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Arboreal singularities

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We introduce a class of combinatorial singularities of Lagrangian skeleta of symplectic manifolds. The link of each singularity is a finite regular cell complex homotopy equivalent to a bouquet of spheres. It is determined by its face poset, which is naturally constructed starting from a tree (nonempty finite acyclic graph). The choice of a root vertex of the tree leads to a natural front projection of the singularity along with an orientation of the edges of the tree. Microlocal sheaves along the singularity, calculated via the front projection, are equivalent to modules over the quiver given by the directed tree.

32S05, 53D37

1 Introduction

This paper is part of a project devoted to combinatorial models of symplectic topology, in particular of singular Lagrangian skeleta. After a summary of our main results immediately below, we discuss in Section 1.2 the subsequent development in Nadler [15] of the theory, which proves a refined version of a conjecture of Kontsevich [11], and has applications to mirror symmetry; see Nadler [18; 17]. On the one hand, this paper introduces the main objects and core calculations and is essential to what follows. On the other hand, by design, this paper can be read independently of further developments and with a minimal amount of geometric background: its constructions are of a combinatorial nature, and its results give elementary realizations of microlocal invariants. Its main results include calculations of microlocal sheaves where the answer can be viewed as an appealing alternative to traditional technical definitions. Low-dimensional examples of the main objects also arise naturally in recent advances in Legendrian knot theory found in Shende, Treumann and Zaslow [21], Ng, Rutherford, Shende, Sivek and Zaslow [19] and related work.

1.1 Summary

We will introduce a class of combinatorial singularities, first as coarse topological spaces, then naturally embedded as Legendrian singularities.

Our starting point is a *tree* T in the sense of a nonempty finite connected acyclic graph.

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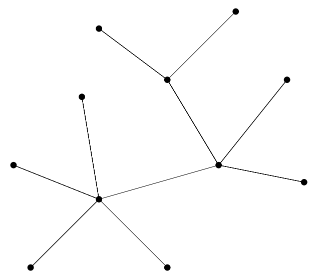

Figure 1: Example of a tree T

To each tree T, we associate a stratified space L_T called an *arboreal singularity*. It is of pure dimension $|T|-1$ where we write $|T|$ for the number of vertices of T. It comes equipped with a compatible metric and contracting $\mathbb{R}_{>0}$–action with a single fixed point. We refer to the compact subspace $\mathsf{L}_T^{\text{link}} \subset \mathsf{L}_T$ of points unit distance from the fixed point as the *arboreal link*. The $\mathbb{R}_{>0}$–action provides a canonical identification

$$\mathsf{L}_T \simeq \operatorname{Cone}(\mathsf{L}_T^{\text{link}})$$

so that one can regard the arboreal singularity L_T and arboreal link $\mathsf{L}_T^{\text{link}}$ as respective local models for a normal slice and normal link to a stratum in a stratified space. It follows easily from the constructions that the arboreal link $\mathsf{L}_T^{\text{link}}$ is homotopy equivalent to a bouquet of $|T|$ spheres each of dimension $|T|-1$.

As a stratified space, the arboreal link $\mathsf{L}_T^{\text{link}}$, and hence the arboreal singularity L_T as well, admits a simple combinatorial description. To each tree T, there is a natural finite poset $\mathfrak{P}_T$ whose elements are correspondences of trees

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T)$$

where i is the inclusion of a subtree and q is a quotient of trees. More precisely, the tree S is the full subgraph (or vertex-induced subgraph) on a subset of vertices of T; the tree R results from contracting a subset of edges of S. Two such correspondences

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T) \quad \text{and} \quad \mathfrak{p}' = (R' \overset{q'}{\twoheadleftarrow} S' \overset{i'}{\hookrightarrow} T')$$

satisfy $\mathfrak{p} \geq \mathfrak{p}'$ if there is another correspondence of the same form

$$\mathfrak{q} = (R \twoheadleftarrow Q \hookrightarrow R')$$

such that $\mathfrak{p} = \mathfrak{q} \circ \mathfrak{p}'$. In particular, the poset $\mathfrak{P}_T$ contains a unique minimum representing the identity correspondence

$$\mathfrak{p}_0 = (T \overset{=}{\twoheadleftarrow} T \overset{=}{\hookrightarrow} T).$$

Recall that a *finite regular cell complex* is a Hausdorff space X with a finite collection of closed cells $c_i \subset X$ whose interiors $c_i^\circ \subset c_i$ provide a partition of X and whose boundaries $\partial c_i \subset X$ are unions of cells. A finite regular cell complex X has the *intersection property* if the intersection of any two cells $c_i, c_j \subset X$ is either another cell or empty. The *face poset* of a finite regular cell complex X is the poset with elements the cells of X with relation $c_i \leq c_j$ whenever $c_i \subset c_j$. The *order complex* of a poset is the natural simplicial complex with simplices the finite totally ordered chains of the poset. (Useful references include Billera and Björner [4], Björner [5] and Wachs [25].)

As topological spaces, arboreal singularities take the following simple combinatorial form. If we were only interested in their topology, we could take the below description as definition. Instead, we will approach them with a geometric construction that leads to their natural realization as Legendrian singularities.

Theorem 1.1 *Let T be a tree.*

The arboreal link $\mathsf{L}_T^{\mathrm{link}}$ is a finite regular cell complex, with the intersection property, with face poset $\mathfrak{P}_T \setminus \{\mathfrak{p}_0\}$, and thus homeomorphic to the order complex of $\mathfrak{P}_T \setminus \{\mathfrak{p}_0\}$.

Remark 1.2 It follows from the theorem and the poset structure on $\mathfrak{P}_T$ that the normal slice to the stratum $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T$ indexed by a partition

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T)$$

is homeomorphic to the arboreal singularity L_R.

Example 1.3 Let us highlight the simplest class of trees.

When T consists of a single vertex, L_T is a single point.

When T consists of two vertices v_1, v_2 (necessarily connected by an edge), L_T is the local trivalent graph given by the cone over the three distinct points $\mathsf{L}_T^{\mathrm{link}}$ representing the three correspondences

$$(\{v_1\} \overset{=}{\twoheadleftarrow} \{v_1\} \hookrightarrow T), \quad (\{v_2\} \overset{=}{\twoheadleftarrow} \{v_2\} \hookrightarrow T) \quad \text{and} \quad (\{v\} \twoheadleftarrow T \overset{=}{\hookrightarrow} T).$$

More generally, consider the class of A_n–trees T_n consisting of n vertices connected by $n-1$ successive edges. The associated arboreal singularity L_{T_n} admits an identification with the cone of the $(n-2)$–skeleton of the n–simplex

$$\mathsf{L}_{T_n} \simeq \mathrm{Cone}(\mathrm{sk}_{n-2}\,\Delta^n)$$

or in a dual realization, the $(n-1)$–skeleton of the polar fan of the n–simplex. This space arises in many places (all intimately related to symplectic topology):

(1) as a tropical hyperplane in n–dimensional tropical projective space (Ardila and Develin [1], Develin and Sturmfels [6] and Speyer and Sturmfels [22]),[1]

(2) as the universal planar tree over the $(n-2)$–dimensional associahedron K_{n-2} (Stasheff [23; 24], Loday [12] and Seidel [20]),

(3) in geometric realizations of Waldhausen's S–construction in K–theory (Dyckerhoff and Kapranov [7], Dyckerhoff and Kapranov [8] and Nadler [14]).

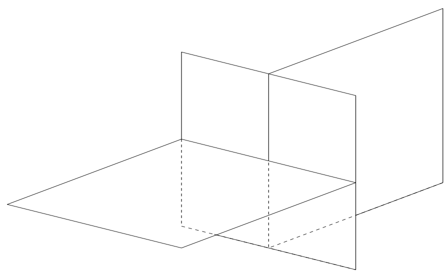

Figure 2: The A_3–arboreal singularity

Example 1.4 The first example beyond A_n–trees is that of the D_4–tree with a central vertex connected to three other vertices. The corresponding arboreal singularity is the union of a Euclidean space $\mathbb{R}^3$ and three closed Euclidean halfspaces $\mathbb{R}_{\geq 0} \times \mathbb{R}^2$ each glued along its boundary $\mathbb{R}^2 = \partial(\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$ to the Euclidean space $\mathbb{R}^3$ along a distinct coordinate hyperplane $\mathbb{R}^2 \subset \mathbb{R}^3$.

Arboreal singularities offer a natural generalization of the above singularities associated to A_n–trees. We will next explain their appearance as Legendrian singularities whose front projections are particularly simple cooriented singular hypersurfaces. (Then in Section 1.2 below we will discuss their related appearance as Lagrangian singularities.)

To this end, our refined starting point is a *rooted tree* $\mathcal{T} = (T, \rho)$ in the sense of a tree T together with a distinguished vertex ρ called the root vertex.

The set of vertices $V(T)$ naturally forms a poset with the root vertex $\rho \in V(T)$ the unique minimum and in general $\alpha < \beta \in V(T)$ if the former is nearer to ρ than the latter.

Let $\mathbb{R}^{\mathcal{T}}$ denote the Euclidean space of real tuples $\{x_\gamma\}$ indexed by vertices $\gamma \in V(T)$. Let us write $S^*\mathbb{R}^{\mathcal{T}}$ for its spherically projectivized cotangent bundle or, equivalently, unit cosphere bundle. Points of $S^*\mathbb{R}^{\mathcal{T}}$ are pairs $(x, [v])$ where $x \in \mathbb{R}^{\mathcal{T}}$ and $[v]$ is the positive ray or, equivalently, unit covector, in the direction of $v \neq 0 \in T^*_x\mathbb{R}^{\mathcal{T}}$. Recall that $S^*\mathbb{R}^{\mathcal{T}}$ is naturally a cooriented contact manifold.

[1] We thank E Zaslow for pointing this out to us, and D Auroux for noting this perspective appears in Kontsevich's expectations [11]. No doubt it holds significance for mirror symmetry.

To each rooted tree $\mathcal{T} = (T, \rho)$, we associate a singular hypersurface $\mathsf{H}_{\mathcal{T}} \subset \mathbb{R}^{\mathcal{T}}$ called an *arboreal hypersurface*. On the one hand, the arboreal hypersurface $\mathsf{H}_{\mathcal{T}} \subset \mathbb{R}^{\mathcal{T}}$ admits a homeomorphism with the rectilinear hypersurface defined by coordinate equalities and inequalities

$$H_{\mathcal{T}} \simeq \bigcup_{\alpha \in V(T)} \{x_\alpha = 0, x_\beta > 0 \text{ for all } \beta < \alpha\} \subset \mathbb{R}^{\mathcal{T}}.$$

On the other hand, the arboreal hypersurface $\mathsf{H}_{\mathcal{T}} \subset \mathbb{R}^{\mathcal{T}}$ is in *good position* in the sense that it has finitely many normal Gauss directions even across its singularities. Thus it defines a *conormal Legendrian* $\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} \subset S^*\mathbb{R}^{\mathcal{T}}$ whose front projection provides a finite surjection

$$\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} \twoheadrightarrow \mathsf{H}_{\mathcal{T}}.$$

The following shows that the arboreal singularity L_T associated to a tree T naturally arises as a Legendrian singularity. The choice of the root vertex $\rho \in V(T)$ plays the role of a polarization enabling this presentation.

Theorem 1.5 *Let $\mathcal{T} = (T, \rho)$ be a rooted tree.*

The conormal Legendrian $\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} \subset S^*\mathbb{R}^{\mathcal{T}}$ of the arboreal hypersurface $\mathsf{H}_{\mathcal{T}} \subset \mathbb{R}^{\mathcal{T}}$ is homeomorphic to the arboreal singularity L_T.

Example 1.6 An instructive example is that of the A_3–tree T_3 with its two possible inequivalent rooted structures. On the one hand, we could take one of the two end vertices as root vertex to obtain a rooted tree. On the other hand, we could take the middle vertex as root vertex to obtain a rooted tree. The resulting arboreal hypersurfaces are quite different though their conormal Legendrians are homeomorphic.

With the theorem in mind, we will write $\mathsf{L}_{\mathcal{T}} \subset S^*\mathbb{R}^{\mathcal{T}}$ in place of $\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} \subset S^*\mathbb{R}^{\mathcal{T}}$, using the subscript $\mathcal{T}$ as opposed to T to emphasize the dependence of the embedding on the poset structure.

We next calculate the categorical quantization of the Legendrian singularity $\mathsf{L}_{\mathcal{T}} \subset S^*\mathbb{R}^{\mathcal{T}}$ in the form of microlocal sheaves supported along it. (We recommend the comprehensive book Kashiwara and Schapira [9] for the general notions that appear in what follows, along with Keller [10] and the references therein for working in a differential graded setting.)

Fix once and for all a field k, and let $\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$ denote the dg category of cohomologically constructible complexes of sheaves of k–vector spaces on $\mathbb{R}^{\mathcal{T}}$. Recall that to any object $\mathcal{F} \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$, one can associate its singular support $\mathrm{ss}(\mathcal{F}) \subset S^*\mathbb{R}^{\mathcal{T}}$. This is a closed Legendrian subspace recording those codirections in which the propagation of sections of $\mathcal{F}$ is obstructed. In particular, one has the vanishing $\mathrm{ss}(\mathcal{F}) = \varnothing$ if and only if the cohomology sheaves of $\mathcal{F}$ are locally constant.

Introduce the dg category $\mathrm{Sh}_{\mathsf{L}_\mathcal{T}}(\mathbb{R}^\mathcal{T})$ of constructible complexes of sheaves of k–vector spaces on $\mathbb{R}^\mathcal{T}$ microlocalized along $\mathsf{L}_\mathcal{T} \subset S^*\mathbb{R}^\mathcal{T}$. Thanks to the simplicity of the situation, we can concretely work with $\mathrm{Sh}_{\mathsf{L}_\mathcal{T}}(\mathbb{R}^\mathcal{T})$ as the full dg subcategory of $\mathrm{Sh}(\mathbb{R}^\mathcal{T})$ consisting of objects $\mathcal{F} \in \mathrm{Sh}(\mathbb{R}^\mathcal{T})$ with the prescribed singular support and vanishing global sections

(1) $\mathrm{ss}(\mathcal{F}) \subset \mathsf{L}_\mathcal{T}$, and
(2) $\mathrm{Hom}_{\mathrm{Sh}(\mathbb{R}^\mathcal{T})}(k_{\mathbb{R}^\mathcal{T}}, \mathcal{F}) \simeq 0$.

Recall that we can regard the set of vertices $V(T)$ of the rooted tree $\mathcal{T} = (T, \rho)$ as a poset with the root vertex $\rho \in V(T)$ the unique minimum. To each nonroot vertex $\alpha \neq \rho \in V(T)$ there is a unique parent vertex $\hat\alpha \in V(T)$ such that $\alpha > \hat\alpha$ and there are no vertices strictly between them.

Now let us regard the rooted tree $\mathcal{T} = (T, \rho)$ as a quiver with a unique arrow pointing from each nonroot vertex $\alpha \neq \rho \in V(T)$ to its parent vertex $\hat\alpha \in V(T)$. Symbolically, we replace the relation $\alpha > \hat\alpha$ with the relation $\alpha \to \hat\alpha$.

Let $\mathrm{Mod}(\mathcal{T})$ denote the dg derived category of finite-dimensional complexes of modules over $\mathcal{T}$ regarded as a quiver. Objects assign to each vertex $\alpha \in V(T)$ a finite-dimensional complex of k–vector spaces $M(\alpha)$, and to each arrow $\alpha \to \hat\alpha$ a degree zero chain map $m_\alpha\colon M(\alpha) \to M(\hat\alpha)$.

Remark 1.7 Let us point out two natural generating collections for $\mathrm{Mod}(\mathcal{T})$. There are the simple modules $S_\alpha \in \mathrm{Mod}(\mathcal{T})$ that assign

$$S_\alpha(\beta) = \begin{cases} k & \text{when } \beta = \alpha, \\ 0 & \text{when } \beta \neq \alpha, \end{cases}$$

with all maps $m_\beta\colon S_\alpha(\beta) \to S_\alpha(\hat\beta)$ necessarily zero. There are also the projective modules $P_\alpha \in \mathrm{Mod}(\mathcal{T})$ that assign

$$P_\alpha(\beta) = \begin{cases} k & \text{when } \beta \leq \alpha, \\ 0 & \text{when } \beta > \alpha, \end{cases}$$

with the maps $m_\beta\colon P_\alpha(\beta) \to P_\alpha(\hat\beta)$ the identity isomorphism whenever both domain and range are nonzero.

The categorical quantization of the Legendrian singularity $\mathsf{L}_\mathcal{T} \subset S^*\mathbb{R}^\mathcal{T}$ admits the following simple description.

Theorem 1.8 *Let $\mathcal{T} = (T, \rho)$ be a rooted tree.*

The dg category $\mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ of constructible complexes microlocalized along $\mathsf{L}_{\mathcal{T}} \subset S^*\mathbb{R}^{\mathcal{T}}$ is canonically equivalent to the dg category of modules $\mathrm{Mod}(\mathcal{T})$.

Remark 1.9 The dg category $\mathrm{Mod}(\mathcal{T})$ is noncanonically independent of the choice of root vertex and resulting quiver structure. Namely, for a different choice of orientations of arrows, reflection functors, Bernstein, Gelfand and Ponomarev [2] provide equivalences between the corresponding module categories. Thus the dg category of microlocal sheaves along the arboreal singularity $\mathsf{L}_{\mathcal{T}}$ is noncanonically independent of its presentation as the conormal Legendrian to a particular arboreal hypersurface.

It is also possible to describe the natural microlocal restriction functors. Recall that the normal slice to the stratum $\mathsf{L}_{\mathcal{T}}(\mathfrak{p}) \subset \mathsf{L}_{\mathcal{T}}$ indexed by a partition

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T)$$

is homeomorphic to the arboreal singularity L_{R}. Note that the quiver structure on $\mathcal{T}$ naturally induces quiver structures on S and R which we denote by $\mathcal{S}$ and $\mathcal{R}$ respectively. Under the equivalence of the theorem, the corresponding microlocal restriction functor is the natural composite quotient functor

$$\mathrm{Mod}(\mathcal{T}) \overset{i^*}{\twoheadrightarrow} \mathrm{Mod}(\mathcal{S}) \overset{q_!}{\twoheadrightarrow} \mathrm{Mod}(\mathcal{R})$$

where i^* kills the projective object $P_\alpha \in \mathrm{Mod}(\mathcal{T})$ attached to $\alpha \in V(T)$ such that $\alpha \notin i(V(S))$, and $q_!$ identifies the projective objects $P_\alpha, P_\beta \in \mathrm{Mod}(\mathcal{S})$ attached to $\alpha, \beta \in V(S)$ such that $q(\alpha) = q(\beta) \in V(R)$.

1.2 Motivation

We briefly discuss here the role of this paper in a broader undertaking. The definitions and discussion of this section will not be used in the rest of the paper.

Our primary aim is to construct a combinatorial model and computational tool for the "quantum category" of A–branes mathematically captured by the Kashiwara–Schapira theory [9] of microlocal sheaves (topology), the Floer–Fukaya–Seidel theory of wrapped and infinitesimal Fukaya categories (analysis), and the theory of holonomic modules over deformation quantizations (algebra), exemplified by $\mathcal{D}$–modules; see Bernstein [3]. In parallel with the cohomology of manifolds, where one has singular complexes (topology), Morse and Hodge theory (analysis), and de Rham complexes (algebra), we seek a parallel to simplicial complexes (combinatorics) in the study of the intersection theory of Lagrangians in symplectic manifolds. The arboreal singularities of the current paper provide a local model for realizing such a combinatorial model.

To explain this further, let us introduce some basic constructions and useful terminology.

Let N be a cooriented contact $(2n+1)$–dimensional manifold with contact field $\xi = \ker(\lambda)$ defined by a contact form λ. By a Legendrian subvariety $L \subset N$, we will mean a closed n–dimensional Whitney stratified subspace (satisfying some mild additional properties spelled out in Nadler [15]) such that $\xi|_Y = 0$ for any submanifold $Y \subset N$ contained within L. By a Legendrian singularity centered at a submanifold $Y \subset N$, we will mean the germ along $Y \subset N$ of a Legendrian subvariety containing Y as a closed stratum.

Recall the contact Darboux theorem that any contact manifold N is locally equivalent to the spherical projectivization $S^*\mathbb{R}^{n+1}$ with its standard contact structure. Thus given a directed tree $\mathcal{T}$, with $|\mathcal{T}| = n+1-k$, we can view the product $\mathsf{L}_{\mathcal{T}} \times \mathbb{R}^k \subset S^*(\mathbb{R}^{\mathcal{T}} \times \mathbb{R}^k)$ as a Legendrian singularity within N.

For the sake of the current discussion, let us proceed with the following definition which is more concrete but less flexible than possible alternatives.

Definition 1.10 A Legendrian subvariety $L \subset N$ is said to have *arboreal singularities* if its singularity at each of its points is equivalent via a contactomorphism to a Legendrian singularity of the form $\mathsf{L}_{\mathcal{T}} \times \mathbb{R}^k$, for a directed tree $\mathcal{T}$, with $|\mathcal{T}| = n+1-k$.

Remark 1.11 If a Legendrian subvariety $L \subset N$ has arboreal singularities, then the dg category of microlocal sheaves on N supported along L can be calculated combinatorially via Theorem 1.8 and the functoriality described thereafter.

In the sequel to this paper, Nadler [15], we study arbitrary Legendrian singularities and prove the following theorem. The term *noncharacteristic* in its statement refers to the property that the dg category of microlocal sheaves supported along the Legendrian singularity is unchanged by the deformation. The phrase *degenerate arboreal singularities* refers to a modest variation on arboreal singularities discussed in [15]. For example, in one dimension, a trivalent vertex of a graph is an arboreal singularity, and a univalent vertex is a degenerate arboreal singularity.

Theorem 1.12 [15] *Any Legendrian singularity admits a noncharacteristic deformation to a Legendrian subvariety with arboreal and degenerate arboreal singularities.*

Roughly speaking, to prove the theorem, given a Legendrian singularity, we expand each of its strata into an irreducible component to arrive at a Legendrian subvariety whose singularities are governed by the combinatorics of the interaction of its irreducible components. With the theorem in hand, the calculation of microlocal sheaves may be performed in terms of finite-dimensional modules over trees, appealing to the results of the current paper. One could compare the situation with Morse theory or resolutions with normal crossing divisors in algebraic geometry, where complicated singularities are reduced to combinatorial assemblages of simple singularities to calculate invariants.

Last, to connect with Lagrangian skeleta, let M be an exact symplectic $2n$–dimensional manifold, with symplectic form ω and primitive α. By an exact Lagrangian subvariety $\Lambda \subset M$, we will mean a closed n–dimensional Whitney stratified subspace (satisfying some mild additional properties), admitting a continuous function $f\colon \Lambda \to \mathbb{R}$, such that for any submanifold $Y \subset M$ contained within Λ, we have $\omega|_Y = 0$, and $f|_Y$ is differentiable with $d(f|_Y) = \alpha|_Y$. By a Lagrangian singularity centered at a submanifold $Y \subset M$, we will mean the germ along $Y \subset M$ of a Lagrangian subvariety containing Y as a closed stratum.

Now let us set $N = M \times \mathbb{R}$ to be the contactification of M with contact field $\xi = \ker(\lambda)$ defined by the contact form $\lambda = dt + \alpha$, where we write t for a coordinate on $\mathbb{R}$. Then any exact Lagrangian subvariety $\Lambda \subset M$, equipped with a primitive $f\colon \Lambda \to \mathbb{R}$, lifts to a Legendrian subvariety given by the graph

$$L_{\Lambda,f} = \{(m, -f(m)) \in \Lambda \times \mathbb{R}\} \subset N.$$

Note that alternative primitives will differ from f by a locally constant function on Λ, and hence the corresponding lift will differ from $L_{\Lambda,f}$ by a locally constant translation. In this way, we can embed the study of exact Lagrangian singularities and subvarieties into that of Legendrian singularities and subvarieties. Notably, we may lift Lagrangian skeleta to Legendrian subvarieties, and then apply the above theory to their singularities.

Definition 1.13 An exact Lagrangian subvariety $\Lambda \subset M$, with primitive $f\colon \Lambda \to \mathbb{R}$, is said to have *arboreal singularities* if the Legendrian subvariety $L_{\Lambda,f} \subset N$ has arboreal singularities.

Remark 1.14 Forming the contactification, or further forming its symplectification, leaves invariants such as microlocal sheaves with prescribed support unchanged.

Example 1.15 A basic example of Lagrangian skeleta are ribbon graphs in punctured Riemann surfaces. Such a Lagrangian skeleton has arboreal and degenerate arboreal singularities if and only if each of its vertices is trivalent or univalent.

Example 1.16 A common example of a Lagrangian singularity is given by the union of two smooth Lagrangian submanifolds intersecting transversely. Thus the geometry is locally modeled by $M = T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$, $L_1 = \mathbb{R}^n \times \{0\}$, $L_2 = \{0\} \times \mathbb{R}^n$ and $L = L_1 \cup L_2$. This is not an arboreal singularity, but we may apply the above theory to it. Depending on choices in the algorithm underlying Theorem 1.12, what results is one of two possible arboreal Lagrangian subvarieties $L_\pm \subset M$ given by the respective unions

$$L_\pm = (L_1 \#_\pm L_2) \cup D^n_\pm,$$

where $L_1 \#_{\pm} L_2 \subset M$ is either the positive or negative Lagrangian surgery, $D^n_{\pm} \subset M$ are the respective vanishing thimbles, and they meet along the respective vanishing spheres $S^{n-1}_{\pm} = \partial D^n_{\pm} \subset L_1 \#_{\pm} L_2$. Thus the arboreal Lagrangian subvariety $L_{\pm} \subset M$ is smooth except along $S^{n-1}_{\pm}$ where its normal geometry is equivalent to the trivalent vertex of the A_2–arboreal singularity. In the basic case of dimension $n = 1$, we recover the two standard trivalent deformations of a four-valent vertex.

As a sample first application in Nadler [18], we apply this circle of ideas to an important example in mirror symmetry: the Landau–Ginzburg A–model with background $M = \mathbb{C}^3$ and superpotential $W = z_1 z_2 z_3$. (Natural generalizations appear in the later work Nadler [16].) Due to the fact that the critical locus $\{dW = 0\} \subset M$ is not smooth or proper, this Landau–Ginzburg A–model is not easily approached with traditional methods. The main theorem of [18] is the calculation of microlocal sheaves along the natural singular Lagrangian thimble $L = \mathrm{Cone}(T^2) \subset M$, and more basically the construction of a deformation of L to a Lagrangian skeleton with arboreal singularities. The description obtained is in the form of a quiver with relations, and immediately equivalent to the B–model of the pair-of-pants $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ as predicted by mirror symmetry.

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2 Arboreal singularities

2.1 Gluing construction

By a *graph* G, we will mean a set of *vertices* $V(G)$ and a set of *edges* $E(G)$ satisfying the simplest convention that $E(G)$ is a subset of the set of two-element subsets of $V(G)$. Thus $E(G)$ records whether pairs of distinct elements of $V(G)$ are connected by an edge or not. We will write $\{\alpha, \beta\} \in E(G)$ and say that $\alpha, \beta \in V(T)$ are *adjacent* if an edge connects them.

By a *tree* T, we will mean a nonempty finite connected acyclic graph. Thus for any vertices $\alpha, \beta \in V(T)$, there is a unique minimal path (nonrepeating sequence of edges) connecting them. Thus it makes sense to call the number of edges in the sequence the *distance* between the vertices.

Fix a tree T with vertex set $V(T)$ and edge set $E(T)$.

Definition 2.1 For each vertex $\alpha \in V(T)$, define $\mathsf{L}_T(\alpha) = \mathbb{R}^{V(T)\setminus\{\alpha\}}$ to be the Euclidean space of tuples of real numbers

$$\{x_\gamma(\alpha)\} \quad \text{with } \gamma \in V(T) \setminus \{\alpha\}.$$

Definition 2.2 For an edge $\{\alpha, \beta\} \in E(T)$, define the $\{\alpha, \beta\}$*–edge gluing* to be the quotient of the disjoint union of Euclidean spaces

$$(\mathsf{L}_T(\alpha) \amalg \mathsf{L}_T(\beta))/\sim$$

where we identify points $\{x_\gamma(\alpha)\} \sim \{x_\gamma(\beta)\}$ whenever the following hold:

$$x_\beta(\alpha) = x_\alpha(\beta) \geq 0 \qquad \text{and} \qquad x_\gamma(\alpha) = x_\gamma(\beta) \quad \text{for all } \gamma \neq \alpha, \beta \in V(T).$$

Definition 2.3 The *arboreal singularity* L_T associated to a tree T is the quotient of the disjoint union of Euclidean spaces

$$\mathsf{L}_T = \Big(\coprod_{\alpha \in V(T)} \mathsf{L}_T(\alpha)\Big)/\sim$$

by the equivalence relation generated by the edge gluings for all edges $\{\alpha, \beta\} \in E(T)$.

Example 2.4 For the tree T with a single vertex, L_T is a single point. For the tree T with two vertices (necessarily) connected by an edge, L_T is the cone over three points. For a general A_n–tree, see Section 2.3 below.

Remark 2.5 Arboreal singularities inherit two natural structures from their Euclidean space constituents.

(1) The Euclidean metric on each $\mathsf{L}_T(\alpha) \subset \mathsf{L}_T$ is respected by the edge gluings and hence induces a metric on L_T whose restriction to each $\mathsf{L}_T(\alpha) \subset \mathsf{L}_T$ is the original Euclidean metric.

(2) The positive dilation on each $\mathsf{L}_T(\alpha) \subset \mathsf{L}_T$ that sends $\{x_\gamma(\alpha)\} \mapsto \{r x_\gamma(\alpha)\}$, for $r \in \mathbb{R}_{>0}$, is also respected by the edge gluings and hence induces a positive dilation on L_T whose restriction to each $\mathsf{L}_T(\alpha) \subset \mathsf{L}_T$ is the original positive dilation.

The two structures satisfy the following evident compatibility.

On the one hand, there is a unique fixed point of positive dilation denoted by $0 \in \mathsf{L}_T$ which we will call the *central point* of L_T. It is contained in $\mathsf{L}_T(\alpha) \subset \mathsf{L}_T$, for all $\alpha \in V(T)$, with coordinates satisfying $x_\gamma(\alpha) = 0$, for all $\gamma \in V(T) \setminus \{\alpha\}$.

On the other hand, by the *arboreal link* $\mathsf{L}_T^{\text{link}} \subset \mathsf{L}_T$, we will mean the compact subspace of points unit distance from $0 \in \mathsf{L}_T$.

Positive dilation provides a canonical homeomorphism

$$\mathsf{L}_T^{\text{link}} \times \mathbb{R}_{>0} \xrightarrow{\sim} \mathsf{L}_T \setminus \{0\}$$

realizing the arboreal singularity as the cone over the link

$$\mathsf{L}_T \simeq \mathrm{Cone}(\mathsf{L}_T^{\text{link}}),$$

where for any space X, the cone is the quotient $\mathrm{Cone}(X) = X \times [0,1) \cup_{X\times\{0\}} \mathrm{pt}$.

Next we will record two useful lemmas regarding arboreal singularities.

For any $\alpha, \beta \in V(T)$, there is a unique minimal path in T connecting them. Suppose the path consists of k edges with successive adjacent vertices

$$\gamma_0 = \alpha,\ \gamma_1, \dots, \gamma_{k-1},\ \gamma_k = \beta \in V(T).$$

When $\alpha, \beta \in V(T)$ are adjacent, so that $k = 1$ and there are no intermediate vertices, the following lemma reduces to the $\{\alpha, \beta\}$–edge gluing.

Lemma 2.6 *The Euclidean spaces $\mathsf{L}_T(\alpha)$ and $\mathsf{L}_T(\beta)$ are glued inside of L_T along the closed quadrants where we identify points $\{x_\gamma(\alpha)\} \sim \{x_\gamma(\beta)\}$ whenever*

$$x_{\gamma_1}(\alpha) = x_\alpha(\beta),\ x_{\gamma_2}(\alpha) = x_{\gamma_1}(\beta), \dots, x_{\gamma_{k-1}}(\alpha) = x_{\gamma_{k-2}}(\beta),\ x_\beta(\alpha) = x_{\gamma_{k-1}}(\beta) \geq 0$$

and

$$x_\gamma(\alpha) = x_\gamma(\beta) \quad \textit{for all } \gamma \neq \alpha, \gamma_1, \dots, \gamma_{k-1}, \beta \in V(T).$$

Proof Note that since T is acyclic, the gluings for other edges play no role.

Let us proceed by induction on k.

For $k = 1$, this is simply the $\{\alpha, \beta\}$–edge gluing.

Suppose the assertion is established for $k-1$ so that $\mathsf{L}_T(\alpha)$ and $\mathsf{L}_T(\gamma_{k-1})$ are glued inside of L_T along the closed quadrants where we identify points $\{x_\gamma(\alpha)\} \sim \{x_\gamma(\gamma_{k-1})\}$ whenever

$$x_{\gamma_1}(\alpha) = x_\alpha(\gamma_{k-1}),\ x_{\gamma_2}(\alpha) = x_{\gamma_1}(\gamma_{k-1}), \dots, x_{\gamma_{k-1}}(\alpha) = x_{\gamma_{k-2}}(\gamma_{k-1}) \geq 0$$

and

$$x_\gamma(\alpha) = x_\gamma(\gamma_{k-1}) \quad \text{for all } \gamma \neq \alpha, \gamma_1, \dots, \gamma_{k-1} \in V(T).$$

Then it suffices to observe that the $\{\gamma_{k-1}, \beta\}$–edge gluing prescribes that $\mathsf{L}_T(\gamma_{k-1})$ and $\mathsf{L}_T(\beta)$ are glued inside of L_T where we identify points $\{x_\gamma(\gamma_{k-1})\} \sim \{x_\gamma(\beta)\}$ whenever

$$x_\beta(\gamma_{k-1}) = x_{\gamma_{k-1}}(\beta) \geq 0 \qquad \text{and} \qquad x_\gamma(\gamma_{k-1}) = x_\gamma(\beta) \quad \text{for all } \gamma \neq \gamma_{k-1}, \beta \in V(T).$$

Composing equations, we immediately obtain the asserted equations. □

By a *terminal vertex* of a tree T, we will mean a vertex contained in a unique edge. By an *internal vertex*, we will mean a vertex that is not a terminal vertex. (By this convention, if T consists of a single vertex alone, then the vertex is an internal vertex.)

Suppose T is a tree with $\tau \in V(T)$ a terminal vertex and $\{\tau, \alpha\} \in E(T)$ the unique edge containing τ. Introduce the tree T_τ where we delete the vertex τ and the edge $\{\tau, \alpha\}$.

Lemma 2.7 *There is a canonical homeomorphism*

$$\mathsf{L}_T \simeq (\mathsf{L}_{T_\tau} \times \mathbb{R}^{\{\tau\}}) \coprod_{\mathsf{L}_{T_\tau}(\alpha) \times \{0\}} (\mathsf{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}_{\le 0}).$$

Proof The edge gluing for the edge $\{\tau, \alpha\} \in E(T)$ attaches the Euclidean space

$$\mathsf{L}_T(\tau) \simeq \mathsf{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}$$

to the product $\mathsf{L}_{T_\tau} \times \mathbb{R}^{\{\tau\}}$ along the closed subspace

$$\mathsf{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}_{\ge 0} \subset \mathsf{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}.$$

The gluing of the lemma results from removing the redundant open subspace

$$\mathsf{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}_{> 0} \subset \mathsf{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}$$

and only attaching the closed complement

$$\mathsf{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}_{\le 0} \subset \mathsf{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}. \qquad \square$$

Remark 2.8 The choice of a terminal vertex is not canonical, but the collection of all terminal vertices is. For a more invariant statement, one could simultaneously apply the above lemma to all terminal vertices (as long as there are three or more vertices).

Corollary 2.9 *The arboreal link $\mathsf{L}_T^{\text{link}}$ is homotopy equivalent to a bouquet of $|V(T)|$ spheres each of dimension $|V(T)| - 2$.*

Proof Let us adopt the setting and notation of the previous lemma.

By induction, $\mathsf{L}_{T_\tau}^{\text{link}}$ is homotopy equivalent to a bouquet of $|V(T)| - 1$ spheres each of dimension $|V(T)| - 3$, and hence the suspension

$$\Sigma(\mathsf{L}_{T_\tau}^{\text{link}}) \simeq (\mathsf{L}_{T_\tau} \times \mathbb{R}^{\{\tau\}})^{\text{link}}$$

is homotopy equivalent to a bouquet of $|V(T)|-1$ spheres each of dimension $|V(T)|-2$.

By the previous lemma, $\mathsf{L}_T^{\text{link}}$ results from starting with $\Sigma(\mathsf{L}_{T_\tau}^{\text{link}})$ and attaching the $(|V(T)|-2)$–cell $L_{T_\tau}(\alpha)$ along the inclusion of its boundary $(|V(T)|-3)$–sphere

$$\mathsf{L}_{T_\tau}(\alpha)^{\text{link}} \subset \Sigma(\mathsf{L}_{T_\tau}^{\text{link}})$$

induced by the inclusion $\mathsf{L}_{T_\tau}(\alpha) \subset \mathsf{L}_{T_\tau}$. Therefore $\mathsf{L}_T^{\mathrm{link}}$ is homotopy equivalent to a bouquet of $|V(T)|$ spheres each of dimension $|V(T)|-2$. □

2.2 Combinatorial description

Let us first review some terminology.

Given a graph G, by a *subgraph* $S \subset G$, we will mean a full subgraph (or vertex-induced subgraph) in the sense that its vertices are a subset $V(S) \subset V(G)$ and its edges are the subset $E(S) \subset E(G)$ such that $\{\alpha, \beta\} \in E(S)$ if and only if $\{\alpha, \beta\} \in E(G)$ and $\alpha, \beta \in V(S)$. By the *complementary subgraph* $G \setminus S \subset G$, we will mean the full subgraph on the complementary vertices $V(T \setminus S) = V(T) \setminus V(S)$.

Given a tree T, any subgraph $S \subset T$ is a disjoint union of trees. By a *subtree* $S \subset T$, we will mean a subgraph that is a tree. The complementary subgraph $T \setminus S \subset T$ is not necessarily a tree but in general a disjoint union of subtrees. Given a subtree $S \subset T$, and a vertex $\alpha \in V(T \setminus S)$, there is a unique vertex $\gamma \in V(S)$ *nearest* to α.

Given a tree T, by a *quotient tree* $T \twoheadrightarrow Q$, we will mean a tree Q with a surjection $V(T) \twoheadrightarrow V(Q)$ such that each fiber comprises the vertices of a subtree of T. We will refer to such subtrees as the *fibers* of the quotient $T \twoheadrightarrow Q$. Given a vertex $\alpha \in V(T)$, we will sometimes write $\bar{\alpha} \in V(Q)$ for its image, and $T_{\bar{\alpha}} \subset T$ for the fiber containing α.

By a *partition* of a tree T, we will mean a collection of subtrees $T_i \in T$, for $i \in I$, that are disjoint $V(T_i) \cap V(T_j) = \varnothing$, for $i \neq j$, and cover $V(T) = \coprod_{i \in I} V(T_i)$. Note that the data of a quotient $T \twoheadrightarrow Q$ is equivalent to the partition of T into the fibers.

Now let T be a tree with arboreal singularity L_T. A point $x \in \mathsf{L}_T$ defines the following invariants.

First, we introduce the function

$$v_x \colon V(T) \to \{\text{yes}, \text{no}\}$$

such that $v_x(\alpha) = \text{yes}$ when $x \in \mathsf{L}_T(\alpha) \subset \mathsf{L}_T$, and $v_x(\alpha) = \text{no}$ when $x \notin \mathsf{L}_T(\alpha) \subset \mathsf{L}_T$.

Define the subgraph $S \subset T$ to consist of those vertices $\alpha \in V(T)$ such that $v_x(\alpha) = \text{yes}$, and those edges $\{\alpha, \beta\} \in E(T)$ such that $v_x(\alpha) = v_x(\beta) = \text{yes}$.

Lemma 2.10 *S is a tree.*

Proof We must show S is connected (since S is a subgraph of T, it is clearly acyclic). Suppose $\alpha, \beta \in V(S)$ so that $x \in \mathsf{L}_T(\alpha) \cap \mathsf{L}_T(\beta) \subset \mathsf{L}_T$. Suppose the unique minimal path in T connecting them consists of k edges with successive vertices

$\gamma_0 = \alpha, \gamma_1, \dots, \gamma_{k-1}, \gamma_k = \beta \in V(T)$. By Lemma 2.6, for all $i = 1, \dots, k-1$, we see that x is contained in the intermediate Euclidean spaces $\mathsf{L}_T(\gamma_i) \subset \mathsf{L}_T$, and thus $\alpha, \beta \in V(S)$ are connected by a path in S. □

Remark 2.11 Consider the complementary graph $T \setminus S$ that consists of those vertices $\alpha \in V(T)$ such that $v_x(\alpha) = \text{no}$, and those edges $\{\alpha, \beta\} \in E(T)$ such that $v_x(\alpha) = v_x(\beta) = \text{no}$. In general, it is the disjoint union of subtrees $N_i \subset T$, for $i \in I$, but not necessarily connected.

Let us continue with the invariants of a point $x \in \mathsf{L}_T$.

Observe that $\{\alpha, \beta\} \in E(S)$ means $x \in \mathsf{L}_T(\alpha) \cap \mathsf{L}_T(\beta) \subset \mathsf{L}_T$, and the $\{\alpha, \beta\}$–edge gluing implies an equality of nonnegative coordinates $x_\beta(\alpha) = x_\alpha(\beta) \geq 0$ evaluated at x.

Next, we introduce the function

$$e_x \colon E(S) \to \{0, +\}$$

such that $e_x(\{\alpha, \beta\}) = 0$ when $x_\beta(\alpha) = x_\alpha(\beta) = 0$ evaluated at x, and $e_x(\{\alpha, \beta\}) = +$ when $x_\beta(\alpha) = x_\alpha(\beta) > 0$ evaluated at x.

Define the tree R to be the quotient of S where we contract those edges $\{\alpha, \beta\} \in E(S)$ such that $e(\{\alpha, \beta\}) = +$. Therefore the vertex set $V(R)$ is the quotient of $V(S)$ where we identify vertices $\alpha, \beta \in V(S)$ that can be connected by a path through edges $\{\alpha, \beta\} \in E(S)$ such that $e_x(\{\alpha, \beta\}) = +$. The edge set $E(R)$ can be taken to consist of those edges $\{\alpha, \beta\} \in E(S)$ such that $e(\{\alpha, \beta\}) = 0$.

Altogether, we see that the point $x \in L_T$ defines a correspondence of trees

$$R \overset{q}{\twoheadleftarrow} S \overset{p}{\hookrightarrow} T$$

where p is the inclusion of a subtree, and q is a quotient map of trees. Here and in what follows, we take such correspondences up to the strictest notion of equivalence: two such correspondences

$$R \overset{q}{\twoheadleftarrow} S \overset{p}{\hookrightarrow} T \quad \text{and} \quad R \overset{q'}{\twoheadleftarrow} S' \overset{p'}{\hookrightarrow} T$$

are equivalent if and only if there is a bijection $b \colon S' \xrightarrow{\sim} S$ such that $q' = q \circ b$ and $p' = p \circ b$.

Here is a useful reformulation of the data of such a correspondence.

Lemma 2.12 *The set of correspondences*

$$R \overset{q}{\twoheadleftarrow} S \overset{p}{\hookrightarrow} T$$

where p is the inclusion of a subtree, and q is a quotient map of trees, is in natural bijection with the set of partitions of T into a collection of subtrees

$$(\{N_i\}_{i\in I}, \{F_j\}_{j\in J})$$

such that J is nonempty, and for any $i \in I$, the complement $T \setminus N_i$ is connected.

Proof Given such a correspondence, define the subtrees $N_i \subset T$, for $i \in I$, to be the connected components of the complementary graph $T \setminus S$. Define the subtrees $F_j \subset T$, for $j \in J$, to be the fibers of the quotient map $q\colon S \twoheadrightarrow R$. Since S is nonempty, J is nonempty.

Suppose there is some $i \in I$ such that $T \setminus N_i$ is disconnected. Since S is connected, it lies in one of the components of $T \setminus N_i$. But then there is another component of $T \setminus N_i$ contained in $T \setminus S$ and connected to N_i, contradicting that N_i itself is a component of $T \setminus S$.

Conversely, given such a partition, the full subgraph $S = \coprod_{j\in J} F_j \subset T$ is nonempty since J is nonempty. Furthermore, S is connected, and hence a subtree, else there is some $i \in I$ such that $T \setminus N_i$ is disconnected. Finally, take the quotient map $q\colon S \twoheadrightarrow R$ to be that with fibers given by $F_j \subset S$, for $j \in J$. □

We will show that the arboreal singularity L_T is the cone over a regular cell complex with each cell the subspace of points leading to a given correspondence. We will arrive at this in Theorem 2.20 below, but first observe that such subspaces and their closures are naturally convex polyhedra.

Definition 2.13 Let T be a tree with associated arboreal singularity L_T.

Define $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T$ to be the subspace of points leading to a given correspondence

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T)$$

where i is the inclusion of a subtree, and q is a quotient map of trees.

Define the rank $\rho(\mathfrak{p}) = |V(T)| - |V(R)|$.

Proposition 2.14 *Let T be a tree with associated arboreal singularity L_T.*

The subspace $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T$ of points leading to a given correspondence

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T)$$

is an open cell of dimension $\rho(\mathfrak{p}) = |V(T)| - |V(R)|$. Its closure is naturally a convex polyhedron in a Euclidean space, and in fact cut out by explicit equalities and inequalities on coordinate functions (appearing in the proof below).

Proof Fix any $\alpha \in V(S)$ and observe that $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T(\alpha)$ since $v_x(\alpha) = \text{yes}$.

Recall that points $x \in \mathsf{L}_T(\alpha)$ consist of tuples of real numbers

$$\{x_\gamma(\alpha)\} \quad \text{with } \gamma \in V(T) \setminus \{\alpha\}.$$

We claim that those points $x \in \mathsf{L}_T(\alpha)$ that lie in $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T(\alpha)$ are precisely cut out by the following equations on their coordinates $x_\gamma(\alpha)$ depending on the location of γ:

(1) Suppose γ lies in the fiber $F_{\overline{\alpha}} \subset S$ containing α. Then we have $x_\gamma(\alpha) > 0$.

(2) Suppose γ lies in S but not in the fiber $F_{\overline{\alpha}} \subset S$ containing α. Suppose also that γ is the nearest vertex to α within the fiber $F_{\overline{\gamma}} \subset S$ containing γ. Then we have $x_\gamma(\alpha) = 0$.

(3) Suppose γ lies in S but not in the fiber $F_{\overline{\alpha}} \subset S$ containing α. Suppose also that γ is not the nearest vertex to α in the fiber $F_{\overline{\gamma}} \subset S$ containing γ. Then we have $x_\gamma(\alpha) > 0$.

(4) Suppose γ lies in $T \setminus S$ so that γ is in some subtree $N_i \subset T \setminus S$. Suppose also that γ is the nearest vertex to α within N_i. Then we have $x_\gamma(\alpha) < 0$.

(5) Suppose γ lies in $T \setminus S$ so that γ is in some subtree $N_i \subset T \setminus S$. Suppose also that γ is not the nearest vertex to α within N_i. Then we allow $x_\gamma(\alpha)$ to be arbitrary.

To confirm this, on the one hand, by Lemma 2.6, if $x_\gamma(\alpha) < 0$, then $x \notin \mathsf{L}_T(\gamma)$, and so γ does not lie in S. On the other hand, suppose γ lies in $T \setminus S$. Consider the minimal path connecting α and γ, and let γ' be the closest point to α that lies on the path and in $T \setminus S$. By Lemma 2.6, we have $x_{\gamma'}(\alpha) < 0$, and $x_\gamma(\alpha)$ can be arbitrary if $\gamma \neq \gamma'$. Thus $x \in \mathsf{L}_T(\alpha)$ leads to the right half of the correspondence,

$$S \hookrightarrow T,$$

if and only if the following coarser equations hold:

(1′) Suppose γ lies in the fiber $F_{\overline{\alpha}} \subset S$ containing α. Then we have $x_\gamma(\alpha) \geq 0$.

(2′) Suppose γ lies in S but not in the fiber $F_{\overline{\alpha}} \subset S$ containing α. Suppose also that γ is the nearest vertex to α within the fiber $F_{\overline{\gamma}} \subset S$ containing γ. Then we have $x_\gamma(\alpha) \geq 0$.

(3′) Suppose γ lies in S but not in the fiber $F_{\overline{\alpha}} \subset S$ containing α. Suppose also that γ is not the nearest vertex to α in the fiber $F_{\overline{\gamma}} \subset S$ containing γ. Then we have $x_\gamma(\alpha) \geq 0$.

(4) Suppose γ lies in $T \setminus S$ so that γ is in some subtree $N_i \subset T \setminus S$. Suppose also that γ is the nearest vertex to α within N_i. Then we have $x_\gamma(\alpha) < 0$.

(5) Suppose γ lies in $T\setminus S$ so that γ is in some subtree $N_i \subset T\setminus S$. Suppose also that γ is not the nearest vertex to α within N_i. Then we allow $x_\gamma(\alpha)$ to be arbitrary.

Now it remains to determine that in cases $(1')$ and $(3')$ the coordinate $x_\gamma(\alpha)$ should be positive as in cases (1) and (3), and in case $(2')$ it should be zero as in case (2).

Again apply Lemma 2.6 to see that for $x \in \mathsf{L}_T(\alpha)$ satisfying the above coarser equations, and for $\gamma \in V(S)$ so that $x \in \mathsf{L}_T(\gamma)$, we have the equality of functions $x_\gamma(\alpha) = x_{\tilde\gamma}(\gamma)$, where $\tilde\gamma \in V(S)$ is the vertex adjacent to γ and one edge nearer to α. Thus by definition $x \in \mathsf{L}_T(\alpha)$ also leads to the left half of the correspondence,

$$R \twoheadleftarrow S,$$

if and only if in cases $(1')$ and $(3')$ the coordinate $x_\gamma(\alpha)$ is positive, and in case $(2')$ it is zero.

Finally, for the dimension assertion, recall that $\dim \mathsf{L}_T(\alpha) = |V(T)| - 1$, and note that there are precisely $|V(R)| - 1$ equalities in the above equations given by case (2). □

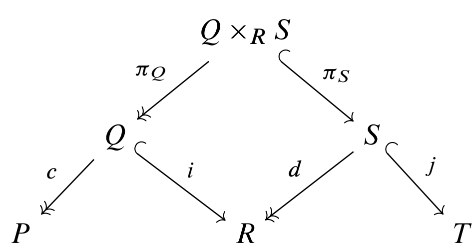
Next we will calculate the closure relations among the cells. Consider a pair of correspondences of trees

$$\mathfrak{q} = (P \overset{c}{\twoheadleftarrow} Q \overset{i}{\hookrightarrow} R) \quad \text{and} \quad \mathfrak{p} = (R \overset{d}{\twoheadleftarrow} S \overset{j}{\hookrightarrow} T)$$

where i and j are inclusions of subtrees, and c and d are quotient maps of trees. Define the composed correspondence

$$\mathfrak{q} \circ \mathfrak{p} = (P \overset{\tilde c}{\twoheadleftarrow} Q \times_R S \overset{\tilde j}{\hookrightarrow} T)$$

by forming the fiber product



and setting $\tilde c = c \circ \pi_Q$ and $\tilde j = j \circ \pi_S$, where π_Q and π_S are the natural projections.

Definition 2.15 Let T be a tree.

(1) Let $\mathfrak{P}_T$ denote the poset whose elements are correspondences of trees

$$\mathfrak{p} = (R \overset{d}{\twoheadleftarrow} S \overset{j}{\hookrightarrow} T)$$

where j is the inclusion of a subtree, and d is a quotient map of trees.

We define the order relation on correspondences

$$\mathfrak{p} = (R \xleftarrow{d} S \xhookrightarrow{j} T) \geq \mathfrak{p}' = (R' \xleftarrow{d'} S' \xhookrightarrow{j'} T)$$

and say the second correspondence refines the first, if there is a third correspondences of trees

$$\mathfrak{q} = (R \xleftarrow{c} Q \xhookrightarrow{i} R')$$

where i is the inclusion of a subtree, and c is a quotient map of trees, such that

$$\mathfrak{p} = \mathfrak{q} \circ \mathfrak{p}'.$$

(2) Let $\mathfrak{Q}_T$ denote the poset whose elements are partitions of T into a collection of subtrees

$$(\{N_i\}_{i \in I}, \{F_j\}_{j \in J})$$

such that J is nonempty, and for any $i \in I$, the complement $T \setminus N_i$ is connected.

We define the order relation on partitions

$$(\{N_i\}_{i \in I}, \{F_j\}_{j \in J}) \geq (\{N'_k\}_{k \in K}, \{F'_\ell\}_{\ell \in L})$$

and say the second partition refines the first, if each N'_k lies in some N_i and each F'_ℓ lies in some N_i or some F_j.

Remark 2.16 There is the unique minimum $\mathfrak{p}_0 \in \mathfrak{P}_T$ given by the identity correspondence

$$\mathfrak{p}_0 = (T \xleftarrow{=} T \xhookrightarrow{=} T).$$

It indexes the cell $\mathsf{L}_{\mathfrak{p}_0} \subset \mathsf{L}_T$ comprising the central point $0 \in \mathsf{L}_T$ alone.

We have the following upgrading of Lemma 2.12.

Lemma 2.17 *The bijection of Lemma 2.12 is a poset isomorphism $\mathfrak{P}_T \simeq \mathfrak{Q}_T$.*

Proof Suppose

$$\mathfrak{p} = (R \xleftarrow{d} S \xhookrightarrow{j} T) \quad \text{and} \quad \mathfrak{p}' = (R' \xleftarrow{d'} S' \xhookrightarrow{j'} T)$$

satisfy $\mathfrak{p} \geq \mathfrak{p}'$ because $\mathfrak{p} = \mathfrak{q} \circ \mathfrak{p}'$ with

$$\mathfrak{q} = (R \xleftarrow{c} Q \xhookrightarrow{i} R').$$

Then the corresponding partitions satisfy

$$(\{N_i\}_{i \in I}, \{F_j\}_{j \in J}) \geq (\{N'_k\}_{k \in K}, \{F'_\ell\}_{\ell \in L})$$

since j factors through j', so each N'_k lies in some N_i, and d factors through the base change of d' by the inclusion i, so each F'_ℓ lies in some F_j.

Conversely, suppose

$$(\{N_i\}_{i\in I}, \{F_j\}_{j\in J}) \geq (\{N'_k\}_{k\in K}, \{F'_\ell\}_{\ell\in L})$$

and consider the corresponding correspondences

$$\mathfrak{p} = (R \overset{d}{\twoheadleftarrow} S \overset{j}{\hookrightarrow} T) \quad \text{and} \quad \mathfrak{p}' = (R' \overset{d'}{\twoheadleftarrow} S' \overset{j'}{\hookrightarrow} T).$$

Since each N'_k lies in some N_i, and $S' = T \setminus \coprod'_{k\in K} N'_k$ and $S = T \setminus \coprod_{i\in I} N_i$, we have $S \subset S'$ or, in other words, j factors through j'. Define the quotient map $S \twoheadrightarrow Q$ to be that with fibers given by $F'_\ell \cap S$, for $\ell \in L$. Then the inclusion $S \hookrightarrow S'$ descends to an inclusion $i\colon Q \hookrightarrow R'$. Finally, the quotient map $d\colon S \twoheadrightarrow R$ factors through $S \twoheadrightarrow Q$, providing a quotient map $c\colon Q \twoheadrightarrow R$, since each F'_ℓ lies in some F_j. Thus $\mathfrak{p} \geq \mathfrak{p}'$ because $\mathfrak{p} = \mathfrak{q} \circ \mathfrak{p}'$ with

$$\mathfrak{q} = (R \overset{c}{\twoheadleftarrow} Q \overset{i}{\hookrightarrow} R'). \qquad \square$$

Proposition 2.18 *Two elements $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}_T$ indexing cells $\mathsf{L}_T(\mathfrak{p}), \mathsf{L}_T(\mathfrak{p}') \subset \mathsf{L}_T$ satisfy $\mathfrak{p} \geq \mathfrak{p}'$ if and only if $\mathsf{L}_T(\mathfrak{p}')$ intersects the closure of $\mathsf{L}_T(\mathfrak{p})$. If this holds, then in fact $\mathsf{L}_T(\mathfrak{p}')$ lies in the closure of $\mathsf{L}_T(\mathfrak{p})$.*

Proof We will proceed in the language of partitions as in Lemmas 2.12 and 2.17 though one could equally well translate the arguments back into the language of correspondences.

Consider two elements $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}_T$ representing respective partitions

$$(\{N_i\}_{i\in I}, \{F_j\}_{j\in J}) \quad \text{and} \quad (\{N'_k\}_{k\in K}, \{F'_\ell\}_{\ell\in L}),$$

comprising subtrees of complementary components and fibers.

Suppose $\mathfrak{p} \geq \mathfrak{p}'$ so that the partition associated to $\mathfrak{p}'$ refines that associated to $\mathfrak{p}$. We will show that $\mathsf{L}_T(\mathfrak{p}')$ is in the closure of $\mathsf{L}_T(\mathfrak{p})$.

Choose any fiber F_j. Then F_j contains some fiber F'_ℓ. Fix any vertex $\alpha \in V(F'_\ell)$. Then we have $\alpha \in V(F_j)$ as well. Thus we have

$$\mathsf{L}_T(\mathfrak{p}), \mathsf{L}_T(\mathfrak{p}') \subset \mathsf{L}_T(\alpha).$$

Returning to the proof of Proposition 2.14, we find explicit equations for these subspaces to contain a point $x \in \mathsf{L}_T(\alpha)$ in terms of its coordinates

$$\{x_\gamma(\alpha)\} \quad \text{with } \gamma \in V(T) \setminus \{\alpha\}.$$

We have the following possibilities depending on the location of γ. The fact that the partition associated to $\mathfrak{p}'$ refines that associated to $\mathfrak{p}$ implies simple constraints on the location of γ with respect to the partitions and α.

(1) Suppose γ lies in the fiber $F_{\overline{\alpha}} \subset S$ containing α so that

$$x \in \mathsf{L}_T(\mathfrak{p}) \implies x_\gamma(\alpha) > 0.$$

Then γ must lie in some fiber $F'_{\overline{\gamma}} \subset F_{\overline{\alpha}}$ so that

$$x \in \mathsf{L}_T(\mathfrak{p}') \implies x_\gamma(\alpha) \geq 0.$$

(2) Suppose γ lies in S but not in the fiber $F_{\overline{\alpha}} \subset S$ containing α. Suppose also that γ is the nearest vertex to α within the fiber $F_{\overline{\gamma}} \subset S$ containing γ so that

$$x \in \mathsf{L}_T(\mathfrak{p}) \implies x_\gamma(\alpha) = 0.$$

Then γ must lie in some fiber $F'_{\overline{\gamma}} \subset F_{\overline{\gamma}}$ and γ must be the nearest vertex to α within the fiber $F'_{\overline{\gamma}} \subset F_{\overline{\gamma}}$ so that

$$x \in \mathsf{L}_T(\mathfrak{p}') \implies x_\gamma(\alpha) = 0.$$

(3) Suppose γ lies in S but not in the fiber $F_{\overline{\alpha}} \subset S$ containing α. Suppose also that γ is not the nearest vertex to α in the fiber $F_{\overline{\gamma}} \subset S$ containing γ so that

$$x \in \mathsf{L}_T(\mathfrak{p}) \implies x_\gamma(\alpha) > 0.$$

Then γ must lie in some fiber $F'_{\overline{\gamma}} \subset F_{\overline{\gamma}}$ so that

$$x \in \mathsf{L}_T(\mathfrak{p}') \implies x_\gamma(\alpha) \geq 0.$$

(4) Suppose γ lies in $T \setminus S$ so that γ is in some subtree $N_i \subset T \setminus S$. Suppose also that γ is the nearest vertex to α within N_i so that

$$x \in \mathsf{L}_T(\mathfrak{p}) \implies x_\gamma(\alpha) < 0.$$

Then either (a) γ must lie in some fiber $F'_{\overline{\gamma}} \subset N_i$ and γ must be the nearest vertex to α within the fiber $F'_{\overline{\gamma}} \subset N_i$ so that

$$x \in \mathsf{L}_T(\mathfrak{p}') \implies x_\gamma(\alpha) = 0$$

or (b) γ must lie in some subtree $N'_k \subset N_i$ and γ must be the nearest vertex to α within the subtree $N'_k \subset N_i$ so that

$$x \in \mathsf{L}_T(\mathfrak{p}') \implies x_\gamma(\alpha) < 0.$$

(5) Suppose γ lies in $T \setminus S$ so that γ is in some subtree $N_i \subset T \setminus S$. Suppose also that γ is not the nearest vertex to α within N_i so that

$$x \in \mathsf{L}_T(\mathfrak{p}) \implies x_\gamma(\alpha) \text{ arbitrary.}$$

Thus we need not worry about the possible constraints imposed by $x \in \mathsf{L}_T(\mathfrak{p}')$.

From the above equations, we conclude that $\mathsf{L}_T(\mathfrak{p}')$ is in the closure of $\mathsf{L}_T(\mathfrak{p})$.

Conversely, suppose $\mathsf{L}_T(\mathfrak{p}')$ intersects the closure of $\mathsf{L}_T(\mathfrak{p})$ at some point $x \in \mathsf{L}_T$. We will show that the partition of $\mathfrak{p}'$ refines that of $\mathfrak{p}$. To begin, choose any $\alpha \in V(T)$ such that $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T(\alpha)$. Hence the closure of $\mathsf{L}_T(\mathfrak{p})$ also lies in $\mathsf{L}_T(\alpha)$. Thus $\mathsf{L}_T(\mathfrak{p}')$ intersects $\mathsf{L}_T(\alpha)$, and hence $\mathsf{L}_T(\mathfrak{p}') \subset \mathsf{L}_T(\alpha)$. Thus we have $x \in \mathsf{L}_T(\alpha)$ with coordinates

$$\{x_\gamma(\alpha)\} \quad \text{with } \gamma \in V(T) \setminus \{\alpha\}.$$

Now we will proceed by contradiction. Suppose some subtree $N'_k \subset T \setminus S'$ is not contained in any subtree $N_i \subset T \setminus S$. Then the vertex β within N'_k closest to α must lie in some fiber F_j. But turning to the equations for $x_\beta(\alpha)$ of Proposition 2.14, and applying case (1), (2) or (3) to $\mathsf{L}_T(\mathfrak{p})$ and case (4) to $\mathsf{L}_T(\mathfrak{p}')$, we find

$$x \in \overline{\mathsf{L}_T(\mathfrak{p})} \implies x_\beta(\alpha) \geq 0 \qquad \text{and} \qquad x \in \mathsf{L}_T(\mathfrak{p}') \implies x_\beta(\alpha) < 0$$

and hence a contradiction.

Next suppose some fiber F'_ℓ is not contained in any subtree N_i or fiber F_j. Then there is a vertex β within F'_ℓ that is not the nearest to α within F'_ℓ but is the nearest to α within whichever subtree N_i or fiber F_j contains β. Again turning to the equations for $x_\beta(\alpha)$ of Proposition 2.14, and applying case (2) or (4) to $\mathsf{L}_T(\mathfrak{p})$ and case (1) or (3) to $\mathsf{L}_T(\mathfrak{p}')$, we find

$$x_\beta(\alpha) \in \overline{\mathsf{L}_T(\mathfrak{p})} \implies x_\beta(\alpha) \leq 0 \qquad \text{and} \qquad x_\beta(\alpha) \in \mathsf{L}_T(\mathfrak{p}') \implies x_\beta(\alpha) > 0$$

and hence a contradiction.

Finally, for the last assertion, we have shown that if $\mathfrak{p} \geq \mathfrak{p}'$ then $\mathsf{L}_T(\mathfrak{p}')$ lies in the closure of $\mathsf{L}_T(\mathfrak{p})$, and also that if $\mathsf{L}_T(\mathfrak{p}')$ intersects the closure of $\mathsf{L}_T(\mathfrak{p})$ then $\mathfrak{p} \geq \mathfrak{p}'$. Thus we conclude that if $\mathsf{L}_T(\mathfrak{p}')$ intersects the closure of $\mathsf{L}_T(\mathfrak{p})$ then $\mathsf{L}_T(\mathfrak{p}')$ lies in the closure of $\mathsf{L}_T(\mathfrak{p})$. □

Recall from Remark 2.5 that the arboreal singularity L_T inherits a natural metric from its Euclidean constituents, and the arboreal link $\mathsf{L}_T^{\text{link}} \subset \mathsf{L}_T$ refers to the compact subspace of points unit distance from the center $0 \in \mathsf{L}_T$. Positive dilation provides a canonical homeomorphism

$$\mathsf{L}_T^{\text{link}} \times \mathbb{R}_{>0} \xrightarrow{\sim} \mathsf{L}_T \setminus \{0\}$$

realizing the arboreal singularity as the cone over the link

$$\mathsf{L}_T \simeq \text{Cone}(\mathsf{L}_T^{\text{link}}).$$

Note that all of the subspaces $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T$ are invariant under positive dilation.

Definition 2.19 Fix an element $\mathfrak{p} \in \mathfrak{P}_T \setminus \{\mathfrak{p}_0\}$.

Introduce the open cell

$$\mathsf{L}_T^{\mathrm{link}}(\mathfrak{p}) = \mathsf{L}_T^{\mathrm{link}} \cap \mathsf{L}_T(\mathfrak{p})$$

of dimension $\rho(\mathfrak{p}) - 1$.

Now the two preceding propositions coupled with general theory [5; 13] immediately imply the following. Recall that a *finite regular cell complex* is a Hausdorff space X with a finite collection of closed cells $c_i \subset X$ whose interiors $c_i^\circ \subset c_i$ provide a partition of X and boundaries $\partial c_i \subset X$ are unions of cells. A finite regular cell complex X has the *intersection property* if the intersection of any two cells $c_i, c_j \subset X$ is either another cell or empty. (This holds for example if the closed cells are convex polyhedra in Euclidean spaces glued along convex subspaces as found in Proposition 2.14.) The *order complex* of a poset is the natural simplicial complex with simplices the finite totally ordered chains of the poset. (Other useful references include [4; 25].)

Theorem 2.20 *Let T be a tree with associated arboreal link $\mathsf{L}_T^{\mathrm{link}}$.*

The decomposition of the arboreal link $\mathsf{L}_T^{\mathrm{link}}$ into the cells $\mathsf{L}_T^{\mathrm{link}}(\mathfrak{p})$, for $\mathfrak{p} \in \mathfrak{P}_T \setminus \{\mathfrak{p}_0\}$, is a regular cell complex with the intersection property. It is homeomorphic to the order complex of $\mathfrak{P}_T \setminus \{\mathfrak{p}_0\}$.

Before continuing on, let us also record the local structure of arboreal singularities.

Definition 2.21 Fix an element $\mathfrak{p} \in \mathfrak{P}_T$.

(1) Introduce the poset

$$\mathfrak{P}_T(\geq \mathfrak{p}) = \{\mathfrak{q} \in \mathfrak{P}_T \mid \mathfrak{q} \geq \mathfrak{p}\}$$

equipped with the induced partial order.

(2) Introduce the open neighborhood

$$\mathsf{L}_T(\geq \mathfrak{p}) = \coprod_{\mathfrak{q} \in \mathfrak{P}_T(\geq \mathfrak{p})} \mathsf{L}_T(\mathfrak{q}) \subset \mathsf{L}_T$$

of the cell $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T$.

Lemma 2.22 *Given an element $\mathfrak{p} \in \mathfrak{P}_T$ representing a correspondence*

$$\mathfrak{p} = (R \overset{d}{\twoheadleftarrow} S \overset{j}{\hookrightarrow} T)$$

the natural poset map is an isomorphism

$$\mathfrak{P}_R \xrightarrow{\sim} \mathfrak{P}_T(\geq \mathfrak{p}) \quad \textit{where } \mathfrak{q} \mapsto \mathfrak{q} \circ \mathfrak{p}.$$

Proof The map is surjective by the definition of the partial order and we must show it is injective. Suppose we have $\mathfrak{q}, \mathfrak{q}' \in \mathfrak{P}_R$ representing correspondences

$$\mathfrak{q} = (P \overset{d}{\twoheadleftarrow} Q \overset{j}{\hookrightarrow} R) \quad \text{and} \quad \mathfrak{q}' = (P' \overset{d}{\twoheadleftarrow} Q' \overset{j}{\hookrightarrow} R)$$

such that $\mathfrak{q} \circ \mathfrak{p} \simeq \mathfrak{q}' \circ \mathfrak{p}$. So we may assume that $P = P'$ and need to show that $Q, Q' \subset R$ are the same subset. But $\mathfrak{q} \circ \mathfrak{p} \simeq \mathfrak{q}' \circ \mathfrak{p}$ implies that $Q \times_R S = Q' \times_R S \subset S$, and hence the surjection $S \twoheadrightarrow R$ implies that $Q = Q' \subset S$. □

Corollary 2.23 *Let T be a tree with associated arboreal singularity L_T.*

Fix an element $\mathfrak{p} \in \mathfrak{P}_T$ indexing the cell $\mathsf{L}_T(\mathfrak{p}) \subset \mathsf{L}_T$ and an open neighborhood $\mathsf{L}_T(\geq \mathfrak{p}) \subset \mathsf{L}_T$.

The poset isomorphism

$$\mathfrak{P}_R \xrightarrow{\sim} \mathfrak{P}_T(\geq \mathfrak{p})$$

induces a homeomorphism

$$\mathsf{L}_T(\mathfrak{p}) \times \mathsf{L}_R \xrightarrow{\sim} \mathsf{L}_T(\geq \mathfrak{p}).$$

2.3 Example: A_n–trees

By the *A_n–tree T_n*, we will mean the tree with n vertices labeled $v_1, \dots, v_n$ and an edge connecting the vertices v_i and v_{i+1}, for all $i = 1, \dots, n-1$.

Let Δ^n denote the n–simplex. Let $[n] = \{0, 1, \dots, n\}$ denote its vertices so that the subsimplices of Δ^n are in natural bijection with nonempty subsets of $[n]$.

Let $\mathrm{sk}_{n-2}\,\Delta^n$ denote the $(n-2)$–skeleton of Δ^n. The subsimplices of $\mathrm{sk}_{n-2}\,\Delta^n$ are in natural bijection with nonempty subsets of $[n]$ containing at most $n-1$ elements.

Proposition 2.24 *There is an identification of regular cell complexes*

$$\mathsf{L}_{T_n} \simeq \mathrm{Cone}(\mathrm{sk}_{n-2}\,\Delta^n).$$

Proof Let $\mathfrak{P}_n$ denote the poset of nonempty subsets of $[n]$ containing at most $n-1$ elements with the standard partial order: for $A, A' \subset [n]$, we set $A \geq A'$ if and only if $A \supset A'$.

By Theorem 2.20, it suffices to establish an isomorphism of posets

$$\varphi\colon \mathfrak{P}_{T_n} \setminus \{\mathfrak{p}_0\} \xrightarrow{\sim} \mathfrak{P}_n.$$

It will be more straightforward to pass to complements and think of $\mathfrak{P}_n$ as the poset of proper subsets of $[n]$ containing at least two elements with the opposite partial order: for $A, A' \subset [n]$, we have $A \geq A'$ if and only if $A \subset A'$.

Thus we will associate a subset $\varphi(\mathfrak{p}) \subset [n] = \{0, 1, \dots, n\}$ of at least two elements to each correspondence

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T_n)$$

where i is the inclusion of a subtree, and q is a quotient map of trees. (The identity correspondence $\mathfrak{p}_0 \in \mathfrak{P}_{T_n}$ will go to the whole subset $[n] \subset [n]$.)

It is useful to introduce the A_{n+2}–tree $\widetilde{T}_n$ with vertices $V(\widetilde{T}_n) = V(T_n) \cup \{v_0, v_{n+1}\}$ and an additional edge connecting v_0 and v_1 and another connecting v_n and v_{n+1}.

Following Lemma 2.17, we can think of elements of $\mathfrak{P}_{T_n}$ equally well as partitions of $\widetilde{T}_n$ into connected subsets $N_0, F_1, \dots, F_r, N_n$ with $v_0 \in N_0$, $v_n \in N_n$, and $i(S) = F_1 \cup \dots \cup F_r$.

We will identify the elements of $[n] = \{0, 1, \dots, n\}$ with the edges of $\widetilde{T}_n$ by matching $i \in [n]$ with the edge connecting v_i and v_{i+1}, for all $i = 0, \dots, n$.

Now we define φ by including the element $i \in \varphi(\mathfrak{p})$ if and only if v_i and v_{i+1} are in different parts of the partition of $\widetilde{T}$ corresponding to $\mathfrak{p}$. (In particular, we have $\varphi(\mathfrak{p}_0) = [n]$.) Note that $\varphi(\mathfrak{p})$ has at least two elements since $i(S)$ is nonempty, so that there is at least one part F_1 as well as the parts N_0 and N_n. If $\mathfrak{p}'$ refines $\mathfrak{p}$ in the sense of Lemma 2.17 so that $\mathfrak{p} \geq \mathfrak{p}'$ then clearly we have $\varphi(\mathfrak{p}) \subset \varphi(\mathfrak{p}')$. Finally, any such partition is uniquely determined by the collection of those edges separating its parts.

This concludes the proof of the proposition. □

3 Arboreal hypersurfaces

3.1 Rectilinear version

By a *rooted tree* $\mathcal{T} = (T, \rho)$, we will mean a tree T equipped with a distinguished vertex $\rho \in V(T)$ called the *root vertex*.

The vertices $V(T)$ of a rooted tree naturally form a poset with the root vertex $\rho \in V(T)$ the unique minimum and $\alpha < \beta \in V(T)$ if the former is nearer to ρ than the latter. To each nonroot vertex $\alpha \neq \rho \in V(T)$ there is a unique *parent vertex* $\hat{\alpha} \in V(T)$ such that $\hat{\alpha} < \alpha$ and there are no vertices strictly between them.

Let us write $\mathbb{R}^{\mathcal{T}} = \mathbb{R}^{V(T)}$ for the Euclidean space of real tuples

$$\{x_\gamma\} \quad \text{with } \gamma \in V(T).$$

Definition 3.1 Fix a rooted tree $\mathcal{T} = (T, \rho)$ and a vertex $\alpha \in V(T)$.

(1) Define the quadrant $Q_\alpha \subset \mathbb{R}^{\mathcal{T}}$ to be the closed subspace
$$Q_\alpha = \{x_\beta \geq 0 \text{ for all } \beta \leq \alpha\}.$$

(2) Define the hypersurface $H_\alpha \subset \mathbb{R}^{\mathcal{T}}$ to be the boundary
$$H_\alpha = \partial Q_\alpha.$$

Remark 3.2 Note that the hypersurface $H_\alpha \subset \mathbb{R}^{\mathcal{T}}$ is homeomorphic (in a piecewise linear fashion) to a Euclidean space of dimension $|V(T)| - 1$.

Definition 3.3 The *rectilinear arboreal hypersurface* $H_{\mathcal{T}}$ associated to a rooted tree $\mathcal{T} = (T, \rho)$ is the union of hypersurfaces
$$H_{\mathcal{T}} = \bigcup_{\alpha \in V(T)} H_\alpha \subset \mathbb{R}^{\mathcal{T}}.$$

Remark 3.4 The rectilinear arboreal hypersurface admits the less redundant presentation
$$H_{\mathcal{T}} = \bigcup_{\alpha \in V(T)} \{x_\alpha = 0, x_\beta > 0 \text{ for all } \beta < \alpha\} \subset \mathbb{R}^{\mathcal{T}}.$$

3.2 Smoothed version

We construct here a smoothed version of the rectilinear arboreal hypersurface of a rooted tree. We will show in the next section that they are homeomorphic as embedded hypersurfaces inside of Euclidean space.

Fix a continuously differentiable function $b\colon \mathbb{R}_{>0} \to \mathbb{R}$ with the properties:

(1) b is nonpositive.
(2) $\lim_{t \to 0} b(t) = 0$.
(3) $\lim_{t \to 0} b'(t) = -\infty$.
(4) $b(t) = 0$, for $t \gg 0$.

Given the above function $b\colon \mathbb{R}_{>0} \to \mathbb{R}$, choose a continuously differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with the properties:

(1) f is a submersion.
(2) $\{f(x_1, x_2) = 0\} = \{x_1 = 0, x_2 \geq 0\} \cup \{x_1 > 0, x_2 = b(x_1)\}$.
(3) $\{f(x_1, x_2) > 0\} = \{x_1 > 0, x_2 > b(x_1)\}$.
(4) $\{f(x_1, x_2) < 0\} = \{x_1 < 0\} \cup \{x_1 = 0, x_2 < 0\} \cup \{x_1 > 0, x_2 < b(x_1)\}$.

Remark 3.5 If preferred, one can fix some N greater than or equal to 1, and arrange that $\lim_{t\to 0} b^{(k)}(t) = -\infty$, for all $1 \leq k \leq N$. Then one can choose f to be correspondingly highly differentiable. One can also take $N = \infty$ and then choose f to be smooth.

Definition 3.6 Fix a rooted tree $\mathcal{T} = (T, \rho)$.

(1) For the root vertex $\rho \in V(T)$, set

$$h_\rho = x_\rho \colon \mathbb{R}^{\mathcal{T}} \to \mathbb{R}.$$

(2) For a nonroot vertex $\alpha \neq \rho \in V(T)$, inductively define

$$h_\alpha \colon \mathbb{R}^{\mathcal{T}} \to \mathbb{R} \quad \text{by } h_\alpha = f(h_{\hat{\alpha}}, x_\alpha),$$

where $\hat{\alpha} \in V(T)$ is the parent vertex of α.

Remark 3.7 (1) Note that h_α depends only on the coordinates x_β, for $\beta \leq \alpha$.

(2) Note also that $h_\alpha \geq 0$ implies $h_\beta \geq 0$, for $\beta \leq \alpha$.

Definition 3.8 Fix a rooted tree $\mathcal{T} = (T, \rho)$ and a vertex $\alpha \in V(T)$.

(1) Define the halfspace $\mathsf{Q}_\alpha \subset \mathbb{R}^{\mathcal{T}}$ to be the closed subspace

$$\mathsf{Q}_\alpha = \{h_\alpha \geq 0\}.$$

(2) Define the hypersurface $\mathsf{H}_\alpha \subset \mathbb{R}^{\mathcal{T}}$ to be the zero-locus

$$\mathsf{H}_\alpha = \{h_\alpha = 0\}.$$

Definition 3.9 The *smoothed arboreal hypersurface* $\mathsf{H}_{\mathcal{T}}$ associated to a rooted tree $\mathcal{T} = (T, \rho)$ is the union of hypersurfaces

$$\mathsf{H}_{\mathcal{T}} = \bigcup_{\alpha \in V(T)} \mathsf{H}_\alpha \subset \mathbb{R}^{\mathcal{T}}.$$

Remark 3.10 The smoothed arboreal hypersurface admits the less redundant presentation

$$\mathsf{H}_{\mathcal{T}} = \bigcup_{\alpha \in V(T)} \{h_{x_\alpha} = 0, h_{x_{\hat{\alpha}}} > 0\} \subset \mathbb{R}^{\mathcal{T}}$$

where $\hat{\alpha} \in V(T)$ is the parent vertex of α.

3.3 Comparison

We next compare the rectilinear and smoothed arboreal hypersurfaces.

Given the function $b\colon \mathbb{R}_{>0} \to \mathbb{R}$, define the continuous function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$\varphi(x_1, x_2) = \begin{cases} x_2 & \text{when } x_1 \le 0, \\ x_2 - b(x_1) & \text{when } x_1 > 0. \end{cases}$$

Definition 3.11 (1) For the root vertex $\rho \in V(T)$, define

$$F_\rho\colon \mathbb{R}^{\mathcal{T}} \to \mathbb{R} \quad \text{by} \quad F_\rho = x_\rho.$$

(2) For a nonroot vertex $\alpha \neq \rho \in V(T)$, define

$$F_\alpha\colon \mathbb{R}^{\mathcal{T}} \to \mathbb{R} \quad \text{by} \quad F_\alpha = \varphi(h_{\hat{\alpha}}, x_\alpha),$$

where $\hat{\alpha} \in V(T)$ is the unique parent of α.

(3) Define the continuous map

$$F_{\mathcal{T}}\colon \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}} \quad \text{by} \quad F_{\mathcal{T}} = \{F_\alpha\}.$$

Remark 3.12 Note that F_α depends only on the coordinates x_β, for $\beta \le \alpha$.

Proposition 3.13 *The map $F_{\mathcal{T}}\colon R^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ is a homeomorphism and restricts to a homeomorphism from the smoothed to rectilinear arboreal hypersurface*

$$F_{\mathcal{T}}\colon \mathsf{H}_{\mathcal{T}} \xrightarrow{\sim} H_{\mathcal{T}}$$

satisfying $F_{\mathcal{T}}(\mathsf{Q}_\alpha) = Q_\alpha$ and $F_{\mathcal{T}}(\mathsf{H}_\alpha) = H_\alpha$, for all $\alpha \in V(T)$.

Proof We will proceed by induction on the size of the vertex set $V(T)$. In the base case when $V(T)$ has a single element, the assertions are evident: $\mathsf{H}_{\mathcal{T}} = H_{\mathcal{T}} = \{0\} \subset \mathbb{R}$, and $F_{\mathcal{T}}\colon \mathbb{R} \to \mathbb{R}$ is the identity.

Now suppose $V(T)$ contains at least two elements. Let $\tau \in V(T)$ be a maximal vertex in the partial order (in particular τ will not be the root vertex ρ). Introduce the rooted tree $\mathcal{T}_\tau = (T_\tau, \rho)$ where we delete the vertex τ and the edge $\{\tau, \hat{\tau}\}$ where $\hat{\tau} \in V(T)$ is the parent vertex of τ. Suppose the assertions are already established for $\mathcal{T}_\tau$.

Let us first show $F_{\mathcal{T}}\colon R^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ is a homeomorphism given that $F_{\mathcal{T}_\tau}\colon R^{\mathcal{T}_\tau} \to \mathbb{R}^{\mathcal{T}_\tau}$ is a homeomorphism. Under the identification $\mathbb{R}^{\mathcal{T}} = \mathbb{R}^{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}$, by definition we have

$$F_{\mathcal{T}} = (F_{\mathcal{T}_\tau}, \varphi(h_{\hat{\tau}}, x_\tau)).$$

By construction, we can regard $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ as a family of homeomorphisms in its second variable depending on its first variable. (In fact, for fixed value of the first variable, the homeomorphism in the second variable is either the identity or a translation.) Since $h_{\hat{\tau}}$ is independent of the variable x_τ, we see that $F_{\mathcal{T}}$ is similarly a parametrized homeomorphism over $\mathcal{F}_{\mathcal{T}_\tau}$, and hence itself a homeomorphism.

To finish the proof, it suffices to show for all $\alpha \in V(T)$ that we have

$$F_{\mathcal{T}}(\mathsf{Q}_\alpha) = F_{\mathcal{T}}(\{h_\alpha \geq 0\}) = \{x_\beta \geq 0 \text{ for all } \beta \leq \alpha\} = Q_\alpha,$$

since passing to boundaries, we will have $F_{\mathcal{T}}(\mathsf{H}_\alpha) = H_\alpha$, for all $\alpha \in V(T)$, and hence passing to the unions of the boundaries $F_{\mathcal{T}}(\mathsf{H}_{\mathcal{T}}) = H_{\mathcal{T}}$.

Thus we will show that for all $\alpha \in V(T)$, and $x \in \mathbb{R}^{\mathcal{T}}$, we have

$$h_\alpha(x) \geq 0 \qquad \iff \qquad F_\beta(x) \geq 0 \quad \text{for all } \beta \leq \alpha.$$

Recall that for all $\alpha \in V(T_\tau)$, the functions h_α and F_α depend only on the subtree $\mathcal{T}_\tau$. Hence by induction, for all $\alpha \in V(T_\tau)$, and $x \in \mathbb{R}^{\mathcal{T}}$, we have

$$h_\alpha(x) \geq 0 \qquad \iff \qquad F_\beta(x) \geq 0 \quad \text{for all } \beta \leq \alpha.$$

Therefore it suffices to show

$$h_\tau(x) \geq 0 \qquad \iff \qquad F_\beta(x) \geq 0 \quad \text{for all } \beta \leq \tau.$$

Recall that $h_\tau(x) \geq 0$ implies $h_\beta(x) \geq 0$, for all $\beta \leq \tau$. Hence by induction, it suffices to assume $h_{\hat{\tau}}(x) \geq 0$, where $\hat{\tau} \in V(T)$ is the parent vertex of τ, and show

$$h_\tau(x) \geq 0 \quad \iff \quad F_\tau(x) \geq 0.$$

Returning to the definitions, on the one hand, we have $h_\tau(x) = f(h_{\hat{\tau}}(x), x_\tau)$. Under the assumption $h_{\hat{\tau}}(x) \geq 0$, the prescribed properties of f ensure

$$h_\tau(x) = f(h_{\hat{\tau}}(x), x_\tau) \geq 0 \quad \iff \quad x_\tau \geq b(h_{\hat{\tau}}(x)).$$

On the other hand, we have $F_\tau(x) = \varphi(h_{\hat{\tau}}(x), x_\tau)$. Under the assumption $h_{\hat{\tau}}(x) \geq 0$, we have $\varphi(h_{\hat{\tau}}(x), x_\tau) = x_\tau - b(h_{\hat{\tau}}(x))$. Thus we similarly conclude

$$F_\tau(x) = \varphi(h_{\hat{\tau}}(x), x_\tau) \geq 0 \quad \iff \quad x_\tau - b(h_{\hat{\tau}}(x)) \geq 0. \qquad \square$$

Remark 3.14 By scaling the original function b by a positive constant, one obtains a family of smoothed arboreal hypersurfaces all compatibly homeomorphic. Moreover, their limit as the scaling constant goes to zero will be the rectilinear arboreal hypersurface. Thus one can view the smoothed arboreal hypersurface as a topologically trivial deformation of the rectilinear arboreal hypersurface.

3.4 Directed hypersurfaces

Let us first review some notions from microlocal geometry.

Let M denote a smooth n–dimensional manifold with $\pi\colon T^*M \to M$ its cotangent bundle. Let P^*M denote the projectivization of T^*M. Points of P^*M are pairs $(x, [v])$ where $x \in M$ and $[v] = \mathbb{R} \cdot v \subset T_x^*M$ is the line through $v \neq 0 \in T_x^*M$. Let S^*M denote the spherical projectivization of T^*M. Points of S^*M are pairs $(x, [v])$ where $x \in M$ and $[v] = \mathbb{R}_{\geq 0} \cdot v \subset T_x^*M$ is the ray through $v \neq 0 \in T_x^*M$.

Given a submanifold $Y \subset M$, let $T_Y^*M \subset T^*M$ denote its conormal bundle. Let $P_Y^*M \subset P^*M$ denote the projectivized conormal bundle. Points of P_Y^*M are pairs $(y, [v])$ where $y \in Y$ and $[v] = \mathbb{R} \cdot v \subset T_Y^*M|_y$ is the line through $v \neq 0 \in T_Y^*M|_y$. Let $S_Y^*M \subset S^*M$ denote the spherically projectivized conormal bundle. Points of S_Y^*M are pairs $(y, [v])$ where $y \in Y$ and $[v] = \mathbb{R}_{\geq 0} \cdot v \subset T_Y^*M|_y$ is the ray through $v \neq 0 \in T_Y^*M|_y$. Suppose M is equipped with a complete Riemannian metric. Then the spherical projectivization S^*M is identified with the unit sphere bundle inside of T^*M.

Throughout what follows, by a hypersurface $H \subset M$, we will mean a closed subspace such that M admits a Whitney stratification with H the closure of the $(n-1)$–dimensional stratum. Thus there exists an open dense locus $H^{\mathrm{sm}} \subset H$ which is a locally closed $(n-1)$–dimensional differentiable submanifold of M.

We have a natural diagram of maps

$$S_{H^{\mathrm{sm}}}^*M \to P_{H^{\mathrm{sm}}}^*M \to H^{\mathrm{sm}}$$

where the first is a two-fold cover and the second is a diffeomorphism.

Definition 3.15 A hypersurface $H \subset M$, with open dense smooth locus $H^{\mathrm{sm}} \subset H$, is said to be in *good position* if the closure

$$\mathcal{L}_H^* = \overline{P_{H^{\mathrm{sm}}}^*M} \subset P^*M$$

is finite over H. If this holds, we refer to $\mathcal{L}_H^*$ as the *coline bundle* of H.

Remark 3.16 Equivalently, we could require the closure

$$\mathcal{R}_H^* = \overline{S_{H^{\mathrm{sm}}}^*M} \subset S^*M$$

be finite over H. If this holds, we refer to $\mathcal{R}_H^*$ as the *coray bundle* of H.

Remark 3.17 If $H \subset M$ is in good position, we have a natural diagram of finite maps

$$\mathcal{R}_H^* \to \mathcal{L}_H^* \to H$$

where the first is a two-fold cover and the second is a diffeomorphism over $H^{\mathrm{sm}} \subset H$.

Example 3.18 (1) All Whitney stratified plane curve singularities are in good position.

(2) The singular quadratic cones

$$\left\{ \sum_{i=1}^{k} x_i^2 - \sum_{j=k+1}^{n} x_j^2 = 0 \right\} \subset \mathbb{R}^n \qquad \text{for } n > 2, n > k > 0$$

are not in good position.

Definition 3.19 (1) By a *coorientation* of a hypersurface $H \subset M$ in good position, we will mean a section

$$\mathcal{R}_H^* \underset{\longrightarrow}{\overset{\sigma}{\longleftarrow}} \mathcal{L}_H^*$$

of the natural two-fold cover from the coray to coline bundle.

(2) By a *directed hypersurface* (H, σ), we will mean a hypersurface $H \subset M$ in good position equipped with a coorientation σ.

(3) By the *positive ray bundle* of a directed hypersurface (H, σ), we will mean the image of the coorientation

$$\mathcal{R}_H^+ = \sigma(\mathcal{L}_H^*) \subset S^* M.$$

Now let us return to a rooted tree $\mathcal{T} = (T, \rho)$ and its smoothed arboreal hypersurface

$$\mathsf{H}_{\mathcal{T}} = \bigcup_{\alpha \in V(T)} \mathsf{H}_\alpha \subset \mathbb{R}^{\mathcal{T}}.$$

Since f is a submersion, each $h_\alpha \colon \mathbb{R}^{\mathcal{T}} \to \mathbb{R}$ is a submersion, hence each hypersurface $\mathsf{H}_\alpha = \{h_\alpha = 0\} \subset \mathbb{R}^{\mathcal{T}}$ is in good position with the natural projection a homeomorphism

$$\pi \colon \mathcal{L}_{\mathsf{H}_\alpha}^* \xrightarrow{\sim} \mathsf{H}_\alpha.$$

Thus the smoothed arboreal hypersurface $\mathsf{H}_T \subset \mathbb{R}^{\mathcal{T}}$ is in good position (since it is a finite union of these hypersurfaces).

Moreover, each hypersurface $\mathsf{H}_\alpha \subset \mathbb{R}^{\mathcal{T}}$ comes equipped with a preferred coorientation σ_α given by the codirection pointing towards the halfspace $\mathsf{Q}_\alpha = \{h_\alpha \geq 0\} \subset \mathbb{R}^{\mathcal{T}}$.

Theorem 3.20 *Let $\mathcal{T} = (T, \rho)$ be a rooted tree with arboreal singularity $\mathsf{L}_{\mathcal{T}}$ and smoothed arboreal hypersurface $\mathsf{H}_{\mathcal{T}} \subset \mathbb{R}^{\mathcal{T}}$.*

(1) *The smoothed arboreal hypersurface $\mathsf{H}_{\mathcal{T}} \subset \mathbb{R}^{\mathcal{T}}$ admits a natural coorientation σ whose restriction to each $\mathsf{H}_\alpha \subset \mathsf{H}_{\mathcal{T}}$ is the coorientation σ_α.*

(2) *Let $\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} \subset P^*\mathbb{R}^{\mathcal{T}}$ be the coline bundle of $\mathsf{H}_{\mathcal{T}} \subset \mathbb{R}^{\mathcal{T}}$. There is a homeomorphism*

$$\varphi\colon \mathsf{L}_{\mathcal{T}} \xrightarrow{\sim} \mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}}$$

whose composition with the natural projection $\pi\colon \mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} \to \mathsf{H}_{\mathcal{T}}$ restricts to homeomorphisms

$$\pi \circ \varphi|_{\mathsf{L}_{\mathcal{T}}(\alpha)}\colon \mathsf{L}_{\mathcal{T}}(\alpha) \xrightarrow{\sim} \mathsf{H}_\alpha \quad \textit{for all } \alpha \in V(T).$$

Remark 3.21 Note that the composition $\pi \circ \varphi\colon \mathsf{L}_{\mathcal{T}} \to \mathsf{H}_{\mathcal{T}}$ will not in general be a homeomorphism but only a finite map. It is possible that distinct points of $\mathsf{L}_{\mathcal{T}}$ will map to the same point of $\mathsf{H}_{\mathcal{T}}$ but correspond to different coorientations at that point.

Proof of Theorem 3.20 The bulk of the proof will be of assertion (2).

We will proceed by induction on the size of the vertex set $V(T)$. In the base case when $V(T)$ has a single element, all assertions are evident: $\mathsf{H}_{\mathcal{T}} = \{0\} \subset \mathbb{R}$, the coorientation points towards the halfspace $\mathbb{R}_{\geq 0} \subset \mathbb{R}$, and $\mathsf{L}_{\mathcal{T}} = \{0\}$.

Now suppose $V(T)$ contains at least two elements. Let $\tau \in V(T)$ be a maximal vertex in the partial order. Introduce the rooted tree $\mathcal{T}_\tau = (T_\tau, \rho)$ where we delete the vertex τ and the edge $\{\tau, \hat{\tau}\}$ where $\hat{\tau} \in V(T)$ is the parent vertex of τ.

Suppose the assertions of the theorem are established for $\mathcal{T}_\tau$. By definition, inside of $\mathbb{R}^{\mathcal{T}}$, we have an identification of hypersurfaces

$$\mathsf{H}_{\mathcal{T}} = (\mathsf{H}_{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}) \cup \mathsf{H}_\tau$$

where each factor in the union is a closed subspace. Therefore inside of $P^*\mathbb{R}^{\mathcal{T}}$, we have an identification of subspaces

$$\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} = \mathcal{L}^*_{\mathsf{H}_{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}} \cup \mathcal{L}^*_{\mathsf{H}_\tau}. \tag{3-1}$$

Note that the first factor in the union admits the presentation

$$\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}} \simeq \mathcal{L}^*_{\mathsf{H}_{\mathcal{T}_\tau}} \times \mathbb{R}^{\{\tau\}}.$$

Now let us more closely analyze the second factor $\mathcal{L}^*_{\mathsf{H}_\tau}$ in the union (3-1). By Proposition 3.13 and Remark 3.2, there is a homeomorphism

$$\mathsf{H}_\tau \simeq \mathbb{R}^{|V(T)|-1}. \tag{3-2}$$

Introduce the subspaces

$$\mathsf{H}_\tau^+ = \mathsf{H}_\tau \cap \{h_{\hat\tau} > 0\}, \quad \mathsf{H}_\tau^0 = \mathsf{H}_\tau \cap \{h_{\hat\tau} = 0\} \quad \text{and} \quad \mathsf{H}_\tau^- = \mathsf{H}_\tau \cap \{h_{\hat\tau} < 0\}.$$

Recall that $h_\tau \geq 0$ implies $h_{\hat\tau} \geq 0$, so that $\mathsf{H}_\tau^- = \varnothing$ and hence

$$\mathsf{H}_\tau = \mathsf{H}_\tau^+ \cup \mathsf{H}_\tau^0.$$

Note as well that

$$\mathsf{H}_\tau^0 \subset \mathsf{H}_{\hat\tau} \subset \mathsf{H}_{\mathcal{T}} \times \mathbb{R}^{\{\tau\}}.$$

Let $\overline{\mathsf{H}}_\tau^+ \subset \mathsf{H}_\tau$ denote the closure of H_τ^+. Then since $\mathsf{H}_\tau^- = \varnothing$ and $\mathsf{H}_\tau^0 \subset \mathsf{H}_{\mathcal{T}} \times \mathbb{R}^{\{\tau\}}$, the union (3-1) admits the refinement

$$\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} = \mathcal{L}^*_{\mathsf{H}_{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}} \cup \mathcal{L}^*_{\overline{\mathsf{H}}_\tau^+} \tag{3-3}$$

where each factor in the union is a closed subspace.

By Proposition 3.13, the homeomorphism (3-2) can be chosen to restrict to a homeomorphism

$$\overline{\mathsf{H}}_\tau^+ \simeq \mathbb{R}^{|V(T)|-2} \times \mathbb{R}_{\geq 0}. \tag{3-4}$$

Furthermore, again by Proposition 3.13, under the above identifications, we have

$$\partial \overline{\mathsf{H}}_\tau^+ \simeq \mathbb{R}^{|V(T)|-2} \times \{0\} \simeq \mathsf{H}_{\hat\tau} \cap \{x_\tau = 0\}. \tag{3-5}$$

Next observe that projection along the τ–direction is a diffeomorphism

$$\mathsf{H}_\tau^+ = \{h_{\hat\tau} > 0, h_\tau = 0\} \xrightarrow{\sim} \{h_{\hat\tau} > 0, x_\tau = 0\},$$

so that we have the nonintersection of coline bundles

$$\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}} \cap \mathcal{L}^*_{\mathsf{H}_\tau^+} = \varnothing.$$

We conclude by the identifications (3-3), (3-4), (3-5), and induction, there is the required homeomorphism

$$\mathsf{L}_T \xrightarrow{\sim} \mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}}$$

since we have the presentation

$$\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}}} \simeq (\mathcal{L}^*_{\mathsf{H}_{\mathcal{T}_\tau}} \times \mathbb{R}^{\{\tau\}}) \coprod_{\mathbb{R}^{|V(T)|-2} \times \{0\}} (\mathbb{R}^{|V(T)|-2} \times \mathbb{R}_{\geq 0})$$

exactly as appears for L_T in Lemma 2.7.

Finally, to see assertion (1), first note that the coorientation of $\mathsf{H}_{\mathcal{T}_\tau}$ naturally extends to a coorientation of $\mathsf{H}_{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}$. Thus by induction, it suffices to return to the union (3-1) and check that the coorientations of $\mathsf{H}_{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}$ and H_τ agree along their intersection

$$\mathsf{H}_\tau^0 = \{h_\tau = h_{\hat{\tau}} = 0\}.$$

Recall that $h_\tau \geq 0$ implies $h_{\hat{\tau}} \geq 0$ or, in other words, $\mathsf{Q}_\tau \subset \mathsf{Q}_{\hat{\tau}}$. By definition, the coorientations of $\mathsf{H}_\tau, \mathsf{H}_{\hat{\tau}} \subset \mathbb{R}^{\mathcal{T}}$ point towards the respective halfspaces $\mathsf{Q}_\tau, \mathsf{Q}_{\hat{\tau}} \subset \mathbb{R}^{\mathcal{T}}$, hence the coorientations of $\mathsf{H}_{\mathcal{T}_\tau} \times \mathbb{R}^{\{\tau\}}$ and H_τ agree along H_τ^0. □

4 Microlocal sheaves

4.1 Stalk calculation

Fix a rooted tree $\mathcal{T} = (T, \rho)$ where as usual T is a tree and $\rho \in V(T)$ is the root vertex. Recall that $\mathbb{R}^{\mathcal{T}}$ denotes the Euclidean space of real tuples

$$\{x_\gamma\} \quad \text{with } \gamma \in V(T).$$

and $S^*\mathbb{R}^{\mathcal{T}}$ denotes its spherically projectivized cotangent bundle. The latter is naturally a cooriented contact manifold.

Recall that we have constructed the smoothed arboreal hypersurface

$$\mathsf{H}_{\mathcal{T}} = \bigcup_{\alpha \in V(T)} \mathsf{H}_\alpha \subset \mathbb{R}^{\mathcal{T}}.$$

It is in good position and comes equipped with a natural coorientation so that its positive ray bundle is homeomorphic to the arboreal singularity

$$\mathsf{L}_T = \bigcup_{\alpha \in V(T)} \mathsf{L}_T(\alpha).$$

Via this identification, we can regard L_T and its Euclidean constituents $\mathsf{L}_T(\alpha)$ as Legendrian subspaces of $S^*\mathbb{R}^{\mathcal{T}}$. When doing so, we will use $\mathcal{T}$ in the notation in place of T.

Fix once and for all a field k, and let $\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$ denote the dg category of cohomologically constructible complexes of sheaves of k–vector spaces on $\mathbb{R}^{\mathcal{T}}$.

Our main object of study will be the dg category $\mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ of constructible complexes of k–vector spaces on $\mathbb{R}^{\mathcal{T}}$ microlocalized along the Legendrian subspace $\mathsf{L}_{\mathcal{T}} \subset S^*\mathbb{R}^{\mathcal{T}}$. There are two equivalent ways to think about $\mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ in terms of $\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$ which we will now explain.

To any object $\mathcal{F} \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$, one can associate its singular support $\mathrm{ss}(\mathcal{F}) \subset S^*\mathbb{R}^{\mathcal{T}}$. This is a closed Legendrian subspace recording those codirections in which the propagation of sections of $\mathcal{F}$ is obstructed. In particular, one has the vanishing $\mathrm{ss}(\mathcal{F}) = \varnothing$ if and only if the cohomology sheaves of $\mathcal{F}$ are locally constant.

Remark 4.1 To fix standard conventions [9], suppose $i\colon U \to \mathbb{R}^{\mathcal{T}}$ is the inclusion of an open subset with a smooth boundary hypersurface $\partial U \subset \mathbb{R}^{\mathcal{T}}$. Then the singular support of the extension by zero of the constant sheaf $i_! k_U \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$ consists of the spherical projectivization of the outward conormal codirection along $\partial U \subset \mathbb{R}^{\mathcal{T}}$.

Abstractly, one can define $\mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ as the dg quotient category of $\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$ by the full subcategory of all objects $\mathcal{F}$ for which $\mathrm{ss}(\mathcal{F}) \cap \mathsf{L}_{\mathcal{T}} = \varnothing$. Equivalently, one can take $\mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ to be the full dg subcategory of $\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$ consisting of objects $\mathcal{F}$ for which

(1) $\mathrm{ss}(\mathcal{F}) \subset \mathsf{L}_{\mathcal{T}}$, and
(2) $\mathrm{Hom}_{\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})}(k_{\mathbb{R}^{\mathcal{T}}}, \mathcal{F}) \simeq 0$.

We will now give two concrete collections of generators for this subcategory, and one could take their triangulated hulls inside of $\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$ as the definition of $\mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$.

Recall that for each $\alpha \in V(T)$, the hypersurface $\mathsf{H}_\alpha \subset \mathbb{R}^{\mathcal{T}}$ is the zero locus of the function

$$h_\alpha\colon \mathbb{R}^{\mathcal{T}} \to \mathbb{R}.$$

Consider the inclusion of the open subspace

$$i_\alpha\colon U_\alpha = \{h_\alpha < 0\} \hookrightarrow \mathbb{R}^{\mathcal{T}}.$$

Introduce the extension by zero

$$\mathcal{P}_\alpha = i_{\alpha!} k_{U_\alpha} \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}}).$$

Observe the elementary properties

(1) $\mathrm{ss}(\mathcal{P}_\alpha) = \mathsf{L}_{\mathcal{T}}(\alpha)$,
(2) $\mathrm{Hom}_{\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})}(k_{\mathbb{R}^{\mathcal{T}}}, \mathcal{P}_\alpha) \simeq 0$.

Alternatively, recall that to each nonroot vertex $\alpha \neq \rho \in V(T)$ there is a unique parent vertex $\hat{\alpha} \in V(T)$ such that $\alpha > \hat{\alpha}$ and there are no vertices strictly between them. Consider the inclusion of the open subspace

$$j_\alpha\colon W_\alpha = \{h_\alpha < 0, h_{\hat{\alpha}} > 0\} \hookrightarrow U_\alpha = \{h_\alpha < 0\}.$$

Introduce the iterated extension

$$\mathcal{S}_\alpha = i_{\alpha!} j_{\alpha*} k_{W_\alpha} \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}}).$$

For the root vertex $\rho \in V(T)$, set

$$\mathcal{S}_\rho = \mathcal{P}_\rho = i_{\alpha!} k_{U_\alpha} \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}}).$$

Observe the collection of canonical exact triangles:

$$\mathcal{P}_{\hat{\alpha}} = i_{\hat{\alpha}!} i_{\hat{\alpha}}^! \mathcal{P}_\alpha \longrightarrow \mathcal{P}_\alpha \longrightarrow i_{\alpha!} j_{\alpha*} j_\alpha^* \mathcal{P}_\alpha = \mathcal{S}_\alpha \xrightarrow{[1]}$$

With the analogous properties recorded above, the exact triangles imply the properties

(1) $\mathrm{ss}(\mathcal{S}_\alpha) \subset \mathsf{L}_{\mathcal{T}}$,
(2) $\mathrm{Hom}_{\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})}(k_{\mathbb{R}^{\mathcal{T}}}, \mathcal{S}_\alpha) \simeq 0$.

Remark 4.2 In fact, the exact triangles imply the precise singular support calculation

$$\mathrm{ss}(\mathcal{S}_\alpha) = \text{ closure of } (\mathsf{L}_{\mathcal{T}}(\alpha) \cup \mathsf{L}_{\mathcal{T}}(\hat{\alpha})) \setminus (\mathsf{L}_{\mathcal{T}}(\alpha) \cap \mathsf{L}_{\mathcal{T}}(\hat{\alpha})).$$

Furthermore, the exact triangles also imply that the triangulated hull of the collection of objects $\mathcal{P}_\alpha \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$, for $\alpha \in V(T)$, coincides with that of the collection of objects $\mathcal{S}_\alpha \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$, for $\alpha \in V(T)$.

Proposition 4.3 *The collection of objects $\mathcal{P}_\alpha \in \mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$, for $\alpha \in V(T)$, or alternatively the collection of objects $\mathcal{S}_\alpha$, for $\alpha \in V(T)$, generates the full dg subcategory $\mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}}) \subset \mathrm{Sh}(\mathbb{R}^{\mathcal{T}})$ consisting of objects $\mathcal{F}$ for which*

(1) $\mathrm{ss}(\mathcal{F}) \subset \mathsf{L}_{\mathcal{T}}$, *and*
(2) $\mathrm{Hom}_{\mathrm{Sh}(\mathbb{R}^{\mathcal{T}})}(k_{\mathbb{R}^{\mathcal{T}}}, \mathcal{F}) \simeq 0$.

Proof It suffices to prove the assertion for the collection of objects $\mathcal{S}_\alpha$, for $\alpha \in V(T)$.

For each $\alpha \in V(T)$, recall the inclusion of the open subspace

$$i_\alpha \colon U_\alpha = \{h_\alpha < 0\} \hookrightarrow \mathbb{R}^{\mathcal{T}}.$$

Introduce the complementary closed inclusion

$$q_\alpha \colon \mathsf{Q}_\alpha = \{h_\alpha \geq 0\} \hookrightarrow \mathbb{R}^{\mathcal{T}}$$

and the open inclusion of its interior

$$q_\alpha^\circ \colon \mathsf{Q}_\alpha^\circ = \{h_\alpha > 0\} \hookrightarrow \mathbb{R}^{\mathcal{T}}.$$

Now let us begin with the first step of an iterative procedure to prove the assertion.

Step (ρ) Fix an object $\mathcal{F} \in \mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$. To start, we have the canonical exact triangle:

$$i_{\rho !} i_{\rho}^{!} \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow q_{\rho *} q_{\rho}^{*} \mathcal{F} \xrightarrow{[1]}$$

Note that $i_{\rho !} i_{\rho}^{!} \mathcal{F}$ is a sum of copies of $\mathcal{S}_{\rho}$. Furthermore, the canonical restriction map

$$q_{\rho *} q_{\rho}^{*} \mathcal{F} \to q_{\rho *}^{\circ} q_{\rho}^{\circ *} \mathcal{F}$$

is a quasi-isomorphism since its cone is an object supported within H_{ρ} but with singular support inside the codirection $\mathsf{L}_{\mathcal{T}}(\rho)$ and hence must vanish.

Set $\mathcal{F}_{>\rho} = q_{\rho}^{\circ *} \mathcal{F} \in \mathrm{Sh}(\mathsf{Q}_{\rho}^{\circ})$.

Consider the descendant set $d(\rho) \subset V(T)$ of those vertices $\alpha \in V(T)$ with parent vertex $\hat{\alpha} = \rho \in V(T)$. Note that the inclusion of $d(\rho)$ into the disjoint union of trees $T \setminus \{\rho\}$ gives a bijection with its connected components. Recall by construction that $\mathsf{L}_{\mathcal{T}}(\rho)$ is the union of $\mathsf{L}_{\mathcal{T}}(\mathfrak{p})$ for those partitions

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T)$$

such that $\rho \in V(S)$. The poset of those partitions $\mathfrak{p}$ such that $\rho \notin V(S)$ is a disjoint union indexed by $d(\rho)$ given by which component of $T \setminus \{\rho\}$ contains S. Thus by Theorem 2.20, the complement $\mathsf{L}_{\mathcal{T}} \setminus \mathsf{L}_{\mathcal{T}}(\rho)$ is a disjoint union indexed by $d(\rho)$, and hence $\mathcal{F}_{>\rho}$ is a corresponding direct sum indexed by $d(\rho)$.

For each $\alpha \in d(\rho)$, set $\mathcal{F}_{\geq\alpha} \subset \mathcal{F}_{>\rho}$ to be the corresponding summand.

This concludes Step (ρ).

Step (α), for each $\alpha \in d(\rho)$ Now repeat the recollement pattern introduced above at Step (ρ), but applied to each summand $\mathcal{F}_{\geq\alpha}$ corresponding to the root vertex $\alpha \in d(\rho)$ of a connected component of the remaining disjoint union of trees $T \setminus \{\rho\}$. It results in the expression of each $\mathcal{F}_{\geq\alpha}$ as an extension built out of sums of copies of $\mathcal{S}_{\alpha}$ and a new sheaf $\mathcal{F}_{>\alpha}$ which is itself a sum of sheaves $\mathcal{F}_{\geq\alpha'}$, for $\alpha' \in d(\alpha)$, to which we can then apply the next iterative step. Here as before we write $d(\alpha) \subset V(T)$ for the descendant set of those vertices $\alpha' \in V(T)$ with parent vertex $\hat{\alpha}' = \alpha \in V(T)$.

Let us spell out the general Step (α) where we start with $\mathcal{F}_{\geq\alpha} \subset \mathcal{F}_{>\hat{\alpha}}$ but do not assume that $\alpha \in d(\rho)$. We have the canonical exact triangle:

$$i_{\alpha !} i_{\alpha}^{!} q_{\hat{\alpha} *}^{\circ} \mathcal{F}_{\geq\alpha} \longrightarrow q_{\hat{\alpha} *}^{\circ} \mathcal{F}_{\geq\alpha} \longrightarrow q_{\alpha *} q_{\alpha}^{*} \mathcal{F}_{\geq\alpha} \xrightarrow{[1]}$$

Note that $i_{\alpha !} i_{\alpha}^{!} q_{\hat{\alpha} *}^{\circ} \mathcal{F}_{\geq\alpha}$ is a sum of copies of $\mathcal{S}_{\alpha}$. Furthermore, the canonical restriction map

$$q_{\alpha *} q_{\alpha}^{*} \mathcal{F}_{\geq\alpha} \to q_{\alpha *}^{\circ} q_{\alpha}^{\circ *} \mathcal{F}_{\geq\alpha}$$

is a quasi-isomorphism since its cone is an object supported within H_α but with singular support inside the codirection $\mathsf{L}_\mathcal{T}(\alpha)$ and hence must vanish.

Set $\mathcal{F}_{>\alpha} = q_\alpha^{\circ *}\mathcal{F}_{\geq\alpha} \in \mathrm{Sh}(\mathsf{Q}_\alpha^\circ)$.

Consider the descendant set $d(\alpha) \subset V(T)$ of those vertices $\alpha' \in V(T)$ with parent vertex $\hat{\alpha}' = \alpha \in V(T)$. By Theorem 2.20, observe that $\mathsf{L}_\mathcal{T} \setminus (\cup_{\gamma \not> \alpha} \mathsf{L}_\mathcal{T}(\gamma))$ is a disjoint union indexed by $d(\alpha)$, and hence $\mathcal{F}_{>\alpha}$ is a corresponding direct sum indexed by $d(\alpha)$.

For each $\alpha' \in d(\alpha)$, set $\mathcal{F}_{\geq\alpha'} \subset \mathcal{F}_{>\alpha}$ to be the corresponding summand.

Now continue inductively vertex by vertex following the partial order. □

Now we will calculate the dg category $\mathrm{Sh}_{\mathsf{L}_\mathcal{T}}(\mathbb{R}^\mathcal{T})$ by calculating the morphisms between the generating objects $\mathcal{P}_\alpha \in \mathrm{Sh}(\mathbb{R}^\mathcal{T})$, for $\alpha \in V(T)$.

Recall we can regard the rooted tree $\mathcal{T} = (T, \rho)$ as a poset with the root vertex $\rho \in V(T)$ the unique minimum. To each nonroot vertex $\alpha \neq \rho \in V(T)$ there is a unique parent vertex $\hat{\alpha} \in V(T)$ such that $\alpha > \hat{\alpha}$ and there are no vertices strictly between them.

Now let us regard the rooted tree $\mathcal{T} = (T, \rho)$ as a quiver with a unique arrow pointing from each nonroot vertex $\alpha \neq \rho \in V(T)$ to its parent vertex $\hat{\alpha} \in V(T)$. Symbolically, we replace the relation $\alpha > \hat{\alpha}$ with the relation $\alpha \to \hat{\alpha}$.

Let $\mathrm{Mod}(\mathcal{T})$ denote the dg derived category of finite-dimensional complexes of modules over $\mathcal{T}$ regarded as a quiver. Objects assign to each vertex $\alpha \in V(T)$ a finite-dimensional complex of k–vector spaces $M(\alpha)$, and to each arrow $\alpha \to \hat{\alpha}$ a degree zero chain map $m_\alpha \colon M(\alpha) \to M(\hat{\alpha})$.

Let us point out two natural generating collections for $\mathrm{Mod}(\mathcal{T})$. There are the simple modules $S_\alpha \in \mathrm{Mod}(\mathcal{T})$ that assign

$$S_\alpha(\beta) = \begin{cases} k & \text{when } \beta = \alpha, \\ 0 & \text{when } \beta \neq \alpha, \end{cases}$$

with all maps $m_\beta \colon S_\alpha(\beta) \to S_\alpha(\hat{\beta})$ necessarily zero. There are also the projective modules $P_\alpha \in \mathrm{Mod}(\mathcal{T})$ that assign

$$P_\alpha(\beta) = \begin{cases} k & \text{when } \beta \leq \alpha, \\ 0 & \text{when } \beta > \alpha, \end{cases}$$

with the maps $m_\beta \colon P_\alpha(\beta) \to P_\alpha(\hat{\beta})$ the identity isomorphism whenever both domain and range are nonzero.

Theorem 4.4 *There is a canonical equivalence*

$$\varphi\colon \mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}}) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{T})$$

such that $\varphi(\mathcal{P}_\alpha) = P_\alpha$ *and* $\varphi(\mathcal{S}_\alpha) = S_\alpha$*, for all* $\alpha \in V(T)$*.*

Proof It suffices to establish the following:

$$\mathrm{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\beta) \simeq 0 \qquad \text{when } \alpha \not\leq \beta \in V(T),$$
$$\mathrm{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\beta) \simeq k \cdot e_\alpha^\beta \quad \text{when } \alpha \leq \beta \in V(T),$$

where e_α^β is a generator of degree zero satisfying

$$e_\alpha^\gamma = e_\beta^\gamma \circ e_\alpha^\beta \quad \text{when } \alpha \leq \beta \leq \gamma \in V(T).$$

To start, for any $\alpha, \beta \in V(T)$, we have

$$\begin{aligned}\mathrm{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\beta) &= \mathrm{Hom}(i_{\alpha!} k_{U_\alpha}, i_{\beta!} k_{U_\beta}) \\ &\simeq \mathrm{Hom}(k_{U_\alpha}, i_\alpha^! i_{\beta!} k_{U_\beta}) \simeq C^*(U_\alpha \cap \overline{U}_\beta, U_\alpha \cap \partial U_\beta; k),\end{aligned}$$

where the last term is the complex of relative singular cochains. Furthermore, the composition of morphisms is the natural cup product

$$\begin{aligned}\cup\colon C^*(U_\alpha \cap \overline{U}_\beta, U_\alpha \cap \partial U_\beta; k) \otimes C^*(U_\beta \cap \overline{U}_\gamma, U_\beta \cap \partial U_\gamma; k) \\ \to C^*(U_\alpha \cap \overline{U}_\gamma, U_\alpha \cap \partial U_\gamma; k)\end{aligned}$$

induced by viewing relative singular cochains as a subcomplex of singular cochains and taking the restriction of the usual cup product.

When $\alpha \not\leq \beta \in V(T)$, then either of two cases hold: (i) $\alpha > \beta$ or (ii) α and β are not comparable. Let us verify in each case the relative cohomology vanishes:

$$C^*(U_\alpha \cap \overline{U}_\beta, U_\alpha \cap \partial U_\beta; k) \simeq 0.$$

We will appeal to Proposition 3.13 in order to assume the rectilinear presentation

$$U_\alpha \simeq \{x_\gamma < 0 \text{ for some } \gamma \leq \alpha\}, \qquad U_\beta \simeq \{x_\gamma < 0 \text{ for some } \gamma \leq \beta\}$$

so that we also have

$$\partial U_\beta \simeq \{x_\gamma \geq 0 \text{ for all } \gamma \leq \beta;\ x_\gamma = 0 \text{ for some } \gamma \leq \beta\}.$$

Since U_β is homeomorphic to an open halfspace with ∂U_β homeomorphic to a hyperplane, it suffices to see that $U_\alpha \cap \partial U_\beta$ is contractible.

Therefore in case (i) when $\alpha > \beta$, we have

$$U_\alpha \cap \partial U_\beta \simeq \{x_\gamma < 0 \text{ for some } \beta < \gamma \leq \alpha;\ x_\gamma \geq 0 \text{ for all } \gamma \leq \beta;\ x_\gamma = 0 \text{ for some } \gamma \leq \beta\},$$

which is clearly contractible: one can contract along straight lines taking $x_\gamma \to -1$, for all $\beta < \gamma \le \alpha$, and $x_\gamma \to 0$, otherwise.

In case (ii) when α and β are not comparable, let $\eta \in V(T)$ be the maximal element such that $\eta \le \alpha, \beta$. Then we have

$$U_\alpha \cap \partial U_\beta \simeq \{x_\gamma < 0 \text{ for some } \eta < \gamma \le \alpha;\ x_\gamma \ge 0 \text{ for all } \gamma \le \beta;\ x_\gamma = 0 \text{ for some } \gamma \le \beta\},$$

which is also clearly contractible: contract along straight lines taking $x_\gamma \to -1$, for all $\eta < \gamma \le \alpha$, and $x_\gamma \to 0$, otherwise.

Now when $\alpha \le \beta \in V(T)$, we have $U_\alpha \subset U_\beta$ with U_α contractible, hence the morphism complex simplifies to

$$\mathrm{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\beta) \simeq C^*(U_\alpha; k) \simeq k \cdot e_\alpha^\beta,$$

where e_α^β denotes the constant cochain of degree zero and value $1 \in k$. Furthermore, for $\alpha \le \beta \le \gamma \in V(T)$, the composition

$$\mathrm{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\beta) \otimes \mathrm{Hom}(\mathcal{P}_\beta, \mathcal{P}_\gamma) \to \mathrm{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\gamma)$$

simplifies to the natural cup product of cochains

$$\cup\colon C^*(U_\alpha; k) \otimes C^*(U_\beta; k) \to C^*(U_\alpha; k)$$

which clearly satisfies $e_\alpha^\beta \cup e_\beta^\gamma = e_\alpha^\gamma$. □

4.2 Restriction functors

We continue with a fixed rooted tree $\mathcal{T} = (T, \rho)$ with smoothed arboreal hypersurface

$$\mathsf{H}_\mathcal{T} = \bigcup_{\alpha \in V(T)} \mathsf{H}_\alpha \subset \mathbb{R}^\mathcal{T}$$

with conormal Legendrian the arboreal singularity

$$\mathsf{L}_\mathcal{T} = \bigcup_{\alpha \in V(T)} \mathsf{L}_\mathcal{T}(\alpha) \subset S^*\mathbb{R}^\mathcal{T}.$$

Recall that the strata $\mathsf{L}_\mathcal{T}(\mathfrak{p}) \subset \mathsf{L}_\mathcal{T}$ are contractible and indexed by correspondences

$$\mathfrak{p} = (R \overset{q}{\twoheadleftarrow} S \overset{i}{\hookrightarrow} T)$$

where i is the inclusion of a subtree, and q is a quotient map of trees. Furthermore, the normal slice to the stratum is homeomorphic to the arboreal singularity L_R.

Fix any point $\lambda \in \mathsf{L}_{\mathcal{T}}(\mathfrak{p})$ with projection $x = \pi(\lambda) \in \mathsf{H}_{\mathcal{T}}$. Choose a small open ball $B(\mathfrak{p}) \subset \mathbb{R}^{\mathcal{T}}$ around x. The open set $\pi^{-}(B) \subset S^*\mathbb{R}^{\mathcal{T}}$ intersects $\mathsf{L}_{\mathcal{T}}$ in possibly many connected components. Let $\Lambda(\mathfrak{p}) \subset \mathsf{L}_{\mathcal{T}}$ denote the connected component containing λ.

Introduce the dg category $\mathrm{Sh}_{\Lambda(\mathfrak{p})}(B(\mathfrak{p}))$ of constructible complexes of k–vector spaces on $B(\mathfrak{p})$ microlocalized along the Legendrian subspace $\Lambda(\mathfrak{p}) \subset S^* B(\mathfrak{p})$. Restriction of sheaves along the open inclusion $B(\mathfrak{p}) \subset \mathbb{R}^{\mathcal{T}}$ induces a natural microlocal restriction functor

$$\mathrm{res}\colon \mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}}) \to \mathrm{Sh}_{\Lambda(\mathfrak{p})}(B(\mathfrak{p})).$$

Let us denote by $\mathcal{N}(\mathfrak{p}) \subset \mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ the full dg subcategory generated by the objects

(1) $\mathcal{P}_\alpha \in \mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ when $\alpha \notin i(V(S))$,

(2) $\mathcal{S}_\alpha \in \mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ when $\alpha, \hat{\alpha} \in i(V(S))$ and $q(\alpha) = q(\hat{\alpha}) \in V(R)$.

Observe that the singular support of any of the above generating objects, and hence any object of $\mathcal{N}(\mathfrak{p})$, is disjoint from $\mathsf{L}_{\mathcal{T}}(\mathfrak{p}) \subset \mathsf{L}_{\mathcal{T}}$. Thus we have the evident vanishing

$$\mathrm{res}(\mathcal{F}) \simeq 0 \quad \text{for any } \mathcal{F} \in \mathcal{N}(\mathfrak{p}).$$

Remark 4.5 Thanks to the canonical exact triangle

$$\mathcal{P}_{\hat{\alpha}} = i_{\hat{\alpha}!} i_{\hat{\alpha}}^! \mathcal{P}_\alpha \xrightarrow{u} \mathcal{P}_\alpha \longrightarrow i_{\alpha!} j_{\alpha*} j_\alpha^* \mathcal{P}_\alpha = \mathcal{S}_\alpha \xrightarrow{[1]}$$

the vanishing $\mathrm{res}(\mathcal{S}_\alpha) \simeq 0$ is equivalent to $\mathrm{res}(u)$ being a quasi-isomorphism.

We will see that the microlocal restriction functor exhibits $\mathrm{Sh}_{\Lambda(\mathfrak{p})}(B(\mathfrak{p}))$ as the dg quotient of $\mathrm{Sh}_{\mathsf{L}_{\mathcal{T}}}(\mathbb{R}^{\mathcal{T}})$ by the dg subcategory $\mathcal{N}(\mathfrak{p})$.

To spell this out, observe that the quiver structure on $\mathcal{T}$ induces one on the subtree S and subsequent quotient tree R. Let us write $\mathcal{S}$ and $\mathcal{R}$ to denote S and R with their respective quiver structures.

Consider the inclusion of the subtree $i\colon S \hookrightarrow T$. Define the quotient functor

$$i^*\colon \mathrm{Mod}(\mathcal{T}) \to \mathrm{Mod}(\mathcal{S})$$

by killing the projective modules $P_\alpha \in \mathrm{Mod}(\mathcal{T})$, when $\alpha \notin i(V(S))$.

Consider the quotient map $q\colon S \twoheadrightarrow R$. Define the quotient functor

$$q_!\colon \mathrm{Mod}(\mathcal{S}) \to \mathrm{Mod}(\mathcal{R})$$

by killing the simple modules $S_\alpha \in \mathrm{Mod}(\mathcal{S})$, when $q(\alpha) = q(\hat{\alpha}) \in V(R)$.

Observe that the composite

$$q_! i^*\colon \mathrm{Mod}(\mathcal{T}) \to \mathrm{Mod}(\mathcal{R})$$

is the quotient functor by the full dg subcategory $N(\mathfrak{p}) \subset \mathrm{Mod}(\mathcal{T})$ generated by

(1) $P_\alpha \in \mathrm{Mod}(\mathcal{T})$ when $\alpha \notin i(V(S))$,

(2) $S_\alpha \in \mathrm{Mod}(\mathcal{T})$ when $\alpha, \hat{\alpha} \in i(V(S))$ and $q(\alpha) = q(\hat{\alpha}) \in V(R)$.

Proposition 4.6 *There is a natural commutative diagram:*

$$\begin{array}{ccc} \mathrm{Sh}_{\mathsf{L}_\mathcal{T}}(\mathbb{R}^\mathcal{T}) & \xrightarrow{\sim} & \mathrm{Mod}(\mathcal{T}) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle q_! i^*} \\ \mathrm{Sh}_{\Lambda(\mathfrak{p})}(B(\mathfrak{p})) & \xrightarrow{\sim} & \mathrm{Mod}(\mathcal{R}) \end{array}$$

Proof The proof is essentially a repeat of the proofs of Proposition 4.3 and Theorem 4.4.

Choose a vertex $\alpha \in V(T)$ in each fiber of $q\colon S \twoheadrightarrow R$, and denote their union by $\widetilde{V}(R) \subset V(T)$. Consider the collection of objects $\mathcal{P}_\alpha \in \mathrm{Sh}(\mathbb{R}^\mathcal{T})$, for $\alpha \in \widetilde{V}(R)$. Denote by $\mathcal{P}_{\overline{\alpha}} \in \mathrm{Sh}(B(\mathfrak{p}))$, for $\alpha \in \widetilde{V}(R)$, their restrictions along the open inclusion $B(\mathfrak{p}) \subset \mathrm{Sh}(\mathbb{R}^\mathcal{T})$.

By the same argument as in the proof of Proposition 4.3, we see that the collection of objects $\mathcal{P}_{\overline{\alpha}} \in \mathrm{Sh}(B(\mathfrak{p}))$, for $\alpha \in \widetilde{V}(R)$, generates $\mathrm{Sh}_{\Lambda(\mathfrak{p})}(B(\mathfrak{p}))$.

By the same argument as in the proof of Theorem 4.4, we see that the generating objects $\mathcal{P}_{\overline{\alpha}} \in \mathrm{Sh}(B(\mathfrak{p}))$, for $\alpha \in \widetilde{V}(R)$, give an equivalence $\mathrm{Sh}_{\Lambda(\mathfrak{p})}(B(\mathfrak{p})) \simeq \mathrm{Mod}(\mathcal{R})$. □

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