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Rational cohomology tori

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We study normal compact varieties in Fujiki's class $\mathcal{C}$ whose rational cohomology ring is isomorphic to that of a complex torus. We call them rational cohomology tori. We classify, up to dimension three, those with rational singularities. We then give constraints on the degree of the Albanese morphism and the number of simple factors of the Albanese variety for rational cohomology tori of general type (hence projective) with rational singularities. Their properties are related to the birational geometry of smooth projective varieties of general type, maximal Albanese dimension, and with vanishing holomorphic Euler characteristic. We finish with the construction of series of examples.

In an appendix, we show that there are no smooth rational cohomology tori of general type. The key technical ingredient is a result of Popa and Schnell on 1–forms on smooth varieties of general type.

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Introduction

Given a compact complex manifold, one fundamental problem is to determine how much information is encoded in its underlying topological space.

Hirzebruch and Kodaira [24] proved that for n *odd*, any compact Kähler manifold which is homeomorphic to $\mathbb{P}^n$ is actually isomorphic to $\mathbb{P}^n$; see also Morrow [32, Theorem 1]. A stronger property is actually conjectured: it should be sufficient to assume that the rings $H^\bullet(X,\mathbb{Z})$ and $H^\bullet(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$ are isomorphic and $c_1(T_X) > 0$ to deduce that X is isomorphic to $\mathbb{P}^n$ (this is known in dimensions at most 6; see Fujita [20, Theorem 1], Libgober and Wood [31, Theorem 1] and Debarre [14]).

Catanese [8, Theorem 70] (see also Theorem 1.1) observed that complex tori X satisfy this stronger property: they can be characterized among compact Kähler manifolds by the fact that there is an isomorphism

$$\bigwedge\nolimits^\bullet H^1(X,\mathbb{Z}) \xrightarrow{\sim} H^\bullet(X,\mathbb{Z}) \tag{1}$$

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of graded rings. The Kähler assumption is essential since one can construct non-Kähler compact complex manifolds (hence not biholomorphic to complex tori) that satisfy (1) (see Example 1.6). If we replace (1) with an isomorphism

$$\textstyle\bigwedge^{\bullet} H^1(X,\mathbb{Q}) \xrightarrow{\sim} H^{\bullet}(X,\mathbb{Q}) \tag{2}$$

of graded $\mathbb{Q}$–algebras, Catanese [8, Conjecture 71] asked whether this property still characterizes complex tori.

We say that a normal compact variety X in Fujiki's class $\mathcal{C}$ is a *rational cohomology torus* if it satisfies (2). The main objective of this article is to study the geometry of these varieties: give restrictive properties and construct examples (some smooth) which are not complex tori, thereby answering Catanese's question negatively.

The Albanese morphism of a smooth rational cohomology torus X is finite (see Catanese [8, Remark 72]). A result of Kawamata (Remark 1.8) then says that there is a morphism $I_X\colon X \to X_1$ which is an Iitaka fibration for X (see Section 1.1 for the definition) such that X_1 is algebraic and has again a finite morphism to a torus. We prove that X_1 is also a rational cohomology torus, but possibly singular. This leads to the following result, proved in Section 1.

Theorem A *Let X be a normal compact class-$\mathcal{C}$ variety with a finite morphism to a torus. Consider the sequence of Iitaka fibrations*

$$X \xrightarrow{I_X} X_1 \xrightarrow{I_{X_1}} X_2 \to \cdots \to X_{k-1} \xrightarrow{I_{X_{k-1}}} X_k,$$

where $X_1,\dots,X_k$ are normal projective varieties. Then X is a rational cohomology torus if, and only if, X_k is a rational cohomology torus. Moreover,

- *either X_k is a point and we say that X is an* Iitaka torus tower*;*
- *or X_k is of general type (of positive dimension).*

It is easy to construct smooth projective surfaces which are Iitaka torus towers but not complex tori (Example 1.11). Since the product of two rational cohomology tori is again a rational cohomology torus, this already gives a negative answer to Catanese's question in any dimension at least 2.

The next question is whether all rational cohomology tori are Iitaka torus towers. By Theorem A, this is the same as asking whether there exist (possibly singular) projective rational cohomology tori of general type. This reduces our problem to the algebraic category; however, the price we have to pay is that we need to deal with singular varieties.

Rational singularities turn out to be the suitable kind of singularities to work with, because they are stable under the construction of the sequence of Iitaka fibrations of Theorem A. Moreover, any desingularization of a projective rational cohomology torus of general type with rational singularities has maximal Albanese dimension and vanishing holomorphic Euler characteristic (Proposition 1.17). These varieties were studied by Chen and Jiang [12] and Chen, Debarre and Jiang [9]. Building on these results, we give a classification, in dimensions up to three, of rational cohomology tori with rational singularities.

Theorem B *Let X be a compact class-$\mathcal{C}$ variety with rational singularities.*

(1) *If X is a surface, X is a rational cohomology torus if, and only if, X is an Iitaka torus tower.*
(2) *If X is a threefold, X is a rational cohomology torus if, and only if,*
   - *either X is an Iitaka torus tower;*
   - *or X has an étale cover which is a Chen–Hacon threefold (X is then singular, of general type).*
(3) *Starting from dimension 4, there exist smooth rational cohomology tori that are not Iitaka torus towers.*

This theorem is proved in Section 2. Chen–Hacon threefolds were constructed in [11, Section 4, Example] (see also Example 2.1 and Proposition 2.2). The n–folds we construct for (3) have Kodaira dimension any number in $\{3, \dots, n-1\}$. In Corollary A.2 of the appendix, William Sawin shows that *smooth* rational cohomology tori of general type do not exist (but we construct in Example 4.4 *singular* rational cohomology tori of general type in any dimension at least 3).

After this classification result, we focus on giving restrictions on rational cohomology tori of general type.

Theorem C *Let X be a projective variety of general type with rational singularities. Assume that X is a rational cohomology torus, with Albanese morphism a_X. There exists a prime number p such that $p^2 \mid \deg(a_X)$.*

Moreover, if $\deg(a_X) = p^2$, the morphism a_X is a $(\mathbb{Z}/p\mathbb{Z})^2$–cover of its image (Corollary 3.8).

We deduce Theorem C and Corollary 3.8 from the analogous restrictions on the degree of the Albanese morphism of a smooth projective variety X of general type, of maximal Albanese dimension and with $\chi(X, \omega_X) = 0$ (Theorems 3.6 and 3.7). Theorem 3.6 is probably the deepest result of this article and a key step in its proof is the description of *minimal primitive varieties of general type with* $\chi = 0$ (Theorem 3.4), *primitive* in the sense that there exist no proper subvarieties with $\chi = 0$ through a general point, and *minimal* with respect to the degree of birational factorizations of the Albanese morphism (Definition 3.2). The most difficult part of the proof of Theorem 3.6 is to realize these varieties as Galois quotients of products of lower-dimensional varieties (Lemma 3.5).

Continuing with the idea of giving constraints to the existence of rational cohomology tori of general type, we prove the following condition on the number of simple factors of their Albanese varieties.

Theorem D *Let X be a projective variety of general type with rational singularities. Assume that X is a rational cohomology torus, with Albanese morphism $a_X\colon X \to A_X$, and let p be the smallest prime divisor of $\deg(a_X)$. Then A_X has at least $p+1$ simple factors.*

Section 4 is devoted to the construction of examples. First, we construct in Example 4.1, for each prime p, minimal primitive varieties X of general type of dimension $p+1$ with $\chi(X, \omega_X) = 0$ whose Albanese morphisms are surjective $(\mathbb{Z}/p\mathbb{Z})^2$–covers of a product of $p+1$ elliptic curves; they are finite Galois quotients of a product of $p+1$ curves. These examples show that the structure of primitive varieties with $\chi = 0$ is much more complicated than expected by Chen and Jiang [12] (see Remark 4.2).

We then use techniques of Pardini [33] to produce (singular) rational cohomology tori of general type in any dimension at least 3 (Examples 4.3 and 4.4). The first of these examples shows that the lower bound on the degree of the Albanese morphism in Theorem C is optimal, and so is the lower bound on the number of factors of the Albanese variety in Theorem D.

The article ends with a series of examples of nonminimal primitive fourfolds with $\chi = 0$ whose Albanese variety has 4 simple factors and whose Albanese morphism has degree 8.

Notation We work over the complex numbers. A (complex) variety is reduced and integral (and possibly singular; manifolds are smooth). A variety is in the (Fujiki) class $\mathcal{C}$ if it is compact and bimeromorphic to a compact Kähler manifold (Fujiki [17, Definition 1.2] and Ancona and Gaveau [2, Part I, Section 7.5]).

A projective variety is of general type if a (hence any) desingularization is of general type, ie has maximal Kodaira dimension.

Given a compact Kähler manifold X, we denote by A_X its Albanese torus and by $a_X\colon X \to A_X$ its Albanese morphism. Given a complex torus A, we denote by $\widehat{A}$ its dual.

A proper morphism $X \to Y$ between normal varieties is a *fibration* if it is surjective with connected fibers; it is *birationally isotrivial* if the general fibers are all birationally isomorphic to a fixed variety F. When X and Y are algebraic, this is equivalent to saying that after a finite (Galois) base change $Y' \to Y$, the product $X \times_Y Y'$ is birationally isomorphic, over Y', to $F \times Y'$ (Bogomolov, Böhning and von Bothmer [4]).

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1 Catanese's theorem and question

Catanese [8] proved the following topological characterization of complex tori.

Theorem 1.1 [8, Theorem 70] *A compact Kähler manifold X is biholomorphic to a complex torus if, and only if, there is an isomorphism*

$$\textstyle\bigwedge^\bullet H^1(X,\mathbb{Z}) \xrightarrow{\sim} H^\bullet(X,\mathbb{Z}) \tag{3}$$

of graded rings.

Let us recall the proof. The Albanese map $a_X\colon X \to A_X$ induces an isomorphism $a_X^{*1}\colon H^1(A_X,\mathbb{Z}) \xrightarrow{\sim} H^1(X,\mathbb{Z})$ and (3) then implies that $a_X^*\colon H^\bullet(A_X,\mathbb{Z}) \to H^\bullet(X,\mathbb{Z})$ is also an isomorphism. Set $n := \dim(X)$; that $a_X^{*n}\colon H^{2n}(X,\mathbb{Z}) \xrightarrow{\sim} H^{2n}(A_X,\mathbb{Z})$ is an isomorphism implies that a_X is birational. Moreover, since we have an isomorphism of the whole cohomology rings and X is Kähler, a_X cannot contract any subvariety of X and is therefore finite. Thus, a_X is an isomorphism.

If we replace (3) with an isomorphism at the level of rational cohomology, Catanese already observed that the Albanese morphism is still surjective and finite [8, Remark 72].

Definition 1.2 Let X be a normal compact class-$\mathcal{C}$ variety. We say that X is a *rational cohomology torus* if there is an isomorphism

$$\textstyle\bigwedge^\bullet H^1(X,\mathbb{Q}) \xrightarrow{\sim} H^\bullet(X,\mathbb{Q}) \tag{$*_X$}$$

of graded $\mathbb{Q}$–algebras.

We also give an a priori slightly different definition.

Definition 1.3 Let X be a normal compact class-$\mathcal{C}$ variety and let $f\colon X \to A$ be a morphism to a torus. We say that X is an *f–rational cohomology torus* if f induces an isomorphism

$$(*_f) \qquad f^*\colon H^\bullet(A,\mathbb{Q}) \xrightarrow{\sim} H^\bullet(X,\mathbb{Q})$$

of graded $\mathbb{Q}$–algebras.

Condition $(*_f)$ certainly implies condition $(*_X)$. But because of singularities, the converse is a priori not clear. However, we show that Definitions 1.2 and 1.3 are in fact equivalent.

Proposition 1.4 *Let X be a normal compact class-$\mathcal{C}$ variety. Assume that X is a rational cohomology torus. There exists a finite morphism $f\colon X \to A$ onto a complex torus such that X is an f–rational cohomology torus. In particular, the Hodge structures on $H^\bullet(X)$ are pure.*

Proof Set $n := \dim(X)$. There are functorial mixed $\mathbb{Q}$–Hodge structures on $H^\bullet(X)$ for which the cup product $\bigwedge^{2n} H^1(X) \to H^{2n}(X)$ is a morphism of mixed Hodge structures [18, Proposition (1.4.1); 2, Part II, Theorem 3.4]. Since X is compact, $H^{2n}(X)$ is a 1–dimensional pure Hodge structure, hence $h^1(X)=2n$, $W_0 H^1(X)=0$ and $H^1(X)$ carries a pure Hodge structure. Therefore, $H^k(X) \simeq \bigwedge^k H^1(X)$ has a pure Hodge structure for each $k \in \{0,\dots,2n\}$.

Let $\mu\colon X' \to X$ be a resolution of singularities with X' Kähler. Since $H^1(X)$ carries a pure Hodge structure, the pullback map $\mu^*\colon H^1(X) \to H^1(X')$ is injective. Considering the Albanese morphism $a_{X'}\colon X' \to A_{X'}$, we note that $a_{X'}^*\colon H^1(A_{X'},\mathbb{Q}) \to H^1(X',\mathbb{Q})$ is an isomorphism. Thus $H^1(A_{X'})$ has a sub-Hodge structure which is isomorphic to $H^1(X)$. Therefore, there exists a quotient $\pi\colon A_{X'} \to A$ of complex tori such that $g^*H^1(A) = \mu^*H^1(X)$ as sub-Hodge structures of $H^1(X')$, where $g := \pi a_{X'}\colon X' \to A$. We then have $\mu^*H^k(X,\mathbb{Q}) = g^*H^k(A,\mathbb{Q})$ as subspaces of $H^k(X',\mathbb{Q})$.

We claim that g contracts every fiber of μ. Otherwise, let F be an irreducible closed subvariety contained in a fiber of μ and assume $\dim(g(F)) > 0$. Let $F' \to F$ be a desingularization with F' Kähler and let $t\colon F' \to F \hookrightarrow X'$ be the composition. Let $\omega \in g^*H^2(A,\mathbb{C}) \subset H^2(X',\mathbb{C})$ be the pullback of a Kähler form on A. Since $\dim(gt(F')) > 0$, the form $t^*\omega \in H^2(F',\mathbb{C})$ is nonzero. This is a contradiction, since ω is in $\mu^*H^2(X,\mathbb{C})$ and $\mu t(F')$ is a point.

Therefore, g contracts every fiber of μ. Moreover, μ is birational and X is normal, hence $\mu_*\mathcal{O}_{X'} = \mathcal{O}_X$. Thus the morphism $g\colon X' \to A$ factors through a morphism $f\colon X \to A$ and X is an f–rational cohomology torus. □

Remark 1.5 If X is a rational cohomology torus, Proposition 1.4 provides a finite morphism $f\colon X \to A$ which has a universal property (up to translations) for morphisms from X to complex tori. We will call f the *Albanese morphism* and A the *Albanese variety* of X.

The hypothesis "X Kähler" in Theorem 1.1 is essential: in dimension at least 3, the topological characterization of tori is not true without this hypothesis, as we show in the following example, whose origins can be traced to [3, page 163] (see also [39, Example 5.1]).

Example 1.6 (a non-Kähler integral cohomology torus) Let E be an elliptic curve, let L be a very ample line bundle on E and let φ and ψ be holomorphic sections of L with no common zeroes on E. Set

$$J_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_3 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad J_4 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Since $\det\bigl(\sum_{i=1}^4 \lambda_i J_i\bigr) = \sum_{i=1}^4 \lambda_i^2$, the group

$$\Gamma := \sum_{i=1}^{4} \mathbb{Z} J_i \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

is a relative lattice in the total space of the rank-2 vector bundle $\mathcal{V} := L \oplus L$ over E. The quotient $M := \mathcal{V}/\Gamma$ is a complex manifold with a surjective holomorphic map $\pi\colon M \to E$. By construction, π is smooth, each fiber of π is a complex 2–dimensional torus and its relative canonical bundle $\omega_{M/E}$ is $\pi^* L^{-2}$.

One checks that M is diffeomorphic to a real torus, hence $H^\bullet(M,\mathbb{Z}) = \bigwedge^\bullet H^1(M,\mathbb{Z})$, but M is not a complex torus, since it is not Kähler: if it were, $\pi_*\omega_{M/E}$ would be semipositive [19, Theorem (2.7)].

Answering a question of Ottem, we note that the hypothesis that $\bigwedge^\bullet H^1(X,\mathbb{Q}) \simeq H^\bullet(X,\mathbb{Q})$ is a ring isomorphism is crucial to get a finite morphism to an abelian variety: if we only assume that the Hodge numbers of X are those of a torus, the Albanese morphism is not even necessarily finite as we show in the following example, though strong constraints on these morphisms were found in [13].

Example 1.7 (a surface with the Hodge numbers of a torus but nonfinite Albanese map) Let $\rho\colon D \to C$ be a double étale cover of smooth projective curves, where C has genus 2, and let τ be the associated involution of D. Let E be an elliptic curve and let σ be the involution of E given by multiplication by -1, with quotient morphism $\rho'\colon E \to \mathbb{P}^1$. Let $S := (D \times E)/\langle \tau \times \sigma \rangle$ be the diagonal quotient. The surface S is smooth and its Hodge numbers are

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \end{array}$$

Indeed, we have

$$\begin{aligned} \rho_*\mathcal{O}_D &= \mathcal{O}_C \oplus L^{-1}, \qquad \text{where } L \in \operatorname{Pic}^0(C) \text{ is a 2–torsion point,} \\ \rho'_*\mathcal{O}_E &= \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2). \end{aligned}$$

If we denote by $g\colon S \to C \times \mathbb{P}^1$ the natural double cover, we have

$$\begin{aligned} g_*\mathcal{O}_S &= \mathcal{O}_{C\times\mathbb{P}^1} \oplus (L^{-1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)), \\ g_*\Omega^1_S &= (\omega_C \boxtimes \mathcal{O}_{\mathbb{P}^1}) \oplus ((\omega_C \otimes L) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)) \oplus (\mathcal{O}_C \boxtimes \omega_{\mathbb{P}^1}) \oplus (L^{-1} \boxtimes \mathcal{O}_{\mathbb{P}^1}), \\ g_*\omega_S &= \omega_{C\times\mathbb{P}^1} \oplus ((\omega_C \otimes L) \boxtimes \mathcal{O}_{\mathbb{P}^1}). \end{aligned}$$

Note that the Albanese variety of S and the Jacobian $J(C)$ are isogenous and that the Albanese morphism of S contracts the elliptic curves E that are the fibers of $S \to C$.

1.1 Iitaka torus towers

By Proposition 1.4, studying rational cohomology tori is equivalent to studying f–rational cohomology tori. For an f–rational cohomology torus X, the property $(*_f)$ implies, since X is Kähler, that f is finite and surjective. A theorem of Kawamata describes Iitaka fibrations for varieties with a finite morphism to a torus.

Recall from [38, Theorem 5.10] that given a normal compact complex variety X of nonnegative Kodaira dimension $\kappa(X)$, there exists a proper modification $X^* \to X$ (with X^* smooth) and a fibration $I_X\colon X^* \to Y^*$ such that $\dim(Y^*) = \kappa(X)$ and the Kodaira dimension of a general fiber of I_X is 0. The fibration I_X is bimeromorphically equivalent to the rational map on X defined by the sections of $\omega_X^{\otimes m}$, for m sufficiently large and divisible. It is in particular unique up to bimeromorphic equivalence. Any fibration $X' \to Y'$ bimeromorphically equivalent to I_X, with X' normal but not necessarily smooth, will be called an *Iitaka fibration* of X.

Remark 1.8 (reduction to algebraic varieties) Let X be a normal compact complex variety and let $f\colon X \to A$ be a finite morphism to a torus. By [27, Theorem 23], there are

- an abelian Galois étale cover $\pi\colon \widetilde{X} \to X$ with group G, induced by an étale cover of A,
- a subtorus K of A,
- a normal projective variety $\widehat{Y}$ of general type, and

- a commutative diagram

$$
\begin{tikzcd}
\widetilde{X} \arrow[r, "\pi"] \arrow[d, "I_{\widetilde{X}}"'] & X \arrow[r, "f", "\text{finite}"'] \arrow[d, "I_X"'] & A \arrow[d] \\
\widehat{Y} \arrow[r, "\text{finite}"'] & Y \arrow[r, "f_Y", "\text{finite}"'] & A/K
\end{tikzcd}
\tag{4}
$$

with the following properties:

- I_X is the Stein factorization of the composition $X \to A \to A/K$. It is an Iitaka fibration of X, with general fiber an étale cover $\widetilde{K}$ of K.
- $I_{\widetilde{X}}$ is the Stein factorization of $I_X\pi\colon \widetilde{X} \to Y$. It is an Iitaka fibration of $\widetilde{X}$ and is an analytic fiber bundle with fiber $\widetilde{K}$. Hence, there is a natural G–action on $\widehat{Y}$ which may not be faithful, $I_{\widetilde{X}}$ is G–equivariant and $Y = \widehat{Y}/G$.

For any finite group G acting on a irreducible projective variety V, we have

$$H^\bullet(V,\mathbb{Q})^G = H^\bullet(V/G,\mathbb{Q});$$

see [6, Chapter III, Theorem 7.2]. Thus, we have isomorphisms

$$
\begin{aligned}
H^\bullet(X,\mathbb{Q}) &\simeq H^\bullet(\widetilde{X},\mathbb{Q})^G \\
&\simeq (H^\bullet(\widehat{Y},\mathbb{Q}) \otimes H^\bullet(\widetilde{K},\mathbb{Q}))^G \\
&\simeq H^\bullet(\widehat{Y},\mathbb{Q})^G \otimes H^\bullet(K,\mathbb{Q}) \\
&\simeq H^\bullet(Y,\mathbb{Q}) \otimes H^\bullet(K,\mathbb{Q})
\end{aligned}
$$

of graded $\mathbb{Q}$–algebras, where the second isomorphism holds by the Leray–Hirsch theorem applied to the fiber bundle $I_{\widetilde{X}}$ [5, Theorem 15.11] and the third isomorphism holds because G acts trivially on $H^\bullet(\widetilde{K},\mathbb{Q})$, which is isomorphic to $H^\bullet(K,\mathbb{Q})$. Thus, $(*_f)$ holds if, and only if, $(*_{f_Y})$ holds. In particular, this allows us to reduce the study of property $(*_f)$ to algebraic varieties.

The following lemma, which implies Theorem A in the introduction, is an easy consequence of the previous remark.

Lemma 1.9 *Let $f\colon X \to A$ be a finite morphism from a normal compact complex variety to a torus. Let*

$$X \xrightarrow{I_X} X_1 \xrightarrow{I_{X_1}} X_2 \to \cdots \to X_{k-1} \xrightarrow{I_{X_{k-1}}} X_k \tag{5}$$

be the tower of Iitaka fibrations as in diagram (4)*, where the general fibers of I_{X_i} are complex tori, the X_i are normal projective varieties with morphisms $f_i\colon X_i \to A_i$*

to quotient tori of A and X_k is of general type or a point. Then X is an f–rational cohomology torus if, and only if, X_k is an f_k–rational cohomology torus. In particular, if X_k is a point, X is an f–rational cohomology torus.

Definition 1.10 We say that X is an *Iitaka torus tower* if, in (5), X_k is a point.

Example 1.11 (an Iitaka torus tower which is not a torus) Let $\rho\colon C \to E$ be a double cover of smooth projective curves, where C has genus $g \geq 2$ and E is an elliptic curve. Let τ be the corresponding involution on C. Let $E' \to E$ be a degree-2 étale cover of elliptic curves and let σ be the corresponding involution on E'. Let X be the smooth surface $(C \times E')/\langle \tau \times \sigma \rangle$. Then X is an Iitaka torus tower but has Kodaira dimension 1, hence is not a torus.

The answer to Catanese's original conjecture [8, Conjecture 70] is therefore negative. Nevertheless, we may still ask the following question.

Question 1.12 Is a compact Kähler manifold which is a rational cohomology torus always an Iitaka torus tower?

To answer (negatively) this question, we study the variety X_k of Lemma 1.9, which is a possibly singular projective variety.

1.2 Rational singularities

Recall the following classical definition.

Definition 1.13 Let X be a compact complex variety and let $\mu\colon X' \to X$ be a desingularization. We say that X has *rational singularities* if $R^i\mu_*\mathcal{O}_{X'} = 0$ for all $i > 0$ and $\mu_*\mathcal{O}_{X'} = \mathcal{O}_X$ (equivalently, X is normal).

The following lemma explains why we work with rational singularities.

Lemma 1.14 *Let $f\colon X \twoheadrightarrow A$ be a finite and surjective morphism from a projective variety X with rational singularities to an abelian variety. Consider a quotient $A \twoheadrightarrow B$ of abelian varieties. If the composition $X \twoheadrightarrow A \twoheadrightarrow B$ factors through a finite morphism $Y \to B$ with Y normal, Y has rational singularities.*

In particular, the lemma applies when $X \to Y \to B$ is the Stein factorization of $X \to B$.

Proof Since $f\colon X \to A$ is finite, so is the induced morphism g in the commutative diagram

$$
\begin{tikzcd}
X \arrow[dr, "g"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& Y \times_B A \arrow[r] \arrow[d] & A \arrow[d, two heads] \\
& Y \arrow[r] & B
\end{tikzcd}
$$

Since $Y \to B$ is finite and surjective, so is $Y \times_B A \to A$, hence the image of g is a component $\widetilde{Y}$ of $Y \times_B A$. Since the induced morphism $X \to \widetilde{Y}$ is finite, the trace operator induces a splitting of the natural morphism $\mathcal{O}_X \to Rg_*\mathcal{O}_{\widetilde{Y}}$. Thus, by [29, Theorem 1], since X has rational singularities, so has $\widetilde{Y}$. Finally, since $A \to B$ is smooth, so is $\widetilde{Y} \hookrightarrow Y \times_B A \to Y$. It follows that Y has rational singularities. $\square$

It follows from the lemma that if X has rational singularities, so do all the X_i in the tower (5). Thus, in order to answer Question 1.12, it suffices to answer the following.

Question 1.15 Can a projective variety with rational singularities which is a rational cohomology torus be of general type?

To answer (positively) this question, we first prove that any desingularization must satisfy very strong numerical properties.

Lemma 1.16 *Let X be a projective variety with rational singularities. For each k, we have an isomorphism*

$$H^k(X, \mathcal{O}_X) \simeq \mathrm{Gr}^0_F H^k(X),$$

where $F^\bullet$ is the Hodge filtration for Deligne's mixed Hodge structure on $H^k(X)$.

Proof By [28, Theorem S], rational singularities are Du Bois. If $\underline{\Omega}^\bullet_X$ is the Deligne–Du Bois complex of X [35, Definition 7.34], this means that $\underline{\Omega}^0_X$ is quasi-isomorphic to $\mathcal{O}_X$. By Deligne's theorem [15, Sections 8.1, 8.2 and 9.3; 30, (4.2.4)], the spectral sequence $E_1^{p,q} = \mathbb{H}^q(X, \underline{\Omega}^p_X) \Rightarrow H^{p+q}(X, \mathbb{C})$ degenerates at E_1 and abuts to the Hodge filtration of Deligne's mixed Hodge structure. Thus, we have $H^k(X, \mathcal{O}_X) \simeq H^k(X, \underline{\Omega}^0_X) = \mathrm{Gr}^0_F H^k(X)$. $\square$

Proposition 1.17 *Let X be a projective variety with rational singularities. Assume that X is a rational cohomology torus and let $\mu\colon X' \to X$ be any desingularization.*

(1) *We have $h^k(X', \omega_{X'}) = h^k(X, \omega_X) = \binom{n}{k}$; in particular, $\chi(X', \omega_{X'}) = 0$.*

(2) *The Albanese morphism $a_{X'}\colon X' \to A_{X'}$ factors through μ and the induced morphism $X \to A_{X'}$ is the Albanese morphism of X in the sense of Remark 1.5.*

Proof By Proposition 1.4, there is a finite morphism $f\colon X \to A$ to an abelian variety such that X is an f–rational cohomology torus. Hence we have isomorphisms of Hodge structures $f^*\colon H^k(A) \xrightarrow{\sim} H^k(X)$ for all k. In particular,

$$\mathrm{Gr}_F^0 H^k(X) \simeq \mathrm{Gr}_F^0 H^k(A) = H^k(A, \mathcal{O}_A).$$

Since X has rational singularities, we have $H^k(X, \mathcal{O}_X) \simeq \mathrm{Gr}_F^0 H^k(X)$ (Lemma 1.16). Hence $h^k(X, \mathcal{O}_X) = \binom{n}{k}$.

Let $\mu\colon X' \to X$ be a desingularization. Since $R\mu_*\omega_{X'} = \omega_X$, we have

$$h^k(X, \omega_X) = h^k(X', \omega_{X'}) = h^{n-k}(X', \mathcal{O}_{X'}) = h^{n-k}(X, \mathcal{O}_X) = \binom{n}{k}.$$

For (2), note that $h^1(X', \mathcal{O}_{X'}) = n$, hence $\dim(A_{X'}) = \dim(X') = \dim(X) = n$. By Proposition 1.4, there is a quotient morphism $A_{X'} \to A$ with connected fibers, hence $A_X = A_{X'}$ and $a_{X'}$ factors through μ. □

This simple but important proposition allows us to use the many known properties of smooth projective varieties X' of maximal Albanese dimension with $\chi(X', \omega_{X'}) = 0$ and $p_g(X') = 1$.

2 Rational cohomology tori in lower dimensions

Thanks to the work of Chen, Debarre and Jiang [9] on smooth varieties of maximal Albanese dimension with $p_g = 1$, we can give a classification of rational cohomology tori up to dimension 3. We first recall some important examples.

Example 2.1 (Ein–Lazarsfeld & Chen–Hacon threefolds) For each $j \in \{1, 2, 3\}$, consider an elliptic curve E_j and a bielliptic curve $C_j \xrightarrow{2:1} E_j$ of genus $g_j \geq 2$, with corresponding involution τ_j of C_j. Set $A := E_1 \times E_2 \times E_3$ and consider the quotient $g\colon C_1 \times C_2 \times C_3 \twoheadrightarrow Z$ by the involution $\tau_1 \times \tau_2 \times \tau_3$ and the tower of Galois covers

$$C_1 \times C_2 \times C_3 \overset{g}{\twoheadrightarrow} Z \overset{f}{\twoheadrightarrow} A$$

of respective degrees 2 and 4. The threefold Z is of general type with rational singularities and it has $2^3 \prod_{j=1}^3 (g_j - 1)$ isolated singular points. We call Z an *Ein–Lazarsfeld threefold* [16, Example 1.13]. If $X \twoheadrightarrow Z$ is any desingularization, we have $\chi(X, \omega_X) = \chi(Z, \omega_Z) = 0$.

A variant of the previous construction gives us varieties with $p_g = 1$, as follows. Keeping the same notation, choose points $\xi_j \in \widehat{E}_j$ of order 2 and consider the induced double étale covers $E'_j \twoheadrightarrow E_j$ and $C'_j \twoheadrightarrow C_j$, with associated involution σ_j of C'_j.

The involution τ_j on C_j pulls back to an involution τ'_j on C'_j (with quotient E'_j). Let $g'\colon C'_1\times C'_2\times C'_3 \twoheadrightarrow Z'$ be the quotient by the group (isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$) of automorphisms generated by $\mathrm{id}_1\times\tau'_2\times\sigma_3$, $\sigma_1\times\mathrm{id}_2\times\tau'_3$, $\tau'_1\times\sigma_2\times\mathrm{id}_3$ and $\tau'_1\times\tau'_2\times\tau'_3$ and consider the tower

$$C'_1\times C'_2\times C'_3 \overset{g'}{\twoheadrightarrow} Z' \overset{f'}{\twoheadrightarrow} A$$

of Galois covers of respective degrees 2^4 and 4. The threefold Z' is of general type and has rational singularities. We call Z' a *Chen–Hacon threefold* [11, Section 4, Example]. For any desingularization $X'\twoheadrightarrow Z'$, one has $p_g(X')=1$.

The étale cover $E'_1\times E'_2\times E'_3\twoheadrightarrow E_1\times E_2\times E_3$ pulls back to an étale cover $Z''\to Z'$, where Z'' is an Ein–Lazarsfeld threefold; in particular, Z' also has isolated singularities. Moreover, the quotient of $C'_1\times C'_2\times C'_3$ by the group of automorphisms generated by $\mathrm{id}_1\times\tau'_2\times\sigma_3$, $\sigma_1\times\mathrm{id}_2\times\tau'_3$ and $\tau'_1\times\sigma_2\times\mathrm{id}_3$ is a smooth double cover of Z', since the group acts freely.

This terminology differs from that of [9]: there, Ein–Lazarsfeld and Chen–Hacon threefolds refer to any of their desingularizations. Our singular threefolds can be obtained from their smooth versions by considering the Stein factorizations of their Albanese morphisms.

We can now prove the classification of rational cohomology tori up to dimension 3 stated in the introduction.

Proof of Theorem B Let X be a compact class-$\mathcal{C}$ variety with rational singularities which is a rational cohomology torus. By Lemma 1.9, we may assume that X is projective. Let $\mu\colon X'\to X$ be a desingularization. By Proposition 1.17, we have $\chi(X',\omega_{X'})=0$ and $p_g(X')=1$.

If $\dim(X)=2$ and X is of general type, we have $\chi(X',\omega_{X'})>0$ by Riemann–Roch, which is a contradiction. If $\kappa(X)=0$, then X is an abelian variety by [27, Corollary 2]. If $\kappa(X)=1$, in the diagram (4) of Remark 1.8, Y is an elliptic curve. Hence X is an Iitaka torus tower. This proves (1).

Assume $\dim(X)=3$. If X' is of general type, we can apply the structure theorem [9, Theorem 6.3]: there exists an abelian étale cover $\widetilde{A}\twoheadrightarrow A_{X'}$ such that, in the Stein factorization $X'\times_{A_{X'}}\widetilde{A}\to\widetilde{X}\to\widetilde{A}$, the variety $\widetilde{X}$ is a Chen–Hacon threefold (Example 2.1). As noted in Proposition 1.17(2), X appears in the Stein factorization of the Albanese morphism of X', hence $\widetilde{X}$ is an étale cover of X and X is singular. We then apply Lemma 1.9 and part (1) to get the first part of (2).

For the second part of (2), it suffices to show that a Chen–Hacon threefold is a rational cohomology torus. This follows from the more general Proposition 2.2 below.

For (3), we note that there exists a smooth projective threefold $\widehat{Y}$ with an involution τ such that $Y := \widehat{Y}/\langle\tau\rangle$ is a Chen–Hacon threefold (Example 2.1), hence a rational cohomology torus. Let σ be a translation of order 2 on any nonzero abelian variety K. The involution $\tau\times\sigma$ acts freely on $\widehat{Y}\times K$ and $X := (\widehat{Y}\times K)/\langle\tau\times\sigma\rangle$ is a smooth projective variety. Moreover, the natural morphism $X\to Y$ is the Iitaka fibration in (4). Thus X is also a rational cohomology torus. Since Y is of general type, X is not an Iitaka torus tower.

Moreover, by Example 4.4, there exists a rational cohomology torus Y with rational singularities and of general type in any dimension at least 3 with a smooth double cover $\widehat{Y}$. By the same construction, whenever $3\le m\le n-1$, there exists a smooth rational cohomology torus X of dimension n with Kodaira dimension m. □

The following proposition, which is further generalized to all abelian covers of abelian varieties in [26], shows in particular that Chen–Hacon threefolds are rational cohomology tori.

Proposition 2.2 *For each $j\in\{1,\dots,n\}$, let $\rho_j\colon C_j\to E_j$ be an abelian Galois cover with group G_j, where C_j is a smooth projective curve and E_j an elliptic curve. Take a subgroup G of $G_1\times\cdots\times G_n$ and set $X := (C_1\times\cdots\times C_n)/G$. Assume $h^0(X,\omega_X)=1$; then X is a rational cohomology torus with rational singularities.*

Proof Set $V := C_1\times\cdots\times C_n$ and $A := E_1\times\cdots\times E_n$, and let

$$\rho\colon V \xrightarrow{f} X \xrightarrow{g} A$$

be the quotient morphisms. The variety X has finite quotient singularities, which are rational singularities. In particular, $H^k(X)$ has a pure Hodge structure for all $k\in\{0,\dots,n\}$ [35, Theorem 2.43]. More precisely, if $\iota\colon X_{\mathrm{reg}}\hookrightarrow X$ is the smooth locus of X and we set $\Omega_X^{[p]} := \iota_*(\Omega^p_{X_{\mathrm{reg}}})$, we have $\Omega_X^{[p]} = (f_*\Omega_V^p)^G$ and $\Omega_X^{[\bullet]}$ is a resolution of the constant sheaf $\mathbb{C}_X$ [35, Lemma 2.46]. Thus, $H^q(X,\Omega_X^{[p]}) = \mathrm{Gr}_F^q\, H^{p+q}(X,\mathbb{C})$ [35, proof of Theorem 2.43].

We may assume that each projection $G\to G_j$ is surjective. Indeed, if we denote by H_j the image of this projection, there are natural morphisms $X\to C_j/H_j\to E_j$. Since X has maximal Albanese dimension, the condition $h^0(X,\omega_X)=1$ implies $h^0(C_j/H_j,\omega_{C_j/H_j})=1$ [25, Lemma 2.3], so C_j/H_j is also an elliptic curve. Then we simply replace G_j with H_j and E_j with C_j/H_j.

Let $j\in\{1,\dots,n\}$. Since ρ_j is an abelian Galois cover, we may write

$$(6)\qquad \rho_{j*}\omega_{C_j} = \mathcal{O}_{E_j}\oplus\bigoplus_{1\ne\chi_j\in G_j^\vee} L_{\chi_j},\quad \rho_{j*}\mathcal{O}_{C_j} = \mathcal{O}_{E_j}\oplus\bigoplus_{1\ne\chi_j\in G_j^\vee} L_{\chi_j}^{-1},$$

where $G_j^\vee$ is the character group of G_j and L_{χ_j} is the line bundle on E_j associated with the character $\chi_j \in G_j^\vee$. Since $G \to G_j$ is surjective, its dual $G_j^\vee \to G^\vee$ is injective.

Since $\rho_{j*}\omega_{C_j}$ is a nef vector bundle on E_j [19, Theorem (3.1)], each line bundle L_{χ_j} is nef and, for $\chi_j \neq 1$, it is either ample or nontrivial torsion in $\widehat{E}_j$. Moreover, if L_{χ_j} is a torsion line bundle, so is $L_{\chi_j^m} = L_{\chi_j}^{\otimes m}$ for each $m \in \mathbb{Z}$. Thus, L_{χ_j} is nontrivial torsion if, and only if, $L_{\chi_j^{-1}} = L_{\chi_j}^{-1}$ is nontrivial torsion.

We compute

$$\begin{aligned} g_*\omega_X &= g_*((f_*\omega_V)^G) = (\rho_*\omega_V)^G \\ &= \Bigg(\bigoplus_{\substack{\chi_j \in G_j^\vee \\ 1\le j\le n}} (L_{\chi_1} \boxtimes \cdots \boxtimes L_{\chi_n})\Bigg)^G \\ &= \bigoplus_{\substack{\chi_j \in G_j^\vee \\ \chi_1\cdots\chi_n = 1\in G^\vee}} (L_{\chi_1} \boxtimes \cdots \boxtimes L_{\chi_n}). \end{aligned}$$

Since $\mathcal{O}_A$ is a direct summand of $g_*\omega_X$ and $h^0(X,\omega_X) = 1$, we conclude that, for any $(\chi_1,\dots,\chi_n) \in G_1^\vee \times \cdots \times G_n^\vee$ not all trivial such that $\chi_1 \cdots \chi_n = 1 \in G^\vee$, at least one of the corresponding line bundles L_{χ_j} is nontrivial torsion.

For any subset $J = \{j_1,\dots,j_p\}$ of $T := \{1,\dots,n\}$, we set $V_J := C_{j_1} \times \cdots \times C_{j_p}$ and we let $p_J\colon V \to V_J$ be the projection. We also denote by $q_j\colon A \to E_j$ the projections. Then

$$\begin{aligned} g_*\Omega_X^{[p]} &= g_*((f_*\Omega_V^p)^G) = (\rho_*\Omega_V^p)^G \\ &= \Bigg(\rho_*\Bigg(\bigoplus_{\substack{J\subset T \\ |J|=p}} p_J^*\omega_{V_J}\Bigg)\Bigg)^G \\ &= \Bigg(\bigoplus_{\substack{J\subset T \\ |J|=p}} \Bigg(\bigoplus_{\substack{\chi_j \in G_j^\vee \\ \text{for all } j\in J}} \Bigg(\bigotimes_{j\in J} q_j^* L_{\chi_j}\Bigg)\Bigg) \otimes \Bigg(\bigoplus_{\substack{\chi_k \in G_k^\vee \\ \text{for all } k\in J^c}} \Bigg(\bigotimes_{k\in J^c} q_k^* L_{\chi_j}^{-1}\Bigg)\Bigg)\Bigg)^G \\ &= \bigoplus_{\substack{J\subset T \\ |J|=p}} \bigoplus_{\substack{\chi_j \in G_j^\vee,\ \chi_k \in G_k^\vee \\ \prod_{j\in J}\chi_j \prod_{k\in J^c}\chi_k^{-1} = 1\in G^\vee}} \Bigg(\bigotimes_{\substack{\chi_j \in G_j^\vee \\ j\in J}} q_j^* L_{\chi_j}\Bigg) \otimes \Bigg(\bigotimes_{\substack{\chi_k \in G_k^\vee \\ k\in J^c}} q_k^* L_{\chi_k}^{-1}\Bigg). \end{aligned}$$

For example, for $J = \{1,\dots,p\}$, the fourth equality reads

$$\rho_* p_J^*\omega_{V_J} = \rho_{1*}\omega_{C_1} \boxtimes \cdots \boxtimes \rho_{p*}\omega_{C_p} \boxtimes \rho_{p+1*}\mathcal{O}_{C_{p+1}} \boxtimes \cdots \boxtimes \rho_{n*}\mathcal{O}_{C_n}.$$

For any nontrivial solution of $\prod_{j\in J}\chi_j\prod_{k\in J^c}\chi_k^{-1}=1\in G^\vee$, we have already seen that the condition $h^0(X,\omega_X)=1$ implies that either there exists $j\in J$ such that L_{χ_j} is nontrivial torsion or there exists $k\in J^c$ such that $L_{\chi_k^{-1}}$ is nontrivial torsion, in which case $L_{\chi_k}^{-1}$ is also nontrivial torsion. Thus, by the Künneth formula, only the trivial direct summands of $g_*\Omega_X^p$ have nontrivial cohomology groups, since all the others contain a nontrivial torsion line bundle. Therefore,

$$H^q(X,\Omega_X^{[p]})=H^q(A,g_*\Omega_X^{[p]})=H^q(A,\Omega_A^p).$$

Since $H^q(X,\Omega_X^{[p]})=\mathrm{Gr}_F^q H^{p+q}(X,\mathbb{C})$, we conclude that X is a rational cohomology torus. □

3 Constraints on the Albanese morphism

In this section, we study how the condition $\chi(X,\omega_X)=0$ gives restrictions on the degree of the Albanese morphism. Note that $\chi(X,\omega_X)$ is a birational invariant. It would be interesting to study the nonbirational conditions $\chi(X,\Omega_X^p)=0$ for $0<p<\dim(X)$ to get further restrictions on the structure of rational cohomology tori.

We first recall some facts about projective varieties X of general type, of maximal Albanese dimension and with $\chi(X,\omega_X)=0$ (see [9; 12] for more details).

Let X be a smooth projective variety and let $f\colon X\to A$ be a generically finite morphism to an abelian variety. We set

$$V^i(f_*\omega_X):=\{[P]\in\widehat{A}\mid H^i(A,f_*\omega_X\otimes P)\neq 0\}.$$

By [21; 22; 37; 23], $V^i(f_*\omega_X)$ is a union of torsion translates of abelian subvarieties of $\widehat{A}$ of codimension $\geq i$. The set

(7) $S_f:=\{T\subset\widehat{A}\mid \exists\, i\geq 1\ \ T \text{ is a component of } V^i(f_*\omega_X) \text{ with } \operatorname{codim}_{\widehat{A}}(T)=i\}$

controls the positivity of the sheaf $f_*\omega_X$ [12; 25, Section 3].

We use the following notation: for any abelian subvariety $\widehat{B}\subset\widehat{A}$, we let

$$X\twoheadrightarrow X_B\xrightarrow{f_B}B \tag{8}$$

be the Stein factorization of the composition $X\xrightarrow{f}A\twoheadrightarrow B$. After birational modifications, we may assume that X_B is also smooth. Note that when f is the Albanese morphism of X and $\widehat{B}\in S_f$, the map f_B is the Albanese morphism of X_B.

Lemma 3.1 [21; 16; 9] *Let X be a smooth projective variety of general type with a generically finite morphism $f\colon X\to A$ to an abelian variety.*

1. *We have $\chi(X,\omega_X)=0$ if, and only if, $V^0(f_*\omega_X)$ is a proper subset of $\widehat{A}$. If these properties hold, the abelian variety A has at least 3 simple factors.*

(2) *If* $\chi(X,\omega_X)=0$ *and* T *is an irreducible component of* $V^0(f_*\omega_X)$, *we have* $T\in S_f$. *More precisely,* T *is an irreducible component of* $V^i(f_*\omega_X)$, *where* $i=\operatorname{codim}_{\widehat{A}}(T)$.

(3) *For any abelian variety* $\widehat{B}\in S_f$, *the variety* X_B *is of general type and we have* $\chi(X_B,\omega_{X_B})>0$.

Proof The equivalence in (1) follows from generic vanishing [21, Theorem 1; 16, Remark 1.6, Theorem 1.2] and the other statement is [9, Corollary 3.4]. For (2), see [16, Claim (1.10)]. For (3), see [9, Theorem 3.1]. □

We introduce the notion of *primitive* and *minimal primitive* varieties. In Section 3.1, we study the structure of minimal primitive varieties and prove Theorem C. We provide examples in Section 4.

Definition 3.2 [12, Definition 6.1] Let X be a smooth projective variety of general type, of maximal Albanese dimension and with $\chi(X,\omega_X)=0$. We say that X is *primitive* if there exist no proper smooth subvarieties F through a general point of X such that $\chi(F,\omega_F)=0$.

We say that X is *minimal primitive* if it is primitive and, for any rational factorization $X\overset{a}{\dashrightarrow}Y\to A_X$ of the Albanese morphism of X through a smooth projective variety Y of general type, the map a is birational.

This definition of primitive is different from [7, Definition 1.24] but is equivalent to [12, Definition 6.1].

We will use the following results about primitive varieties.

Lemma 3.3 [9; 12] *Let* X *be a smooth projective variety of general type, of maximal Albanese dimension and with* $\chi(X,\omega_X)=0$.

(1) *If* $f\colon X\to A$ *is a morphism to an abelian variety, then there exists a quotient* $A\twoheadrightarrow B$ *of abelian varieties such that the general fiber* F_B *of the induced morphism* $X\twoheadrightarrow X_B$ *is primitive with* $\chi(F_B,\omega_{F_B})=0$.

Assume now that X *is primitive.*

(2) *The Albanese morphism* $a_X\colon X\to A_X$ *is surjective.*

(3) *For any quotient* $A_X\twoheadrightarrow B$ *to a simple abelian variety, with connected fibers, the composition* $X\xrightarrow{a_X}A_X\twoheadrightarrow B$ *is a fibration.*

(4) *If the abelian variety* A *has* m *simple factors, then* $V^0(f_*\omega_X)$ *has at least* m *irreducible components; each component is a torsion translate of an abelian variety with* $m-1$ *simple factors and the intersection of these components has dimension* 0.

Proof For the first assertion in (1), see [9, Theorem 3.1]. The second follows from [12, Proposition 6.2]. For (2), see [9, Lemma 4.6] and the definition of primitive. For (3), see [12, Lemma 6.4]. Statement (4) also follows from [12, Lemma 6.4], since for any quotient $\widehat{A}_X \twoheadrightarrow \widehat{K}$ of abelian varieties, the composition $V^0(f_*\omega_X) \hookrightarrow A_X \twoheadrightarrow \widehat{K}$ is surjective. □

3.1 The structure of minimal primitive varieties

We describe the structure of minimal primitive varieties X: we prove that each simple factor K_j of the Albanese variety A_X has a birational model K'_j that admits a Galois cover $F_j \to K'_j$ with finite Galois group G_j such that X is a quotient of the product of the F_j by a subgroup of the product of the G_j. When the F_j are curves, these quotient varieties already played an important role in Proposition 2.2.

Theorem 3.4 *Let X be a smooth projective variety of general type, of maximal Albanese dimension and with $\chi(X,\omega_X)=0$. We assume that X is minimal primitive. For some $m \geq 3$, there exist*

- *smooth projective varieties $F_1,\dots,F_m$ of general type,*
- *nontrivial finite groups G_j acting faithfully on F_j such that the quotient F_j/G_j is birationally isomorphic to a simple (nonzero) abelian variety K_j,*
- *an isogeny $K_1\times\cdots\times K_m \to A_X$ which induces an étale cover $\widetilde{X}\to X$,*
- *a subgroup G of $G_1\times\cdots\times G_m$,*

such that $\widetilde{X}$ is birationally isomorphic to $(F_1\times\cdots\times F_m)/G$.

Furthermore, we can assume that the projections

$$\pi_{ij}\colon G \to G_1\times\cdots\times G_{i-1}\times G_{i+1}\times\cdots\times G_{j-1}\times G_{j+1}\times\cdots\times G_m$$

are injective and the projections $G \twoheadrightarrow G_i$ are surjective whenever $1\leq i<j\leq m$.

We summarize part of the conclusions of the theorem in a commutative diagram:

$$\begin{tikzcd}
F_1\times\cdots\times F_m \arrow[d, dashed, two heads, "/G"] \arrow[dr, two heads, "/G_1\times\cdots\times G_m"] & \\
\widetilde{X} \arrow[r, two heads] \arrow[d, two heads, "\text{étale}"] \arrow[dr, phantom, "\square"] & K_1\times\cdots\times K_m \arrow[d, two heads, "\text{étale}"] \\
X \arrow[r, two heads, "a_X"] & A_X
\end{tikzcd}$$

Minimal primitive varieties with $\chi=0$ will be constructed in Examples 4.1 and 4.3.

Proof The proof is divided into four steps.

Step 1 (reduction via étale covers) Taking if necessary an étale cover of A_X (which induces an étale cover of X), we may assume that each element of S_{a_X} (see (7)), which by Lemma 3.1(1)–(2) is nonempty, contains the origin $0_{\widehat{A}_X}$.

Assume that A_X has m simple factors. Since we are assuming that X is *primitive*, there exist by Lemma 3.3(4) irreducible components $\widehat{A}_1, \dots, \widehat{A}_m$ of $V^0(a_{X*}\omega_X)$ such that each $\widehat{A}_j$ has $m-1$ simple factors and

$$\dim\Bigl(\bigcap_{1\le j\le m} \widehat{A}_j\Bigr) = 0. \tag{9}$$

The quotient $\widehat{K}_j := \widehat{A}_X/\widehat{A}_j$ is a simple abelian variety and we consider the dual injective morphism $K_j \hookrightarrow A_X$. By (9), the sum morphism

$$\pi\colon A' := K_1 \times \cdots \times K_m \to A_X$$

is an isogeny.

If $X' \twoheadrightarrow X$ is the étale cover induced by π, then $\pi^*(\widehat{A}_j) = \widehat{K}_1 \times \cdots \times \{0_{\widehat{K}_j}\} \times \cdots \times \widehat{K}_m$. Thus $V^0(a_{X'*}\omega_{X'})$ contains at least the m components $\widehat{K}_1 \times \cdots \times \{0_{\widehat{K}_j}\} \times \cdots \times \widehat{K}_m$. Moreover, $A_{X'} = K_1 \times \cdots \times K_m$.

Thus, we have constructed the following elements from the statement of the theorem: the simple abelian varieties K_i and the isogeny $K_1 \times \cdots \times K_m \to A_X$. We still need to identify the fibers F_j and the groups G_j and G.

Step 2 (a special property of fiber products) By Step 1, we can suppose that $\widehat{A}_j := \widehat{K}_1 \times \cdots \times \{0_{\widehat{K}_j}\} \times \cdots \times \widehat{K}_m$ is a component of $V^0(a_{X*}\omega_X)$ and $A_X = K_1 \times \cdots \times K_m$.

For each $1 \le i < j \le m$, set $\widehat{A}_{ij} := \widehat{A}_i \cap \widehat{A}_j$. Using the notation (8), we have a commutative diagram

$$\begin{tikzcd}
 & X \arrow[ddl, two heads, "f_i"'] \arrow[d, two heads, "f_{ij}"] \arrow[ddr, two heads, "f_j"] & \\
 & Y_{ij} \arrow[dl, two heads] \arrow[dr, two heads] & \\
X_{A_i} \arrow[dr, two heads, "g_{ij}"] & & X_{A_j} \arrow[dl, two heads, "g_{ji}"'] \\
 & X_{A_{ij}} &
\end{tikzcd} \tag{10}$$

where Y_{ij} is a desingularization of the main component of $X_{A_i} \times_{X_{A_{ij}}} X_{A_j}$. Since $A_X = K_1 \times \cdots \times K_m$, we have $A_i \times_{A_{ij}} A_j \simeq A_X$. Hence, the Albanese morphism factors as

$$a_X\colon X \xrightarrow{f_{ij}} Y_{ij} \to A_X. \tag{11}$$

Since X_{A_i} and X_{A_j} are of general type by Lemma 3.1(3), Y_{ij} is of general type by Viehweg's subadditivity theorem [40, Corollary IV]. Moreover, the assumption that X is *minimal* implies that f_{ij} is birational.

In other words, X is birationally isomorphic to the fiber product of any two of the fibrations induced by the simple factors of the Albanese variety.

Step 3 (the f_j are all birationally isotrivial fibrations) We use the notation of (10). Since f_{12} is birational, for a general point $x \in X_{A_1}$ the fiber F_x of f_1 is birationally isomorphic to $g_{21}^{-1}(g_{12}(x))$. Hence F_x is birationally isomorphic to F_y for $y \in g_{12}^{-1}(g_{12}(x))$ general. Similarly, F_x is birationally isomorphic to F_y for $y \in g_{1j}^{-1}(g_{1j}(x))$ general for all $j \in \{2, \dots, m\}$. Any two points of X_{A_1} can be connected by a chain of fibers of g_{12}, $g_{13}, \dots$ or g_{1m}. For general points x and y of X_{A_1}, the fiber F_x is therefore birationally isomorphic to F_y and f_1 is a birationally isotrivial fibration.

By the same argument, we see that f_j is a birationally isotrivial fibration for each $j \in \{1, \dots, m\}$. We denote by F_j its general fiber; since X is of general type, so is F_j.

We have now constructed the varieties F_j in the statement of the theorem. It remains to see that there are finite groups G_j acting faithfully on F_j such that F_j/G_j is birationally isomorphic to K_j and X is birationally isomorphic to the quotient $(F_1 \times \cdots \times F_m)/G$ for some subgroup G of $G_1 \times \cdots \times G_m$.

Step 4 (the finite groups $G_1, \dots, G_m$ and the subgroup G of $G_1 \times \cdots \times G_m$) The following lemma allows us to characterize varieties which are finite group quotients of a product of varieties and finishes the proof of Theorem 3.4.

Lemma 3.5 *Let $f\colon X \to V_1 \times \cdots \times V_m$ be a generically finite and surjective morphism between normal projective varieties. Assume that X is of general type and that, for each $j \in \{1, \dots, m\}$, there is a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[d, two heads, "f_j"'] \arrow[dr, two heads, "\varphi"] \arrow[rr, bend left, two heads, "g_j"] & V_1 \times \cdots \times V_m \arrow[r, two heads] \arrow[d, two heads] & V_j \\
X_j \arrow[r, two heads] & V_1 \times \cdots \times V_{j-1} \times V_{j+1} \times \cdots \times V_m &
\end{tikzcd}$$

where $X \xrightarrow{f_j} X_j$ is the Stein factorization of φ, the variety X_j is of general type, f_j is a birationally isotrivial fibration with general fiber F_j and g_j is a fibration.

Then there exists a finite group G_j acting faithfully on F_j such that F_j/G_j is birationally isomorphic to V_j. Moreover, X is birationally isomorphic to a quotient $(F_1 \times \cdots \times F_m)/G$, where G is a subgroup of $G_1 \times \cdots \times G_m$ with surjective projections $G \twoheadrightarrow G_j$.

Before giving the proof of the lemma, note that it applies to our situation

$$f\colon X \to K_1 \times \cdots \times K_m,$$

thanks to Step 3 and Lemma 3.3(3), which ensures that the $g_j\colon X \to K_j$ are fibrations.

Proof Let $\widetilde{X}_1$ be a general fiber of $g_1\colon X \twoheadrightarrow V_1$ and let $f|_{\widetilde{X}_1}\colon \widetilde{X}_1 \twoheadrightarrow V_2 \times \cdots \times V_m$ be the induced generically finite morphism. For $j \in \{2, \dots, m\}$, denote by $\widetilde{X}_1 \twoheadrightarrow V'_j \twoheadrightarrow V_j$ the Stein factorization of the natural morphism $\widetilde{X}_1 \twoheadrightarrow V_j$. The induced morphism $f'\colon \widetilde{X}_1 \to V'_2 \times \cdots \times V'_m$ is generically finite and surjective. For each $j \in \{2, \dots, m\}$, let

$$\widetilde{X}_1 \xrightarrow{f'_j} Y_j \twoheadrightarrow V'_2 \times \cdots \times V'_{j-1} \times V'_{j+1} \times \cdots \times V'_m$$

be the Stein factorization of the natural morphism. We summarize these constructions in the commutative diagram

$$\begin{tikzcd}
\widetilde{X}_1 \arrow[r, two heads, "f'_j"] \arrow[d, hook] & Y_j \arrow[r, two heads] \arrow[d] & V'_2 \times \cdots \times V'_{j-1} \times V'_{j+1} \times \cdots \times V'_m \arrow[r, two heads] \arrow[d] & \{*\} \arrow[d, hook] \\
X \arrow[r, two heads, "f_j"] \arrow[rrr, bend right, two heads, "g_1"'] & X_j \arrow[r, two heads] & V_1 \times \cdots \times V_{j-1} \times V_{j+1} \times \cdots \times V_m \arrow[r, two heads] & V_1
\end{tikzcd}$$

where the second and third vertical arrows are finite. Since the images of the Y_j in X_j cover X_j, and X_j is of general type by hypothesis, Y_j is also of general type; similarly, $\widetilde{X}_1$ is also of general type. Moreover, f'_j is also a birationally isotrivial fibration with general fiber F_j.

The morphism $f'\colon \widetilde{X}_1 \to V'_2 \times \cdots \times V'_m$ satisfies again the hypotheses of the lemma. Thus, by induction on m, we obtain that $\widetilde{X}_1$ is birationally isomorphic to a quotient of $F_2 \times \cdots \times F_m$. Since a fixed variety can only dominate finitely many birational classes of varieties of general type, $g_1\colon X \to V_1$ is birationally isotrivial.

Thus, after a suitable finite Galois base change $F'_1 \to V_1$ with Galois group G_1, where F'_1 is normal, we have a birational isomorphism $F'_1 \times \widetilde{X}_1 \overset{\sim}{\dashrightarrow} F'_1 \times_{V_1} X$. Since $\widetilde{X}_1$ is of general type, its birational automorphism group is finite. The action of $G_1 \times \mathrm{id}$ on $F_1' \times_{V_1} X$ therefore induces a birational action of G_1 on $\widetilde{X}_1$ such that X is birationally isomorphic to the quotient of $F_1' \times \widetilde{X}_1$ by the diagonal action of G_1.

Note that G_1 acts on the canonical models of $\widetilde{X}_1$ and F_1'. After an equivariant resolution of singularities [1, Theorem 0.1], we may assume that $\widetilde{X}_1$ and F_1' are smooth, still with G_1–actions, and that G_1 acts faithfully on F_1'. We have the commutative diagram

$$\begin{tikzcd} F_1' \times \widetilde{X}_1 \arrow[r, two heads] \arrow[rr, bend right, "\pi"'] & (F_1' \times \widetilde{X}_1)/G_1 \arrow[r, dashed, "\sim"] & X \arrow[r, two heads, "f"] \arrow[d, two heads, "f_1"] & V_1 \times \cdots \times V_m \arrow[d, two heads] \\ & & X_1 \arrow[r, two heads] & V_2 \times \cdots \times V_m \end{tikzcd}$$

Let $x \in F_1'$ be a general point; there is a dominant rational map $f_{1x} = f_1 \circ \pi|_{\{x\}\times \widetilde{X}_1}\colon \widetilde{X}_1 \dashrightarrow X_1$. Since any family of dominant maps between varieties of general type is locally constant, we have $f_{1x} = f_{1y}$ for x and y general points of F_1'. Thus, f_1 contracts the image of F_1' in X and $F_1' \overset{\sim}{\dashrightarrow} F_1$, hence $F_1 \to V_1$ is a birational Galois cover with Galois group G_1.

Similarly, for each $j \in \{1, \dots, m\}$, the map $F_j \to V_j$ is a birational Galois cover with Galois group G_j and there are dominant maps

$$\begin{tikzcd} F_1 \times \cdots \times F_m \arrow[r, dashed] \arrow[rr, bend left, "G_1\times\cdots\times G_m", "\text{Galois}"'] & X \arrow[r] & V_1 \times \cdots \times V_m \end{tikzcd}$$

Thus, there is a subgroup G of $G_1 \times \cdots \times G_m$ such that X is birationally isomorphic to $(F_1 \times \cdots \times F_m)/G$. Since $g_j\colon X \to V_j$ is a fibration, the projection $G \to G_j$ is surjective for each $j \in \{1, \dots, m\}$. $\square$

To finish the proof of Theorem 3.4, it only remains to prove the injectivity assertion of the projection to the product with two factors missing. This follows from the minimality assumption: the morphisms $f_{ij}\colon X \to Y_{ij}$ are birational, thus a general fiber of $X \to X_{A_{ij}}$ is birationally isomorphic to $F_i \times F_j$. This implies that the projections π_{ij} are injective. $\square$

3.2 Divisibility properties of the degree of the Albanese morphism

The main result of this section is the following theorem.

Theorem 3.6 *Let X be a smooth projective variety of general type and maximal Albanese dimension and let a_X be its Albanese morphism. If $\chi(X, \omega_X) = 0$, there exists a prime number p such that p^2 divides the degree of a_X onto its image.*

By Proposition 1.17, if X is a rational cohomology torus of general type with rational singularities, we have $\chi(X,\omega_X)=\chi(X',\omega_{X'})=0$ for any desingularization $X'\to X$. Thus Theorem C in the introduction directly follows from Theorem 3.6.

Proof We first reduce to the case of minimal primitive varieties (see Definition 3.2) using induction on the dimension. Then, we apply Theorem 3.4 and study the numerical properties of the degree of the Albanese morphism.

Step 1 (reduction to minimal primitive varieties) We may assume that X is primitive with $\chi(X,\omega_X)=0$. Otherwise, by Lemma 3.3(1) (see also (8)), there exists a quotient $A_X \twoheadrightarrow B := A_X/K$ such that the general fiber F of the induced fibration $X \twoheadrightarrow X_B$ is primitive with $\chi(F,\omega_F)=0$. The restriction $a_X|_F\colon F\to K$ factors through the (surjective) Albanese morphism of F and we can argue by induction on the dimension.

Moreover, if a_X factors as $X \dashrightarrow Y \to A_X$ through a variety of general type Y, after birational modifications, we may assume that we have morphisms between smooth projective varieties

$$a_X\colon X \twoheadrightarrow Y \xrightarrow{a_Y} A_X = A_Y.$$

Therefore, we can replace X with Y and study the structure of a_Y. Note that Y may not be primitive and we need to reapply induction on the dimension as before. Finally, we get an X which is a minimal primitive variety of general type.

The structure of a_X remains the same after taking abelian étale covers of X. Thus, using Theorem 3.4, we can assume that X is birationally isomorphic to

$$(F_1\times\cdots\times F_m)/G,$$

where F_j is a smooth projective variety acted on faithfully by the finite group G_j such that F_j/G_j is birationally isomorphic to a simple abelian variety K_j, the group G is a subgroup of $G_1\times\cdots\times G_m$ and

$$A_X = K_1\times\cdots\times K_m.$$

Furthermore, we can assume that, for all $1\le i<j\le m$, the projection

$$G\to G_1\times\cdots\times G_{j-1}\times G_{j+1}\times\cdots\times G_{j-1}\times G_{j+1}\times\cdots\times G_m$$

is injective and the projection $G\twoheadrightarrow G_j$ is surjective.

Step 2 (computation of $\deg(a_X)$) Set $g:=|G|$. For each $j\in\{1,\dots,m\}$, set $g_j:=|G_j|$. Since $A_X=K_1\times\cdots\times K_m$ and K_j is birationally isomorphic to F_j/G_j, we have

$$\deg(a_X)=\frac{1}{g}\prod_{1\le j\le m} g_j.$$

Moreover, since the projection $G \twoheadrightarrow G_j$ is surjective, we have $g_j \,|\, g$, hence

$$\deg(a_X) \Big| \prod_{2 \le j \le m} g_j . \tag{12}$$

Finally, since the projections $G \to G_1 \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_{k-1} \times G_{k+1} \times \cdots \times G_m$ are injective, we have

$$g_j g_k \,|\, \deg(a_X) \quad \text{for all } 1 \le j < k \le m. \tag{13}$$

Let now p be a prime factor of g_1. Then $p \,|\, \deg(a_X)$ by (13), hence $p \,|\, g_j$ for some $j \in \{2, \dots, m\}$ by (12), and $p^2 \,|\, g_1 g_j \,|\, \deg(a_X)$ by (13) again. This finishes the proof of Theorem 3.6. □

Given a smooth projective variety X of general type, of maximal Albanese dimension and primitive with $\chi(X, \omega_X) = 0$, we know by Theorem 3.6 that $p^2 \,|\, \deg(a_X)$ for some prime number p. Using the proofs of Theorems 3.6 and 3.4, we study the extremal case $\deg(a_X) = p^2$.

Theorem 3.7 *Let X be a smooth projective variety of general type, of maximal Albanese dimension and with $\chi(X, \omega_X) = 0$. If $\deg(a_X) = p^2$ for some prime number p, the morphism a_X is birationally a $(\mathbb{Z}/p\mathbb{Z})^2$–cover of its image.*

Proof We use the same notation as in Theorem 3.4.

There exists by Lemma 3.3(1) a quotient $A_X \twoheadrightarrow B = A_X/K$ such that the general fiber F of the induced fibration $X \twoheadrightarrow X_B$ is primitive with $\chi(F, \omega_F) = 0$. There is a factorization

$$a_X|_F \colon F \overset{a_F}{\twoheadrightarrow} A_F \xrightarrow{h} K \hookrightarrow A_X .$$

On the other hand, we have

$$p^2 = \deg(a_X) = \deg(h a_F) \deg(X_B \xrightarrow{a_{X_B}} B).$$

By Theorem 3.6, $\deg(a_F)$ is divisible by the square of a prime number. It follows that this prime number must be p and that a_{X_B} is birational onto its image.

Let $\eta \in X_B$ be the generic point. The geometric generic fiber $X_{\overline{\eta}}$ of $X \twoheadrightarrow X_B$ is then primitive and satisfies $\chi(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}) = 0$ and $\deg(X_{\overline{\eta}} \to A_{\overline{\eta}}) = p^2$. We are therefore reduced to the case where X is primitive.

Note that $a_X \colon X \to A_X$ is *minimal* (see Definition 3.2). We can therefore apply Theorem 3.4. Keeping its notation, we see G has index $\deg(a_X) = p^2$ in $G_1 \times \cdots \times G_m$; since π_{ij} is injective whenever $1 \le i < j \le m$, we obtain that each G_j is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and that $a_X \colon X \to A_X$ is a $(\mathbb{Z}/p\mathbb{Z})^2$–cover. □

Corollary 3.8 *Let X be a projective variety of general type with rational singularities. If X is a rational cohomology torus and $\deg(a_X) = p^2$ for some prime number p, then a_X is a $(\mathbb{Z}/p\mathbb{Z})^2$–cover.*

Proof Let $\mu\colon X' \to X$ be a desingularization. By Proposition 1.17, $A_X = A_{X'}$ and X is the Stein factorization of the Albanese morphism $X' \to A_X$. Hence a_X is a $(\mathbb{Z}/p\mathbb{Z})^2$–cover. □

3.3 Simple factors of the Albanese variety

Using the description of minimal primitive varieties of general type with $\chi = 0$, we obtain restrictions on the number of simple factors of their Albanese varieties (which we already know is at least 3 by Lemma 3.1(1)).

Proposition 3.9 *Let X be a smooth projective variety of general type, of maximal Albanese dimension, with $\chi(X, \omega_X) = 0$. If p is the smallest prime divisor of the degree of the Albanese map a_X, the Albanese variety A_X has at least $p+1$ simple factors.*

Proof We argue by induction on $\dim(X)$. As in Step 1 of the proof of Theorem 3.6, we can assume that X is minimal primitive (in the sense of Definition 3.2). Then, applying Theorem 3.4, we may assume, after taking an étale cover of X, that X is birationally isomorphic to a quotient $(F_1 \times \cdots \times F_m)/G$, the abelian variety A_X has m simple factors, $V^0(a_{X*}\omega_X)$ has m irreducible components

$$\begin{aligned}
\widehat{B}_1 &:= \{0_{\widehat{K}_1}\} \times \widehat{K}_2 \times \cdots \times \widehat{K}_m, \\
\widehat{B}_2 &:= \widehat{K}_1 \times \{0_{\widehat{K}_2}\} \times \cdots \times \widehat{K}_m, \\
&\ \vdots \\
\widehat{B}_m &:= \widehat{K}_1 \times \cdots \times \widehat{K}_{m-1} \times \{0_{\widehat{K}_m}\}
\end{aligned}$$

and all elements of S_{a_X} contain the origin $0_{\widehat{A}_X}$.

By the decomposition theorem [12, Theorems 1.1 and 3.5], we have

$$a_{X*}\omega_X = \bigoplus_{\widehat{B} \in S_{a_X}} p_B^* \mathcal{F}_B,$$

where $p_B\colon A_X \twoheadrightarrow B$ is the natural quotient and $\mathcal{F}_B$ is a coherent sheaf supported on B.

On the other hand, by [12, Lemma 3.7], we have, for each $j \in \{1, \ldots, m\}$,

$$a_{X_{B_j}*}\omega_{X_{B_j}} = \bigoplus_{\substack{\widehat{B} \in S_{a_X} \\ \widehat{B} \subset \widehat{B}_j}} p_{B,j}^* \mathcal{F}_B,$$

where $p_{B,j}\colon B_j \twoheadrightarrow B$ is the natural quotient, hence

$$\deg(a_{X_{B_j}}) = 1 + \sum_{\substack{\widehat{B} \in S_{a_X} \\ \widehat{B} \subset \widehat{B}_j}} \operatorname{rank}(\mathcal{F}_B).$$

Since all elements of S_{a_X} are contained in $\bigcup_{1 \le j \le m} \widehat{B}_j$, we have

$$\deg(a_X) - 1 \le \sum_{1 \le j \le m} (\deg(a_{X_{B_j}}) - 1). \tag{14}$$

With the notation of the proof of Theorem 3.4 ($g = |G|$ and $g_j = |G_j|$), we get

$$\deg(a_X) = \frac{1}{g} \prod_{1 \le j \le m} g_j \quad \text{and} \quad \deg(a_{X_{B_k}}) = \frac{1}{g} \prod_{j \ne k} g_j.$$

We may assume that $\deg(a_{X_{B_1}})$ is maximal among all $\deg(a_{X_{B_k}})$. Using (14), we then obtain $m \deg(a_{X_{B_1}}) \ge \deg(a_X) + m - 1 > g_1 \deg(a_{X_{B_1}})$. Hence $m \ge g_1 + 1 \ge p + 1$. □

By Proposition 1.17, we obtain Theorem D in the introduction as a direct corollary.

4 Construction of examples

We show that the varieties in Theorem 3.7 and Corollary 3.8 actually exist. The lower bounds on the degree of the Albanese morphisms in Theorem C, Theorem 3.6 and Proposition 3.9 are therefore optimal.

We first construct, for every prime p, a series of examples of (smooth) minimal primitive varieties X of general type with $\chi(X, \omega_X) = 0$, such that X is a finite quotient of a product of $p+1$ curves and the Albanese morphism a_X is a $(\mathbb{Z}/p\mathbb{Z})^2$–cover. Then we show that a slight modification of this construction leads to rational cohomology tori.

Example 4.1 ((smooth) minimal primitive varieties of general type with $\chi = 0$ whose Albanese morphisms are $(\mathbb{Z}/p\mathbb{Z})^2$–covers of a product of $p+1$ elliptic curves) Let p be a prime number. For each $j \in \{1, \ldots, p+1\}$, let $\rho_j\colon C_j \to E_j$ be a $(\mathbb{Z}/p\mathbb{Z})$–cover, where C_j is a smooth projective curve of genus $g_j \geq 2$ and E_j is an elliptic curve. For each $j \in \{1, \ldots, p+1\}$, write, as in (6),

$$\rho_{j*}\omega_{C_j} = \mathcal{O}_{E_j} \oplus \bigoplus_{1 \leq m \leq p-1} L_{\chi_j^m},$$

where $\chi_j \in (\mathbb{Z}/p\mathbb{Z})^\vee$ is a generator and $L_{\chi_j^m} = L_{\chi_j}^{\otimes m}$ is an ample line bundle on E_j for each $m \in \{1, \ldots, p-1\}$. Let $\rho\colon V = C_1 \times \cdots \times C_{p+1} \to A = E_1 \times \cdots \times E_{p+1}$ be the corresponding G–cover, where $G := (\mathbb{Z}/p\mathbb{Z})^{p+1}$.

We now construct a subgroup H of G isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{p-1}$. Actually, we will describe dually the quotient morphism of character groups. Let $p_j\colon G \to \mathbb{Z}/p\mathbb{Z}$ be the projection to the j^{th} factor, let $\chi \in (\mathbb{Z}/p\mathbb{Z})^\vee$ be a generator and set $\chi_j := \chi \circ p_j\colon G \to \mathbb{C}^*$. Then $(\chi_1, \ldots, \chi_{p+1})$ is a generating family of $G^\vee \simeq (\mathbb{Z}/p\mathbb{Z})^{p+1}$. On the other hand, let $(e_1, \ldots, e_{p-1})$ be the canonical basis for $(\mathbb{Z}/p\mathbb{Z})^{p-1}$. Define the quotient morphism $G^\vee \twoheadrightarrow H^\vee$ by

$$\pi\colon G^\vee \to (\mathbb{Z}/p\mathbb{Z})^{p-1}, \quad \chi_j \mapsto \begin{cases} e_j & \text{for } \leq j \leq p-1, \\ \sum_{1 \leq j \leq p-1} e_j & \text{for } j = p, \\ \sum_{1 \leq j \leq p-1} je_j & \text{for } j = p+1. \end{cases}$$

Let H be the corresponding subgroup of G and set $X := V/H$. We consider

$$\rho\colon V \xrightarrow{f} X \xrightarrow{g} A = E_1 \times \cdots \times E_{p+1}$$

and compute

$$\rho_*\omega_V = \bigoplus_{\tau \in G^\vee} L_\tau = \bigoplus_{(m_1,\ldots,m_{p+1}) \in (\mathbb{Z}/p\mathbb{Z})^{p+1}} (L_{\chi_1^{m_1}} \boxtimes \cdots \boxtimes L_{\chi_{p+1}^{m_{p+1}}}).$$

Moreover, as in the proof of Proposition 2.2, we have

$$\begin{aligned} g_*\omega_X = (\rho_*\omega_V)^H &= \bigoplus_{\substack{\tau \in G^\vee \\ \pi(\tau) = 1 \in H^\vee}} L_\tau \\ &= \bigoplus_{\substack{(m_1,\ldots,m_{p+1}) \in (\mathbb{Z}/p\mathbb{Z})^{p+1} \\ m_j + m_p + jm_{p+1} = 0 \text{ for all } j \in \{1,\ldots,p-1\}}} (L_{\chi_1^{m_1}} \boxtimes \cdots \boxtimes L_{\chi_{p+1}^{m_{p+1}}}) \\ &= \bigoplus_{(a,b) \in (\mathbb{Z}/p\mathbb{Z})^2} (L_{\chi_1^{-a-b}} \boxtimes L_{\chi_2^{-a-2b}} \boxtimes \cdots \boxtimes L_{\chi_{p-1}^{-a-(p-1)b}} \boxtimes L_{\chi_p^a} \boxtimes L_{\chi_{p+1}^b}). \end{aligned}$$

For a and b both nontrivial, there exists a unique $j \in \{1, \ldots, p-1\}$ such that $a + jb = 0 \in \mathbb{Z}/p\mathbb{Z}$. Thus, $\chi(X, \omega_X) = \chi(A, g_*\omega_X) = 0$.

Moreover, we see that

$$V^0(g_*\omega_X)=\bigcup_{1\le j\le p+1}\widehat{E}_1\times\cdots\times\{0_{\widehat{E}_j}\}\times\cdots\times\widehat{E}_{p+1}. \tag{15}$$

By [10, Theorem 1], X is of general type.

By studying the fibrations induced by the components of $V^0(g_*\omega_X)$, we deduce that X is primitive by Lemma 3.3(1). Since $\deg(a_X)=p^2$, Theorem 3.6 implies that X is minimal.

Remark 4.2 We saw that X is primitive in the sense of Definition 3.2 (see also [12, Section 6]). However, $g\colon X\to A$ is a $(\mathbb{Z}/p\mathbb{Z})^2$–cover. This shows that [12, Conjecture 6.6] is false: the structure of primitive varieties with $\chi=0$ is more complicated than expected.

Example 4.3 (rational cohomology tori of general type with finite quotient singularities whose Albanese morphisms are $(\mathbb{Z}/p\mathbb{Z})^2$–covers of a product of $p+1$ elliptic curves) We first recall some constructions of Pardini. Let X be a normal projective variety, let A be a smooth variety, let G be a finite abelian group and let $g\colon X\to A$ be a G–cover. We write, as in (6),

$$g_*\mathcal{O}_X=\bigoplus_{\tau\in G^\vee}L_\tau^{-1},$$

where the L_τ are line bundles on A. The algebra structure on $g_*\mathcal{O}_X$ gives rise to effective divisors $(D_{\tau,\tau'})_{(\tau,\tau')\in G^\vee\times G^\vee}$ such that $L_\tau\otimes L_{\tau'}\simeq L_{\tau\cdot\tau'}(D_{\tau,\tau'})$. Conversely, the data $(L_\tau)_{\tau\in G^\vee}$ and $(D_{\tau,\tau'})_{(\tau,\tau')\in G^\vee\times G^\vee}$ define a G–cover as above; see [33, Theorem 2.1].

Let p be a prime number, set $G:=(\mathbb{Z}/p\mathbb{Z})^2$ and consider the G–cover $g\colon X\to A$ from Example 4.1 with its associated data $(L_\tau)_{\tau\in G^\vee}$ and $(D_{\tau,\tau'})_{(\tau,\tau')\in G^\vee\times G^\vee}$.

Let P_j and P'_j be p–torsion line bundles on E_j such that their classes $[P_j]$ and $[P'_j]$ generate the group $\widehat{E}_j[p]\simeq(\mathbb{Z}/p\mathbb{Z})^2$ of p–torsion line bundles on E_j. Set $P:=P_1\boxtimes\cdots\boxtimes P_{p+1}$ and $P':=P'_1\boxtimes\cdots\boxtimes P'_{p+1}$.

We pick generators τ_1 and τ_2 of $G^\vee\simeq(\mathbb{Z}/p\mathbb{Z})^2$ and, for $(a,b)\in(\mathbb{Z}/p\mathbb{Z})^2$, we set

$$L'_{\tau_1^a\cdot\tau_2^b}:=L_{\tau_1^a\cdot\tau_2^b}\otimes P^{\otimes a}\otimes P'^{\otimes b}.$$

The relations $L'_\tau\otimes L'_{\tau'}\simeq L'_{\tau\cdot\tau'}(D_{\tau,\tau'})$ hold for all $(\tau,\tau')\in G^\vee\times G^\vee$. Thus, by [33, Theorem 2.1], we get a G–cover $g'\colon X'\to A$ such that

$$g'_*\mathcal{O}_{X'}=\bigoplus_{\tau\in G^\vee}L'^{-1}_\tau.$$

Both covers have the same branch divisors $D_{\tau,\tau'}$, so X' has the same singularities as X. Therefore, X' has finite quotient singularities and we have by duality (as in (6))

$$g'_*\omega_{X'} = \bigoplus_{\tau\in G^\vee} L'_\tau.$$

Hence, $V^0(g_*\omega_{X'})$ still generates A, so X' is also of general type by [10, Theorem 1].

We saw in Example 4.1 that for any τ nontrivial, L_τ is trivial when restricted to some E_j. However, since P_j and P'_j generate $\widehat{E}_j[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2$, L'_τ restricted to E_j is a nontrivial torsion line bundle. Therefore, $H^k(A, L'_\tau) = 0$ for any $k\in\mathbb{Z}$ and τ nontrivial. This implies $h^0(X', \omega_{X'}) = 1$.

Let $A' \to A$ be the abelian cover induced by P and P'. Since they have the same building data, the Galois covers $X\times_A A' \to A'$ and $X'\times_A A' \to A'$ are isomorphic [33, Theorem 2.1]. This shows that $X'\times_A A'$ is a quotient of a product of curves as in Proposition 2.2, hence so is X'. It follows from Proposition 2.2 that X' is a rational cohomology torus.

Now we show that there exist rational cohomology tori of general type with mild singularities in any dimension at least 3.

Example 4.4 (rational cohomology tori of general type with finite quotient singularities of any dimension at least 3) For each $j\in\{1,2,3\}$, consider a nonzero abelian variety A_j, an ample line bundle L_j on A_j, a smooth divisor $D_j\in|2L_j|$ and a nontrivial line bundle $P_j\in\widehat{A}_j$ of order two. Let $X_j\to A_j$ be the double cover associated with the data L_j and D_j, with involution τ_j, and let $X'_j\to X_j$ be the étale cover defined by P_j, with involution σ_j. Moreover, let τ'_j be a lifting of τ_j to X'_j.

Set $G := (\mathbb{Z}/2\mathbb{Z})^2$. Let H_1, H_2 and H_3 be the nontrivial cyclic subgroups of G and let χ_1, χ_2 and χ_3 be the nontrivial characters of G, with $\operatorname{Ker}(\chi_j) = H_j$. We now define data in order to construct a G–cover of $A := A_1\times A_2\times A_3$. Let $p_j\colon A\to A_j$ be the projections. We define

$$\begin{aligned} L_{\chi_1} &:= P_1\boxtimes L_2\boxtimes(L_3\otimes P_3), & D_{H_1} &:= p_1^*D_1,\\ L_{\chi_2} &:= (L_1\otimes P_1)\boxtimes P_2\boxtimes L_3, & D_{H_2} &:= p_2^*D_2,\\ L_{\chi_3} &:= L_1\boxtimes(L_2\otimes P_2)\boxtimes P_3, & D_{H_3} &:= p_3^*D_3. \end{aligned}$$

For $\{i,j,k\}=\{1,2,3\}$, we have $L_{\chi_i}\otimes L_{\chi_i}\simeq\mathcal{O}_A(D_{H_j}+D_{H_k})$ and $L_{\chi_i}\otimes L_{\chi_j}\simeq L_{\chi_k}(D_k)$. By [33, Theorem 2.1], there exists a G–cover $f\colon X\to A$ such that

$$f_*\mathcal{O}_X = \mathcal{O}_A\oplus L_{\chi_1}^{-1}\oplus L_{\chi_2}^{-1}\oplus L_{\chi_3}^{-1},$$

with branch locus $D = D_{H_1} + D_{H_2} + D_{H_3}$. Since D is a normal crossing divisor, a local computation shows that X has finite group quotient singularities. In particular, X has rational singularities. Actually, X is isomorphic to the quotient of $X'_1 \times X'_2 \times X'_3$ by the automorphism group generated by $\mathrm{id}_1 \times \tau'_2 \times \sigma_3$, $\sigma_1 \times \mathrm{id}_2 \times \tau'_3$, $\tau'_1 \times \sigma_2 \times \mathrm{id}_3$ and $\tau'_1 \times \tau'_2 \times \tau'_3$.

As in Proposition 2.2, we set $\Omega_X^{[s]} = \iota_*(\Omega^s_{X_{\mathrm{reg}}})$, where $\iota\colon X_{\mathrm{reg}} \hookrightarrow X$ is the open subset of smooth points. By [33, Theorem 4.1] or [34, Proposition 1.2], we have

$$\begin{aligned} f_*\Omega_X^{[s]} = \Omega^s_A &\oplus (\Omega^s_A(\log(D_{H_2} + D_{H_3})) \otimes L^{-1}_{\chi_1}) \\ &\oplus (\Omega^s_A(\log(D_{H_3} + D_{H_1})) \otimes L^{-1}_{\chi_2}) \\ &\oplus (\Omega^s_A(\log(D_{H_1} + D_{H_2})) \otimes L^{-1}_{\chi_3}), \end{aligned}$$

hence

$$H^t(X, \Omega_X^{[s]}) \simeq H^t(A, \Omega^s_A) \quad \text{for all } s, t \geq 0,$$

and X is a rational cohomology torus of general type with finite quotient singularities. Moreover, let Y be the quotient of $X'_1 \times X'_2 \times X'_3$ by the automorphism group generated by $\mathrm{id}_1 \times \tau'_2 \times \sigma_3$, $\sigma_1 \times \mathrm{id}_2 \times \tau'_3$ and $\tau'_1 \times \sigma_2 \times \mathrm{id}_3$. Then $Y \to X$ is a double cover and Y is smooth.

The following example exhibits (smooth) primitive fourfolds with $\chi = 0$ whose Albanese varieties have 4 simple factors and whose Albanese morphisms have degree 8, which is not a square. Thus, we have constructed two essentially different series of examples of (smooth) primitive varieties with $\chi = 0$ whose Albanese varieties have 4 simple factors: the minimal varieties provided by Example 4.1 taking $p = 3$ and the following nonminimal primitive varieties.

Example 4.5 (nonminimal (smooth) primitive fourfolds of general type with $\chi = 0$ whose Albanese morphisms are $(\mathbb{Z}/2\mathbb{Z})^3$–covers of a product of 4 elliptic curves) Let $\rho_1\colon C_1 \to E_1$ be a $(\mathbb{Z}/2\mathbb{Z})^2$–cover, where C_1 is a smooth projective curve of genus $g_1 \geq 2$ and E_1 is an elliptic curve. Let σ and τ be generators of the Galois group and let $\sigma^\vee$ and $\tau^\vee$ be the dual characters. For $j \in \{2, 3, 4\}$, let $\rho_j\colon C_j \to E_j$ be a $(\mathbb{Z}/2\mathbb{Z})$–cover with associated involution ι_j, where C_j is a smooth projective curve of genus at least 2 and E_j an elliptic curve.

Thus, we are considering the case where $G_1 = (\mathbb{Z}/2\mathbb{Z})^2$ and $G_2 = G_3 = G_4 = \mathbb{Z}/2\mathbb{Z}$ in Proposition 2.2 or Theorem 3.4. We have

$$\begin{aligned} \rho_{1*}\omega_{C_1} &= \mathcal{O}_{E_1} \oplus L_{\sigma^\vee} \oplus L_{\tau^\vee} \oplus L_{\sigma^\vee\tau^\vee}, \\ \rho_{j*}\omega_{C_j} &= \mathcal{O}_{E_j} \oplus L_{\iota_j^\vee} \quad \text{for } j \in \{2, 3, 4\}, \end{aligned}$$

where $L_{\chi^\vee}$ is an ample line bundle corresponding to the character $\chi^\vee$.

We then set $X := (C_1 \times C_2 \times C_3 \times C_4)/\langle \sigma \times \iota_1 \times \iota_2 \times \mathrm{id}_{F_3}, \tau \times \iota_1 \times \mathrm{id}_{F_2} \times \iota_3 \rangle$ and consider the $(\mathbb{Z}/2\mathbb{Z})^2$–quotient

$$f\colon C_1 \times C_2 \times C_3 \times C_4 \to X.$$

With the notation of Theorem 3.4 or Proposition 2.2, we have $G = (\mathbb{Z}/2\mathbb{Z})^2$.

Let $g\colon X \to A = E_1 \times E_2 \times E_3 \times E_4$ be the morphism such that the composition

$$\rho\colon Z = C_1 \times C_2 \times C_3 \times C_4 \xrightarrow[4:1]{f} X \xrightarrow[8:1]{g} A$$

is the quotient by $G_1 \times G_2 \times G_3 \times G_4$.

Abusing the notation, we can describe the quotient $(G_1 \times G_2 \times G_3 \times G_4)^\vee \to G^\vee$ by

$$\begin{aligned} \sigma^\vee &\mapsto (1,0), \\ \tau^\vee &\mapsto (0,1), \end{aligned} \qquad \begin{aligned} \iota_1^\vee &\mapsto (1,0), \\ \iota_2^\vee &\mapsto (0,1), \\ \iota_3^\vee &\mapsto (1,1). \end{aligned}$$

One checks that

$$\begin{aligned} g_*\omega_X &\simeq (\rho_*\omega_Z)^G \\ &\simeq \mathcal{O}_A \oplus (\mathcal{O}_{E_1} \boxtimes L_{\iota_1^\vee} \boxtimes L_{\iota_2^\vee} \boxtimes L_{\iota_3^\vee}) \\ &\qquad \oplus (L_{\sigma^\vee} \boxtimes L_{\iota_1^\vee} \boxtimes \mathcal{O}_{E_2} \boxtimes \mathcal{O}_{E_3}) \oplus (L_{\sigma^\vee} \boxtimes \mathcal{O}_{E_1} \boxtimes L_{\iota_2^\vee} \boxtimes L_{\iota_3^\vee}) \\ &\qquad \oplus (L_{\tau^\vee} \boxtimes \mathcal{O}_{E_1} \boxtimes L_{\iota_2^\vee} \boxtimes \mathcal{O}_{E_3}) \oplus (L_{\tau^\vee} \boxtimes L_{\iota_1^\vee} \boxtimes \mathcal{O}_{E_2} \boxtimes L_{\iota_3^\vee}) \\ &\qquad \oplus (L_{\sigma^\vee\tau^\vee} \boxtimes \mathcal{O}_{E_1} \boxtimes \mathcal{O}_{E_2} \boxtimes L_{\iota_3^\vee}) \oplus (L_{\sigma^\vee\tau^\vee} \boxtimes L_{\iota_1^\vee} \boxtimes L_{\iota_2^\vee} \boxtimes \mathcal{O}_{E_3}). \end{aligned}$$

Thus, $\chi(X,\omega_X) = \chi(A, g_*\omega_X) = 0$. Moreover, we obtain

$$V^0(g_*\omega_X) = \bigcup_{1 \le j \le 4} \widehat{E}_1 \times \cdots \times \{0_{\widehat{E}_j}\} \times \cdots \times \widehat{E}_4.$$

By [10, Theorem 1], X is of general type and, by studying the fibrations induced by the components of $V^0(g_*\omega_X)$, we obtain that X is primitive by Lemma 3.3(1).

Note that X is not minimal primitive: if we consider the quotient

$$(G_1 \times G_2 \times G_3 \times G_4)^\vee \to H^\vee \simeq (\mathbb{Z}/2\mathbb{Z})^3$$

defined by

$$\begin{aligned} \sigma^\vee &\mapsto (1,0,0), \\ \tau^\vee &\mapsto (0,1,0), \end{aligned} \qquad \begin{aligned} \iota_1^\vee &\mapsto (1,0,1), \\ \iota_2^\vee &\mapsto (0,1,1), \\ \iota_3^\vee &\mapsto (1,1,1), \end{aligned}$$

and the varieties $Z := C_1 \times C_2 \times C_3 \times C_4$ and $Y := Z/H$, we have a factorization

$$\rho\colon Z \xrightarrow[4:1]{f} X \xrightarrow[2:1]{} Y \xrightarrow[4:1]{h} A.$$

One checks that

$$\begin{aligned} h_*\omega_Y = (\rho_*\omega_Z)^H = \mathcal{O}_A &\oplus (L_{\sigma^\vee} \boxtimes \mathcal{O}_{E_1} \boxtimes L_{\iota_2^\vee} \boxtimes L_{\iota_3^\vee}) \\ &\oplus (L_{\tau^\vee} \boxtimes L_{\iota_1^\vee} \boxtimes \mathcal{O}_{E_2} \boxtimes L_{\iota_3^\vee}) \\ &\oplus (L_{\sigma^\vee\tau^\vee} \boxtimes L_{\iota_1^\vee} \boxtimes L_{\iota_2^\vee} \boxtimes \mathcal{O}_{E_3}). \end{aligned}$$

Thus, $\chi(Y, \omega_Y) = \chi(A, h_*\omega_Y) = 0$. Moreover, we have $V^0(g_*\omega_X) = V^0(h_*\omega_Y)$, so by [10, Theorem 1], Y is of general type.

Appendix: Nonexistence of smooth rational cohomology tori of general type
by William F Sawin

Theorem A.1 *Let $f\colon X \to A$ be a finite morphism from a smooth projective variety of general type X to an abelian variety A, all over $\mathbb{C}$. Let n be the dimension of X. Then*

$$(-1)^n \chi_{\mathrm{top}}(X) > 0.$$

Proof Recall that $(-1)^n \chi_{\mathrm{top}}(X)$ is the top Chern class of the cotangent bundle, or, equivalently, the intersection number of a section of the cotangent bundle and the zero section. We will compute this by taking a generic 1–form of A and pulling it back to X. We will show that its vanishing locus is 0–dimensional and nonempty, which implies that the intersection number is positive.

First we will show that the vanishing locus is 0–dimensional. Let

$$Z \subseteq X \times H^0(A, \Omega^1_A)$$

be the locus of pairs of a point $x \in X$ and a 1–form ω on A such that $f^*\omega$ vanishes at x. Let m be the dimension of A. Then the dimension of Z is at most m: because it is a closed subset, it is sufficient to check that for each subvariety $Y \subseteq X$ of dimension k with generic point η, the fiber Z_η has dimension at most $m-k$. The map f remains finite when restricted to Y and finite morphisms in characteristic 0 are generically unramified, so the map

$$H^0(A, \Omega^1_A) \otimes_{\mathbb{C}} \mathbb{C}(\eta) = \Omega^1_{A, f(\eta)} \otimes_{\mathbb{C}(f(\eta))} \mathbb{C}(\eta) \to \Omega^1_{Y,\eta}$$

from the cotangent space of $f(\eta)$ to the cotangent space of η is surjective, hence its kernel has dimension $m-k$. Then the kernel of the natural map

$$H^0(A,\Omega^1_A)\otimes_{\mathbb{C}}\mathbb{C}(\eta)\to\Omega^1_{X,\eta}$$

has dimension at most $m-k$, because it is contained in the previous kernel. But Z_η is precisely the affine space corresponding to this kernel, viewed as a vector space over $\mathbb{C}(\eta)$. So the dimension of Z_η equals the dimension of the kernel and is at most $m-k$, and thus the dimension of Z is at most m, as desired. Hence the vanishing locus of a generic 1–form from A is 0–dimensional.

By a result of Popa and Schnell [36, Conjecture 1], any 1–form on X vanishes at some point. So the vanishing locus is nonempty. Now the Chern number $c_n(\Omega^1_X)$ is the intersection number of the zero section with this generic 1–form. Because the intersection consists of finitely many points, the intersection number is a sum of contributions at those points, which is 1 if they are transverse but is always positive in general, so the total intersection number is positive. Thus

$$(-1)^n\chi_{\mathrm{top}}(X)=c_n(\Omega^1_X)>0. \qquad \square$$

Corollary A.2 *Let X be a smooth projective variety of general type. Then X is not a rational cohomology torus.*

Proof If it is, then by a remark of Catanese [8, Remark 72], its Albanese morphism $X\to A_X$ is finite. So by Theorem A.1, its topological Euler characteristic is nonzero. But because its rational cohomology is the same as that of an abelian variety, its Euler characteristic must be the same as that of an abelian variety, which is zero. This is a contradiction, so X is not a rational cohomology torus. $\square$

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References

[1] **D Abramovich**, **J Wang**, *Equivariant resolution of singularities in characteristic* 0, Math. Res. Lett. 4 (1997) 427–433 MR

[2] **V Ancona**, **B Gaveau**, *Differential forms on singular varieties: de Rham and Hodge theory simplified*, Pure and Applied Mathematics 273, Chapman & Hall/CRC, Boca Raton, FL (2006) MR

[3] **A Blanchard**, *Sur les variétés analytiques complexes*, Ann. Sci. Ecole Norm. Sup. 73 (1956) 157–202 MR

[4] **F A Bogomolov**, **C Böhning**, **H-C G von Bothmer**, *Birationally isotrivial fiber spaces*, Eur. J. Math. 2 (2016) 45–54 MR

[5] **R Bott**, **L W Tu**, *Differential forms in algebraic topology*, Graduate Texts in Mathematics 82, Springer, New York (1982) MR

[6] **G E Bredon**, *Introduction to compact transformation groups*, Pure and Applied Mathematics 46, Academic Press, New York (1972) MR

[7] **F Catanese**, *Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations*, Invent. Math. 104 (1991) 263–289 MR

[8] **F Catanese**, *Topological methods in moduli theory*, Bull. Math. Sci. 5 (2015) 287–449 MR

[9] **J A Chen**, **O Debarre**, **Z Jiang**, *Varieties with vanishing holomorphic Euler characteristic*, J. Reine Angew. Math. 691 (2014) 203–227 MR

[10] **J A Chen**, **C D Hacon**, *Pluricanonical maps of varieties of maximal Albanese dimension*, Math. Ann. 320 (2001) 367–380 MR

[11] **J A Chen**, **C D Hacon**, *On the irregularity of the image of the Iitaka fibration*, Comm. Algebra 32 (2004) 203–215 MR

[12] **J A Chen**, **Z Jiang**, *Positivity in varieties of maximal Albanese dimension*, J. Reine Angew. Math. (online publication July 2015)

[13] **J Chen**, **Z Jiang**, **Z Tian**, *Irregular varieites with geometric genus one, theta divisors, and fake tori*, preprint (2016) arXiv

[14] **O Debarre**, *Cohomological characterizations of the complex projective space*, preprint (2015) arXiv

[15] **P Deligne**, *Théorie de Hodge, III*, Inst. Hautes Études Sci. Publ. Math. 44 (1974) 5–77 MR

[16] **L Ein**, **R Lazarsfeld**, *Singularities of theta divisors and the birational geometry of irregular varieties*, J. Amer. Math. Soc. 10 (1997) 243–258 MR

[17] **A Fujiki**, *Closedness of the Douady spaces of compact Kähler spaces*, Publ. Res. Inst. Math. Sci. 14 (1978/79) 1–52 MR

[18] **A Fujiki**, *Duality of mixed Hodge structures of algebraic varieties*, Publ. Res. Inst. Math. Sci. 16 (1980) 635–667 MR

[19] **T Fujita**, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan 30 (1978) 779–794 MR

[20] **T Fujita**, *On topological characterizations of complex projective spaces and affine linear spaces*, Proc. Japan Acad. Ser. A Math. Sci. 56 (1980) 231–234 MR

[21] **M Green**, **R Lazarsfeld**, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*, Invent. Math. 90 (1987) 389–407 MR

[22] **M Green**, **R Lazarsfeld**, *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc. 4 (1991) 87–103 MR

[23] **C D Hacon**, *A derived category approach to generic vanishing*, J. Reine Angew. Math. 575 (2004) 173–187 MR

[24] **F Hirzebruch**, **K Kodaira**, *On the complex projective spaces*, J. Math. Pures Appl. 36 (1957) 201–216 MR Reprinted in *Kunihiko Kodaira: collected works, II*, Princeton University Press (1975) 744–759

[25] **Z Jiang**, **M Lahoz**, **S Tirabassi**, *On the Iitaka fibration of varieties of maximal Albanese dimension*, Int. Math. Res. Not. 2013 (2013) 2984–3005 MR

[26] **Z Jiang**, **Q Yin**, *On the Chow ring of certain rational cohomology tori*, preprint (2016) arXiv

[27] **Y Kawamata**, *Characterization of abelian varieties*, Compositio Math. 43 (1981) 253–276 MR

[28] **S J Kovács**, *Rational, log canonical, Du Bois singularities: on the conjectures of Kollár and Steenbrink*, Compositio Math. 118 (1999) 123–133 MR

[29] **S J Kovács**, *A characterization of rational singularities*, Duke Math. J. 102 (2000) 187–191 MR

[30] **S J Kovács**, **K E Schwede**, *Hodge theory meets the minimal model program: a survey of log canonical and Du Bois singularities*, from "Topology of stratified spaces" (G Friedman, E Hunsicker, A Libgober, L Maxim, editors), Math. Sci. Res. Inst. Publ. 58, Cambridge Univ. Press (2011) 51–94 MR

[31] **A S Libgober**, **J W Wood**, *Uniqueness of the complex structure on Kähler manifolds of certain homotopy types*, J. Differential Geom. 32 (1990) 139–154 MR

[32] **J A Morrow**, *A survey of some results on complex Kähler manifolds*, from "Global analysis: papers in honor of K Kodaira" (D C Spencer, S Iyanaga, editors), Univ. Tokyo Press (1969) 315–324 MR

[33] **R Pardini**, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. 417 (1991) 191–213 MR

[34] **R Pardini**, *Infinitesimal Torelli and abelian covers of algebraic surfaces*, from "Problems in the theory of surfaces and their classification" (F Catanese, C Ciliberto, M Cornalba, editors), Sympos. Math. XXXII, Academic Press, London (1991) 247–257 MR

[35] **C A M Peters**, **J H M Steenbrink**, *Mixed Hodge structures*, Ergeb. Math. Grenzgeb. 52, Springer (2008) MR

[36] **M Popa**, **C Schnell**, *Kodaira dimension and zeros of holomorphic one-forms*, Ann. of Math. 179 (2014) 1109–1120 MR

[37] **C Simpson**, *Subspaces of moduli spaces of rank one local systems*, Ann. Sci. École Norm. Sup. 26 (1993) 361–401 MR

[38] **K Ueno**, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics 439, Springer, Berlin (1975) MR

[39] **K Ueno**, *Bimeromorphic geometry of algebraic and analytic threefolds*, from "Algebraic threefolds" (A Conte, editor), Lecture Notes in Math. 947, Springer, Berlin (1982) 1–34 MR

[40] **E Viehweg**, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, from "Algebraic varieties and analytic varieties" (S Iitaka, editor), Adv. Stud. Pure Math. 1, North-Holland, Amsterdam (1983) 329–353 MR

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