

Qualitative aspects of counting real rational curves on real K3 surfaces

VIATCHESLAV KHARLAMOV
RAREȘ RĂSDEACONU

We study qualitative aspects of the Welschinger-like \mathbb{Z} -valued count of real rational curves on primitively polarized real K3 surfaces. In particular, we prove that, with respect to the degree of the polarization, at logarithmic scale, the rate of growth of the number of such real rational curves is, up to a constant factor, the rate of growth of the number of complex rational curves. We indicate a few instances when the lower bound for the number of real rational curves provided by our count is sharp. In addition, we exhibit various congruences between real and complex counts.

14N99; 14P99, 14J28

“В цифрах есть нечто, чего в словах, даже крикнув их, нет.”
Иосиф Бродский, Полдень в комнате.^(†)

Introduction

The discovery by J-Y Welschinger [23] of a deformation-invariant \mathbb{Z} -valued count of real rational curves interpolating real collections of points on a real rational surface has led to an affirmative response to the long-standing problem of existence of real solutions in this enumerative problem. Moreover, the lower bound on the number of real solutions provided by the Welschinger invariants has been so powerful that it allowed Itenberg, Kharlamov and Shustin [11] to uncover a remarkable new *phenomenon of abundance*: in the logarithmic scale, when the degree of curves is growing, the number of real solutions happens to be of about the same growth rate as the number of complex ones. Later on, similar abundance phenomena were observed in a few other, even more classical, enumerative problems, like enumerating linear subspaces on projective hypersurfaces (see Finashin and Kharlamov [8] and references therein). All this motivates further natural questions, which are essential for applications: *what are the asymptotic and arithmetical properties of the lower bounds provided by such an invariant \mathbb{Z} -valued count; how nontrivial and sharp are these lower bounds?*

(†) “Numbers have something that words, even if yelled, don’t have.” Joseph Brodsky, *Noon in a Room*.

A response to these questions requires a comparison with the behavior of the numbers of solutions over the complex field, which in the above-mentioned problems are given by some Gromov–Witten and Schubert numbers, respectively. To our knowledge, the corresponding aspects of the complex enumerative algebraic geometry are rarely treated in the literature; for related information we refer the interested reader to Di Francesco and Itzykson [6], Grünberg and Moree [9] and Itenberg, Kharlamov and Shustin [12]. The sharpness of Welschinger lower bounds is also little studied; here, we can cite only Welschinger [24].

In our previous paper [15], we considered the problem of counting real rational curves on primitively polarized real K3 surfaces, introduced an appropriate invariant \mathbb{Z} -valued count and expressed the answer in a closed form, which can be viewed as a real version of the Yau–Zaslow formula (see Section 1 below). Our aim here is to show that thus-obtained invariant lower bounds have similar peculiar asymptotic and arithmetic properties as those that were observed in the previously studied real enumerative problems, and to indicate some instances where the lower bounds are sharp.

The note is organized as follows. First of all we recall the precise statement of the real version of the Yau–Zaslow formula. Then, in Section 2.1, we start the qualitative analysis by relating our formula with the Dedekind eta function and use one of Jacobi identities to establish some positivity property of the Welschinger invariants. In Section 2.2 we apply results of Hardy and Ramanujan [10] and Uspensky [21] on the asymptotic behavior of the number of partitions to determine the asymptotic behavior of the Welschinger invariants in the logarithmic scale and to exhibit an abundance phenomenon. Section 2.3 is devoted to the comparison modulo 2, 3, 4 and 8 of our Welschinger-type invariants, with the corresponding reduced Gromov–Witten invariants in the complex case, computed by the Yau–Zaslow formula. Finally, in Section 2.4 we apply Kulikov’s type I and II degenerations [17] to establish the sharpness of our lower bounds for certain real deformation types of real K3 surfaces. Section 3 contains a couple of concluding remarks. Some numerical data collected to illustrate the results obtained is shown in the table in the appendix.

1 Real version of the Yau–Zaslow formula

Let X be a generic real K3 surface admitting a complete real g -dimensional linear system of curves of genus g . If $g \geq 2$, assume, in addition, that X is of Picard rank 1 and the curves in the linear system belong to a primitive divisor class. Let c_g denote the number of complex rational curves in this linear system, and $w_g = n_+ - n_-$ the number of real rational curves in the same linear system counted with Welschinger

sign; that is, with the sign + if the number of real solitary points is even and with the sign – otherwise (recall that, by Chen’s theorem [3, Theorem 1.1], the genericity assumption ensures that all the rational curves in our linear system are nodal).

The numbers c_g depend only on g and not on a specific choice of the surface X , and obey the Yau–Zaslow formula [25]

$$(1-1) \quad \sum_{g \geq 0} c_g q^g = \prod_{s \geq 1} \frac{1}{(1 - q^s)^{e_{\mathbb{C}}}},$$

where $e_{\mathbb{C}} = 24$ is the Euler characteristic of X . As we proved in [15], w_g depends only on the Euler characteristic $e_{\mathbb{R}}$ of the real part $X_{\mathbb{R}}$ of X , and for fixed $e_{\mathbb{R}}$ the generating function for w_g is

$$(1-2) \quad \sum_{g \geq 0} w_g q^g = \prod_{r \geq 1} \frac{1}{(1 + q^r)^{e_{\mathbb{R}}}} \prod_{s \geq 1} \frac{1}{(1 - q^{2s})^{(e_{\mathbb{C}} - e_{\mathbb{R}})/2}}.$$

Note that $e_{\mathbb{R}}$ is always even and its values vary between -18 and 20 (for references and more details on the topology of real K3 surfaces, see [5]).

2 Analysis of the real version

2.1 Positivity

Theorem 2.1 *For each fixed $e_{\mathbb{R}} < 0$, all w_g are positive and form a strictly increasing sequence $|e_{\mathbb{R}}| = w_1 < w_2 < w_3 < \dots$. For $e_{\mathbb{R}} = 0$, all w_g with odd g are zero, while those with even g are positive and form a strictly increasing sequence $12 = w_2 < w_4 < \dots$. For each fixed $e_{\mathbb{R}} > 0$, all the terms of the sequence $(-1)^g w_g$ are positive and form a strictly increasing sequence $e_{\mathbb{R}} = -w_1 < w_2 < -w_3 < \dots$.*

Proof To prove these statements in the case $e_{\mathbb{R}} \leq 0$ it is sufficient to notice that the second product power series in (1-2) has nonzero coefficients only in even degrees and each of these coefficients is positive, while the first product power series has all coefficients positive as soon as $e_{\mathbb{R}} < 0$. The strict monotonicity claim follows immediately by noticing that multiplying positive power series whose coefficients form increasing sequences yields a positive series with coefficients in increasing sequence.

Assume now that $e_{\mathbb{R}} > 0$. First, we rewrite our formula (1-2) in terms of the Dedekind eta function

$$\eta(z) = e^{\pi iz/12} \prod_{n \geq 1} (1 - e^{2\pi inz}) = q^{1/24} \prod_{n \geq 1} (1 - q^n),$$

and of the modular discriminant

$$\Delta(z) = \eta^{24}(z) = q \prod_{n \geq 1} (1 - q^n)^{24},$$

where z is in the upper half-plane and $q = e^{2\pi iz}$. In this notation, formula (1-2) can be written as

$$(2-1) \quad \sum_{g \geq 0} w_g q^g = \frac{q}{\sqrt{\Delta(2z)}} \left(\frac{\eta^2(z)}{\eta(2z)} \right)^{\frac{e_{\mathbb{R}}}{2}}$$

For the above eta quotient there is the following remarkable Gauss identity (see [16, Corollary 1.3], for example):

$$(2-2) \quad \frac{\eta^2(z)}{\eta(2z)} = 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2}.$$

To finish the proof it is sufficient now to notice that the property to have nonpositive coefficients in odd degrees and nonnegative coefficients in even degrees is preserved under multiplication of power series (with this property) and that the power series

$$(2-3) \quad \frac{q}{\sqrt{\Delta(2z)}} = \prod_{n \geq 1} (1 - q^{2n})^{-12}$$

has positive coefficients in each even degree and zero coefficients in each odd degree. Another possible approach is to replace q by $-q$, which makes all the power series involved have positive coefficients, and then to apply the same arguments as above in the case $e_{\mathbb{R}} < 0$. This argument yields the strict monotonicity claim. \square

2.2 Asymptopia

Lemma 2.2 *Let $\sum a_n q^n$ and $\sum b_n q^n$ be two power series with positive coefficients, and $\sum p_n q^n = (\sum a_n q^n)(\sum b_n q^n)$ the product power series. If, at a logarithmic scale, the coefficients a_n and b_n have the asymptotic behavior $\log a_n \sim (an)^\alpha$ and $\log b_n \sim (bn)^\alpha$ for some real constant $0 < \alpha < 1$, then $\log p_n \sim (cn)^\alpha$, where $c = (a^{\alpha/(1-\alpha)} + b^{\alpha/(1-\alpha)})^{(1-\alpha)/\alpha}$.*

Proof The result follows from

$$\max_{0 \leq k \leq n} \log a_k b_{n-k} \leq \log p_n \leq \log n + \max_{0 \leq k \leq n} \log a_k b_{n-k}$$

and

$$\max_{0 \leq k \leq n} \log a_k b_{n-k} \sim ((a^{\alpha/(1-\alpha)} + b^{\alpha/(1-\alpha)})^{(1-\alpha)/\alpha} n)^\alpha.$$

To derive the latter relation it is sufficient to bound, from above and from below, the sequences a_n and b_n by sequences of the form $\exp(k_{\pm} + (1 \pm \epsilon)(an)^{\alpha})$ and $\exp(k_{\pm} + (1 \pm \epsilon)(bn)^{\alpha})$, respectively, and then let ϵ go to 0. The bounds needed here can be found by using the fact that $((a^{\alpha/(1-\alpha)} + b^{\alpha/(1-\alpha)})^{(1-\alpha)/\alpha}n)^{\alpha}$ is exactly the maximal value of the function $(at)^{\alpha} + (b(n-t))^{\alpha}$ on the interval $[0, n]$. \square

Theorem 2.3 *The following asymptotic relations hold:*

(i) *If $e_{\mathbb{R}} < 0$, then*

$$\log w_n \sim \pi \sqrt{\frac{4(e_{\mathbb{C}} - 3e_{\mathbb{R}})}{e_{\mathbb{C}}}} \cdot n \sim \sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{4e_{\mathbb{C}}}} \log c_n.$$

(ii) *If $e_{\mathbb{R}} > 0$, then*

$$\log |w_n| \sim 2\pi \sqrt{n} \sim \frac{1}{2} \log c_n.$$

(iii) *If $e_{\mathbb{R}} = 0$, then*

$$\log w_{2n} \sim \frac{1}{2} \log c_{2n}.$$

Proof The Hardy–Ramanujan and Uspensky results [10; 21] on the asymptotic behavior of the coefficients in the power series expansions

$$\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}} = \sum_{n \geq 1} Q(n)q^n \quad \text{and} \quad \prod_{n \geq 1} \frac{1}{1 - q^n} = \sum_{n \geq 1} P(n)q^n$$

give us the following equivalence relations:

$$(2-4) \quad Q(n) \sim \frac{e^{\pi \sqrt{n/3}}}{4 \cdot 3^{1/4} \cdot n^{3/4}}$$

and

$$(2-5) \quad P(n) \sim \frac{e^{\pi \sqrt{2n/3}}}{4n \sqrt{3}}.$$

Thus, in the logarithmic scale, $\log P(n) \sim \log Q(2n)$ and $\log Q(n) \sim \pi \sqrt{n/3}$. Then Lemma 2.2 implies that $\log c_n \sim \pi \sqrt{e_{\mathbb{C}} \cdot 2n/3} = 4\pi \sqrt{n}$.

If $e_{\mathbb{R}} < 0$, then the coefficients u_r in the power expansion $\sum u_r q^r$ of the first product in the formula (1-2) are positive, and, according to Lemma 2.2 and formula (2-4), they grow in the log scale as $\pi \sqrt{-e_{\mathbb{R}} r/3}$. The coefficients v_r in the power expansion $\sum v_r q^r$ of the second product are vanishing in odd degree and they are positive in even degree. Lemma 2.2 and formula (2-5) imply that the coefficients v_{2s} grow in the log scale as $\pi \sqrt{(e_{\mathbb{C}} - e_{\mathbb{R}})/2 \cdot 2s/3}$. It follows that

$$\log w_{2n} \sim \pi \sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{2} \cdot \frac{2n}{3}}.$$

Wherefrom, by monotonicity of the sequence w_n ,

$$\log w_n \sim \pi \sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{2} \cdot \frac{n}{3}} \sim \sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{4e_{\mathbb{C}}}} \log c_n.$$

If $e_{\mathbb{R}} \geq 0$, then the proof is similar. This time we start from formula (2-1). Notice that, according to formula (2-3) and Lemma 2.2, the coefficients of the first factor in the odd degrees are zero, while the coefficients in degree $2n$ grow in the log scale as $\pi \sqrt{12 \cdot 2n/3} \sim \frac{1}{2} \log c_{2n}$. Thus, it remains to notice that, due to formula (2-2), the coefficients of the second factor have polynomial growth, and then, if $e_{\mathbb{R}} > 0$, to apply the monotonicity. \square

Corollary 2.4 *The number $r_g(X)$ of real rational curves in the divisor class of the primitive polarization of X satisfies the following bounds:*

$$\phi(g) = |w_g| \leq r_g(X) \leq c_g = \psi(g),$$

where

$$\log \psi(g) = 4\pi \sqrt{g} + o(\sqrt{g}),$$

and, for a fixed $e_{\mathbb{R}} \neq 0$,

$$\log \phi(g) = 4\pi \rho \sqrt{g} + o(\sqrt{g})$$

with $\rho = \frac{1}{2}$ if $e_{\mathbb{R}} > 0$ and $\rho = \sqrt{(e_{\mathbb{C}} - 3e_{\mathbb{R}})/(4e_{\mathbb{C}})}$ if $e_{\mathbb{R}} < 0$, while, for $e_{\mathbb{R}} = 0$,

$$\log \phi(2g) = 2\pi \sqrt{2g} + o(\sqrt{g}).$$

Similar abundance of real solutions phenomena are observed in several other real enumerative problems; see [11; 13; 1; 7; 8]. There, as for $e_{\mathbb{R}} \geq 0$ in the above corollary, a magic factor $\frac{1}{2}$ occurs in quite a few cases.

2.3 Congruences

The modularity of the generating series for the real and complex counting described in (1-1) and (1-2) allows us to exhibit noteworthy congruences that go much further than $w_g \equiv c_g \pmod{2}$, which is straightforward from definitions.

Theorem 2.5 *The following congruences hold:*

- (i) *We have $w_g \equiv c_g \pmod{2}$ for any $g \geq 1$, and $w_g \equiv c_g \equiv 0 \pmod{2}$ for every g with $g \not\equiv 0 \pmod{8}$.*
- (ii) *If $e_{\mathbb{R}} \equiv 0 \pmod{4}$ then $w_g \equiv c_g \pmod{4}$ for any $g \geq 1$, and if in addition $g \not\equiv 0 \pmod{4}$ then $w_g \equiv c_g \equiv 0 \pmod{4}$.*

(iii) If $e_{\mathbb{R}} \equiv 0 \pmod{8}$ then $w_g \equiv c_g \pmod{8}$ for any $g \geq 1$, and if in addition $g \not\equiv 0 \pmod{2}$ then $w_g \equiv c_g \equiv 0 \pmod{8}$.

(iv) If $e_{\mathbb{R}} \equiv 0 \pmod{3}$ and $g \not\equiv 0 \pmod{3}$ then $w_g \equiv c_g \equiv 0 \pmod{3}$.

Proof As a consequence of (2-2) and (2-3) we get

$$(2-6) \quad \sum_{g \geq 0} w_g q^g \equiv \prod_{n \geq 1} \frac{1}{(1 - q^{2n})^{12}} \equiv \prod_{n \geq 1} \frac{1}{(1 - q^{8n})^3} \pmod{2}.$$

In particular, it implies that w_g is even for every g with $g \not\equiv 0 \pmod{8}$.

Likewise,

$$(2-7) \quad \sum_{g \geq 0} c_g q^g = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{24}} \equiv \prod_{n \geq 1} \frac{1}{(1 - q^{8n})^3} \equiv \sum_{g \geq 0} w_g q^g \pmod{2}.$$

Making the additional assumption that $e_{\mathbb{R}} \equiv 0 \pmod{4}$, we get, in a similar way,

$$\begin{aligned} \sum_{g \geq 0} w_g q^g &\equiv \left(1 + e_{\mathbb{R}} \left(\sum_{n=1}^{\infty} (-1)^n q^{n^2} \right) \right) \prod_{n \geq 1} \frac{1}{(1 - q^{2n})^{12}} \pmod{4} \\ &\equiv \prod_{n \geq 1} \frac{1}{(1 - q^{2n})^{12}} \pmod{4} \\ &\equiv \sum_{g \geq 0} c_g q^g \pmod{4}. \end{aligned}$$

We conclude that w_g and c_g are congruent modulo 4 for any g , and that they are divisible by 4 for each $g \geq 1$ with $g \not\equiv 0 \pmod{4}$, as follows from

$$(1 - q^{2n})^{12} \equiv (1 + 2q^{4n} + q^{8n})^3 \pmod{4}.$$

Assuming now that $e_{\mathbb{R}} \equiv 0 \pmod{8}$, the same proof as in the previous case yields

$$\sum_{g \geq 0} w_g q^g \equiv \prod_{n \geq 1} \frac{1}{(1 - q^{2n})^{12}} \equiv \sum_{g \geq 0} c_g q^g \pmod{8},$$

and so we conclude that $w_g \equiv c_g \pmod{8}$ for any $g \geq 1$. Moreover, since

$$(1 - q^{2n})^{12} \equiv (1 + 4q^{2n} + 6q^{4n} + 4q^{6n} + q^{8n})^3 \pmod{8},$$

we notice that $w_g \equiv c_g \equiv 0 \pmod{8}$ for any $g \equiv 1 \pmod{2}$.

If $e_{\mathbb{R}} \equiv 0 \pmod{3}$, we get, in a similar way,

$$\sum_{g \geq 0} w_g q^g \equiv \left(1 + \sum_{n=1}^{\infty} (-1)^{n+1} q^{3n^2} \right)^{\frac{e_{\mathbb{R}}}{6}} \prod_{n \geq 1} \frac{1}{(1 - q^{6n})^4} \pmod{3}$$

and

$$\sum_{g \geq 0} c_g q^g \equiv \prod_{n \geq 1} \frac{1}{(1 - q^{3n})^8} \pmod{3}.$$

We conclude that $w_g \equiv c_g \equiv 0 \pmod{3}$ for each $g \geq 1$ with $g \not\equiv 0 \pmod{3}$. \square

2.4 Sharpness

In [15, Section 5.2] we observed that the lower bound for the count of real rational curves on primitively polarized K3 surfaces given by the absolute value of w_g is sharp for any g as soon as this bound is restricted to surfaces without real points. The reasoning is simple: there is no real nodal rational curve in the corresponding linear system when g is odd, since such a curve would have at least one real point among its singular points, which is impossible; while, when g is even, the singular points of such a curve split into pairs of conjugate ones, and therefore the curve counts with positive Welschinger sign. Here we prove that the lower bound is optimal in a few more cases.

Theorem 2.6 *For any g , the lower bound on the number of real solutions by the absolute value of w_g is sharp for surfaces whose real locus is a torus or a pair of tori.*

Proof First, we treat the case of a pair of tori and g odd. Let $\pi: Y \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the double covering ramified along a real nonsingular curve of bidegree $(4, 4)$ without real points (compare [22; 20]), where $\mathbb{P}^1 \times \mathbb{P}^1$ is considered with the standard product real structure (ie the hyperboloid). Such a double covering carries two real structures that differ by the automorphism of the covering, and we pick the one for which the real locus is formed by two tori. We denote by F_1 and F_2 the generators of $\mathbb{P}^1 \times \mathbb{P}^1$. The linear system $|F_1 + nF_2|$ embeds $\mathbb{P}^1 \times \mathbb{P}^1$ into \mathbb{P}^{2n+1} , while its pull-back $|\pi^*(F_1 + nF_2)|$ provides a representation of Y as a hyperelliptic K3 surface in \mathbb{P}^{2n+1} . In such a representation the pull-back of hyperplane sections form a complete $2n+1$ -dimensional linear system of curves of genus $g = 2n + 1$. Finally, we take as X an embedding into \mathbb{P}^{2n+1} of a real K3 surface obtained by a small generic real perturbation of Y (its existence follows from the period space description; see [20], for example). Notice now that $Y_{\mathbb{R}}$ consists of a pair of disjoint tori which are noncontractible in $\mathbb{P}_{\mathbb{R}}^{2n+1}$, since the real locus of the starting ruled surface is noncontractible in $\mathbb{P}_{\mathbb{R}}^{2n+1}$. This implies that $X_{\mathbb{R}}$ consists of a pair of disjoint, noncontractible tori as well. Hence, every real hyperplane section of $X_{\mathbb{R}}$ has at least 2 components, and thus cannot be rational. This proves the sharpness claim, since according to formula (1-2) we have $w_g = 0$ for odd values of g whenever $e_{\mathbb{R}}$ is zero.

To treat the case of a pair of tori and g even, we consider a degeneration of a K3 to a double covering of the blown-up projective plane. Namely, we start from $\mathbb{P}^2(1)$, the

projective plane blown up at a real point, and consider a double covering $\varpi: Y \rightarrow \mathbb{P}^2(1)$ ramified along the proper transform of a one-nodal sextic with a real solitary point at the center of the blow-up and two ovals surrounding this solitary point. The standard real structure on $\mathbb{P}^2(1)$ lifts to two real structures on Y , and we pick the one for which the real locus consists of two tori. We embed now $\mathbb{P}^2(1)$ into \mathbb{P}^{2n} by the linear system $|E + nF|$, where E is the exceptional divisor and F stands for the straight lines through the center of the blow-up. As above, we take $X \subset \mathbb{P}^{2n}$ to be a generic small real perturbation of Y . The K3 surface Y carries a natural real elliptic fibration given by the pull-back of the pencil of lines through the center of the blow-up. Since the starting sextic admits no real tangents passing through the node, all the singular fibers of this elliptic fibration are imaginary. This implies that every real rational hyperplane section of X , which, as is known (compare [2, Proposition 4.1]), is a perturbation of the section $\varpi^{-1}(E)$, and a collection of — possibly multiple — singular fibers, has no real singular points. Therefore, all the inputs into w_g in such a construction are positive, wherefrom the sharpness for this other particular case: $X_{\mathbb{R}}$ is a pair of tori and g is even, $g = 2n$. Notice that in this case $w_g > 0$ and it grows fast, as discussed in Theorems 2.1 and 2.3, respectively.

In the case of one torus and g even, we start again from $\mathbb{P}^2(1)$, the projective plane blown up at a real point, and consider a double covering $\varpi: Y \rightarrow \mathbb{P}^2(1)$ ramified in a proper transform of a one-nodal sextic with a real solitary point at the center of the blow-up and, this time, no other real points. As a real structure on Y , we select that lift of the standard real structure on $\mathbb{P}^2(1)$ for which the real locus of Y is a torus. We embed $\mathbb{P}^2(1)$ into \mathbb{P}^{2n} by the linear system $|E + nF|$, where E is the exceptional divisor and F stands for the linear system of lines through the center of the blow-up. As above, we can assume that all the singular fibers of the associated elliptic fibration are imaginary. For that it is sufficient to make our sextic generic, since a generic nodal sextic has no double tangents passing through the node. Hence, we can argue as in the previous case, that is to use the pull-back $|\varpi^*(E + nF)|$ to represent the double covering under consideration by a hyperelliptic K3 surface in \mathbb{P}^{2n} , and then to take as X a generic small real perturbation of Y . Once more all the inputs in such a construction are positive. Wherefrom the sharpness for this other particular case: $X_{\mathbb{R}}$ is a torus and g is even, $g = 2n$. Notice again that in this case again $w_g > 0$ and it grows fast.

To construct sharp examples for the remaining case, a K3 surface in \mathbb{P}^{2n+1} with a torus as the real locus, we start from a suitable type II degeneration of such a K3 surface [17]. Namely, we start, as in the case of a pair of tori, from the same real rational ruled surface $\mathbb{P}^1 \times \mathbb{P}^1 = \Sigma_0 \subset \mathbb{P}^{2n+1}$ and consider an elliptic normal curve E without real points that is cut on this ruled surface by a real rational ruled surface Σ_2 having empty real locus

and intersecting Σ_0 transversally along E with $E^2 = -2$ on this Σ_2 . To construct an explicit example (compare [4, Lemma 1, page 643]) one can start from the elliptic curve $\mathbb{C}^2/(\mathbb{Z} + i\mathbb{Z})$ equipped with the real structure $z \mapsto \bar{z} + \frac{1}{2}$, embed it into \mathbb{P}^{2n+1} by means of the linear system $n(A + B) + (\frac{1}{2} + A) + (\frac{1}{2} + B) \sim (n + 1)(A + B)$, where $A = 0$ and $B = \frac{i}{2}$, and choose as a scroll $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ containing E , the scroll corresponding to the hyperelliptic involution defined by the divisor $A + C$ with $C = \frac{1}{2}$. Under such choices the scroll Σ_0 becomes real and has a torus as its real part, while the scroll corresponding to the divisor $A + B$ gives us a real Σ_2 with an empty real part, as required. After that, there remains to pick as X a small generic real perturbation of $\Sigma_0 \cup \Sigma_2$. The existence of a smooth such surface X is guaranteed by [4, Theorem 1, page 644]. The generic such surface X is primitively polarized [4, Theorem 2, page 646], and, as always due to the surjectivity of the period map, it can be deformed further to a surface generic in the sense of Chen's theorem [3], which guarantees that the rational curves in the primitive polarization are all nodal.

Under such a choice, X does not have any real rational hyperplane section. Indeed, if such a section exists, then, by the compactness of Kontsevich's space of stable curves, there would exist a real projective connected nodal curve Z of arithmetic genus 0 and a real regular map $f: Z \rightarrow \Sigma_0 \cup \Sigma_2$ that realizes a hyperplane section of $\Sigma_0 \cup \Sigma_2$. On the other hand, the hyperplane sections of Σ_0 form the linear system $|F_1 + nF_2|$, where F_1 and F_2 are generators of $\mathbb{P}^1 \times \mathbb{P}^1 = \Sigma_0$, while the hyperplane sections of Σ_2 form the linear system $|F'_1 + (n + 1)F'_2|$, where F'_1 stands for the (-2) -section and F'_2 for the fiber of Σ_2 . The (-2) -section and the standard generators, F_1 and F_2 , are complex conjugation-invariant. Hence, every real rational map $f: Z \rightarrow \Sigma_0 \cup \Sigma_2$ as above, which represents a hyperplane section of $\Sigma_0 \cup \Sigma_2$, should have in its source, Z , an irreducible real component of type $F_1 + aF_2$ and an irreducible real component of type $F'_1 + aF'_2$. Since Z is rational and connected, each two real components should be connected by a chain of real components. This is impossible for real components belonging one to Σ_0 and another to Σ_2 , since any two such components intersect only at pairs of complex conjugated points. This contradiction ends the proof. \square

3 Concluding remarks

3.1 On asymptotics

Hardy and Ramanujan have obtained not only an asymptotic approximation for coefficients $P(n)$ and $Q(n)$ (see formulas (2-4) and (2-5)), but also a full asymptotic expansion, which later was even improved by Rademacher up to a convergent series expression. These can be applied to get similar expansions for w_g . We have restricted

ourselves here to asymptotic approximations, since for our aim — to demonstrate the abundance phenomenon — it is not necessary to go further. Moreover, it would only obscure the presentation by a much heavier and lengthy analysis.

3.2 On congruences modulo 2

Table A below, which gives the values of w_g and c_g for $g \leq 20$, shows that these values are odd if $g = 8$ and 16 . Thus, it may give the impression that they should be odd for all $g \equiv 0 \pmod{8}$ (compare Section 2.3). In fact, the situation is much different.

Let $\{i_n\}_{n \geq 0}$ be the parity sequence given by $i_n \equiv w_{8n} \equiv c_{8n} \pmod{2}$ for $n \geq 0$.

Proposition 3.1 *The sequence $\{i_n\}_{n \geq 0}$ contains infinitely many 0's and infinitely many 1's.*

Proof As noticed in (2-6),

$$\sum_{n \geq 0} i_n q^{8n} \equiv \sum_{g \geq 0} w_g q^g \equiv \prod_{n \geq 1} \frac{1}{(1 - q^{8n})^3} \pmod{2}.$$

Moreover, we have

$$\prod_{n \geq 1} \frac{1}{(1 - q^{8n})^3} \equiv \prod_{n \geq 1} \frac{(1 - q^{16n})^2}{1 - q^{8n}} \prod_{n \geq 1} \frac{1}{(1 - q^{16n})^3} \pmod{2}.$$

Since

$$\prod_{n \geq 1} \frac{1}{(1 - q^{16n})^3} \equiv \prod_{n \geq 1} \frac{1}{(1 - q^n)^{48}} \equiv \left(\sum_{n \geq 0} i_n q^{8n} \right)^2 \pmod{2},$$

we obtain

$$(3-1) \quad \sum_{n \geq 0} i_n q^{8n} \equiv \prod_{n \geq 1} \frac{(1 - q^{16n})^2}{1 - q^{8n}} \sum_{n \geq 0} i_n q^{16n} \pmod{2}.$$

Furthermore, from the Jacobi identity [16, Corollary 1.4]

$$\prod_{n \geq 1} \frac{(1 - q^{16n})^2}{1 - q^{8n}} \equiv \prod_{n \geq 1} (1 - q^{8n})^3 \equiv \sum_{n=0}^{\infty} q^{4n(n+1)} \pmod{2},$$

we notice that the power series development of the term $\prod_{n \geq 1} (1 - q^{16n})^2 / (1 - q^{8n})$ in (3-1) contains infinitely many odd and infinitely many even coefficients and the gaps between odd coefficients are growing quadratically.

Suppose now that the sequence $\{i_n\}_{n \geq 0}$ contains finitely many 1's. That would imply (in arithmetic over $\mathbb{Z}/2$) the equality between a nonzero polynomial and a product of an infinite series with a polynomial, which is impossible. Furthermore, if we suppose now that the sequence $\{i_n\}_{n \geq 0}$ contains finitely many 0's, to get a contradiction it is sufficient to write $\sum_{n \geq 0} i_n q^{8n}$ as a sum of a polynomial with $\sum_{n \geq C} q^{8n}$, and to observe that the coefficients of $\sum_{n \geq 0} i_n q^{8n}$ in powers $4(4k + 1)(4k + 2)$ and $4(4k + 2)(4k + 3)$ have opposite parities for all k sufficiently large with respect to C , which is impossible. \square

3.3 Other congruences

In Section 2.3, we discussed congruences between the real and complex invariants, modulo primes and their powers that divide 24, ie congruences modulo 2, 3, 4 and 8. In fact, when the powers of 2 and 3 do not divide 24, interesting congruences are still expected to occur. We discuss below some congruences modulo 9 and 16.

Proposition 3.2 *The following congruences hold:*

- (i) *If $e_{\mathbb{R}} \equiv 0 \pmod{9}$ then $w_g \equiv c_g \equiv 0 \pmod{9}$ for all $g \equiv 4 \pmod{6}$.*
- (ii) *If $e_{\mathbb{R}} \equiv 0 \pmod{16}$ then $w_g \equiv c_g \equiv 0 \pmod{16}$ for all odd $g > 1$.*

Proof We assume first that $e_{\mathbb{R}}$ is divisible by 9 (it happens with $e_{\mathbb{R}} = -18, 0$ and 18), and discuss the congruence of the two invariants modulo 9.

As Guo-Niu Han kindly pointed to us, the Jacobi identity [16, Corollary 1.4] shows that

$$\prod_{n \geq 1} (1 - q^n)^3 = A(q^3) + 3qB(q^3),$$

where $A, B \in \mathbb{Z}[[q]]$ and $A(0) = 1$, which implies that, for any integer k ,

$$(3-2) \quad \prod_{n \geq 1} (1 - q^n)^{3k} = A_k(q^3) + 3qB_k(q^3) + 9q^2C_k(q^3)$$

for some power series $A_k, B_k, C_k \in \mathbb{Z}[[q]]$ with $A_k(0) = 1$. In particular, when $k = -8$, it makes evident the congruence $c_g \equiv 0 \pmod{9}$ for all $g \equiv 2 \pmod{3}$.

To prove the claim for the real invariant, observe first that

$$\left(1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2}\right)^9 \equiv E(q^3) \pmod{9},$$

where $E \in \mathbb{Z}[[q]]$ and $E(0) = 1$. This implies that

$$\left(1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2}\right)^{9l} \equiv E_l(q^3) \pmod{9},$$

for any $l \in \mathbb{Z}$, where $E_l \in \mathbb{Z}[[q]]$ with $E_l(0) = 1$. Thus, using (2-2), (2-3) and (3-2), we find that

$$\sum_{g \geq 0} w_g q^g \equiv E_{e_{\mathbb{R}}/18}(q^3)(A_{-4}(q^6) + 3q^2 B_{-4}(q^6) + 9q^4 C_{-4}(q^6)) \pmod{9}.$$

The multiplication of $A_{-4}(q^6) + 3q^2 B_{-4}(q^6)$ by $E_{e_{\mathbb{R}}/18}(q^3)$ does not produce any term of exponent $6k + 4$. Therefore, $w_g \equiv 0 \pmod{9}$ for all g such that $g \equiv 4 \pmod{6}$, and claim (i) is proved.

Congruences modulo 16 for the complex invariants c_g can be detected by using Klein’s modular function $j(z)$ and the classical formula

$$j(q) = \frac{1}{q} + \sum_{n=0}^{\infty} a(n)q^n = \frac{(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n)^3}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}},$$

where $\sigma_3(n) = \sum_{d|n} d^3$. As $(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n)^3 \equiv 1 \pmod{16}$, we have

$$(3-3) \quad \sum_{g=0} c_g q^g \equiv qj(q) \pmod{16}.$$

On the other hand, according to Lehner [18], one has $a(2k) \equiv 0 \pmod{2^{11}}$ for all $k > 0$. Hence, c_g is divisible by 16 for all $g > 1$ with $g \equiv 1 \pmod{2}$.

If $e_{\mathbb{R}} \equiv 0 \pmod{16}$ (it happens with $e_{\mathbb{R}} = 0, 16$ and -16), then the same arguments as in the proof of congruences modulo 4 and 8 show that

$$w_g \equiv \prod_{n \geq 1} \frac{1}{(1 - q^{2n})^{12}} \pmod{16}.$$

This implies $w_g \equiv 0 \pmod{16}$ for any odd $g > 1$, and claim (ii) is proved. □

Remark 3.3 The congruence (3-3) holds modulo 9 as well, and $a(3k) \equiv 0 \pmod{3^5}$ for all $k > 0$ according to [18]. Hence $c_g \equiv 0 \pmod{9}$ if $g > 1$ and $g \equiv 1 \pmod{3}$.

3.4 On sharpness

A couple of other instances of real K3 surfaces where the lower bound for the number of real rational curves given by $|w_g|$ is optimal were already pointed in our previous paper [15]. One such example was the case of Harnack surfaces of degree 4 in \mathbb{P}^3 .

On the other hand, for real nonsingular K3 surfaces of degree 4 in \mathbb{P}^3 having the real locus consisting of 6 spheres and a sphere with 5 handles the lower bound given by $|w_g|$, which is equal to 48 in this special case, is not sharp.

Proposition 3.4 *The number of real rational hyperplane sections of a generic real K3 surface of degree 4 in \mathbb{P}^3 is at least 272 if the real locus of the surface consists of 6 spheres and a sphere with 5 handles.*

Proof The Euler characteristic $e_{\mathbb{R}}$ of the real part of such a surface is 4, and thus, for such surfaces, $w_g = -48$ (here $g = 3$). But, in fact, there are always at least 272 real rational hyperplane sections on such a surface. Indeed, each real line and each real plane intersect the component of genus 5, as such a component is not homotopy trivial in $\mathbb{R}\mathbb{P}^3$ (see [14, Theorem II]). By Bézout's theorem, this implies that each of the 6 spheres is convex (that is, they bound convex balls) and each pair of them is contained in a convex set disjoint from other spheres. By genus argument, a real hyperplane through 3 of the 6 spheres does not intersect the 3 others, and thus, these 3 spheres — as any 3 disjoint convex spheres in a real affine 3-space such that each 2 of them are contained in a convex set disjoint from the third — have 8 common supporting planes. Each of the supporting planes gives us a real rational curve with 3 solitary points, hence of Welschinger weight -1 . Since the total number of supporting planes obtained in such a way is equal to $8 \times \binom{6}{3} = 160$ and $w_g = -48$, the total number of real rational curves is at least $160 + (160 - 48) = 272$. \square

The same argument can be applied to other deformation classes of real nonsingular K3 surfaces of degree 4 in \mathbb{P}^3 whose real locus contains a noncontractible component and $m \geq 3$ components contractible in $\mathbb{R}\mathbb{P}^3$ (for a full deformation classification of real nonsingular K3 surfaces of degree 4 in \mathbb{P}^3 one can look at the survey [5] or at Nikulin's original paper [19]; some of them have $e_{\mathbb{R}} = 0$ and are different from tori). More precisely, as above, by Bézout's theorem each of the contractible components is an affine convex sphere and each 2 of them are contained in a convex set disjoint of the third, while, as is easy to show, any 3 affine convex spheres with such a property are contained in a common affine space. Thus, as above, each 3 contractible components provide 8 tritangent planes and give an input -8 into w_g . In particular, for this class of K3 surfaces, if $e_{\mathbb{R}} = 0$ we get $8 \times \binom{m}{3}$ as an improved lower bound.

Appendix

The table below is based on formulas (1-2) and (1-1). It provides the number of real rational curves counted with the Welschinger sign on primitively polarized K3 surfaces of degrees up to twenty, in the cases when $e_{\mathbb{R}} = 0, -18, \text{ and } 20$. The last column gives, for comparison, the corresponding number of complex curves.

g	real case			complex case
	$e_{\mathbb{R}} = 0$	$e_{\mathbb{R}} = -18$	$e_{\mathbb{R}} = 20$	
0	1	1	1	1
1	0	18	-20	24
2	12	192	192	324
3	0	1 536	-1 200	3 200
4	90	10 152	5 630	25 650
5	0	58 284	-21 744	176 256
6	520	299 776	73 600	1 073 720
7	0	1 410 048	-226 688	5 930 496
8	2 535	6 155 079	648 195	30 178 575
9	0	25 207 736	-1 742 320	143 184 000
10	10 908	97 675 200	4 446 912	639 249 300
11	0	360 471 552	-10 863 840	2 705 114 880
12	42 614	1 273 876 088	25 553 402	10 914 317 934
13	0	4 329 852 624	-58 129 280	42 189 811 200
14	153 960	14 207 361 792	128 365 440	156 883 829 400
15	0	45 144 664 064	-276 044 032	563 116 739 584
16	521 235	139 288 329 729	579 574 795	1 956 790 259 235
17	0	418 257 062 220	-1 190 636 016	6 599 620 022 400
18	1 669 720	1 224 808 431 104	2 397 710 720	21 651 325 216 200
19	0	3 503 958 594 048	-4 740 978 480	69 228 721 526 400
20	5 098 938	9 808 358 121 720	9 217 285 614	216 108 718 571 250

Table A: Numbers of real rational curves versus complex curves on K3 surfaces

Acknowledgements Both authors would like to thank Vanderbilt University for its hospitality while this work was finalized. Răşdeaconu acknowledges the support of the Simons Foundation’s “Collaboration Grant for Mathematicians”, award number 281266. We thank Alex Degtyarev, who provided the computer program exhibiting the numbers

included in the [appendix](#), and Guo-Niu Han for invaluable help with proving congruences modulo 9. We are grateful to the anonymous referee for pointing out to us several discords in the first version of the note.

References

- [1] **E Brugallé, G Mikhalkin**, *Enumeration of curves via floor diagrams*, C. R. Math. Acad. Sci. Paris 345 (2007) 329–334 [MR](#)
- [2] **J Bryan, N C Leung**, *The enumerative geometry of $K3$ surfaces and modular forms*, J. Amer. Math. Soc. 13 (2000) 371–410 [MR](#)
- [3] **X Chen**, *A simple proof that rational curves on $K3$ are nodal*, Math. Ann. 324 (2002) 71–104 [MR](#)
- [4] **C Ciliberto, A Lopez, R Miranda**, *Projective degenerations of $K3$ surfaces, Gaussian maps, and Fano threefolds*, Invent. Math. 114 (1993) 641–667 [MR](#)
- [5] **A I Degtyarev, V M Kharlamov**, *Topological properties of real algebraic varieties: Rokhlin’s way*, Uspekhi Mat. Nauk 55 (2000) 129–212 [MR](#) In Russian; translated in *Russian Math. Surveys* 55 (2000) 735–814
- [6] **P Di Francesco, C Itzykson**, *Quantum intersection rings*, from “The moduli space of curves” (R Dijkgraaf, C Faber, G van der Geer, editors), Progr. Math. 129, Birkhäuser, Boston (1995) 81–148 [MR](#)
- [7] **S Finashin, V Kharlamov**, *Abundance of real lines on real projective hypersurfaces*, Int. Math. Res. Not. 2013 (2013) 3639–3646 [MR](#)
- [8] **S Finashin, V Kharlamov**, *Abundance of 3–planes on real projective hypersurfaces*, Arnold Math. J. 1 (2015) 171–199 [MR](#)
- [9] **DB Grünberg, P Moree**, *Sequences of enumerative geometry: congruences and asymptotics*, Experiment. Math. 17 (2008) 409–426 [MR](#)
- [10] **GH Hardy, S Ramanujan**, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) 17 (1918) 75 [MR](#)
- [11] **I Itenberg, V Kharlamov, E Shustin**, *Welschinger invariant and enumeration of real rational curves*, Int. Math. Res. Not. 2003 (2003) 2639–2653 [MR](#)
- [12] **I Itenberg, V Kharlamov, E Shustin**, *Logarithmic asymptotics of the genus zero Gromov–Witten invariants of the blown up plane*, Geom. Topol. 9 (2005) 483–491 [MR](#)
- [13] **I Itenberg, V Kharlamov, E Shustin**, *Welschinger invariants of real del Pezzo surfaces of degree ≥ 2* , Internat. J. Math. 26 (2015) art. id. 1550060 [MR](#)
- [14] **VM Kharlamov**, *Additional congruences for the Euler characteristic of even-dimensional real algebraic varieties*, Funkcional. Anal. i Priložen. 9 (1975) 51–60 [MR](#) In Russian; translated in *Funct. Anal. Appl.* 9 (1975) 134–141

- [15] **V Kharlamov, R Răşdeaconu**, *Counting real rational curves on $K3$ surfaces*, Int. Math. Res. Not. 2015 (2015) 5436–5455 [MR](#)
- [16] **G Köhler**, *Eta products and theta series identities*, Springer, Heidelberg (2011) [MR](#)
- [17] **V S Kulikov**, *Degenerations of $K3$ surfaces and Enriques surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977) 1008–1042 [MR](#) In Russian; translated in *Math. USSR-Izv.* 11 (1977) 957–989
- [18] **J Lehner**, *Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$* , Amer. J. Math. 71 (1949) 373–386 [MR](#)
- [19] **V V Nikulin**, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979) 111–177, 238 [MR](#) In Russian; translated in *Math. USSR-Izv.* 14 (1980) 103–167
- [20] **V V Nikulin, S Saito**, *Real $K3$ surfaces with non-symplectic involution and applications*, Proc. London Math. Soc. 90 (2005) 591–654 [MR](#)
- [21] **J V Uspensky**, *Asymptotic formulae for numerical functions which occur in the theory of partitions*, Bull. Acad. Sci. URSS (6) 14 (1920) 199–218
- [22] **O Ya Viro**, *Construction of multicomponent real algebraic surfaces*, Dokl. Akad. Nauk SSSR 248 (1979) 279–282 [MR](#) In Russian; translated in Soviet Math. Dokl. 20 (1979) 991–995
- [23] **J-Y Welschinger**, *Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry*, C. R. Math. Acad. Sci. Paris 336 (2003) 341–344 [MR](#)
- [24] **J-Y Welschinger**, *Optimalité, congruences et calculs d’invariants des variétés symplectiques réelles de dimension quatre*, preprint (2007) [arXiv](#)
- [25] **S-T Yau, E Zaslow**, *BPS states, string duality, and nodal curves on $K3$* , Nuclear Phys. B 471 (1996) 503–512 [MR](#)

IRMA UMR 7501, Université de Strasbourg
7 Rue René-Descartes, 67084 Strasbourg Cedex, France

Department of Mathematics, Vanderbilt University
1326 Stevenson Center, Nashville, TN 37240, United States

viatcheslav.kharlamov@math.unistra.fr, rares.rasdeaconu@Vanderbilt.edu

Proposed: Lothar Göttsche

Received: 9 June 2015

Seconded: Gang Tian, Yasha Eliashberg

Revised: 26 January 2016

