Geometry & Topology 21 (2017) 315–343

Shift operators and toric mirror theorem

HIROSHI IRITANI

We give a new proof of Givental's mirror theorem for toric manifolds using shift operators of equivariant parameters. The proof is almost tautological: it gives an A–model construction of the I–function and the mirror map. It also works for noncompact or nonsemipositive toric manifolds.

14N35, 53D45; 14J33, 53D37

1 Introduction

In 1995, Seidel [31] introduced an invertible element of quantum cohomology associated to a Hamiltonian circle action. This has had many applications in symplectic topology. Seidel himself used it to construct nontrivial elements of π_1 of the group of Hamiltonian diffeomorphisms. McDuff and Tolman [26] calculated Seidel's elements in a more general setting and obtained Batyrev's ring presentation of quantum cohomology of toric manifolds. Their method, however, does not yield explicit structure constants of quantum cohomology, ie genus-zero Gromov–Witten invariants.

Recently, Braverman, Maulik and Okounkov [4], Maulik and Okounkov [25] and Okounkov and Pandharipande [29] introduced a shift operator of equivariant parameters for equivariant quantum cohomology. Their shift operators reduce to Seidel's invertible elements under the nonequivariant limit. In this paper, we show that equivariant genus-zero Gromov–Witten invariants of toric manifolds are reconstructed *only from formal properties of shift operators*. This means that the equivariant quantum topology of toric manifolds is determined by its classical counterpart.

More specifically, we give a new proof of Givental's mirror theorem for toric manifolds, which is stated as follows:

Theorem 1.1 (Givental [14], Lian, Liu and Yau [24], Iritani [19] and Brown [5]; see Section 4.2 for more details) *Let X_Σ be a semiprojective toric manifold having a torus fixed point. Let $I(y,z)$ be the cohomology-valued hypergeometric series defined by*

$$I(y,z) = ze^{\sum_{i=1}^m u_i \log y_i/z} \sum_{d\in \mathrm{Eff}(X_\Sigma)} \left(\prod_{i=1}^m \frac{\prod_{c=-\infty}^{0}(u_i + cz)}{\prod_{c=-\infty}^{u_i\cdot d}(u_i+cz)} \right) Q^d y_1^{u_1\cdot d} \cdots y_m^{u_m \cdot d},$$

Published: 10 February 2017 DOI: 10.2140/gt.2017.21.315

where u_i for $i=1,\dots,m$ is the class of a prime toric divisor. Then $I(y,-z)$ lies in Givental's Lagrangian cone $\mathcal{L}_{X_\Sigma}$ associated to X_Σ.

We prove this theorem in the following way. Recall that equivariant genus-zero Gromov–Witten invariants of a T–variety X can be encoded by an infinite-dimensional Lagrangian submanifold $\mathcal{L}_X$ of the symplectic vector space (see Givental [15])

$$\mathcal{H}_X = H_T^*(X) \otimes_{H_T^*(\mathrm{pt})} \mathrm{Frac}(H_T^*(\mathrm{pt})[z]).$$

The space $\mathcal{H}_X$ is called the Givental space and $\mathcal{L}_X$ is called the Givental cone. By the general theory, each $\mathbb{C}^\times$–subgroup $k\colon \mathbb{C}^\times \to T$ defines a shift operator $\mathcal{S}_k$ acting on the Givental space $\mathcal{H}_X$ and induces a vector field on $\mathcal{L}_X$,

$$\boldsymbol{f} \mapsto z^{-1}\mathcal{S}_k \boldsymbol{f} \in T_{\boldsymbol{f}}\mathcal{L}_X.$$

The operator $\mathcal{S}_k$ is determined by T–fixed loci in X and their normal bundles (see Definition 3.13). For toric manifolds, we have a shift operator $\mathcal{S}_i$ for each torus-invariant prime divisor. Then we identify the I–function $I(y,z)$ with an integral curve of the commuting vector fields $\boldsymbol{f} \mapsto z^{-1}\mathcal{S}_i \boldsymbol{f}$.

Theorem 1.2 *Givental's I–function $I(y,z)$ is the unique integral curve which satisfies the differential equation*

$$\frac{\partial I(y,z)}{\partial y_i} = z^{-1}\mathcal{S}_i I(y,z), \quad i=1,\dots,m,$$

and is of the form $I(y,z) = z e^{\sum_{i=1}^m u_i \log y_i/z}\bigl(1+\sum_{d\in \mathrm{Eff}(X_\Sigma)\setminus\{0\}} I_d Q^d y^d\bigr)$, where we set $y^d = \prod_{i=1}^m y_i^{u_i\cdot d}$.

The I–function defines a mirror map $y \mapsto \tau(y) \in H_T^*(X)$ via Birkhoff factorization; see Coates and Givental [8] and Iritani [19]. As a corollary to our proof, we obtain the following relationship between the equivariant Seidel elements $S_i(\tau)$ and the mirror map. This generalizes a previous result in the semipositive case obtained in joint work with González [16].

Theorem 1.3 *The mirror map $\tau(y)$ associated to the I–function is the unique integral curve which satisfies the differential equation*

$$\frac{\partial \tau(y)}{\partial y_i} = S_i(\tau(y)), \quad i=1,\dots,m,$$

and is of the form $\tau(y) = \sum_{i=1}^m u_i \log y_i + \sum_{d\in\mathrm{Eff}(X_\Sigma)\setminus\{0\}} \tau_d Q^d y^d$.

The mirror map and the I–function are related by the formula

$$I(y,z) = zM(\tau(y),z)\Upsilon(y,z),$$

where $M(\tau,z)$ is a fundamental solution for the quantum differential equation (see Proposition 2.2) and $\Upsilon(y,z)$ is an $H_T^*(X)[z]$–valued function. We can also characterize $\Upsilon(y,z)$ by the differential equation

$$\frac{\partial \Upsilon(y,z)}{\partial y_i} = [z^{-1}\mathbb{S}_i(\tau(y))]_+\Upsilon(y,z),$$

where $\mathbb{S}_i(\tau)$ is the shift operator acting on quantum cohomology. The most technical point in our proof is to show the existence of solutions $\tau(y)$ and $\Upsilon(y,z)$ with prescribed asymptotics (see Proposition 4.7).

Since we do not assume that $c_1(X_\Sigma)$ is nef, the mirror map $\tau(y)$ does not necessarily lie in $H_T^{\le 2}(X)$. For this reason, we need to generalize shift operators to big quantum cohomology. We also observe that shift operators are closely related to the $\widehat{\Gamma}$–integral structure introduced by Coates, Iritani and Jiang [9], Iritani [20] and Katzarkov, Kontsevich and Pantev [22]. We show that a flat section of the quantum connection associated to an equivariant vector bundle in the formalism of $\widehat{\Gamma}$–integral structure is invariant under shift operators (Proposition 3.18).

This paper is structured as follows. In Section 2, we review equivariant quantum cohomology and in Section 3, we study shift operators for big quantum cohomology. In Section 4, we prove a mirror theorem for toric manifolds.

1.1 Notation

Unless otherwise stated, we consider cohomology groups with complex coefficients. We use the following notation throughout the paper:

- $T \cong (\mathbb{C}^\times)^m$ is an algebraic torus;
- X is a smooth T–variety; X_Σ is a smooth toric variety associated to a fan Σ;
- $\widehat{T} = T \times \mathbb{C}^\times$;
- $\lambda \in \mathrm{Lie}(T)$ and $z \in \mathrm{Lie}(\mathbb{C}^\times)$ are equivariant parameters for $\widehat{T}$;
- the Givental space is

$$\begin{aligned} H_{\widehat{T}}(X)_{\mathrm{loc}} &:= H_{\widehat{T}}^*(X) \otimes_{H_{\widehat{T}}^*(\mathrm{pt})} \mathrm{Frac}(H_{\widehat{T}}^*(\mathrm{pt})) \\ &= H_T^*(X) \otimes_{H_T^*(\mathrm{pt})} \mathrm{Frac}(H_T^*(\mathrm{pt})[z]). \end{aligned}$$

Acknowledgments The author thanks Tom Coates, Alessio Corti, Eduardo Gonzalez, Hiraku Nakajima and Hsian-Hua Tseng for very helpful discussions on shift operators, Seidel representations and toric mirror symmetry. This work is supported by JSPS Kakenhi Grant Number 16K05127, 16H06337, 16H06335, 25400069, 26610008, 23224002.

2 Equivariant quantum cohomology

2.1 Hypotheses on a T–space

Let $T \cong (\mathbb{C}^\times)^m$ be an algebraic torus. Let X be a smooth variety over $\mathbb{C}$ equipped with an algebraic T–action. We assume the following conditions:

(1) X is semiprojective, ie the natural map $X \to X_0 := \operatorname{Spec} H^0(X, \mathcal{O})$ is projective;

(2) all T–weights appearing in the T–representation $H^0(X, \mathcal{O})$ are contained in a strictly convex cone in $\operatorname{Hom}(T, \mathbb{C}^\times) \otimes \mathbb{R}$ and $H^0(X, \mathcal{O})^T = \mathbb{C}$.

A T–space X with these assumptions has nice cohomological properties; see eg [18]. These conditions ensure that the T–fixed set X^T is projective. We also note the following:

Proposition 2.1 *A smooth T–variety X satisfying the conditions (1) and (2) is equivariantly formal, ie $H_T^*(X)$ is a free module over $H_T^*(\mathrm{pt})$ and there is a noncanonical isomorphism $H_T^*(X) \cong H^*(X) \otimes H_T^*(\mathrm{pt})$ as an $H_T^*(\mathrm{pt})$–module.*

Proof We use the argument of Kirwan [23, Proposition 5.8] (see also [27, Section 5.1]). Choose a one-parameter subgroup $k\colon \mathbb{C}^\times \to T$ such that k is negative on every nonzero weight of $H^0(X, \mathcal{O})$. This defines a $\mathbb{C}^\times$–action on X. Let $L \to X$ be a very ample line bundle. The $\mathbb{C}^\times$–action on X lifts to a $\mathbb{C}^\times$–linearization on L, possibly after replacing L with its power $L^{\otimes i}$ [12, Corollary 7.2]. Then L defines a $\mathbb{C}^\times$–equivariant closed embedding $X \hookrightarrow X_0 \times \mathbb{P}^n$, where $\mathbb{P}^n$ is equipped with a linear $\mathbb{C}^\times$–action. By assumption, we can embed the affine variety $X_0 = \operatorname{Spec}(H^0(X, \mathcal{O}))$ equivariantly into a $\mathbb{C}^\times$–representation V which has only positive[1] weights. Thus we have a $\mathbb{C}^\times$–equivariant closed embedding $X \hookrightarrow V \times \mathbb{P}^n$. The associated S^1–action on $V \times \mathbb{P}^n$ admits, with respect to the standard Kähler metric, a moment map μ which is proper and bounded from below. These properties allow us to use Morse theory for the moment map $\mu|_X$. The argument in [23; 27] shows that $\mu|_X$ is a perfect Bott–Morse function and X is equivariantly formal. □

[1] We use the (usual) convention that $t \in \mathbb{C}^\times$ acts on *functions* by $f(x) \mapsto f(t^{-1}x)$.

2.2 Gromov–Witten invariants

For a second homology class $d \in H_2(X, \mathbb{Z})$ and a nonnegative integer $n \geq 0$, we denote by $X_{0,n,d}$ the moduli stack of genus-zero stable maps to X of degree d with n marked points. The T–action on X induces a T–action on $X_{0,n,d}$. It has a virtual fundamental class $[X_{0,n,d}]_{\mathrm{vir}} \in H_*(X_{0,n,d}, \mathbb{Q})$ of dimension $D = \dim X + n - 3 + c_1(X) \cdot d$. For equivariant cohomology classes $\alpha_1, \dots, \alpha_n \in H_T^*(X, \mathbb{Q})$ and nonnegative integers $k_1, \dots, k_n$, the genus-zero T–equivariant Gromov–Witten invariant is defined by

$$\langle \alpha_1 \psi^{k_1}, \dots, \alpha_n \psi^{k_n} \rangle_{0,n,d}^{X,T} = \int_{[X_{0,n,d}]_{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i) \psi_i^{k_i}.$$

Here $\mathrm{ev}_i \colon X_{0,n,d} \to X$ is the evaluation map at the i^{th} marked point and ψ_i denotes the equivariant first Chern class of the i^{th} universal cotangent line bundle L_i over $X_{0,n,d}$. When the moduli space $X_{0,n,d}$ is not compact, the right-hand side is defined via the Atiyah–Bott localization formula [1; 17] and belongs to the fraction field $\mathrm{Frac}(H_T^*(\mathrm{pt}))$ of $H_T^*(\mathrm{pt})$.

2.3 Quantum cohomology

Let $\mathrm{Eff}(X) \subset H_2(X, \mathbb{Z})$ denote the semigroup of homology classes of effective curves. We write Q for the Novikov variable and define $M[\![Q]\!]$ to be the space of formal power series,

$$M[\![Q]\!] = \left\{ \sum_{d \in \mathrm{Eff}(X)} a_d Q^d : a_d \in M \right\},$$

with coefficients in a module M. When M is a ring, $M[\![Q]\!]$ is also a ring. Let $(\cdot, \cdot)$ denote the T–equivariant Poincaré pairing on $H_T^*(X)$

$$(\alpha, \beta) = \int_X \alpha \cup \beta.$$

If X is not compact, we define the right-hand side via the localization formula. Therefore $(\cdot, \cdot)$ takes values in $\mathrm{Frac}(H_T^*(\mathrm{pt}))$ in general. Let $\{\phi_i\}_{i=0}^N$ be a basis of $H_T^*(X)$ over $H_T^*(\mathrm{pt})$. We write $\{\tau^i\}_{i=0}^N$ for the dual coordinates on $H_T^*(X)$ and $\tau = \sum_{i=0}^N \tau^i \phi_i$ for a general point on $H_T^*(X)$. The (big) quantum product $\star$ is defined by the formula

$$(\phi_i \star \phi_j, \phi_k) = \sum_{d \in \mathrm{Eff}(X)} \sum_{n=0}^{\infty} \frac{Q^d}{n!} \langle \phi_i, \phi_j, \phi_k, \tau, \dots, \tau \rangle_{0,n+3,d}^{X,T}.$$

We note that the quantum product $\phi_i \star \phi_j$ is defined without localization:

$$\phi_i \star \phi_j \in H_T^*(X)[\![Q]\!][\![\tau^0,\ldots,\tau^N]\!].$$

In fact, $\phi_i \star \phi_j$ can be written as the push-forward

$$\sum_{d\in\mathrm{Eff}(X)}\sum_{n=0}^{\infty}\frac{Q^d}{n!}\,\mathrm{PD}\,\mathrm{ev}_{3*}\Big(\mathrm{ev}_1^*(\phi_i)\,\mathrm{ev}_2^*(\phi_j)\prod_{l=4}^{n+3}\mathrm{ev}_l^*(\tau)\cap[X_{0,n+3,d}]_{\mathrm{vir}}\Big) \tag{2-1}$$

along the *proper* evaluation map ev_3, and hence the localization is not necessary. The properness of ev_3 follows from the assumption that X is semiprojective.

2.4 Quantum connection and fundamental solution

The quantum connection is the operator

$$\nabla_i\colon H_T^*(X)[z][\![Q]\!][\![\tau^0,\ldots,\tau^N]\!]\to z^{-1}H_T^*(X)[z][\![Q]\!][\![\tau^0,\ldots,\tau^N]\!]$$

defined by

$$\nabla_i=\frac{\partial}{\partial\tau^i}+\frac{1}{z}(\phi_i\star).$$

The quantum connection has a parameter z: we identify it with the equivariant parameter for an additional $\mathbb{C}^\times$–action. Set $\widehat{T}=T\times\mathbb{C}^\times$ and consider the $\widehat{T}$–action on X induced by the projection $\widehat{T}\to T$. Then we have $H_{\widehat{T}}^*(X)\cong H_T^*(X)[z]$. The quantum connection is known to be flat, and admits a fundamental solution

$$M(\tau)\colon H_{\widehat{T}}^*(X)[\![Q]\!][\![\tau^0,\ldots,\tau^N]\!]\to H_{\widehat{T}}^*(X)_{\mathrm{loc}}[\![Q]\!][\![\tau^0,\ldots,\tau^N]\!]$$

satisfying the quantum differential equation

$$z\frac{\partial}{\partial\tau^i}M(\tau)=M(\tau)(\phi_i\star),$$

or equivalently

$$\frac{\partial}{\partial\tau^i}\circ M(\tau)=M(\tau)\circ\nabla_i,$$

where $H_{\widehat{T}}^*(X)_{\mathrm{loc}}:=H_{\widehat{T}}^*(X)\otimes_{H_{\widehat{T}}^*(\mathrm{pt})}\mathrm{Frac}(H_{\widehat{T}}^*(\mathrm{pt}))$ is the localized equivariant cohomology. The following proposition is well-known; see [13, Section 1; 30, Proposition 2]:

Proposition 2.2 *A fundamental solution is given by*

$$(M(\tau)\phi_i,\phi_j)=(\phi_i,\phi_j)+\sum_{\substack{d\in\mathrm{Eff}(X),n\geq 0\\(d,n)\neq(0,0)}}\frac{Q^d}{n!}\Big\langle\phi_i,\tau,\ldots,\tau,\frac{\phi_j}{z-\psi}\Big\rangle_{0,n+2,d}^{X,T}.$$

Remark 2.3 Expanding $1/(z-\psi) = \sum_{n=0}^{\infty} \psi^n/z^{n+1}$, we find that $M(\tau)\phi_i$ takes values in $H_T^*(X)[\![z^{-1}]\!]$. By the localization calculation, it also follows that $M(\tau)\phi_i$ takes values in $H_{\hat{T}}^*(X)_{\mathrm{loc}}$. The localized $\hat{T}$–equivariant cohomology $H_{\hat{T}}^*(X)_{\mathrm{loc}}$ is also called the *Givental space* [15].

3 Shift operator

The shift operator for equivariant quantum cohomology has been introduced by Okounkov and Pandharipande [29], Braverman, Maulik and Okounkov [4] and Maulik and Okounkov [25]. We discuss its (straightforward) extension to the big quantum cohomology.

3.1 Twisted homomorphism

We write $\hat{T} = T \times \mathbb{C}^\times$. For a group homomorphism $k\colon \mathbb{C}^\times \to T$, we consider the $\hat{T}$–action ρ_k on X defined by

$$\rho_k(t,u)x = tu^k \cdot x,$$

where $(t,u) \in \hat{T}$ and $x \in X$, and $u^k \in T$ denotes the image of $u \in \mathbb{C}^\times$ under k. Let $\lambda \in \mathrm{Lie}(T)$ denote the equivariant parameter for T and let $z \in \mathrm{Lie}(\mathbb{C}^\times)$ denote the equivariant parameter for $\mathbb{C}^\times$. The identity map id: $(X,\rho_0) \to (X,\rho_k)$ is equivariant with respect to the group automorphism

$$\phi_k\colon \hat{T} \to \hat{T}, \quad \phi_k(t,u) = (tu^{-k}, u).$$

Therefore the identity map induces an isomorphism

$$\Phi_k\colon H^*_{\hat{T},\rho_0}(X) \cong H^*_{\hat{T},\rho_k}(X)$$

such that

$$\Phi_k(f(\lambda,z)\alpha) = f(\lambda+kz,z)\Phi_k(\alpha), \tag{3-1}$$

where $\alpha \in H^*_{\hat{T},\rho_0}(X)$ and $f(\lambda,z) \in H^*_{\hat{T}}(\mathrm{pt})$ is a polynomial function on $\mathrm{Lie}(\hat{T})$. Referring to the property (3-1), we say that Φ_k is a k–*twisted* homomorphism.

Notation 3.1 We write $H^*_{\hat{T},\rho}(X)$ for the $\hat{T}$–equivariant cohomology of X with respect to the $\hat{T}$–action ρ on X. When ρ is omitted, $H^*_{\hat{T}}(X)$ means $H^*_{\hat{T},\rho_0}(X)$.

3.2 Bundle associated to a $\mathbb{C}^\times$–subgroup

Definition 3.2 (associated bundle) Let $k\colon \mathbb{C}^\times \to T$ be a group homomorphism. Consider the $\mathbb{C}^\times$–action on $X \times (\mathbb{C}^2 \setminus \{0\})$ given by $s \cdot (x, (v_1, v_2)) = (s^k \cdot x, (s^{-1} v_1, s^{-1} v_2))$. Let E_k denote the quotient space

$$E_k := X \times (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^\times.$$

We have a natural projection $\pi\colon E_k \to \mathbb{P}^1$ given by $\pi([x, (v_1, v_2)]) = [v_1, v_2]$ and E_k is a fiber bundle over $\mathbb{P}^1$ with fiber X. We consider the $\widehat{T}$–action on E_k given by $(t, u) \cdot [x, (v_1, v_2)] = [t \cdot x, (v_1, u v_2)]$. Let X_0 denote the fiber of $E_k \to \mathbb{P}^1$ at $[1, 0]$ and let X_∞ denote the fiber at $[0, 1]$. Note that we have

$$X_0 \cong (X, \rho_0) \quad \text{and} \quad X_\infty \cong (X, \rho_k)$$

as $\widehat{T}$–spaces.

Definition 3.3 A group homomorphism $k\colon \mathbb{C}^\times \to T$ is said to be *seminegative* if k is nonpositive on each T–weight of $H^0(X, \mathcal{O})$. We say that k is *negative* if k is negative on each nonzero T–weight of $H^0(X, \mathcal{O})$.

Remark 3.4 When X is complete, every $\mathbb{C}^\times$–subgroup is negative.

Suppose that $k\colon \mathbb{C}^\times \to T$ is seminegative and consider the $\mathbb{C}^\times$–action on X induced by k. Let L be a very ample line bundle on X. As discussed in the proof of Proposition 2.1, we may assume that L admits a $\mathbb{C}^\times$–linearization. By tensoring L with a $\mathbb{C}^\times$–character, we may assume that all the $\mathbb{C}^\times$–weights on $H^0(X, L^{\otimes n})$ are negative for $n > 0$. Let $p\colon X \times \mathbb{C}^2 \to X$ be the natural projection. Then $p^* L$ is a $\mathbb{C}^\times$–equivariant line bundle on $X \times \mathbb{C}^2$, where $\mathbb{C}^\times$ acts on the base by $s \cdot (x, (v_1, v_2)) = (s^k \cdot x, (s^{-1} v_1, s^{-1} v_2))$. We can see that

$$H^0(X \times \mathbb{C}^2, (p^* L)^{\otimes n}) = \bigoplus_{i=0}^{\infty} H^0(X, L^{\otimes n})^{(-i)} \otimes \mathbb{C}[v_1, v_2]^{(i)},$$

where the superscript (l) means the component of $\mathbb{C}_s^\times$–weight l. The unstable locus for the $\mathbb{C}^\times$–action on $(X \times \mathbb{C}^2, p^* L)$, in the sense of geometric invariant theory (GIT), is $X \times \{0\}$ and therefore we find that E_k is the GIT quotient of $X \times \mathbb{C}^2$, ie $E_k = \operatorname{Proj}\big(\bigoplus_{n=0}^{\infty} H^0(X \times \mathbb{C}^2, (p^* L)^{\otimes n})\big)$. This proves:

Lemma 3.5 *If k is seminegative, E_k is semiprojective.*

Let $k\colon \mathbb{C}^\times \to T$ be a seminegative subgroup and consider the $\mathbb{C}^\times$–action on X induced by k. A $\mathbb{C}^\times$–fixed point $x \in X$ defines a section of $E_k \to \mathbb{P}^1$,

$$\sigma_x = (\{x\} \times \mathbb{P}^1) \subset E_k. \tag{3-2}$$

We now define a minimal section among all such sections associated to fixed points. Using the argument in the proof of Proposition 2.1, we obtain a $\mathbb{C}^\times$–equivariant closed embedding $X \hookrightarrow \mathbb{P}^n \times \mathbb{C}^l$, where $\mathbb{C}^l$ is a $\mathbb{C}^\times$–representation with only nonnegative weights. In particular, for every point $x \in X$, the limit $\lim_{s\to 0} s^k \cdot x$ exists. This implies the existence of the Białynicki-Birula decomposition [3, Theorem 4.1] for X: if $X^{\mathbb{C}^\times} = \bigsqcup_i F_i$ is the decomposition of the $\mathbb{C}^\times$–fixed locus $X^{\mathbb{C}^\times}$ into connected components, we have the induced decomposition of X,

$$X = \bigsqcup_i U_i, \quad U_i = \Bigl\{x \in X : \lim_{s\to 0} s^k \cdot x \in F_i\Bigr\},$$

into locally closed smooth subvarieties U_i. In particular there exists a unique $\mathbb{C}^\times$–fixed component $F_{\min} \subset X$ such that all the $\mathbb{C}^\times$–weights on the normal bundle to $F_{\min}$ are positive. The moment map μ for the associated S^1–action attains a global minimum on $F_{\min}$. We call the class of a section $\sigma_{\min}$ of E_k associated to a point in $F_{\min}$ the *minimal section class*. We write

$$\begin{aligned} H_2^{\mathrm{sec}}(E_k, \mathbb{Z}) &= \{d \in H_2(E_k, \mathbb{Z}) : \pi_*(d) = [\mathbb{P}^1]\}, \\ \mathrm{Eff}(E_k)^{\mathrm{sec}} &= \mathrm{Eff}(E_k) \cap H_2^{\mathrm{sec}}(E_k, \mathbb{Z}). \end{aligned}$$

Lemma 3.6 *If k is seminegative, we have* $\mathrm{Eff}(E_k)^{\mathrm{sec}} = \sigma_{\min} + \mathrm{Eff}(X)$.

Proof The compact case was discussed in [16, Lemma 2.2]. Take a negative one-parameter subgroup $l\colon \mathbb{C}^\times \to T$ and consider the $\mathbb{C}^\times$–action on E_k induced by $\mathbb{C}^\times \xrightarrow{l} T \times \{1\} \subset \widehat{T}$. Observe that all nonzero $\mathbb{C}^\times$–weights on $H^0(E_k, \mathcal{O})$ are negative. This means that $E_{k,0} := \operatorname{Spec} H^0(E_k, \mathcal{O})$ has a unique $\mathbb{C}^\times$–fixed point 0 and $\lim_{s\to 0} s \cdot x = 0$ for all $x \in E_{k,0}$. Therefore every curve can be deformed, via the $\mathbb{C}^\times$–action, to a stable curve in the fiber K of $E_k \to E_{k,0}$ at $0 \in E_{k,0}$ in the same homology class. Since $\widehat{T}$–action on E_k preserves K and K is compact, we may further deform a curve in K to a $\widehat{T}$–invariant stable curve. A $\widehat{T}$–invariant stable curve in E_k is a union of a section class σ_x associated to a T–fixed point $x \in X$ and effective curves in $X_0 \sqcup X_\infty$. Suppose that two different fixed points x, $y \in X^T$ are connected by a $k(\mathbb{C}^\times)$–orbit, ie $x = \lim_{s\to\infty} s^k \cdot p$ and $y = \lim_{s\to 0} s^k \cdot p$ for some $p \in X$. The closure $C = \overline{k(\mathbb{C}^\times) \cdot p}$ is isomorphic to $\mathbb{P}^1$, and σ_x and σ_y are contained in a Hirzebruch surface

$$C \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times \subset E_k.$$

Then one finds $\sigma_x = \sigma_y + a[C]$ for some $a > 0$. Using the Białynicki-Birula decomposition for the $k(\mathbb{C}^\times)$–action on X, we find that every T–fixed point is connected to a T–fixed point on $F_{\min}$ by a chain of $k(\mathbb{C}^\times)$–orbits. The conclusion follows. □

Lemma 3.7 *We have an isomorphism*

$$H^*_{\hat{T}}(E_k) \cong \{(\alpha,\beta) \in H^*_{\hat{T},\rho_0}(X) \oplus H^*_{\hat{T},\rho_k}(X) : \alpha - \Phi_k^{-1}(\beta) \equiv 0 \bmod z\}$$

which sends τ *to* $(\tau|_{X_0}, \tau|_{X_\infty})$. *Recall that* z *is the equivariant parameter for* $\mathbb{C}^\times$ *and we have a canonical isomorphism* $H^*_{\hat{T},\rho_0}(X) \cong H^*_T(X)[z]$.

Proof Consider the Mayer–Vietoris exact sequence associated to the covering $E_k = U_0 \cup U_\infty$ with $U_0 = \pi^{-1}(\mathbb{C})$ and $U_\infty = \pi^{-1}(\mathbb{P}^1 \setminus \{0\})$. We have

$$H^*_{\hat{T}}(U_0) \cong H^*_{\hat{T},\rho_0}(X), \quad H^*_{\hat{T}}(U_\infty) \cong H^*_{\hat{T},\rho_k}(X), \quad H^*_{\hat{T}}(U_0 \cap U_\infty) \cong H_T(X).$$

The map $H^*_{\hat{T}}(U_0) \oplus H^*_{\hat{T}}(U_\infty) \to H^*_{\hat{T}}(U_0 \cap U_\infty)$ is surjective and is given by $(\alpha,\beta) \mapsto (\alpha - \Phi_k^{-1}\beta)|_{z=0}$. □

Notation 3.8 By Lemma 3.7, for $\tau \in H^*_T(X)$, there exists $\hat{\tau} \in H^*_{\hat{T}}(E_k)$ such that $\hat{\tau}|_{X_0} = \tau$ and $\hat{\tau}|_{X_\infty} = \Phi_k(\tau)$. This defines a map $\hat{\ }\colon H^*_T(X) \to H^*_{\hat{T}}(E_k)$. This is not $H^*_T(\mathrm{pt})$–linear.

3.3 Shift operator

Definition 3.9 (shift operator) Let $k\colon \mathbb{C}^\times \to T$ be a seminegative group homomorphism. For $\tau \in H^*_T(X)$, we define $\widetilde{\mathbb{S}}_k(\tau)\colon H^*_{\hat{T},\rho_0}(X)[\![Q]\!] \to H^*_{\hat{T},\rho_k}(X)[\![Q]\!]$ by

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha,\beta) = \sum_{\hat{d}\in \mathrm{Eff}(E_k)^{\mathrm{sec}}} \frac{Q^{\hat{d}-\sigma_{\min}}}{n!} \langle \iota_{0*}\alpha, \iota_{\infty*}\beta, \hat{\tau},\dots,\hat{\tau}\rangle^{E_k,\hat{T}}_{0,n+2,\hat{d}},$$

where $(\cdot,\cdot)$ in the left-hand side is the $\hat{T}$–equivariant Poincaré pairing on $H^*_{\hat{T},\rho_k}(X)$, $\alpha \in H^*_{\hat{T},\rho_0}(X)$, $\beta \in H^*_{\hat{T},\rho_k}(X)$, $\sigma_{\min}$ is the minimal section class for E_k, and $\iota_0\colon X_0 \to E_k$ and $\iota_\infty\colon X_\infty \to E_k$ are the natural inclusions. We also define

$$\mathbb{S}_k(\tau) = \Phi_k^{-1} \circ \widetilde{\mathbb{S}}_k(\tau)\colon H^*_{\hat{T}}(X)[\![Q]\!] \to H^*_{\hat{T}}(X)[\![Q]\!].$$

Note that $\widetilde{\mathbb{S}}_k$ is untwisted but $\mathbb{S}_k$ is $(-k)$–twisted (see (3-1)).

Remark 3.10 When k is seminegative, E_k is semiprojective by Lemma 3.5 and thus the shift operator $\mathbb{S}_k$ is defined without localization: we may rewrite $\widetilde{\mathbb{S}}_k$ as the push-forward along an evaluation map (see (2-1)). When k is not seminegative, we can still define $\mathbb{S}_k$ over $\mathrm{Frac}(H^*_T(\mathrm{pt}))$ after choosing a suitable section class $\sigma_{\min}$.

Remark 3.11 Since the map $\tau \mapsto \hat{\tau}$ is not $H_T^*(\mathrm{pt})$–linear, $\mathbb{S}(\tau)$ cannot be written as a formal power series in the $H_T^*(\mathrm{pt})$–valued variables $\tau^0, \dots, \tau^N$. For $\alpha_1, \dots, \alpha_l \in H_T^*(X)$ and $\mathbb{C}$–*valued* variables $t^1, \dots, t^l$, the shift operator $\mathbb{S}(\tau)$ with $\tau = \sum_{i=1}^l t^i \alpha_i$ is a formal power series in $t^1, \dots, t^l$.

Remark 3.12 (divisor equation) Suppose that $\tau = h + \tau'$ with $h \in H_T^2(X)$. Using the divisor equation, we have

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha, \beta) = e^{-h(k)} \sum_{d \in \mathrm{Eff}(X)} \frac{Q^d e^{h \cdot d}}{n!} \langle \iota_{0*}\alpha, \iota_{\infty*}\beta, \hat{\tau}', \dots, \hat{\tau}' \rangle_{0,n+2,\sigma_{\min}+d}^{E_k, \hat{T}},$$

where $h(k)$ is the pairing between k and the restriction $h|_x \in H_T^2(\mathrm{pt}) \cong \mathrm{Lie}(T)^*$ of h to a fixed point x in the minimal fixed component $F_{\min}$ (with respect to k). Note that $\hat{h} \cdot \sigma_{\min} = -h(k)$.

By the localization theorem of equivariant cohomology [1], the restriction to the T–fixed subspace X^T induces an isomorphism

$$\iota^*\colon H_{\hat{T}}^*(X)_{\mathrm{loc}} \xrightarrow{\cong} H_{\hat{T}}^*(X^T)_{\mathrm{loc}} = H^*(X^T) \otimes \mathrm{Frac}(H_{\hat{T}}^*(\mathrm{pt})).$$

We use this to define the shift operator on the Givental space $H_{\hat{T}}^*(X)_{\mathrm{loc}}$.

Definition 3.13 (shift operator on the Givental space) Let $X^T = \bigsqcup_i F_i$ be the decomposition of X^T into connected components. Let N_i be the normal bundle to F_i in X. Let $N_i = \bigoplus_\alpha N_{i,\alpha}$ denote the T–eigenbundle decomposition, where T acts on $N_{i,\alpha}$ by the character $\alpha \in \mathrm{Hom}(T, \mathbb{C}^\times)$. Let $\rho_{i,\alpha,j}$ for $j = 1, \dots, \mathrm{rank}(N_{i,\alpha})$ denote the Chern roots of $N_{i,\alpha}$. For a seminegative $k \in \mathrm{Hom}(\mathbb{C}^\times, T)$, we define

$$\Delta_i(k) = Q^{\sigma_i - \sigma_{\min}} \prod_\alpha \prod_{j=1}^{\mathrm{rank}(N_{i,\alpha})} \frac{\prod_{c=-\infty}^{0} (\rho_{i,\alpha,j} + \alpha + cz)}{\prod_{c=-\infty}^{-\alpha\cdot k} (\rho_{i,\alpha,j} + \alpha + cz)} \in H_{\hat{T}}^*(F_i)_{\mathrm{loc}}[\![Q]\!],$$

where α is regarded as an element of $H_T^2(\mathrm{pt}, \mathbb{Z})$, σ_i is the section class of E_k associated to a fixed point in F_i and $\sigma_{\min}$ is the minimal section class of E_k. Note that all but finitely many factors in the infinite product cancel. We define the operator $\mathcal{S}_k\colon H_{\hat{T}}^*(X)_{\mathrm{loc}} \to H_{\hat{T}}^*(X)_{\mathrm{loc}}$ by the commutative diagram

$$\begin{array}{ccc} H_{\hat{T}}^*(X)_{\mathrm{loc}} & \xrightarrow{\quad \mathcal{S}_k \quad} & H_{\hat{T}}^*(X)_{\mathrm{loc}} \\ \iota^* \downarrow & & \downarrow \iota^* \\ H_{\hat{T}}^*(X^T)_{\mathrm{loc}} & \xrightarrow{\bigoplus_i \Delta_i(k) e^{-zk\partial_\lambda}} & H_{\hat{T}}^*(X^T)_{\mathrm{loc}} \end{array} \tag{3-3}$$

where we use the decomposition $H^*_{\widehat{T}}(X^T)_{\mathrm{loc}} \cong \bigoplus_i H^*(F_i) \otimes \mathrm{Frac}(H_{\widehat{T}}(\mathrm{pt}))$ in the bottom arrow and $e^{-kz\partial_\lambda}$ acts on $\mathrm{Frac}(H_{\widehat{T}}(\mathrm{pt}))$ by $f(\lambda, z) \mapsto f(\lambda - kz, z)$. The operator $\mathcal{S}_k$ is a $(-k)$–twisted homomorphism.

The following is a key property of the shift operator:

Theorem 3.14 *We have $M(\tau) \circ \mathbb{S}_k(\tau) = \mathcal{S}_k \circ M(\tau)$, where $M(\tau)$ is the fundamental solution in Proposition 2.2.*

Proof A similar intertwining property has been discussed in [29; 4; 25]. We calculate $\widetilde{\mathbb{S}}_k(\tau)$ using $\widehat{T}$–equivariant localization. We refer the reader to [17; 10] for localization arguments in Gromov–Witten theory. Fix a section class $\hat{d} \in \mathrm{Eff}(E_k)^{\mathrm{sec}}$. A $\widehat{T}$–fixed stable map $f\colon (C, x_1, \dots, x_{n+2}) \to E_k$ of degree $\hat{d}$ is of the form

- $C = C_0 \cup C_{\mathrm{sec}} \cup C_\infty$ with $C_{\mathrm{sec}} \cong \mathbb{P}^1$;
- $f_0 = f|_{C_0}$ is a T–fixed stable map to X_0;
- $f_\infty = f|_{C_\infty}$ is a T–fixed stable map to X_∞;
- $f_{\mathrm{sec}} = f|_{C_{\mathrm{sec}}}$ is a section of E_k associated to a T–fixed point in X (see (3-2)).

Recall that the tangent space T^1 and the obstruction space T^2 at the stable map f fit into the exact sequence

$$0 \to \mathrm{Ext}^0(\Omega^1_C(\boldsymbol{x}), \mathcal{O}_C) \to H^0(C, f^*T_{E_k}) \to T^1 \\ \to \mathrm{Ext}^1(\Omega^1_C(\boldsymbol{x}), \mathcal{O}_C) \to H^1(C, f^*T_{E_k}) \to T^2 \to 0,$$

where $\boldsymbol{x} = x_1 + \cdots + x_{n+2}$. The virtual normal bundle at f is

$$\mathcal{N}^{\mathrm{vir}} = T^{1,\mathrm{mov}} - T^{2,\mathrm{mov}} = \chi(f^*T_{E_k})^{\mathrm{mov}} - \chi(\Omega^1_C(\boldsymbol{x}), \mathcal{O}_C)^{\mathrm{mov}}$$

where "mov" means the moving part with respect to the $\widehat{T}$–action and $\chi(\mathcal{E}) = H^0(C, \mathcal{E}) - H^1(C, \mathcal{E})$ and $\chi(\mathcal{E}, \mathcal{F}) = \mathrm{Ext}^0(\mathcal{E}, \mathcal{F}) - \mathrm{Ext}^1(\mathcal{E}, \mathcal{F})$ denote the Euler characteristics. Let p and q denote the nodal intersection points $C_0 \cap C_{\mathrm{sec}}$ and $C_\infty \cap C_{\mathrm{sec}}$, respectively. Using the normalization exact sequence $0 \to \mathcal{O}_C \to \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_{\mathrm{sec}}} \oplus \mathcal{O}_{C_\infty} \to \mathbb{C}_p \oplus \mathbb{C}_q \to 0$, we find

$$\chi(f^*T_{E_k})^{\mathrm{mov}} = \chi(f_0^*T_{X_0})^{\mathrm{mov}} + \chi(f_\infty^*T_{X_\infty})^{\mathrm{mov}} + \chi(f_{\mathrm{sec}}^*T_{E_k}) \\ + \xi + \xi^{-1} - (T_{f(p)}E)^{\mathrm{mov}} - (T_{f(q)}E)^{\mathrm{mov}}, \tag{3-4}$$

where ξ is the one-dimensional $\mathbb{C}^\times$–representation of weight one. We write $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{x}_\infty$, where $\boldsymbol{x}_0$ and $\boldsymbol{x}_\infty$ are divisors on C_0 and C_∞, respectively. Then we have

$$-\chi(\Omega^1_C(\boldsymbol{x}), \mathcal{O}_C)^{\mathrm{mov}} = T_pC_0 \otimes T_pC_{\mathrm{sec}} + T_qC_\infty \otimes T_qC_{\mathrm{sec}} \\ - \chi(\Omega^1_{C_0}(\boldsymbol{x}_0 + p), \mathcal{O}_{C_0})^{\mathrm{mov}} \\ - \chi(\Omega^1_{C_\infty}(\boldsymbol{x}_\infty + q), \mathcal{O}_{C_\infty})^{\mathrm{mov}}. \tag{3-5}$$

The $\widehat{T}$–fixed locus in the moduli space $(E_k)_{0,n+2,\hat{d}}$ is given by

$$\bigsqcup_i \bigsqcup_{I_1 \sqcup I_2 = \{1,\dots,n+2\}} \bigsqcup_{d_0+d_\infty+\sigma_i=\hat{d}} ((X_0)_{0,I_1\cup p,d_0})^T \times_{F_i} ((X_\infty)_{0,I_2\cup q,d_\infty})^T,$$

where F_i and σ_i are as in Definition 3.13. Combining (3-4) and (3-5), we find that the virtual normal bundle $\mathcal{N}_i^{\mathrm{vir}}$ on the component $((X_0)_{0,I_1\cup p,d_0})^T \times_{F_i} ((X_\infty)_{0,I_2\cup q,d_\infty})^T$ is

$$\mathcal{N}_i^{\mathrm{vir}} = \mathcal{N}_0^{\mathrm{vir}} + \mathcal{N}_\infty^{\mathrm{vir}} + \mathcal{N}_{\mathrm{sec},i} - N_{F_i/X_0} - N_{F_i/X_\infty} + L_p^{-1}\otimes\xi + L_q^{-1}\otimes\xi^{-1},$$

where $\mathcal{N}_0^{\mathrm{vir}}$ is the virtual normal bundle of $(X_0)^T_{0,I_1\cup p,d_0}$ in $(X_0)_{0,I_1\cup p,d_0}$, $\mathcal{N}_\infty^{\mathrm{vir}}$ is the virtual normal bundle of $(X_\infty)^T_{0,I_2\cup q,d_\infty}$ in $(X_\infty)_{0,I_2\cup q,d_\infty}$, L_p (resp. L_q) is the universal cotangent line bundle at p (resp. q) and $\mathcal{N}_{\mathrm{sec},i}$ is the vector bundle with fiber $\chi(f_{\mathrm{sec}}^* T_{E_k})^{\mathrm{mov}}$. Let $N_{F_i/X} = N_i = \bigoplus_\alpha N_{i,\alpha}$ be decomposition as in Definition 3.13. The normal bundle of $F_i \times \mathbb{P}^1$ in E_k is

$$\bigoplus_\alpha N_{i,\alpha} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-\alpha\cdot k).$$

Thus we find

$$\mathcal{N}_{\mathrm{sec},i} = \xi \oplus \xi^{-1} \oplus \bigoplus_\alpha N_{i,\alpha} \otimes \Bigl(\bigoplus_{c\le 0} \xi^c - \bigoplus_{c<\alpha\cdot k} \xi^c\Bigr). \tag{3-6}$$

The virtual localization formula gives

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha,\beta) = \sum_{i,k,l,a,b} \sum_{d_0+d_\infty+\sigma_i=\hat{d}} \Bigl\langle z\alpha,\tau,\dots,\tau,\frac{(\iota_{0,i})_*\phi_{i,a}}{z-\psi}\Bigr\rangle^{X_0,\widehat{T}}_{0,k+2,d_0} \frac{Q^{d_0}}{k!}$$
$$\times \Bigl(\int_{F_i} \frac{Q^{\sigma_i-\sigma_{\min}}}{e_{\widehat{T}}(\mathcal{N}_{\mathrm{sec},i})}\phi_i^a\phi_i^b\Bigr)\Bigl\langle \frac{(\iota_{\infty,i})_*\phi_{i,b}}{-z-\psi},\tau',\dots,\tau',-z\beta\Bigr\rangle^{X_\infty,\widehat{T}}_{0,l+2,d_\infty}\frac{Q^{d_\infty}}{l!},$$

where $\alpha\in H^*_{\widehat{T}}(X_0)$, $\beta\in H^*_{\widehat{T}}(X_\infty)$, $\tau'=\Phi_k(\tau)$, the maps $\iota_{0,i}\colon F_i\to X_0$ and $\iota_{\infty,i}\colon F_i\to X_\infty$ are the natural inclusions, $\{\phi_{i,a}\}\subset H^*(F_i)$ is a basis, and $\{\phi_i^a\}$ is the dual basis such that $\int_{F_i}\phi_{i,a}\cup\phi_i^b=\delta_a^b$. Note that we have, by (3-6),

$$\frac{Q^{\sigma_i-\sigma_{\min}}}{e_{\widehat{T}}(\mathcal{N}_{\mathrm{vir},i})} = \frac{1}{z(-z)}\frac{1}{e_{\widehat{T}}(N_{F_i/X_\infty})}(e^{kz\partial_\lambda}\Delta_i(k)).$$

Combining these equations, we conclude

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha,\beta) = (\widetilde{\mathcal{S}}_k M(\tau,z)\alpha, M'(\tau',-z)\beta),$$

where we write the argument z in the fundamental solution explicitly and

- $\widetilde{\mathcal{S}}_k\colon H_{\widehat{T}}(X_0)_{\mathrm{loc}} \to H_{\widehat{T}}(X_\infty)_{\mathrm{loc}}$ is a map defined similarly to $\mathcal{S}_k$ by replacing $\bigoplus_i \Delta_i(k)e^{-kz\partial_\lambda}$ in the diagram (3-3) with $\bigoplus_i (e^{kz\partial_\lambda}\Delta_i(k))$;
- $M'(\tau', z)$ is defined similarly to Proposition 2.2 by replacing T–equivariant Gromov–Witten invariants there with $(\widehat{T}, \rho_k)$–equivariant invariants.

Note that $M'(\tau', z) = \Phi_k \circ M(\tau, z) \circ \Phi_k^{-1}$ and $\widetilde{\mathcal{S}}_k = \Phi_k \circ \mathcal{S}_k$. The conclusion follows from the so-called "unitarity" $M(\tau, -z)^* = M(\tau, z)^{-1}$ of the fundamental solution (see [13, Section 1]). □

Theorem 3.14 and the differential equation $\partial_i \circ M(\tau) = M(\tau) \circ \nabla_i$ show:

Corollary 3.15 *The shift operator commutes with the quantum connection, that is,* $[\nabla_i, \mathbb{S}_k(\tau)] = 0$ *for* $i = 0, \dots, N$.

This corollary is shown in [25, Section 8] in the case where $\tau = 0$. We also remark that the shift operators commute each other.

Corollary 3.16 *We have* $\mathcal{S}_k \circ \mathcal{S}_l = Q^{d(k,l)}\mathcal{S}_{k+l}$ *for some* $d(k,l) \in H_2(X, \mathbb{Z})$ *which is symmetric in* k *and* l. *In particular,* $\mathbb{S}_k \circ \mathbb{S}_l = Q^{d(k,l)}\mathbb{S}_{k+l}$, $[\mathcal{S}_k, \mathcal{S}_l] = [\mathbb{S}_k, \mathbb{S}_l] = 0$.

Proof Consider the X–bundle $E_{k,l}$ over $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$E_{k,l} = X \times (\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\}) \big/ \mathbb{C}^\times \times \mathbb{C}^\times,$$

where $(s_1, s_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$ acts on $X \times \mathbb{C}^2 \times \mathbb{C}^2$ by $(s_1, s_2) \cdot (x, (a_1, a_2), (b_1, b_2)) = (s_1^k s_2^l, (s_1^{-1}a_1, s_1^{-1}a_2), (s_2^{-1}b_1, s_2^{-1}b_2))$. Note $E_{k,l}|_{\mathbb{P}^1 \times [1:0]} \cong E_k$, $E_{k,l}|_{[1:0]\times\mathbb{P}^1} \cong E_l$ and $E_{k,l}|_{\Delta(\mathbb{P}^1)} \cong E_{k+l}$, where $\Delta(\mathbb{P}^1) \subset \mathbb{P}^1 \times \mathbb{P}^1$ denotes the diagonal. The addition in $H_2(E_{k,l}, \mathbb{Z})$ defines a map $\#\colon H_2^{\mathrm{sec}}(E_l, \mathbb{Z}) \times H_2^{\mathrm{sec}}(E_k, \mathbb{Z}) \to H_2^{\mathrm{sec}}(E_{k+l}, \mathbb{Z})$. For any T–fixed point x, the section class σ_x (see (3-2)) associated to x satisfies $\sigma_x \# \sigma_x = \sigma_x$. A straightforward computation now shows that $\mathcal{S}_k \circ \mathcal{S}_l = Q^{\sigma_{\min}(k+l) - \sigma_{\min}(k)\#\sigma_{\min}(l)}\mathcal{S}_{k+l}$, where $\sigma_{\min}(k)$ denotes the minimal section class of E_k. The conclusion follows by setting $d(k,l) = \sigma_{\min}(k+l) - \sigma_{\min}(k)\#\sigma_{\min}(l)$ and the commutativity of $\#$. □

3.4 Relation to the Seidel representation

Taking the $z \to 0$ limit of shift operators, we obtain a big quantum cohomology version of the Seidel representation [31]. The author learned the idea of big Seidel elements from Eduardo González during joint work with him [16].

Definition 3.17 (Seidel elements) Let $k \in \mathrm{Hom}(\mathbb{C}^\times, T)$ be a seminegative homomorphism. The element $S_k(\tau) := \lim_{z\to 0} \mathbb{S}_k(\tau)1$ of $H_T^*(X)[\![Q]\!][\![\tau^0, \dots, \tau^m]\!]$ is called the *Seidel element*.

By Corollary 3.15, the $z \to 0$ limit of the operator $\mathbb{S}_k(\tau)$ commutes with the quantum multiplication, and therefore coincides with the quantum multiplication by $S_k(\tau)$ (see also [25, Section 8]). By Corollary 3.16, we have

$$S_k(\tau) \star S_l(\tau) = Q^{d(k,l)} S_{k+l}(\tau).$$

This is called the *Seidel representation*.

3.5 Relation to the $\widehat{\Gamma}$–integral structure

We note a relationship between the shift operator and the $\widehat{\Gamma}$–integral structure introduced in [20; 22; 9]. For quantum cohomology of the Hilbert scheme of points on $\mathbb{C}^2$, it has been observed in [29] that certain Γ–factors play an important role in the difference equation associated to the shift operators.

We recall the $\widehat{\Gamma}$–class of X. Let $\delta_1, \dots, \delta_D$ denote the T–equivariant Chern roots of the tangent bundle TX such that $c^T(TX) = (1+\delta_1)\cdots(1+\delta_D)$. The *$T$–equivariant $\widehat{\Gamma}$–class* of X is the class

$$\widehat{\Gamma}_X = \widehat{\Gamma}(TX) = \prod_{i=1}^{D} \Gamma(1+\delta_i)$$

in $H_T^{**}(X) = \prod_{p=0}^{\infty} H_T^p(X)$. Here $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ is Euler's Γ–function. By Taylor expansion, the right-hand side becomes a symmetric formal power series in $\delta_1, \dots, \delta_D$ and thus can be expressed in terms of the equivariant Chern classes of TX.

The *$\widehat{\Gamma}$–integral structure* assigns the following homogeneous flat section $\mathfrak{s}(E)$ of the quantum connection to a T–equivariant vector bundle $E \to X$:

$$\mathfrak{s}(E) = (2\pi)^{-D/2} M(\tau)^{-1} z^{-\mu} z^{c_1(X)} \widehat{\Gamma}_X (2\pi \mathtt{i})^{\deg/2} \operatorname{ch}^T(E),$$

where $D = \dim_{\mathbb{C}} X$, $M(\tau)$ is the fundamental solution given in Proposition 2.2, $\mu \in \operatorname{End}_{\mathbb{C}}(H_T^*(X))$ is the Hodge grading operator $\mu(\phi_i) = \left(\frac{1}{2}\deg\phi_i - \frac{1}{2}D\right)\phi_i$, $z^{c_1(X)} = e^{c_1(X)\log z}$ and $(2\pi\mathtt{i})^{\deg/2}\operatorname{ch}^T(E) = \sum_{p=0}^{\infty} (2\pi\mathtt{i})^p \operatorname{ch}_p^T(E)$. The section $\mathfrak{s}(E)$ is flat, ie $\nabla_i \mathfrak{s}(E) = 0$ and is homogeneous in the sense that

$$\left[z\frac{\partial}{\partial z} + \mu + \sum_{i=0}^{N}\left(1 - \frac{1}{2}\deg\phi_i\right)\tau^i \frac{\partial}{\partial \tau^i} + \sum_{i=0}^{N} \rho^i \frac{\partial}{\partial \tau^i}\right]\mathfrak{s}(E) = 0,$$

where we set $c_1(X) = \sum_{i=0}^{N} \rho^i \phi_i$. A key property of $\mathfrak{s}(E)$ is that the pairing

$$(\mathfrak{s}(E)(\tau, e^{-\pi\mathtt{i}} z), \mathfrak{s}(F)(\tau, z))$$

equals the T–equivariant Euler pairing $z^{-\deg/2}(2\pi\mathtt{i})^{\deg/2}\chi(E,F)$, where $\chi(E,F) = \sum_{i=0}^{D}(-1)^i \operatorname{ch}^T(\operatorname{Ext}^i(E,F)) \in H_T^{**}(\mathrm{pt})$. This follows from an appropriate equivariant Hirzebruch–Riemann–Roch formula. See [9, Section 2–3] for more details.

The T–equivariant K–group is a module over $K_T^0(\mathrm{pt}) = \mathbb{C}[T]$ and the Chern character $\operatorname{ch}^T\colon K_T^0(\mathrm{pt}) \to H_T^{**}(\mathrm{pt})$ can be viewed as the pull-back by the universal covering $\exp\colon \operatorname{Lie}(T) = \mathbb{C}^m \to T = (\mathbb{C}^\times)^m$. A deck-transformation of this covering is given by the shift[2] of equivariant parameters $\lambda_j \to \lambda_j + 2\pi\mathtt{i}$. This suggests that $\mathfrak{s}(E)$ should be "invariant" under integral shifts of equivariant parameters.

Proposition 3.18 *When the Novikov variable Q is set to be one, the flat section $\mathfrak{s}(E)$ is invariant under the shift operator:*

$$\mathbb{S}_k \mathfrak{s}(E) = \mathfrak{s}(E)$$

for every seminegative $k \in \operatorname{Hom}(\mathbb{C}^\times, T)$.

Proof As is discussed in [9, Section 3], the divisor equation shows that the specialization $Q=1$ of the Novikov variable is well-defined for $\mathfrak{s}(E)$. In view of the intertwining property in Theorem 3.14, it suffices to show that

$$\widetilde{\mathcal{S}}_k(z^{-\mu} z^{c_1(X)} \widehat{\Gamma}_X (2\pi\mathtt{i})^{\deg/2} \operatorname{ch}(E)) = z^{-\mu} z^{c_1(X)} \widehat{\Gamma}_X (2\pi\mathtt{i})^{\deg/2} \operatorname{ch}(E).$$

The restriction to the T–fixed component F_i gives

$$[z^{-\mu} z^{c_1(X)} \widehat{\Gamma}_X (2\pi\mathtt{i})^{\deg/2} \operatorname{ch}(E)]_{F_i}$$
$$= z^{D/2} z^{c_1(F_i)/z} (z^{-\frac{\deg}{2}} \widehat{\Gamma}_{F_i}) \left(\prod_{\alpha} \prod_{j=1}^{\operatorname{rank} N_{\alpha,i}} z^{(\rho_{i,\alpha,j}+\alpha)/z} \Gamma\left(1 + \frac{\rho_{i,\alpha,j}}{z} + \frac{\alpha}{z}\right)\right) \sum_{\epsilon} e^{2\pi\mathtt{i}\epsilon/z},$$

where ϵ ranges over T–equivariant Chern roots of E and we use the notation from Definition 3.13. The conclusion easily follows from the identity $\Gamma(1+z) = z\Gamma(z)$. □

4 Toric mirror theorem

In this section we give a new proof of a mirror theorem [14] for toric manifolds.

[2] The shift by $2\pi\mathtt{i}$ is superseded by the shift by z because of the operators $z^{-\mu}$ and $(2\pi\mathtt{i})^{\deg/2}$.

4.1 Toric manifolds

We fix notation for toric manifolds. For background materials on toric manifolds, we refer the reader to [28; 2; 11]. Let $N \cong \mathbb{Z}^D$ denote a lattice. A toric manifold is given by a rational simplicial fan Σ in the vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$. We assume that

- each cone σ of Σ is generated by part of a $\mathbb{Z}$–basis of N;
- the support $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ of Σ is convex and full-dimensional;
- Σ admits a strictly convex piecewise linear function $\eta\colon |\Sigma| \to \mathbb{R}$.

These assumptions ensure that the corresponding toric variety X_Σ is smooth and satisfies the hypotheses in Section 2.1. We do not require that X is compact, or $c_1(X)$ is semipositive. Let $b_1, \dots, b_m \in N$ be primitive integral generators of one-dimensional cones of Σ. Let $\beta\colon \mathbb{Z}^m \to N$ be the homomorphism sending the standard basis vector $e_i \in \mathbb{Z}^m$ to b_i. The *fan sequence* is the exact sequence

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\beta} N \to 0$$

with $\mathbb{L} = \mathrm{Ker}(\beta)$. Set $K = \mathbb{L} \otimes \mathbb{C}^\times$. The inclusion $\mathbb{L} \hookrightarrow \mathbb{Z}^m$ induces the inclusion $K \hookrightarrow (\mathbb{C}^\times)^m$ of tori and defines a linear K–action on $\mathbb{C}^m$. The toric variety associated to Σ is given by the GIT quotient

$$X_\Sigma = U/K, \quad U = \mathbb{C}^m \setminus Z,$$

where $Z \subset \mathbb{C}^m$ is the common zero set of monomials $z^I = z_{i_1} \cdots z_{i_k}$ with $I = \{i_1, \dots, i_k\}$ such that $\{b_i : 1 \le i \le m,\ i \notin I\}$ spans a cone in Σ. We consider the T–action on X_Σ induced by the $T = (\mathbb{C}^\times)^m$–action on $\mathbb{C}^m$.

Let $\lambda_i \in H^2_T(\mathrm{pt}) \cong \mathrm{Lie}(T)^*$ denote the class corresponding to the i^{th} projection $T \to \mathbb{C}^\times$. We have

$$H^*_T(\mathrm{pt}) = \mathbb{C}[\lambda_1, \dots, \lambda_m].$$

All the T–weights of $H^0(X_\Sigma, \mathcal{O})$ are contained in the cone $\sum_{i=1}^m \mathbb{R}_{\ge 0}(-\lambda_i)$ and therefore the condition (2) in Section 2.1 is satisfied. A cocharacter $k\colon \mathbb{C}^\times \to T$ is seminegative in the sense of Definition 3.3 if $\lambda_i \cdot k \ge 0$ for all $i = 1, \dots, m$.

Let $u_i \in H^2_T(X_\Sigma)$ denote the class of the torus-invariant divisor $\{z_i = 0\}$ defined as the vanishing set of the i^{th} coordinate z_i on $\mathbb{C}^m$. The T–equivariant cohomology ring of X_Σ is generated by these classes:

$$H^*_T(X_\Sigma) \cong H^*_T(\mathrm{pt})[u_1, \dots, u_m]/(\mathfrak{I}_1 + \mathfrak{I}_2),$$

where $\mathfrak{I}_1$ is the ideal generated by $\prod_{i \in I} u_i$ such that $\{b_i : i \in I\}$ does not span a cone in Σ and $\mathfrak{I}_2$ is the ideal generated by $\sum_{i=1}^m \chi(b_i)(u_i - \lambda_i)$ with $\chi \in \mathrm{Hom}(N, \mathbb{Z})$.

4.2 Mirror theorem

Define a cohomology-valued hypergeometric series $I(y,z)$ by the formula

$$I(y,z) = z e^{\sum_{i=1}^m u_i \log y_i / z} \sum_{d \in \mathrm{Eff}(X_\Sigma)} \left(\prod_{i=1}^m \frac{\prod_{c=-\infty}^0 (u_i + cz)}{\prod_{c=-\infty}^{u_i \cdot d} (u_i + cz)} \right) Q^d y_1^{u_1 \cdot d} \cdots y_m^{u_m \cdot d}.$$

This formula defines an element of $H^*_{\hat{T}}(X_\Sigma)_{\mathrm{loc}}[\![Q]\!][\![\log y]\!]$. We may write $I(y,z)$ as a sum over $H_2(X_\Sigma, \mathbb{Z})$ since the summand automatically vanishes if $d \notin \mathrm{Eff}(X_\Sigma)$.

Givental's mirror theorem [14] (generalized later in [24; 19; 5]) states the following:

Theorem 4.1 *The function $I(y,-z)$ lies on the Givental cone associated to genus-zero Gromov–Witten theory of X_Σ.*

We explain the meaning of the statement. The *Givental cone* $\mathcal{L}$ [15] is a subset of $H^*_{\hat{T}}(X_\Sigma)_{\mathrm{loc}}[\![Q]\!]$ consisting of points of the form

$$\text{(4-1)} \qquad -z + \boldsymbol{t}(z) + \sum_{i=0}^N \sum_{n=0}^\infty \sum_{d \in \mathrm{Eff}(X_\Sigma)} \frac{Q^d}{n!} \left\langle \frac{\phi^i}{-z-\psi}, \boldsymbol{t}(\psi), \dots, \boldsymbol{t}(\psi) \right\rangle_{0,n+1,d}^{X,T} \phi_i$$

with $\boldsymbol{t}(z) \in H^*_{\hat{T}}(X_\Sigma)[\![Q]\!] = H^*_T(X_\Sigma)[z][\![Q]\!]$. The Givental cone $\mathcal{L}$ can be written as the graph of the differential of the genus-zero descendant Gromov–Witten potential, and encodes all genus-zero descendant Gromov–Witten invariants. Theorem 4.1 says that $I(y,z)$ is of the form (4-1) for some $\boldsymbol{t}(z) \in H^*_T(X_\Sigma)[z][\![Q]\!][\![\log y]\!]$ with $\boldsymbol{t}(z)|_{Q=\log y=0} = 0$. For toric manifolds, the above I–function determines the Givental cone and hence all the genus-zero Gromov–Witten invariants completely.

In this paper, we use an alternative description [15] of the Givental cone $\mathcal{L}$. We can write $\mathcal{L}$ as the union

$$\mathcal{L} = \bigcup_{\tau \in H^*_T(X_\Sigma)[\![Q]\!]} z T_\tau$$

of the semi-infinite subspaces $T_\tau = M(\tau,-z) H_T(X_\Sigma)[z][\![Q]\!]$, where $M(\tau,-z)$ denotes the fundamental solution from Proposition 2.2 with the sign of z flipped. The subspace T_τ is a (common) tangent space to $\mathcal{L}$ along $zT_\tau \subset \mathcal{L}$. Therefore, it suffices to show that $I(y,z)$ can be written in the form

$$I(y,z) = zM(\tau(y), z)\Upsilon(y,z)$$

for some $\tau(y) \in H^*_T(X_\Sigma)[\![Q]\!][\![\log y]\!]$ and $\Upsilon(y,z) \in H^*_T(X_\Sigma)[z][\![Q]\!][\![\log y]\!]$.

4.3 Proof

The idea of the proof is as follows. Let e_i denote the cocharacter $\mathbb{C}^\times \to T = (\mathbb{C}^\times)^m$ given by the inclusion of the i^{th} factor. Let $\mathbb{S}_i = \mathbb{S}_{e_i}$ and $\mathcal{S}_i = \mathcal{S}_{e_i}$ denote the corresponding shift operators. In view of Theorem 3.14, the shift operator $\mathcal{S}_i$ defines a vector field on the Givental cone $\mathcal{L}$,

$$\boldsymbol{f} \mapsto z^{-1}\mathcal{S}_i \boldsymbol{f} \in T_{\boldsymbol{f}}\mathcal{L}. \tag{4-2}$$

These vector fields define commuting flows by Corollary 3.16. We will identify the I–function with an integral submanifold of these vector fields.

Consider the $\mathbb{C}^\times$–action on X_Σ induced by the cocharacter $e_i \in \mathrm{Hom}(\mathbb{C}^\times, T)$. The minimal fixed component $F_{\min}$ for this $\mathbb{C}^\times$–action is the toric divisor $\{z_i = 0\}$. Let $E_i = E_{e_i}$ denote the associated bundle. For a fixed point $x \in X_\Sigma^T$, we set $d_i(x) = \sigma_x - \sigma_{\min} \in H_2(X_\Sigma, \mathbb{Z})$, where $\sigma_x \in H_2^{\sec}(E_k)$ is the section (3-2) of E_i associated to x and $\sigma_{\min} \in H_2^{\sec}(E_k)$ is the minimal section class of E_i. We write $u_j(x) \in H_T^2(\mathrm{pt})$ for the restriction of u_j to x.

Lemma 4.2 *With the notation as above, we have*

$$u_j(x) \cdot e_i = \delta_{ij} - u_j \cdot d_i(x).$$

Proof Consider the $\widehat{T}$–invariant divisor $\{z_j = 0\} \times \mathbb{P}^1$ in E_i and let $\hat{u}_j$ denote the $\widehat{T}$–equivariant Poincaré dual of the divisor. Then we have $\hat{u}_j|_{(x,[1,0])} = u_j(x)$ and $\hat{u}_j|_{(x,[0,1])} = u_j(x) + (u_j(x) \cdot e_i)z$. The localization formula gives

$$\hat{u}_j \cdot \sigma_x = \frac{\hat{u}_j|_{(x,[1,0])}}{z} + \frac{\hat{u}_j|_{(x,[0,1])}}{-z} = -u_j(x) \cdot e_i.$$

Similarly we have $\hat{u}_j \cdot \sigma_{\min} = -u_j(y) \cdot e_i$ for any T–fixed point y in the divisor $F_{\min} = \{z_i = 0\}$. If $i \neq j$, taking y away from $\{z_j = 0\}$, we get $u_j(y) = 0$. If $i = j$, $u_j(y) \cdot e_i = 1$. Therefore $\hat{u}_j \cdot \sigma_{\min} = -\delta_{ij}$. The conclusion follows. □

Lemma 4.3 *The I–function is an integral curve of the vector field (4-2), that is, for $i \in \{1, \dots, m\}$, we have*

$$z\frac{\partial}{\partial y_i} I(y,z) = \mathcal{S}_i I(y,z).$$

Proof Note that all the T–fixed points on X_Σ are isolated. Let $x \in X^T$ be a fixed point. It suffices to show that

$$z\frac{\partial}{\partial y_i} I_x(y,z) = \Delta_x(e_i) e^{-z\partial_{\lambda_i}} I_x(y,z),$$

where $I_x(y,z)$ is the restriction of the I–function to x and

$$\Delta_x(e_i) = Q^{d_i(x)} \prod_{j=1}^m \frac{\prod_{c=-\infty}^0 (u_j(x)+cz)}{\prod_{c=-\infty}^{-u_j(x)\cdot e_i}(u_j(x)+cz)}.$$

Using Lemma 4.2, we have

$$\begin{aligned}
&\Delta_x(e_i)e^{-z\partial_{\lambda_i}} I_x(y,z)\\
&= z e^{\sum_{j=1}^m u_j(x)\log y_j/z} e^{-\log y_i + \sum_{j=1}^m (u_j\cdot d_i(x))\log y_j} Q^{d_i(x)}\\
&\quad\times \sum_{d\in H_2(X_\Sigma,\mathbb{Z})} \Bigg(\prod_{j=1}^m \frac{\prod_{c=-\infty}^0 (u_j(x)+cz)}{\prod_{c=-\infty}^{-u_j(x)\cdot e_i}(u_j(x)+cz)} \frac{\prod_{c=-\infty}^{-u_j(x)\cdot e_i}(u_j(x)+cz)}{\prod_{c=-\infty}^{u_j\cdot d - u_j(x)\cdot e_i}(u_j(x)+cz)}\Bigg) Q^d y^d,
\end{aligned}$$

where $y^d = \prod_{j=1}^m y_j^{u_j\cdot d}$. Changing variables $d \to d - d_i(x)$ and again using Lemma 4.2, we find that this equals $z\,\partial I(y,z)/\partial y_i$. □

We identify the classical shift operators:

Notation 4.4 We set $v_i := u_i - \lambda_i \in H_T^2(X_\Sigma)$ and write $v_i(x) \in H_T^2(\mathrm{pt})$ for the restriction of v_i to a T–fixed point x.

Lemma 4.5 *Let $f(v,\lambda)$ be a cohomology class in $H_T^*(X_\Sigma)$ expressed as a polynomial in $v_1,\dots,v_m$ and $\lambda_1,\dots,\lambda_m$. When we write $\tau \in H_T^*(X_\Sigma)$ as a polynomial $\tau(v,\lambda)$ in $v_1,\dots,v_m$ and $\lambda_1,\dots,\lambda_m$, we have*

$$\lim_{Q\to 0} \mathbb{S}_i(\tau) f(v,\lambda) = u_i e^{(\tau(v,\lambda - e_i z) - \tau(v,\lambda))/z} f(v, \lambda - z e_i),$$

where $\lambda - z e_i = (\lambda_1,\dots,\lambda_{i-1},\lambda_i - z, \lambda_{i+1},\dots,\lambda_m)$. In particular, the classical Seidel elements are given by

$$\lim_{Q\to 0} S_i(\tau) = u_i e^{-\partial\tau(v,\lambda)/\partial\lambda_i}.$$

Proof Recall from Theorem 3.14 that we have $\mathcal{S}_i \circ M(\tau) = M(\tau)\circ \mathbb{S}_i(\tau)$. Since $\lim_{Q\to 0} M(\tau) = e^{\tau/z}$, we have

$$\lim_{Q\to 0} \mathbb{S}_i(\tau) f(v,\lambda) = e^{-\tau/z}\big(\lim_{Q\to 0} \mathcal{S}_i\big) e^{\tau/z} f(v,\lambda).$$

By definition of $\mathcal{S}_i$, this vanishes when restricted to a fixed point outside of the minimal fixed component $\{z_i = 0\}$ with respect to e_i. On the other hand, for any T–fixed point x in $\{z_i = 0\}$, Lemma 4.2 implies that $u_j(x)\cdot e_i = \delta_{ij}$ and $v_j(x)\cdot e_i = 0$, and thus

$$\begin{aligned}
\lim_{Q\to 0} \mathbb{S}_i(\tau) f(v,\lambda)\big|_x &= e^{-\tau(v(x),\lambda)/z} u_i(x) e^{-z\partial_{\lambda_i}} [e^{\tau(v(x),\lambda)/z} f(v(x),\lambda)]\\
&= u_i(x) e^{(\tau(v(x),\lambda - e_i z) - \tau(v(x),\lambda))/z} f(v(x), \lambda - e_i z),
\end{aligned}$$

where we set $v(x) = (v_1(x), \dots, v_m(x))$. The conclusion follows. □

Lemma 4.6 *Let x be a T–fixed point on X_Σ. The restriction $u_j(x)$ is a linear combination of λ_i such that x does not lie on the divisor $\{z_i = 0\}$.*

Proof Note that if x does not lie on the divisor $\{z_i = 0\}$, we have $u_i(x) = 0$ and thus $v_i(x) = -\lambda_i$. This together with the linear relation $\sum_{i=1}^m \chi(b_i) v_i = 0$, $\chi \in \operatorname{Hom}(N, \mathbb{Z})$ determines $v_1(x), \dots, v_m(x)$ uniquely. This implies the conclusion. □

Let $\overline{\mathcal{L}} = \mathcal{L}|_{z \to -z}$ denote the Givental cone with the sign of z flipped. By the description in Section 4.2, we have a parametrization of the Givental cone $\overline{\mathcal{L}}$ by $(\tau, \Upsilon) \in H_T^*(X) \times H_{\hat{T}}^*(X) = H_T^*(X) \times H_T^*(X)[z]$ as

$$(\tau, \Upsilon) \mapsto zM(\tau, z)\Upsilon \in \overline{\mathcal{L}}.$$

The vector field (4-2) on $\overline{\mathcal{L}}$ corresponds to the vector field on $H_T^*(X) \times H_T^*(X)[z]$,

$$(V_i)_{\tau,\Upsilon} = (S_i(\tau), [z^{-1}\mathbb{S}_i(\tau)]_+ \Upsilon),$$

where $S_i(\tau)$ is the Seidel element in Definition 3.17 and $[\cdots]_+$ means the projection to the polynomial part in z, ie $[z^{-1}\mathbb{S}_i(\tau)]_+\Upsilon = z^{-1}\mathbb{S}_i(\tau)\Upsilon - z^{-1}S_i(\tau)\star_\tau \Upsilon$. In fact, if we have a curve $t \mapsto (\tau(t), \Upsilon(t))$ with $\tau'(0) = S_i(\tau(0))$ and $\Upsilon'(0) = [z^{-1}\mathbb{S}_i(\tau(0))]_+\Upsilon(0)$, the corresponding curve $f(t) = zM(\tau(t), z)\Upsilon(t)$ on $\overline{\mathcal{L}}$ satisfies

$$\begin{aligned} f'(0) &= M(\tau(0), z)(S_i(\tau(0)) \star_{\tau(0)} \Upsilon(0)) + zM(\tau(0), z)[z^{-1}\mathbb{S}_i(\tau(0))]_+ \Upsilon(0) \\ &= M(\tau(0), z)\mathbb{S}_i(\tau(0))\Upsilon(0) = z^{-1}\mathcal{S}_i f(0), \end{aligned}$$

where we used $z\partial_i M(\tau, z) = M(\tau, z)(\phi_i \star_\tau)$ in the first line and Theorem 3.14 in the second line. Since the vector fields (4-2) commute each other, the corresponding vector fields V_i for $i = 1, \dots, m$ also commute each other. In what follows, we show the existence of an integral curve for the vector field V_i with prescribed asymptotics.

Proposition 4.7 *There exist unique functions*

$$\tau(y) \in H_T^*(X_\Sigma)[\![Q]\!][\![\log y]\!] \quad \textit{and} \quad \Upsilon(y, z) \in H_T^*(X_\Sigma)[z][\![Q]\!][\![\log y]\!]$$

which are of the form

$$\begin{aligned} \tau(y) &= \sum_{i=1}^m u_i \log y_i + \sum_{d \in \operatorname{Eff}(X_\Sigma), d \neq 0} Q^d y^d \tau_d, \\ \Upsilon(y, z) &= 1 + \sum_{d \in \operatorname{Eff}(X_\Sigma), d \neq 0} Q^d y^d \Upsilon_d, \end{aligned}$$

with $y^d = \prod_{j=1}^m y_j^{u_j\cdot d}$ *and give an integral curve for the vector field* V_i *:*

$$\frac{\partial\tau(y)}{\partial y_i} = S_i(\tau(y)) \quad \textit{and} \quad \frac{\partial\Upsilon(y,z)}{\partial y_i} = [z^{-1}\mathbb{S}_i(\tau(y))]_+\Upsilon(y,z)$$

for all $1 \le i \le m$*.*

Proof Write $\tau(y) = \sum_{j=1}^m u_j \log y_j + \tau'$. The divisor equation in Remark 3.12 gives

$$\mathbb{S}_i(\tau(y)) = y_i^{-1}\mathbb{S}_i(\tau'; Qy),$$

where $\mathbb{S}_i(\sigma; Qy)$ is obtained from $\mathbb{S}_i(\sigma)$ by replacing Q^d with $Q^d y^d$. Therefore we need to solve the differential equations

$$y_i\frac{\partial\tau'}{\partial y_i} = S_i(\tau'; Qy) - u_i \quad \text{and} \quad y_i\frac{\partial\Upsilon}{\partial y_i} = [z^{-1}\mathbb{S}_i(\tau'; Qy)]_+\Upsilon. \tag{4-3}$$

We expand

$$\tau' = \sum_{d\in\mathrm{Eff}(X_\Sigma), d\neq 0} \tilde{\tau}_d(y)Q^d, \quad \Upsilon = \sum_{d\in\mathrm{Eff}(X_\Sigma)} \tilde{\Upsilon}_d(y)Q^d,$$

with $\tilde{\Upsilon}_0(y) = 1$ and solve for the coefficients $\tilde{\tau}_d(y)$ and $\tilde{\Upsilon}_d(y)$ recursively. Note that (4-3) holds true mod Q by Lemma 4.5.

First we solve for τ'. Choose a Kähler class ω such that $\omega\cdot d_1 = \omega\cdot d_2$ for $d_1, d_2 \in \mathrm{Eff}(X_\Sigma)$ if and only if $d_1 = d_2$. This defines a positive real grading on the Novikov ring $\mathbb{C}[\![Q]\!]$ such that $\deg Q^d = \omega\cdot d$. Take $d_0 \in \mathrm{Eff}(X_\Sigma)\setminus\{0\}$. Suppose by induction that there exist $\tilde{\tau}_d$ for all d with $\omega\cdot d < \omega\cdot d_0$ such that $\tilde{\tau}_d = \tau_d y^d$ for some $\tau_d \in H_T^*(X)$ and that $\tau' = \sum_{\omega\cdot d<\omega\cdot d_0} \tilde{\tau}_d Q^d$ satisfies the differential equation (4-3) modulo terms of degree $\ge \omega\cdot d_0$. We write τ_d as a polynomial in $v_1,\dots,v_m$ and $\lambda_1,\dots,\lambda_m$. Comparing the coefficients of Q^{d_0} of the differential equation, we obtain using Lemma 4.5 that

$$y_i\frac{\partial\tilde{\tau}_{d_0}}{\partial y_i} + u_i\frac{\partial\tilde{\tau}_{d_0}}{\partial\lambda_i} = \begin{pmatrix}\text{an expression in } \tilde{\tau}_d \\ \text{with } \omega\cdot d < \omega\cdot d_0\end{pmatrix}.$$

Here the right-hand side is of the form $g_i(v,\lambda)y^{d_0}$ by the induction hypothesis, where $g_i(v,\lambda)$ is a polynomial in $v_1,\dots,v_m$ and $\lambda_1,\dots,\lambda_m$. Setting $\tilde{\tau}_{d_0} = \tau_{d_0}y^{d_0}$, we obtain

$$(u_i\cdot d_0)\tau_{d_0} + (v_i+\lambda_i)\frac{\partial\tau_{d_0}}{\partial\lambda_i} = g_i(v,\lambda).$$

The Kähler class can be written as a nonnegative linear combination of u_i, and thus there exists i_0 such that $u_{i_0}\cdot d_0 > 0$. Then we can solve for the polynomial $\tau_{d_0} = \tau_{d_0}(v,\lambda)$

from the above equation with $i = i_0$ recursively from the highest-order term in λ_{i_0}. Setting $\tau(y) = \sum_i u_i \log y_i + \sum_{\omega\cdot d \le \omega\cdot d_0} \tau_d y^d Q^d$, we have

$$\frac{\partial \tau(y)}{\partial y_i} \equiv S_i(\tau(y))$$

modulo terms of degree $\ge \omega \cdot d_0$ for $i \ne i_0$ and modulo terms of degree $> \omega \cdot d_0$ for $i = i_0$. The commutativity of the flow implies that we have, for $i \ne i_0$,

$$\begin{aligned} \frac{\partial}{\partial y_{i_0}}\left(\frac{\partial \tau}{\partial y_i} - S_i(\tau(y))\right) &= \frac{\partial^2 \tau(y)}{\partial y_i \partial y_{i_0}} - (d_{\frac{\partial \tau(y)}{\partial y_{i_0}}} S_i)(\tau(y)) \\ &\equiv \frac{\partial S_{i_0}(\tau(y))}{\partial y_i} - (d_{S_{i_0}(\tau(y))} S_i)(\tau(y)) \\ &= (d_{\frac{\partial \tau(y)}{\partial y_i}} S_{i_0})(\tau(y)) - (d_{S_i(\tau(y))} S_{i_0})(\tau(y)) \\ &= (d_{\frac{\partial \tau(y)}{\partial y_i} - S_i(\tau(y))} S_{i_0})(\tau(y)) \end{aligned} \tag{4-4}$$

modulo terms of degree $> \omega \cdot d_0$. Using the divisor equation again, we have

$$y_i\left(\frac{\partial \tau(y)}{\partial y_i} - S_i(\tau(y))\right) = u_i + y_i \frac{\partial \tau'}{\partial y_i} - S_i(\tau'; Qy).$$

Modulo terms of degree $> \omega \cdot d_0$, this is $\alpha(Qy)^{d_0}$ for some $\alpha = \alpha(v, \lambda) \in H_T^*(X)$. Now the coefficient of Q^{d_0} of (4-4) gives (by Lemma 4.5)

$$(u_{i_0} \cdot d_0)\alpha + u_{i_0} \frac{\partial \alpha}{\partial \lambda_{i_0}} = 0.$$

We want to show that $\alpha = 0$ as a cohomology class. Consider the restriction $\alpha(x)$ of α to a T–fixed point $x \in X_\Sigma$. If x lies in the divisor $\{z_{i_0} = 0\}$, then $v_j(x) \in H_T^2(\mathrm{pt})$ is a linear combination of $\lambda_{j'}$ with $j' \ne i_0$ by Lemma 4.6. Thus

$$\left.\frac{\partial \alpha}{\partial \lambda_{i_0}}\right|_x = \frac{\partial \alpha(x)}{\partial \lambda_{i_0}}. \tag{4-5}$$

If x is not in the divisor $\{z_{i_0} = 0\}$, then $u_{i_0}(x) = 0$. Therefore, by restricting to x, we have

$$(u_{i_0} \cdot d)\alpha(x) + u_{i_0}(x) \frac{\partial \alpha(x)}{\partial \lambda_{i_0}} = 0.$$

This shows that $\alpha(x) = 0$ recursively from the highest-order term in λ_{i_0}. Note that the same argument shows the uniqueness of τ_{d_0}. This completes the induction.

Next we solve for Υ assuming that τ' is already solved. Let ω be a Kähler class as above and $d_0 \in \mathrm{Eff}(X_\Sigma)$ be a nonzero effective class. Suppose by induction that there exist $\widetilde{\Upsilon}_d$ for all d with $\omega \cdot d < \omega \cdot d_0$ such that $\widetilde{\Upsilon}_d = \Upsilon_d y^d$ and that $\Upsilon = \sum_{\omega\cdot d < \omega\cdot d_0} \widetilde{\Upsilon}_d Q^d$

satisfies the differential equation (4-3) modulo terms of degree $\ge \omega\cdot d_0$. We regard Υ_d as a polynomial in $v_1,\dots,v_m$ and $\lambda_1,\dots,\lambda_m$. Comparing the coefficients of Q^{d_0} of the differential equation and using Lemma 4.5, we obtain

$$y_i\frac{\partial\widetilde{\Upsilon}_{d_0}(v,\lambda)}{\partial y_i}-(v_i+\lambda_i)z^{-1}(\widetilde{\Upsilon}_{d_0}(v,\lambda-e_iz)-\widetilde{\Upsilon}_{d_0}(v,\lambda))=\begin{pmatrix}\text{an expression in } \widetilde{\Upsilon}_d\\ \text{with } \omega\cdot d<\omega\cdot d_0\end{pmatrix}.$$

Here the right-hand side is of the form $g_i(v,\lambda)y^{d_0}$ for some polynomial $g_i(v,\lambda)$ in $v_1,\dots,v_m$ and $\lambda_1,\dots,\lambda_m$. Setting $\widetilde{\Upsilon}_{d_0}=\Upsilon_{d_0}y^{d_0}$, we have

$$(u_i\cdot d_0)\Upsilon_{d_0}(v,\lambda)-(v_i+\lambda_i)z^{-1}(\Upsilon_{d_0}(v,\lambda-e_iz)-\Upsilon_{d_0}(v,\lambda))=g_i(v,\lambda).$$

As before, we can find i_0 such that $u_{i_0}\cdot d_0>0$. We can solve for $\Upsilon_{d_0}(v,\lambda)$ recursively from the highest order term in λ_{i_0} using this equation with $i=i_0$. Setting $\Upsilon=\sum_{\omega\cdot d\le\omega\cdot d_0}\Upsilon_d Q^d$, we have

$$\frac{\partial\Upsilon(y)}{\partial y_i}\equiv[z^{-1}\mathbb{S}_i(\tau(y))]_+\Upsilon(y)$$

modulo terms of degree $\ge\omega\cdot d_0$ for $i\ne i_0$ and modulo terms of degree $>\omega\cdot d_0$ for $i=i_0$. We have, for $i\ne i_0$,

$$\begin{aligned}
&\frac{\partial}{\partial y_{i_0}}\left(\frac{\partial\Upsilon(y)}{\partial y_i}-[z^{-1}\mathbb{S}_i(\tau(y))]_+\Upsilon(y)\right)\\
&=\frac{\partial^2\Upsilon(y)}{\partial y_i\partial y_{i_0}}-\frac{\partial}{\partial y_{i_0}}[z^{-1}\mathbb{S}_i(\tau(y))]_+\Upsilon(y)\\
&\equiv\frac{\partial}{\partial y_i}[z^{-1}\mathbb{S}_{i_0}(\tau(y))]_+\Upsilon(y)-\frac{\partial}{\partial y_{i_0}}[z^{-1}\mathbb{S}_i(\tau(y))]_+\Upsilon(y)\\
&\equiv[z^{-1}(d_{\mathbb{S}_i(\tau(y))}\mathbb{S}_{i_0})(\tau(y))]_+\Upsilon(y)+[z^{-1}\mathbb{S}_{i_0}(\tau(y))]_+\frac{\partial\Upsilon(y)}{\partial y_i}\\
&\qquad-[z^{-1}(d_{\mathbb{S}_{i_0}(\tau(y))}\mathbb{S}_i)(\tau(y))]_+\Upsilon(y)-[z^{-1}\mathbb{S}_i(\tau(y))]_+[z^{-1}\mathbb{S}_{i_0}(\tau(y))]_+\Upsilon(y)
\end{aligned}$$

modulo terms of degree $>\omega\cdot d_0$. The commutativity of the flows $\boldsymbol{V}_i$ for $i=1,\dots,m$ implies, for $i\ne j$,

$$\begin{aligned}
&[z^{-1}(d_{\mathbb{S}_i(\tau)}\mathbb{S}_j)(\tau)]_+\Upsilon+[z^{-1}\mathbb{S}_j(\tau)]_+[z^{-1}\mathbb{S}_i(\tau)]_+\Upsilon\\
&\qquad=[z^{-1}(d_{\mathbb{S}_j(\tau)}\mathbb{S}_i)(\tau)]_+\Upsilon+[z^{-1}\mathbb{S}_i(\tau)]_+[z^{-1}\mathbb{S}_j(\tau)]_+\Upsilon.
\end{aligned}$$

Therefore we have

$$\frac{\partial}{\partial y_{i_0}}\left(\frac{\partial \Upsilon(y)}{\partial y_i} - [z^{-1}\mathbb{S}_i(\tau(y))]_+\Upsilon(y)\right) \equiv [z^{-1}\mathbb{S}_{i_0}(\tau(y))]_+\left(\frac{\partial \Upsilon(y)}{\partial y_i} - [z^{-1}S_i(\tau(y))]_+\Upsilon(y)\right) \tag{4-6}$$

modulo terms of degree $> \omega \cdot d_0$. By the divisor equation, we have

$$y_i\left(\frac{\partial \Upsilon(y)}{\partial y_i} - [z^{-1}\mathbb{S}_i(\tau(y))]_+\Upsilon(y)\right) = y_i\frac{\partial \Upsilon(y)}{\partial y_i} - [z^{-1}\mathbb{S}_i(\tau'; Qy)]_+\Upsilon(y).$$

This is of the form $\alpha(Qy)^{d_0}$ for some $\alpha = \alpha(v, \lambda, z) \in H^*_{\widehat{T}}(X_\Sigma)$, modulo terms of degree $> \omega \cdot d_0$. Hence the differential equation (4-6) implies via Lemma 4.5 that

$$(u_{i_0} \cdot d_0)\alpha - u_{i_0} z^{-1}(\alpha(v, \lambda - e_{i_0} z, z) - \alpha(v, \lambda, z)) = 0.$$

We want to show that $\alpha = 0$ in the cohomology group. By restricting this to a T–fixed point x and using a similar argument as before (see (4-5)), we obtain

$$(u_{i_0} \cdot d_0)\alpha(x) - (v_{i_0}(x) + \lambda_{i_0}) z^{-1}(e^{-z\partial_{\lambda_{i_0}}}\alpha(x) - \alpha(x)) = 0$$

for the restriction $\alpha(x) \in H^*_{\widehat{T}}(\mathrm{pt})$ of α to x. We can easily see that $\alpha(x) = 0$ recursively from the highest-order term in λ_{i_0}. Therefore $\alpha = 0$. Note that the same argument also shows the uniqueness of Υ_{d_0}. This completes the induction and the proof. □

We now come to the final step of the proof. Let $\tau(y)$, $\Upsilon(y, z)$ be as in Proposition 4.7. Then, as discussed in the paragraph preceding Proposition 4.7,

$$y \mapsto \boldsymbol{f}(y) := zM(\tau(y), z)\Upsilon(y, z)$$

defines an integral manifold for the vector fields in (4-2). We shall show that $\boldsymbol{f}(y) = I(y, z)$. Using the divisor equation for $M(\tau, z)$, we find that $\boldsymbol{f}(y)$ is of the form

$$\boldsymbol{f}(y) = z e^{\sum_{i=1}^m u_i \log y_i / z}\left(1 + \sum_{d \in \mathrm{Eff}(X_\Sigma)\setminus\{0\}} f_d Q^d y^d\right) \tag{4-7}$$

with $f_d \in H_{\widehat{T}}(X)_{\mathrm{loc}}$. In view of Lemma 4.3, the following lemma shows that $\boldsymbol{f}(y) = I(y, z)$ and completes the proof of Theorem 4.1.

Lemma 4.8 *The family of elements $y \mapsto \boldsymbol{f}(y)$ of the form (4-7) satisfying $\partial_{y_i}\boldsymbol{f}(y) = z^{-1}\mathcal{S}_i\boldsymbol{f}(y)$ for $i = 1, \ldots, m$ is unique.*

Proof Suppose that we have two families $\boldsymbol{f}_1(y)$ and $\boldsymbol{f}_2(y)$ of elements of the form (4-7) satisfying $\partial_{y_i}\boldsymbol{f}_j(y) = z^{-1}\mathcal{S}_i\boldsymbol{f}_j(y)$ for $j = 1, 2$ and $i = 1, 2, \ldots, m$. The difference $g(y) = f_1(y) - f_2(y)$ satisfies the same differential equation and is of the form

$$g(y) = ze^{\sum_{i=1}^m u_i \log y_i/z} \sum_{d \in \mathrm{Eff}(X_\Sigma)\setminus\{0\}} g_d Q^d y^d.$$

Choose a Kähler class ω and suppose by induction that we know $g_d = 0$ for all $d \in \mathrm{Eff}(X_\Sigma)$ with $\omega \cdot d < \omega \cdot d_0$ for some $d_0 \in \mathrm{Eff}(X_\Sigma)\setminus\{0\}$. Let x be a T–fixed point. Let δ be the set of indices i such that x does not lie on the toric divisor $\{z_i = 0\}$. The Kähler class ω can be written as a positive linear combination of nonequivariant limits of u_i with $i \in \delta$. Therefore, there exists $i_0 \in \delta$ such that $u_{i_0} \cdot d_0 > 0$. The coefficient in front of Q^{d_0} of the equation $\partial_{y_{i_0}} g(y) = z^{-1} \mathcal{S}_{i_0} g(y)$ restricted to the fixed point x gives

$$(u_{i_0} \cdot d_0) g_{d_0}(x) = 0$$

since x does not lie on the minimal fixed component $\{z_{i_0} = 0\}$ with respect to e_{i_0}. Therefore $g_{d_0}(x) = 0$. Since x is arbitrary, $g_{d_0} = 0$. This completes the induction and the proof. □

4.4 Example

Consider the toric variety $X_\Sigma = \mathbb{P}^{m-1}$. In this case we have m shift operators $\mathbb{S}_1, \dots, \mathbb{S}_m$ corresponding to m toric divisors. It is well known that the mirror map $\tau(y)$ and the function $\Upsilon(y)$ are trivial:

$$\tau(y) = \sum_{i=1}^m u_i \log y_i, \qquad \Upsilon(y) = 1.$$

Generalizing the differential equation in Lemma 4.3, we can show that

$$\mathcal{S}_{i_1} \cdots \mathcal{S}_{i_a} I(y, z) = z\partial_{y_{i_1}} \cdots z\partial_{y_{i_a}} I(y, z)$$

when $i_1, \dots, i_a$ are distinct. This together with the intertwining property $\mathcal{S}_i \circ M(\tau, z) = M(\tau, z) \circ \mathbb{S}_i(\tau)$ and the divisor equation $\mathbb{S}_i(\tau(y)) = y_i^{-1} \mathbb{S}_i(0; Qy)$ implies

$$\mathbb{S}_{i_1}(0; Qy) \cdots \mathbb{S}_{i_a}(0; Qy) 1 = z\nabla_{u_{i_1}} \cdots z\nabla_{u_{i_a}} 1\big|_{\tau(y)} = \begin{cases} u_{i_1} \cdots u_{i_a} & \text{if } a < m, \\ Q y_1 \cdots y_m & \text{if } a = m, \end{cases}$$

where $i_1, \dots, i_a$ are distinct and $\mathbb{S}_i(0; Qy)$ means $\mathbb{S}_i(0)|_{Q \to Qy_1 \cdots y_m}$. This determines the action of $\mathbb{S}_i(0)$ completely. Since the one-parameter subgroup $e_1 + \cdots + e_m$ acts on $\mathbb{P}^{m-1}$ trivially, we have a relation $\mathbb{S}_1(\tau) \circ \cdots \circ \mathbb{S}_m(\tau) = Q$ by Corollary 3.16. Writing $u_i = v + \lambda_i$ for $i = 1, \dots, m$, we recover the relation

$$(z\nabla_v + \lambda_1) \cdots (z\nabla_v + \lambda_m) 1\big|_{\tau=0} = Q$$

in the equivariant small quantum D–module of $\mathbb{P}^{m-1}$.

4.5 Remarks

We first note a relation to the results in [16]. Let X_Σ be a compact toric manifold such that $c_1(X_\Sigma)$ is nef. In this case, the mirror map $\tau(y)$ takes values in $H^2_T(X)$. We write

$$\tau(y) = \sum_{i=1}^{m} (\log y_i - g^i(y)) u_i$$

for some $\mathbb{C}$–valued functions $g^i(y)$. Using the divisor equation from Remark 3.12, the differential equation in Proposition 4.7 implies:

$$y_i \frac{\partial \tau(y)}{\partial y_i} = e^{g^i(y)} S_i(0; Qe^{\tau(y)}),$$

where we set $S_i(0; Qe^{\tau(y)}) = S_i(0)|_{Q \to Qe^{\tau(y)}}$. The left-hand side is called the Batyrev element in [16] and this recovers the relationship between the Seidel and the Batyrev elements in [16, Theorem 1.1].

We should also recover a mirror theorem for the extended I–function [7] by considering the shift operators corresponding to general seminegative cocharacters $k \in (\mathbb{Z}_{\geq 0})^m \subset \mathrm{Hom}(\mathbb{C}^\times, T)$. It would be also interesting to see if our method can be generalized to toric orbifolds [7; 6], toric fibrations [5] or other T–varieties.

Notes added in proof The extended I–function for toric manifolds has been recovered in [21] by considering shift operators for arbitrary cocharacters k.

References

[1] **M F Atiyah**, **R Bott**, *The moment map and equivariant cohomology*, Topology 23 (1984) 1–28 MR

[2] **M Audin**, *Torus actions on symplectic manifolds*, revised edition, Progress in Mathematics 93, Birkhäuser, Basel (2004) MR

[3] **A Białynicki-Birula**, *Some theorems on actions of algebraic groups*, Ann. of Math. 98 (1973) 480–497 MR

[4] **A Braverman**, **D Maulik**, **A Okounkov**, *Quantum cohomology of the Springer resolution*, Adv. Math. 227 (2011) 421–458 MR

[5] **J Brown**, *Gromov–Witten invariants of toric fibrations*, Int. Math. Res. Not. 2014 (2014) 5437–5482 MR

[6] **D Cheong**, **I Ciocan-Fontanine**, **B Kim**, *Orbifold quasimap theory*, Math. Ann. 363 (2015) 777–816 MR

[7] **T Coates**, **A Corti**, **H Iritani**, **H-H Tseng**, *A mirror theorem for toric stacks*, Compos. Math. 151 (2015) 1878–1912 MR

[8] **T Coates**, **A Givental**, *Quantum Riemann–Roch, Lefschetz and Serre*, Ann. of Math. 165 (2007) 15–53 MR

[9] **T Coates**, **H Iritani**, **Y Jiang**, *The crepant transformation conjecture for toric complete intersections*, preprint (2014) arXiv

[10] **D A Cox**, **S Katz**, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs 68, Amer. Math. Soc., Providence, RI (1999) MR

[11] **D A Cox**, **J B Little**, **H K Schenck**, *Toric varieties*, Graduate Studies in Mathematics 124, Amer. Math. Soc., Providence, RI (2011) MR

[12] **I Dolgachev**, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series 296, Cambridge University Press (2003) MR

[13] **A Givental**, *Elliptic Gromov–Witten invariants and the generalized mirror conjecture*, from "Integrable systems and algebraic geometry" (M-H Saito, Y Shimizu, K Ueno, editors), World Sci. Publ., River Edge, NJ (1998) 107–155 MR

[14] **A Givental**, *A mirror theorem for toric complete intersections*, from "Topological field theory, primitive forms and related topics" (M Kashiwara, A Matsuo, K Saito, I Satake, editors), Progr. Math. 160, Birkhäuser, Boston (1998) 141–175 MR

[15] **A Givental**, *Symplectic geometry of Frobenius structures*, from "Frobenius manifolds: quantum cohomology and singularities" (C Hertling, M Marcolli, editors), Aspects Math. E36, Friedr. Vieweg, Wiesbaden (2004) 91–112 MR

[16] **E González**, **H Iritani**, *Seidel elements and mirror transformations*, Selecta Math. 18 (2012) 557–590 MR

[17] **T Graber**, **R Pandharipande**, *Localization of virtual classes*, Invent. Math. 135 (1999) 487–518 MR

[18] **T Hausel**, **F Rodriguez Villegas**, *Cohomology of large semiprojective hyperkähler varieties*, from "De la géométrie algébrique aux formes automorphes, II" (J-B Bost, P Boyer, A Genestier, L Lafforgue, S Lysenko, S Morel, B C Ngo, editors), Astérisque 370, Société Mathématique de France, Paris (2015) 113–156 MR

[19] **H Iritani**, *Quantum D–modules and generalized mirror transformations*, Topology 47 (2008) 225–276 MR

[20] **H Iritani**, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. 222 (2009) 1016–1079 MR

[21] **H Iritani**, *A mirror construction for the big equivariant quantum cohomology of toric manifolds*, Math. Ann. (online publication June 2016)

[22] **L Katzarkov**, **M Kontsevich**, **T Pantev**, *Hodge theoretic aspects of mirror symmetry*, from "From Hodge theory to integrability and TQFT tt*-geometry" (R Y Donagi, K Wendland, editors), Proc. Sympos. Pure Math. 78, Amer. Math. Soc., Providence, RI (2008) 87–174 MR

[23] **F C Kirwan**, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes 31, Princeton University Press (1984) MR

[24] **B H Lian**, **K Liu**, **S-T Yau**, *Mirror principle, III*, Asian J. Math. 3 (1999) 771–800 MR

[25] **D Maulik**, **A Okounkov**, *Quantum groups and quantum cohomology*, preprint (2012) arXiv

[26] **D McDuff**, **S Tolman**, *Topological properties of Hamiltonian circle actions*, Int. Math. Res. Pap. (2006) art. id. 72826, 1–77 MR

[27] **H Nakajima**, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series 18, Amer. Math. Soc., Providence, RI (1999) MR

[28] **T Oda**, *Convex bodies and algebraic geometry: an introduction to the theory of toric varieties*, Ergeb. Math. Grenzgeb. 15, Springer, Berlin (1988) MR

[29] **A Okounkov**, **R Pandharipande**, *The quantum differential equation of the Hilbert scheme of points in the plane*, Transform. Groups 15 (2010) 965–982 MR

[30] **R Pandharipande**, *Rational curves on hypersurfaces (after A Givental)*, from "Séminaire Bourbaki, Vol. 1997/98 (Exposé 848)", Astérisque 252, Société Mathématique de France, Paris (1998) 307–340 MR

[31] **P Seidel**, *π_1 of symplectic automorphism groups and invertibles in quantum homology rings*, Geom. Funct. Anal. 7 (1997) 1046–1095 MR

Department of Mathematics, Graduate School of Science, Kyoto University
Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

iritani@math.kyoto-u.ac.jp

Proposed: Jim Bryan Received: 4 January 2015
Seconded: Richard Thomas, Yasha Eliashberg Accepted: 26 February 2016