(n-1)$–very ampleness of L implies there is a surjection $H^0(S, L) \otimes \mathcal{O}_P \to L^{[n]}$, inducing a morphism

$$P \to \mathrm{Gr}\big(H^0(S, L), n\big).$$

Choosing a subspace $V \subseteq H^0(S, L)$, the intersection of $\Gamma(\phi)$ with $D_r \setminus D_{r-1}$ corresponds to the intersection of P with a certain smooth subset of $\mathrm{Gr}\big(H^0(S, L), n\big)$. By the Kleiman–Bertini transversality theorem [17], for a general $V \subseteq H^0(S, L)$, the intersection of each stratum Y with this set will be smooth of the expected dimension, and hence ϕ is r–general.

Hence the formula of [33, Theorem 2.10] applies, and by Theorem 1.1(ii), the statement of the lemma follows. $\square$

7.4 A bootstrap

Let $A = \mathbb{Q}[x_{c_1(S)^2}, x_{c_1(S)c_1(L)}, x_{c_1(L)^2}, x_{c_2(S)}]$. For any $F \in A$, write $F(S, L)$ for the value obtained by assigning the Chern numbers of (S, L) to the x_i. We have shown above that there are $G, H \in A$ such that $G(S, L)/H(S, L)$ computes the Euler characteristic of $\chi(\mathbb{P}^k \cap |L|_m)$ when L is N–very ample. Furthermore, H is contained in the subring $\mathbb{Q}[x_g]$, where $x_g = \frac{1}{2}(x_{c_1(L)^2} - x_{c_1(L)c_1(S)}) - 1$.

The claim of Proposition 7.9 is that we may take $H = 1$ here, or equivalently that H divides G. Lemma 7.18 below shows that this is indeed the case.

Lemma 7.16 *Suppose $F \in \mathbb{Q}(x)$ is such that $F(n)$ is an integer for all $n \gg 0$. Then $F \in \mathbb{Q}[x]$.*

Proof We can write $F = Q + R/H$ with $H, Q, R \in \mathbb{Q}[x]$ and with $\deg R < \deg H$. Let $N \in \mathbb{Z}$ be such that $NQ \in \mathbb{Z}[x]$; then $NF(n) - NQ(n)$ is integral for $n \gg 0$. This equals $NR(n)/H(n)$, which tends to 0 as $n \to \infty$. Hence $R(n) = 0$ for $n \gg 0$, and so $R = 0$. □

Lemma 7.17 *The set of quadruples*

$$\bigl(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)\bigr) \in \mathbb{Z}^4,$$

where S is a smooth, connected, projective surface and L is an ample line bundle forms a Zariski dense subset of $\mathbb{C}^4$.

Proof Let $\mathcal{S}$ be the set of surfaces such that the Picard rank is ≥ 2 and $c_1(S)$ is numerically nontrivial. Let $S \in \mathcal{S}$ and let $\mathrm{NS}(S)$ be the Néron–Severi group of S. Since the ample classes in $\mathrm{NS}(S) \otimes \mathbb{R}$ form an open cone, we see that $\{[L] \mid L \text{ ample}\}$ is a Zariski dense subset of $\mathrm{NS}(S) \otimes \mathbb{C}$. One checks that the set

$$\{(\alpha^2, \alpha c_1(S)) \mid \alpha \in \mathrm{NS}(S) \otimes \mathbb{C}\}$$

is Zariski dense in $\mathbb{C}^2$, and hence

$$\{(c_1(L)^2, c_1(L)c_1(S)) \mid L \text{ ample}\}$$

is Zariski dense in $\mathbb{C}^2$.

Therefore the Zariski closure of the set in the statement of the lemma contains

$$\{(a, b, c_1(S)^2, c_2(S)) \mid a, b \in \mathbb{C}, S \in \mathcal{S}\}.$$

Within $\mathcal{S}$ is the set $\mathcal{S}'$ of surfaces birational to the product of two curves of genera ≥ 2. One checks that $\{(c_1^2(S), c_2(S)) \mid S \in \mathcal{S}'\}$ is Zariski dense in $\mathbb{C}^2$. The claim follows. □

Lemma 7.18 *Assume $G \in A$ and $H \in \mathbb{Q}[x_g] \subset A$ are such that if L is N–very ample, then $G(S,L)/H(S,L)$ is an integer. Then H divides G.*

Proof Consider the map $A \to A[t]$ under which

$$x_{c_1(L)^2} \mapsto t^2 x_{c_1(L)^2} \quad \text{and} \quad x_{c_1(L)c_1(S)} \mapsto t x_{c_1(L)c_1(S)}$$

and which leaves the other generators fixed. Let $G', H' \in A[t]$ be the images of G, H under this map. Then if $t \in \mathbb{Z}$ we have $G'(S,L)(t) = G(S,tL)$ and likewise for H.

Ordering by t–degree, the leading term of H' is proportional to $t^{2k} x^k_{c_1(L)^2}$ for some k. We may thus write $G' = QH' + R$, where $Q, R \in A[x^{-1}_{c_1(L)^2}, t]$ and the t–degree of R is less than that of H.

Let (S,L) be such that L is ample. We have

$$G'(S,L)(t)/H'(S,L)(t) = G(S,tL)/H(S,tL) \quad \text{for } t \in \mathbb{Z}.$$

If $t \gg 0$, then tL is N–very ample, and so $G(S,tL)/H(S,tL)$ is an integer. By Lemma 7.16, $H'(S,L)$ must then divide $G'(S,L)$. It follows that $R(S,L) = 0$. Since this is true for all pairs (S,L) with L ample, Lemma 7.17 implies that $R = 0$.

We thus have $G' = QH'$. Setting $t = 1$ gives $G = PH$, where $P = Q|_{t=1} \in A[x^{-1}_{c_1(L)^2}]$. Since $H \in \mathbb{Q}[x_g]$, it is not divisible by $x_{c_1(L)^2}$, and it follows that $P \in A$. This proves the claim. □

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All Souls College, Oxford
OX1 4AL, United Kingdom

jorgen.rennemo@all-souls.ox.ac.uk

Proposed: Jim Bryan
Seconded: Lothar Göttsche, Ronald Stern

Received: 25 November 2014
Revised: 15 December 2015