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Existence of Lefschetz fibrations on Stein and Weinstein domains

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We show that every Stein or Weinstein domain may be presented (up to deformation) as a Lefschetz fibration over the disk. The proof is an application of Donaldson's quantitative transversality techniques.

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1 Introduction

In this paper, we prove the existence of *Lefschetz fibrations* (certain singular fibrations with Morse-type singularities) on *Stein domains* (from complex geometry) and on *Weinstein domains* (from symplectic geometry). These two results are linked (in fact, logically so) by the close relationship between Stein and Weinstein structures established in the book by Cieliebak and Eliashberg [7] building on earlier work of Eliashberg [12]. Nevertheless, they can be understood independently from either a purely complex geometric viewpoint or from a purely symplectic viewpoint.

1.1 Lefschetz fibrations on Stein domains

We begin by explaining our results for Stein domains.

Definition 1.1 A real-valued function ϕ on a complex manifold V is called J–*convex* (or *strictly plurisubharmonic*) if and only if $(i\,d'd''\phi)(v, Jv) > 0$ for every nonzero (real) tangent vector v.

Definition 1.2 A *Stein manifold* is a complex manifold V which admits a smooth exhausting J–convex function $\phi\colon V \to \mathbb{R}$.

Definition 1.3 A *Stein domain* is a compact complex manifold with boundary V which admits a smooth J–convex function $\phi\colon V \to \mathbb{R}$ with $\partial V = \{\phi{=}0\}$ as a regular level set.

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For us, a complex manifold with boundary (or corners) shall mean one equipped with a germ of (codimension zero) embedding into an (open) complex manifold. A holomorphic function on a complex manifold with boundary (or corners) is one which extends holomorphically to an open neighborhood in the ambient (open) complex manifold.

For example, if $\overline{V}$ is a Stein manifold with smooth exhausting J–convex function $\phi\colon \overline{V} \to \mathbb{R}$ with $\{\phi=0\}$ as a regular level set, then $V := \{\phi \leq 0\}$ is a Stein domain. In fact, it is not hard to see that every Stein domain is of this form.

Definition 1.4 Let $D^2 \subseteq \mathbb{C}$ denote the closed unit disk. A *Stein Lefschetz fibration* is a holomorphic map $\pi\colon V \to D^2$, where V is a compact complex manifold with corners, such that:

- **Singular fibration** The map π is a (smooth) fibration with manifold with boundary fibers, except for a finite number of critical points $\operatorname{crit}(\pi)$ in the interior of V.
- **Nondegenerate critical points** Near each critical point $p \in \operatorname{crit}(\pi)$, there are local holomorphic coordinates in which π is given by
$$(z_1, \dots, z_n) \mapsto \pi(p) + \sum_{i=1}^{n} z_i^2$$
(according to the complex Morse lemma, this holds if and only if the complex Hessian at p is nondegenerate). Furthermore, all critical values are distinct.
- **Stein fibers** There exists a J–convex function $\phi\colon V \to \mathbb{R}$ with $\partial_h V = \{\phi=0\}$ as a regular level set, where $\partial_h V := \bigcup_{p\in D^2} \partial(\pi^{-1}(p))$ denotes the "horizontal boundary" of V.

Note that the boundary of V is the union of the horizontal boundary $\partial_h V$ and the "vertical boundary" $\partial_v V := \pi^{-1}(\partial D^2)$, whose intersection $\partial_h V \cap \partial_v V$ is the corner locus.

The total space V of any Stein Lefschetz fibration may be smoothed out to obtain a Stein domain V^{sm}, unique up to deformation. Specifically, for any function $g\colon \mathbb{R}_{<0} \to \mathbb{R}$ satisfying $g' > 0$, $g'' > 0$, and $\lim_{x\to 0^-} g(x) = \infty$, the function
$$\Phi_g := g(|\pi|^2 - 1) + g(\phi)$$
is an exhausting J–convex function on V°. Moreover, the critical locus of Φ_g stays away from ∂V as g varies in any compact family; this follows from the obvious

inclusion

$$\operatorname{crit}(\Phi_g) \subseteq \bigcup_{p \in D^2} \operatorname{crit}(\phi|_{\pi^{-1}(p)})$$

and the fact that the latter is a compact subset of $V \setminus \partial_h V$. As a result, the sublevel set $\{\Phi_g \le M\}$ is a Stein domain which, up to deformation, is independent of the choice of g and the choice of M larger than all critical values of Φ_g. We denote this (deformation class of) Stein domain by V^{sm}, which, of course, depends not only on V, but also on π.

The simplest (and weakest) version of our existence result is the following.

Theorem 1.5 *Let V be a Stein domain. There exists a (Stein) Lefschetz fibration $\pi\colon V' \to D^2$ with $(V')^{\mathrm{sm}}$ deformation equivalent to V.*

Deformation is meant in the sense of a real 1–parameter family of Stein domains. The nature of the deformation required is made explicit by considering the following stronger version of our existence result.

Theorem 1.6 *Let V be a Stein domain. For every sufficiently large real number k, there exists a holomorphic function $\pi\colon V \to \mathbb{C}$ such that:*

- *For $|\pi(p)| \ge 1$, we have $d \log \pi(p) = k \cdot d'\phi(p) + O(k^{\frac{1}{2}})$.*
- *For $|\pi(p)| \le 1$ and $p \in \partial V$, we have $d\pi(p)|_\xi \ne 0$.*

We may, in addition, require that $\pi^{-1}(D^2)$ contain any given compact subset of V°.

Theorem 1.5 follows from Theorem 1.6 by smoothing out the deformation of Stein domains $\{\pi^{-1}(D_r^2)\}_{1 \le r < \infty}$ (this argument is given in detail in Section 5). Theorem 1.6 is a corollary of the following, which is the main technical result of the paper.

Theorem 1.7 *Let $\overline{V}$ be a Stein manifold, equipped with a smooth exhausting J–convex function $\phi\colon \overline{V} \to \mathbb{R}$. For every sufficiently large real number k, there exists a holomorphic function $f\colon \overline{V} \to \mathbb{C}$ such that*

- *$|f(p)| \le e^{\frac{1}{2}k\phi(p)}$ for $p \in \{\phi \le 1\}$,*
- *$|f(p)| + k^{-\frac{1}{2}} |df(p)|_\xi| > \eta$ for $p \in \{\phi = 0\}$ (with df measured in the metric induced by ϕ),*

where ξ denotes the Levi distribution on $\{\phi = 0\} \subseteq \overline{V}$, and $\eta > 0$ is a constant depending only on the dimension of $\overline{V}$.

To prove Theorem 1.6 (for $V := \{\phi \leq 0\}$) from Theorem 1.7, we take π to be (a small perturbation of) $\eta^{-1} \cdot f$, which works once k is sufficiently large (the details of this argument are given in Section 5). To prove Theorem 1.7, we use methods introduced by Donaldson [10] (this proof occupies Sections 2–4). A closely related result was obtained by Mohsen [20] also using Donaldson's techniques.

1.2 Lefschetz fibrations on Weinstein domains

Next, we turn to our result for Weinstein domains.

Definition 1.8 A *Weinstein domain* $(W, \omega, \lambda, \phi)$ is a compact symplectic manifold with boundary, equipped with a 1–form λ satisfying $d\lambda = \omega$, and a Morse function $\phi\colon W \to \mathbb{R}$ which has $\partial W = \{\phi = 0\}$ as a regular level set and for which X_λ (defined by the condition $\omega(X_\lambda, \cdot) = \lambda$) is gradient-like.

Definition 1.9 An *abstract Weinstein Lefschetz fibration* is a tuple

$$W = (W_0; L_1, \dots, L_m)$$

consisting of a Weinstein domain W_0^{2n-2} (the "central fiber") along with a finite sequence of exact parametrized[1] Lagrangian spheres $L_1, \dots, L_m \subseteq W_0$ (the "vanishing cycles").

From any abstract Weinstein Lefschetz fibration $W = (W_0; L_1, \dots, L_m)$, we may construct a Weinstein domain $|W|$ (its "total space") by attaching critical Weinstein handles to the stabilization $W_0 \times D^2$ along Legendrians

$$\Lambda_j \subseteq W_0 \times S^1 \subseteq \partial(W_0 \times D^2)$$

near $2\pi j/m \in S^1$ obtained by lifting the exact Lagrangians L_j. We give this construction in detail in Section 6.

We will prove the following existence result.

Theorem 1.10 *Let W be a Weinstein domain. There exists an abstract Weinstein Lefschetz fibration $W' = (W_0; L_1, \dots, L_m)$ whose total space $|W'|$ is deformation equivalent to W.*

[1] Parametrized shall mean equipped with a diffeomorphism $S^{n-1} \xrightarrow{\sim} L$ defined up to precomposition with elements of $O(n)$.

Deformation is meant in the sense of a 1–parameter family of Weinstein domains, but where the requirement that ϕ be Morse is relaxed to allow birth–death critical points. Theorem 1.10 is deduced from Theorem 1.5 using the existence theorem for Stein structures on Weinstein domains proved by Cieliebak and Eliashberg [7, Theorem 1.1(a)]. The main step is thus to show that a Stein Lefschetz fibration $\pi\colon V \to D^2$ naturally gives rise to an abstract Weinstein Lefschetz fibration whose total space is deformation equivalent to V^{sm} (the details of this argument are given in Section 6).

In current work in progress, we hope to apply Donaldson's techniques directly in the Weinstein setting to produce on any Weinstein domain W an approximately holomorphic function $f\colon W \to \mathbb{C}$ satisfying conditions similar to those in Theorem 1.7, and thus give a proof of Theorem 1.10 which does not appeal to the existence of a compatible Stein structure.

Given Theorem 1.10, it is natural to ask whether every deformation equivalence between the total spaces of two abstract Weinstein Lefschetz fibrations is induced by a finite sequence of moves of some simple type. Specifically, applying any of the following operations to an abstract Weinstein Lefschetz fibration preserves the total space up to canonical deformation equivalence, and it is natural to ask whether they are enough:

- **Deformation** Simultaneous Weinstein deformation of W_0 and exact Lagrangian isotopy of $(L_1,\dots,L_m)$.
- **Cyclic permutation** Replace $(L_1,\dots,L_m)$ with $(L_2,\dots,L_m,L_1)$.
- **Hurwitz moves** Let τ_L denote the symplectic Dehn twist around L, and replace $(L_1,\dots,L_m)$ with either
$$(L_2,\tau_{L_2}L_1,L_3,\dots,L_m) \quad \text{or} \quad (\tau_{L_1}^{-1}L_2,L_1,L_3,\dots,L_m).$$
- **Stabilization** For a parametrized Lagrangian disk $D^{n-1} \hookrightarrow W_0$ with Legendrian boundary $S^{n-2} = \partial D^{n-1} \hookrightarrow \partial W_0$ such that
$$0 = [\lambda_0] \in H^1(D^{n-1},\partial D^{n-1}),$$
replace W_0 with $\widetilde{W}_0$, obtained by attaching a Weinstein handle to W_0 along ∂D^{n-1}, and replace $(L_1,\dots,L_m)$ with $(\widetilde{L},L_1,\dots,L_m)$, where $\widetilde{L} \subseteq \widetilde{W}_0$ is obtained by gluing together D^{n-1} and the core of the handle.

It would be very interesting if the methods of this paper could be brought to bear on this problem as well.

Remark 1.11 The reader is likely to be familiar with more geometric notions of symplectic Lefschetz fibrations (eg as in Seidel [22, Section 15d] or Bourgeois, Ekholm and Eliashberg [5, Section 8.1] and the references therein), and may prefer these to the notion of an abstract Weinstein Lefschetz fibration used to state Theorem 1.10. We believe, though, that the reader wishing to construct a symplectic Lefschetz fibration in their preferred setup with the same total space as a given abstract Weinstein Lefschetz fibration will have no trouble doing so (eg see Seidel [22, Section 16e]).

Seidel [21; 22; 23; 24] has developed powerful methods for calculations in and of Fukaya categories coming from Lefschetz fibrations, in particular relating the Fukaya category of the total space to the vanishing cycles and the Fukaya category of the central fiber. Our existence result shows that these methods are applicable to any Weinstein domain. We should point out, however, that, while our proof of existence of Lefschetz fibrations is in principle effective, it does not immediately lead to any practical way of computing a Lefschetz presentation of a given Weinstein manifold.

1.3 Remarks about the proof

We outline briefly the proof of Theorem 1.7 (the main technical result of the paper), which occupies Sections 2–4. As mentioned earlier, the proof is an application of Donaldson's quantitative transversality techniques, first used to construct symplectic divisors inside closed symplectic manifolds [10] (somewhat similar ideas appeared earlier in Cheeger and Gromov [6]).

The J–convex function $\phi\colon \overline{V} \to \mathbb{R}$ determines a positive line bundle L on $\overline{V}$. We consider the high tensor powers L^k of this positive line bundle. Using L^2–methods of Hörmander [16] and Andreotti and Vesentini [1], one may construct "peak sections" of L^k, that is, holomorphic sections $s\colon \overline{V} \to L^k$ which are "concentrated" over the ball of radius $k^{-\frac{1}{2}}$ centered at any given point $p_0 \in V := \{\phi \le 0\}$ and have decay $|s(p)| = O(e^{-\epsilon\cdot k\cdot d(p,p_0)^2})$ for $p \in \{\phi \le 1\}$.

Donaldson introduced a remarkable method to, given enough localized holomorphic sections, construct a linear combination $s\colon \overline{V} \to L^k$ which satisfies, quantitatively, any given holomorphic transversality condition which is generic. The key technical ingredient for Donaldson's construction is a suitably quantitative version of Sard's theorem, and this step was simplified considerably by Auroux [4]. The function f asserted to exist in Theorem 1.7 is simply the quotient of such a quantitatively transverse section $s\colon \overline{V} \to L^k$ by a certain tautological section "1": $\overline{V} \to L^k$.

We take advantage of the fact that we are in the holomorphic category by working with genuinely holomorphic functions, instead of the approximately holomorphic functions which are the standard context of Donaldson's techniques. This allows us to use simplified arguments at various points in the proof, and this is the reason for our passage from the Weinstein setting to the Stein setting. It is not clear whether one should expect to be able to generalize our arguments to apply directly to Weinstein manifolds.

Note that in most applications of quantitative transversality techniques in symplectic/contact geometry, the result in the integrable case requires only generic transversality, and the passage from integrable to nonintegrable J is what necessitates quantitative transversality. Here, quantitative transversality is needed in both the integrable and nonintegrable settings (although indeed, one would need more quantitative transversality in the nonintegrable case).

Besides Donaldson's original paper [10], which is the best place to first learn the methods introduced there, let us mention a few other papers where approximately holomorphic techniques have been used to obtain results similar to Theorem 1.10. In addition to constructing symplectic divisors [10], Donaldson also constructed Lefschetz pencils on closed symplectic manifolds [11]. Auroux [2; 3] further generalized and refined Donaldson's techniques to 1–parameter families of sections and to high twists $E \otimes L^k$ of a given Hermitian vector bundle E. In particular, he showed that Donaldson's symplectic divisors are all isotopic for fixed sufficiently large k, and that symplectic four-manifolds can be realized as branched coverings of $\mathbb{C}P^2$. Ibort, Martínez-Torres and Presas [17] obtained analogues for contact manifolds of Donaldson's and Auroux's results, and these were used in Giroux [15] to construct open books on contact manifolds in any dimension. Mohsen [19; 20] extended the techniques of Donaldson and Auroux to construct sections whose restrictions to a given submanifold satisfy certain quantitative transversality conditions. He also showed that this result implies both the uniqueness theorem of Auroux on symplectic divisors and the contact theorem of Ibort, Martínez-Torres and Presas. His main observation is that the quantitative Sard theorem applies to real (not just to complex) polynomials. This plays an important role in the present work; it makes it possible to obtain quantitative transversality for the restriction of a holomorphic section to a real hypersurface.

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2 Review of complex geometry

We now provide for the reader a review of some classical results in complex geometry which we need. Our specific target is the solution of the d''–operator on Stein manifolds via the L^2 methods of Hörmander [16] and Andreotti and Vesentini [1]. This will be used later to construct the localized "peak sections" necessary for Donaldson's construction. The reader may refer to [10, Proposition 34] for an analogous discussion in the case of compact Kähler manifolds.

2.1 Kähler geometry

For a complex vector bundle E with connection d over a complex manifold M, we denote by $d'\colon E\otimes\Omega^{p,q}\to E\otimes\Omega^{p+1,q}$ and $d''\colon E\otimes\Omega^{p,q}\to E\otimes\Omega^{p,q+1}$ the complex linear and complex conjugate linear parts of the exterior derivative $d\colon E\otimes\Omega^k\to E\otimes\Omega^{k+1}$. When M is equipped with a Kähler metric and E is equipped with a Hermitian metric, we let d'^* and d''^* denote the formal adjoints of d' and d'' respectively, and we let $\Delta' := d'^*d' + d'd'^*$ and $\Delta'' := d''^*d'' + d''d''^*$ denote the corresponding Laplacians.

Recall that on any holomorphic vector bundle with a Hermitian metric, there exists a unique connection compatible with the metric and the holomorphic structure, called the Chern connection.

Lemma 2.1 (Bochner–Kodaira–Nakano identity) *Let E be a holomorphic Hermitian vector bundle over a Kähler manifold. Then we have*

$$\Delta''_E = \Delta'_E + [i\,\Theta(E),\Lambda], \tag{2-1}$$

where $\Theta(E)$ is the curvature of E and Λ is the adjoint of $L := \cdot\wedge\omega$.

For a holomorphic Hermitian vector bundle E over a Kähler manifold, there is an induced Hermitian metric on $E\otimes\Omega^{0,q}$. The operator

$$d'\colon E\otimes\Omega^{0,q}\to E\otimes\Omega^{1,q} = E\otimes\Omega^{0,q}\otimes\Omega^{1,0}$$

further equips $E \otimes \Omega^{0,q}$ with an anti-holomorphic structure. Together these induce a Chern connection on $E \otimes \Omega^{0,q}$. We denote this connection by $\nabla = \nabla' + \nabla''$, where $\nabla' = d'$, and we denote the corresponding Laplacians by $\Box'$ and $\Box''$, where $\Box' = \Delta'$. Applying (2-1) to $E \otimes \Omega^{0,q}$ gives

$$\Box''_{E\otimes\Omega^{0,q}} = \Box'_{E\otimes\Omega^{0,q}} + [i\,\Theta(E \otimes \Omega^{0,q}), \Lambda]. \tag{2-2}$$

Now since $\Box'_{E\otimes\Omega^{0,q}} = \Delta'_E$ operating on $E \otimes \Omega^{0,q}$, we may combine (2-1) and (2-2) to produce the following Weitzenböck formula operating on $E \otimes \Omega^{0,q}$:

$$\Delta''_E = \Box''_{E\otimes\Omega^{0,q}} + \Lambda i\,\Theta(E \otimes \Omega^{0,q}) - \Lambda i\,\Theta(E). \tag{2-3}$$

We remark, for clarity, that the first composition is of maps $E\otimes\Omega^{0,q} \rightleftarrows E\otimes\Omega^{0,q}\otimes\Omega^{1,1}$ and the second composition is of maps $E \otimes \Omega^{0,q} \rightleftarrows E \otimes \Omega^{1,q+1}$. We have followed Donaldson [9, page 36] in the derivation of this identity.

Lemma 2.2 (Morrey–Kohn–Hörmander formula) *Let E be a holomorphic Hermitian vector bundle over a Kähler manifold M. For any $u \in C_c^\infty(M, E \otimes \Omega^{0,q})$, we have*

$$\int |d''u|^2 + |d''^*u|^2 = \int |\nabla''u|^2 + \int \langle u, \Lambda i\,\Theta(E\otimes\Omega^{0,q})u\rangle - \langle u, \Lambda i\,\Theta(E)u\rangle. \tag{2-4}$$

Proof By the definition of the adjoint, integrating by parts gives

$$\int |d''u|^2 + |d''^*u|^2 = \int \langle u, \Delta''u\rangle. \tag{2-5}$$

The same integration by parts with ∇ in place of d gives

$$\int |\nabla''u|^2 = \int \langle u, \Box''u\rangle. \tag{2-6}$$

Now we take the difference of these two identities and use (2-3) to obtain (2-4). □

2.2 L^2 theory of the d''–operator

The L^2 theory that we review here is due to Hörmander [16] and Andreotti and Vesentini [1].

Lemma 2.3 *Let E be a holomorphic Hermitian vector bundle over a complete Kähler manifold M. We consider sections u of $E \otimes \Omega^{p,q}$.*

- *If $u, d''u \in L^2$ (in the sense of distributions), then there exists a sequence $u_i \in C_c^\infty$ such that $(u_i, d''u_i) \to (u, d''u)$ in L^2.*

- *If $u, d''u, d''^*u \in L^2$ (in the sense of distributions), then there exists a sequence $u_i \in C_c^\infty$ such that $(u_i, d''u_i, d''^*u_i) \to (u, d''u, d''^*u)$ in L^2.*

Proof This is essentially a special case of Friedrichs' result [14], which applies more generally to any first order differential operator. We outline the argument, which is also given in Hörmander [16, Proposition 2.1.1] and Andreotti and Vesentini [1, Lemma 4, Proposition 5].

We prove the first statement only, as the proof of the second is identical. Let u be given. Composing the distance function to a specified point in M with the cutoff function $x \mapsto \max(1-\epsilon x, 0)$, we get a function $f_\epsilon\colon M \to \mathbb{R}$ with $\sup|f_\epsilon| \le 1$ and $\sup|df_\epsilon| \le \epsilon$, such that $f_\epsilon \to 1$ uniformly on compact subsets of M as $\epsilon \to 0$. Using these properties, it follows that $f_\epsilon u \to u$ in L^2 and that $d''(f_\epsilon u) \to d''u$ in L^2. Since M is complete, f_ϵ is compactly supported. Hence we may assume without loss of generality that u is compact supported.

Since u is compactly supported, we may use a partition of unity argument to reduce to the case when u is supported in a given small coordinate chart of M. Now in a small coordinate chart, choosing trivializations of the bundles in question, the operator d'' is a first order differential operator D with smooth coefficients. It can now be checked (and this is the key point) that $\|D(u * \varphi_\epsilon) - Du * \varphi_\epsilon\|_2 \to 0$, where $\varphi_\epsilon := \epsilon^{-n}\varphi(x/\epsilon)$ is a smooth compactly supported approximation to the identity. It follows that the convolutions $u * \varphi_\epsilon$ give the desired approximation of u by smooth functions of compact support. □

Proposition 2.4 *Let E be a holomorphic Hermitian vector bundle over a complete Kähler manifold M. Fix q, and suppose that for all $u \in C_c^\infty(M, E \otimes \Omega^{0,q})$, we have*

$$\int |u|^2 \le A \int |d''u|^2 + |d''^*u|^2. \tag{2-7}$$

Then for any $u \in L^2(M, E \otimes \Omega^{0,q})$ satisfying $d''u = 0$, there is $\xi \in L^2(M, E \otimes \Omega^{0,q-1})$ satisfying $d''\xi = u$ and

$$\int |\xi|^2 \le A \int |u|^2 \tag{2-8}$$

(d'' is taken in the sense of distributions).

Proof We follow an argument from notes by Demailly [8, page 33, (8.4) Theorem].

We wish to find ξ such that $d''\xi = u$, or, equivalently, $\int \langle d''^*\varphi, \xi\rangle = \int \langle \varphi, u\rangle$ for all $\varphi \in C_c^\infty(M, E \otimes \Omega^{0,q})$. We claim that the existence of such a ξ with $\int |\xi|^2 \le B$ is equivalent to the estimate

$$\left|\int \langle \varphi, u\rangle\right|^2 \le B\int |d''^*\varphi|^2 \tag{2-9}$$

for all $\varphi \in C_c^\infty(M, E\otimes\Omega^{0,q})$. Indeed, given (2-9), the map $d''^*\varphi \mapsto \int\langle\varphi,u\rangle$ on $d''^*(C_c^\infty(M,E\otimes\Omega^{0,q}))$ is well-defined and L^2 bounded, and thus it is of the form $\int\langle d''^*\varphi,\xi\rangle$ for a unique ξ in the closure of

$$d''^*(C_c^\infty(M,E\otimes\Omega^{0,q}))\subseteq L^2(M,E\otimes\Omega^{0,q-1})$$

satisfying $\int|\xi|^2\le B$. Thus we are reduced to showing (2-9) for $B = A\int|u|^2$.

To prove (2-9), argue as follows. Since L^2 convergence implies distributional convergence, the kernel (in the sense of distributions) $\ker d''\subseteq L^2(M,E\otimes\Omega^{0,q})$ is a closed subspace. Hence for any $\varphi\in C_c^\infty(M,E\otimes\Omega^{0,q})$, we may write $\varphi=\varphi_1+\varphi_2$, where $\varphi_1\in\ker d''$ and $\varphi_2\in(\ker d'')^\perp$. Now since $u\in\ker d''$, we have

$$\left|\int\langle\varphi,u\rangle\right|^2=\left|\int\langle\varphi_1,u\rangle\right|^2\le\int|u|^2\cdot\int|\varphi_1|^2. \tag{2-10}$$

Since $\varphi_2\perp\ker d''\supseteq\operatorname{im} d''$, it follows that $\varphi_2\in\ker d''^*$ (in the sense of distributions). Hence

$$\int |d''\varphi_1|^2+|d''^*\varphi_1|^2=\int|d''^*\varphi|^2. \tag{2-11}$$

Combining (2-10) and (2-11), we see that to prove (2-9) with $B=A\int|u|^2$, it suffices to show that

$$\int|\varphi_1|^2\le A\int|d''\varphi_1|^2+|d''^*\varphi_1|^2. \tag{2-12}$$

This is true by hypothesis (2-7) for $\varphi_1\in C_c^\infty(M,E\otimes\Omega^{0,q})$, and hence by Lemma 2.3 it holds given just that $\varphi_1, d''\varphi_1, d''^*\varphi_1\in L^2$. □

2.3 Stein manifolds and solving the d''–operator

Let V be a Stein manifold or a Stein domain. A smooth J–convex function $\phi\colon V\to\mathbb{R}$ induces a symplectic form $\omega_\phi := i\,d'd''\phi$ and a Riemannian metric $g_\phi(X,Y):=\omega_\phi(X,JY)$ (so $h_\phi:=g_\phi-i\omega_\phi$ is a Hermitian metric) whose distance function we denote by $d_\phi(\cdot,\cdot)$. The function ϕ also gives rise to a holomorphic Hermitian line bundle L^ϕ over V, namely the trivial complex line bundle $\mathbb{C}$ equipped with its standard holomorphic structure $d''_{\mathbb{C}}$ and the Hermitian metric $|\cdot|_{L^\phi}:=e^{-\frac{1}{2}\phi}|\cdot|_{\mathbb{C}}$. Then the resulting Chern connection on L^ϕ is given by

$$d_{L^\phi}=d_{\mathbb{C}}-d'\phi, \tag{2-13}$$

with curvature $\Theta(L^\phi) = d'd''\phi = -i\omega_\phi$. (Equivalently, L^ϕ is the trivial complex line bundle equipped with its standard Hermitian metric and the holomorphic structure $d''_{\mathbb{C}} + \frac{1}{2}d''\phi$, with resulting Chern connection $d_{\mathbb{C}} + \frac{1}{2}iJ^*d\phi$. This is equivalent to the first definition via multiplication by $e^{\frac{1}{2}\phi}$.)

The following result (due to Hörmander [16] and Andreotti and Vesentini [1]) allows us to produce many holomorphic sections of L^ϕ for sufficiently J–convex ϕ.

Proposition 2.5 *For every Stein manifold V with complete Kähler metric g, there exists a continuous function $c\colon V \to \mathbb{R}_{>0}$ with the following property. Let $\phi\colon V \to \mathbb{R}$ be J–convex and satisfy $g_\phi \geq c\cdot g$ (pointwise inequality of quadratic forms). Then for any $u \in L^2(V, L^\phi \otimes \Omega^{0,q})$ (with $q > 0$) satisfying $d''u = 0$, there exists $\xi \in L^2(V, L^\phi \otimes \Omega^{0,q-1})$ satisfying $d''\xi = u$ and*

$$\int |\xi|^2 \leq \int |u|^2. \tag{2-14}$$

Proof By Proposition 2.4, it suffices to show the estimate

$$\int |d''u|^2 + |d''^*u|^2 \geq \int |u|^2 \tag{2-15}$$

for all $u \in C_c^\infty(V, L\otimes \Omega^{0,q})$. Applying the Morrey–Kohn–Hörmander identity (2-4) to the left-hand side, it suffices to show the pointwise curvature estimate

$$\langle u, \Lambda i\,\Theta(L\otimes\Omega^{0,q})u\rangle - \langle u, \Lambda i\,\Theta(L)u\rangle \geq |u|^2. \tag{2-16}$$

Expanding $\Theta(L\otimes\Omega^{0,q}) = \Theta(L)\otimes \mathrm{id}_{\Omega^{0,q}} + \mathrm{id}_L \otimes \Theta(\Omega^{0,q})$, it suffices to show that

$$\langle u, \Lambda i\,(\Theta(L)\otimes\mathrm{id})u\rangle - \langle u, \Lambda i\,\Theta(L)u\rangle \geq \bigl(1+|\Lambda|\,|\Theta(\Omega^{0,q})|\bigr)|u|^2. \tag{2-17}$$

We remark for clarity that the first composition is of maps $L\otimes\Omega^{0,q} \rightleftarrows L\otimes\Omega^{0,q}\otimes\Omega^{1,1}$ and the second composition is of maps $L\otimes\Omega^{0,q} \rightleftarrows L\otimes\Omega^{1,q+1}$. Let $\alpha_1,\dots,\alpha_n$ denote the scaling factors associated to a simultaneous diagonalization of g and g_ϕ, meaning that $|v_i|^2_{g_\phi} = \alpha_i|v_i|^2_g$ for a simultaneous orthogonal basis $v_1,\dots,v_n$. We may now calculate (see Voisin [25, Lemma 6.19])

$$\Lambda i\,(\Theta(L)\otimes\mathrm{id})u = \biggl(\sum_{i=1}^n \alpha_i\biggr)u. \tag{2-18}$$

The operator $\Lambda i\,\Theta(L)$ has an orthonormal basis of eigenvectors with eigenvalues $\sum_{i\in I}\alpha_i$ for all $I\subseteq\{1,\dots,n\}$ with $|I| = n-q$. Thus to ensure (2-17), it suffices to have $q\min\alpha_i \geq 1+|\Lambda|\,|\Theta(\Omega^{0,q})|$, which can be achieved by choosing $c = 1+|\Lambda|\,|\Theta(\Omega^{0,q})|$ since $q > 0$. □

Lemma 2.6 *Let $\phi\colon (\mathbb{C}^n, 0) \to \mathbb{R}$ be a germ of a smooth J–convex function. For all $\epsilon > 0$, there exists a germ of a holomorphic function $u\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ satisfying*

$$\left|\operatorname{Re} u(z) - [\phi(z) - \tfrac{1}{2} d_\phi(z,0)^2]\right| \le \epsilon \cdot d_\phi(z,0)^2 \tag{2-19}$$

in a neighborhood of zero.

Proof The statement depends only on ϕ up to second order, so we may assume without loss of generality that ϕ is a real degree two polynomial on $\mathbb{C}^n$. Any real polynomial on $\mathbb{C}^n$ may be expressed uniquely as a polynomial in z_i and $\bar{z}_i$ with coefficients $c_{i_1,\dots,i_k,\bar{i}_1,\dots,\bar{i}_\ell} \in \mathbb{C}$ satisfying $c_{i_1,\dots,i_k,\bar{i}_1,\dots,\bar{i}_\ell} = \overline{c_{i_1,\dots,i_\ell,\bar{i}_1,\dots,\bar{i}_k}}$. In the case of degree two, we thus have

$$\phi(z) = a + \sum_i \operatorname{Re} a_i z_i + \sum_{i,j} \operatorname{Re} a_{ij} z_i z_j + \sum_{i,j} b_{ij} z_i \bar{z}_j, \tag{2-20}$$

where $a \in \mathbb{R}$, $a_i, a_{ij}, b_{ij} \in \mathbb{C}$ and $b_{ij} = \overline{b_{ji}}$. The statement is also unaffected by adding the real part of a holomorphic function to ϕ, so we may assume that $a = a_i = a_{ij} = 0$. Finally, the statement is unaffected by precomposing ϕ with a germ of biholomorphism of $\mathbb{C}^n$ near zero, so we may apply an element of $\mathrm{GL}_n(\mathbb{C})$ so that the positive definite Hermitian matrix (b_{ij}) becomes the identity matrix. Hence we have without loss of generality that $\phi(z) = |z|^2$, for which we may take $u \equiv 0$. □

3 Donaldson's construction

We now prove Theorem 1.7.

Let us begin by fixing some notation/terminology. We fix a Stein manifold $\bar{V}$ and a smooth exhausting J–convex function $\phi\colon \bar{V} \to \mathbb{R}$. We let $V := \{\phi \le 0\}$, so $\partial V = \{\phi{=}0\}$. We denote by $g := g_\phi$ the induced metric on $\bar{V}$, with associated distance function $d := d_\phi$. We denote by $L := L^\phi$ the associated line bundle. For any positive real number k, we let

$$g_k := g_{k\phi} = kg, \quad d_k := d_{k\phi} = k^{\frac{1}{2}} d \quad \text{and} \quad L^k := L^{k\phi}.$$

In what follows, we treat k as a fixed real parameter, and most statements — in particular, the notations $O(\cdot)$ and $o(\cdot)$ — are meant in the limit $k \to \infty$ (ie for k sufficiently large). Most implied constants are independent of $(\bar{V}, \phi)$ (unless stated otherwise), however how large k must be may (and almost always will) depend on $(\bar{V}, \phi)$.

Near any point $p_0 \in \bar{V}$, there exist a holomorphic coordinate chart $\Psi\colon (U, 0) \to (\bar{V}, p_0)$, where $U \subseteq \mathbb{C}^n$ is an open subset containing zero, and a holomorphic function $u\colon \Psi(U) \to \mathbb{C}$, satisfying

- $B(r) \subseteq U$ for $r^{-1} = O(1)$,
- $\Psi^* \phi = a \operatorname{Re} z_1 + O(|z|^2)$ if $p_0 \in \partial V$, where $a = |d\phi(p_0)|$,
- $\Psi^* g = g_{\mathbb{C}^n} + O(|z|)$,
- $\phi(p) - \frac{3}{4} d(p, p_0)^2 \leq \operatorname{Re} u(p) \leq \phi(p) - \frac{1}{4} d(p, p_0)^2$.

(For the existence of u, we appeal to Lemma 2.6.) There exists such a triple (U, Ψ, u) for which the implied constants above are bounded as p_0 varies over any compact subset of $\overline{V}$. It is convenient to also have at our disposal the rescaled coordinates $\Psi_k\colon (B(2), 0) \to (\overline{V}, p_0)$ defined by $\Psi_k(\,\cdot\,) = \Psi(k^{-\frac{1}{2}}\,\cdot)$ and the rescaled function ku (for sufficiently large k), which satisfy

- $\Psi_k^* \phi = a k^{-\frac{1}{2}} \operatorname{Re} z_1 + O(k^{-1}|z|^2)$ if $p_0 \in \partial V$, where $a = |d\phi(p_0)|$,
- $\Psi_k^* g_k = g_{\mathbb{C}^n} + O(k^{-\frac{1}{2}}|z|)$,
- $k\phi(p) - \frac{3}{4} d_k(p, p_0)^2 \leq \operatorname{Re} ku(p) \leq k\phi(p) - \frac{1}{4} d_k(p, p_0)^2$.

Now the section $\sigma := e^{\frac{1}{2}ku}$ of L^k satisfies

$$e^{-\frac{3}{8} d_k(p,p_0)^2} \leq |\sigma(p)| \leq e^{-\frac{1}{8} d_k(p,p_0)^2} \tag{3-1}$$

over its domain of definition $\Psi(U)$. This "reference section" provides a convenient local holomorphic trivialization of L^k over $\Psi_k(B(2))$. We also need holomorphic sections of L^k defined on all of $\overline{V}$ which satisfy a decay bound similar to (3-1) over $\{\phi \leq 1\}$ and which approximate σ over $\Psi_k(B(2))$. That such sections exist is the content of the following lemma.

Lemma 3.1 *Let $(\overline{V}, \phi)$ be as above. Fix $p_0 \in \{\phi = 0\}$ and consider the associated coordinates Ψ and reference section σ as above. There are holomorphic sections $\tilde{\sigma}, \tilde{\sigma}_1, \dots, \tilde{\sigma}_n\colon \overline{V} \to L^k$ satisfying*

- $|\tilde{\sigma}(p)| \leq e^{-\frac{1}{9} d_k(p,p_0)^2} + e^{-\epsilon k}$ *over* $\{\phi \leq 1\}$,
- $|\tilde{\sigma}_r(p)| \leq e^{-\frac{1}{9} d_k(p,p_0)^2} + e^{-\epsilon k}$ *over* $\{\phi \leq 1\}$ *for* $r = 1, \dots, n$,
- $|(\tilde{\sigma}/\sigma) \circ \Psi_k - 1| \leq e^{-\epsilon k}$ *over* $B(2)$,
- $|(\tilde{\sigma}_r/\sigma) \circ \Psi_k - z_r| \leq e^{-\epsilon k}$ *over* $B(2)$ *for* $r = 1, \dots, n$,

for some $\epsilon > 0$ depending on $(\overline{V}, \phi)$ and sufficiently large k.

Proof Fix a smooth cutoff function $\beta\colon \overline{V} \to [0, 1]$ supported inside $\Psi(U)$ which equals 1 in a neighborhood of p_0. Now $\|d''(\beta\sigma)\|_2 \leq e^{-\epsilon k}$ in the fixed metric g for sufficiently large k and some $\epsilon > 0$ depending on $(\overline{V}, \phi)$.

Fix a smooth exhausting J–convex function $\phi_1\colon \overline{V} \to \mathbb{R}$ which coincides with ϕ over $\{\phi \le 2\}$ and for which $g_{k\phi_1} \ge c \cdot g$ for sufficiently large k (for c as in Proposition 2.5). We apply Proposition 2.5 to $(\overline{V}, g, k\phi_1)$ and conclude that there exists a section ξ of L^k for which $\beta\sigma + \xi$ is holomorphic and $\|e^{\frac{1}{2}k\cdot(\phi-\phi_1)}\xi\|_2 \le \|d''(\beta\sigma)\|_2 \le e^{-\epsilon k}$.

Let us now show that $\tilde{\sigma} := \beta\sigma + \xi$ satisfies the desired properties. Over the set where $\beta = 1$, the section ξ is holomorphic. In particular, the function $(\xi/\sigma) \circ \Psi_k$ is holomorphic over $B(3)$ (for sufficiently large k). We have $\|(\xi/\sigma) \circ \Psi_k\|_{B(3),2} \le e^{-\epsilon k}$, from which it follows that $|(\xi/\sigma) \circ \Psi_k| \le e^{-\epsilon k}$ over $B(2)$ (for a possibly smaller $\epsilon > 0$ and larger k) since $(\xi/\sigma) \circ \Psi_k$ is holomorphic. Thus we have $|(\tilde{\sigma}/\sigma) \circ \Psi_k - 1| \le e^{-\epsilon k}$ over $B(2)$.

Now let $p \in \{\phi \le 1\}$ and consider the associated coordinates Ψ' and reference section σ' as above. We have $\|(\tilde{\sigma}/\sigma') \circ \Psi'_k\|_{B(3),2} = O(e^{-\frac{1}{8}d_k(p,p_0)^2} + e^{-\epsilon k})$, from which it follows that

$$|\tilde{\sigma}(p)| = O(e^{-\frac{1}{8}d_k(p,p_0)^2} + e^{-\epsilon k})$$

(since $(\tilde{\sigma}/\sigma') \circ \Psi'_k$ is holomorphic), which gives the desired decay bound on $\tilde{\sigma}$.

The argument for $\{\tilde{\sigma}_r\}_{1 \le r \le n}$ is identical, with $(z_r \circ \Psi_k^{-1}) \cdot \sigma$ in place of σ. □

It is helpful to rephrase Theorem 1.7 as follows in terms of the line bundle L^k and the rescaled metric g_k on $\overline{V}$.

Theorem 3.2 *Let $\overline{V}$ be a Stein manifold, equipped with a smooth exhausting J–convex function $\phi\colon \overline{V} \to \mathbb{R}$. For every sufficiently large real number k, there exists a holomorphic section $s\colon \overline{V} \to L^k$ such that*

- $|s(p)| \le 1$ *for* $p \in \{\phi \le 1\}$,
- $|s(p)| + |ds(p)|_\xi| > \eta$ *for* $p \in \{\phi = 0\}$ *(with ds measured in the metric induced by $k\phi$),*

where ξ denotes the Levi distribution on $\{\phi = 0\} \subseteq \overline{V}$, and $\eta > 0$ is a constant depending only on the dimension of $\overline{V}$.

Proof The proof follows Donaldson [10, Section 3], as simplified by Auroux [4].

Part I Fix a maximal collection of points $p_1, \dots, p_N \in \partial V$ whose pairwise d_k–distances are ≥ 1. Since this collection is maximal, the unit d_k–balls B_i centered at the p_i cover ∂V. The d_k–balls of radius $\frac{1}{2}$ centered at the p_i are disjoint, so by volume considerations, the total number of points satisfies $N = O_{(\overline{V},\phi)}(k^{2n-1})$, where n is the complex dimension of $\overline{V}$.

We now specify the form of the section $s\colon \overline{V} \to L^k$ we will construct. For each p_i, we will define a holomorphic section $s_i\colon \overline{V} \to L^k$ satisfying the bound

$$|s_i(p)| \le e^{-\frac{1}{9}d_k(p,p_i)^2} + e^{-\epsilon k} \quad \text{for } p \in \{\phi \le 1\} \tag{3-2}$$

for some $\epsilon > 0$ depending on $(\overline{V}, \phi)$, and we will let

$$s := \sum_{i=1}^{N} s_i. \tag{3-3}$$

Let us observe immediately that this bound on $|s_i|$ implies that

$$\begin{aligned} |s(p)| &\le \sum_{i=1}^{N} e^{-\frac{1}{9}d_k(p,p_i)^2} + e^{-\epsilon k} \\ &\approx \int_{\partial V} e^{-\frac{1}{9}d_k(p,p_0)^2} dg_k(p_0) + O_{(\overline{V},\phi)}(k^{2n-1}e^{-\epsilon k}) = O(1) \end{aligned}$$

for $p \in \{\phi \le 1\}$. In particular, this ensures the first condition $|s(p)| \le 1$ (after dividing by a constant factor depending only on $n = \dim \overline{V}$).

Remark 3.3 (C^0–bounds imply C^∞–bounds for holomorphic functions) For a holomorphic function f defined on $B(1+\epsilon) \subseteq \mathbb{C}^n$, we have

$$\|f\|_{C^\ell(B(1))} \le c_{n,\ell}\Big(1 + \frac{1}{\epsilon^\ell}\Big)\|f\|_{C^0(B(1+\epsilon))}. \tag{3-4}$$

(Indeed, we have $|D^\ell f(0)| \le c_{n,\ell} \sup_{B(1)} f$ by the Cauchy integral formula, and applying this to balls of radius $\epsilon > 0$ along with the maximum principle yields the above estimate.)

For simplicity of notation, we have stated the upper bounds in (3-1), Lemma 3.1, (3-2), and (3-5) below only in the C^0–norm, though of course we will often need to use the resulting bounds on higher derivatives implied by (3-4). If we were working in the approximately holomorphic setting, we would need to explicitly bound the higher derivatives up to some appropriate fixed finite order.

Definition 3.4 A section $s\colon V \to L^k$ will be called *η–transverse at $p \in \partial V$* if and only if $|s(p)| + \big|ds(p)|_\xi\big| > \eta$. The property of being η–transverse is obviously stable under C^1–perturbation, and for holomorphic sections it is in fact stable under C^0–perturbation by (3-4) with $\ell = 1$, as long as the perturbation is defined in a fixed neighborhood of p.

Remark 3.5 This particular quantitative transversality condition was first considered by Mohsen [20], and is closely related to those used by Donaldson and Auroux. Donaldson [10] called a section $s\colon V \to L^k$ η–transverse at p if and only if either $|s(p)| \geq \eta$ or $|ds(p)| \geq \eta$ (this is equivalent, up to a constant, to requiring that $|s(p)|+|ds(p)| \geq \eta$). Mohsen [20] generalized this notion to quantitative transversality relative to a given submanifold Y. Specifically, he called a section η–transverse relative to Y if and only if either $|s(p)| \geq \eta$ or $ds(p)|_{TY}$ has a right inverse of norm $\leq \eta^{-1}$. In the case of the submanifold $\partial V \subseteq \overline{V}$ and an (approximately) holomorphic section s, this condition is equivalent, up to a constant, to our formulation $|s(p)| + |ds(p)|_{\xi}| > \eta$ (see [20, Section 2]). Thus, Theorem 3.2 can be thought of as a holomorphic version of Mohsen's transversality theorem for hypersurfaces.

Part II Our goal is to construct sections s_i satisfying the decay bound (3-2) so that s is η–transverse over ∂V for some $\eta > 0$ depending only on n.

We will define the sections s_i in a series of steps, at each step achieving (quantitative) transversality over some new part of ∂V, while maintaining (quantitative) transversality over the part of ∂V already dealt with. The most naive version of this procedure, choosing s_i to achieve transversality over B_i while maintaining transversality over $B_1, \dots, B_{i-1}$, runs into trouble, essentially due to the rather large number of steps. Instead, we first construct a suitable coloring of the points p_i, and then in the inductive procedure we choose the s_i for the p_i of a particular color simultaneously (so there is one step per color). For this to work, we must ensure that points of the same color are sufficiently far apart.

Let $D < \infty$ be a (large) positive real number, to be fixed (depending only on n) at the end of the proof. We color the p_i so that the d_k–distance between any pair of points of the same color is at least D. More precisely, we construct such a coloring by iteratively choosing a maximal collection of as yet uncolored points p_i with pairwise distances $\geq D$ and then coloring this collection with a new color. Because each color was chosen from a maximal collection of as yet uncolored points, it follows that the ball of radius D centered at any point colored with the final color contains points of every other color. Hence by volume considerations, it follows that the total number of colors M is $O(D^{2n-1})$. Let us denote the coloring function by $c\colon \{1, \dots, N\} \to \{1, \dots, M\}$.

Part III Let $\boldsymbol{p} < \infty$ and $A < \infty$ be (large) positive real numbers, to be fixed (depending only on n) later in the proof. To be precise, we must first choose A (depending on n), then choose $\boldsymbol{p}$ (depending on n and A), and finally choose D (depending on n, A, and $\boldsymbol{p}$).

It suffices to construct sections s_i so that:

- For all $j \in \{1, \dots, M\}$ and $c(i) = j$, we have

$$|s_i(p)| \leq \frac{1}{A}\eta_{j-1}\big[e^{-\frac{1}{9}d_k(p,p_i)^2} + e^{-\epsilon k}\big] \quad \text{for} \quad p \in \{\phi \leq 1\}. \tag{3-5}$$

- For all $j \in \{1, \dots, M\}$, we have

$$s^j := \sum_{i=1;\, c(i)\leq j}^{N} s_i \quad \text{is } \eta_j\text{–transverse over} \quad X_j := \bigcup_{i=1;\, c(i)\leq j}^{N} B_i. \tag{3-6}$$

Here $\frac{1}{4} = \eta_0 > \eta_1 > \dots > \eta_M > 0$ are defined by $\eta_j = \eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$ (the reason for this particular choice will become apparent later).

We construct such sections s_i by induction on j. More precisely, it suffices to suppose that sections s_i are given for $c(i) \leq j-1$ (satisfying the above in the range $\{1, \dots, j-1\}$) and to construct sections s_i for $c(i) = j$ (satisfying the above in the range $\{1, \dots, j\}$).

Part IV As a first step, let us fix an index i with $c(i) = j$, and construct a section s_i satisfying (3-5) so that $s^{j-1} + s_i$ is $\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$–transverse over B_i (for some $\boldsymbol{p} < \infty$ depending on n and A).

Fix a triple (U, Ψ, u) based at $p_i \in \partial V$ (as discussed at the beginning of this section), with rescaling Ψ_k and reference section $\sigma = e^{(1/2)ku}$. We will use the local coordinates Ψ_k and the reference section σ to measure the transversality of $s^{j-1} + s_i$ over B_i. Precisely, we claim that it suffices to construct s_i satisfying (3-5) so that

$$\frac{s^{j-1} + s_i}{\sigma} \circ \Psi_k \tag{3-7}$$

is $\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$–transverse over $B\big(\frac{3}{2}\big) \cap \Psi_k^{-1}(\partial V)$. Indeed, σ is bounded above and below by (3-1), so using (3-4) with $\ell = 1$ this implies that the section $s^{j-1} + s_i$ is $\frac{1}{C}\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$–transverse over B_i for some constant $C < \infty$ depending only on n (which we can absorb into the last factor by increasing $\boldsymbol{p}$).

Now as $k \to \infty$, the real hypersurface $B\big(\frac{3}{2}\big) \cap \Psi_k^{-1}(\partial V)$ approaches $B\big(\frac{3}{2}\big) \cap \{\operatorname{Re} z_1 = 0\}$ in C^∞, uniformly over the choice of $p_i \in \partial V$. Since (3-7) is bounded uniformly over $B(2)$, using (3-4) with $\ell = 2$ we see that η–transversality over $B\big(\frac{3}{2}\big) \cap \{\operatorname{Re} z_1 = 0\}$ implies $(\eta - o(1))$–transversality over $B\big(\frac{3}{2}\big) \cap \Psi_k^{-1}(\partial V)$ (of course, the condition of η–transversality over a real hypersurface is with respect to its own Levi distribution). Since the number of colors M is bounded independently of k, it follows that η_{j-1} is bounded away from zero as $k \to \infty$. Hence it suffices to show that the section (3-7) is $\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$–transverse over $B\big(\frac{3}{2}\big) \cap \{\operatorname{Re} z_1 = 0\}$ (we again lose a constant on the transversality estimate, but as before it can be absorbed into the exponent $\boldsymbol{p}$).

For any vector $w = (w_0, w_2, \dots, w_n) \in \mathbb{C}^n$, we consider the holomorphic function on $B(2)$ given by

$$\frac{s^{j-1}}{\sigma} \circ \Psi_k + w_0 + w_2 z_2 + \dots + w_n z_n. \tag{3-8}$$

A quantitative transversality theorem, Proposition 4.1 (whose proof we defer to later) says that for $\frac{1}{3} > \eta > 0$, there exists a vector $w = (w_0, w_2, \dots, w_n) \in \mathbb{C}^n$ with $|w| \le \eta$ such that (3-8) is $\eta|\log \eta|^{-\boldsymbol{p}}$–transverse over $B\big(\frac{3}{2}\big) \cap \{\operatorname{Re} z_1 = 0\}$ (for some $\boldsymbol{p} < \infty$ depending only on n). This fact that with a perturbation of size η we can achieve $\eta|\log \eta|^{-\boldsymbol{p}}$–transversality is what forces the choice of recursion $\eta_j = \eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$ declared above.

Let $\widetilde{\sigma}$ and $\{\widetilde{\sigma}_r\}_{1 \le r \le n}$ denote the "peak sections" based at $p_0 = p_i$ from Lemma 3.1. We define $s_i := w_0 \widetilde{\sigma} + w_2 \widetilde{\sigma}_2 + \dots + w_n \widetilde{\sigma}_n$ (for w to be determined), so now (3-7) equals

$$\frac{s^{j-1}}{\sigma} \circ \Psi_k + w_0 \frac{\widetilde{\sigma}}{\sigma} \circ \Psi_k + w_2 \frac{\widetilde{\sigma}_2}{\sigma} \circ \Psi_k + \dots + w_n \frac{\widetilde{\sigma}_n}{\sigma} \circ \Psi_k. \tag{3-9}$$

There is a constant $C < \infty$ (depending only on n) such that for $|w| \le \frac{1}{A \cdot C} \eta_{j-1}$, the section s_i satisfies the decay bound (3-5). By Proposition 4.1, there exists $|w| \le \frac{1}{A \cdot C} \eta_{j-1}$ for which (3-8) is $\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$–transverse over $B\big(\frac{3}{2}\big) \cap \{\operatorname{Re} z_1 = 0\}$ (absorbing constants into $\boldsymbol{p}$). It follows that (3-9), so also (3-7), is $\big(\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}} - O(e^{-\epsilon k})\big)$–transverse over $B\big(\frac{3}{2}\big) \cap \{\operatorname{Re} z_1 = 0\}$, which is enough.

Part V We have constructed sections s_i for $c(i) = j$ with the property that $s^{j-1} + s_i$ is $\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$–transverse over B_i (for some $\boldsymbol{p} < \infty$ depending on n and A). Now let us argue that with this choice of sections, s^j is η_j–transverse over X_j (for some possibly different $\boldsymbol{p} < \infty$ depending on n and A).

We know that s^j differs from s^{j-1} over X_{j-1} by $O\big(\frac{1}{A}\eta_{j-1}\big)$ and that s^{j-1} is η_{j-1}–transverse over X_{j-1}. Hence s^j is $\big(1 - O\big(\frac{1}{A}\big)\big)\eta_{j-1}$–transverse over X_{j-1}, which gives η_j–transversality over X_{j-1} once A and $\boldsymbol{p}$ are large.

We know that s^j differs from $s^{j-1} + s_i$ over B_i by $O(\eta_{j-1} e^{-\frac{1}{9}D^2})$ and that $s^{j-1} + s_i$ is $\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}}$–transverse over B_i, so s^j is $(\eta_{j-1}|\log \eta_{j-1}|^{-\boldsymbol{p}} - O(\eta_{j-1} e^{-\frac{1}{9}D^2}))$–transverse over B_i. This gives η_j–transversality over B_i (increasing $\boldsymbol{p}$ to make up for the lost constant factor) as long as we have

$$e^{-\frac{1}{9}D^2} \le \frac{1}{B}|\log \eta_{j-1}|^{-\boldsymbol{p}} \tag{3-10}$$

for some constant $B < \infty$ depending only on n.

Hence we conclude that the entire construction succeeds as long as (3-10) holds for $j = 1, \dots, M$. It is elementary to observe that the recursive definition of η_j yields rough asymptotics $\eta_j \approx e^{-c \cdot j \log j}$ (with c depending on $\boldsymbol{p}$). Thus it suffices to ensure that, for some $B' < \infty$ depending on n and $\boldsymbol{p}$,

$$e^{-\frac{1}{9}D^2} \le \frac{1}{B'}(M \log M)^{-\boldsymbol{p}}. \tag{3-11}$$

We observed earlier that $M = O(D^{2n-1})$, so this inequality is satisfied once D is sufficiently large. □

Remark 3.6 A common theme in h–principle arguments à la Gromov, in which we want to construct some structure globally on a given manifold X, is to extend the desired structure to larger and larger subsets $\cdots \subseteq X_{j-1} \subseteq X_j \subseteq \cdots$ in a series of steps. This reduces the desired result to an extension problem from X_{j-1} to X_j (see for example Eliashberg and Mishachev [13]). For example, X_j is usually taken to be (an open neighborhood of) the j–skeleton of X (under a fixed triangulation), the point being that now the *topology* governing the extension from X_{j-1} to X_j is easy to understand. Donaldson's method, used in the proof above, employs a similar inductive procedure, but where one instead controls the *geometry* governing the extension from X_{j-1} to X_j (the key point being that we can do local modifications *independently* at any collection of points which are sufficiently far away from each other).

4 Quantitative transversality theorem

We now prove the quantitative transversality theorem (Proposition 4.1), which was the key technical ingredient in Donaldson's construction, as used in the proof of Theorem 3.2. The statement and proof are similar to Auroux [3, Section 2.3]; see also [4]. A key ingredient is an upper bound on the volume of tubular neighborhoods of real algebraic sets (Lemma 4.4) due to Wongkew [27].

Proposition 4.1 *Let $B(1) \subseteq B(1+\epsilon) \subseteq \mathbb{C}^n$ be the balls centered at zero. Fix a holomorphic function $f\colon B(1+\epsilon) \to \mathbb{C}$ with $|f| \le 1$. For a vector $w = (w_0, w_2, \dots, w_n)$ in $\mathbb{C}^n$, we define*

$$f_w := f + w_0 + w_2 z_2 + \cdots + w_n z_n. \tag{4-1}$$

For all $\frac{1}{3} > \eta > 0$, there exists a vector $w \in \mathbb{C}^n$ satisfying $|w| \le \eta |\log \eta|^{\boldsymbol{p}}$ such that

$$|f_w(z)| + |df_w(z)|_\xi| > \eta \quad \text{for } z \in B(1) \text{ with } \operatorname{Re} z_1 = 0,$$

where ξ denotes the Levi distribution of $\{z \in B(1) : \operatorname{Re} z_1 = 0\}$, and $\boldsymbol{p} < \infty$ depends only on the dimension n and $\epsilon > 0$.

Remark 4.2 A stronger version of Proposition 4.1 (a true quantitative Sard theorem where we only perturb f by a constant, ie $w_2 = \cdots = w_n = 0$ above) is due to Donaldson [10; 11] and Mohsen [20] with a rather more difficult proof. Mohsen's result could be used in Section 3 in place of Proposition 4.1, resulting in a simpler definition $s_i := w_0\tilde{\sigma}$, eliminating the need for the remaining $\tilde{\sigma}_2, \dots, \tilde{\sigma}_n$. We have chosen instead to present the argument following Auroux's observation that the weaker Proposition 4.1, whose proof is more elementary, is sufficient for the argument in Section 3.

Proof For a given $z \in B(1)$ with $\operatorname{Re} z_1 = 0$, the quantity $|f_w(z)| + \big|df_w(z)|_\xi\big|$ vanishes for exactly one value of w. The function $F\colon \{z \in B(1) : \operatorname{Re} z_1 = 0\} \to \mathbb{C}^n$ which associates to a given z this unique w is the restriction of a holomorphic function $F\colon B(1+\epsilon) \to \mathbb{C}^n$. Explicitly,

$$F(z) = \left(-f + z_2\frac{\partial f}{\partial z_2} + \cdots + z_n\frac{\partial f}{\partial z_n}, -\frac{\partial f}{\partial z_2}, \dots, -\frac{\partial f}{\partial z_n}\right). \tag{4-2}$$

In fact, the quantity $|f_w(z)| + |df_w(z)|_\xi|$ is bounded below by (a constant depending only on n, times) the distance from w to $F(z)$. Hence it suffices to show that

$$B(\delta) \setminus N_\eta\big(F(\{z \in B(1) : \operatorname{Re} z_1 = 0\})\big)$$

is nonempty for $\delta = \eta|\log\eta|^{O(1)}$.

We may approximate F to within error $\le \eta$ on $B(1)$ by a polynomial $\tilde{F}$ of degree $O(|\log\eta|)$. Indeed, the error in the degree m Taylor approximation of F is exponentially small in m, uniformly over $B(1)$, since F is holomorphic and bounded effectively on $B\big(1+\frac{\epsilon}{2}\big)$ by (3-4) with $\ell = 1$. To see this, observe that (by the $U(n)$ symmetry) it is enough to prove an effective exponential upper bound on the error over $B(1) \cap (\mathbb{C} \times \{0\}^{n-1})$, and this is just the well-known single-variable case (proved using the Cauchy integral formula).

It thus suffices to show that $B(\delta) \setminus N_{2\eta}\big(\tilde{F}(\{z \in B(1) : \operatorname{Re} z_1 = 0\})\big)$ is nonempty for $\delta = \eta|\log\eta|^{O(1)}$. Since $\tilde{F}$ is a polynomial of degree $O(|\log\eta|)$, a pigeonhole principle argument (Lemma 4.3 below) implies that its image is contained in a real algebraic hypersurface $X \subseteq \mathbb{C}^n$ of degree $\le |\log\eta|^{O(1)}$. Hence it suffices to show that $B(\delta) \setminus N_{2\eta}(X)$ is nonempty for $\delta = \eta|\log\eta|^{O(1)}$ and any real hypersurface $X \subseteq \mathbb{C}^n$ of degree $\le |\log\eta|^{O(1)}$.

Wongkew's estimate [27] (Lemma 4.4 below) on the volume of a tubular neighborhood of a real algebraic variety gives

$$\operatorname{vol}_{2n}(N_{2\eta}(X) \cap B(\delta)) = \delta^{2n} \cdot O\Big(\frac{\eta}{\delta}|\log\eta|^{O(1)}\Big). \tag{4-3}$$

For $\delta = \eta|\log\eta|^{O(1)}$, this is less than the total volume of $B(\delta)$, which is enough. □

Lemma 4.3 (Auroux [3, page 565]) *Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be a real polynomial map of degree $\le d$, where $n < m$. Then the image of F is contained in a real algebraic hypersurface of degree $D \le \lceil (\frac{m!}{n!} d^n)^{1/(m-n)} \rceil$.*

Proof The space of real polynomials G of degree $\le D$ on $\mathbb{R}^m$ has dimension $\binom{m+D}{m}$. The composition $G \circ F$ has degree $\le dD$. Hence there exists a nonzero G for which the composition is zero provided $\binom{m+D}{m} > \binom{n+dD}{n}$, or equivalently,

$$\frac{(D+1)\cdots(D+m)}{(dD+1)\cdots(dD+n)} > \frac{m!}{n!}.$$

The left-hand side is bounded below by $D^m/(dD)^n$, and so there exists a suitable G as long as $D^{m-n} \ge \frac{m!}{n!} d^n$. □

Lemma 4.4 (Wongkew [27]) *Let $X \subseteq \mathbb{R}^n$ be a real algebraic variety of codimension m defined by polynomials of degree $\le d$. Then we have the estimate*

$$\operatorname{vol}_n(N_\epsilon(X) \cap [0,1]^n) = O((\epsilon d)^m), \tag{4-4}$$

where the implied constant depends only on n.

It can be seen via simple examples that this bound is sharp, up to the implied constant. For completeness, we reproduce Wongkew's argument below.

Proof We proceed by induction on n, the case $n = 0$ being clear. All implied constants depend only on n. We assume for convenience that $\epsilon \le 1$ (otherwise the desired estimate is clear).

Let $\boldsymbol{H}$ be the collection of hyperplanes $H \subseteq \mathbb{R}^n$ given by constraining any one of the coordinates to lie in $[-2\epsilon, 1+2\epsilon] \cap (\epsilon\mathbb{Z} + \delta)$, where δ is chosen so that X intersects each $H \in \boldsymbol{H}$ properly (ie $X \cap H$ has codimension m inside H). Such a δ exists by Bertini's theorem. Clearly $\#\boldsymbol{H} = O(\epsilon^{-1})$. This set of hyperplanes partitions $\mathbb{R}^n$ into some unbounded components and some cubes of side length ϵ. We denote the set of such cubes by $\mathbf{C}$.

We call a cube $C \in \boldsymbol{C}$ *exceptional* if and only if X intersects the interior of C but not its boundary. The number of exceptional cubes is clearly bounded by $\dim H_0(X)$, which by a result of Milnor [18] is bounded by $d(2d-1)^{n-1} = O(d^n)$.

It is straightforward to check that

$$N_\epsilon(X) \cap [0,1]^n \subseteq \Big[N_{(1+\sqrt{n})\epsilon}\Big(X \cap \bigcup_{H \in \boldsymbol{H}} H \Big) \cap [0,1]^n \Big] \cup \Big[\bigcup_{\substack{C \in \boldsymbol{C} \\ C \text{ exceptional}}} N_\epsilon(C) \Big]. \tag{4-5}$$

Indeed, suppose $p \in [0,1]^n$ and $d(p,X) \le \epsilon$. There exists $x \in X$ with $d(p,x) \le \epsilon$, and $x \in C$ for some (closed) cube $C \in \boldsymbol{C}$. If C is exceptional, then p lies in the second term above. If C is not exceptional, then $X \cap \partial C$ is nonempty. It thus follows that $d(p, \partial C \cap X) \le \epsilon + d(x, \partial C \cap X) \le \epsilon + \epsilon\sqrt{n}$, and so p lies in the first term above.

Now the inclusion (4-5) implies the following inequality on volumes:

$$\mathrm{vol}_n(N_\epsilon(X) \cap [0,1]^n) \le \sum_{H \in \boldsymbol{H}} 2(1+\sqrt{n})\epsilon \,\mathrm{vol}_{n-1}(N_{(1+\sqrt{n})\epsilon}(X \cap H) \cap H \cap [0,1]^n) + \sum_{\substack{C \in \boldsymbol{C} \\ C \text{ exceptional}}} (3\epsilon)^n$$

If $m = n$, then the first term vanishes (each $X \cap H$ is empty by assumption), and Milnor's bound on the second term gives the desired result. If $m \le n-1$, then we apply the induction hypothesis to the first term and Milnor's result to the second term. The result is

$$\mathrm{vol}_n(N_\epsilon(X) \cap [0,1]^n) = O\big((\epsilon d)^m + (\epsilon d)^n\big). \tag{4-6}$$

This implies the desired estimate for $\epsilon d \le 1$, and for $\epsilon d \ge 1$ the desired estimate is trivial. □

5 Lefschetz fibrations on Stein domains

We now show how the function f, guaranteed to exist by Theorem 1.7, gives rise to a Lefschetz fibration. To be precise, we will show that Theorem 1.7 implies Theorem 1.6 and that Theorem 1.6 implies Theorem 1.5.

Proof of Theorem 1.6 from Theorem 1.7 Fix an embedding $V \hookrightarrow \overline{V}$ of the Stein domain V into a Stein manifold $\overline{V}$ of the same dimension, and fix an exhausting J–convex function $\phi\colon \overline{V} \to \mathbb{R}$ with $V = \{\phi \le 0\}$.

By Theorem 1.7 there exists, for sufficiently large k, a holomorphic function $f\colon \overline{V} \to \mathbb{C}$ such that

- $|f(p)| + k^{-\frac{1}{2}}|df(p)|_\xi| > \eta$ for $p \in \partial V$,
- $|f(p)| \le e^{\frac{1}{2}k\phi(p)}$ for $p \in \{\phi \le 1\}$.

We claim that the bound $|f(p)| \le e^{\frac{1}{2}k\phi(p)}$ implies

- $|df(p) - k \cdot f(p) \cdot d'\phi(p)| = O(k^{\frac{1}{2}} e^{\frac{1}{2}k\phi(p)})$ for $p \in V$.

To see this, argue as follows. Fix a point $p \in V$ and choose a holomorphic function u defined in a neighborhood of p such that $\operatorname{Re} u(q) = \phi(q) + O(d(p,q)^2)$. It follows that $f(q) \cdot e^{-\frac{1}{2}ku(q)} = O(1)$ for $d(p,q) = O(k^{-\frac{1}{2}})$, and hence it follows that

$$d(f \cdot e^{-\frac{1}{2}ku})(p) = O(k^{\frac{1}{2}}).$$

Expanding the left-hand side and using the fact that $du(p) = 2d' \operatorname{Re} u(p) = 2d'\phi(p)$, the claim follows.

Now we take $\pi := \eta^{-1} \cdot f$, which satisfies the desired properties. □

Proof of Theorem 1.5 from Theorem 1.6 By Theorem 1.6 there exists, for sufficiently large k, a holomorphic function $\pi\colon V \to \mathbb{C}$ such that

- for $|\pi(p)| \geq 1$, we have $d \log \pi(p) = k \cdot d'\phi(p) + O(k^{\frac{1}{2}})$,
- for $|\pi(p)| \leq 1$ and $p \in \partial V$, we have $d\pi(p)|_\xi \neq 0$.

Note that these conditions together imply that the critical locus of π is contained in the interior of $\pi^{-1}(D^2)$. Both conditions are preserved under small perturbations of π, hence we may perturb π so that

- all critical points of π on V are nondegenerate and have distinct critical values.

Indeed, the existence of such a perturbation follows from the standard fact that global holomorphic functions on any Stein manifold $\overline{V}$ generate $\mathcal{O}_{\overline{V}}$ and $\Omega^1_{\overline{V}}$ at every point (this follows from Cartan's Theorems A and B, or by properly embedding $\overline{V}$ in $\mathbb{C}^N$).

Now $\pi\colon \pi^{-1}(D^2) \to D^2$ is a Stein Lefschetz fibration, so it suffices to construct a deformation of Stein domains from V to $\pi^{-1}(D^2)^{\mathrm{sm}}$. Let $g\colon \mathbb{R}_{<0} \to \mathbb{R}$ satisfy $g' > 0$, $g'' > 0$, and $\lim_{x \to 0^-} g(x) = \infty$. Consider the family $\{\pi^{-1}(D^2_r)\}_{1 \leq r < \infty}$, and consider its smoothing $\{r^{-3} g(|\pi|^2 - r^2) + g(\phi) \leq M\}_{1 \leq r < \infty}$ for some large $M < \infty$. Since $\pi^{-1}(D^2_r)$ is cut out by the inequalities $\phi \leq 0$ and $\operatorname{Re} \log \pi \leq \log r$, this smoothing gives the desired deformation as long as for every point $p \in V$ with $|\pi(p)| \geq 1$, the differentials $d\phi(p)$ and $\operatorname{Re} d \log \pi(p)$ are either linearly independent or positively proportional. Since $d \log \pi(p) = k \cdot d'\phi(p) + O(k^{\frac{1}{2}})$, this condition is clearly satisfied for sufficiently large k. □

6 Lefschetz fibrations on Weinstein domains

We now show how the existence of Lefschetz fibrations on Stein domains (Theorem 1.5) may be used to deduce the same for Weinstein domains (Theorem 1.10). For this implication, we use the result of Cieliebak and Eliashberg [7, Theorem 1.1(a)] that every Weinstein domain may be deformed to carry a compatible Stein structure. The main step (Proposition 6.2) is thus to show that for any Stein Lefschetz fibration $\pi\colon V \to D^2$, there exists an abstract Weinstein Lefschetz fibration whose total space is deformation equivalent to V^{sm}.

6.1 From Stein structures to Weinstein structures

We give a very brief review of the relationship between Stein and Weinstein structures; for a complete treatment, the reader may consult [7, Section 1]. Let (V,ϕ) be a pair consisting of a Stein domain V and a smooth J–convex function $\phi\colon V \to \mathbb{R}$ with $\partial V = \{\phi{=}0\}$ as a regular level set. If ϕ is Morse (which can be achieved by small perturbation), then it induces the structure of a Weinstein domain on V, namely taking the 1–form $\lambda_\phi := -J^* d\phi$ and the function ϕ itself. This Weinstein domain is denoted $\mathfrak{W}(V,\phi)$. For any deformation of Stein domains $(V_t,\phi_t)_{t\in[0,1]}$ where every ϕ_t is generalized Morse (any $\{\phi_t\}_{t\in[0,1]}$ may be perturbed to satisfy this condition), the associated family $\mathfrak{W}(V_t,\phi_t)_{t\in[0,1]}$ is a deformation of Weinstein domains. In particular, the deformation class of $\mathfrak{W}(V,\phi)$ is independent of ϕ, so we may denote it by $\mathfrak{W}(V)$. Now a decisive result is the following (we state a simplified version which is sufficient for our purpose).

Theorem 6.1 (Cieliebak and Eliashberg [7, Theorem 1.1(a)]) *Every deformation class of Weinstein domain is of the form $\mathfrak{W}(V)$ for a Stein domain V.*

6.2 From Stein Lefschetz fibrations to abstract Weinstein Lefschetz fibrations

Theorem 1.10 follows from Theorem 1.5, Theorem 6.1, and the following proposition.

Proposition 6.2 *Let $\pi\colon V \to D^2$ be a Stein Lefschetz fibration. There exists an abstract Weinstein Lefschetz fibration $W = (W_0; L_1, \dots, L_m)$ whose total space $|W|$ is deformation equivalent to $\mathfrak{W}(V^{\mathrm{sm}})$.*

The abstract Weinstein Lefschetz fibration associated to a Stein Lefschetz fibration may be described as follows. The "central fiber" W_0 is the Weinstein domain associated to a regular fiber $\pi^{-1}(p)$ of $\pi\colon V \to D^2$, and the "vanishing cycles" $L_1, \dots, L_m$ are the images of the critical points of π under symplectic parallel transport along a set of disjoint paths from the critical values of π to the regular value p. Hence the content of the proposition is that (as a Weinstein manifold) V^{sm} may be described as a small product neighborhood of a regular fiber with Weinstein handles attached along the vanishing cycles.

We now give a detailed definition of the total space of an abstract Weinstein Lefschetz fibration.

Definition 6.3 Let $W = ((W_0, \lambda_0, \phi_0); L_1, \dots, L_m)$ be an abstract Weinstein Lefschetz fibration. Its *total space* $|W|$ is defined as follows. We equip $W_0 \times \mathbb{C}$ with the Liouville form $\lambda_0 - J^* d\big(\frac{1}{2}|z|^2\big)$ and the Morse function $\phi_0 + |z|^2$ for which the resulting Liouville vector field $X_{\lambda_0} + \frac{1}{2}\big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\big)$ is gradient-like. Fix Legendrian lifts $\Lambda_j \subseteq (W_0 \times S^1, \lambda_0 + N d\theta)$ of the exact Lagrangians $L_j \subseteq W_0$ such that Λ_j projects to a small interval around $2\pi j/m \in S^1$ (here we choose $N < \infty$ sufficiently large so that these intervals are disjoint). Now the embedding $S^1 \hookrightarrow \mathbb{C}$ as the circle of radius $\sqrt{N}$ pulls back the Liouville form $-J^* d\big(\frac{1}{2}|z|^2\big)$ to the contact form $N d\theta$. Hence we may think of Λ_j as lying inside $W_0 \times \mathbb{C}$ as a Legendrian on the level set $\{|z| = \sqrt{N}\}$. The downward Liouville flow applied to Λ_j gives rise to a map $\Lambda_j \times \mathbb{R}_{\geq 0} \to W_0 \times \mathbb{C}$, which intersects the level set $\{\phi_0 + |z|^2 = 0\}$ in a Legendrian Λ_j' (here we choose $N < \infty$ so that the projection of $\{\phi_0 + |z|^2 \leq 0\}$ to $\mathbb{C}$ is contained inside the disk of radius $\sqrt{N}$). The total space $|W|$ is defined as the result of attaching Weinstein handles [26] to the Weinstein domain $\{\phi_0 + |z|^2 \leq 0\}$ along the Legendrians Λ_j' (marked via the maps $S^{n-1} \to L_j \xrightarrow{\sim} \Lambda_j \xrightarrow{\sim} \Lambda_j'$). It is easy to see that $|W|$ is well-defined up to canonical deformation (we will remark in detail on the well-definedness of Weinstein handle attachment in Lemma 6.6).

We now introduce a variant of the above construction, which will be used in the proof of Proposition 6.2.

Definition 6.4 Let $W = (\pi\colon V \to D^2, \phi, g; L_1, \dots, L_m)$ consist of a Stein Lefschetz fibration $\pi\colon V \to D^2$, a J–convex function $\phi\colon V \to \mathbb{R}$ with $\partial_h V = \{\phi = 0\}$ as a regular level set, a function $g\colon \mathbb{R}_{<0} \to \mathbb{R}$ with $g > 0$, $g' > 0$, $g'' > 0$, and $\lim_{x \to 0^-} g(x) = \infty$, and a collection of exact parametrized Lagrangian (with respect to $\lambda_{g(\phi)}$) spheres $L_j \subseteq V_{p_j} := \pi^{-1}(p_j)$, for distinct points $p_1, \dots, p_m \in S^1 = \partial D^2$, ordered counterclockwise. Define its *total space* $|W|$ as follows. Consider the J–convex function $\epsilon g(\phi) + \frac{1}{2}|\pi|^2$ on V. The induced contact form on $\pi^{-1}(\partial D^2)$ may be written as $\epsilon\lambda_{g(\phi)} + d\theta$. Let us center the S^1–coordinate at $p_j \in S^1$, rescale it by ϵ^{-1}, and rescale the contact form by ϵ^{-1}. In the limit $\epsilon \to 0$, this rescaling of $\pi^{-1}(\partial D^2)$ converges to the contact manifold $(V_{p_j} \times \mathbb{R}, \lambda_{g(\phi)} + dt)$. In $V_{p_j} \times \mathbb{R}$, there is a unique (up to translation) Legendrian Λ_j projecting to L_j. During the deformation of $V_{p_j} \times \mathbb{R}$ back to $\pi^{-1}(\partial D^2)$ for small $\epsilon > 0$, there clearly exists a simultaneous Legendrian isotopy $\Lambda_j^\epsilon \subseteq \pi^{-1}(\partial D^2)$ starting at $\Lambda_j^0 = \Lambda_j$. Now the downward Liouville flow applied to Λ_j^ϵ intersects $\big\{\epsilon g(\phi) + \frac{1}{2}(|\pi|^2 - 1) = 0\big\}$ in a Legendrian $\Lambda_j^{\epsilon\prime}$. The total space $|W|$ is defined as the result of attaching Weinstein handles to the Weinstein domain $\big\{\epsilon g(\phi) + \frac{1}{2}(|\pi|^2 - 1) \leq 0\big\}$

along these Legendrians. This total space is independent of the choice of sufficiently small $\epsilon > 0$ and the family $\{\Lambda_i^\epsilon\}_{\epsilon\geq 0}$ up to canonical deformation.

Definition 6.4 reduces to Definition 6.3 in the special case of a product fibration, in the sense that there is a canonical deformation equivalence

$$\big|(V_0 \times D^2 \to D^2, \phi_0, g; L_1 \times \{\alpha_1\}, \dots, L_m \times \{\alpha_m\})\big| = \big|(\mathfrak{W}(V_0, g(\phi_0)); L_1, \dots, L_m)\big|, \tag{6-1}$$

where $\phi_0\colon V_0 \to \mathbb{R}$ is J–convex with $\partial V_0 = \{\phi_0{=}0\}$ as a regular level set, $L_1, \dots, L_m$ are exact parametrized Lagrangian spheres in V_0 with respect to $\lambda_{g(\phi_0)}$, and the points $\alpha_1, \dots, \alpha_m \in S^1 = \partial D^2$ are ordered counterclockwise. The right-hand side of (6-1) is a slight abuse of notation, as we should really write $\mathfrak{W}\big(\{g(\phi_0) \leq M\}, g(\phi_0)\big)$ for sufficiently large M.

Proof of Proposition 6.2 We assume that $0 \in D^2$ is a regular value of π and that each critical value of π has a distinct complex argument (this may be achieved by post-composing π with a generic Schwarz biholomorphism $D^2 \to D^2$).

Fix a smooth J–convex function $\phi\colon V \to \mathbb{R}$ with $\partial_h V = \{\phi{=}0\}$ as a regular level set (as is guaranteed to exist by Definition 1.4). We let $V_0 := \pi^{-1}(0)$ denote the central fiber, and we assume that $\phi_0 := \phi|_{V_0}$ is Morse (this can be achieved by a small perturbation of ϕ).

By Lemma 6.5 below, there exists a smooth function $g\colon \mathbb{R}_{<0} \to \mathbb{R}$ satisfying $g > 0$, $g' > 0$, $g'' > 0$ and $\lim_{x\to 0^-} g(x) = \infty$ such that the symplectic connection on $\pi\colon V \setminus \partial_h V \to D^2$ induced by $\omega_{g(\phi)}$ is *complete*. Fix one such g.

We consider parallel transport along radial paths in D^2 with respect to the symplectic connection induced by $\omega_{g(\phi)}$. Under this parallel transport, each critical point of π sweeps out a Lagrangian disk called a *Lefschetz thimble* (to see this, apply the stable manifold theorem to the Hamiltonian vector field $X_{\operatorname{Im}\log\pi}$, and recall that the critical values of π have distinct complex arguments). The fiber over $0 \in D^2$ of a Lefschetz thimble is an exact Lagrangian sphere called a *vanishing cycle*. Let $L_1, \dots, L_m \subseteq V_0$ denote the vanishing cycles of all the critical points of π, ordered by angle. As stable manifolds of the vector field $X_{\operatorname{Im}\log\pi}$, they come equipped with parametrizations $S^{n-1} \to L_j$, which are well-defined in $\operatorname{Diff}(S^{n-1}, L_j)/O(n)$ up to contractible choice.

Now $W := (\mathfrak{W}(V_0, g(\phi_0)); L_1, \dots, L_m)$ is an abstract Weinstein Lefschetz fibration, and it remains to show that its total space $|W|$ is deformation equivalent to $\mathfrak{W}(V^{\mathrm{sm}})$.

We consider the J–convex function $\epsilon g(\phi) + h(|\pi|/\delta)$ on $V \setminus \partial_h V$ for small $\epsilon, \delta > 0$, where

$$h(r) := \begin{cases} \log r & \text{if } r \geq 1, \\ \frac{1}{2}(r^2 - 1) & \text{if } r \leq 1. \end{cases} \tag{6-2}$$

We claim that for $(\epsilon, \delta) \to (0, 0)$, the sublevel set

$$\left\{\epsilon g(\phi) + h\left(\frac{|\pi|}{\delta}\right) \leq \log \frac{1}{\delta}\right\} \subseteq V \tag{6-3}$$

is deformation equivalent to V^{sm}. Indeed, consider the $\leq \log(1/\delta)$ sublevel set of the linear interpolation between $\epsilon g(\phi) + h(|\pi|/\delta)$ and $\epsilon g(\phi) + \epsilon g(|\pi|^2 - 1)$. As $(\epsilon, \delta) \to (0, 0)$, the boundary of this deformation stays arbitrarily close to ∂V, and the critical locus of the linear interpolation stays away from ∂V (note that this critical locus is always contained in the fiberwise critical locus of ϕ). Thus (6-3) is deformation equivalent to V^{sm} as claimed.

As $(\epsilon, \delta) \to (0, 0)$, the critical points of $\epsilon g(\phi) + h(|\pi|/\delta)$ over $D^2 \setminus D^2_\delta$ are in bijective correspondence with $\operatorname{crit}(\pi)$ (note that the critical locus is contained in the fiberwise critical locus of ϕ). Over $D^2 \setminus D^2_\delta$, the stable manifolds of these critical points approach the Lefschetz thimbles as $\epsilon \to 0$ and $\delta > 0$ is fixed. Indeed, h is harmonic over $D^2 \setminus D^2_\delta$, and hence the Liouville vector field of $\epsilon g(\phi) + h(|\pi|/\delta)$ is given by $X_{g(\phi)} + \epsilon^{-1} X_{\operatorname{Im}\log\pi}$ over $D^2 \setminus D^2_\delta$, where $X_{g(\phi)}$ is the Liouville vector field of $g(\phi)$ and $X_{\operatorname{Im}\log\pi}$ is the Hamiltonian vector field with respect to $\omega_{g(\phi)}$ of $\operatorname{Im}\log\pi$.

Denote by $\overline{\Lambda}_j^{\epsilon,\delta} \subseteq \pi^{-1}(\partial D^2_\delta)$ the intersections of the stable manifolds of $\epsilon g(\phi) + h(|\pi|/\delta)$ with $\pi^{-1}(\partial D^2_\delta)$. Thus $\overline{\Lambda}_j^{\epsilon,\delta}$ is Legendrian with respect to the contact form $\epsilon\lambda_{g(\phi)} + d\theta$. Let L_j^δ denote the intersections of the Lefschetz thimbles with $\pi^{-1}(\partial D^2_\delta)$. Thus as $\epsilon \to 0$ and $\delta > 0$ is fixed, we have that $\overline{\Lambda}_j^{\epsilon,\delta} \to L_j^\delta$ in C^∞. Now we claim that $\overline{\Lambda}_j^{\epsilon,\delta}$ is in fact (Legendrian isotopic to) the Legendrian lift $\Lambda_j^{\epsilon,\delta}$ of L_j^δ (as in Definition 6.4) for sufficiently small $\epsilon > 0$. In the rescaled limit as $\epsilon \to 0$, the projection of $\overline{\Lambda}_j^{\epsilon,\delta}$ to V_{p_j} approaches L_j^δ in C^∞, and this is enough to show that it converges (up to translation) to $\Lambda_j^{0,\delta}$ as $\epsilon \to 0$. Hence the claim is valid, so we conclude that $\mathfrak{W}(V^{\mathrm{sm}})$ is deformation equivalent to

$$|(\pi\colon \pi^{-1}(D^2_\delta) \to D^2_\delta; L_1^\delta, \ldots, L_m^\delta)|. \tag{6-4}$$

We have used Lemma 6.6 below to show that the Weinstein cobordism

$$\left\{0 \leq \epsilon g(\phi) + h\left(\frac{|\pi|}{\delta}\right) \leq \log \frac{1}{\delta}\right\}$$

is a Weinstein handle attachment.

In the limit $\delta \to 0$, rescaling D^2_δ to D^2, clearly (6-4) converges to

$$|(V_0 \times D^2 \to D^2; L_1 \times \{\alpha_1\}, \dots, L_m \times \{\alpha_m\})|, \tag{6-5}$$

where $\alpha_j \in S^1 = \partial D^2$ are the angles of the critical points of π. Hence using (6-1) we have shown the desired deformation equivalence between $\mathfrak{W}(V^{\mathrm{sm}})$ and $|(\mathfrak{W}(V_0, g(\phi_0)); L_1, \dots, L_m)|$. □

Lemma 6.5 *Let $\pi\colon V \to D^2$ be a Stein Lefschetz fibration, and let $\phi\colon V \to \mathbb{R}$ be J–convex with $\partial_h V = \{\phi{=}0\}$ as a regular level set. There exists a smooth function $g\colon \mathbb{R}_{<0} \to \mathbb{R}$ satisfying $g' > 0$, $g'' > 0$ and $\lim_{x\to 0^-} g(x) = \infty$ such that the symplectic connection on $\pi\colon V \setminus \partial_h V \to D^2$ induced by $\omega_{g(\phi)}$* is complete, *in the sense that parallel transport along a smooth path in the base D^2 gives rise to a diffeomorphism between the corresponding fibers (away from the critical points of π).*

Proof We will in fact show that there exists a natural contractible family of functions g which satisfy the desired conclusion for all (V, ϕ).

Let $g\colon \mathbb{R}_{<0} \to \mathbb{R}$ be such that $g' > 0$, $g'' > 0$, and $\lim_{x\to 0^-} g(x) = \infty$. Let $\perp_\phi$ (resp. $\perp_{g(\phi)}$) denote orthogonal complement with respect to ω_ϕ (resp. $\omega_{g(\phi)}$), so the horizontal distribution of the symplectic connection induced by $\omega_{g(\phi)}$ is $(\ker d\pi)^{\perp_{g(\phi)}}$.

Our first goal is to show that in a neighborhood of $\partial_h V$, every horizontal vector field X satisfies

$$|X\phi| = O\left(\frac{g'(\phi)}{g''(\phi)}\right) \cdot |\pi_* X|, \tag{6-6}$$

as long as $g'(\phi)/g''(\phi)$ is sufficiently small. Note that in a neighborhood of $\partial_h V$, there is a direct sum decomposition

$$\begin{aligned} TV = (\ker d\pi \cap \ker d'\phi) \oplus (\ker d\pi \cap \ker d'\phi)^{\perp_\phi} \cap \ker d\pi \\ \oplus (\ker d\pi \cap \ker d'\phi)^{\perp_\phi} \cap \ker d'\phi \end{aligned} \tag{6-7}$$

into subspaces of real dimension $2n-4$, 2, 2, respectively. Now suppose that $X = X_1 \oplus X_2 \oplus X_3 \in TV$ is horizontal, ie $X \perp_{g(\phi)} \ker d\pi$. Note the explicit form

$$\omega_{g(\phi)} = g'(\phi) \cdot \omega_\phi + g''(\phi) \cdot i\, d'\phi \wedge d''\phi. \tag{6-8}$$

We may choose a vector $v \in (\ker d\pi \cap \ker d'\phi)^{\perp_\phi} \cap \ker d\pi$ with $|v|_{g_\phi} = 1$ such that $|(i\, d'\phi \wedge d''\phi)(v, X_2)| \asymp |X_2|_{g_\phi}$ (where g_ϕ denotes the metric induced by ϕ). Now since $v \in \ker d\pi$, it pairs to zero with X under $\omega_{g(\phi)}$, so we have

$$0 = g'(\phi) \cdot \omega_\phi(v, X_2 + X_3) + g''(\phi) \cdot (i\, d'\phi \wedge d''\phi)(v, X_2). \tag{6-9}$$

It follows from this that

$$|X_2|_{g_\phi} = O\left(\frac{g'(\phi)}{g''(\phi)}\right) \cdot |X_3|_{g_\phi}$$

for $g'(\phi)/g''(\phi)$ sufficiently small. This implies the desired estimate (6-6) since $|\pi_* X| \asymp |X_3|_{g_\phi}$ and $|X\phi| \asymp |X_2|_{g_\phi}$.

It now follows that the connection is complete as long as

$$\limsup_{x\to 0^-} \frac{g'(x)}{|x| g''(x)} < \infty. \tag{6-10}$$

Indeed, by (6-6) this condition guarantees that the derivative of $\log(-\phi)$ is bounded along the horizontal lift of a smooth curve in the base D^2.

We now just need to exhibit a function $g\colon \mathbb{R}_{<0} \to \mathbb{R}$ satisfying $g' > 0$, $g'' > 0$, $\lim_{x\to 0^-} g(x) = \infty$, and (6-10), which we may write as

$$\liminf_{x\to 0^-} (\log g'(x))' |x| > 0. \tag{6-11}$$

For example, we may take

$$g(x) := \int_{-\infty}^{x} e^{-t^2 - t^{-1}}\, dt. \tag{6-12}$$

Moreover, the space of such functions is contractible, since the map $g \mapsto (g(-1), \log g')$ gives a bijection with a convex set. □

6.3 Uniqueness of Weinstein handle attachment

We record here a proof of the fact that an elementary Weinstein cobordism is "the same" as a Weinstein handle attachment (the precise statement is Lemma 6.6), as was used in the proof of Proposition 6.2. We were unable to find a precise reference for this standard fact, though it is of course implicit in Weinstein's original paper [26], as well as in Cieliebak and Eliashberg [7].

Recall that a Weinstein cobordism (W, λ, ϕ) is called *elementary* if and only if there is no trajectory of $X = X_\lambda$ between any two critical points. For a critical point $p \in W$, we denote by $T_p^\pm W$ the positive/negative eigenspaces of $d_p X\colon T_p W \to T_p W$, and we denote the stable manifold by W_p^-. For an elementary cobordism, each stable manifold W_p^- intersects the negative boundary $\partial_- W$ in an isotropic sphere $\Lambda_p \subseteq \partial_- W$; note that $\Lambda_p = (W_p^- \setminus p)/\mathbb{R}$ via the Liouville flow. A choice of exponential coordinates $\exp_p\colon T_p^- W \to W_p^-$ and a small sphere centered at zero in $T_p^- W$ determines a diffeomorphism $(W_p^- \setminus p)/\mathbb{R} = (T_p^- W \setminus 0)/\mathbb{R}$. We thus obtain a diffeomorphism $\rho_p \in \mathrm{Diff}((T_p^- W \setminus 0)/\mathbb{R}, \Lambda_p)$, which is well-defined up to contractible choice.

Lemma 6.6 *Let (Y^{2n-1},λ) be a contact manifold with contact form, let $\Lambda_1,\dots,\Lambda_m$ be disjoint Legendrian spheres in Y, and let $\sigma_j\in\operatorname{Diff}(S^{n-1},\Lambda_j)/O(n)$. The space of triples (W,i,q) consisting of*

- *an elementary Weinstein cobordism (W^{2n},λ,ϕ) with critical points p_j and stable manifolds V_{p_j},*
- *an isomorphism $i\colon(\partial_-W,\lambda)\xrightarrow{\sim}(Y,\lambda)$ sending $V_{p_j}\cap\partial_-W$ to Λ_j,*
- *a path q between σ_j and the image of ρ_{p_j} in $\operatorname{Diff}(S^{n-1},\Lambda_j)/O(n)$,*

is weakly contractible, in the sense that for all $k\geq 0$, any family of such objects (W,i,q) over ∂D^k can be extended to a family over D^k.

There is also a version of Lemma 6.6 for any critical points of any index, though it is more complicated to state since subcritical handle attachment requires an additional piece of data (a framing of the symplectic normal bundle of the attaching sphere). In this paper, we only need the case of critical handle attachment, so we omit the more general statement and its proof. We thank Yasha Eliashberg for useful discussions regarding the proof.

Proof Let a family over ∂D^k be given ($k\geq 0$).

We first equip the family with local Darboux charts near the critical points, and homotope it so that the Liouville vector field coincides with a certain standard model in these charts. We phrase this part of the argument as if there were just a single triple (W,i,q) and a single critical point p, but it is clear that each step also works in families and for multiple critical points. The details are as follows.

Fix a local symplectomorphism (Darboux chart) $\exp_p\colon(T_pW,0)\to(W,p)$ whose derivative at zero is the identity. On the symplectic vector space T_pW, the vector field $d_pX\colon T_pW\to T_pW$ is Liouville (this is just the linearization of the Liouville structure of W near p); it follows that the positive/negative eigenspaces $T_p^{\pm}W$ of d_pX are Lagrangian [7, Proposition 11.9].

We first homotope the function ϕ so that

$$\exp_p^*\phi=\phi_{\mathrm{std}}\quad\text{near zero,}\tag{6-13}$$

where $\phi_{\mathrm{std}}\colon T_pW\to\mathbb{R}$ is given by $\phi_{\mathrm{std}}(v):=|v^+|^2-|v^-|^2$. Here we fix positive definite quadratic forms on $T_p^{\pm}W$ such that the Liouville vector field $\exp_p^*X$ is gradient-like for ϕ_{std} near zero. Note that the space of such quadratic forms is clearly open and convex, and it is seen to be nonempty by considering quadratic forms which are diagonal with respect to a basis which puts d_pX into Jordan normal form. Now we consider the homotopy $\{\phi + (\phi_{\mathrm{std}} - \phi)t\chi\}_{t\in[0,1]}$ for some smooth compactly supported cutoff function $\chi\colon T_pW \to [0,1]$ which equals 1 in a neighborhood of zero. Its differential equals $(1-t\chi)d\phi + t\chi d\phi_{\mathrm{std}} + t(\phi_{\mathrm{std}}-\phi)d\chi$. We have $\phi_{\mathrm{std}}-\phi = O(|v|^2)$, so to ensure that $\exp_p^* X$ is gradient-like throughout the homotopy, it suffices to choose χ so that $|d\chi|$ is much smaller than $|v|^{-1}$. Such a cutoff function exists (supported in any given neighborhood of zero) since $\int_0^1 r^{-1}\,dr$ diverges. Thus we have achieved (6-13).

We next homotope the Liouville vector field X so that

$$\exp_p^* X = X_{\mathrm{std}} \quad \text{near zero,} \tag{6-14}$$

where $X_{\mathrm{std}}\colon T_pW \to T_pW$ acts by $-\mathrm{id}$ on $T_p^- W$ and by $2\,\mathrm{id}$ on $T_p^+ W$ (observe that this is indeed a Liouville vector field). Note that both vector fields $\exp_p^* X$ and X_{std} are gradient-like with respect to $\exp_p^* \phi = \phi_{\mathrm{std}}$ near zero. Write the Liouville form for $\exp_p^* X$ as λ, write the Liouville form for X_{std} as λ_{std}, and write $\lambda_{\mathrm{std}} - \lambda = df$ for a function f vanishing at zero. We consider the homotopy $\{\lambda + d(t\chi f)\}_{t\in[0,1]}$ for χ as above. We may write this as $(1-t\chi)\lambda + t\chi\lambda_{\mathrm{std}} + tf\,d\chi$. We have $f = O(|v|^2)$, so in order to guarantee that the resulting Liouville vector field remains gradient-like for $\exp_p^* \phi = \phi_{\mathrm{std}}$, it is again enough to choose χ so that $|d\chi|$ is much smaller than $|v|^{-1}$, which exists as before. This achieves (6-14).

We have now homotoped X and ϕ near p so that they coincide via the chosen Darboux chart $\exp_p\colon (T_pW, 0) \to (W, p)$ with X_{std} and ϕ_{std} as above near zero. Since $\exp_p^* X = X_{\mathrm{std}}$ in a neighborhood of zero, there is an induced diffeomorphism $\rho\colon (T_p^- W \setminus 0)/\mathbb{R}_{>0} \to \Lambda$.

Now $\mathrm{Diff}(S^{n-1}, \Lambda)/O(n)$ classifies vector bundles V along with a fiberwise diffeomorphism from the sphere bundle $S(V) := (V \setminus 0)/\mathbb{R}_{>0}$ to Λ. Hence the data of q determines an extension of the vector bundle $T_p^- W$ from ∂D^k to D^k (which we also denote by $T_p^- W$), an extension of the fiberwise diffeomorphism $\rho\colon (T_p^- W\setminus 0)/\mathbb{R}_{>0} \to \Lambda$ to D^k, and an extension of q itself from ∂D^k to D^k. We may also extend $T_p^+ W$ from ∂D^k to D^k by observing that $T_p^+ W = (T_p^- W)^*$ over ∂D^k (by virtue of the symplectic form) and so defining $T_p^+ W := (T_p^- W)^*$ over D^k. Hence $T_pW := T_p^- W \oplus T_p^+ W$ is a symplectic vector bundle over D^k. We conclude that it suffices to extend W from ∂D^k to D^k so that it has the chosen tangent spaces T_pW, has exponential charts $\exp_p$ satisfying (6-13) and (6-14) above, and so that it induces the chosen diffeomorphisms $\rho\colon (T_p^- W \setminus 0)/\mathbb{R}_{>0} \to \Lambda$.

Over any point in D^k, we have a co-oriented contact manifold $(T_pW \setminus T_p^+ W)/\mathbb{R}$ (quotient by the Liouville flow), and a Legendrian submanifold $(T_p^- W \setminus 0)/\mathbb{R}_{>0}$ (quotient by dilation, which coincides with the Liouville flow). Over any point in ∂D^k, the Liouville flow on W determines a germ of co-orientation-preserving contactomorphism $\widetilde{\rho}$ between a neighborhood of this Legendrian submanifold and a neighborhood of $\Lambda_p \subseteq Y$, restricting to ρ. Conversely, a neighborhood of W_p^- in W is determined by $T_pW = T_p^- W \oplus T_p^+ W$ and the germ of co-orientation-preserving contactomorphism $\widetilde{\rho}$. Note that W always deforms down to a neighborhood of $\partial_- W \cup W_p^-$. Thus it suffices to extend $\widetilde{\rho}$ from ∂D^k to D^k such that it restricts to ρ (such an extension determines for us an extension of W from ∂D^k to D^k).

To show that $\widetilde{\rho}$ extends to D^k, it suffices to show that for any closed manifold M, the restriction map from germs of co-orientation-preserving contactomorphisms of J^1M mapping the zero section to itself to diffeomorphisms of M is a weak homotopy equivalence (we will apply this to $M = S^{n-1}$). Equivalently, it suffices to show that the space of germs of co-orientation-preserving contactomorphisms of J^1M fixing the zero section pointwise is weakly contractible. Write $J^1M = T^*M \times \mathbb{R}$ with contact form $\lambda - ds$, and write h_t for the flow of the contact vector field $X_\lambda + s\frac{\partial}{\partial s}$. Fix any germ of co-orientation-preserving contactomorphism $f\colon J^1M \to J^1M$ fixing the zero section pointwise, and we will define a canonical path from f to the identity (clearly this is enough). We first consider the limit as $t \to \infty$ of the conjugation $h_t \circ f \circ h_t^{-1}$, which is nothing other than the vertical projection of the derivative of f along the zero section. We are thus reduced to considering a co-orientation-preserving contactomorphism $f_0\colon J^1M \to J^1M$ which is a linear map of bundles over M. Now a general such linear map has the form

$$(\alpha, g) \mapsto (A\alpha + Bg, C\alpha + Dg), \tag{6-15}$$

where $A\colon M \to \mathrm{End}(T^*M)$, $B\colon M \to T^*M$, $C\colon M \to TM$ and $D\colon M \to \mathbb{R}$ are sections over M. As a contactomorphism, f_0 preserves the Legendrian sections (dg, g) of J^1M over M, which means that

$$A(dg) + gB = d(Cg) + gdD + Ddg \tag{6-16}$$

for all functions $g\colon M \to \mathbb{R}$. Since $d(Cg)$ is the only second-order term, we conclude that $C \equiv 0$. Comparing first-order terms shows that $A = D \cdot \mathrm{id}$, and finally we may solve for $B = dD$. Thus $f_0\colon J^1M \to J^1M$ is given by $(\alpha, g) \mapsto (D\cdot\alpha + g\cdot dD, D\cdot g)$ for some function $D\colon M \to \mathbb{R}$. Since f_0 is a diffeomorphism, D is nonvanishing, and since f_0 is co-orientation-preserving, $D > 0$ everywhere. Finally, we may connect f_0 to the identity using the obvious linear homotopy from D to the constant function 1. □

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