Distinguishing geometries using finite quotients

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We prove that the profinite completion of the fundamental group of a compact 3–manifold M satisfies a Tits alternative: if a closed subgroup H does not contain a free pro-p subgroup for any p, then H is virtually soluble, and furthermore of a very particular form. In particular, the profinite completion of the fundamental group of a closed, hyperbolic 3–manifold does not contain a subgroup isomorphic to $\hat{\mathbb{Z}}^2$. This gives a profinite characterization of hyperbolicity among irreducible 3–manifolds. We also characterize Seifert fibred 3–manifolds as precisely those for which the profinite completion of the fundamental group has a nontrivial procyclic normal subgroup. Our techniques also apply to hyperbolic, virtually special groups, in the sense of Haglund and Wise. Finally, we prove that every finitely generated pro-p subgroup of the profinite completion of a torsion-free, hyperbolic, virtually special group is free pro-p.

In a heuristic commonly used to distinguish 3–manifolds M and N, one computes all covers $M_1, \dots, M_m$ of M and $N_1, \dots, N_n$ of N up to some small degree, and then compares the resulting finite lists (M_i) and (N_j). If they can be distinguished, then one has a proof that M and N are not homeomorphic.

It is natural to ask whether this method always works. A more precise question was formulated by Calegari, Freedman and Walker [6], who asked whether the fundamental group of a 3–manifold is determined by its finite quotients or, equivalently, by its profinite completion. (A standard argument shows that two finitely generated groups have the same set of finite quotients if and only if their profinite completions are isomorphic.) Bridson, Conder and Reid [5] have answered the corresponding question for Fuchsian groups positively, while Long and Reid [29] have given a positive answer to a related question. We refer the reader to Reid [40, Section 8] for a discussion of this and related problems.

Funar [9] used work of Stebe [46] to exhibit Sol manifolds that answer the Calegari–Freedman–Walker question in the negative, and more recently Hempel [21] has exhibited Seifert fibred examples. Nevertheless, because of the effectiveness of the above heuristic,

it remains natural to ask how much information about a given 3–manifold is contained in the profinite completion of its fundamental group.

Our first theorem shows that the hyperbolicity of a closed 3–manifold M is determined by this profinite completion. Recall that the sphere theorem and the Kneser–Milnor decomposition imply that a closed, orientable 3–manifold M is aspherical if and only if M is irreducible and its fundamental group is infinite.

Theorem A Let M be a closed, orientable, aspherical 3–manifold. Then M is hyperbolic if and only if the profinite completion $\widehat{\pi_1 M}$ does not contain a subgroup isomorphic to $\widehat{\mathbb{Z}}^2$.

This can be thought of as a profinite analogue of the hyperbolization theorem, which asserts that M is hyperbolic if and only if $\pi_1 M$ does not contain a subgroup isomorphic to $\mathbb{Z}^2$. This is one case of the geometrization theorem, proved by Perelman [37; 38; 39] (see also Kleiner and Lott [28] and Morgan and Tian [33; 34]). We also have the following profinite characterization of Seifert fibred 3–manifolds:

Theorem B Let M be a closed, orientable, aspherical 3–manifold. Then M is Seifert fibred if and only if the profinite completion $\widehat{\pi_1 M}$ has a nontrivial procyclic normal subgroup.

The Seifert conjecture, which follows from the geometrization theorem but was proved earlier by Casson and Jungreis [7] and Gabai [10], building on work of Mess [30] and Tukia [47], asserts that if $\pi_1 M$ is infinite then M is Seifert fibred if and only if $\pi_1 M$ has an infinite cyclic normal subgroup.

In fact, the geometrization theorem is equivalent to the hyperbolization theorem and the Seifert conjecture, together with the elliptization theorem, which asserts that M is spherical if and only if $\pi_1 M$ is finite; see Scott [44]. Since 3–manifold groups are residually finite (see Hempel [20]), $\pi_1 M$ is finite if and only if its profinite completion is, so there is no distinct profinite analogue of the elliptization theorem. Theorems A and B can therefore be thought of as providing a complete profinite analogue of the geometrization theorem (although one should note, of course, that our proofs rely essentially on geometrization). In Theorem 8.4, we proceed to show that the profinite completion of the fundamental group detects the geometry of a closed, orientable, irreducible 3–manifold.

Alternatively, Theorem A can be thought of as a classification result for the abelian subgroups of profinite completions of fundamental groups of hyperbolic manifolds. In fact, we prove a much more general "Tits alternative" for profinite completions of 3–manifold groups. We use the notation $\mathbb{Z}_\pi$ to denote $\prod_{p \in \pi} \mathbb{Z}_p$.

Theorem C *If M is any compact 3–manifold and H is a closed subgroup of $\widehat{\pi_1 M}$ that does not contain a free nonabelian pro-p subgroup for any prime p then either*

(1) *H is conjugate to the completion of a virtually soluble subgroup of $\pi_1 M$, or*

(2) *H is isomorphic to $((\mathbb{Z}_\sigma \times \mathbb{Z}_\pi) \rtimes C)$, where π and σ are (possibly empty) sets of primes with $\pi \cap \sigma = \varnothing$ and C is procyclic (possibly finite).*

Note that virtually soluble 3–manifold groups are classified, so the first possibility is well understood; see Aschenbrenner, Friedl and Wilton [3, Theorem 1.11.1]. For the second possibility, there are in fact additional constraints on the structure of H, depending on the geometry of M. We refer the reader to Theorem 5.2 and Proposition 9.4 for details.

The proof of Theorem A uses the dramatic recent developments of Agol [1], Kahn and Markovic [27] and Wise [50], whose work implies that the fundamental groups of closed hyperbolic 3–manifolds are also fundamental groups of compact, virtually special cube complexes (see [3] for a summary of these developments). Indeed, we prove the following theorem:

Theorem D *Let G be hyperbolic, virtually special group. If H is a closed subgroup of $\widehat{G}$ that does not contain a free nonabelian pro-p subgroup for any p then H is virtually isomorphic to $\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$, where π and ρ are disjoint (possibly empty) sets of primes. If G is torsion-free, then "virtually" can be omitted.*

As well as hyperbolic 3–manifold groups, many other classes of hyperbolic groups are now known to be virtually special: word-hyperbolic Coxeter groups (see Haglund and Wise [15]); $C'(\frac{1}{6})$ small-cancellation groups (see Wise [49] and Agol [1]); random groups at density less than $\frac{1}{6}$ (see Ollivier and Wise [36] and Agol [1]); and one-relator groups with torsion (see Wise [50]). Most of these examples are torsion-free, with the exception of Coxeter groups and one-relator groups. In the one-relator case, we can improve our classification:

Theorem E *Let G be a one-relator group with torsion and H a closed subgroup of $\widehat{G}$ that does not contain a free nonabelian pro-p subgroup for any p. Then H is virtually soluble and has one of the following forms:*

(1) *$H \cong \mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$, where π and ρ are disjoint sets of primes.*

(2) *H is a profinite dihedral group $\mathbb{Z}_\pi \rtimes C_2$.*

(3) *H is a profinite Frobenius group $\mathbb{Z}_\pi \rtimes C_n$, ie the order of every prime divisor p of n divides $q - 1$ for some $q \in \pi$ and the centralizers of nonidentity elements of C_n coincide with C_n.*

We then turn to analyse the pro-p subgroups of $\widehat{G}$, whose structure turns out to be amazingly restricted: they are all free whenever G is torsion-free, hyperbolic and virtually special (such as the fundamental group of a hyperbolic 3–manifold).

Theorem F *Let G be a torsion-free, hyperbolic, virtually special group (such as the fundamental group of a compact hyperbolic 3–manifold). Any finitely generated pro-p subgroup H of $\widehat{G}$ is free pro-p.*

Also, if G has torsion, then a finitely generated pro-p subgroup H of $\widehat{G}$ is virtually free pro-p and therefore, by the main result of Herfort and Zalesskii [23], admits a pro-p analogue of Karras, Pietrowsky and Solitar's description, ie as the pro-p fundamental group of a finite graph of finite p–groups (see Corollary 11.3).

We conclude this introduction by outlining the structures of the proofs in this paper. Since Theorem A follows immediately from Theorem D, we will first illustrate the proof of Theorem D by sketching the proof that the profinite completion of a hyperbolic virtually special group cannot contain a subgroup isomorphic to $\widehat{\mathbb{Z}}^2$.

Any hyperbolic virtually special group G has a subgroup G_0 of finite index that admits a malnormal quasiconvex hierarchy (see Definition 1.3). There is a well-known geometric proof that hyperbolic groups cannot contain subgroups isomorphic to $\mathbb{Z}^2$, but one can give an alternative proof of this fact for hyperbolic virtually special groups using the malnormal hierarchy, as follows: since the edge groups in the hierarchy are malnormal, the corresponding action of G_0 on the Bass–Serre tree is 1–acylindrical (see Definition 7.1); but $\mathbb{Z}^2$ does not admit an acylindrical action on a tree, so any $\mathbb{Z}^2$ subgroup of G_0 is conjugate into a lower level of the hierarchy, and we conclude by induction. Note that, in this proof, only the malnormality of the hierarchy was used; this makes it suitable for translation to the profinite setting, where malnormality makes sense.

The proof of Theorem D is a profinite analogue of this argument. Classical Bass–Serre theory is replaced by the theory of groups acting on profinite trees (see Sections 5 and 6). As in the discrete setting, the key fact needed is that the edge stabilizers should be malnormal. This is provided by the following theorem, which is of independent interest. (See Theorem 3.3 for a more comprehensive statement.)

Theorem G *Let G be a hyperbolic virtually special group and let H be a quasiconvex, almost malnormal subgroup. Then $\widehat{H}$ is also almost malnormal in $\widehat{G}$.*

Theorem G is proved using detailed properties of the *canonical completion* for immersions of special cube complexes, developed by Haglund and Wise [14; 16] (see

Section 2 for details). This completes our sketch of the proof of Theorem D (and hence Theorem A).

The above argument implies in particular that, after passing to a finite-index subgroup, the profinite completion of the fundamental group of a closed hyperbolic 3–manifold admits an acylindrical action on a profinite tree (see Definition 7.2). In fact, the same holds true for every closed 3–manifold M which is neither Seifert fibred nor a Sol manifold: if M is reducible then the profinite tree is induced by the Kneser–Milnor decomposition, and if M is irreducible and not geometric then the profinite tree is induced by the JSJ decomposition. The bulk of the proof of Theorem B consists of these observations, together with a result (Proposition 6.6) which implies that a profinite group acting acylindrically on a profinite tree cannot have a nontrivial procyclic normal subgroup. Again, this is the profinite analogue of a well-known lemma from classical Bass–Serre theory. A separate argument is needed to show that profinite completions of Sol manifolds do not admit nontrivial procyclic normal subgroups. With Theorem B in hand, we go on to show that the profinite completion of the fundamental group of a geometric 3–manifold determines its geometry (see Theorem 8.4).

As in the case of Theorem D, the proof of Theorem C is by induction on a suitable hierarchy. By applying first the (profinite) Kneser–Milnor decomposition and then the (profinite) JSJ decomposition, the theorem is reduced to the cases of Seifert fibred and hyperbolic manifolds, possibly with cusps. The main difficulty at this point is provided by hyperbolic manifolds with cusps; as in the closed case, one needs to know that (after passing to a finite-sheeted cover) there is a suitable hierarchy in which the corresponding actions on profinite trees are profinitely acylindrical. We resolve this difficulty with a combinatorial Dehn filling argument (see Section 9A), which reduces the problem to the setting of hyperbolic virtually special groups, where Theorem G applies.

The proof of Theorem E amounts to a careful analysis of the hierarchy used by Wise to show that one-relator groups with torsion are virtually special. Finally, the main additional ingredient of Theorem F is Theorem 11.1, of interest in its own right, which asserts that a pro-p group acting 1–acylindrically on a profinite tree actually splits as a free (pro-p) product.

1   Hierarchies for virtually special groups

Haglund and Wise [14] defined a nonpositively curved cube complex to be *special* if its hyperplanes do not exhibit a certain finite list of pathologies (they are not one-sided, and do not self-intersect, directly self-osculate, or inter-osculate). The reader is referred to that paper for details, but we will not need them here.

We will call a group G *virtually special* if some finite-index subgroup is the fundamental group of a compact, special cube complex. (Note that, in some other contexts, the compactness hypothesis is omitted. This leads to a different class of groups.)

Since the hyperplanes in a special cube complex are embedded and two-sided, one can cut along them to obtain a hierarchical decomposition of the fundamental group. This can be thought of as a more geometric version of the hierarchy admitted by a Haken 3–manifold. Indeed, in the word-hyperbolic case, results of Haglund and Wise characterize virtually special groups as precisely those that admit a sufficiently well behaved hierarchy.

In order to state their results, we first need to describe the sorts of hierarchies that we are interested in.

Definition 1.1   The class of word-hyperbolic groups with a *quasiconvex hierarchy* is the smallest class of groups, closed under isomorphism, that contains the trivial group and is such that, if

(1) $G = A *_C B$, and A and B each have a quasiconvex hierarchy, or

(2) $G = A *_C$ and A has a quasiconvex hierarchy,

and C is quasiconvex in G, then G also has a quasiconvex hierarchy.

The *quasiconvex* subgroups referred to above are the "geometrically well-behaved" subgroups of a word-hyperbolic group. The following, algebraic, notion of "good behaviour" will also concern us:

Definition 1.2   A subgroup $H \subseteq G$ is called *malnormal* if

$$H^\gamma \cap H = 1$$

whenever $\gamma \notin H$. In groups G with torsion, it is often more appropriate to consider *almost malnormal* subgroups: a subgroup H is almost malnormal if $H^\gamma \cap H$ is finite whenever $\gamma \notin H$; in particular, if G is torsion-free then almost malnormal subgroups are malnormal.

It is frequently also useful to generalize this definition to families of subgroups: a family $\mathcal{H} = \{H, \ldots, H_n\}$ of subgroups of a group G is called *malnormal* if

$$H_i \cap H_j^\gamma \neq 1 \quad \Longrightarrow \quad i = j \text{ and } \gamma \in H_i$$

for any $\gamma \in G$ and any indices i and j. As above, one may also relax the definition of a malnormal family of subgroups to obtain a notion of an *almost malnormal* family of subgroups.

We can use this definition to define a more restrictive notion of hierarchy.

Definition 1.3 The class of word-hyperbolic groups with a *malnormal quasiconvex hierarchy* is the smallest class of groups, closed under isomorphism, that contains the trivial group and is such that, if

(1) $G = A *_C B$, and A and B each have a malnormal quasiconvex hierarchy, or

(2) $G = A*_C$ and A has a malnormal quasiconvex hierarchy,

and C is malnormal and quasiconvex in G, then G also has a malnormal quasiconvex hierarchy.

With these definitions in hand, we can summarize some deep results of Haglund, Hsu and Wise in the following theorem:

Theorem 1.4 [16; 26; 50] *Let G be a word-hyperbolic group. The following are equivalent:*

(1) *G is virtually special;*

(2) *G has a subgroup G_0 of finite index with a malnormal quasiconvex hierarchy;*

(3) *G has a subgroup G_1 of finite index with a quasiconvex hierarchy.*

Proof The equivalence of (1) and (2) follows from theorems of Hsu and Wise [26] and Haglund and Wise [16]. Wise used these results in his proof of the equivalence of (1) and (3) [50]. □

Although quasiconvex subgroups of word-hyperbolic groups may not be (almost) malnormal, they enjoy a weaker algebraic property that is almost as useful, first studied by Gitik, Mitra, Rips and Sageev [11].

Definition 1.5 Let H be a subgroup of a group G. The *width* of H is the maximal cardinality of a set of distinct right cosets

$$\{H\gamma_i\} \subseteq H\backslash G$$

such that $|H^{\gamma_i} \cap H^{\gamma_j}| = \infty$ for all i and j.

In particular, note that the width of H is 0 if and only if H is finite, and the width of H is 1 if and only if H is infinite and almost malnormal.

Theorem 1.6 [11] *If G is hyperbolic and H is quasiconvex in G then the width of H is finite.*

Recall that a subgroup H is *separable* if it is closed in the profinite topology on G. The following lemma is often useful when combined with Theorem 1.6:

Lemma 1.7 *If a subgroup H of a torsion-free group G is both separable and has finite width, then there is a subgroup G_0 of finite index in G that contains H and such that H is malnormal in G_0.*

Therefore, as long as we know that H is separable (as is the case for quasiconvex subgroups of virtually special groups, for instance), we may promote finite width to malnormality, by passing to a finite-index subgroup.

The connection to 3–manifolds arises from Agol's solution to the virtual Haken conjecture [1] which, in addition to the above theorems of Haglund, Hsu and Wise, makes essential use of the work of Kahn and Markovic [27].

Theorem 1.8 (Agol) *If M is a closed hyperbolic 3–manifold then $\pi_1 M$ is hyperbolic and virtually special.*

In summary, the virtual Haken conjecture asserts that, after passing to a finite-index subgroup, every closed hyperbolic 3–manifold admits a Haken hierarchy. However, Agol proved the even stronger result that, after passing to a finite-index subgroup, every closed hyperbolic 3–manifold admits a quasiconvex malnormal hierarchy.

The malnormality of the hierarchy will be crucial to our argument, since we will be working in the profinite setting, where there is no well-developed notion of quasiconvexity. Malnormality, however, is a purely algebraic notion, and therefore makes perfect sense for subgroups of profinite completions.

2 Haglund and Wise's canonical completion

One important element of the proof of Theorem 1.4 is Haglund and Wise's combination theorem for virtually special cube complexes [16]. Their principal tool is the *canonical completion* of a quasiconvex subgroup. In this section, we extract an algebraic property of the canonical completion, which we will then be able to apply in the profinite setting. Our main result is as follows:

Theorem 2.1 *Let G be a hyperbolic group and the fundamental group of a compact, virtually special cube complex, and let $\mathcal{H} = \{H, K\}$ be a malnormal family of quasiconvex subgroups. For any finite-index subgroup G_1 of G there exists a finite-index subgroup $G_0 \subseteq G_1$ and a retraction map $\rho \colon G_0 \to H \cap G_1$ with the following properties:*

(1) $\rho(H^\gamma \cap G_0) = 1$ *unless* $\gamma \in HG_1$; *and*

(2) $\rho(K^\gamma \cap G_0) = 1$ *for all* $\gamma \in G$.

To prove this we assemble some definitions and results of Haglund and Wise. We refer to their paper and the references therein for definitions.

Let X be a compact special cube complex, and let $A, B \subseteq X$ be subcomplexes. Then Haglund and Wise define the *wall projection* $W_X(A \to B)$, a subcomplex of X. We refer to their paper for the definition, the details of which we will not need, but we note that $W_X(A \to B)$ is said to be *trivial* if every loop in $W_X(A \to B)$ is null-homotopic in X.

Theorem 2.2 *Let A, B be locally convex subcomplexes of a special cube complex X. There is finite-sheeted covering space $C(A, X) \to X$ to which the inclusion $A \to X$ lifts, and a retraction $r \colon C(A, X) \to A$. Furthermore, if B_0 is the full preimage of B in $C(A, X)$, then*

$$r(B_0) \subseteq W_X(B \to A).$$

Proof The existence of $C(A, X)$ and r were proved in [14]. The inclusion is [16, Lemma 3.16]. □

The covering space $C(A, X)$ is called the *canonical completion* of A, and the map r is the *canonical retraction*. The above result shows that the image of the canonical retraction is controlled by the wall projection. Ensuring that wall projections are trivial is a delicate problem, and much of the difficulty of [16] lies in its resolution. In order to resolve the problem, we need to recall the definition of an *elevation*.

Definition 2.3 Let $f\colon Y \to X$ be a continuous map of topological spaces and let $p\colon X' \to X$ be a covering map. Recall that the fibre product

$$Y \times_X X' = \{(y, x') \in Y \times X' \mid f(y) = p(x')\}$$

is naturally a covering space of Y and is equipped with a natural lift $f'\colon Y \times_X X' \to X'$. The restriction of f' to a path component Y' of $Y \times_X X'$ is called an *elevation of f to X'*.

Remark 2.4 By standard covering space theory, after choosing base points there is a natural bijection between the set of double cosets

$$f_* \pi_1 Y \backslash \pi_1 X / \pi_1 X'$$

and the path components of $Y \times_X X'$. If this bijection is denoted by

$$(f_* \pi_1 Y) \gamma \pi_1 X' \mapsto Y_\gamma$$

then

$$f'_* \pi_1 Y_\gamma = (f_* \pi_1 Y)^\gamma \cap \pi_1 X',$$

where the subscript star means the induced homomorphism of the fundamental groups.

We are now ready to state a theorem of Haglund and Wise, which guarantees that we can make wall projections trivial in a finite-sheeted cover.

Theorem 2.5 [16, Corollary 5.8] *Let X be a compact, virtually special cube complex with $\pi_1 X$ word-hyperbolic. Suppose that $A \to X$ and $B \to X$ are local isometries (ie $\pi_1 A$ and $\pi_1 B$ are quasiconvex subgroups of X) and that $\{\pi_1 A, \pi_1 B\}$ is a malnormal family of subgroups of $\pi_1 X$. There is a finite-sheeted covering space $A_0 \to A$ such that any further finite-sheeted covering space $\bar{A} \to A_0 \to A$ can be completed to a finite-sheeted covering space $\bar{X} \to X$ with the following properties:*

(1) *Every elevation of $A \to X$ and $B \to X$ to $\bar{X}$ is injective.*

(2) *Every elevation A' of $A \to X$ to $\bar{X}$ distinct from $\bar{A}$ has trivial $W_{\bar{X}}(A' \to \bar{A})$.*

(3) *Every elevation $\bar{B}$ of $B \to X$ to $\bar{X}$ has trivial $W_{\bar{X}}(\bar{B} \to \bar{A})$.*

We will also need the simple observation that the property of trivial wall projections is preserved under passing to finite-sheeted covers.

Lemma 2.6 [16, Lemma 5.2] *Suppose that $X' \to X$ is a covering map of connected cube complexes, and that $A, B \subseteq X$ are connected sub-complexes. If $W_X(B \to A)$ is trivial then, for any pair of elevations A' and B' (of A and B, respectively) to X', $W_{X'}(B' \to A')$ is also trivial.*

We can combine these to obtain the result that we need:

Proof of Theorem 2.1 Let X be a virtually special cube complex with $G \cong \pi_1 X$. Since H and K are quasiconvex subgroups, they are realized by local isometries of cube complexes $A \to X$ and $B \to X$ respectively. Let G_1 be any finite-index subgroup of G and let X^1 be the corresponding covering space of X.

We first prove the following: for any $\gamma \notin HG_1$, there exists a finite-sheeted covering space $X' \to X$ such that

(1) the trivial elevation $A' \to X'$ of $A \to X$ is injective;

(2) the elevation $A'_\gamma \to X'$ of $A \to X$ corresponding to the double coset $H\gamma\pi_1 X'$ is injective;

(3) the wall projections $W_{X'}(A'_\gamma \to A')$ are trivial.

To prove this, let $X^1 \to X$ be the covering space corresponding to G_1, let $A^1 \to X^1$ be the elevation of $A \to X$ to X^1 that corresponds to the trivial double coset, and let $A^1_\gamma \to X^1$ be the elevation corresponding to $H\gamma G_1$. Since $\{H \cap G_1, H^\gamma \cap G_1\}$ is a malnormal pair of subgroups of G_1, we may apply Theorem 2.5 with X^1 in the role of X, A^1 in the role of A and A^1_γ in the role of B to obtain a finite-sheeted covering space $A' \to A^1$ (such as the one denoted by A_0 in the statement of the theorem) and a corresponding "completion" to a finite covering space X' of X^1. Now, by Theorem 2.5(1), the maps $A' \to X'$ and $A'_\gamma \to X'$ are injective and, by (3), $W_{X'}(A'_\gamma \to A')$ is trivial.

Since the above covering space X' depends on the element γ, we will denote it by X_1^γ (although note that it actually only depends on the double coset $H\gamma G_1$).

In exactly the same way, we may also prove the following: for any $\gamma \in G$, there exists a finite-sheeted covering space $X' \to X$ such that

(1) the trivial elevation $A' \to X'$ of $A \to X$ is injective;

(2) the elevation $B'_\gamma \to X'$ of $B \to X$ corresponding to the double coset $K\gamma\pi_1 X'$ is injective;

(3) the wall projections $W_{X'}(B'_\gamma \to A')$ are trivial.

We will denote this covering space X' by X_2^γ.

Now let $\overline{X}$ be the finite-sheeted covering space of X such that

$$\pi_1 \overline{X} = \bigcap_{\gamma \notin HG_1} \pi_1 X_1^\gamma \cap \bigcap_{\gamma \in G} \pi_1 X_1^\gamma,$$

which has the property that it covers every X_1^γ and X_2^γ. For any $\gamma \in G$, let $\bar{A}_\gamma \to \bar{X}$ be the elevation of $A \to X$ corresponding to $H\gamma\pi_1\bar{X}$ and let $\bar{B}_\gamma \to \bar{X}$ be the elevation of $B \to X$ corresponding to $H\gamma\pi_1\bar{X}$. Let $\bar{A} \to \bar{X}$ be the elevation of $A \to X$ corresponding to the trivial coset. By the construction of $\bar{X}$ and by Lemma 2.6, we have that $W_{\bar{X}}(\bar{A}_\gamma \to \bar{A})$ is trivial whenever $\gamma \notin HG_1$, and $W_{\bar{X}}(\bar{B}_\gamma \to \bar{A})$ is trivial for all γ.

Finally, let $G_0 = \pi_1 C(\bar{A}, \bar{X})$ and let ρ be the map induced on fundamental groups by the canonical retraction r. Theorem 2.1 now follows immediately from Theorem 2.2. □

3 Malnormality in the profinite completion

In this section we prove that, for quasiconvex subgroups of hyperbolic, virtually special groups, malnormality passes to the profinite closure. In preparation, we need to observe that the profinite completion of a torsion-free, virtually special group is itself torsion-free. For this we need to explain Serre's notion of a good group, which we will need in the proof.

Definition 3.1 A group G is *good* if the natural homomorphism $G \to \hat{G}$ of the group in its profinite completion induces an isomorphism on cohomology with finite coefficients.

In particular, the profinite completion of a good group of finite cohomological dimension is of finite cohomological dimension and so is torsion-free. It follows quickly from the results of [13] and the standard theory of virtually special groups that a torsion-free, virtually special group is good.

Proposition 3.2 *If a group G has a finite-index subgroup G_0 which is the fundamental group of a compact, special cube complex then G is good. If G is torsion-free then so is $\hat{G}$.*

Proof Let G_0 be a subgroup of G which is the fundamental group of a compact, special cube complex. The results of [14] combined with [13, Theorem 1.4] imply that G_0 is good, and therefore G is also good by [13, Lemma 3.2]. If G is torsion-free then it is of finite cohomological dimension; by goodness, $\hat{G}$ is also of finite cohomological dimension and hence torsion-free. □

We are now ready to prove the main theorem of this section.

Theorem 3.3 *Let G be word-hyperbolic and the fundamental group of a compact, virtually special cube complex. Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a malnormal family of quasiconvex subgroups. Then the family $\widehat{\mathcal{H}} = \{\widehat{H}_1, \ldots, \widehat{H}_n\}$ is a malnormal family of subgroups of the profinite completion $\widehat{G}$.*

Proof It suffices to take $n = 2$. First, we prove that $\widehat{H} = \widehat{H}_1$ is a malnormal subgroup of $\widehat{G}$.

Let $\widehat{\gamma} \in \widehat{G} \smallsetminus \widehat{H}$ and suppose that $\widehat{\delta} \in \widehat{H} \cap \widehat{H}^{\widehat{\gamma}}$, so $\widehat{\delta} = \widehat{\epsilon}^{\widehat{\gamma}}$ for $\widehat{\epsilon} \in \widehat{H}$. Since $\widehat{H}$ is closed, there exists a finite quotient $q \colon G \to Q$ whose continuous extension $\widehat{q} \colon \widehat{G} \to Q$ satisfies $\widehat{q}(\widehat{\gamma}) \notin q(H)$. Let $G_1 = \ker q$, and let G_0 be the finite-index subgroup and ρ the retraction guaranteed by Theorem 2.1. Let n be such that $\widehat{\delta}^n \in \widehat{G}^0$. If $\widehat{\rho}$ is the continuous extension of ρ to $\widehat{G}^0$ then

$$\widehat{\delta}^n = \widehat{\rho}(\widehat{\delta}^n) = \widehat{\rho}((\widehat{\epsilon}^n)^{\widehat{\gamma}})) = 1,$$

where the final equality follows from Theorem 2.1(1) by continuity, using that the closure $\overline{HG_1}$ is clopen in $\widehat{G}$. So $\widehat{\delta}$ is torsion and therefore trivial, since $\widehat{G}$ is torsion-free by Proposition 3.2. This proves that $\widehat{H}_1$ is malnormal.

To complete the proof that $\widehat{\mathcal{H}}$ is a malnormal family, suppose that $\widehat{\delta} = \widehat{\epsilon}^{\widehat{\gamma}}$, where $\widehat{\delta} \in \widehat{H}_1$ and $\widehat{\epsilon} \in \widehat{H}_2$. Let $G_1 = G$ and, as before, let G_0 be the finite-index subgroup guaranteed by Theorem 2.1 and let n be such that $\widehat{\delta}^n \in \widehat{G}^0$. Then, as before, we have that

$$\widehat{\delta}^n = \widehat{\rho}(\widehat{\delta}^n) = \widehat{\rho}((\widehat{\epsilon}^n)^{\widehat{\gamma}})) = 1,$$

where the final inequality follows from Theorem 2.1(2) by continuity. Again, since $\widehat{G}$ is torsion-free, we deduce that $\widehat{\delta} = 1$, which proves the theorem. □

In the statement of Theorem 3.3, the group G is assumed to be the fundamental group of a compact, virtually special cube complex. In particular, G is torsion-free. However, we can weaken the hypotheses on G in a small but significant way, and instead assume that G is merely virtually special, meaning that it has a subgroup of finite index which is the fundamental group of a compact special cube complex. In particular, such a G may have torsion. As mentioned above, in this context it is preferable to work with almost malnormal families of subgroups.

We can quickly deduce a similar result in this setting, which will be useful later.

Corollary 3.4 *Let G be word-hyperbolic and virtually special. Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be an (almost) malnormal family of quasiconvex subgroups. Then the family $\widehat{\mathcal{H}} = \{\widehat{H}_1, \ldots, \widehat{H}_n\}$ is an (almost) malnormal family of subgroups of the profinite completion $\widehat{G}$.*

Proof Let G_0 be a subgroup of finite index in G that is the fundamental group of a compact special cube complex. For each i let $\{\gamma_{ij}\}$ be a set of representatives for the double coset space $H_i \backslash G / G_0$. For each i and j, let $K_{ij} = H_i^{\gamma_{ij}} \cap G_0$. Then $\mathcal{K} = \{K_{ij}\}$ is a malnormal family of quasiconvex subgroups G_0. By Theorem 3.3 it follows that $\widehat{\mathcal{K}} = \{\widehat{K}_{ij}\}$ is a malnormal family of subgroups of the profinite completion $\widehat{G}_0$. It follows that $\widehat{\mathcal{H}}$ is an (almost) malnormal family of subgroups of $\widehat{G}$. □

Likewise, we can deduce that the notion of *finite width* also passes to the profinite completion.

Corollary 3.5 *Let G be word-hyperbolic and virtually special, and let H be a quasi-convex subgroup of G. Then $\widehat{H}$ has finite width in the profinite completion $\widehat{G}$.*

Proof Since H is separable in G, by Lemma 1.7, we may pass to a subgroup G_0 of finite index in G in which H is malnormal. Therefore, by Corollary 3.4, $\widehat{H}$ is malnormal in $\widehat{G}_0$. Suppose now that $\{\widehat{H}\widehat{\gamma}_i\}$ is a subset of $\widehat{H} \backslash \widehat{G}$ and $\widehat{H}^{\widehat{\gamma}_i} \cap \widehat{H}^{\widehat{\gamma}_j}$ is infinite for all i and j. Then $\widehat{H}^{\widehat{\gamma}_i \widehat{\gamma}_j^{-1}} \cap \widehat{H}$ is also infinite, whence $\widehat{\gamma}_i \widehat{\gamma}_j^{-1} \notin \widehat{G}_0$ if $i \neq j$. Therefore, the map

$$\{\widehat{H}\widehat{\gamma}_i\} \to \widehat{G}_0 \backslash \widehat{G} \cong G_0 \backslash G$$

induced by the inclusion $\widehat{H} \to \widehat{G}_0$ is an injection. This completes the proof. □

4 Malnormality in the relative case

In order to deal with cusped hyperbolic manifolds, we will also need relative versions of the results of the previous section. A *toral relatively hyperbolic* is a group that is torsion-free and hyperbolic relative to sets of finitely generated abelian subgroups. We refer the reader to [25] for a survey of the various equivalent definitions of relative hyperbolicity.

There is a notion of a *relatively quasiconvex* subgroup of a relatively hyperbolic group. Again, the reader is referred to [25] for various equivalent definitions. We will also be interested in a relatively hyperbolic version of malnormality.

Definition 4.1 Suppose that a group G is hyperbolic relative to a collection of *parabolic* subgroups $\{P_1, \ldots, P_n\}$. A subgroup H of G is called *relatively malnormal* if, whenever an intersection of conjugates $H^\gamma \cap H$ is not conjugate into some P_i, we have $\gamma \in H$.

The next theorem, which is the main result of this section, is an analogue of Theorem 3.3 in the toral relatively hyperbolic setting.

Theorem 4.2 *Suppose that G is a virtually compact special group which is also toral relatively hyperbolic with parabolic subgroups $P_1, \ldots, P_n$. Let H be a subgroup which is relatively malnormal and relatively quasiconvex. Then $\widehat{H}$ is also a relatively malnormal subgroup of $\widehat{G}$, in the sense that $\widehat{H} \cap \widehat{H}^{\widehat{\gamma}}$ is conjugate into $\widehat{P}_i$ (for some i) whenever $\widehat{\gamma} \notin \widehat{H}$.*

One could envisage a proof of this theorem along the lines of the techniques of Section 3, again using Haglund and Wise's canonical completion. However, this would require a generalization of Theorem 2.5 to the relatively hyperbolic setting, which is not currently in the literature. Instead, we will appeal to [50, Lemma 16.13], which is a relatively hyperbolic version of the malnormal special quotient theorem of Wise [50, Theorem 12.3].

Theorem 4.3 *Suppose G is toral relatively word-hyperbolic and virtually compact special and $\mathcal{P} = \{P_1, \ldots, P_k\}$ is an almost malnormal family of quasiconvex subgroups of G. There are subgroups of finite index $K_i \subseteq P_i$ such that, for all subgroups of finite index $L_i \subseteq K_i$, the quotient*

$$G/\langle\!\langle L_1, \ldots, L_n \rangle\!\rangle$$

is word-hyperbolic and virtually the fundamental group of a compact special cube complex.

Proof This is a special case of [50, Lemma 16.13].   □

We will also need to make use of a relatively hyperbolic extension of the results of Agol, Groves and Manning from [2].

Theorem 4.4 [12] *Let G be a toral relatively hyperbolic group, with parabolic subgroups $\{P_1, \ldots, P_n\}$ and let H be a subgroup which is relatively quasiconvex and relatively malnormal. There exist subgroups of finite index $K_i' \subseteq P_i$ (for all i) such that, for all subgroups of finite index $L_i \subseteq K_i'$, if*

$$\eta \colon G \to Q = G/\langle\!\langle L_1, \ldots, L_n \rangle\!\rangle$$

is the quotient map, the quotient Q is word-hyperbolic and the image $\eta(H)$ in Q is quasiconvex and almost malnormal.

Proof of Theorem 4.2 Suppose that $\widehat{h} \in \widehat{H}$, $\widehat{\gamma} \notin \widehat{H}$ and $\widehat{h}^{\widehat{\gamma}} \in \widehat{H}$. Since $\widehat{\gamma} \notin \widehat{H}$, there exists a finite quotient $q_0 \colon G \to Q_0$ such that, when extended to the profinite

completion, $\hat{q}_0(\hat{\gamma}) \notin q_0(H)$ and also $\hat{q}_0(\hat{h}) \neq 1$. Let $q\colon G \to Q$ be any finite quotient of G such that q_0 factors through q. For each i, let

$$L_i = \ker q \cap K_i \cap K_i'$$

for K_i as in Theorem 4.3 and K_i' as in Theorem 4.4. Now we Dehn fill to obtain

$$\eta\colon G \to G/\langle\!\langle L_1, \dots, L_n \rangle\!\rangle =: \Delta,$$

and let $K = \eta(H)$. By Theorem 4.3, Δ is virtually special. Note that q_0 factors through η, so $\hat{\eta}(\hat{\gamma}) \notin \hat{K}$ and $\hat{\eta}(\hat{h}) \neq 1$. By Theorem 4.4, K is quasiconvex and almost malnormal in Δ.

Hence, by Corollary 3.4, $\hat{K}$ is also almost malnormal in $\hat{\Delta}$. But $\hat{\eta}(\hat{h}) \in \hat{K} \cap \hat{K}^{\hat{\eta}(\hat{\gamma})^{-1}}$, and so $\hat{\eta}(\hat{h})$ has finite order, and hence is conjugate into $\hat{\eta}(\hat{P}_i)$ for some i. In particular, $\hat{q}(\hat{h})$ is conjugate into $\hat{q}(\hat{P}_i)$ for some i.

Since q was an arbitrarily deep finite quotient of G, it follows that $\hat{h}$ is conjugate into $\hat{P}_i$ for some i. This completes the proof. $\qquad\qquad\square$

In the case of cusped hyperbolic manifolds, it is also an important fact that the peripheral subgroups form a malnormal family. We will need the profinite version of this fact (Lemma 4.5). This result was also needed in [19], but the result stated there [19, Lemma 4.7] is slightly weaker, as noticed independently by Gareth Wilkes and the second author. We therefore provide a strengthened result here, with thanks to Emily Hamilton.

Lemma 4.5 (cf [19, Lemma 4.7]) *Let Γ be the fundamental group of a cusped hyperbolic manifold and let $\{P_i\}$ be conjugacy representatives of the cusp subgroups. Then the set of their closures $\{\overline{P}_i\}$ in the profinite completion $\hat{\Gamma}$ forms a malnormal family; that is, if $\overline{P}_i \cap \overline{P}_j^{\hat{\gamma}}$ is nontrivial for some $\hat{\gamma} \in \hat{\Gamma}$ then $i = j$ and $\hat{\gamma} \in \overline{P}_i$.*

Note that it is enough to consider the case of two cusps, so we adopt the notation $P = P_1$ and $Q = P_2$. We first consider the intersection $\overline{P} \cap \overline{Q}^{\hat{\gamma}}$. The next lemma is a slight modification of [19, Lemma 4.6].

Lemma 4.6 *Let $M = \mathbb{H}^3/\Gamma$ be a cusped hyperbolic 3–manifold of finite volume. Let P and Q be nonconjugate cusp subgroups of Γ. Then there exists a positive integer n with the following property: for each integer $m \geq n$, there exist finite fields F_1 and F_2 and group homomorphisms $f_1\colon \Gamma \to \mathrm{PSL}(2, F_1)$ and $f_2\colon \Gamma \to \mathrm{PSL}(2, F_2)$ such that*

(1) *the image of P under $f_1 \times f_2$ is isomorphic to $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$;*

(2)   *for every element $p \in P$ and every i, if $f_i(p)$ is nontrivial, then $\operatorname{tr} f_i(p) \neq \pm 2$;*

(3)   *for every element $q \in Q$ and every i, $\operatorname{tr} f_i(q) = \pm 2$.*

In particular, if every $f_i(p)$ is conjugate into $f_i(Q)$, then $(f_1 \times f_2)(p)$ is trivial.

Proof   Since P and Q are nonconjugate, they correspond to distinct cusps of M. Let T be the cusp of M corresponding to P. The group P is free abelian of rank 2. By Thurston's hyperbolic Dehn surgery theorem, there exist a basis $\{p_1, p_2\}$ of P and complete hyperbolic 3–manifolds M_1 and M_2 of finite volume, obtained by Dehn surgery on M along T, such that if

$$\phi_1 \colon \Gamma \to \pi_1(M_1) \quad \text{and} \quad \phi_2 \colon \Gamma \to \pi_1(M_2)$$

are the homomorphisms induced by inclusion, then $\phi_1(p_1)$ is a loxodromic isometry of $\pi_1(M_1)$, $\phi_1(p_2)$ is trivial, $\phi_2(p_1)$ is trivial and $\phi_2(p_2)$ is a loxodromic isometry of $\pi_1(M_2)$.

Since M_1 has finite volume and $\phi_1(p_1)$ is loxodromic, there exists a discrete, faithful representation

$$\rho_1 \colon \pi_1(M_1) \to \operatorname{PSL}(2, \mathbb{C})$$

such that $\rho_1(\pi_1(M_1)) \subset \operatorname{PSL}(2, L_1)$ for some number field L_1 and

$$\rho_1(\phi_1(p_1)) = \pm \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad |\omega| \neq 1.$$

Let R_1 be the ring in L_1 generated by the coefficients of the generators of $\rho_1(\pi_1(M_1))$ over $\mathbb{Z}$. Then $\rho_1(\pi_1(M_1)) \subset \operatorname{PSL}(2, R_1) \subset \operatorname{PSL}(2, L_1)$. By [18, Corollary 2.5], there exists a positive integer n with the following property: for each integer $m \geq n$, there exist a finite field F_1 and a ring homomorphism $\eta_1 \colon R_1 \to F_1$ such that the multiplicative order of $\eta_1(\omega)$ is equal to $2m$. This ring homomorphism induces a group homomorphism

$$\psi_1 \colon \rho_1(\pi_1(M_1)) \hookrightarrow \operatorname{PSL}(2, R_1) \to \operatorname{PSL}(2, F_1).$$

Let

$$f_1 \colon \Gamma \to \operatorname{PSL}(2, F_1)$$

denote the composition $\psi_1 \circ \rho_1 \circ \phi_1$. Then $f_1(p_1)$ has order m and $f_1(p_2)$ is trivial. For every element $q \in Q$, $\rho_1 \circ \phi_1(q)$ is parabolic. Therefore, $\operatorname{tr} f_1(q) = \pm 2$. For every element $p \in P$,

$$\rho_1(\phi_1(p)) = \pm \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{pmatrix} \quad \text{for some } k \in \mathbb{Z}.$$

If $\eta_1(\omega^k + \omega^{-k}) = 2$, then $\eta_1(\omega^k) = \eta_1(\omega^{-k}) = 1$. If $\eta_1(\omega^k + \omega^{-k}) = -2$, then $\eta_1(\omega^k) = \eta_1(\omega^{-k}) = -1$. Therefore, if $f_1(p)$ is nontrivial, then $\operatorname{tr} f_1(p) \neq \pm 2$.

In a similar way, we can choose n such that for each integer $m \geq n$ there exist a finite field F_2 and a group homomorphism

$$f_2 \colon \Gamma \to \mathrm{PSL}(2, F_2)$$

such that $f_2(p_1)$ is trivial, $f_2(p_2)$ has order m, $\operatorname{tr} f_2(q) = \pm 2$ for every $q \in Q$, and, for every $p \in P$, if $f_2(p)$ is nontrivial then $\operatorname{tr} f_2(p) \neq \pm 2$. The homomorphisms f_1 and f_2 then satisfy the three conditions above. $\qquad\square$

We can now easily prove that the peripheral subgroups form a profinite malnormal family.

Proof of Lemma 4.5 Consider first the intersection $\bar{P} \cap \bar{Q}^{\hat{\gamma}}$. Let $\hat{p}$ be an element of the intersection $\bar{P} \cap \bar{Q}^{\hat{\gamma}}$. Let f be an arbitrary homomorphism from Γ to a finite group, and let f_0 be the restriction of f to P. Choose f_1 and f_2 as in Lemma 4.6 so that f_0 factors through $(f_1 \times f_2)|_P$, and extend them by continuity to homomorphisms $\hat{f}_i$ from the profinite completion. Choose $p \in P$ so that $f_i(p) = \hat{f}_i(\hat{p})$ for all i. Since $\hat{p}$ is conjugate into $\bar{Q}$, (3) implies that $\operatorname{tr} f_i(p) = \operatorname{tr} \hat{f}_i(\hat{p}) = 2$ for every i. Item (2) then implies that $f_i(p) = 1$ for every i, and so by (1), $\hat{f}_0(\hat{p}) = f_0(p) = 1$. Thus, every finite quotient of Γ kills $\hat{p}$, and so $\hat{p} = 1$ by the definition of the profinite completion. We have shown that the intersection $\bar{P} \cap \bar{Q}^{\hat{\gamma}}$ is trivial.

We now consider the intersection $\bar{P} \cap \bar{P}^{\hat{\gamma}}$, and suppose that $\hat{\gamma} \notin \bar{P}$. Since $\bar{P}$ is closed, there is a subgroup Γ_0 of finite index in Γ that contains P but such that $\hat{\Gamma}_0 \subseteq \hat{\Gamma}$ does not contain $\hat{\gamma}$. Let $\hat{\gamma} = \gamma \hat{\gamma}_0$, where $\gamma \in \Gamma \smallsetminus \Gamma_0$ and $\hat{\gamma}_0 \in \hat{\Gamma}_0$. Then Γ_0 is the fundamental group of a closed hyperbolic manifold with nonconjugate cusp subgroups P and $Q = P^\gamma \cap \Gamma_0$ (since P is malnormal in Γ). By the argument of the previous paragraph applied to Γ_0, the intersection $\bar{P} \cap \bar{Q}^{\hat{\gamma}_0}$ is trivial. Since Q is of finite index in P^γ, it follows that the intersection $\bar{P} \cap \bar{P}^{\hat{\gamma}}$ is finite. But 3–manifold groups are torsion-free and good, whence their profinite completions are also torsion-free. In particular, $\bar{P} \cap \bar{P}^{\hat{\gamma}}$ is trivial. $\qquad\square$

5 Profinite trees

In order to prove our main theorems, our strategy is to promote the action of a group on a tree to an action of its profinite completion on a *profinite tree*. In this section, we recall the necessary elements of the theory of profinite trees.

A graph Γ is a disjoint union $E(\Gamma) \cup V(\Gamma)$ with two maps $d_0, d_1 \colon \Gamma \to V(\Gamma)$ that are the identity on the set of vertices $V(\Gamma)$. For an element e of the set of edges $E(\Gamma)$, $d_0(e)$ is called the initial and $d_1(e)$ the terminal vertex of e.

Definition 5.1 A *profinite graph* Γ is a graph such that

(1) Γ is a profinite space (ie an inverse limit of finite discrete spaces);

(2) $V(\Gamma)$ is closed; and

(3) the maps d_0 and d_1 are continuous.

Note that $E(\Gamma)$ is not necessary closed.

By [54, Proposition 1.7] every profinite graph Γ is an inverse limit of finite quotient graphs of Γ.

For a profinite space X that is the inverse limit of finite discrete spaces X_j, $[\![\widehat{\mathbb{Z}}X]\!]$ is the inverse limit of $[\widehat{\mathbb{Z}}X_j]$, where $[\widehat{\mathbb{Z}}X_j]$ is the free $\widehat{\mathbb{Z}}$–module with basis X_j. For a pointed profinite space $(X, *)$ that is the inverse limit of pointed finite discrete spaces $(X_j, *)$, $[\![\widehat{\mathbb{Z}}(X, *)]\!]$ is the inverse limit of $[\widehat{\mathbb{Z}}(X_j, *)]$, where $[\widehat{\mathbb{Z}}(X_j, *)]$ is the $\widehat{\mathbb{Z}}$–module with basis $X_j \setminus \{*\}$ [42, Chapter 5.2].

For a profinite graph Γ define the pointed space $(E^*(\Gamma), *)$ as $\Gamma/V(\Gamma)$ with the image of $V(\Gamma)$ as a distinguished point $*$, and denote the image of $e \in E(\Gamma)$ by $\bar{e}$. By definition a profinite tree Γ is a profinite graph with a short exact sequence

$$0 \to [\![\widehat{\mathbb{Z}}(E^*(\Gamma), *)]\!] \xrightarrow{\delta} [\![\widehat{\mathbb{Z}}V(\Gamma)]\!] \xrightarrow{\epsilon} \widehat{\mathbb{Z}} \to 0,$$

where $\delta(\bar{e}) = d_1(e) - d_0(e)$ for every $e \in E(\Gamma)$ and $\epsilon(v) = 1$ for every $v \in V(\Gamma)$. If v and w are elements of a profinite tree T, we denote by $[v, w]$ the smallest profinite subtree of T containing v and w and call it geodesic.

By definition a profinite group G acts on a profinite graph Γ if we have a continuous action of G on the profinite space Γ that commutes with the maps d_0 and d_1.

If a profinite group G acts on a profinite tree T then by [52, Lemma 1.5] there exists a minimal G–invariant subtree D of T (note that in the classical Bass–Serre theory there is another possibility, namely the group can have an invariant end; this possibility is missing in the profinite case because of the compactness of T); moreover D is unique if it is not a vertex. If D is finite then it is a vertex and so G stabilizes a vertex. Therefore, if G does not stabilize a vertex, D is infinite.

We state now the general Tits alternative-type result from [52] that is the key result for our use.

Theorem 5.2 [52] *Let H be a profinite group acting on a profinite tree T. Suppose H does not possess a nonabelian free pro-p subgroup for every prime p. Then either H stabilizes a vertex or there exists a unique infinite minimal H–invariant subtree D of T such that the quotient group $L = H/K$ modulo the kernel of the action on D is soluble and isomorphic to one of the following groups:*

(1) $\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$, *where π and ρ are disjoint sets of primes.*

(2)   *A profinite dihedral group $\mathbb{Z}_\pi \rtimes C_2$.*

(3)   *A profinite Frobenius group $\mathbb{Z}_\pi \rtimes C_n$, ie the order of every prime divisor p of n divides $q - 1$ for some $q \in \pi$ and the centralizers of nonidentity elements of C_n coincide with C_n.*

Note that the group of automorphisms $\mathrm{Aut}(\mathbb{Z}_\pi)$ coincides with the group of units of the ring $\mathbb{Z}_\pi = \prod_{p \in \pi} \mathbb{Z}_p$ and so is the direct product $\prod_{p \in \pi} \mathbb{Z}_p^*$ of groups of units of $\mathbb{Z}_p$. Note also that $\mathbb{Z}_p^* \cong \mathbb{Z}_p \times C_{p-1}$ for $p \neq 2$ and $\mathbb{Z}_2^* \cong \mathbb{Z}_2 \times C_2$.

Remark 5.3 If in the notation of Theorem 5.2 H acts k–acylindrically for some natural k (see Definition 7.2 below) then $K = 1$. If in addition H is torsion-free we have just the first case of a projective soluble group $H \cong \mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$, where π and ρ are disjoint sets of primes. In this case any torsion-free profinite group containing H as an open subgroup has a similar structure.

6   Graphs of groups and their profinite analogues

In this section, we describe the basic theory of profinite graphs of groups in the case when underlying graph is finite (see [54, Section 3]). We shall need only this case here; in the general case the definitions and terminology are much more involved (see [55]).

Let Δ be a connected finite graph. A *graph of groups* $(\mathcal{G}, \Delta)$ over Δ assigns a group $\mathcal{G}(m)$ to each $m \in \Delta$, and monomorphisms $\partial_i \colon \mathcal{G}(e) \to \mathcal{G}(d_i(e))$ for each edge $e \in E(\Delta)$. If each $\mathcal{G}(m)$ is a profinite group and the monomorphisms ∂_i are continuous, we say that $(\mathcal{G}, \Delta)$ is a *graph of profinite groups*.

The *abstract fundamental group*

$$\Pi^{\mathrm{abs}} = \Pi_1^{\mathrm{abs}}(\mathcal{G}, \Delta)$$

of the graph of groups $(\mathcal{G}, \Delta)$ can be defined by means of a universal property. Fix a maximal subtree T of Δ. Then Π^{abs} is an abstract group equipped with a collection of homomorphisms

$$\nu_m \colon \mathcal{G}(m) \to \Pi^{\mathrm{abs}} \quad (m \in \Delta),$$

and a map $E(\Delta) \to \Pi^{\text{abs}}$, denoted $e \mapsto t_e$ $(e \in E(\Delta))$, such that $t_e = 1$ if $e \in E(T)$, and

$$(v_{d_0(e)}\partial_0)(x) = t_e(v_{d_1(e)}\partial_1)(x)t_e^{-1} \quad \text{for all } x \in \mathcal{G}(e), \ e \in E(\Delta).$$
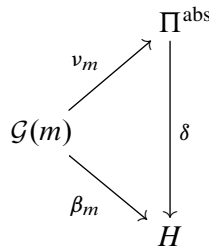
The defining universal property of Π^{abs} is then as follows.

If one has the following data:

- H an abstract group;

- $\beta_m \colon \mathcal{G}(m) \to H$ $(m \in \Delta)$ a collection of homomorphisms; and

- a map $e \mapsto s_e \in H$ $(e \in E(\Delta))$, with $s_e = 1$ if $e \in E(T)$, satisfying

$$(\beta_{d_0(e)}\partial_0)(x) = s_e(\beta_{d_1(e)}\partial_1)(x)s_e^{-1} \quad \text{for all } x \in \mathcal{G}(e), \ e \in E(\Delta);$$

then there exists a unique homomorphism $\delta \colon \Pi^{\text{abs}} \to H$ such that $\delta(t_e) = s_e$ $(e \in E(\Delta))$, and for each $m \in \Delta$ the diagram

commutes.

In [8, Chapter I, Definition 7.3 and Corollary 7.5] and in [45, Part I, Sections 5.1 and 5.2] the fundamental group Π^{abs} is defined explicitly in terms of generators and relations; there it is also proved that the definition given above is independent of the choice of the maximal subtree T, and furthermore it is proved that the homomorphisms $v_m \colon \mathcal{G}(m) \to \Pi^{\text{abs}}$ are injective for every $m \in \Delta$. We use the notation $\Pi^{\text{abs}}(m) = \text{Im}(v_m)$; so $\Pi^{\text{abs}}(m) \cong \mathcal{G}(m)$ for $m \in \Delta$.

The definition of the *profinite fundamental group*

$$\Pi = \Pi_1(\mathcal{G}, \Delta)$$

of a graph $(\mathcal{G}, \Delta)$ of profinite groups over a finite graph Δ is formally as above: one simply assumes that all the conditions take place in the category of profinite groups, ie all groups involved are profinite and all homomorphisms are assumed to be continuous. The explicit construction of Π is as follows; see [54, (3.3)]. Consider the family

$$\mathcal{N} = \{N \triangleleft_f \Pi_1^{\text{abs}}(\mathcal{G}, \Delta) \mid N \cap G(v) \triangleleft_o \mathcal{G}(v)\}$$

of all normal subgroups of finite index whose intersection with vertex groups are open. The profinite fundamental group $\Pi_1(\mathcal{G}, \Delta)$ is just

$$\Pi_1(\mathcal{G}, \Delta) = \varprojlim_{N \in \mathcal{N}} \Pi_1^{\mathrm{abs}}(\mathcal{G}, \Delta)/N,$$

which is to say the completion of $\Pi_1^{\mathrm{abs}}(\mathcal{G}, \Delta)$ with respect to the topology defined by $\mathcal{N}$.

There is one important difference with the abstract case: in the profinite setting, the canonical homomorphisms $\nu_m \colon \mathcal{G}(m) \to \Pi$ $(m \in \Delta)$ are not embeddings in general (see [42, Examples 9.2.9 and 9.2.10]). We use the notation $\Pi(m) = \mathrm{Im}(\nu_m)$ for $m \in \Delta$.

Associated with the graph of groups $(\mathcal{G}, \Delta)$ there is a corresponding *standard graph* (or universal covering graph)

$$S^{\mathrm{abs}} = \bigcup_{m \in \Delta} \Pi^{\mathrm{abs}}/\Pi^{\mathrm{abs}}(m).$$

The set of vertices of S^{abs} is given by

$$V(S^{\mathrm{abs}}) = \bigcup_{v \in V(\Delta)} \Pi^{\mathrm{abs}}/\Pi^{\mathrm{abs}}(v),$$

and the incidence maps of S^{abs} are given by the following formulae:

$$d_0(g\,\Pi^{\mathrm{abs}}(e)) = g\,\Pi^{\mathrm{abs}}(d_0(e));$$
$$d_1(g\,\Pi^{\mathrm{abs}}(e)) = gt_e\,\Pi^{\mathrm{abs}}(d_1(e)) \quad (e \in E(\Delta)).$$

In fact S^{abs} is a tree, usually called the *Bass–Serre tree* (see [8, Chapter I, Theorem 7.6] or [45, Part I, Section 5.3]). There is a natural left action of Π^{abs} on S^{abs}, and clearly $\Pi^{\mathrm{abs}} \backslash S^{\mathrm{abs}} = \Delta$.

Analogously, there is a profinite standard graph $S = \bigcup \Pi/\Pi(m)$ associated with a graph of profinite groups $(\mathcal{G}, \Delta)$, with the space of vertices and edges and with incidence maps defined as above. In fact, S is a profinite tree (see [54, Proposition 3.8]), and Π acts continuously on S with $\Pi \backslash S = \Delta$.

Given an abstract graph of groups $(\mathcal{G}, \Delta)$, there is a naturally associated graph of profinite groups $(\bar{\mathcal{G}}, \Delta)$. Under certain natural hypotheses, we shall further see that $(\bar{\mathcal{G}}, \Delta)$ is closely related to $(\mathcal{G}, \Delta)$.

First, suppose that $\Pi^{\mathrm{abs}} = \Pi^{\mathrm{abs}}(\mathcal{G}, \Delta)$ is residually finite, and let Π be the profinite completion of Π^{abs}. For each $m \in \Delta$, the profinite topology of Π^{abs} induces on $\Pi^{\mathrm{abs}}(m)$ a certain profinite topology (which is not necessarily its full profinite topology) and so on $\mathcal{G}(m)$. Define $\bar{\mathcal{G}}(m)$ to be the completion of $\mathcal{G}(m)$ with respect

to this topology. Then the monomorphisms $\partial_i \colon \mathcal{G}(e) \to \mathcal{G}(d_i(e))$ induce continuous monomorphisms which are again denoted by $\partial_i \colon \bar{\mathcal{G}}(e) \to \bar{\mathcal{G}}(d_i(e))$ $(i = 0, 1)$. We then have a graph $(\bar{\mathcal{G}}, \Delta)$ of profinite groups over Δ. The canonical injection $\mathcal{G}(m) \to \Pi^{\mathrm{abs}}$ induces an injection $\bar{\mathcal{G}}(m) \to \Pi$ $(m \in \Delta)$; furthermore, if we denote by $\Pi(m)$ the image of $\bar{\mathcal{G}}(m)$ on Π under this injection, then $\Pi(m) = \overline{\Pi^{\mathrm{abs}}(m)}$, the closure of $\Pi^{\mathrm{abs}}(m)$ in Π.

Clearly

$$\partial_0(g) = t_e \partial_1(g) t_e^{-1} \quad (g \in \bar{\mathcal{G}}(e), \ e \in E(\Delta))$$

in Π (there a certain abuse of notation here, as we are identifying $\bar{\mathcal{G}}(v)$ with its image in Π, and similarly we are denoting both the original elements t_e $(e \in E(\Delta))$ and their images in Π, which is justified since with our assumptions Π^{abs} injects into Π). Furthermore, one checks immediately the following result:

Proposition 6.1 *The profinite completion Π of Π^{abs} is the fundamental profinite group $\Pi_1(\bar{\mathcal{G}}, \Delta)$ of the graph of profinite groups $(\bar{\mathcal{G}}, \Delta)$. The canonical homomorphisms $\bar{\mathcal{G}}(m) \to \Pi = \Pi_1(\bar{\mathcal{G}}, \Delta)$ are injective $(m \in \Delta)$.*

We make a further assumption, namely that $(\mathcal{G}, \Delta)$ is *sufficient*.

Definition 6.2 A graph of groups $(\mathcal{G}, \Delta)$ is *sufficient* if
  (1)   the fundamental group $\Pi^{\mathrm{abs}} = \Pi_1^{\mathrm{abs}}(\mathcal{G}, \Delta)$ is residually finite; and
  (2) $\Pi^{\mathrm{abs}}(m)$ is closed in the profinite topology of Π^{abs} for each $m \in \Delta$ (or, equivalently, $\Pi(m) \cap \Pi^{\mathrm{abs}} = \Pi^{\mathrm{abs}}(m)$).

Consider the natural morphism of graphs

$$\varphi \colon S^{\mathrm{abs}} \to S,$$

which on vertices and edges is

$$g\Pi^{\mathrm{abs}}(v) \mapsto g\Pi(v), \quad g\Pi^{\mathrm{abs}}(e) \mapsto g\Pi(e) \quad (g \in \Pi^{\mathrm{abs}}, \ v \in V(\Delta), \ e \in E(\Delta)).$$

If $(\mathcal{G}, \Delta)$ is sufficient then φ is an injection of graphs; we think of S^{abs} as a subgraph of S. Moreover it is clear that S^{abs} is dense in S. We collect all of this in the following proposition:

Proposition 6.3 *Let $(\mathcal{G}, \Delta)$ be a sufficient graph of abstract groups over a finite connected graph Δ, and consider the graph $(\bar{\mathcal{G}}, \Delta)$ of profinite groups over Δ such that each $\bar{\mathcal{G}}(m)$ is the completion of $\mathcal{G}(m)$ with respect to the topology induced by the profinite topology of Π^{abs}. Then the standard tree $S^{\mathrm{abs}} = S^{\mathrm{abs}}(\mathcal{G}, \Delta)$ of the graph of groups $(\mathcal{G}, \Delta)$ is canonically embedded in the standard profinite tree $S = S(\bar{\mathcal{G}}, \Delta)$ of the graph of profinite groups $(\bar{\mathcal{G}}, \Delta)$, and S^{abs} is dense in S.*

The situation is particularly attractive if the profinite topology induced on the vertex and edge groups $\mathcal{G}(m)$ coincides with their intrinsic profinite topology.

Definition 6.4 Suppose that $(\mathcal{G}, \Delta)$ is sufficient and that Π^{abs} is the fundamental group. If, furthermore, for each $m \in \Delta$, each finite-index subgroup H of $\mathcal{G}(m)$ is closed in the profinite topology on Π^{abs}, then $(\mathcal{G}, \Delta)$ is called *efficient*.

In this case, the vertex and edge stabilizers in the profinite tree are conjugate to the profinite completions of the vertex and edge stabilizers in the standard tree. Results of Haglund and Wise imply that the graphs of groups arising in quasiconvex hierarchies of virtually special groups are efficient [14].

Definition 6.5 We say that $(\mathcal{G}, \Delta)$ is *of dihedral type* if Π^{abs} has a normal subgroup V contained in a vertex group such that Π^{abs}/N is either infinite dihedral or cyclic.

We shall need the following proposition for the proof of Theorem B:

Proposition 6.6 *Let* $\Pi^{\text{abs}} = \Pi_1^{\text{abs}}(\mathcal{G}, \Delta)$ *be the fundamental group of an efficient finite graph of groups* $(\mathcal{G}, \Delta)$ *not of dihedral type and* K *a closed normal subgroup of* $\widehat{\Pi} = \Pi_1(\bar{\mathcal{G}}, \Delta)$ *having no nonabelian free profinite subgroup. Then* K *is contained in the completion of a vertex group of* $\widehat{\Pi}$.

Proof Use a decomposition

$$\widehat{\Pi} = \varprojlim_{N} \Pi_1^{\text{abs}}(\mathcal{G}, \Delta)/N,$$

where N ranges over normal subgroups of finite index in Π^{abs}. For any such N define a graph of group $(\mathcal{G}_N, \Delta)$ replacing $\mathcal{G}(m)$ by $\mathcal{G}(m)N/N$ for all $m \in \Delta$. Note that

$$\Pi_1(\mathcal{G}_N, \Delta) = \widehat{\Pi_1^{\text{abs}}(\mathcal{G}_N, \Delta)}$$

and so contains a free profinite group as an open subgroup. It is clear also that it is a quotient group of $\widehat{\Pi}$ and

$$\widehat{\Pi} = \varprojlim_{N} \Pi_1(\mathcal{G}_N, \Delta).$$

We denote by K_N the image of K in $\Pi_1(\mathcal{G}_N, \Delta)$. Since $(\mathcal{G}, \Delta)$ is not of dihedral type, $(\mathcal{G}_N, \Delta)$ is not of dihedral type for some N as well. Hence

$$\widehat{\Pi_1^{\text{abs}}(\mathcal{G}_N, \Delta)}$$

contains an open nonabelian free profinite subgroup F. Then, by [42, Theorem 8.6.6], $K_N \cap F = 1$ and so K_N is finite. Hence, by [54, Theorem 3.10], K_N is contained in a vertex group of $\Pi_1(\mathcal{G}_N, \Delta)$. It follows then from the decomposition

$$\widehat{\Pi} = \varprojlim_N \Pi_1(\mathcal{G}_N, \Delta)$$

that K is contained in a vertex group of $\Pi_1(\mathcal{G}, \Delta)$, as needed. □

7 Virtually special hyperbolic groups

In this section we prove Theorem D. By Theorem 1.4, a hyperbolic, virtually special group Γ has a subgroup Γ_0 of finite index that admits a malnormal quasiconvex hierarchy. (Note that this statement does not need the most difficult parts of Theorem 1.4; it is a consequence of the fact that quasiconvex subgroups of hyperbolic groups have finite width, together with the fact that hyperplane stabilizers in virtually special groups are separable.) Our proof of Theorem D will be by induction on the length of the hierarchy for Γ_0.

Recall the following standard definition:

Definition 7.1 The action of a group Γ on a tree T is said to be *k–acylindrical*, for k a constant, if the set of fixed points of γ has diameter at most k whenever $\gamma \neq 1$.

We may make the analogous definition in the profinite setting.

Definition 7.2 The action of a profinite group $\widehat{\Gamma}$ on a profinite tree T is said to be *k–acylindrical*, for k a constant, if the set of fixed points of γ has diameter at most k whenever $\gamma \neq 1$.

A malnormal hierarchy for Γ_0 implies that Γ_0 has a 1–acylindrical action on a tree. This result carries over to the profinite setting.

Lemma 7.3 *Suppose that Π is the (profinite) fundamental group of a graph of profinite groups $(\bar{\mathcal{G}}, \Delta)$ with one edge e, and suppose that the edge group $\Pi(e)$ is malnormal in Π. Then the action of Π on the standard tree S is 1–acylindrical.*

Proof For an action to be 1–acylindrical means that the intersection of the stabilizers of any two distinct edges is trivial. The hypothesis that there is only one edge e means that any distinct pair of edges of S correspond to distinct left cosets $g\Pi(e)$ and $h\Pi(e)$, and their stabilizers are the conjugates $g\Pi(e)g^{-1}$ and $h\Pi(e)h^{-1}$. Malnormality implies that the intersection of these is trivial. □

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We are now ready to prove that closed subgroups of the profinite completions of virtually special hyperbolic groups satisfy a Tits alternative.

Proof of Theorem D Put $H_0 = \widehat{\Gamma}_0 \cap H$, where Γ_0 is the subgroup from Theorem 1.4. By Proposition 3.2, $\widehat{\Gamma}_0$ is torsion-free. We prove that $H_0 \cong \mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$, with $\pi \cap \rho = \varnothing$, by induction on the length of the malnormal hierarchy. By Theorem 3.3 and Lemma 7.3, $\widehat{\Gamma}_0$ acts 1–acylindrically on a profinite tree and therefore so does H_0. By the inductive hypothesis, the vertex stabilizers in H_0 have the claimed structure. If H_0 stabilizes a vertex by the induction hypothesis we are done. Otherwise, by Remark 5.3, H_0 has the claimed structure. $\square$

Theorem A follows as a quick consequence:

Proof of Theorem A By Theorem 1.8, if M is a closed hyperbolic 3–manifold then $\pi_1 M$ is virtually special, and hence Theorem D applies to show that there are no $\widehat{\mathbb{Z}}^2$ subgroups. Conversely, if $\widehat{\pi_1 M}$ contains no subgroups isomorphic to $\widehat{\mathbb{Z}}^2$ then, by Hamilton's result that abelian subgroups of 3–manifold groups are separable [17], $\pi_1 M$ contains no subgroups isomorphic to $\mathbb{Z}^2$ and so, by the hyperbolization theorem, M is hyperbolic. $\square$

8 Seifert fibred 3–manifolds

This section is devoted to characterizing certain features of Seifert fibred 3–manifolds using the profinite completions of their fundamental groups. We start by proving Theorem B, which shows that the profinite completion of the fundamental group distinguishes Seifert fibred 3–manifolds among all 3–manifolds.

Proof of Theorem B Suppose that M is closed, orientable and irreducible. If M is Seifert fibred then $\pi_1 M$ has a cyclic normal subgroup Z. Since every finitely generated subgroup of $\pi_1 M$ is separable, [43] $\widehat{\pi_1 M}$ has a procyclic normal subgroup, as claimed.

Suppose therefore that M is not Seifert fibred. If M is a Sol manifold, then $\pi_1 M \cong \mathbb{Z}^2 \rtimes \mathbb{Z}$, where the action of $\mathbb{Z} = \langle z \rangle$ on $\mathbb{Z}^2$ is Anosov, in the sense that no nontrivial power of z fixes a nontrivial element of $\mathbb{Z}^2$. This induces $\widehat{\pi_1 M} \cong \widehat{\mathbb{Z}}^2 \rtimes \widehat{\mathbb{Z}}$.

We next show that z does not fix a nontrivial element of $\widehat{\mathbb{Z}}^2$. Since $\widehat{\mathbb{Z}}^2 = \prod_p \mathbb{Z}_p^2$ it suffices to show that z does not fix a nontrivial vector in $\mathbb{Z}_p^2$. We show in fact that z does not fix a vector in $\mathbb{Q}_p^2$. Indeed, if it does then $\det(z - I) = 0$ and so z has a nonzero fixed point in $\mathbb{Q}^2$ and hence in $\mathbb{Z}^2$, a contradiction.

Thus any procyclic normal subgroup of $\widehat{\pi_1 M} \cong \widehat{\mathbb{Z}}^2 \rtimes \widehat{\mathbb{Z}}$ intersects $\widehat{\mathbb{Z}}^2$ trivially and so is in the kernel of the natural homomorphism $\langle z \rangle \to \mathrm{Aut}(\widehat{\mathbb{Z}}^2) \cong \mathrm{GL}_2(\widehat{\mathbb{Z}})$. Therefore to conclude that $\widehat{\pi_1 M}$ does not have procyclic normal subgroups in this case, it suffices to observe now that the closure of $\langle z \rangle$ in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ is isomorphic to $\widehat{\mathbb{Z}}$. To see this it is enough to show that all Sylow subgroups of $\langle z \rangle$ are infinite. As $\mathrm{GL}_2(\widehat{\mathbb{Z}}) = \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ and $\mathrm{GL}_2(\mathbb{Z})$ embeds in the virtually pro-p group $\mathrm{GL}_2(\mathbb{Z}_p)$, the closure of $\langle z \rangle$ in $\mathrm{GL}_2(\mathbb{Z}_p)$ has infinite Sylow p–subgroup for each p and therefore so does $\langle \widehat{z} \rangle$.

If M is nongeometric or hyperbolic, then $\widehat{\pi_1 M}$ acts acylindrically on a profinite tree. (In the hyperbolic case, this follows from the results of the previous section. In the nongeometric case, this follows from the results of [19; 48].) By Proposition 6.6, a procyclic normal subgroup $\widehat{\mathbb{Z}}$ of $\widehat{\pi_1 M}$ is contained in a vertex stabilizer, and hence acts trivially on the profinite tree [54, Theorem 2.12]. This would contradict the fact that the action is acylindrical unless $\widehat{\mathbb{Z}} = 1$. □

A compact Seifert fibred 3–manifold M can admit any of six different geometries. We next explain how these geometric structures are also distinguished by the profinite completion of the fundamental group. The reader is referred to [44] for details of the theory of Seifert fibred 3–manifolds. Consider the corresponding short exact sequence

$$1 \to Z \to \pi_1 M \to \pi_1 O \to 1,$$

where Z is cyclic and O is a compact, cone-type 2–orbifold. The geometric structure of M is determined by two invariants: the geometric structure of O; and the "Euler number" $e(M)$ [44, Table 4.1].

We first note that the geometry of O is detected by the profinite completion of $\pi_1 M$.

Lemma 8.1 *Let M be a compact, Seifert fibred 3–manifold, as above.*

(1) *O is spherical if and only if $\widehat{\pi_1 M}$ is virtually procyclic.*

(2) *O is Euclidean if and only if $\widehat{\pi_1 M}$ is virtually nilpotent.*

Proof This follows immediately from the following facts: O is spherical if and only if $\pi_1 M$ is virtually cyclic; O is Euclidean if and only if $\pi_1 M$ is virtually nilpotent; $\pi_1 M$ is residually finite. □

If M is not spherical then O is a good orbifold, and so is covered by a compact, orientable surface Σ. The short exact sequence for $\pi_1 M$ then pulls back along the covering map $\Sigma \to O$ to a short exact sequence

$$1 \to Z \to \pi_1 N \to \pi_1 \Sigma \to 1,$$

where N is a finite-sheeted covering space of M. We next explain how to the Euler number of M manifests itself in the structure of $\pi_1 M$. The following lemma can be deduced from [44]:

Lemma 8.2 *Let M be a compact, Seifert fibred 3–manifold.*

(1) *If O is spherical then $e(M) = 0$ if and only $\pi_1 M$ is infinite.*

(2) *If O is not spherical then $e(M) = 0$ if and only if, for some finite-sheeted surface cover Σ of O, the short exact sequence*

$$1 \to Z \to \pi_1 N \to \pi_1 \Sigma \to 1$$

splits.

Finally, we explain how the splitting of the short exact sequence

$$1 \to Z \to \pi_1 N \to \pi_1 \Sigma \to 1$$

is detected by the profinite completion.

Lemma 8.3 *Let Σ be a compact, orientable surface. Then a short exact sequence*

(1) $$1 \to \mathbb{Z} \to G \to \pi_1 \Sigma \to 1$$

splits if and only if the induced exact sequence of profinite completions

(2) $$1 \to \hat{\mathbb{Z}} \to \hat{G} \to \widehat{\pi_1 \Sigma} \to 1$$

splits.

Proof If (1) splits then clearly so does (2). Suppose therefore that (1) does not split.

The exact sequence $1 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 1$ induces a long exact sequence in cohomology, part of which is

$$\cdots \to H^2(\pi_1 \Sigma, \mathbb{Z}) \cong \mathbb{Z} \to H^2(\pi_1 \Sigma, \mathbb{Z}/n) \to H^1(\pi_1 \Sigma, \mathbb{Z}) \to \cdots .$$

But $H^2(\pi_1 \Sigma, \mathbb{Z}/n) \cong \mathbb{Z}/n$, whereas $H^1(\pi_1 \Sigma, \mathbb{Z})$ is isomorphic to the abelianization of $\pi_1 \Sigma$, which is torsion-free. Thus the induced map $H^2(\pi_1 \Sigma, \mathbb{Z}) \to H^2(\pi_1 \Sigma, \mathbb{Z}/n)$ is surjective. In particular, if (1) does not split then neither does the corresponding extension $1 \to \mathbb{Z}/n \to G \to \pi_1 \Sigma \to 1$, for some n.

Since $\pi_1 \Sigma$ is good, we have that $H^2(\widehat{\pi_1 \Sigma}, \mathbb{Z}/n) \to H^2(\pi_1 \Sigma, \mathbb{Z}/n)$, and so the corresponding sequence $1 \to \mathbb{Z}/n \to \hat{G} \to \widehat{\pi_1 \Sigma} \to 1$ does not split. Therefore, (2) does not split, as required.   □

We can summarize our results so far in the following theorem, which asserts that the profinite completion detects the geometric structure of a closed, orientable, irreducible 3–manifold.

Theorem 8.4 *Let M and N be closed, orientable, irreducible 3–manifolds, and suppose that $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$. Then M admits one of Thurston's eight geometric structures if and only if N does, and in this case both M and N admit the same geometric structure.*

Proof By Theorem A, M is hyperbolic if and only if N is. Since 3–manifold groups are residually finite and admit Sol geometry if and only if they are solvable but not virtually nilpotent, it follows that M admits Sol geometry if and only if N does. Finally, by Theorem B, M is Seifert fibred if and only if N is, and by Lemmas 8.1, 8.2 and 8.3, they are of the same geometric type. $\square$

9 3–manifold groups

In this section, we complete the proof of Theorem C. We start by addressing the cusped hyperbolic case, and then deal with the Seifert fibred case, before concluding with the irreducible nongeometric and reducible cases.

9A The cusped case

We need to generalize the results of the previous section to the case of noncompact hyperbolic 3–manifolds of finite volume. As in [50, Corollary 14.16], the strategy is to cut along a family of surfaces, in such a way that the pieces are hyperbolic without cusps. We next exhibit the family of surfaces that we will need (cf [50, Corollary 14.16]).

Theorem 9.1 *A hyperbolic 3–manifold M with cusps has a finite-sheeted covering space $N \to M$ that contains a disjoint family of connected, geometrically finite, incompressible subsurfaces $\{\Sigma_1, \ldots, \Sigma_n\}$ such that*

(1) *each cusp of N contains a boundary component of some Σ_i;*

(2) *each $\pi_1 \Sigma_i \subseteq \pi_1 N$ is relatively malnormal.*

Proof By the "half lives, half dies" lemma, there is a homomorphism $\pi_1 M \to \mathbb{Z}$ such that each peripheral subgroup maps nontrivially. One can realize this by a smooth map $M \to S^1$, and the preimage of a generic point is a (possibly disconnected) surface that cuts every cusp, which can be compressed to obtain an incompressible surface with the

same property. Passing to a finite-sheeted cover and applying an argument with the Thurston norm, one can further ensure that each component is geometrically finite.

Let Σ be such a component. Geometrical finiteness implies that there are finitely many double cosets $\pi_1 \Sigma g_i \pi_1 \Sigma$ such that the intersection $\pi_1 \Sigma^{g_i} \cap \pi_1 \Sigma$ is nonperipheral. By subgroup separability, we can pass to a finite-sheeted covering space to which Σ_i lifts but the elements g_i do not. Doing this for every component, and passing to a deeper regular covering space (still of finite index), we obtain the required covering space.   □

Wise used a similar result to prove that cusped hyperbolic manifolds are virtually compact special [50, Theorem 14.29]. We state this result here for future use.

Theorem 9.2 (Wise) *Let M be a cusped hyperbolic 3–manifold of finite volume. Then $\pi_1 M$ is virtually compact special.*

Cutting along the family of surfaces given by Theorem 9.1 produces a graph-of-groups decomposition $(\mathcal{G}, \Delta)$ for $\pi_1 N$, with the property that every vertex and edge group is hyperbolic and virtually special, and the stabilizer of any infinite subtree of the Bass–Serre tree is cyclic. Note also that $(\mathcal{G}, \Delta)$ is efficient. Passing to the corresponding graph of profinite groups $(\bar{\mathcal{G}}, \Delta)$, it follows from Theorem 4.2 that the stabilizer of any infinite subtree of the standard profinite tree is procyclic (possibly trivial).

We next state our main theorem in the cusped case.

Theorem 9.3 *Let M be a cusped hyperbolic 3–manifold, $\pi_1 M$ its fundamental group and $\widehat{\pi_1 M}$ its profinite completion. If H is a closed subgroup of $\widehat{\pi_1 M}$ that does not contain a free nonabelian pro-p subgroup for any p then H is isomorphic to $\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$. Furthermore, if H is not projective (ie $\pi \cap \rho \neq \varnothing$) then H is conjugate into the closure of a cusp subgroup of $\pi_1 M$.*

Proof Put $H_0 = \widehat{\pi_1 N} \cap H$, where N is the cover from Theorem 9.1. We first prove that H_0 has the claimed semidirect product structure. Consider the action of H_0 on the standard profinite tree. By Theorem D the stabilizers of vertices in H_0 have the claimed structure. Since the stabilizers of infinite subtrees are procyclic, the kernel K of the action is torsion-free procyclic.

Suppose first that K is nontrivial. The closure of any peripheral subgroup $\widehat{P}_i$ is malnormal in $\widehat{\pi_1 M}$ by Lemma 4.5, and since up to conjugation K is in $\widehat{P}_i$, we deduce that K is central in H and H is abelian. So H is a subgroup of $\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$ in this case.

Alternatively, if K is trivial then H_0 is isomorphic to a soluble projective group $\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$ by Theorem 5.2. Note now that $\widehat{\pi_1 M}$ is torsion free, since Γ is good (see Definition 3.1 and [13, Theorem 1.4] combined with Theorem 9.2). So H is torsion-free and hence, by Remark 5.3, a semidirect product of the claimed form.

Finally, if H is not projective then K is necessarily nontrivial by Theorem 5.2, and so the above argument shows that H is conjugate into the closure of some $\widehat{P}_i$, as claimed. □

9B Seifert fibred 3–manifolds

In this section we prove Theorem C when M is Seifert fibred. For many of the cases (if M has Euclidean, spherical, Nil or $S^2 \times \mathbb{R}$ geometry), $\pi_1 M$ is virtually soluble, and there is nothing to prove. The next proposition addresses the remaining geometries — $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\mathrm{SL}_2 \mathbb{R}}$ — in which the base orbifold is hyperbolic.

Proposition 9.4 *Suppose that M is any compact, Seifert fibred 3–manifold and that $\pi_1 M$ is not virtually soluble. If H is a closed subgroup of $\widehat{\pi_1 M}$ that does not contain a free nonabelian pro-p subgroup for any p then H is isomorphic to $(\mathbb{Z}_\sigma \times \mathbb{Z}_\pi) \rtimes \mathbb{Z}_\rho$, where $\pi \cap \rho = \varnothing$ and $\mathbb{Z}_\rho$ acts on $\mathbb{Z}_\sigma$ by inversion.*

Proof The fundamental group $G = \pi_1 M$ fits into a short exact sequence

$$1 \to Z \longrightarrow G \xrightarrow{p} \pi_1(O) \to 1,$$

where $Z = \langle z \rangle$ is cyclic and O is a cone-type 2–orbifold (see [44], for instance).

Since G is not virtually soluble, O is a hyperbolic orbifold, which is finitely covered by an orientable hyperbolic surface Σ; passing to a further cover if necessary, we may further assume that Σ admits an essential simple-closed γ which is not boundary parallel and that $\pi_1 \Sigma$ is torsion-free.

The exact sequence above induces an exact sequence

$$1 \to \widehat{Z} \longrightarrow \widehat{G} \xrightarrow{\widehat{p}} \widehat{\pi_1(O)} \to 1$$

of profinite completions. Then $\widehat{\pi_1 \Sigma}$ is naturally an open subgroup of $\widehat{\pi_1(O)}$, which pulls back to an open subgroup $\widehat{\pi_1 N}$ of $\widehat{G}$. Cutting Σ along γ induces a splitting of $\pi_1 \Sigma$ in which the edge group is malnormal; this in turn induces a graph of profinite groups for $\widehat{\pi_1 \Sigma}$ with procyclic malnormal edge group.

Since $\widehat{\pi_1 \Sigma}$ is torsion-free, by Theorem 5.2 $\widehat{p}(H) \cong \mathbb{Z}_{\bar{\pi}} \rtimes \mathbb{Z}_{\bar{\rho}}$ is a projective soluble group with $\bar{\pi} \cap \bar{\rho} = \varnothing$. Let δ be the set of primes dividing the orders of torsion elements

of $\widehat{\pi_1(O)}$. Then $\widehat{Z} = \mathbb{Z}_\sigma \times \mathbb{Z}_\delta$ and $H/\mathbb{Z}_\sigma$ is a torsion-free soluble group containing a projective group U. Hence $H/\mathbb{Z}_\sigma$ is soluble projective (projective groups are groups of cohomological dimension 1 and torsion-free overgroups of finite index preserve cohomological dimension — cf [42, Theorem 7.3.7]) and hence isomorphic to $\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$ for some $\pi \cap \rho = \varnothing$ (see [42, Exercise 7.7.8]). Thus $H \cong \mathbb{Z}_\sigma \rtimes H/\mathbb{Z}_\sigma \cong \mathbb{Z}_\sigma \rtimes (\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho)$. Since the action on $\widehat{Z}$ is induced from the action on Z, which is either trivial or by inversion, moving the prime 2 from π to ρ if necessary we can rewrite H as $H \cong (\mathbb{Z}_\sigma \times \mathbb{Z}_\pi) \rtimes \mathbb{Z}_\rho$. $\qquad\square$

9C Irreducible 3–manifolds

We next consider the general case in which M is an arbitrary closed, orientable, irreducible 3–manifold. The only remaining case to consider is when M has nontrivial torus decomposition. Our previous results give a 4–acylindrical action on a profinite tree, and the argument will now go through as before.

Proposition 9.5 *Let M be a compact, orientable, irreducible 3–manifold whose torus decomposition is nontrivial. If H is a closed subgroup of $\widehat{\pi_1 M}$ that does not contain a free nonabelian pro-p subgroup for any p then H is isomorphic to $(\mathbb{Z}_\sigma \times \mathbb{Z}_\pi) \rtimes \mathbb{Z}_\rho$, where $\mathbb{Z}_\rho$ acts on $\mathbb{Z}_\sigma$ by inversion and $\pi \cap \rho = \varnothing$.*

Proof By the results of [19; 48], $\widehat{\pi_1 M}$ acts 4–acylindrically on a profinite tree. Therefore so does H, and by Proposition 9.4 and Theorem 9.3 the stabilizers of vertices in H have the claimed structure. So if H stabilizes a vertex we are done. Otherwise, since the action is 4–acylindrical the kernel K is trivial. Then H is isomorphic to a soluble projective group $\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho$ by Remark 5.3 since $\widehat{\pi_1 M}$ is torsion-free. $\qquad\square$

Since we have already dealt with the Seifert fibred and hyperbolic cases, and the fundamental group of a Sol manifold is soluble, Theorem C follows whenever M is closed, orientable and irreducible.

9D Reducible 3–manifolds

To finish the proof of Theorem C we double to obtain a retract of a closed 3–manifold M. Now we argue as before, using the Grushko decomposition. After doubling, we may assume that M is closed. We then have

$$\pi_1 M \cong A_1 * \cdots * A_m * B_1 * \cdots * B_n * F,$$

where the A_i are all finite, the B_i are all as considered in the previous section, and F is free.

Proof of Theorem C   We consider the action of

$$\widehat{\pi_1 M} = \widehat{A}_1 \amalg \cdots \amalg \widehat{A}_m \amalg \widehat{B}_1 \amalg \cdots \amalg \widehat{B}_n \amalg \widehat{F}$$

on the standard profinite tree S associated to the free profinite product decomposition of $\widehat{\pi_1 M}$. If H stabilizes a vertex then H is conjugate to a subgroup of a free factor, so the structure of H is described in the previous results. Otherwise, it follows from Theorem 5.2 combined with Remark 5.3 that H is as in Theorem C(2). □

10   One-relator groups with torsion

One-relator groups with torsion are hyperbolic, and Wise proved further that they are virtually special. Theorem D therefore applies. However, by carefully examining the hierarchy that Wise used in his proof, we can improve the conclusions and prove Theorem E.

Proof of Theorem E   Every one-relator group G embeds naturally into a free product $G' = G * \mathbb{Z}$ which is an HNN extension $\mathrm{HNN}(L, M, t)$ of a simpler one-relator group H, where M is a free subgroup generated by subsets of the generators of the presentation of G (cf the Magnus–Moldavanskii construction in [50, Section 18b]). The hierarchy terminates at a virtually free group of the form $\mathbb{Z}/n * F$, where F is free. We use induction on the length of this hierarchy.

If G has torsion, this hierarchy is quasiconvex (see [50, Lemma 18.8]) and so (at each step of the hierarchy) the edge group K has finite width by Theorem 1.6. By Corollary 3.5, $\widehat{K}$ has finite width in $\widehat{G}'$. It follows that the stabilizer in $\widehat{G}'$ of any infinite subtree of the corresponding profinite standard tree S is trivial.

Consider the action of H on its minimal invariant profinite subtree. If H stabilizes a vertex, then by induction on the length of the hierarchy we are done. Otherwise, by Theorem 5.2, H is of one of the claimed forms. This finishes the proof. □

11   Pro-*p* subgroups

Finally, we study the pro-p subgroups of profinite groups acting acylindrically on profinite trees. The following general theorem is the main result of this section:

Theorem 11.1   *Let G be a finitely generated pro-p subgroup of a profinite group Γ acting 1–acylindrically on a profinite tree T. Then G is a free pro-p product of vertex stabilizers and a free pro-p group.*

We start with the following:

Lemma 11.2 *Let G be a profinite group acting on a profinite tree T and let e be an edge of T. Suppose the stabilizers of the vertices v, w of the edge e are prosoluble and do not coincide with G_e. Then the group $H = \langle G_v, G_w \rangle$ has a free prosoluble amalgamated product $G_v \amalg_{G_e} G_w$ as a quotient. In particular, H is not pro-p.*

Proof Let $D = H(e \cup v \cup w)$. Then for any open subgroup U of H the quotient graph $U \backslash D$ is a finite connected quotient graph of D and since $D = \varprojlim U \backslash D$ it is a connected profinite subgraph of T. Then D is a profinite tree [54, (1.15)]. Since H is generated by the vertex stabilizers, $H \backslash D$ is a profinite tree [54, Proposition 2.5] and so $\{v, w\}$ maps to $H \backslash D$ injectively, ie $H \backslash D$ is isomorphic to $v \cup e \cup w$. Let $f: H \to H_s$ be the maximal prosoluble quotient and let K be the kernel of f. Then $K \backslash D$ is a profinite tree [51, Proposition 2.9(b)] and in particular is a prosoluble, simply connected graph. Therefore by the prosoluble version of [55, Proposition 4.4] (it is remarked in Section 5.4 there that it is valid in the prosoluble case) H/K is isomorphic to the prosoluble fundamental group $\Pi_1^s(\mathcal{H}, H \backslash D)$ of the graph of groups $(\mathcal{H}, H \backslash D)$, where edge and vertex groups are the stabilizers of e and vertices v and w. This shows that H/K is a free prosoluble amalgamated product $G_v \amalg_{G_e}^s G_w$. $\square$

We are now ready to prove the main theorem of this section.

Proof of Theorem 11.1 First note that a nontrivial stabilizer of any edge e coincides with one of its vertex stabilizers G_v or G_w, say G_w, since otherwise by Lemma 11.2 the stabilizers of two vertices of e do not generate a pro-p group. Moreover, if e' is another edge with vertices w and u then $G_{e'} = \{1\}$ since, because of 1–acylindricity, $G_w = G_e \cap G_{e'} = 1$. It follows that the connected components of the abstract subgraphs of points with nontrivial stabilizers are at most stars. Let $\tilde{G}$ be the subgroup of G generated by all vertex stabilizers of G. Then, by [54, Theorem 2.6], $G/\tilde{G}$ is a free pro-p group and we denote its retract in G by F_0.

Now since G is finitely generated, its Frattini series $\Phi^n(G)$ is a fundamental system of neighbourhoods of 1. Let $\widetilde{\Phi^n(G)}$ be the subgroup of $\Phi^n(G)$ generated be all vertex stabilizers of $\Phi^n(G)$. Then

$$G_n = G/\widetilde{\Phi^n(G)}$$

acts on a profinite tree $T_n = \widetilde{\Phi^n(G)} \backslash T$ [54, Proposition 2.5] and $\Phi^n(G)/\widetilde{\Phi^n(G)}$ acts freely on T_n and therefore is free pro-p [54, Theorem 2.6]. Note that the vertex and edge stabilizers of G_n acting on T_n are finite epimorphic images of the corresponding vertex and edge stabilizers of G acting on T and so the images in T_n of edges of T

with trivial edge stabilizers have trivial stabilizers. Therefore the (abstract) connected components of the subgraph of points of T_n with nontrivial stabilizers in G_n are still at most stars. Indeed, let S_n be the image of a star S of T with nontrivial edge stabilizers and let e_n be an edge not in S_n having a vertex $v_n \in S_n$ with nontrivial stabilizer. Let v be a vertex of S whose image in T_n is v_n. Then $G_v \neq \{1\}$ and there exists an edge e incident to v whose image in T_n is e_n. But G_e is trivial and so G_{e_n} is trivial.

By [22, Lemma 8] a virtually free pro-p group has only finitely many finite subgroups up to conjugation and so T_n has only finitely many edges with nontrivial edge stabilizers up to translation. Therefore the subgraph Σ_n of points with nontrivial edge stabilizers is closed in T_n, ie is a profinite subgraph (forest) of T_n. Collapsing all connected components (stars) of Σ_n in T_n, by [53, Proposition, page 486] we get a pro-p tree T_n on which G_n acts with trivial edge stabilizers, so, by [24, Proposition 2.4], G_n is a free pro-p product

$$G_n = G/\widetilde{\Phi^n(G)} = \left(\coprod_v G_{nv} \right) \amalg F_{0n}$$

of representatives (chosen arbitrarily) of the nontrivial vertex stabilizers and of the isomorphic image F_{0n} of F_0 in G_n.

Since $G = \varprojlim G_n$ and G is finitely generated, by choosing n large enough we may assume that the number of free factors is the same for every $m > n$, ie v ranges over a finite set V. By [41, Theorem 4.2] every finite subgroup of a free pro-p product is conjugate to a subgroup of a free factor. Therefore, the free factors of G_{m+1} are mapped onto the free factors of G_m up to conjugation. But in a free pro-p product decomposition replacing any free factor by its conjugate does not change the group. So, starting from n, we can inductively choose G_{m+1v} in such a way that its image in G_m is G_{mv}. Let G_v be the inverse limit of G_{mv}. Then, by [42, Lemma 9.1.5], $G = \coprod_{v \in V} G_v \amalg F_0$.

It remains to observe that G_{mv} is a stabilizer of a vertex in T_m so the set of fixed points $T^{G_{mv}}$ is nonempty and closed [41, Theorem 3.7]. Therefore, G_v is the stabilizer of a nonempty set of vertices $\varprojlim T^{G_{mv}}$. This finishes the proof. $\qquad\square$

Using Theorem 11.1 we can precisely describe the finitely generated pro-p subgroups of the profinite completions of torsion-free hyperbolic virtually special groups.

Proof of Theorem F By Theorems 1.4 and 3.3 and Lemma 7.3, G has a subgroup of finite index Γ_0 whose profinite completion acts 1–acylindrically on a profinite tree. Then by Theorem 11.1 $H \cap \widehat{\Gamma}_0$ is a free pro-p product of vertex stabilizers and a free pro-p group. By induction on the hierarchy vertex stabilizers are free pro-p, therefore so is $H \cap \widehat{\Gamma}_0$. Finally by Serre's theorem (see [42, Theorem 7.3.7]) H is free pro-p. $\qquad\square$

Using the main result of [23] that describes finitely generated virtually free pro-p groups, we can deduce a description of the finitely generated pro-p subgroups of the profinite completion of a hyperbolic virtually special group.

Corollary 11.3 *Let G be a hyperbolic, virtually special group. Any finitely generated pro-p subgroup H of $\hat{G}$ is the fundamental pro-p group of a finite graph of finite p–groups.*

Proof The group G contains a virtually special, torsion-free subgroup of finite index U. Hence $H \cap \hat{U}$ is free pro-p. The result then follows from [23, Theorem 1.1]. □

Of course, Corollary 11.3 applies to one-relator groups with torsion by the work of Wise [50]. However, a careful analysis gives a more refined classification in that case.

Theorem 11.4 *Let G be a one-relator group with torsion and H a finitely generated pro-p subgroup of $\hat{G}$. Then H is a free pro-p product of finite cyclic p–groups and a free pro-p group.*

Proof The group G contains a virtually special, torsion-free subgroup of finite index U. Hence every finitely generated pro-p subgroup of $\hat{U}$ is free pro-p.

Now we show that centralizers of torsion elements of $\hat{G}$ are finite. First observe that all torsion elements of $\hat{G}$ are conjugate to elements of G (see [4, Theorem 2.1]). But, by [32, Theorem 1.1] combined with [31, Proposition 3.2], the centralizer of any element of G is dense in the centralizer of this element in $\hat{G}$. Thus, since the centralizer of any torsion element of G is finite [35, Theorem 2], the centralizers of torsion elements in $\hat{G}$ are finite as well.

So any finitely generated pro-p subgroup of $\hat{G}$ is virtually free and has finite centralizers of torsion elements. Therefore by [22, Theorem 1] it has the claimed structure. □

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