Equidistribution for sequences of line bundles on normal Kähler spaces

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We study the asymptotics of Fubini–Study currents and zeros of random holomorphic sections associated to a sequence of singular Hermitian line bundles on a compact normal Kähler complex space.

1 Introduction

In this paper we continue the study of equidistribution of Fubini–Study currents and zeros of sequences of holomorphic sections of singular Hermitian holomorphic bundles started by Coman and Marinescu [10; 11; 12]. We generalize our previous results in two directions. On the one hand, we allow the base space to be singular and work over Kähler spaces. On the other hand, we consider sequences (L_p, h_p), $p \geq 1$, of singular Hermitian holomorphic line bundles whose Chern curvature satisfy a natural growth condition, instead of sequences of powers (L^p, h^p) of a fixed line bundle (L, h).

Recall that, by the results of Tian [57] (see also Ma and Marinescu [40, Section 5.3]), if (X, ω) is a compact Kähler manifold whose Kähler form is integral and (L, h) is a prequantum line bundle (ie the Chern curvature form $c_1(L, h)$ equals ω), then the normalized Fubini–Study forms $\frac{1}{p} \gamma_p$ associated to $H^0(X, L^p)$ converge in the $\mathscr{C}^2$ topology to ω. This result can be applied to describe the asymptotic distribution of zeros of random sequences of holomorphic sections. Indeed, Shiffman and Zelditch [55] showed (see also Dinh and Sibony [22], Dinh, Marinescu and Schmidt [20], Nonnenmacher and Voros [47], Shiffman [54] and Shiffman and Zelditch [56]) that for almost all sequences $\{\sigma_p \in H^0(X, L^p)\}_{p \geq 1}$ the normalized zero-currents $\frac{1}{p} [\sigma_p = 0]$ converge weakly to ω on X. This means that the Kähler form ω can be approximated by various algebraic/analytic objects in the semiclassical limit $p \to \infty$. Some important technical tools for higher dimensions used in the these works were introduced by Fornæss and Sibony [27]. We note that the statistics of zeros of sections

and hypersurfaces have been studied also in the context of real manifolds and real vector bundles; see eg Gayet and Welschinger [28] and Nicolaescu and Savale [46].

In [10; 11; 12] we relaxed the smoothness condition on ω and assumed that ω is merely an integral Kähler current. Then there exists a holomorphic line bundle (L, h) endowed with a singular Hermitian metric such that $c_1(L, h)$ equals ω as currents. We established the above approximation results in the sense of currents by working with the spaces of square integrable holomorphic sections $H^0_{(2)}(X, L^p)$. The setting in [10; 11; 12] allows to deal with several singular Kähler metrics, such as singular metrics on big line bundles, metrics with Poincaré growth, Kähler–Einstein metrics singular along a divisor (eg good metrics in the sense of Mumford on toroidal compactifications of arithmetic quotients; see Ash, Mumford, Rapoport and Tai [2]) or metrics on orbifold line bundles.

We consider in this paper the following setting:

(A) (X, ω) is a compact (reduced) normal Kähler space of pure dimension n, X_{reg} is the set of regular points of X and X_{sing} is the set of singular points of X.

(B) (L_p, h_p), $p \geq 1$, is a sequence of holomorphic line bundles on X with singular Hermitian metrics h_p whose curvature currents satisfy

$$(1) \qquad c_1(L_p, h_p) \geq a_p \omega \quad \text{on } X, \qquad \text{where} \quad a_p > 0 \quad \text{and} \quad \lim_{p \to \infty} a_p = \infty.$$

Let $A_p = \int_X c_1(L_p, h_p) \wedge \omega^{n-1}$. If $X_{\mathrm{sing}} \neq \varnothing$ we assume moreover that

$$(2) \qquad \exists\, T_0 \in \mathscr{T}(X) \ \ \forall p \geq 1 \quad c_1(L_p, h_p) \leq A_p T_0.$$

Here we consider currents on the analytic space X in the sense of Demailly [16], and $\mathscr{T}(X)$ is the set of positive closed currents of bidegree $(1, 1)$ on X which have a local plurisubharmonic (PSH) potential in the neighborhood of each point of X (see Section 2).

We let $H^0_{(2)}(X, L_p)$ be the Bergman space of L^2–holomorphic sections of L_p relative to the metric h_p and the volume form induced by ω on X,

$$(3) \qquad H^0_{(2)}(X, L_p) = \left\{ S \in H^0(X, L_p) \ \middle| \ \|S\|_p^2 := \int_{X_{\mathrm{reg}}} |S|^2_{h_p} \frac{\omega^n}{n!} < \infty \right\},$$

endowed with the obvious inner product. Furthermore, we denote by P_p and γ_p the Bergman kernel function and Fubini–Study current, respectively, of the space $H^0_{(2)}(X, L_p)$, which are defined as follows: For $p \geq 1$, let $S_1^p, \ldots, S_{d_p}^p$ be an orthonormal basis of $H^0_{(2)}(X, L_p)$. If $x \in X$ and e_p is a local holomorphic frame of L_p

in a neighborhood U_p of x, we write $S_j^p = s_j^p e_p$, where $s_j^p \in \mathcal{O}_X(U_p)$. Then

$$(4) \qquad P_p(x) = \sum_{j=1}^{d_p} |S_j^p(x)|_{h_p}^2, \quad \gamma_p|_{U_p} = \frac{1}{2} d d^c \log\left(\sum_{j=1}^{d_p} |s_j^p|^2 \right),$$

where $d^c = \frac{1}{2\pi i}(\partial - \bar\partial)$. Note that P_p and γ_p are independent of the choice of basis.

Our main results are the following theorems:

Theorem 1.1 *If (X, ω) and (L_p, h_p), $p \geq 1$, satisfy assumptions (A)–(B), then:*

(i) $\frac{1}{A_p} \log P_p \to 0$ *as* $p \to \infty$ *in* $L^1(X, \omega^n)$.

(ii) $\frac{1}{A_p}(\gamma_p - c_1(L_p, h_p)) \to 0$ *as* $p \to \infty$ *in the weak sense of currents on* X.

Moreover, if $\frac{1}{A_p} c_1(L_p, h_p) \to T$ *for some positive closed current* T *of bidegree* $(1,1)$ *on X, then* $\frac{1}{A_p} \gamma_p \to T$ *as* $p \to \infty$ *in the weak sense of currents on* X.

When X is smooth we obtain:

Theorem 1.2 *Let (X, ω) be a compact Kähler manifold of dimension n and (L_p, h_p) for $p \geq 1$ be a sequence of singular Hermitian holomorphic line bundles on X which satisfy $c_1(L_p, h_p) \geq a_p \omega$, where $a_p > 0$ and $\lim_{p\to\infty} a_p = \infty$. If $A_p = \int_X c_1(L_p, h_p) \wedge \omega^{n-1}$ then $\frac{1}{A_p} \log P_p \to 0$ in $L^1(X, \omega^n)$ and $\frac{1}{A_p}(\gamma_p - c_1(L_p, h_p)) \to 0$ weakly on X as $p \to \infty$.*

If $(L_p, h_p) = (L^p, h^p)$, where (L, h) is a fixed singular Hermitian holomorphic line bundle, we obtain analogues of the equidistributions results from [10; 11; 12], Dinh and Sibony [22] and Shiffman and Zelditch [55] for compact normal Kähler spaces. Note that in this case hypothesis (2) is automatically satisfied as $c_1(L^p, h^p) = p c_1(L, h)$, so we can take $A_p = p$ and $T_0 = c_1(L, h)$.

In the case $(L_p, h_p) = (L^p, h^p)$, Theorem 1.1 gives semiclassical approximation results for integral Kähler currents. In order to deal with the nonintegral case, we consider those currents T which can be approximated by the curvatures of a sequence (L_p, h_p), $p \geq 1$, of singular Hermitian holomorphic line bundles. Such a sequence can be thought as a "prequantization" of the nonintegral positive closed $(1,1)$ current T. Theorems 1.1–1.2 and their consequences are a manifestation of the quantum ergodicity in this context. For details, see Sections 4 and 5, and Theorem 4.7. Another interesting situation is when L_p equals a product of tensor powers of several holomorphic line bundles, $L_p = F_1^{m_{1,p}} \otimes \cdots \otimes F_k^{m_{k,p}}$, where $\{m_{j,p}\}_p$, $1 \leq j \leq k$, are sequences in $\mathbb{N}$ such that $m_{j,p} = r_j p + o(p)$ as $p \to \infty$, where $r_j > 0$ are given. This means that

$(m_{1,p}, \ldots, m_{k,p}) \in \mathbb{N}^k$ approximate the semiclassical ray $\mathbb{R}_{>0} \cdot (r_1, \ldots, r_k) \in \mathbb{R}^k_{>0}$ with a remainder $o(p)$ as $p \to \infty$. For details see Corollary 5.11.

We consider further the situation when the metrics h_p on the bundles L_p are smooth. In the case of powers $(L_p, h_p) = (L^p, h^p)$ of an ample line bundle (L, h), the first-order asymptotics of the Bergman kernel function was showed by Tian [57]. A simple proof for this was given by Berndtsson [5, Section 2]. Adapting his methods to our situation we prove the following:

Theorem 1.3 *Let (X, ω) be a compact Kähler manifold of dimension n. Let (L_p, h_p), $p \geq 1$, be a sequence of holomorphic line bundles on X with Hermitian metrics h_p of class $\mathscr{C}^3$ whose curvature forms satisfy (1) and such that*

$$(5) \qquad\qquad \varepsilon_p := \|h_p\|_3^{1/3} a_p^{-1/2} \to 0 \quad \text{as } p \to \infty.$$

Then there exist $C > 0$ depending only on (X, ω) and $p_0 \in \mathbb{N}$ such that

$$(6) \qquad\qquad \left| P_p(x) \frac{\omega_x^n}{c_1(L_p, h_p)_x^n} - 1 \right| \leq C \varepsilon_p^{2/3}$$

holds for every $x \in X$ and $p > p_0$.

Here $\|h_p\|_3$ is the sup norm of the derivatives of h_p of order at most three with respect to a reference cover of X as defined in Section 2.5. Theorem 1.3 is a generalization of the first-order asymptotic expansion of the Bergman kernel; see Berndtsson [5], Catlin [9], Dai, Liu and Ma [13], Ma and Marinescu [40; 41], Ruan [52], Tian [57] and Zelditch [59] for $(L_p, h_p) = (L^p, h^p)$, where (L, h) is a positive line bundle with smooth metric h (see Remark 3.3).

The paper is organized as follows. In Section 2 we collect the necessary technical facts about complex spaces, plurisubharmonic functions, currents and singular Hermitian metrics on holomorphic line bundles. Section 3 is devoted to the proofs of Theorems 1.1–1.3. In Section 4 we give some applications of these theorems to the equidistribution of zeros of random sequences of holomorphic sections and to the approximation of certain positive closed currents on X by currents of integration along zeros of holomorphic sections. In Section 5 we specialize these results to the cases when (L_p, h_p) are the powers of a single line bundle, or tensor products of powers of several line bundles. We also show that some interesting cases, like Satake–Baily–Borel compactifications, singular Kähler–Einstein metrics on varieties of general type or Kähler metrics with conical singularities, fit into this framework.

2 Preliminaries

We recall here a few notions and results that will be needed throughout the paper.

2.1 Plurisubharmonic functions and currents on analytic spaces

Let X be a complex space. A chart (U, τ, V) on X is a triple consisting of an open set $U \subset X$, a closed complex space $V \subset G \subset \mathbb{C}^N$ in an open set G of $\mathbb{C}^N$ and a biholomorphic map $\tau \colon U \to V$ (in the category of complex spaces). The map $\tau \colon U \to G \subset \mathbb{C}^N$ is called a local embedding of the complex space X. We write

$$X = X_{\text{reg}} \cup X_{\text{sing}},$$

where X_{reg} and X_{sing} are the sets of regular and singular points of X, respectively. Recall that a reduced complex space $(X, \mathscr{O})$ is called normal if for every $x \in X$ the local ring $\mathscr{O}_x$ is integrally closed in its quotient field $\mathscr{M}_x$. Every normal complex space is locally irreducible and locally pure-dimensional (see [32, page 125]), and X_{sing} is a closed complex subspace of X with codim $X_{\text{sing}} \geq 2$. Moreover, Riemann's second extension theorem holds on normal complex spaces [32, page 143]. In particular, every holomorphic function on X_{reg} extends uniquely to a holomorphic function on X.

Let X be a complex space. A continuous (resp. smooth) function on X is a function $\varphi \colon X \to \mathbb{C}$ such that for every $x \in X$ there exists a local embedding $\tau \colon U \to G \subset \mathbb{C}^N$ with $x \in U$ and a continuous (resp. smooth) function $\widetilde{\varphi} \colon G \to \mathbb{C}$ such that $\varphi|_U = \widetilde{\varphi} \circ \tau$.

A *(strictly) plurisubharmonic (PSH)* function on X is a function $\varphi \colon X \to [-\infty, \infty)$ such that for every $x \in X$ there exists a local embedding $\tau \colon U \to G \subset \mathbb{C}^N$ with $x \in U$ and a (strictly) PSH function $\widetilde{\varphi} \colon G \to [-\infty, \infty)$ such that $\varphi|_U = \widetilde{\varphi} \circ \tau$. If $\widetilde{\varphi}$ can be chosen continuous (resp. smooth), then φ is called a continuous (resp. smooth) PSH function. The definition is independent of the chart, as is seen from [45, Lemma 4]. It is clear that a continuous PSH function is continuous; by a theorem of Richberg [51] the converse also holds true, ie a continuous function which is (strictly) PSH is also continuous (strictly) PSH. The analogue of Riemann's second extension theorem for PSH functions holds on normal complex spaces [31, Satz 4]. In particular, every PSH

function on X_{reg} extends uniquely to a PSH function on X. We let $\mathrm{PSH}(X)$ denote the set of PSH functions on X, and refer to [31; 45; 26; 16] for the properties of PSH functions on X. We recall here that PSH functions on X are locally integrable with respect to the area measure on X given by any local embedding $\tau\colon U \to G \subset \mathbb{C}^N$ [16, Proposition 1.8].

Let X be a complex space of pure dimension n. We consider currents on X as defined in [16]. The sheaf of smooth (p,q)–forms on X is defined at first locally. Let $\tau\colon U \to G \subset \mathbb{C}^N$ be a local embedding. We define $\Omega^{p,q}(U)$ to be the image of the morphism $\tau^*\colon \Omega^{p,q}(G) \to \Omega^{p,q}(U_{\mathrm{reg}})$. It can be easily seen that there exists a sheaf $\Omega^{p,q}$ on X whose space of sections on any domain U of local embedding is $\Omega^{p,q}(U)$. Let $\mathcal{D}^{p,q}(X) \subset \Omega^{p,q}(X)$ be the space of forms with compact support, endowed with the inductive limit topology. The dual $\mathcal{D}'_{p,q}(X)$ of $\mathcal{D}_{p,q}(X)$ is the space of currents of bidimension (p,q), or bidegree $(n-p, n-q)$, on X. In particular, if $v \in \mathrm{PSH}(X)$ then $dd^c v \in \mathcal{D}'_{n-1,n-1}(X)$ is positive and closed.

Let $\mathcal{T}(X)$ be the space of positive closed currents of bidegree $(1,1)$ on X which have local PSH potentials: $T \in \mathcal{T}(X)$ if every $x \in X$ has a neighborhood U (depending on T) such that there exists a PSH function v on U with $T = dd^c v$ on $U \cap X_{\mathrm{reg}}$. Most of the currents considered here, like the curvature currents $c_1(L_p, h_p)$ and the Fubini–Study currents γ_p from Theorem 1.1, belong to $\mathcal{T}(X)$. Suppose now that Y is a normal analytic space, $f\colon Y \to X$ is a holomorphic map, and $T \in \mathcal{T}(X)$ is such that if v is a local PSH potential of T then $v \circ f$ is not identically $-\infty$ on an open set of Y. Then the pull-back $f^*T \in \mathcal{T}(Y)$ is a well-defined current whose local PSH potentials are $v \circ f$. Some interesting open questions that we will not pursue here are the following: Does every positive closed current of bidegree $(1,1)$ on X belong to $\mathcal{T}(X)$? Is $\mathcal{T}(X)$ closed in the weak* topology on currents? If $T_k, T \in \mathcal{T}(X)$ and $T_k \to T$ weakly on X, does $\{f^*T_k\}$ converge to f^*T weakly on Y?

A Kähler form on X is a current $\omega \in \mathcal{T}(X)$ whose local potentials extend to smooth strictly PSH functions in local embeddings of X to Euclidean spaces. We call X a Kähler space if X admits a Kähler form (see also [30, page 346; 48; 25, Section 5]). A Kähler form is a particular case of a Hermitian form on a complex space. Recall that a Hermitian form on a complex manifold is a smooth positive $(1,1)$–form and can be identified to a Hermitian metric. Now, a Hermitian form on a complex space X is defined as a smooth $(1,1)$–form ω on X such that for every point $x \in X$ there exists a local embedding $\tau\colon U \to G \subset \mathbb{C}^N$ with $x \in U$ and a Hermitian form $\widetilde{\omega}$ on G with $\omega = \tau^*\widetilde{\omega}$ on $U \cap X_{\mathrm{reg}}$. A Hermitian form on a paracompact complex space X is constructed as usual by a partition of unity argument. A Hermitian form ω on X clearly induces a Hermitian form in the usual sense (and thus a Hermitian metric) on X_{reg}. Note that $\omega^n/n!$ gives locally an area measure on X.

2.2 Singular Hermitian holomorphic line bundles on analytic spaces

Let L be a holomorphic line bundle on a normal Kähler space (X, ω). The notion of singular Hermitian metric h on L is defined exactly as in the smooth case (see [18; 40, page 97]): if e_α is a holomorphic frame of L over an open set $U_\alpha \subset X$ then $|e_\alpha|^2_h = e^{-2\varphi_\alpha}$, where $\varphi_\alpha \in L^1_{\mathrm{loc}}(U_\alpha, \omega^n)$. If $g_{\alpha\beta} = e_\beta/e_\alpha \in \mathcal{O}^*_X(U_\alpha \cap U_\beta)$ are the transition functions of L then $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$. The curvature current $c_1(L, h) \in \mathcal{D}'_{n-1,n-1}(X)$ of h is defined by $c_1(L, h) = dd^c\varphi_\alpha$ on $U_\alpha \cap X_{\mathrm{reg}}$. We will denote by h^p the singular Hermitian metric induced by h on $L^p := L^{\otimes p}$. If $c_1(L, h) \geq 0$ then the weight φ_α is PSH on $U_\alpha \cap X_{\mathrm{reg}}$ and since X is normal it extends to a PSH function on U_α [31, Satz 4], hence $c_1(L, h) \in \mathcal{T}(X)$. Let L be a holomorphic line bundle on a compact normal Kähler space (X, ω). Then the space $H^0(X, L)$ of holomorphic sections of L is finite-dimensional. This is a special case of the Cartan–Serre finiteness theorem; an elementary proof using the Schwarz lemma can be found in [1, Théorème 1, page 27]. The space $H^0_{(2)}(X, L)$ defined as in (3) is therefore also finite-dimensional.

If P_p and γ_p are the Bergman kernel functions and Fubini–Study currents of the spaces $H^0_{(2)}(X, L_p)$ from Theorem 1.1, it follows from (4) that $\log P_p \in L^1(X, \omega^n)$ and

$$(7) \qquad \gamma_p - c_1(L_p, h_p) = \tfrac{1}{2}dd^c \log P_p.$$

Moreover, as in [10; 12], one has the variational formula

$$P_p(x) = \max\{|S(x)|^2_{h_p} : S \in H^0_{(2)}(X, L_p), \; \|S\|_p = 1\}.$$

This is valid for all $x \in X$ such that $\varphi_p(x) > -\infty$, where φ_p is a local weight of the metric h_p near x.

2.3 Resolution of singularities

Bierstone and Milman constructed a resolution $\pi\colon \widetilde{X} \to X$ of singularities of a compact analytic space X by a finite sequence of blow-ups with smooth center $\sigma_j\colon X_{j+1} \to X_j$, $X_0 = X$, with the property that for any local embedding $X|_U \hookrightarrow \mathbb{C}^N$ this sequence of blow-ups is induced by the embedded desingularization of $X|_U$ [6, Theorem 13.2]. In [29, Section 6] it is shown that the embedded desingularization of $X|_U \hookrightarrow \mathbb{C}^N$ by a finite sequence of blow-ups with smooth center is equivalent to a single blow-up along a coherent sheaf of ideals $\mathcal{I}$ whose support is $X_{\mathrm{sing}}|_U$. It follows that every point $x \in X$ has a neighborhood $U \subset X$ for which there exists an ideal $\mathcal{I}_U$ generated by finitely many holomorphic functions on U such that $\pi\colon \pi^{-1}(U) \to U$ is equivalent to the blow-up of $X|_U$ along $\mathcal{I}_U$.

We fix throughout the paper a resolution of singularities $\pi\colon \widetilde{X} \to X$ of our compact normal space X as described above, and we denote by $\Sigma = \pi^{-1}(X_{\mathrm{sing}})$ the exceptional divisor. Note that $\pi\colon \widetilde{X} \setminus \Sigma \to X_{\mathrm{reg}}$ is a biholomorphism. We will need to consider the singular Hermitian holomorphic line bundles $(\pi^* L_p, \pi^* h_p)$ obtained by pulling back (L_p, h_p) to $\widetilde{X}$ by the map π, and their spaces of L^2–holomorphic sections

$$H^0_{(2)}(\widetilde{X}, \pi^* L_p) = \left\{ \widetilde{S} \in H^0(\widetilde{X}, \pi^* L_p) \,\Big|\, \int_{\widetilde{X}} |\widetilde{S}|^2_{\pi^* h_p} \frac{\pi^* \omega^n}{n!} < \infty \right\}.$$

Lemma 2.1 *The map*

$$\pi^*\colon H^0_{(2)}(X, L_p) \to H^0_{(2)}(\widetilde{X}, \pi^* L_p)$$

is an isometry and the Bergman kernel function of $H^0_{(2)}(\widetilde{X}, \pi^* L_p)$ is $\widetilde{P}_p = P_p \circ \pi$.

Proof Let $S^p_1, \ldots, S^p_{d_p}$ be an orthonormal basis of $H^0_{(2)}(X, L_p)$ and let $\widetilde{S}^p_j = \pi^* S^p_j$ be the induced sections of $\pi^* L^p$. Then $|\widetilde{S}^p_j|_{\pi^* h_p} = |S^p_j|_{h_p} \circ \pi$ and

$$\int_{\widetilde{X}} |\widetilde{S}^p_j|^2_{\pi^* h_p} \frac{\pi^* \omega^n}{n!} = \int_{\widetilde{X} \setminus \Sigma} |\widetilde{S}^p_j|^2_{\pi^* h_p} \frac{\pi^* \omega^n}{n!} = \int_{X_{\mathrm{reg}}} |S^p_j|^2_{h_p} \frac{\omega^n}{n!} = 1.$$

Suppose now that $\widetilde{S} \in H^0_{(2)}(\widetilde{X}, \pi^* L_p)$ is orthogonal to all $\widetilde{S}^p_j$ and let S be the induced section on X_{reg}. Then, since X is normal, Riemann's second extension theorem [32, page 143] shows that S extends to a section $S \in H^0(X, L_p)$. By the preceding argument, $S \in H^0_{(2)}(X, L_p)$ is orthogonal to all S^p_j, so $S = 0$ and $\widetilde{S} = 0$. It follows that $\{\widetilde{S}^p_j\}$ is an orthonormal basis of $H^0_{(2)}(\widetilde{X}, \pi^* L_p)$ and

$$\widetilde{P}_p = \sum_{j=1}^{d_p} |\widetilde{S}^p_j|^2_{\pi^* h_p} = \sum_{j=1}^{d_p} |S^p_j|^2_{h_p} \circ \pi = P_p \circ \pi. \qquad \square$$

Adapting the proof of [43, Lemma 1] to our situation we obtain the following lemma, whose proof is included for the convenience of the reader.

Lemma 2.2 *Let (X, ω) be a compact (reduced) Hermitian space and $\pi\colon \widetilde{X} \to X$ be a resolution of singularities as described above, with exceptional divisor Σ. Then there exists a smooth Hermitian metric θ on $F = \mathcal{O}_{\widetilde{X}}(-\Sigma)$ and $C > 0$ such that $\Omega = C\pi^*\omega + c_1(F, \theta)$ is a Hermitian form on $\widetilde{X}$ and $\Omega \geq \pi^*\omega$. If ω is Kähler, then Ω is Kähler, too.*

Proof As in the proof of [43, Lemma 1], we can find an open cover $X = \bigcup_{k=1}^{N} U_k$ with the following properties:

(i) There exist a local embedding $\tau_k\colon U_k \to B_k$ into a ball $B_k \subset \mathbb{C}^{l_k}$ and a smooth Hermitian form ω_k on B_k such that $\omega = \tau_k^* \omega_k$ on $U_k \cap X_{\mathrm{reg}}$. Hence we may assume that $U_k \subset B_k$ and ω is the restriction of ω_k to $U_k \cap X_{\mathrm{reg}}$. Choose a strictly PSH function η_k on B_k such that $\omega_k \geq dd^c \eta_k$ on B_k. If ω is Kähler, we choose η_k such that $\omega_k = dd^c \eta_k$ on B_k, hence $\omega = dd^c \eta_k$ on $U_k \cap X_{\mathrm{reg}}$.

(ii) There exist finitely many holomorphic functions $f_{0k}, \dots, f_{N_k k} \in \mathcal{O}_X(U_k)$ such that over U_k the map $\pi\colon \pi^{-1}(U_k) \to U_k$ and the line bundle $\mathcal{O}_{\widetilde{X}}(\Sigma)|_{\pi^{-1}(U_k)}$ are described as follows: if $\Gamma_k \subset U_k \times \mathbb{P}^{N_k}$ is the graph of the meromorphic map

$$x \mapsto [f_{0k}(x) : \cdots : f_{N_k k}(x)]$$

with canonical projections $\pi_{1k}\colon \Gamma_k \to U_k$ and $\pi_{2k}\colon \Gamma_k \to \mathbb{P}^{N_k}$, then $\pi\colon \pi^{-1}(U_k) \to U_k$ coincides with $\pi_{1k}\colon \Gamma_k \to U_k$ and $\mathcal{O}_{\widetilde{X}}(\Sigma)|_{\pi^{-1}(U_k)}$ can be identified with the line bundle $\pi_{2k}^* \mathcal{O}_{\mathbb{P}^{N_k}}(-1)$ on Γ_k.

Let $[y_{0k} : \cdots : y_{N_k k}]$ be the homogeneous coordinates on $\mathbb{P}^{N_k}$, $G_{ik} = \{y_{ik} \neq 0\} \subset \mathbb{P}^{N_k}$ and $\widetilde{G}_{ik} = \pi_{2k}^{-1}(G_{ik}) \subset \Gamma_k = \pi^{-1}(U_k)$. The transition functions of $\mathcal{O}_{\mathbb{P}^{N_k}}(-1)$ are y_{ik}/y_{jk} and we endow $\mathcal{O}_{\mathbb{P}^{N_k}}(-1)$ with a smooth Hermitian metric with weights ψ_{ik}' on G_{ik} such that $-\psi_{ik}'$ is strictly PSH. The corresponding transition functions of $\mathcal{O}_{\widetilde{X}}(\Sigma)|_{\pi^{-1}(U_k)}$ are $\widetilde{g}_{ij,k} = (f_{ik} \circ \pi_{1k})/(f_{jk} \circ \pi_{1k})$ on $\widetilde{G}_{ik} \cap \widetilde{G}_{jk}$, and we denote by ψ_{ik} the weights of the induced Hermitian metric. Then $-\psi_{ik}$ is a smooth PSH function on $\widetilde{G}_{ik}$ and $\psi_{ik} = \psi_{jk} + \log|\widetilde{g}_{ij,k}|$ on $\widetilde{G}_{ik} \cap \widetilde{G}_{jk}$.

Consider the open cover $\{\widetilde{G}_{ik} \mid 1 \leq k \leq N, \, 0 \leq i \leq N_k\}$ of $\widetilde{X}$. Note that $\pi^{-1}(U_k) = \bigcup_{i=0}^{N_k} \widetilde{G}_{ik}$ and that $\mathcal{O}_{\widetilde{X}}(\Sigma)$ is trivial on $\widetilde{G}_{ik}$. We denote the transition functions of $\mathcal{O}_{\widetilde{X}}(\Sigma)$ by $g_{i_1 k_1, i_2 k_2} \in \mathcal{O}(\widetilde{G}_{i_1 k_1} \cap \widetilde{G}_{i_2 k_2})$. Hence

$$\psi_{i_1 k} = \psi_{i_2 k} + \log|g_{i_1 k, i_2 k}|, \quad g_{i_1 k, i_2 k} = \widetilde{g}_{i_1 i_2, k}.$$

We now construct a smooth Hermitian metric on $\mathcal{O}_{\widetilde{X}}(\Sigma)$ with weights s_{ik} on $\widetilde{G}_{ik}$. Let $\{\rho_k\}_{1 \leq k \leq N}$ be a smooth partition of unity on X so that $\operatorname{supp}\rho_k \subset U_k$. For a fixed $k_1 \in \{1, \dots, N\}$ let $\varphi_{k_1}^{ik}$ be a function defined on $\pi^{-1}(U_{k_1}) \cap \widetilde{G}_{ik}$ as follows: if $x \in \widetilde{G}_{i_1 k_1}$ then

$$\varphi_{k_1}^{ik}(x) = \psi_{i_1 k_1}(x) + \log|g_{ik, i_1 k_1}(x)|.$$

Note that $\varphi_{k_1}^{ik}$ is well-defined and smooth on $\pi^{-1}(U_{k_1}) \cap \widetilde{G}_{ik}$. Indeed, if x is in $\widetilde{G}_{i_1 k_1} \cap \widetilde{G}_{i_1' k_1} \cap \widetilde{G}_{ik}$ then $g_{i_1' k_1, i_1 k_1} \cdot g_{i_1 k_1, ik} \cdot g_{ik, i_1' k_1} = 1$ so

$$\psi_{i_1' k_1}(x) + \log|g_{ik, i_1' k_1}(x)| = \psi_{i_1 k_1}(x) + \log|g_{i_1' k_1, i_1 k_1}(x)| + \log|g_{ik, i_1' k_1}(x)|$$
$$= \psi_{i_1 k_1}(x) + \log|g_{ik, i_1 k_1}(x)|.$$

Next we define

$$s_{ik} = \sum_{k_1=1}^{N} (\rho_{k_1} \circ \pi) \varphi_{k_1}^{ik} \quad \text{on } \widetilde{G}_{ik}.$$

We claim that $s_{ik} = s_{i'k'} + \log |g_{ik,i'k'}|$ on $\widetilde{G}_{ik} \cap \widetilde{G}_{i'k'}$, so $\{s_{ik}\}$ defines a smooth Hermitian metric on $\mathcal{O}_{\widetilde{X}}(\Sigma)$. For this we show that if $k_1 \in \{1, \ldots, N\}$ then

$$(\rho_{k_1} \circ \pi) \varphi_{k_1}^{ik} = (\rho_{k_1} \circ \pi)(\varphi_{k_1}^{i'k'} + \log |g_{ik,i'k'}|).$$

Let $x \in \pi^{-1}(U_{k_1})$ and assume $x \in \widetilde{G}_{i_1 k_1} \cap \widetilde{G}_{ik} \cap \widetilde{G}_{i'k'}$. Since

$$g_{i'k',i_1 k_1} \cdot g_{i_1 k_1,ik} \cdot g_{ik,i'k'} = 1$$

we obtain

$$\begin{aligned}
\varphi_{k_1}^{ik}(x) &= \psi_{i_1 k_1}(x) + \log |g_{ik,i_1 k_1}(x)| \\
&= \psi_{i_1 k_1}(x) + \log |g_{i'k',i_1 k_1}(x)| + \log |g_{ik,i'k'}(x)| \\
&= \varphi_{k_1}^{i'k'}(x) + \log |g_{ik,i'k'}(x)|,
\end{aligned}$$

which proves our claim.

We finally show that the desired metric θ on F is the metric defined by the weights $\{-s_{ik}\}$. By a standard compactness argument it suffices to prove that for every $x \in \widetilde{G}_{ik}$ there exists $C_x > 0$ such that for $C > C_x$ the function $C\eta_k \circ \pi - s_{ik}$ is strictly PSH at x. We write $T_x \widetilde{X} = E_x \oplus F_x$, where $E_x = \ker d\pi(x)$. Note that the Levi form (see eg [36, page 228] for the definition) of the function $\eta_k \circ \pi$ is given by

$$\mathcal{L}(\eta_k \circ \pi)(x)(t, t') = \mathcal{L}\eta_k(\pi(x))(d\pi(x)(t), d\pi(x)(t')),$$

so

(8) $$\mathcal{L}(\eta_k \circ \pi)(x)(t, t') = 0 \quad \text{for all } t \in T_x \widetilde{X}, \, t' \in E_x.$$

Moreover, since η_k is strictly PSH at $\pi(x) \in B_k$ and $d\pi(x)$ is injective on F_x we deduce that

(9) $$\mathcal{L}(\eta_k \circ \pi)(x)(t, t) > 0 \quad \text{for all } t \in F_x \setminus \{0\}.$$

The formula of s_{ik} implies that, for each $t \in E_x$,

$$-\mathcal{L}s_{ik}(x)(t, t) = -\sum_{k_1=1}^{N} \rho_{k_1}(\pi(x)) \mathcal{L}\psi_{i_1 k_1}(x)(t, t) \geq 0,$$

since each function $-\psi_{i_1 k_1}$ is PSH on $\widetilde{G}_{i_1 k_1}$. If $\rho_{k_1}(\pi(x)) > 0$, we may assume that $x \in \widetilde{G}_{i_1 k_1} \cap \widetilde{G}_{ik}$ and we will show that $-\mathcal{L}\psi_{i_1 k_1}(x)(t, t) > 0$ for all $t \in E_x \setminus \{0\}$. As $\pi^{-1}(U_{k_1}) = \Gamma_{k_1} \subset U_{k_1} \times \mathbb{P}^{N_{k_1}} \subset B_{k_1} \times \mathbb{P}^{N_{k_1}}$, there is a neighborhood $\widetilde{U} \subset \widetilde{X}$

of x such that $\widetilde{U} \subset B_{k_1} \times G_{i_1 k_1}$. Recall that $\psi_{i_1 k_1} = \psi'_{i_1 k_1} \circ \pi_{2k_1}$ and $\pi = \pi_{1k_1}$ on $\widetilde{U}$. We consider π_{1k_1} and π_{2k_1} as restrictions of the canonical projections $\pi_{1k_1} \colon B_{k_1} \times G_{i_1 k_1} \to B_{k_1}$ and $\pi_{2k_1} \colon B_{k_1} \times G_{i_1 k_1} \to G_{i_1 k_1}$. Since $t \in E_x \subset \ker d\pi_{1k_1}(x)$ and $t \neq 0$ it follows that $d\pi_{2k_1}(x)(t) \neq 0$. Therefore

$$-\mathcal{L}\psi_{i_1 k_1}(x)(t, t) = -\mathcal{L}\psi'_{i_1 k_1}(\pi_{2k_1}(x))(d\pi_{2k_1}(x)(t), d\pi_{2k_1}(x)(t)) > 0,$$

as $-\psi'_{i_1 k_1}$ is strictly PSH on $G_{i_1 k_1}$. This yields

$$(10) \qquad\qquad -\mathcal{L}s_{ik}(x)(t, t) > 0 \quad \text{for all } t \in E_x \setminus \{0\}.$$

By (8), (9) and (10) we conclude that there exists $C_x > 0$ such that if $C > C_x$ then $\mathcal{L}(C\eta_k \circ \pi - s_{ik})(x)(t, t) > 0$ for all $t \in T_x \widetilde{X} \setminus \{0\}$. This finishes the proof. $\qquad \square$

We look now at the nature of the base space X as implied by the hypotheses made on the curvature of the bundles involved in our results. Recall that a compact irreducible complex space X of dimension n is called Moishezon if X possesses n algebraically independent meromorphic functions, ie if the transcendence degree of the field of meromorphic functions on X equals the complex dimension of X. Let X' and X be compact irreducible spaces and $h \colon X' \to X$ be a proper modification. Then h induces an isomorphism of the fields of meromorphic functions on X' and X, respectively [40, Theorem 2.1.18], hence X' is Moishezon if and only if X is too. In [43], Moishezon showed that if X is a Moishezon space, then there exists a proper modification $h \colon X' \to X$, obtained by a finite number of blow-ups with smooth centers, such that X' is a projective algebraic manifold (for a proof see also [40, Theorem 2.2.16]).

Lemma 2.2 yields in particular the following:

Proposition 2.3 *If (X, ω) is a compact (reduced) Hermitian space endowed with a singular Hermitian holomorphic line bundle (L, h) such that $c_1(L, h) \geq \varepsilon\omega$ for some constant $\varepsilon > 0$ then X is Moishezon.*

Proof Let $\pi \colon \widetilde{X} \to X$ be a resolution of singularities as in Lemma 2.2, with exceptional divisor Σ and Hermitian form $\Omega = C\pi^*\omega + c_1(F, \theta)$, where $F = \mathcal{O}_{\widetilde{X}}(-\Sigma)$. Consider the line bundles $E_p = \pi^*L^p \otimes F$ with singular Hermitian metrics $\theta_p = \pi^*h^p \otimes \theta$. Then

$$c_1(E_p, \theta_p) = p\pi^*c_1(L, h) + c_1(F, \theta) \geq p\varepsilon\pi^*\omega + c_1(F, \theta) \geq \Omega$$

provided that $p\varepsilon \geq C$. Hence $\widetilde{X}$ carries a singular Hermitian holomorphic line bundle with strictly positive curvature in the sense of currents. By [38] (see also [40, Theorem 2.3.8]) it follows that $\widetilde{X}$ is Moishezon and hence X is, too. If (X, ω) is

Kähler, then $\widetilde{X}$ is already projective. Indeed, $(\widetilde{X}, \Omega)$ is Kähler and Moishezon, so by a theorem of Moishezon is projective (see eg [40, Theorem 2.2.26]). □

The paper [42] (see also [40, Theorems 3.4.10 and 3.4.14]) gives an integral criterion for a complex space with isolated singularities to be Moishezon, generalizing the criterion of Siu and Demailly from the smooth case.

We recall next the following projectivity criterion:

Proposition 2.4 (Grauert [30]) *If (X, ω) is a compact (reduced) Hermitian space endowed with a $\mathscr{C}^2$ Hermitian holomorphic line bundle (L, h) such that $c_1(L, h) \geq \varepsilon \omega$ for some constant $\varepsilon > 0$, then L is ample and X is projective.*

This follows from [30, Satz 2, page 343]; see also [30, Satz 3, page 346]. If X is normal, Grauert actually shows more: if Ω is a continuous Kähler metric whose de Rham cohomology class is integral, then there exists a Hermitian holomorphic line bundle (L, h) with $\mathscr{C}^2$ Hermitian metric and $c_1(L, h) = \Omega$; hence L is ample and X is projective.

2.4 L^2–estimates for $\overline{\partial}$

The following version of Demailly's estimates for the $\overline{\partial}$ operator [15, Théorème 5.1] will be needed in our proofs:

Theorem 2.5 [15] *Let Y, with $\dim Y = n$, be a complete Kähler manifold and let Ω be a Kähler form on Y (not necessarily complete) such that its Ricci form $\mathrm{Ric}_\Omega \geq -2\pi B \Omega$ on Y for some constant $B > 0$. Let (L_p, h_p) be singular Hermitian holomorphic line bundles on Y such that $c_1(L_p, h_p) \geq 2 a_p \Omega$, where $a_p \to \infty$ as $p \to \infty$, and fix p_0 such that $a_p \geq B$ for all $p > p_0$. If $p > p_0$ and $g \in L^2_{0,1}(Y, L_p, \mathrm{loc})$ satisfies $\overline{\partial} g = 0$ and $\int_Y |g|^2_{h_p} \Omega^n < \infty$, then there exists $u \in L^2_{0,0}(Y, L_p, \mathrm{loc})$ such that $\overline{\partial} u = g$ and $\int_Y |u|^2_{h_p} \Omega^n \leq \frac{1}{a_p} \int_Y |g|^2_{h_p} \Omega^n$.*

Proof We write $L_p = F_p \otimes K_Y$, where $F_p = L_p \otimes K_Y^{-1}$, and the canonical line bundle K_Y is endowed with the metric h^{K_Y} induced by Ω. If $\theta_p = h_p \otimes h^{K_Y^{-1}}$ is the metric induced on F_p then

$$c_1(F_p, \theta_p) = c_1(L_p, h_p) - c_1(K_Y, h^{K_Y})$$

$$= c_1(L_p, h_p) + \frac{1}{2\pi} \mathrm{Ric}_\Omega \geq (2a_p - B)\Omega \geq a_p \Omega$$

for $p > p_0$. The theorem follows by using the isometries

$$L^2_{0,j}(Y, L_p, \mathrm{loc}) \cong L^2_{n,j}(Y, F_p, \mathrm{loc}) \quad \text{for } j = 0, 1$$

and applying [15, Théorème 5.1] (see also [10, Corollaries 4.2 and 4.3]). □

2.5 Special weights of Hermitian metrics on reference covers

Let (X, ω) be a compact Kähler manifold of dimension n. Let (U, z), $z = (z_1, \ldots, z_n)$, be local coordinates centered at a point $x \in X$. For $r > 0$ and $y \in U$ we denote by

$$\Delta^n(y, r) = \{z \in U : |z_j - y_j| \leq r, \ j = 1, \ldots, n\}$$

the (closed) polydisk of polyradius $(r, \ldots, r)$ centered at y. The coordinates (U, z) are called Kähler at $y \in U$ if

$$(11) \qquad \omega_z = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j + O(|z - y|^2) \quad \text{on } U.$$

Definition 2.6 A *reference cover* of X consists of the following data: for $j = 1, \ldots, N$, a set of points $x_j \in X$ and

   (1) Stein open simply connected coordinate neighborhoods $(U_j, w^{(j)})$ centered at $x_j \equiv 0$;

   (2) $R_j > 0$ such that $\Delta^n(x_j, 2R_j) \Subset U_j$ and for every $y \in \Delta^n(x_j, 2R_j)$ there exist coordinates on U_j which are Kähler at y;

   (3) $X = \bigcup_{j=1}^{N} \Delta^n(x_j, R_j)$.

Given the reference cover as above we set $R = \min R_j$.

We can construct a reference cover as follows: for $x \in X$ fix a Stein open simply connected neighborhood U of $x \equiv 0 \in \mathbb{C}^n$ and fix $R > 0$ such that the polydisk $\Delta^n(x, 2R) \Subset U$ and for every $y \in \Delta^n(x, 2R)$ there exist coordinates (U, z) which are Kähler at y. By compactness there exist $x_1, \ldots, x_N \in X$ such that the above conditions are fulfilled.

On U_j we consider the partial derivatives D_w^α of order $|\alpha|$, with $\alpha \in \mathbb{N}^{2n}$, corresponding to the real coordinates associated to $w = w^{(j)}$. For a function $\varphi \in \mathscr{C}^k(U_j)$ we set

$$(12) \qquad \|\varphi\|_k = \|\varphi\|_{k,w} = \sup\{|D_w^\alpha \varphi(w)| : w \in \Delta^n(x_j, 2R_j), |\alpha| \leq k\}.$$

Let (L, h) be a Hermitian holomorphic line bundle on X, where the metric h is of class $\mathscr{C}^l$. Note that $L|_{U_j}$ is trivial. For $k \leq l$ set

$$(13) \qquad \begin{aligned} \|h\|_{k,U_j} &= \inf\{\|\varphi_j\|_k : \varphi_j \in \mathscr{C}^l(U_j) \text{ is a weight of } h \text{ on } U_j\}, \\ \|h\|_k &= \max\{1, \|h\|_{k,U_j} : 1 \leq j \leq N\}. \end{aligned}$$

Recall that φ_j is a weight of h on U_j if there exists a holomorphic frame e_j of L on U_j such that $|e_j|_h = e^{-\varphi_j}$.

Lemma 2.7 *There exists $C > 1$ (depending on the reference cover) with the following property: Given any Hermitian line bundle (L, h) on X, where h is of class $\mathscr{C}^3$, any $j \in \{1, \ldots, N\}$ and any $x \in \Delta^n(x_j, R_j)$ there exist coordinates $z = (z_1, \ldots, z_n)$ on $\Delta^n(x, R)$ which are centered at $x \equiv 0$ and Kähler coordinates for x such that:*

(i) *$dm \le (1 + Cr^2) \omega^n/n!$ and $\omega^n/n! \le (1 + Cr^2) dm$ hold on $\Delta^n(x, r)$ for any $r < R$, where $dm = dm(z)$ is the Euclidean volume relative to the coordinates z.*

(ii) *(L, h) has a weight φ on $\Delta^n(x, R)$ with $\varphi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + \widetilde{\varphi}(z)$, where $\lambda_j \in \mathbb{R}$ and $|\widetilde{\varphi}(z)| \le C \|h\|_3 |z|^3$ for $z \in \Delta^n(x, R)$.*

Proof By the properties of a reference cover there exist coordinates z on U_j which are Kähler for $x \in \Delta^n(x_j, R_j)$, so $\omega = \frac{i}{2} \sum_{l=1}^n dz_l \wedge d\overline{z}_l + O(|z - x|^2)$ and (i) holds with a constant C_j uniform for $x \in \Delta^n(x_j, R_j)$. Let e_j be a frame of L on U_j and φ' be a weight of h on U_j with $|e_j|_h = e^{-\varphi'}$ and $\|\varphi'\|_{3,w} \le 2\|h\|_3$; see (12)–(13). By translation we may assume $x = 0$ and write $\varphi'(z) = \operatorname{Re} f(z) + \varphi'_2(z) + \varphi'_3(z)$, where $f(z)$ is a holomorphic polynomial of degree ≤ 2 in z, $\varphi'_2(z) = \sum_{k,l=1}^n \mu_{kl} z_k \overline{z}_l$, and $\operatorname{Re} f(z) + \varphi'_2(z)$ is the Taylor polynomial of order 2 of φ' at 0. Note that $\|\varphi'\|_{3,z} \le C'_j \|\varphi'\|_{3,w} \le 2C'_j \|h\|_3$, where $\|\varphi'\|_{3,z}$ is the sup norm on $\Delta^n(x, R)$ of the derivatives of order 3 of φ' in the coordinates z and C'_j is a constant uniform for $x \in \Delta^n(x_j, R_j)$. This follows from the fact that z and w are coordinates on $U_j \supseteq \Delta^n(x_j, 2R_j)$. We conclude that $|\varphi'_3(z)| \le 2C'_j \|h\|_3 |z|^3$ for $z \in \Delta^n(x, R)$.

Consider the frame $\widetilde{e}_j = e^f e_j$ of L on U_j. Then $|\widetilde{e}_j|_h = e^{\operatorname{Re} f - \varphi'} = e^{-\varphi}$, so $\varphi(z) := \varphi'_2(z) + \varphi'_3(z)$ is a weight of h on U_j. By a unitary change of coordinates we may assume $\varphi(z) = \sum_{l=1}^n \lambda_l |z_l|^2 + \widetilde{\varphi}(z)$. In these new coordinates, $\omega^n/n!$ and $\widetilde{\varphi}(z)$ satisfy the desired estimates with a constant C_j uniform for $x \in \Delta^n(x_j, R_j)$. Finally we let $C = \max_{1 \le j \le N} C_j$. $\qquad \square$

3 Proofs of Theorems 1.1, 1.2 and 1.3

We use here the notations introduced in Section 2 and we start with two lemmas that will be needed in the proof of Theorem 1.1:

Lemma 3.1 *Let D be a divisor in a complex manifold Y and θ be a smooth Hermitian metric on $\mathcal{O}(-D)$ over Y with weight φ over $Y \setminus D$. Then $\lim_{y \to x, \, y \in Y \setminus D} \varphi(y) = -\infty$ for every $x \in D$.*

Proof Let U_α be a neighborhood of x where D has defining function $f_\alpha \in \mathcal{O}(U_\alpha)$ and θ has weight $\varphi_\alpha \in \mathscr{C}^\infty(U_\alpha)$. The transition function of $\mathcal{O}(-D)$ on $U_\alpha \cap (Y \setminus D)$ is $g = 1/f_\alpha$, so $\varphi_\alpha = \varphi + \log|g|$ and $\varphi = \varphi_\alpha + \log|f_\alpha|$. $\qquad \square$

Lemma 3.2 *Let (X, ω) and (L_p, h_p) satisfy assumptions (A)–(B) and let $F = \mathcal{O}_{\tilde{X}}(-\Sigma)$, θ and $\Omega = C\pi^*\omega + c_1(F, \theta)$ be as in Lemma 2.2. Then there exist $\alpha \in (0, 1)$, $b_p \in \mathbb{N}$ and singular Hermitian metrics $\tilde{h}_p$ on $\pi^*L_p|_{\tilde{X}\setminus\Sigma}$ such that $a_p \geq Cb_p$, $b_p \to \infty$ and $b_p/A_p \to 0$ as $p \to \infty$, $\tilde{h}_p \geq \alpha^{b_p}\pi^*h_p$ and $c_1(\pi^*L_p, \tilde{h}_p) \geq b_p\Omega$ on $\tilde{X} \setminus \Sigma$. Moreover, for every open relatively compact subset $\tilde{U}$ of $\tilde{X} \setminus \Sigma$ there exists a constant $\beta_{\tilde{U}} > 1$ such that $\tilde{h}_p \leq \beta_{\tilde{U}}^{b_p}\pi^*h_p$ on $\tilde{U}$.*

Proof If h' and h'' are singular Hermitian metrics on some holomorphic line bundle G then by $h' \geq ch''$ we mean that $|e|^2_{h'} \geq c|e|^2_{h''}$ for any $e \in G$. Consider the line bundles $E_p = \pi^*L_p \otimes F^{b_p}$ with metrics $\theta_p = \pi^*h_p \otimes \theta^{b_p}$, where $b_p \in \mathbb{N}$. Then

$$c_1(E_p, \theta_p) = \pi^*c_1(L_p, h_p) + b_p c_1(F, \theta) \geq a_p\pi^*\omega + b_p c_1(F, \theta) \geq b_p\Omega$$

provided that $a_p \geq Cb_p$. Since F is trivial on $\tilde{X} \setminus \Sigma$, we have $\pi^*L_p|_{\tilde{X}\setminus\Sigma} \cong E_p|_{\tilde{X}\setminus\Sigma}$ and we can find a smooth weight φ of θ on $\tilde{X} \setminus \Sigma$ by setting $|f|^2_\theta = e^{-2\varphi}$, where f is a holomorphic frame of F on $\tilde{X} \setminus \Sigma$. Let $\tilde{h}_p$ be the metric of $\pi^*L_p|_{\tilde{X}\setminus\Sigma}$ defined by $\tilde{h}_p = e^{-2b_p\varphi}\pi^*h_p$. Then $c_1(\pi^*L_p, \tilde{h}_p) = c_1(E_p, \theta_p) \geq b_p\Omega$ on $\tilde{X} \setminus \Sigma$. Since $\varphi \in \mathscr{C}^\infty(\tilde{X} \setminus \Sigma)$ and, by Lemma 3.1, $\varphi(y) \to -\infty$ as $y \to \Sigma$, it follows that there exists $\alpha \in (0, 1)$ such that $e^{-2\varphi} \geq \alpha$ on $\tilde{X} \setminus \Sigma$. Moreover, if $\tilde{U}$ is an open relatively compact subset of $\tilde{X} \setminus \Sigma$, there exists $\beta_{\tilde{U}} > 1$ such that $e^{-2\varphi} \leq \beta_{\tilde{U}}$ on $\tilde{U}$. These imply that $\tilde{h}_p \geq \alpha^{b_p}\pi^*h_p$ on $\tilde{X} \setminus \Sigma$ and $\tilde{h}_p \leq \beta_{\tilde{U}}^{b_p}\pi^*h_p$ on $\tilde{U}$. The lemma follows if we choose $b_p \in \mathbb{N}$ such that $a_p \geq Cb_p$, $b_p \to \infty$ and $b_p/A_p \to 0$ as $p \to \infty$. $\square$

Proof of Theorem 1.1 Note that (ii) follows at once from (i) by using (7). The proof of (i) will be done in two steps.

Step 1 We show here that $\frac{1}{A_p}\log P_p \to 0$ as $p \to \infty$ in $L^1_{\text{loc}}(X_{\text{reg}}, \omega^n)$. Fix $x \in X_{\text{reg}}$, $W \Subset X_{\text{reg}}$ a contractible Stein coordinate neighborhood of x and $r_0 > 0$ such that the (closed) ball $V := B(x, 2r_0)$ is a subset of W, and set $U := B(x, r_0)$.

Note that on a Stein manifold M we have $H^1(M, \mathcal{O}^*) \cong H^2(M, \mathbb{Z})$ due to Cartan's Theorem B (see eg [35, page 201]), thus any holomorphic line bundle L over a Stein contractible manifold Ω is holomorphically trivial (this is due to [50], and is of course a special case of the Oka–Grauert principle). Thus there exist local holomorphic frames $e'_p: W \to L_p$, for all p. Let ψ'_p be the corresponding PSH weights of h_p on W, $|e'_p|^2_{h_p} = e^{-2\psi'_p}$. The sequence of currents $\{\frac{1}{A_p}c_1(L_p, h_p)\}$ has uniformly bounded mass, so from [23, Proposition A.16] (see also [21]) it follows that there exist PSH functions ψ_p on int V such that $dd^c\psi_p = c_1(L_p, h_p)$ and the sequence $\{\frac{1}{A_p}\psi_p\}$ is bounded, hence relatively compact [36, Theorem 3.2.12], in $L^1_{\text{loc}}(\text{int } V, \omega^n)$. Since $\psi'_p - \psi_p$ is pluriharmonic, $\psi'_p - \psi_p = \text{Re } f_p$ for some function $f_p \in \mathcal{O}(\text{int } V)$.

Considering the local frames $e_p = e^{f_p} e_p'$ of $L_p|_{\mathrm{int}\, V}$, we obtain

$$|e_p|^2_{h_p} = e^{2\,\mathrm{Re}\, f_p} |e_p'|^2_{h_p} = e^{-2\psi_p},$$

ie ψ_p is the PSH weight of h_p relative to the frame e_p.

Let $\{b_p\}$ be as in Lemma 3.2. We prove that there exist $C' > 1$ and $p_0 \in \mathbb{N}$ such that

$$(14) \qquad -\frac{b_p \log C'}{A_p} \le \frac{\log P_p(z)}{A_p} \le \frac{\log(C' r^{-2n})}{A_p} + \frac{2}{A_p}\Big(\max_{B(z,r)} \psi_p - \psi_p(z)\Big)$$

holds for all $p > p_0$, $0 < r < r_0$ and $z \in U$ with $\psi_p(z) > -\infty$. The upper bound in (14) is proved exactly as the corresponding upper bound from the proof of [12, Theorem 5.1]. For the lower bound, we show that there exist $c \in (0,1)$ and $p_0 \in \mathbb{N}$ such that if $p > p_0$ and $z \in U$ with $\psi_p(z) > -\infty$, then there exists a section $S_{z,p} \in H^0_{(2)}(X, L_p)$ satisfying $S_{z,p}(z) \ne 0$ and

$$(15) \qquad\qquad c^{b_p} \|S_{z,p}\|^2_p \le |S_{z,p}(z)|^2_{h_p}.$$

This implies that

$$\frac{1}{A_p} \log P_p(z) = \frac{1}{A_p} \max_{\|S\|_p = 1} \log |S(z)|^2_{h_p} \ge \frac{b_p \log c}{A_p}.$$

To prove (15) we work on $\widetilde{X} \setminus \Sigma$ and recall that $\pi \colon \widetilde{X} \setminus \Sigma \to X_{\mathrm{reg}}$ is a biholomorphism. Let $\Omega \ge \pi^* \omega$ be the Kähler form on $\widetilde{X}$ constructed in Lemma 2.2, and b_p and $\widetilde{h}_p$ be as in Lemma 3.2. Then $b_p \to \infty$ and

$$c_1(\pi^* L_p|_{\widetilde{X} \setminus \Sigma}, \widetilde{h}_p) \ge b_p \Omega \quad \text{on } \widetilde{X} \setminus \Sigma.$$

Since Ω is a Kähler form on $\widetilde{X}$ we have $\mathrm{Ric}_\Omega \ge -2\pi B \Omega$ on $\widetilde{X}$ for some constant $B > 0$. Moreover, since $\widetilde{X}$ is a compact Kähler manifold, $\widetilde{X} \setminus \Sigma$ has a complete Kähler metric (see [15; 48]). Repeating the argument in the proof of [12, Theorem 5.1] (see also [10, Theorems 4.2 and 4.3]) one applies the Ohsawa–Takegoshi extension theorem [49] and then solves a suitable $\bar\partial$–equation using Theorem 2.5, as in [17, Proposition 3.1; 19, Section 9], to show the following: there exist $C'' > 1$ and $p_0 \in \mathbb{N}$ such that if $p > p_0$ and $\widetilde{z} \in \pi^{-1}(U)$ with $\psi_p \circ \pi(\widetilde{z}) > -\infty$, then there exists a section $\widetilde{S} \in H^0(\widetilde{X} \setminus \Sigma, \pi^* L_p)$ satisfying $\widetilde{S}(\widetilde{z}) \ne 0$ and

$$\int_{\widetilde{X} \setminus \Sigma} |\widetilde{S}|^2_{\widetilde{h}_p} \frac{\Omega^n}{n!} \le C'' |\widetilde{S}(\widetilde{z})|^2_{\widetilde{h}_p}.$$

It is important to recall here that the weight of $\widetilde{h}_p$ near $\widetilde{z}$ is the sum of $\psi_p \circ \pi$ and a smooth function. By Lemma 3.2 we have for all p that $\widetilde{h}_p \ge \alpha^{b_p} \pi^* h_p$ on $\widetilde{X} \setminus \Sigma$ and

$\widetilde{h}_p \leq \beta^{b_p} \pi^* h_p$ on $\pi^{-1}(U)$ for some constant $\beta > 1$. Since $\Omega \geq \pi^* \omega$, these imply that

$$\alpha^{b_p} \int_{\widetilde{X} \setminus \Sigma} |\widetilde{S}|^2_{\pi^* h_p} \frac{\pi^* \omega^n}{n!} \leq C'' \beta^{b_p} |\widetilde{S}(\widetilde{z})|^2_{\pi^* h_p}.$$

Fix a constant $c \in (0, 1)$ with $C'' c^{b_p} \leq (\alpha/\beta)^{b_p}$ for all p and let $S_{z,p}$, where $z = \pi(\widetilde{z})$, be the section of $L_p|_{X_{\text{reg}}}$ induced by $\widetilde{S}$. Since X is normal it follows that $S_{z,p}$ extends to a holomorphic section of L_p on X. Moreover, $S_{z,p}(z) \neq 0$ and (15) holds for $S_{z,p}$ and c. The proof of (14) is now complete.

To conclude Step 1, it suffices to show that every subsequence of $\{\frac{1}{A_p} \log P_p\}$ has a subsequence convergent to 0 in $L^1(U, \omega^n)$. Without loss of generality, we prove that $\{\frac{1}{A_p} \log P_p\}$ has a subsequence convergent to 0 in $L^1(U, \omega^n)$. Since $\{\frac{1}{A_p} \psi_p\}$ is locally uniformly upper bounded in int V and relatively compact in $L^1_{\text{loc}}(\text{int } V, \omega^n)$, there exists a subsequence $\{\psi_{p_j}\}$ such that

$$\frac{1}{A_{p_j}} \psi_{p_j} \to \psi = \left(\limsup \frac{1}{A_{p_j}} \psi_{p_j} \right)^*$$

in $L^1_{\text{loc}}(\text{int } V, \omega^n)$ and a.e. on int V, where $\psi \in \text{PSH}(\text{int } V)$. Moreover, by the Hartogs lemma,

$$\limsup \frac{1}{A_{p_j}} \max_{B(z,r)} \psi_{p_j} \leq \max_{B(z,r)} \psi$$

for each $z \in U$ and $r < r_0$ (see eg [36, Theorem 3.2.13]). Letting $p_j \to \infty$ in (14) we get, since $b_p/A_p \to 0$, that

$$0 \leq \liminf \frac{\log P_{p_j}(z)}{A_{p_j}} \leq \limsup \frac{\log P_{p_j}(z)}{A_{p_j}} \leq 2\left(\max_{B(z,r)} \psi - \psi(z) \right)$$

for a.e. $z \in U$ and every $r < r_0$. Letting $r \searrow 0$ and using the upper semicontinuity of ψ we deduce that $\frac{1}{A_{p_j}} \log P_{p_j} \to 0$ a.e. on U. Since $\{\frac{1}{A_p} \psi_p\}$ is locally uniformly upper bounded in int V, it follows by (14) that there exists a constant $C'' > 0$ such that

$$\left| \frac{1}{A_{p_j}} \log P_{p_j} \right| \leq C'' - \frac{2}{A_{p_j}} \psi_{p_j} \quad \text{a.e. on } U.$$

Since $\psi_{p_j}, \psi \in L^1(U, \omega^n)$, $\frac{1}{A_{p_j}} \psi_{p_j} \to \psi$ a.e. on U and in $L^1(U, \omega^n)$ and, since $\frac{1}{A_{p_j}} \log P_{p_j} \to 0$ a.e. on U, the generalized Lebesgue dominated convergence theorem gives that $\frac{1}{A_{p_j}} \log P_{p_j} \to 0$ in $L^1(U, \omega^n)$.

Step 2 To complete the proof of (i) we show here that there exists a compact set $K \subset X$ such that $X_{\text{sing}} \subset \text{int } K$ and

$$\frac{1}{A_p} \int_K |\log P_p| \omega^n \to 0 \quad \text{as } p \to \infty.$$

Let $H^0_{(2)}(\widetilde{X}, \pi^*L_p)$ be the Bergman spaces from Lemma 2.1. We note that there exists $M > 0$ such that

$$(16) \qquad \int_{\widetilde{X}} c_1(\pi^*L_p, \pi^*h_p) \wedge \Omega^{n-1} \le MA_p \quad \text{for all } p \ge 1.$$

Indeed, since $c_1(L_p, h_p)$, $T_0 \in \mathscr{T}(X)$ and $c_1(L_p, h_p) \le A_p T_0$, we have

$$c_1(\pi^*L_p, \pi^*h_p) = \pi^*c_1(L_p, h_p) \le A_p \pi^* T_0.$$

This yields (16) with $M = \int_{\widetilde{X}} \pi^* T_0 \wedge \Omega^{n-1}$.

We now fix $y \in \Sigma$ and $\widetilde{W}$ an open neighborhood of y in $\widetilde{X}$ biholomorphic to a ball in $\mathbb{C}^n$. Using (16) and repeating an argument from Step 1, we find holomorphic frames $\widetilde{e}_p$ of $\pi^*L_p|_{\widetilde{W}}$ for which the corresponding PSH weights $\widetilde{\psi}_p$ of π^*h_p are such that $\{\frac{1}{A_p}\widetilde{\psi}_p\}$ is bounded in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$, hence locally uniformly upper bounded on $\widetilde{W}$ and relatively compact in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$. If $\{\widetilde{S}^p_j : 1 \le j \le d_p\}$ is an orthonormal basis of $H^0_{(2)}(\widetilde{X}, \pi^*L_p)$ we write $\widetilde{S}^p_j = \widetilde{s}^p_j \widetilde{e}_p$ and let

$$\widetilde{v}_p = \frac{1}{2} \log\left(\sum_{j=1}^{d_p} |\widetilde{s}^p_j|^2 \right) \in \text{PSH}(\widetilde{W}).$$

By Lemma 2.1, $P_p \circ \pi$ is the Bergman kernel function of $H^0_{(2)}(\widetilde{X}, \pi^*L_p)$, hence

$$\frac{1}{A_p}\widetilde{v}_p - \frac{1}{A_p}\widetilde{\psi}_p = \frac{1}{2A_p} \log P_p \circ \pi.$$

We claim that $\frac{1}{A_p} \log P_p \circ \pi \to 0$ in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$. As in Step 1, it suffices to produce a subsequence with this property. Since $\{\frac{1}{A_p}\widetilde{\psi}_p\}$ is relatively compact in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$ there is a subsequence $\{\frac{1}{A_{p_j}}\widetilde{\psi}_{p_j}\}$ convergent in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$ to a PSH function $\widetilde{\psi}$ on $\widetilde{W}$. By Step 1, we have $\frac{1}{A_p} \log P_p \to 0$ as $p \to \infty$ in $L^1_{\text{loc}}(X_{\text{reg}}, \omega^n)$ so, since $\pi \colon \widetilde{X} \setminus \Sigma \to X_{\text{reg}}$ is biholomorphic, $\frac{1}{A_p} \log P_p \circ \pi \to 0$ in $L^1_{\text{loc}}(\widetilde{W} \setminus \Sigma, \Omega^n)$. Thus $\frac{1}{A_{p_j}}\widetilde{v}_{p_j} \to \widetilde{\psi}$ in $L^1_{\text{loc}}(\widetilde{W} \setminus \Sigma, \Omega^n)$. By the proof of [10, Theorem 1.1(i)], $\{\frac{1}{A_{p_j}}\widetilde{v}_{p_j}\}$ is locally uniformly upper bounded in $\widetilde{W}$ and converges to $\widetilde{\psi}$ in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$. Hence $\frac{1}{A_{p_j}} \log P_{p_j} \circ \pi \to 0$ in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$, which proves our claim.

Since $y \in \Sigma$ was arbitrary and Σ is compact we can find an open set $\widetilde{U} \supset \Sigma$ such that $\frac{1}{A_p} \log P_p \circ \pi \to 0$ in $L^1(\widetilde{U}, \Omega^n)$. Then we fix a compact set $K \subset X$ such that $X_{\text{sing}} \subset \text{int } K$ and $\pi^{-1}(K) \subset \widetilde{U}$. As $\pi \colon \widetilde{X} \setminus \Sigma \to X_{\text{reg}}$ is biholomorphic and $\pi^*\omega \le \Omega$, we have

$$\frac{1}{A_p} \int_K |\log P_p| \, \omega^n = \frac{1}{A_p} \int_{K \cap X_{\text{reg}}} |\log P_p| \, \omega^n$$

$$= \frac{1}{A_p} \int_{\pi^{-1}(K)\setminus\Sigma} |\log P_p \circ \pi| \, \pi^* \omega^n$$

$$\leq \frac{1}{A_p} \int_{\pi^{-1}(K)} |\log P_p \circ \pi| \, \Omega^n \to 0 \quad \text{as } p \to \infty.$$

This concludes Step 2 and finishes the proof of Theorem 1.1. $\qquad\square$

Theorem 1.1 holds under the weaker hypotheses obtained if one replaces the domination assumption (2) by (16), where $\pi \colon \widetilde{X} \to X$ is the resolution of singularities fixed in Section 2. The proof goes through without change. Due to the presence of singularities of X, it is not clear whether (16) holds true without the domination condition (2), ie if the mass of the pull-back currents $\pi^* c_1(L_p, h_p)$ on $\widetilde{X}$ is dominated by the mass of the currents $c_1(L_p, h_p)$ on X, uniformly in p.

Proof of Theorem 1.2 We repeat the argument in Step 1 from the previous proof, working directly on (X, L_p, h_p) with the Kähler form ω. $\qquad\square$

Proof of Theorem 1.3 We use methods from [5, Section 2]. Let us consider a reference cover of X as in Definition 2.6. We fix $x \in X$, so $x \in \Delta^n(x_j, R_j)$ for some j, and we pick coordinates z centered at x as in Lemma 2.7. Let

$$\varphi_p(z) = \varphi_p'(z) + \widetilde{\varphi}_p(z), \quad \varphi_p'(z) = \sum_{l=1}^{n} \lambda_l^p |z_l|^2,$$

be a weight of h_p on $\Delta^n(x, R)$ such that $\widetilde{\varphi}_p$ satisfies Lemma 2.7(ii) and let e_p be a frame of L_p on U_j with $|e_p|_{h_p} = e^{-\varphi_p}$. Finally, let $r_p \in \left(0, \frac{R}{2}\right)$ be an arbitrary number, which will be specified later.

We begin by estimating the norm of a section $S \in H^0(X, L_p)$ at x. Writing $S = s e_p$, where $s \in \mathcal{O}(\Delta^n(x, R))$, we obtain, by the subaveraging inequality for PSH functions,

$$|S(x)|_{h_p}^2 = |s(0)|^2 \leq \frac{\int_{\Delta^n(0, r_p)} |s|^2 e^{-2\varphi_p'} \, dm}{\int_{\Delta^n(0, r_p)} e^{-2\varphi_p'} \, dm}.$$

If $C > 1$ is the constant from Lemma 2.7, we have

$$\int_{\Delta^n(0, r_p)} |s|^2 e^{-2\varphi_p'} \, dm \leq (1 + C r_p^2) \exp\left(2 \max_{\Delta^n(0, r_p)} \widetilde{\varphi}_p\right) \int_{\Delta^n(0, r_p)} |s|^2 e^{-2\varphi_p} \, \frac{\omega^n}{n!}$$

$$\leq (1 + C r_p^2) \exp(2C \|h_p\|_3 r_p^3) \|S\|_p^2.$$

Set

$$E(r) := \int_{|\xi| \leq r} e^{-2|\xi|^2} \, dm(\xi) = \frac{\pi}{2}(1 - e^{-2r^2}),$$

where dm is the Lebesgue measure on $\mathbb{C}$. Since $\lambda_j^p \geq a_p$ we obtain

$$\frac{E(r_p\sqrt{a_p})^n}{\lambda_1^p \cdots \lambda_n^p} \leq \int_{\Delta^n(0,r_p)} e^{-2\varphi_p'}\, dm \leq \int_{\mathbb{C}^n} e^{-2\varphi_p'}\, dm = \frac{(\pi/2)^n}{\lambda_1^p \cdots \lambda_n^p}.$$

Combining these estimates it follows that

$$(17) \qquad |S(x)|_{h_p}^2 \leq \frac{(1+Cr_p^2)\exp(2C\|h_p\|_3 r_p^3)}{E(r_p\sqrt{a_p})^n}\lambda_1^p \cdots \lambda_n^p \|S\|_p^2.$$

By taking the supremum in (17) over all $S \in H^0(X, L_p)$ with $\|S\|_p = 1$, we get

$$(18) \qquad \frac{P_p(x)}{\lambda_1^p \cdots \lambda_n^p} \leq \frac{(1+Cr_p^2)\exp(2C\|h_p\|_3 r_p^3)}{E(r_p\sqrt{a_p})^n} \quad \text{for all } r_p \in \left(0, \frac{R}{2}\right).$$

For the lower estimate on P_p, let $0 \leq \chi \leq 1$ be a cut-off function on $\mathbb{C}^n$ with support in $\Delta^n(0,2)$ and $\chi \equiv 1$ on $\Delta^n(0,1)$, and set $\chi_p(z) = \chi(z/r_p)$. Then $F = \chi_p e_p$ is a section of L_p and $|F(x)|_{h_p} = |e_p(x)|_{h_p} = e^{-\varphi_p(0)} = 1$. We have

$$(19) \qquad \|F\|_p^2 \leq \int_{\Delta^n(0,2r_p)} e^{-2\varphi_p} \frac{\omega^n}{n!}$$

$$\leq (1+4Cr_p^2)\exp(16C\|h_p\|_3 r_p^3)\int_{\Delta^n(0,2r_p)} e^{-2\varphi_p'}\, dm$$

$$\leq \left(\frac{\pi}{2}\right)^n \frac{(1+4Cr_p^2)\exp(16C\|h_p\|_3 r_p^3)}{\lambda_1^p \cdots \lambda_n^p}.$$

Set $\alpha = \overline{\partial}F$. Since $\|\overline{\partial}\chi_p\|^2 = \|\overline{\partial}\chi\|^2/r_p^2$, where $\|\overline{\partial}\chi\|$ is the maximum of $|\overline{\partial}\chi|$, we obtain, as above,

$$\|\alpha\|_p^2 = \int_{\Delta^n(0,2r_p)} |\overline{\partial}\chi_p|^2 e^{-2\varphi_p} \frac{\omega^n}{n!} \leq \frac{\|\overline{\partial}\chi\|^2}{r_p^2}\left(\frac{\pi}{2}\right)^n \frac{(1+4Cr_p^2)\exp(16C\|h_p\|_3 r_p^3)}{\lambda_1^p \cdots \lambda_n^p}.$$

Since $a_p \to \infty$ there exists $p_0 \in \mathbb{N}$ such that for $p > p_0$ we can solve the $\overline{\partial}$–equation by Theorem 2.5. We get a smooth section G of L_p with $\overline{\partial}G = \alpha = \overline{\partial}F$ and

$$(20) \qquad \|G\|_p^2 \leq \frac{2}{a_p}\|\alpha\|_p^2 \leq \frac{2\|\overline{\partial}\chi\|^2}{a_p r_p^2}\left(\frac{\pi}{2}\right)^n \frac{(1+4Cr_p^2)\exp(16C\|h_p\|_3 r_p^3)}{\lambda_1^p \cdots \lambda_n^p}.$$

Since $F = e_p$ is holomorphic on $\Delta^n(0,r_p)$, G is holomorphic on $\Delta^n(0,r_p)$ as $\overline{\partial}G = \overline{\partial}F = 0$ there. So the estimate (17) applies to G on $\Delta^n(0,r_p)$ and gives

$$|G(x)|_{h_p}^2 \leq \frac{(1 + Cr_p^2)\exp(2C\|h_p\|_3 r_p^3)}{E(r_p\sqrt{a_p})^n} \lambda_1^p \cdots \lambda_n^p \|G\|_p^2$$

$$\leq \frac{2\|\bar{\partial}\chi\|^2}{a_p r_p^2 E(r_p\sqrt{a_p})^n} \left(\frac{\pi}{2}\right)^n (1 + 4Cr_p^2)^2 \exp(18C\|h_p\|_3 r_p^3).$$

Let $S = F - G \in H^0(X, L_p)$. Then

$$|S(x)|_{h_p}^2 \geq \left(|F(x)|_{h_p} - |G(x)|_{h_p}\right)^2 = \left(1 - |G(x)|_{h_p}\right)^2$$

$$\geq \left(1 - \left(\frac{\pi}{2}\right)^{n/2} \frac{\sqrt{2}\|\bar{\partial}\chi\|(1 + 4Cr_p^2)}{r_p\sqrt{a_p} E(r_p\sqrt{a_p})^{n/2}} \exp(9C\|h_p\|_3 r_p^3)\right)^2 =: K_1(r_p).$$

Moreover, by (19) and (20),

$$\|S\|_p^2 \leq (\|F\|_p + \|G\|_p)^2 \leq \left(\frac{\pi}{2}\right)^n \frac{K_2(r_p)}{\lambda_1^p \cdots \lambda_n^p},$$

where

$$K_2(r_p) = (1 + 4Cr_p^2)\exp(16C\|h_p\|_3 r_p^3)\left(1 + \frac{\sqrt{2}\|\bar{\partial}\chi\|}{r_p\sqrt{a_p}}\right)^2.$$

Therefore,

$$(21) \qquad P_p(x) \geq \frac{|S(x)|_{h_p}^2}{\|S\|_p^2} \geq \frac{\lambda_1^p \cdots \lambda_n^p}{(\pi/2)^n} \frac{K_1(r_p)}{K_2(r_p)}.$$

Note that at x, we have $\omega_x = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ and $c_1(L_p, h_p)_x = dd^c\varphi_p(0) = \frac{i}{\pi}\sum_{j=1}^n \lambda_j^p dz_j \wedge d\bar{z}_j$, thus

$$\frac{c_1(L_p, h_p)_x^n}{\omega_x^n} = \left(\frac{2}{\pi}\right)^n \lambda_1^p \cdots \lambda_n^p.$$

By (18) and (21) we conclude that

$$(22) \qquad \frac{K_1(r_p)}{K_2(r_p)} \leq P_p(x)\frac{\omega_x^n}{c_1(L_p, h_p)_x^n} \leq K_3(r_p)$$

holds for every $x \in X$, $r_p < \frac{R}{2}$ and $p > p_0$, where

$$K_3(r_p) = \left(\frac{\pi/2}{E(r_p\sqrt{a_p})}\right)^n (1 + Cr_p^2)\exp(2C\|h_p\|_3 r_p^3).$$

By (5) we have that $\varepsilon_p = \|h_p\|_3^{1/3} a_p^{-1/2} \to 0$. We set

$$r_p := \varepsilon_p^{1/3}\|h_p\|_3^{-1/3} = \varepsilon_p^{-2/3} a_p^{-1/2}, \quad \text{so} \quad \|h_p\|_3 r_p^3 = \varepsilon_p, \quad r_p\sqrt{a_p} = \varepsilon_p^{-2/3}.$$

As $\|h_p\|_3 \geq 1$, we have $r_p \leq \varepsilon_p^{1/3}$, thus $r_p \to 0$ as $p \to \infty$. With this choice for r_p we obtain

$$K_3(r_p) \leq \left(\frac{\pi/2}{E(\varepsilon_p^{-2/3})}\right)^n (1+C\varepsilon_p^{2/3}) \exp(2C\varepsilon_p) \leq 1+C'\varepsilon_p^{2/3}, \quad \frac{K_1(r_p)}{K_2(r_p)} \geq 1-C'\varepsilon_p^{2/3},$$

where $C' > 0$ is a constant depending only on the reference cover. Therefore (6) follows from (22) and the proof is complete. $\qquad\square$

Remark 3.3 Theorem 1.3 shows that

$$\lim_{p\to\infty} P_p(x)\frac{\omega_x^n}{c_1(L_p, h_p)_x^n} = 1 \quad \text{uniformly on } X.$$

This is a generalization of the asymptotic expansion of the Bergman kernel [9; 13; 37; 40; 41; 52; 57; 59] for $(L_p, h_p) = (L^p, h^p)$, where (L, h) is a positive line bundle with smooth metric h. Indeed, if $(L_p, h_p) = (L^p, h^p)$, we have $a_p = p$ and $\|h_p\|_3 \leq C_h p$, where C_h is a constant depending only on h and the reference cover. Hence

$$\varepsilon_p = \|h_p\|_3^{1/3} a_p^{-1/2} \leq C_h^{1/3} p^{-1/6},$$

so condition (5) is fulfilled. Estimate (6) yields

$$(23) \qquad\qquad \left|\frac{P_p(x)}{p^n}\frac{\omega_x^n}{c_1(L, h)_x^n} - 1\right| \leq \frac{CC_h^{2/9}}{p^{1/9}},$$

hence

$$(24) \qquad |P_p(x) - b_0(x)p^n| \leq C'p^{n-1/9}, \quad \text{where } b_0(x) := \frac{c_1(L, h)_x^n}{\omega_x^n}.$$

By the above-mentioned papers there exists $C > 0$ such that $|P_p(x) - b_0(x)p^n| \leq Cp^{n-1}$ on X, which gives a sharper estimate of $P_p(x)$ than (23)–(24). On the other hand, the method used here can handle the much more general case of sequences of line bundles (L_p, h_p) satisfying the minimal hypotheses of Theorem 1.3.

4 Zeros of holomorphic sections and approximation results

We assume as before that (X, ω) and $(L_p, h_p) \to X$ satisfy conditions (A) and (B). Consider the unit sphere $\mathcal{S}^p \subset H^0_{(2)}(X, L_p)$, with $d_p = \dim H^0_{(2)}(X, L_p)$. We identify the unit sphere $\mathcal{S}^p$ to the unit sphere $\mathbf{S}^{2d_p-1}$ in $\mathbb{C}^{d_p}$ by

$$\mathbf{S}^{2d_p-1} \to \mathcal{S}^p, \quad a = (a_1, \ldots, a_{d_p}) \mapsto S_a = \sum_{j=1}^{d_p} a_j S_j^p,$$

and we let λ_p be the probability measure on $\mathcal{S}^p$ induced by the normalized surface measure on S^{2d_p-1}, denoted also by λ_p (ie $\lambda_p(S^{2d_p-1})=1$). Consider the probability space $\mathcal{S}_\infty = \prod_{p=1}^\infty \mathcal{S}^p$ endowed with the probability measure $\lambda_\infty = \prod_{p=1}^\infty \lambda_p$. We denote by $[\sigma=0]$ the current of integration (with multiplicities) over the zero set of a nontrivial section $\sigma \in H^0_{(2)}(X, L_p)$.

Theorem 4.1 *Let* (X, ω) *and* (L_p, h_p), $p \geq 1$, *satisfy assumptions* (A)–(B) *and assume that* $\sum_{p=1}^\infty \frac{1}{A_p^2} < \infty$. *Then for* λ_∞–*a.e. sequence* $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$ *we have in the weak sense of currents on* X *that*

$$\lim_{p\to\infty} \frac{1}{A_p}([\sigma_p=0] - c_1(L_p, h_p)) = 0.$$

Moreover, if $\lim_{p\to\infty} \frac{1}{A_p} c_1(L_p, h_p) = T$ *for some positive closed current* T *of bidegree* $(1, 1)$ *on* X, *then, for* λ_∞–*a.e. sequence* $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$,

$$\lim_{p\to\infty} \frac{1}{A_p}[\sigma_p=0] = T \quad \textit{weakly on } X.$$

Proof The arguments in [55; 56] (see also [10, Section 5.2]; in all these papers $A_p = p$) imply that, for λ_∞–a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$,

$$\lim_{p\to\infty} \frac{1}{A_p}([\sigma_p=0] - \gamma_p) = 0$$

weakly in the sense of currents on X. Indeed, by working with a countable set of test forms and since

$$\int_X [\sigma_p=0] \wedge \omega^{n-1} = \int_X \gamma_p \wedge \omega^{n-1} = \int_X c_1(L_p, h_p) \wedge \omega^{n-1} = A_p,$$

it suffices to show that, for a fixed test form θ, one has

$$(25) \qquad \lim_{p\to\infty} \frac{1}{A_p}\langle[\sigma_p=0] - \gamma_p, \theta\rangle = 0$$

for λ_∞–a.e. $\sigma = \{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$. Let

$$Y_p: \mathcal{S}_\infty \to \mathbb{C}, \quad Y_p(\sigma) = \frac{1}{A_p}\langle[\sigma_p=0] - \gamma_p, \theta\rangle.$$

The calculations in [55, Sections 3.1–3.3] show that

$$\int_{\mathcal{S}_\infty} Y_p \, d\lambda_\infty = 0, \quad \int_{\mathcal{S}_\infty} |Y_p|^2 \, d\lambda_\infty \leq \frac{AC_\theta}{A_p^2},$$

with

$$A = \frac{1}{\pi^2} \int_{\mathbb{C}^2} (\log|z_1|)^2 e^{-|z_1|^2-|z_2|^2} \, dz,$$

where dz is the Lebesgue measure on $\mathbb{C}^2$, and C_θ is a constant depending only on θ. Then (25) follows since

$$\int_{\mathcal{S}_\infty} \left(\sum_{p=1}^\infty |Y_p|^2 \right) d\lambda_\infty \le AC_\theta \sum_{p=1}^\infty \frac{1}{A_p^2} < +\infty.$$

We now conclude by Theorem 1.1. $\square$

We show next that equidistribution results hold not only for the zeros of random sequences of holomorphic sections but also for the logarithms of their pointwise norms.

Theorem 4.2 *Let* (X, ω) *and* (L_p, h_p), $p \ge 1$, *satisfy assumptions* (A)–(B) *and assume that*

$$(26) \qquad\qquad \liminf_{p \to \infty} \frac{\log d_p}{A_p} = 0.$$

Then there exists an increasing sequence of natural numbers $\{p_j\}_{j \ge 1}$ *such that for* λ_∞*–a.e. sequence* $\{\sigma_p\}_{p \ge 1} \in \mathcal{S}_\infty$ *we have*

$$\lim_{j \to \infty} \frac{\log |\sigma_{p_j}|_{h_{p_j}}}{A_{p_j}} = 0 \quad \text{in } L^1(X, \omega^n).$$

To prove the theorem we need the following elementary lemma, whose proof is included for the convenience of the reader.

Lemma 4.3 *If* S^{2k-1} *is the unit sphere in* $\mathbb{C}^k$ *with surface measure* dA *and*

$$I(k) := -\frac{1}{\operatorname{area}(S^{2k-1})} \int_{S^{2k-1}} \log |z_k| \, dA,$$

where $z = (z_1, \dots, z_k) \in \mathbb{C}^k$, *then there exist numbers* a, $b > 1$ *such that*

$$I(k) \le a \log k + b \quad \text{for all } k \ge 1.$$

Proof We use spherical coordinates $(\theta_1, \dots, \theta_{2k-2}, \varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]^{2k-2} \times [0, 2\pi]$ on S^{2k-1} such that

$$z_k = \sin \theta_{2k-3} \cos \theta_{2k-2} + i \sin \theta_{2k-2},$$

$$dA = \cos \theta_1 \cos^2 \theta_2 \cdots \cos^{2k-2} \theta_{2k-2} \, d\theta_1 \cdots d\theta_{2k-2} \, d\varphi.$$

We obtain

$$I(k) = \frac{J(k)}{2C_{2k-2}C_{2k-3}}, \quad \text{where } C_k = \int_0^{\pi/2} \cos^k t \, dt,$$

and, after performing the change of variables $x = \sin\theta_{2k-2}$ and $y = \sin\theta_{2k-3}$,

$$J(k) = -\int_0^1\int_0^1 (1-x^2)^{k-3/2}(1-y^2)^{k-2}\log(x^2+y^2-x^2y^2)\,dx\,dy.$$

Since $2x^2y^2 \le x^2 + y^2$ for $0 \le x,\, y \le 1$, we have

$$-\log(x^2+y^2-x^2y^2) \le -\log\frac{x^2+y^2}{2},$$

$$(1-x^2)^{k-3/2}(1-y^2)^{k-2} \le (1-x^2-y^2+x^2y^2)^{k-2} \le \left(1-\frac{x^2+y^2}{2}\right)^{k-2},$$

provided that $k \ge 2$. If $D = \{(x,y) \in \mathbb{R}^2 : x \ge 0,\ y \ge 0,\ x^2+y^2 \le 2\}$ it follows that

$$J(k) \le -\iint_D \left(1-\frac{x^2+y^2}{2}\right)^{k-2}\log\frac{x^2+y^2}{2}\,dx\,dy$$

$$= -\frac{\pi}{2}\int_0^{\sqrt{2}} r\left(1-\frac{r^2}{2}\right)^{k-2}\log\frac{r^2}{2}\,dr = -\frac{\pi}{2}\int_0^1 (1-t)^{k-2}\log t\,dt.$$

One shows by induction on $k \ge 2$ that

$$-\int_0^1 (1-t)^{k-2}\log t\,dt = \frac{1}{k-1}\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}\right) \le \frac{1+\log(k-1)}{k-1}.$$

The lemma follows since there exist constants $A,\, B > 0$ such that $A \le C_k\sqrt{k} \le B$ for all $k \ge 1$. Indeed, an integration by parts shows that

$$C_{k+2} = \frac{k+1}{k+2}C_k, \quad \text{so} \quad C_{2k} = \frac{\pi(2k)!}{2^{2k+1}(k!)^2}, \quad C_{2k+1} = \frac{2^{2k}(k!)^2}{(2k+1)!} \quad \text{for } k \ge 0.$$

By Stirling's formula, $k! \sim (k/e)^k\sqrt{k}$ and $(2k)! \sim (2k/e)^{2k}\sqrt{2k}$, which implies our claim. $\qquad\square$

Proof of Theorem 4.2 Using (26) we can find a sequence of integers $p_j \nearrow \infty$ such that $\sum_{j=1}^\infty (\log d_{p_j})/A_{p_j} < \infty$. We define

$$Y_p\colon \mathcal{S}_\infty \to \mathbb{R}, \quad Y_p(\sigma) = \frac{1}{A_p}\int_X \log\frac{|\sigma_p|_{h_p}}{\sqrt{P_p}}\,\omega^n, \quad \text{where } \sigma = \{\sigma_p\}_{p\ge 1}.$$

By Theorem 1.1 we have $\frac{1}{A_p}\log P_p \to 0$ as $p \to \infty$, in $L^1(X,\omega^n)$. Since

$$\log\frac{|\sigma_p|_{h_p}}{\sqrt{P_p}} \le 0 \quad \text{on } X$$

for $\sigma = \{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$, it suffices to show that $Y_{p_j}(\sigma) \to 0$ as $j \to \infty$ for λ_∞–a.e. $\sigma \in \mathcal{S}_\infty$. By Tonelli's theorem we have

$$\int_{\mathcal{S}_\infty} Y_p(\sigma)\, d\lambda_\infty = \frac{1}{A_p} \int_{\mathcal{S}^p} \left(\int_X \log \frac{|\sigma_p|_{h_p}}{\sqrt{P_p}}\, \omega^n \right) d\lambda_p$$

$$= \frac{1}{A_p} \int_X \left(\int_{\mathcal{S}^p} \log \frac{|\sigma_p|_{h_p}}{\sqrt{P_p}}\, d\lambda_p \right) \omega^n.$$

For a fixed $x \in X$, we write $S_l^p = s_l^p e_p$ for some holomorphic frame e_p of L_p near x and we set

$$u = (u_1, \ldots, u_{d_p}), \quad u_l = \frac{s_l^p}{\sqrt{|s_1^p|^2 + \cdots + |s_{d_p}^p|^2}}.$$

Then the integral

$$\int_{\mathcal{S}^p} \log \frac{|\sigma_p(x)|_{h_p}}{\sqrt{P_p(x)}}\, d\lambda_p = \int_{\mathbf{S}^{2d_p-1}} \log |a \cdot u(x)|\, d\lambda_p(a) = -\boldsymbol{I}(d_p)$$

is independent of x, where $a \cdot u = a_1 u_1 + \cdots + a_{d_p} u_{d_p}$ and $\boldsymbol{I}(k)$ is as defined in Lemma 4.3. Using Lemma 4.3 it follows that

$$\int_{\mathcal{S}_\infty} Y_p(\sigma)\, d\lambda_\infty \geq -\frac{a \log d_p + b}{A_p} \int_X \omega^n.$$

The definition of the sequence $\{p_j\}_{j\geq 1}$ shows that

$$\sum_{j=1}^{\infty} \int_{\mathcal{S}_\infty} Y_{p_j}(\sigma)\, d\lambda_\infty > -\infty.$$

Since $Y_p \leq 0$, this implies that $\sum_{j=1}^{\infty} Y_{p_j}$ converges in $L^1(\mathcal{S}_\infty, \lambda_\infty)$, hence $Y_{p_j}(\sigma) \to 0$ as $j \to \infty$ for λ_∞–a.e. $\sigma \in \mathcal{S}_\infty$. $\qquad\square$

Let us give two general examples in which condition (26) holds true.

Proposition 4.4 Let (X, ω), (L_p, h_p), $p \geq 1$, satisfy assumptions (A)–(B) and assume that X is smooth and that each line bundle L_p has a continuous metric h'_p with the following property: every $x \in X$ has a contractible Stein coordinate neighborhood W_x on which each metric h'_p has a weight ψ'_p such that the family $\{\psi'_p / A_p\}_{p\geq 1}$ is equicontinuous on W_x. Then

$$\lim_{p \to \infty} \frac{\log \dim H^0(X, L_p)}{A_p} = 0.$$

Proof Let $\varepsilon > 0$ and let P_p' be the Bergman kernel function of the space $H^0(X, L_p)$ with respect to the metrics h_p' and ω. For $x \in X$ fix $r_x > 0$ so that the (closed) ball $B(x, 2r_x)$ is a subset of W_x and let $U_x = B(x, r_x)$. The proof of the upper bound in (14) works for any metric on L_p (see also [12, (7)]) and shows that

$$P_p'(z) \leq C_x r^{-2n} \exp\left(2 \max_{B(z, r)} \psi_p' - 2\psi_p'(z)\right),$$

for any $p \geq 1$, $r < r_x$ and $z \in U_x$, where C_x is a constant depending only on x. The equicontinuity assumption implies that there exists $r_1 = r_1(x, \varepsilon) < r_x$ such that $2 \max_{B(z, r_1)} \psi_p' - 2\psi_p'(z) \leq A_p \varepsilon$ for all $p \geq 1$ and $z \in U_x$, hence $P_p'(z) \leq C_x r_1^{-2n} \exp(A_p \varepsilon)$. A standard compactness argument now shows that there exists a constant $C' = C'(\varepsilon) > 0$ such that $P_p' \leq C' \exp(A_p \varepsilon)$ holds on X for all $p \geq 1$. It follows that

$$\dim H^0(X, L_p) = \int_X P_p' \frac{\omega^n}{n!} \leq C' \exp(A_p \varepsilon) \int_X \frac{\omega^n}{n!} \quad \text{for all } p \geq 1,$$

which implies the conclusion of the proposition. □

The second general example is provided by the class of semiample line bundles. Recall that a line bundle L on X is called semiample if L^k is globally generated for some $k > 0$, or, equivalently, the space $H^0(X, L^k)$ has no base locus.

Proposition 4.5 *Let (X, ω) and (L_p, h_p), $p \geq 1$, satisfy assumptions (A)–(B) and assume that X is smooth and that each line bundle L_p is semiample. Then there exist an integer $N > 0$ and a constant $C > 0$ depending only on ω such that*

$$\dim H^0(X, L_p) \leq C A_p^N \quad \text{for all } p \geq 1.$$

Proof Since L_p is big, X is Moishezon and hence projective, since it is Kähler. By the main theorem in [39], there exists a polynomial $Q(y, z)$ depending only on $\dim X$ such that for any semiample line bundle L on X one has that

$$\dim H^0(X, L) \leq Q\left(\int_X c_1(L)^n, \int_X c_1(L)^{n-1} \wedge c_1(X)\right).$$

Lemma 4.6 below implies that there exists a constant $C' > 0$ depending only on (X, ω) such that

$$\int_X c_1(L_p)^n \leq C' A_p^n, \quad \int_X c_1(L_p)^{n-1} \wedge c_1(X) \leq C' A_p^{n-1}.$$

The conclusion now follows. □

Lemma 4.6 *Let (X, ω) be a compact Kähler manifold of dimension n and let β be a real-valued closed form of type $(1, 1)$ on X. Then there exists $C > 0$ depending only*

on ω and β such that for any pseudoeffective class $\alpha \in H^{1,1}(X, \mathbb{R})$ we have

$$\int_X \alpha^k \wedge \beta^{n-k} \le C \|\alpha\|^k \quad \text{for } k = 1, \dots, n, \qquad \text{where } \|\alpha\| = \int_X \alpha \wedge \omega^{n-1}.$$

Proof If $\|\alpha\| = 0$ then $\alpha = 0$ since α is pseudoeffective, so we can assume $\|\alpha\| > 0$. Let $\theta \in \alpha$ be a smooth form and $T = \theta + dd^c\varphi$ be a positive closed current, where φ is a θ–PSH function. The Lelong numbers satisfy $\nu(T, x) < C_1\|\alpha\|$ for all $x \in X$, where C_1 is a constant depending only on ω (see eg [8, Lemma 2.5]). Demailly's regularization theorem [17] shows that there exists a sequence of smooth functions $\varphi_k \searrow \varphi$ such $\theta + dd^c\varphi_k \ge -C_2\lambda_k\omega$, where λ_k are continuous functions on X, $\lambda_k(x) \searrow \nu(T, x)$ as $k \to \infty$ for every $x \in X$, and C_2 is a constant depending only on ω. We fix k such that $\lambda_k(x) < C_1\|\alpha\|$ for every $x \in X$ and let $R = \theta + dd^c\varphi_k$ and $R' = R + C_3\|\alpha\|\omega$, where $C_3 = C_1C_2$, so $R' \ge 0$. Next we set $\beta' = \beta + c\,\omega$, where $c > 0$ is chosen so that $\beta' \ge 0$. Since R', $\beta' \ge 0$ we obtain

$$\int_X \alpha^k \wedge \beta^{n-k} = \int_X R^k \wedge \beta^{n-k} = \int_X (R' - C_3\|\alpha\|\omega)^k \wedge (\beta' - c\omega)^{n-k}$$

$$\le \int_X (R' + C_3\|\alpha\|\omega)^k \wedge (\beta' + c\omega)^{n-k} \le C_4 \|R' + C_3\|\alpha\|\omega\|^k,$$

where C_4 is a constant depending only on the Kähler form $\beta' + c\omega$ (hence on β and ω). The lemma follows since

$$\|R' + C_3\|\alpha\|\omega\| = \int_X (R + 2C_3\|\alpha\|\omega) \wedge \omega^{n-1} = \left(1 + 2C_3 \int_X \omega^n\right) \|\alpha\|. \qquad \square$$

We conclude this section by discussing an application of the above results to the problem of approximation of positive closed currents of bidegree $(1, 1)$ on X by currents of integration along analytic hypersurfaces of X. Let $\mathscr{A}(X)$ be the space of positive closed currents $T \in \mathcal{D}'_{n-1,n-1}(X)$ with the property that there exist a sequence of singular Hermitian holomorphic line bundles $\{(F_p, h^{F_p})\}_{p \ge 1}$ with $c_1(F_p, h^{F_p}) \ge 0$ and a sequence of natural numbers $N_p \to \infty$ such that

$$\lim_{p \to \infty} \frac{1}{N_p} c_1(F_p, h^{F_p}) = T.$$

If $X_{\text{sing}} \ne \varnothing$ we require in addition that there exists a current $T_0 \in \mathscr{T}(X)$ (depending on T) such that for all $p \ge 1$ we have $\frac{1}{N_p} c_1(F_p, h^{F_p}) \le T_0$.

When X is smooth the space $\mathscr{A}(X)$ is the closure in $\mathcal{D}'_{n-1,n-1}(X)$ of the convex cone generated by positive closed integral currents. Recall that a real closed current $T \in \mathcal{D}'_{n-1,n-1}(X)$ is called integral if its de Rham cohomology class $[T]$ belongs to $H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$. A current T is integral if and only if there exists a

singular Hermitian holomorphic line bundle (L, h) on X with $c_1(L, h) = T$ (see eg [40, Lemma 2.3.5]).

Theorem 4.7 Let (X, ω) be a compact normal Kähler space and (L, h) be a singular Hermitian holomorphic line bundle on X such that $c_1(L, h) \geq \varepsilon\omega$ for some $\varepsilon > 0$. If $T \in \mathscr{A}(X)$ then there exist a sequence of singular Hermitian holomorphic line bundles $\{(L_p, h_p)\}_{p \geq 1}$ with $c_1(L_p, h_p) \geq 0$ and a sequence of natural numbers $N_p \to \infty$ such that, for λ_∞–a.e. sequence $\{\sigma_p\}_{p \geq 1} \in \mathcal{S}_\infty$,

$$\lim_{p \to \infty} \frac{1}{N_p}[\sigma_p = 0] = T \quad \text{weakly on } X.$$

Here the probability space $(\mathcal{S}_\infty, \lambda_\infty)$ is associated to the sequence $\{(L_p, h_p)\}_{p \geq 1}$, as above.

Proof Since $T \in \mathscr{A}(X)$ there exist line bundles $\{(F_p, h^{F_p})\}_{p \geq 1}$ with $c_1(F_p, h^{F_p}) \geq 0$ and a sequence of natural numbers $N_p \to \infty$ such that $\lim_{p \to \infty} \frac{1}{N_p} c_1(F_p, h^{F_p}) = T$. Moreover, if $X_{\text{sing}} \neq \varnothing$ there exists a current $T_0 \in \mathscr{T}(X)$ such that $c_1(F_p, h^{F_p}) \leq N_p T_0$ for all $p \geq 1$. We can assume without loss of generality that $N_p \geq p$; otherwise replace (F_p, h^{F_p}) by $(F_p^{m_p}, (h^{F_p})^{m_p})$ and N_p by $m_p N_p$, with a convenient $m_p \in \mathbb{N}$. We fix a sequence $b_p \in \mathbb{N}$ such that $b_p \leq N_p$ for all $p \geq 1$, and $b_p \to \infty$ and $b_p/N_p \to 0$ as $p \to \infty$. Let

$$L_p = F_p \otimes L^{b_p}, \quad h_p = h^{F_p} \otimes h^{b_p}.$$

The conclusion follows from Theorem 4.1 since $c_1(L_p, h_p) \geq b_p c_1(L, h) \geq b_p \varepsilon\omega$ and

$$c_1(L_p, h_p) \leq N_p T_0 + b_p c_1(L, h) \leq N_p(T_0 + c_1(L, h)),$$

$$\frac{c_1(L_p, h_p)}{N_p} \to T, \quad \frac{\|c_1(L_p, h_p)\|}{N_p} \to \|T\| \quad \text{as } p \to \infty. \qquad \square$$

For related approximation results on compact Kähler manifolds we refer to [19; 14; 33; 11]. We note that Dinh and Sibony [22] developed a general method to obtain the asymptotic distribution with speed of convergence of zeros of random sequences of holomorphic sections.

5 Applications

5.1 Powers of a line bundle

Theorems 1.1, 4.1 and 4.2 apply to the sequence $(L_p, h_p) = (L^p, h^p)$, where (L, h) is a singular Hermitian holomorphic line bundle on X with strictly positive curvature current.

Corollary 5.1 *Let (X, ω) be a compact normal Kähler space and (L, h) be a singular Hermitian holomorphic line bundle on X such that $c_1(L, h) \geq \varepsilon\omega$ for some $\varepsilon > 0$. Then, as $p \to \infty$:*

(i) $\frac{1}{p} \log P_p \to 0$ *in* $L^1(X, \omega^n)$.

(ii) $\frac{1}{p}\gamma_p \to c_1(L, h)$ *weakly on X.*

(iii) $\frac{1}{p}[\sigma_p{=}0] \to c_1(L, h)$ *weakly on X, for λ_∞–a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$.*

Indeed, assumptions (A)–(B) are satisfied with $a_p = p\varepsilon$ and $T_0 = c_1(L, h)/\|c_1(L, h)\|$, where $\|c_1(L, h)\| := \int_X c_1(L, h) \wedge \omega^{n-1}$. Moreover, $A_p = p\|c_1(L, h)\|$.

We consider now the case when the curvature current of the singular metric is not necessarily Kähler.

Corollary 5.2 *Let (L, h) be a singular Hermitian holomorphic line bundle on the compact normal Kähler space (X, ω) such that $c_1(L, h) \geq 0$ and assume that L has a singular metric h_0 with $c_1(L, h_0) \geq \varepsilon\omega$ for some $\varepsilon > 0$. Let $\{n_p\}_{p\geq 1}$ be a sequence of natural numbers such that*

$$(27) \qquad\qquad n_p \to \infty \quad \text{and} \quad \frac{n_p}{p} \to 0 \quad \text{as } p \to \infty.$$

Let

$$(28) \qquad\qquad\qquad h_p = h^{p-n_p} \otimes h_0^{n_p}$$

and let P_p, γ_p and $\mathcal{S}^p$ be the Bergman kernel function, Fubini–Study current and unit sphere associated to the spaces $H^0_{(2)}(X, L_p) = H^0_{(2)}(X, L^p, h_p)$. Then, as $p \to \infty$, we have $\frac{1}{p} \log P_p \to 0$ in $L^1(X, \omega^n)$, $\frac{1}{p}\gamma_p \to c_1(L, h)$ and $\frac{1}{p}[\sigma_p{=}0] \to c_1(L, h)$ in the sense of currents on X for λ_∞–a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$.

Proof Note that

$$c_1(L^p, h_p) = (p - n_p)c_1(L, h) + n_p c_1(L, h_0) \geq \varepsilon n_p \omega,$$

$$\frac{A_p}{p} = \|c_1(L, h)\| = \|c_1(L, h_0)\| > 0.$$

Hence

$$c_1(L^p, h_p) \leq p(c_1(L, h) + c_1(L, h_0)) = A_p T_0,$$

with

$$T_0 = \frac{1}{\|c_1(L, h)\|}(c_1(L, h) + c_1(L, h_0)),$$

and Theorems 1.1 and 4.1 apply in this setting and conclude the proof. $\qquad\square$

Corollary 5.1 applies to the case of the Satake–Baily–Borel compactifications of arithmetic quotients. Let D be a bounded symmetric domain in $\mathbb{C}^n$ and let Γ be a torsion-free arithmetic lattice. Then $U := D/\Gamma$ is a smooth quasiprojective variety, called an arithmetic variety. The Satake–Baily–Borel compactification of U is a normal compact analytic space X containing U as a Zariski open dense set and which is minimal with this property, that is, given any normal compactification $U \subset X'$, the identity map on U extends to a holomorphic map $X' \to X$ (see Satake [53] and Baily and Borel [3]). Moreover, $\mathrm{codim}(X \setminus U) \geq 2$.

The Bergman metric β_D on D descends to a complete Kähler metric β_U on U, which is Kähler–Einstein with $\mathrm{Ric}_{\beta_U} = -\beta_U$. We denote by h^{K_U} the Hermitian metric induced by β on K_U. Then $c_1(K_U, h^{K_U}) = -\mathrm{Ric}_{\beta_U} = \beta_U$.

Lemma 5.3 *Assume that Γ is neat. There exists an ample line bundle $F \to X$ such that $F|_U = K_U$ and a singular metric h^F on F such that $h^F|_U = h^{K_U}$, such that $c_1(F, h^F)$ is a Kähler current on X and $c_1(F, h^F)|_U = \beta_U$. Hence the Bergman Kähler metric β_U extends to a Kähler current β_X on X.*

Proof The existence of an ample line bundle $F \to X$ such that $F|_U = K_U$ is shown in [44, Proposition 3.4(b)]; the bundle F is the bundle used in [3, Theorem 10.11] to embed X into a projective space. Since $c_1(K_U, h^{K_U}) = \beta_U$, the local weights of h^{K_U} are PSH, so they extend to X. Thus, the metric h^{K_U} extends to a metric h^F on $F \to X$ and $c_1(K_U, h^{K_U})$ extends to a positive closed current $c_1(F, h^F)$ on X.

To see that $c_1(F, h^F)$ is a Kähler current, one can describe F in the following way. By [2], U admits a smooth toroidal compactification $\widetilde{X}$. In particular, $\Sigma := \widetilde{X} \setminus U$ is a divisor with normal crossings. Let $L := K_{\widetilde{X}} \otimes \mathscr{O}_{\widetilde{X}}(\Sigma)$. Then the metric h^{K_U} defines a singular metric h^L on L such that $c_1(L, h^L)$ is a closed positive current on $\widetilde{X}$ which extends $\beta_U/(2\pi)$; see [12, Lemma 6.8]. It follows from [44, Proposition 3.4] (see also [12, Section 6.4]) that h^{K_U} is a good metric in the sense of Mumford, so that β_U has Poincaré growth near Σ. Therefore, $c_1(L, h^L)$ is a Kähler current on $\widetilde{X}$. If $\tau : \widetilde{X} \to X$ is the unique holomorphic map extending the identity on U, we have $L = \tau^* F$. Hence $c_1(F, h^F) = \tau_* c_1(L, h^L)$, so $c_1(F, h^F)$ is a Kähler current on X. $\square$

Corollary 5.4 *Let X be the Satake–Baily–Borel compactification of a smooth arithmetic variety $U = D/\Gamma$, where Γ is neat. Let (F, h^F) be the extension of (K^U, h^{K_U}) given by Lemma 5.3, where h^{K_U} is induced by the Bergman metric β_U. Then the conclusions of Corollary 5.1 hold for $c_1(F, h^F) = \beta_X$.*

Note that

$$H^0_{(2)}(X, F^p) = H^0_{(2)}(U, K_U^p) = H^0(\widetilde{X}, K_{\widetilde{X}}^p \otimes \mathscr{O}_{\widetilde{X}}(\Sigma)^{p-1}) = H^0(\widetilde{X}, L^p \otimes \mathscr{O}_{\widetilde{X}}(\Sigma)^{-1})$$

is the space of cusp forms of weight p; see [12, Lemma 6.11]. Hence Corollary 5.4 says that the Bergman kernel P_p of the space of cusps forms satisfies $\frac{1}{p} \log P_p \to 0$ in $L^1(X, \omega^n)$, where ω is any smooth Kähler form on X. Moreover, the normalized zero-currents of random cusp forms distribute on X to the extension β_X of the Bergman metric β_U on U.

We consider next a more general situation as above. Let X be a projective variety of general type with only canonical singularities such that the canonical divisor $\mathcal{K}_X$ is an ample $\mathbb{Q}$–divisor. There exists an integer $l > 0$ such that $\mathcal{K}_X^l$ is integral and the holomorphic line bundle $L = \mathcal{O}_X(\mathcal{K}_X^l) \to X$ admits a smooth Hermitian metric h^L such that $c_1(L, h^L) = \omega$ is a smooth Kähler form on X. By [25, Theorem 7.8] there exists a unique $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ such that

$$(29) \qquad\qquad c_1(L, e^{-2\varphi} h^L) = \omega + dd^c\varphi = \omega_\varphi$$

is a singular Kähler–Einstein metric in the sense of [25, Definition 7.3]. In particular, ω_φ is a closed positive current on X and its restriction to X_{reg} is a smooth Kähler–Einstein metric of negative curvature. Corollary 5.2 yields the following:

Corollary 5.5 *Let X be a projective variety of general type with only canonical singularities and let ω_φ be the singular Kähler–Einstein metric (29). Let h_p be the metric on L^p constructed as in (28) using a sequence $\{n_p\}_{p\geq 1}$ as in (27) and the metrics $h_0 = h^L$ and $h = e^{-2\varphi} h^L$ on L. Let γ_p and $\mathcal{S}^p$ be the Fubini–Study current and the unit sphere associated to the space $H^0_{(2)}(X, L^p, h_p)$. Then $\frac{1}{p}\gamma_p \to \omega_\varphi$ and $\frac{1}{p}[\sigma_p{=}0] \to \omega_\varphi$ as $p \to \infty$ weakly on X for λ_∞–a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$.*

A similar discussion applies to canonically polarized KLT compact Kähler pairs (X, Δ) considered in [25, Theorem 7.12] or to $\mathbb{Q}$–Fano Kähler spaces (that is, KLT compact Kähler spaces with $-\mathcal{K}_X$ ample $\mathbb{Q}$–divisor).

5.2 Powers of ample line bundles

We specialize in the sequel the results of the previous corollary to the case when (X, ω) is a projective manifold with a polarization (L, h_0), where L is a positive line bundle on X endowed with a smooth Hermitian metric h_0 such that $c_1(L, h_0) = \omega$. The set of singular Hermitian metrics h on L with $c_1(L, h) \geq 0$ is in one-to-one correspondence to the set $\text{PSH}(X, \omega)$ of ω–plurisubharmonic (ω–PSH) functions on X, by associating to $\varphi \in \text{PSH}(X, \omega)$ the metric $h_\varphi = e^{-2\varphi} h_0$ (see eg [18; 34]). Note that

$$c_1(L, h_\varphi) = \omega + dd^c\varphi =: \omega_\varphi.$$

Corollary 5.6 *Let (X, ω) be a compact Kähler manifold and (L, h_0) be a positive line bundle on X with $c_1(L, h_0) = \omega$. Let $\varphi \in \mathrm{PSH}(X, \omega)$ and h_p be a metric on L^p constructed as in (28) using a sequence $\{n_p\}_{p \geq 1}$ as in (27) and the metrics h_0 and $h = h_\varphi$ on L. Let γ_p and $\mathcal{S}^p$ be the Fubini–Study current and the unit sphere associated to the space $H^0_{(2)}(X, L^p, h_p)$. Then $\frac{1}{p}\gamma_p \to \omega_\varphi$ and $\frac{1}{p}[\sigma_p{=}0] \to \omega_\varphi$ as $p \to \infty$ weakly on X for λ_∞–a.e. sequence $\{\sigma_p\}_{p \geq 1} \in \mathcal{S}_\infty$. Moreover, if φ is continuous then*

$$\frac{1}{p^k}\gamma_p^k \to \omega_\varphi^k = (\omega + dd^c\varphi)^k \quad \text{weakly on } X \text{ for } k \leq n.$$

Proof The first conclusion follows directly from Corollary 5.2. If φ is continuous and P_p is the Bergman kernel function of the space $H^0_{(2)}(X, L^p, h_p)$ one can proceed as in the proof of [12, Theorem 5.3] to show that $\frac{1}{p}\log P_p \to 0$ uniformly on X. This implies the second conclusion of the corollary, as in [12, Theorem 5.4]. $\qquad\square$

Note that since h_0 is smooth we have that $H^0_{(2)}(X, L^p, h^p) \subset H^0_{(2)}(X, L^p, h_p)$. Moreover, if the metric $h = h_\varphi$ is bounded (ie φ is bounded) then equality holds, $H^0_{(2)}(X, L^p, h^p) = H^0_{(2)}(X, L^p, h_p) = H^0(X, L^p)$.

We remark now that, instead of working with random sections of spheres, one can identify $H^0_{(2)}(X, L_p)$ to $\mathbb{C}^{d_p}$ by

$$\mathbb{C}^{d_p} \to H^0_{(2)}(X, L_p), \quad a = (a_1, \ldots, a_{d_p}) \mapsto S_a = \sum_{j=1}^{d_p} a_j S_j^p,$$

and one can consider a_j, $1 \leq j \leq d_p$, as independent identically distributed Gaussian random variables on $\mathbb{C}$. Thus the probability space $(\mathcal{S}^p, \lambda_p)$ is replaced by $(H^0_{(2)}(X, L_p), \mu_p)$, where

$$d\mu_p(z) = \pi^{-d_p} e^{-(|z_1|^2 + \cdots + |z_{d_p}|^2)} dm(z)$$

and $dm(z)$ is the Lebesgue measure on $\mathbb{C}^{d_p}$. Let $\mu_\infty = \prod_{p=1}^\infty \mu_p$ be the product measure on the space $\mathcal{H} = \prod_{p=1}^\infty H^0_{(2)}(X, L_p)$. Since the measure μ_p is unitary invariant, Theorems 4.1 and 4.2 hold in this context with similar proofs. More precisely, we have the following:

Theorem 5.7 *Let (X, ω), (L_p, h_p), $p \geq 1$, satisfy assumptions (A)–(B).*

(i) *If $\sum_{p=1}^\infty \frac{1}{A_p^2} < \infty$ then for μ_∞–a.e. sequence $\{\sigma_p\}_{p \geq 1} \in \mathcal{H}$ we have that*

$$\lim_{p \to \infty} \frac{1}{A_p}([\sigma_p{=}0] - c_1(L_p, h_p)) = 0 \quad \text{in the weak sense of currents on } X.$$

(ii) *If condition* (26) *holds then there exists an increasing sequence of natural numbers* $\{p_j\}_{j \geq 1}$ *such that for* μ_∞*–a.e. sequence* $\{\sigma_p\}_{p \geq 1} \in \mathcal{H}$ *we have*

$$\lim_{j \to \infty} \frac{\log |\sigma_{p_j}|_{h_{p_j}}}{A_{p_j}} = 0 \quad \text{in } L^1(X, \omega^n).$$

Hence Corollary 5.6 can be seen as a generalization of [7, Theorem 5.2], which deals with the special case when $\varphi = \mathcal{V}^*_{K,q}$ is the weighted ω–PSH global extremal function of a compact $K \subset X$. Using different methods and a different inner product on $H^0(X, L^p)$ (note that $\mathcal{V}^*_{K,q}$ is bounded) it is shown in [7, Theorem 5.2] that

$$\frac{1}{p}[\sigma_p = 0] \to \omega + dd^c \mathcal{V}^*_{K,q} \quad \text{for } \mu_\infty\text{–a.e. sequences } \{\sigma_p\}_{p \geq 1} \in \prod_{p=1}^{\infty} H^0(X, L^p).$$

On the other hand [7, Theorem 5.2] holds for more general probability measures than μ_∞ (see [7, (2.1)–(2.2)]).

Remark 5.8 A particularly interesting case is when $X = \mathbb{P}^n$, $L = \mathcal{O}(1)$ and $\omega = \omega_{\mathrm{FS}}$ is the Fubini–Study metric in Corollary 5.6. In this case the class $\mathrm{PSH}(\mathbb{P}^n, \omega_{\mathrm{FS}})$ is in one-to-one correspondence with the Lelong class of PSH functions on $\mathbb{C}^n$ of logarithmic growth, and the sections in $H^0(\mathbb{P}^n, \mathcal{O}(p))$ can be identified to polynomials of degree $\leq p$ on $\mathbb{C}^n$ (see eg [7, Section 5]). Therefore Corollary 5.6 yields a general equidistribution result for the zeros of a random sequence of polynomials on $\mathbb{C}^n$. For related results see [4; 7] and references therein.

We consider now approximation of metrics with conic singularities [24; 58]. Let X be a Fano manifold, that is, the anticanonical line bundle K_X^{-1} is ample. Fix a Hermitian metric h_0 on K_X^{-1}, such that $\omega := c_1(K_X^{-1}, h_0)$ is a Kähler metric. Recall that Hermitian metrics on K_X^{-1} can be identified to volume forms on X. Let D be a smooth divisor in the linear system defined by K_X^{-l}, so there exists a section $s \in H^0(X, K_X^{-l})$ with $D = \mathrm{Div}(s)$. We fix a smooth metric h on the bundle $\mathcal{O}_X(D)$. Let $\beta \in [0, 1)$.

A Kähler metric $\hat{\omega}$ on X with cone angle $2\pi(1 - \beta)$ along D is a current $\hat{\omega} \in c_1(X)$ such that

$$\hat{\omega} = \omega_\varphi = \omega + dd^c \varphi,$$

where $\varphi = \psi + |s|_h^{2-2\beta} \in \mathrm{PSH}(X, \omega)$ and $\psi \in \mathscr{C}^\infty(X) \cap \mathrm{PSH}(X, \omega)$. In a neighborhood of a point of D where D is given by $z_1 = 0$ the metric $\hat{\omega}$ is equivalent to the cone metric $\frac{i}{2}(|z_1|^{-2\beta} dz_1 \wedge d\bar{z}_1 + \sum_{j=2}^{n} dz_j \wedge d\bar{z}_j)$. The Riemannian metric determined by $\hat{\omega}$ has conic singularity along D of conic angle $2\pi(1 - \beta)$.

The metric $\hat{\omega}$ defines a singular metric $h_{\hat{\omega}}$ on K_X^{-1} with curvature current $\mathrm{Ric}_{\hat{\omega}} := c_1(K_X^{-1}, h_{\hat{\omega}})$. The metric $\hat{\omega}$ is called Kähler–Einstein with conic singularities of cone

angle $2\pi(1 - \beta)$ along D if $\text{Ric}_{\widehat{\omega}} = (1 - l\beta)\widehat{\omega} + \beta[D]$, where $[D]$ is the current of integration on D.

Corollary 5.9 *Let X be a Fano manifold and let $\widehat{\omega}$ be a Kähler metric with cone angle β. Let h_p be the metric on K_X^{-p} constructed as in Corollary 5.2 using the metric $h = h_{\widehat{\omega}}$ on K_X^{-1}. Let γ_p and $\mathcal{S}^p$ be the Fubini–Study current and the unit sphere associated to the space $H^0_{(2)}(X, K_X^{-p}, h_p)$. Then $\frac{1}{p}\gamma_p \to \text{Ric}_{\widehat{\omega}}$ and $\frac{1}{p}[\sigma_p = 0] \to \text{Ric}_{\widehat{\omega}}$ as $p \to \infty$ weakly on X, for λ_∞–a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$. If $\widehat{\omega}$ is a Kähler current, then we can take $h_p = h_{\widehat{\omega}}^p$ above.*

Remark 5.10 If $\widehat{\omega}$ is a Kähler current, [37, Theorem 1.8] shows that the Bergman kernel of $H^0_{(2)}(X, K_X^{-p}, h_{\widehat{\omega}}^p)$ has a full asymptotic expansion in powers p^{n-j} for $j = 0, 1, \ldots$ and the Fubini–Study forms converge to $\text{Ric}_{\widehat{\omega}}$ in the $\mathscr{C}^\infty$–topology on compact sets of $X \setminus D$.

5.3 Tensor products of powers of several line bundles

Another typical application of Theorem 1.1 is the following:

Corollary 5.11 *Let (X, ω) be a compact normal Kähler space. Assume that (F_j, h^{F_j}), $1 \leq j \leq k$, are singular Hermitian holomorphic line bundles with $c_1(F_j, h^{F_j}) \geq 0$ and $c_1(F_1, h^{F_1}) \geq \varepsilon\omega$ for some $\varepsilon > 0$. Let $T = \sum_{j=1}^k r_j c_1(F_j, h^{F_j})$, where $r_j \geq 0$, and let $\{m_{j,p}\}_p$, $1 \leq j \leq k$, be sequences of natural numbers such that*

$$m_{1,p} \to \infty, \quad \frac{m_{j,p}}{p} \to r_j, \quad 1 \leq j \leq k, \quad \text{as } p \to \infty.$$

Let P_p, γ_p and $\mathcal{S}^p$ be the Bergman kernel function, Fubini–Study current and unit sphere associated to $H^0_{(2)}(X, L_p)$, where

$$L_p = F_1^{m_{1,p}} \otimes \cdots \otimes F_k^{m_{k,p}}, \quad h_p = (h^{F_1})^{m_{1,p}} \otimes \cdots \otimes (h^{F_k})^{m_{k,p}}.$$

Then, as $p \to \infty$, we have $\frac{1}{p}\log P_p \to 0$ in $L^1(X, \omega^n)$, $\frac{1}{p}\gamma_p \to T$ and $\frac{1}{p}[\sigma_p = 0] \to T$ in the weak sense of currents on X for λ_∞–a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$.

Proof We may assume that $m_{j,p}/p < r_j + 1$ for all $1 \leq j \leq k$ and $p \geq 1$, so

$$c_1(L_p, h_p) = \sum_{j=1}^k m_{j,p} c_1(F_j, h^{F_j}) \geq \varepsilon m_{1,p}\omega,$$

$$c_1(L_p, h_p) \leq pT_0, \quad \text{where } T_0 = \sum_{j=1}^k (r_j + 1)c_1(F_j, h^{F_j}).$$

Moreover, $c_1(L_p, h_p)/p \to T$ as $p \to \infty$. Note that

$$\frac{A_p}{p} = \frac{1}{p} \sum_{j=1}^{k} m_{j,p} \|c_1(F_j, h^{F_j})\| \to \sum_{j=1}^{k} r_j \|c_1(F_j, h^{F_j})\|.$$

The conclusion follows from Theorems 1.1 and 4.1, since

$$\frac{\gamma_p}{p} = \frac{A_p}{p} \frac{1}{A_p} (\gamma_p - c_1(L_p, h_p)) + \frac{1}{p} c_1(L_p, h_p).$$

The fact that $\frac{1}{p}[\sigma_p{=}0] \to T$ weakly on X, for λ_∞–a.e. sequence $\{\sigma_p\}_{p \geq 1} \in \mathcal{S}_\infty$, follows as in the proof of Theorem 4.1, where we normalize all currents dividing by p, by using that $A_p \lesssim p$ and $\sum_{p=1}^{\infty} p^{-2}$ is finite.   □

Let us remark that Theorem 4.2 holds in the context of Corollaries 5.1, 5.2, 5.6 and 5.11, since the condition (26) holds in all of these situations.

5.4 Covering manifolds

Let (X, ω) be a compact Kähler manifold of dimension n and (L_p, h_p), $p \geq 1$, be a sequence of singular Hermitian holomorphic line bundles on X satisfying condition (1). Recall that $A_p = \int_X c_1(L_p, h_p) \wedge \omega^{n-1}$. Let $q \colon \widetilde{X} \to X$ be a (paracompact) Galois covering of X, where q is the canonical projection. Let $\widetilde{\omega} = q^*\omega$ and $\widetilde{L}_p = q^*(L_p)$, and let $\widetilde{h}_p$ be the metric on $\widetilde{L}_p$ which is the pull-back of the metric h_p. We let $H^0_{(2)}(\widetilde{X}, \widetilde{L}_p)$ be the Bergman space of L^2–holomorphic sections of $\widetilde{L}_p$ relative to the metric $\widetilde{h}_p$ and the volume form $\widetilde{\omega}^n/n!$ on $\widetilde{X}$, defined as in (3), endowed with the obvious inner product. We define the Bergman kernel function $\widetilde{P}_p$ and Fubini–Study currents $\widetilde{\gamma}_p$ associated to $H^0_{(2)}(\widetilde{X}, \widetilde{L}_p)$ as in (4). In this context, $d_p \in \mathbb{N} \cup \{\infty\}$, and these objects are well-defined even for $d_p = \infty$; see [12, Lemmas 3.1–3.2].

Note that $\widetilde{\omega}$ is a complete Kähler metric on $\widetilde{X}$ and $c_1(\widetilde{L}_p, \widetilde{h}_p) = q^*c_1(L_p, h_p) \geq a_p\widetilde{\omega}$. Since $\mathrm{Ric}_\omega \geq -B\omega$ holds on X for some constant $B > 0$ and q is a local biholomorphism, we have $\mathrm{Ric}_{\widetilde{\omega}} \geq -B\widetilde{\omega}$ on $\widetilde{X}$. Moreover, if $K \Subset \widetilde{X}$, there exists a constant $C_K > 0$ such that

$$\int_K c_1(\widetilde{L}_p, \widetilde{h}_p) \wedge \widetilde{\omega}^{n-1} \leq C_K A_p.$$

A straightforward adaptation of the proofs given above yields

(30)
$$\frac{1}{A_p} \log \widetilde{P}_p \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\widetilde{X}, \widetilde{\omega}^n),$$

$$\frac{1}{A_p} (\widetilde{\gamma}_p - c_1(\widetilde{L}_p, \widetilde{h}_p)) \to 0 \quad \text{weakly on } \widetilde{X}.$$

Assume moreover that the metrics h_p are of class $\mathscr{C}^3$ and condition (5) is fulfilled, that is, $\varepsilon_p := \|h_p\|_3^{1/3} a_p^{-1/2} \to 0$ as $p \to \infty$. Then there exists $p_0 \in \mathbb{N}$ such that, for

all $p > p_0$,

$$(31) \qquad \left| \widetilde{P}_p \frac{\widetilde{\omega}^n}{c_1(\widetilde{L}_p, \widetilde{h}_p)^n} - 1 \right| \leq C \varepsilon_p^{2/3} \quad \text{on } \widetilde{X},$$

where $C > 0$ is a constant depending only on (X, ω). Estimates (6) and (31) show that the asymptotics of the Bergman kernels $\widetilde{P}_p(\widetilde{x})$ and $P_p(x)$, where $x = q(\widetilde{x})$, are the same. For $(L_p, h_p) = (L^p, h^p)$ with a smooth metric h satisfying $c_1(L, h) \geq a\omega$, this follows from [40, Theorem 6.1.4].

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