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Geometry & Topology 21 (2017) 1805–1835

Strong accessibility for finitely presented groups

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A hierarchy of a group is a rooted tree of groups obtained by iteratively passing to vertex groups of graphs of groups decompositions. We define a (relative) slender JSJ hierarchy for (almost) finitely presented groups and show that it is finite, provided the group in question doesn't contain any slender subgroups with infinite dihedral quotients and satisfies an ascending chain condition on certain chains of subgroups of edge groups.

As a corollary, slender JSJ hierarchies of finitely presented subgroups of $\mathrm{SL}_n(\mathbb{Z})$ or of hyperbolic groups which are (virtually) without 2–torsion are finite.

20E08, 20F65, 20F67, 57M60

1 Introduction

A group G is said to be *accessible* over a family of subgroups $\mathcal{C}$ if there is an upper bound to the size of reduced graphs of groups decompositions of G with edge groups in $\mathcal{C}$. The classic theorem is due to Grushko and Neumann: If $G = A * B$ is a nontrivial decomposition of G as a free product, then $\mathrm{rk}(A) + \mathrm{rk}(B) = \mathrm{rk}(G)$, where $\mathrm{rk}(G)$ is the minimal number of elements needed to generate G. This implies that there is an upper bound to the size of reduced graphs of groups decompositions of a given finitely generated group G over trivial edge groups. As a consequence, every finitely generated group G admits a free product decomposition $G \cong G_1 * \cdots * G_p * \mathbb{F}_q$, where each G_i is freely indecomposable and $\mathbb{F}_q$ is free.

Finitely generated groups are not accessible over the class of small subgroups, as Dunwoody [7] and Bestvina and Feighn [2] have produced counterexamples (with finite and small edge groups, respectively). Finitely presented groups, on the other hand, are accessible over the class of small subgroups; see Dunwoody [6] and Bestvina and Feighn [1]. In particular, any sequence of reduced refinements of graphs of groups decompositions of a finitely presented group over small edge groups must terminate. This was shown for two-generated torsion-free hyperbolic groups (see Kapovich and Weidmann [14]), and a similar theorem holds for sequences of minimal graph of groups decompositions of Coxeter groups (see Mihalik and Tschantz [16]).

Published: 10 May 2017 DOI: 10.2140/gt.2017.21.1805

Dunwoody's theorem, along with Stallings' theorem on groups with infinitely many ends, implies that any finitely presented group admits a graph of groups decomposition with finite or one-ended vertex groups. (We will call this the Grushko–Stallings–Dunwoody, or GSD, decomposition.)

The slender JSJ decomposition of a finitely presented group is the natural generalization of the GSD decomposition to splittings over slender subgroups, and it is natural to ask if the process of iteratively passing to vertex groups of slender JSJ decompositions terminates, or in other words, if a group is *strongly accessible*.

We know of two nonartificial classes of groups which have hierarchies that must be finite. A Haken hierarchy of a three-manifold gives a finite hierarchy in this sense: an incompressible two-sided surface in a three-manifold corresponds to a splitting of its fundamental group over the fundamental group of the surface. Sela [18] and, independently, Kharlampovich, Myasnikov and Remeslennikov [15] have shown that the hierarchy, or *analysis lattice* in Sela's terminology, of a limit group obtained by alternatingly passing to vertex groups of the Grushko or abelian JSJ decomposition is finite. It should be noted that finiteness of analysis lattices is used to prove finite presentability of limit groups, rather than the other way around.

Delzant and Potyagailo claim in [5] that finitely presented groups admit finite hierarchies over elementary families (see Definition 5.1), but unfortunately, the proof of [5, Lemma 4.10] is not correct. See Section 5. We believe that any proof which attempts to assign a complexity to each group in a hierarchy is unlikely to work.

2 Definitions and results

A group is *small* if it doesn't contain a nonabelian free subgroup. An action of a small group on a tree is either *elliptic* (fixes a point in the tree), *hyperbolic* (has an axis and acts by translations), *dihedral* (has an axis and acts dihedrally) or *parabolic* (fixes a point in the boundary but has no axis) [1, page 453]. A group is *slender* if all its subgroups are finitely generated. An action of a slender group on a tree is either elliptic or stabilizes an axis.

Definition 2.1 (hierarchy) A *hierarchy* for a group H is a rooted tree of groups $\mathcal{H}$, with H at the root, such that the descendants of a group $L \in \mathcal{H}$ are the vertex groups of a nontrivial graph of groups decomposition Δ_L of L. A group $L \in \mathcal{H}$ is *terminal* if L has no descendants.

A hierarchy is *slender* if all graphs of groups decompositions Δ_L, for $L \in \mathcal{H}$, are over slender edge groups. A slender hierarchy is *hyperbolic* if for all $L \in \mathcal{H}$, if $E < L$ is an edge group of Δ_L, then for every $L' \in \mathcal{H}$ and every conjugate E^g of E, the action of $E^g \cap L'$ on $T_{\Delta_{L'}}$, the tree associated to $\Delta_{L'}$, is either elliptic or hyperbolic, but not dihedral.

Subgroups which are conjugate to elliptic subgroups in every level of the hierarchy are particularly important.

Definition 2.2 ($\mathcal{H}$–elliptic) Denote the groups at level n of the hierarchy by $\mathcal{H}^n$. A subgroup $V < L \in \mathcal{H}^n$ is *$\mathcal{H}$–elliptic* if there is a chain $L = L_n > L_{n+1} > L_{n+2} > \cdots$ such that $L_i \in \mathcal{H}^i$ is a vertex group of $\Delta_{L_{i-1}}$ and if there are elements $h_i \in L_i$ such that

$$V < L_n \cap L_{n+1}^{h_n} \cap L_{n+2}^{h_{n+1}h_n} \cap L_{n+3}^{h_{n+2}h_{n+1}h_n} \cap \cdots ,$$

where by L^h we mean hLh^{-1}.

We denote the collection of $\mathcal{H}$–elliptic subgroups by $\mathcal{H}^\infty$. Let $\mathcal{EG}_{\mathcal{H}}^n$ be the collection of conjugates of edge groups of graphs of groups decompositions Δ_L as L varies over all groups in $\mathcal{H}^n$, let $\mathcal{EG}_{\mathcal{H}} = \bigcup_n \mathcal{EG}_{\mathcal{H}}^n$, let $\mathcal{C}_{\mathcal{H}}^n$ be the collection

$$(\mathrm{Subgroups}(\mathcal{EG}_{\mathcal{H}}^n \setminus \mathcal{H}^\infty)) \cap \mathcal{H}^\infty,$$

and let $\mathcal{C}_{\mathcal{H}} = \bigcup_n \mathcal{C}_{\mathcal{H}}^n$. In plain English, $\mathcal{C}_{\mathcal{H}}$ is the collection of $\mathcal{H}$–elliptic subgroups of non-$\mathcal{H}$–elliptic edge groups.

Definition 2.3 (ascending chain condition) We say that $\mathcal{H}$ satisfies the *ascending chain condition, or acc, on $\mathcal{C}_{\mathcal{H}}$* if every ascending chain

$$S_n \le S_{n+1} \le S_{n+2} \le \cdots ,$$

where $S_i \in \mathcal{C}_{\mathcal{H}}^i$, stabilizes.

Recall that a group is *almost finitely presented* if it acts freely and cocompactly on a connected simplicial complex X with $H^1(X, \mathbb{Z}_2) = 0$; see [6]. We will use a slightly less restrictive notion of almost finitely presented.

Definition 2.4 ($\mathcal{H}$–almost finitely presented) Let H be a finitely generated group and $\mathcal{E}$ a family of subgroups of H. We say that H is *almost finitely presented relative to $\mathcal{E}$* if H acts cocompactly on a connected triangular complex X such that $H^1(X, \mathbb{Z}_2) = 0$ and cell stabilizers in X are either slender or conjugate into $\mathcal{E}$.

Let H be finitely generated and let $\mathcal{H}$ be a hierarchy of H. We say that H is *$\mathcal{H}$–almost finitely presented* if H is almost finitely presented relative to $\mathcal{H}^\infty$, the family of $\mathcal{H}$–elliptic subgroups.

Results

Theorem 2.5 (Main theorem) *Let H be a finitely generated group and let $\mathcal{H}$ be a slender hyperbolic hierarchy of H such that $\mathcal{H}$ satisfies the acc on $\mathcal{C}_{\mathcal{H}}$. If H is $\mathcal{H}$–almost finitely presented, then for some N there is a constant C such that each $L \in \mathcal{H}^m$, for $m \geq N$, has a finite hierarchy $\mathcal{X}_L$ of height at most C whose terminal groups are either $\mathcal{H}$–elliptic or slender.*

We apply the main theorem by imposing various conditions on finitely presented groups which guarantee that their slender JSJ hierarchies are hyperbolic. Call a slender group E with an infinite dihedral ($D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$) quotient *$D_\infty$–slender*. If E has no D_∞–slender subgroups, it is *$\mathbb{Z}$–slender*. Then a hierarchy for which every edge group is $\mathbb{Z}$–slender is hyperbolic. Note that finite groups and Tarski monsters are $\mathbb{Z}$–slender. Call a graph of groups decomposition $\mathbb{Z}$–slender if its edge groups are $\mathbb{Z}$–slender. For the construction of the (relative) slender JSJ decomposition, see [9] and Sections 4 and 9 of this paper.

Corollary 2.6 *Let H be finitely presented (relative to a family of subgroups $\mathcal{E}$) without any D_∞–slender subgroups. Let $\mathcal{H}$ be the hierarchy such that for each $L \in \mathcal{H}$, Δ_L is a slender* JSJ *decomposition (relative to $\mathcal{E}$) of L, and such that if $L \in \mathcal{H}$ is slender, then L is terminal. If $\mathcal{H}$ satisfies the acc on $\mathcal{C}_{\mathcal{H}}$, then $\mathcal{H}$ is finite.*

In Section 9 we show that the JSJ hierarchy of a relatively hyperbolic group satisfies the acc on $\mathcal{C}_{\mathcal{H}}$. Also note that a relatively hyperbolic group contains a two-ended D_∞–slender group if and only if it contains a noncentral element of order two.

Corollary 2.7 ((relatively) hyperbolic groups; see [14, Theorem C]) *Suppose that G is relatively hyperbolic, finitely generated, and without a noncentral element of order two. Then the hierarchy $\mathcal{H}$ such that Δ_L is the slender* JSJ *decomposition relative to peripheral subgroups which are not two-ended is finite. If G is a virtually without 2–torsion (for example, if G is residually finite) hyperbolic group, then the slender* JSJ *hierarchy of G is finite.*

If G is toral relatively hyperbolic, the same holds for the full abelian JSJ *decomposition.*

If $[G : G_1] < \infty$ and $H < G$ acts nontrivially on some tree T with slender edge stabilizers, then $H \cap G_1$ has finite index in H and acts on T with slender edge stabilizers; furthermore, if T corresponds to the (relative) slender JSJ decomposition of H, then $T/(H \cap G_1)$ is obtained from the (relative) slender JSJ decomposition of $H \cap G_1$ by possibly removing valence-two slender vertex groups, or by cutting enclosing vertex groups of the (relative) slender JSJ decomposition of $H \cap G_1$ along essential simple closed curves. In particular, the nonslender nonenclosing vertex groups of $T/(H \cap G_1)$ are finite-index subgroups of the nonslender nonenclosing vertex groups of the JSJ of G_1. Thus if the slender JSJ hierarchy of G has a nonslender nonenclosing vertex group at level n, then so does G_1. The corollary then follows for residually finite hyperbolic groups by observing that any residually finite hyperbolic group is virtually torsion free, so in particular, virtually without two-torsion.

Similarly, suppose G is a finitely presented subgroup of $\mathrm{SL}_n(\mathbb{Z})$. Since $\mathrm{SL}_n(\mathbb{Z})$ is virtually torsion free, so is G. The union S of a chain

$$S_1 \leq S_2 \leq \cdots$$

of slender subgroups of G is virtually solvable by Tits' alternative, and by [17, Section 2, Corollary 1], S is virtually polycyclic, hence slender. Any slender hierarchy $\mathcal{H}$ of G therefore satisfies the acc on $\mathcal{C}_{\mathcal{H}}$. Since strong accessibility passes to finite index overgroups we have:

Corollary 2.8 *The slender JSJ hierarchy of any finitely presented subgroup of* $\mathrm{SL}_n(\mathbb{Z})$ *is finite.*

As stated below, [5, Théorème 3.2] holds if we impose the ascending chain condition on finite subgroups of elements in an elementary family; see Definition 5.1.

Theorem 2.9 (cf [5, Théorème 3.2]) *Let G be finitely presented and let $\mathcal{C}$ be an elementary family of subgroups of G. Suppose that any ascending chain of finite subgroups of elements of $\mathcal{C}$ eventually stabilizes, and that two-ended subgroups of G are $\mathbb{Z}$–slender. Then G has a hierarchy $\mathcal{H}$ over edge groups in $\mathcal{C}$ such that terminal groups of $\mathcal{H}$ are either in $\mathcal{C}$ or don't split over an element of $\mathcal{C}$.*

Note that the hierarchy $\mathcal{H}$ in Theorem 2.9 is not a priori canonical, whereas the nonslender vertex groups appearing in the slender JSJ hierarchy are.

3 Dunwoody/Delzant–Potyagailo resolution

Given a simplicial complex X with a free G–action and a G–tree T, there is always a G–equivariant map from X to T. If T is simplicial and the map is chosen reasonably well, preimages of midpoints of edges form a subset of X called a *pattern*, the connected components of which are two-sided *tracks*. Patterns were introduced by Dunwoody in [6] to show that (almost) finitely presented groups are accessible, and used in [8] to construct a JSJ decomposition for finitely presented groups over slender edge groups.

If the action on X is not free, there is, in general, no G–equivariant map $X \to T$. The construction below is a generalization of [5, Section 4], after [6]. Like them, we construct a class of spaces such that if T is a (suitable) G–tree, then there are G–equivariant maps $X \to \widehat{T} = T \cup \partial T$, where ∂T is the boundary at infinity of T.

$\mathcal{H}$–complexes

All complexes in the sequel are at most two-dimensional.

Definition 3.1 ($\mathcal{H}$–complex) Let H be a finitely generated group with a hierarchy $\mathcal{H}$ over a class of groups $\mathcal{C}$. For $G < H$, an *$\mathcal{H}$–complex* for G is a connected simplicial complex X with $H^1(X, \mathbb{Z}_2) = 0$, X/G compact, and with cell stabilizers in $\mathcal{C}$ or $\mathcal{H}^\infty$. An $\mathcal{H}$–complex is *nondegenerate* if it contains a triangle and is *degenerate* if it doesn't.

The number of orbits of triangles in an $\mathcal{H}$–complex X is denoted by $\operatorname{covol}(X)$.

Denote the stabilizer of a cell $c \subset X$ by $\operatorname{Stab}_X(c)$ and the pointwise stabilizer by $\operatorname{Stab}^+_X(c)$, and if K acts on a space Z, denote the fixed point set of K by $\operatorname{Fix}_Z(K)$. If X and Z are clear from the context we will omit them.

Let X be a triangular CW–complex (a CW–complex whose two-cells have at most three sides). If X is not simplicial, let Y be the triangular complex whose vertices are the vertices of X, whose edges are determined by (unordered) pairs of distinct vertices which are the endpoints of some edge in X, and whose triangles are determined by (again unordered) triples of distinct vertices of X which are contained in a triangle. There is a continuous map $X \to Y$ which maps cells to cells of equal or lower dimension, and if the dimensions are the same then it maps interiors of cells homeomorphically to their images.

The next lemma is obvious.

Lemma 3.2 *If X is simply connected, then Y is simply connected. If $H^1(X, \mathbb{Z}_2) = 0$, then $H^1(Y, \mathbb{Z}_2) = 0$.*

If $X \to Y$ is not a homeomorphism, then we say that X is *reducible*; if it is, then X is *reduced*. The space Y constructed above is said to be obtained from X by *reducing*. If X is equipped with an (combinatorial) action of a group G, then Y naturally inherits an action of G, and if cell stabilizers in X are in some class $\mathcal{C}$ (for example if X is an $\mathcal{H}$–complex and $\mathcal{C}$ is the collection of slender or $\mathcal{H}$–elliptic subgroups) which is closed under passing to subgroups and extensions by subgroups of S_3, then cell stabilizers in Y are elements of $\mathcal{C}$ as well.

Remark 3.3 It is *not* necessarily the case that $\operatorname{Stab}^+(c) = \operatorname{Stab}(c)$ for a cell c in Y, even if this is the case in X.

Cut trees

A *cutpoint* in a simplicial complex X is a vertex v such that $X \setminus v$ has more than one component, and a *cutpoint-free component* of X is a maximal connected subcomplex which isn't separated by a cutpoint. Any simplicial complex is a union of cutpoint-free components which are either disjoint or meet in a single vertex. Suppose X is connected and let $\{Y_\alpha\}$ be the collection of cutpoint-free components in X, let $\{v_\beta\}$ be the collection of cutpoints, and finally let T_X be the tree whose vertex set is the collection of cutpoint-free components and cutpoints of X, and whose edges are given by pairs (Y_α, v_β) such that $v_\beta \in Y_\alpha$.

Suppose that X is connected, $H^1(X, \mathbb{Z}_2) = 0$, and that X doesn't have any cutpoints. A *cut-edge* is an edge e such that $X \setminus e$ has at least two components. We mimic the definition above and let the cut-edge tree S_X be the tree whose vertices are the maximal connected cut-edge-free components $\{Y_\alpha\}$ of X and cut-edges $\{e_\gamma\}$ in X, and whose edges are given by pairs (Y_α, e_γ) such that $e_\gamma \subset Y_\alpha$.

Resolving actions on trees

Suppose X is an $\mathcal{H}$–complex for $G < G' \in \mathcal{H}$, and let T be the tree associated to $\Delta_{G'}$. Cell stabilizers in X might not act elliptically in T, and there is therefore no G–equivariant map $X \to T$. If $\mathcal{H}$ is a slender hyperbolic hierarchy, then each cell stabilizer in X fixes a point in $\widehat{T} = T \cup \partial T$. We exploit this fact to produce a G–equivariant map $X \to \widehat{T}$, a la Dunwoody and Delzant–Potyagailo.

Definition 3.4 Let X be a reduced triangular complex with an action of a group G, and let G act on a tree T without inversions. A G–equivariant map $\rho\colon X \to \widehat{T}$ such that vertices are mapped to vertices or points in ∂T, interiors of edges are mapped homeomorphically to interiors of arcs in T, and each intersection of a triangle t and a connected component of the preimage of a midpoint of an edge of T is an embedded closed arc connecting distinct edges of t is a *resolution*.

Lemma 3.5 *Let G act on a triangular complex X, and let T be the tree associated to a graph of groups decomposition of G. Suppose that*

- *vertex stabilizers in X act either elliptically, hyperbolically, or parabolically on T, and*
- *if $W \subset X$ is a connected subset of X^1 such that $\mathrm{Stab}_X(e)$ acts hyperbolically or parabolically in T for all cells $e \subset W$, then all stabilizers of vertices in W have a common fixed point in ∂T.*

Then there is a resolution $\rho\colon X \to \widehat{T}$.

Remark 3.6 Lemma 3.5 is sharp, which is one reason our proof of strong accessibility only works for splittings over slender edge groups or edge groups in an elementary family. These are seemingly the only natural hypotheses which guarantee the lemma holds.

Proof Since cell stabilizers in X don't act dihedrally on T, $\mathrm{Fix}_{\hat{T}}(\mathrm{Stab}(c))$ is not empty for all cells c in X.

Let v be a representative of an orbit of vertices in X. Choose arbitrarily a point $\rho(v) \in \mathrm{Fix}_{\hat{T}}(\mathrm{Stab}(v)) \in \hat{T}$, provided that if $\mathrm{Stab}(v)$ is elliptic then $\rho(v) \in T^0$, and that if v and w are contained in a connected subset $W \subset X^1$ as in the second bullet, then $\rho(v) = \rho(w)$ is a point fixed by all stabilizers of cells in W. For each $g \cdot v$ in the orbit of v, set $\rho(g \cdot v) = g \cdot \rho(v)$. Repeat over all orbits of vertices.

Let e be a representative of an orbit of edges of X, and let v and w be the endpoints of e. Suppose that $\mathrm{Stab}_X(e)$ inverts e and acts elliptically in T. Let b_e be the fixed point of the stabilizer of e, and choose $c \in \mathrm{Fix}_T(\mathrm{Stab}_X(e))$. Let r_v and r_w be the two arcs/rays in T connecting c to $\rho(v)$ and $\rho(w)$, respectively, and set $\rho(b_e)$ to be the furthest point in the nonempty, but possibly degenerate, arc $r_v \cap r_w$ from c. In particular, if $\rho(v) = \rho(w) \in \partial T$, then set $\rho(e) = \rho(v)$. Then $\mathrm{Stab}(e)$ stabilizes the (possibly degenerate) arc in T connecting $\rho(v)$ and $\rho(w)$. Map $[b_e, v]$ homeomorphically to the (possibly degenerate) interval connecting $\rho(v)$ and $\rho(b_e)$ in $\hat{T}$, and extend equivariantly. If $\mathrm{Stab}(e)$ doesn't invert e, then map e in the obvious way to the (again, possibly degenerate) arc connecting $\rho(v)$ to $\rho(w)$. Repeat over all orbits of edges.

The edges of a triangle t determine a (possibly degenerate) tripod in T. Map t to T equivariantly (t may have nontrivial stabilizer), and extend equivariantly to all translates of t. See Figure 1. Repeat over all orbits of triangles. (As with edges which are inverted, some care must be taken since $\mathrm{Stab}(t)$ might not fix t.) □

Corollary 3.7 (cf [5, Section 4.1]) *Given a slender hyperbolic hierarchy $\mathcal{H}$ of a group H, an $\mathcal{H}$–complex X for $G < H$, and a slender G–tree T from $\mathcal{H}$, there is a resolution $\rho\colon X \to \hat{T}$.*

If $\mathcal{C}$ is an elementary family in a group G, X a reduced G–complex with cell stabilizers in $\mathcal{C}$, and T is a G–tree with edge stabilizers in $\mathcal{C}$ such that no element of $\mathcal{C}$ acts dihedrally on T, then there is a resolution $\rho\colon X \to \hat{T}$.

Let G, X, and T be as in the first paragraph of Corollary 3.7. We divide our treatment of the map ρ constructed in Lemma 3.5 into two cases.

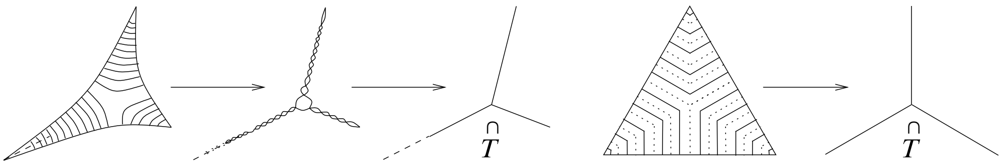

Figure 1: The first sequence represents the map on a typical triangle. The vertex in the lower left corner is sent to ∂T, which is indicated by the dotted lines. The center picture represents the triangle after crushing preimages of midpoints of edges, which introduces bigons and creates a new triangle. This is essentially [5, Dessins 1 et 2]. The second map illustrates a case where $\mathrm{Stab}(t) \neq \mathrm{Stab}^+(t)$. Preimages of vertices are represented by dotted lines and preimages of midpoints of edges are solid lines. If the stabilizer of t acts as S_3 on t, then ρ must send a tripod connecting the centers of edges of t to a vertex in T.

Type 1 $\rho^{-1}(\partial T)$ does not contain an edge of X Let $X^* = X \setminus \rho^{-1}(\partial T)$. For each edge e of T, let m_e be the midpoint of e, and let Λ' be the one-complex $\rho^{-1}(\bigcup_e \{m_e\}) \subset X^*$. Call a connected component λ of Λ' *essential* if both components of $X \setminus \lambda$ are unbounded and λ is not parallel to the link of a vertex, and let Λ be the union of all essential components of Λ'. For the remainder of the paper Λ' and Λ will be used to indicate patterns constructed in the manner described above.

Let X^*/Λ be the space obtained by collapsing each connected component of Λ to a point, and let X_T be the space obtained by reducing; see [5, Proposition 4.2]. The stabilizers of the vertices corresponding to connected components of Λ are slender, and there is a G–equivariant map $X^*/\Lambda \to X_T$. A simple case of this procedure is illustrated in Figure 3.

Lemma 3.8 (cf [5, Lemma 4.9]) *If $\pi_1(X) = 1$, then we have $\pi_1(X^*/\Lambda) = 1$. If $H^1(X, \mathbb{Z}_2) = 0$, then $H^1(X^*/\Lambda, \mathbb{Z}_2) = 0$.*

See Figure 2.

Proof Let Z be a connected component of X^*, let Y be its closure in X, and let W be the connected component of X^*/Λ corresponding to Z. Let B be the second barycentric subdivision of Y, C the union of simplices in B which miss $\rho^{-1}(\partial T)$, A the union of simplices in B which meet $\rho^{-1}(\partial T)$, and let $L = A \cap C$.

Consider the Mayer–Vietoris sequence for the pair of subspaces A and C:

$$\cdots \to H^1(Y, \mathbb{Z}_2) \to H^1(A, \mathbb{Z}_2) \oplus H^1(C, \mathbb{Z}_2) \to H^1(L, \mathbb{Z}_2) \to \cdots .$$

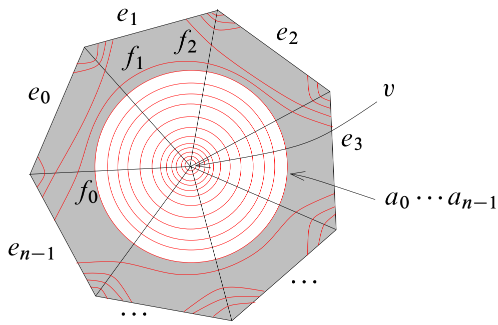

Figure 2: Illustration for Lemma 3.8. The outer loop represents the path $e_0 \cdots e_{n-1}$, which is homotopic (with the homotopy represented by the shaded annulus) to the path $a_0 \cdots a_{n-1}$, which is a path in Λ; hence d is trivial in X^*/Λ.

Each connected component of A is contractible, the inclusion $C \hookrightarrow Z$ is a homotopy equivalence, and since $H^1(Y, \mathbb{Z}_2) = 0$, there is an exact sequence

$$0 \to H^1(Z, \mathbb{Z}_2) \to H^1(L, \mathbb{Z}_2).$$

It therefore suffices to show that any closed path in L dies under the map $Z \to W$. The situation is similar for the fundamental group: all π_1 is carried by L.

Let d be a reduced edge path in L. Then there is a vertex $v \in Y \cap \rho^{-1}(\partial T)$ such that d is homotopic in the star of v to an edge path $e_0 \cdots e_{n-1}$ in the link of v. Let $t_0, \dots, t_{n-1}$ be the triangles in Y with $e_i, \{v\} \subset t_i$, and let $f_0, \dots, f_{n-1}$ be the edges connecting v to e_i such that the boundary of t_i is formed by e_i, f_i and f_{i+1}. Since $\rho(e_i) \neq \rho(v)$, there is an edge e of T such that $\rho^{-1}(m_e) \cap t_i$ is a single arc a_i connecting f_i to f_{i+1} in t_i for each i. This implies the collection of arcs $\{a_i\}$ forms a closed loop, and d is homotopic *in* Z to the path $a_0 \cdots a_{n-1}$ in $\Lambda \subset Z$. Thus d has nullhomotopic image in W. Hence if X is acyclic, then W is as well, and if X has trivial fundamental group, then so does W. □

Type 2 $\rho^{-1}(\partial T)$ contains an edge of X Let X_T be the complex obtained by collapsing each connected component of $\rho^{-1}(\partial T)$ to a point and reducing.

Lemma 3.9 *Vertex stabilizers in X_T either*

- *are vertex stabilizers from X,*
- *act hyperbolically in T and are HNN extensions of subgroups of edge groups in T/G, or*
- *act parabolically in T and are strictly ascending HNN extensions of subgroups of edge groups in T/G.*

Each edge stabilizer in X_T is, or has an index two subgroup which is, a subgroup of a conjugate of an edge group in T/G.

If X is an $\mathcal{H}$–complex for $G < L \in \mathcal{H}$ and $T = T_{\Delta_L}$, then vertex stabilizers in the third category are small but not slender. If X is a G–complex with stabilizers in an elementary family $\mathcal{C}$, then stabilizers in X_T are in $\mathcal{C}$.

Let $\mathcal{H}$ be a slender hierarchy and suppose X, G and T are as above. Since X_T potentially has small but not slender vertex stabilizers, ie $\rho\colon X \to \widehat{T}$ is Type 2, it is *not*, in general, an $\mathcal{H}$–complex.

Proof Let v be a vertex in X_T. Denote Stab(v) by G_v. If v is a vertex from X, then clearly G_v is a vertex stabilizer from X. Suppose v corresponds to a connected component $V \subset X$ of $\rho^{-1}(p)$ with $p \in \partial T$. Clearly Stab$(V) = G_v$. If $gV \cap V \neq \varnothing$, then $g \cdot p = p$, and therefore, $gV = V$. Hence $V/G_v \to X/G$ is an embedding, and G_v fixes p.

Suppose X is an $\mathcal{H}$–complex. Since X/G is compact, G_v acts cocompactly on V, and since V has slender cell stabilizers, G_v is finitely generated. Let $T' \subset T$ be the union of axes of elements of G_v, and consider the quotient T'/G_v. Since G_v fixes an end in T, it fixes an end in T', and T'/G_v is therefore an ascending HNN extension with slender edge groups, hence is either small or slender, and if small, it acts parabolically on T.

Let e be an edge in X_T. The stabilizer of e either fixes the endpoints v and w of e or has an index two subgroup which does. Let V and W be the preimages of v and w in X. Then Stab$^+(e)$ stabilizes V and W. If V and W are connected components of preimages of points in ∂T then Stab$^+(e)$ fixes a pair of distinct points in ∂T, hence is a subgroup of an edge group of T/G. If V is not a connected component of a preimage of any point in ∂T, then V is a vertex, and Stab$^+(e)$ stabilizes a half-line in T, hence is (conjugate into) a subgroup of an edge group of T/G in this case as well. □

4 Remarks on accessibility

Kneser finiteness, existence of a Haken hierarchy, and Dunwoody/Bestvina–Feighn accessibility all rely on uniform upper bounds to the number of disjoint nonparallel tracks in two-complexes.

Theorem 4.1 *Let Y be a finite two-dimensional simplicial complex. There is a constant $C = C(Y)$ such that if $\Lambda \subset Y$ is a pattern with at least C connected components, then two connected components of Λ are parallel.*

Bestvina and Feighn's accessibility theorem for finitely presented groups is used to show that (almost) finitely presented groups have slender JSJ decompositions.

Theorem [1, Main theorem] *Let G be a finitely presented group. Then there exists an integer $\gamma(G)$ such that the following holds: if T is a reduced G–tree with small edge stabilizers, then the number of vertices in T/G is bounded by $\gamma(G)$.*

They remark that this holds for almost finitely presented groups, and that the proof goes through without change. In fact, slightly more is true:

- Let G be finitely generated and let $\mathcal{E}$ be a collection of subgroups of G. The conclusion holds if G acts cocompactly on a simplicial complex X with $H^1(X, \mathbb{Z}_2) = 0$ and cell stabilizers which are either slender, ascending HNN extensions of slender groups, or conjugate into $\mathcal{E}$, provided that elements of $\mathcal{E}$ act elliptically in T.
- Similarly, if G has a finite hierarchy $\mathcal{X}$ over edge groups which are slender or are ascending HNN extensions of slender subgroups, and such that each terminal leaf of $\mathcal{X}$ is either slender, an ascending HNN extension of a slender group, or of the form in the previous bullet, then the conclusion holds provided that elements of $\mathcal{E}$ act elliptically in T.

Accessibility of (almost) (relatively) finitely presented groups ensures the existence of a (relative) JSJ decomposition. In Section 9, we will use the above to define a (relative) JSJ hierarchy of (almost) (relatively) finitely presented groups. Since slender subgroups of finitely presented groups are not necessarily finitely presented, we must work in the category of almost finitely presented groups.

5 Counterexample to the proof of [5]

This section illustrates some of the problems with the approach to strong accessibility taken by [5]. We sketch their proof below, and try to make clear why such an approach is unlikely to be successful.

Definition 5.1 [5, Section 1.1] An *elementary family* in a group G is a family $\mathcal{C}$ of subgroups with the following properties:

- $\mathcal{C}$ is closed under conjugation and passing to infinite subgroups.
- Each infinite subgroup of $\mathcal{C}$ is contained in a unique maximal element of $\mathcal{C}$, and each ascending union of finite elements of $\mathcal{C}$ is an element of $\mathcal{C}$.

- Elements of $\mathcal{C}$ are *small*, ie if $A \in \mathcal{C}$, then for every subgroup $B \le A$ acting minimally on an infinite tree T, either B fixes a point in ∂T or stabilizes a pair of distinct points in ∂T. (Equivalently, no element of $\mathcal{C}$ contains a free subgroup.)
- If $C \in \mathcal{C}$ is infinite and maximal in $\mathcal{C}$, and if $C^g = C$, then $g \in C$. In particular, for a maximal C and $C' < C$, the normalizer of C' is contained in C.

Elementary families are designed to mimic the family of elementary subgroups of a (relatively) hyperbolic group, ie the class of virtually cyclic (or peripheral, in the relative case) subgroups.

Delzant and Potyagailo claim:

Theorem 5.2 [5, Théorème 3.2] *Let G be finitely presented, and let $\mathcal{C}$ be an elementary family of subgroups of G. Then G has a hierarchy $\mathcal{H}$ over edge groups in $\mathcal{C}$ such that terminal groups of $\mathcal{H}$ are either in $\mathcal{C}$ or don't split over an element of $\mathcal{C}$.*

Note that we are only able to prove Theorem 2.9 with the additional hypotheses that the collection of finite subgroups of elements of $\mathcal{C}$ satisfies the ascending chain condition.

Let G be finitely presented, and suppose G acts simplicially, cocompactly, and without inversions on a simply connected triangular complex X with cell stabilizers in an elementary family $\mathcal{C}$. The quotient X/G is then a complex of groups. The *T–invariant* of G is the ordered pair

$$T(G) = \min\big\{\big(|X/G|, \mathrm{b}_1(X/G)\big) \,\big|\, X \text{ as above}\big\},$$

where $|X/G|$ is the number of triangles in X/G. The set of such ordered pairs is ordered lexicographically.

Suppose the action of G on X achieves the T–invariant. Let T be a G–tree with edge stabilizers in $\mathcal{C}$, and let $\varphi\colon X \to \widehat{T}$ be the map constructed above. Suppose that X^* is not connected. Then G splits as a graph of groups over edge groups in $\mathcal{C}$ so that vertex groups have strictly lower T–invariant; therefore, we may assume that X^* is connected. They first construct X_T and its cutpoint tree T_{X_T}. Edge stabilizers in T_{X_T} are elements of $\mathcal{C}$, and the quotient graph of groups decomposition has vertex groups G_i, each of which acts as above on a cutpoint-free component X_i of X_T.

In order to conclude that the G_i have lower T–invariants than G, they erroneously claim in [5, Lemma 4.10] that there is a map $X/G \to X_T/G$ with connected fibers, inducing an isomorphism on fundamental groups, hence that

$$\mathrm{b}_1(X/G) \ge \sum_i \mathrm{b}_1(X_i/G_i) + \mathrm{b}_1(T_{X_T}/G), \tag{1}$$

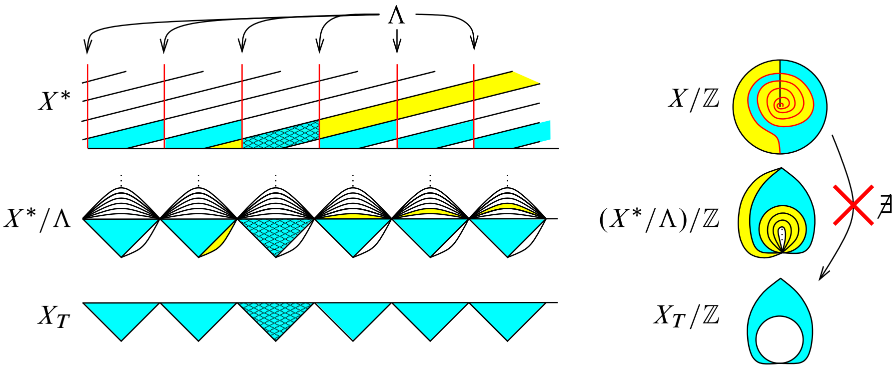

Figure 3: [5, Lemma 4.10] violates the no retraction theorem.

where each X_i is a representative of an orbit of cutpoint-free components of X_T, and G_i is its stabilizer. They then argue that if $\mathrm{b}_1(T_{X_T}/G) = 0$, then there is more than one orbit of cutpoint-free components in X_T. In particular, $|X/G| > |X_i/G_i|$, and if $\mathrm{b}_1(T_{X_T}/G) > 0$ and there is only one orbit of cutpoint-free components, then $|X/G| \geq |X_1/G_1|$ and $\mathrm{b}_1(X/G) > \mathrm{b}_1(X_1/G_1)$, hence $T(G) > T(G_1)$.

The argument used to prove (1) is incorrect. See Figure 3. Consider a disk $X/\mathbb{Z}$ with one orbifold point, labeled $\mathbb{Z}$, and two edges, such that the boundary of the disk defines a generator. Then X is the (orbihedral) universal cover of the disk. The cyclic group $\mathbb{Z}$ acts on the line T, and $\pi_1(X_T/\mathbb{Z}) = \mathbb{Z}$. Any continuous map from a disk to a circle, however, has nullhomotopic image; hence there is in general no G–equivariant map $X \to X_T$.

It is important to note that this is only a counterexample to the proof of [5, Lemma 4.10], not its conclusion: we know a priori that the disk $X/\mathbb{Z}$ doesn't achieve the T–invariant of $\mathbb{Z}$. Their proof however, never actually uses the hypothesis that X achieves $T(G)$. Any such proof *must* either show that (1) holds or that $|X/G|$ is not minimal. We think it's unlikely that a proof of strong accessibility along these lines exists.

6 Products of trees

Let G be a group, and let T and T' be a pair of G–trees with $T/G = \Delta$ and $T'/G = \Omega$. Then G acts diagonally on the product $T \times T'$. If $S \subset T \times T'$ is a simply connected G–invariant subcomplex, the quotient S/G is a square complex, which, after [9], should be thought of as a complex of groups. Denote the projections $T \times T' \to T$ and $T \times T' \to T'$ by π_T and $\pi_{T'}$, respectively.

Let $S \subset T \times T'$ be a simply connected G–invariant subset and suppose that point preimages under π_T are connected. For v a vertex of Δ, let $\tilde{v}$ be a lift of v to T.

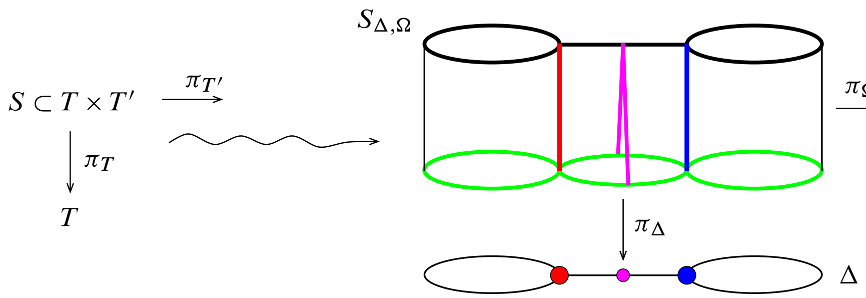

Figure 4: Projections of $S_{\Delta,\Omega}$ to Δ and Ω

Then $G_v \cong \mathrm{Stab}(\widetilde{v})$ acts on the tree $\pi_T^{-1}(\widetilde{v})$, and $\pi_T^{-1}(\widetilde{v})/G_v$ is a graph of groups decomposition of G_v. Similarly, if m_e is a midpoint of an edge of Δ, then $\pi_T^{-1}(\widetilde{m}_e)/G_e$ is a graph of groups decomposition of G_e.

Theorem 6.1 (cf [12, Théorème principal, Corollaire 8.2]) *There is a connected, simply connected, G–equivariant square complex $S \subset T \times T'$ of minimal covolume such that the projections $S \to T'$ and $S \to T$ have connected point preimages.*

Moreover, if G and all vertex and edge stabilizers are finitely generated and the G–trees T and T' are cocompact, then S/G may be taken to be compact.

Recall that a G–tree is *minimal* if it has no proper invariant subtrees, and that if a G–tree doesn't have a global fixed point (elliptic) or fixed end (parabolic), then there is a unique minimal invariant subtree. Though it is customary to assume that all G–trees are minimal, it is necessary to relax this restriction.

Let $S_{\Delta,\Omega} = S/G$, and denote the projections $S_{\Delta,\Omega} \to \Delta, \Omega$ by π_Δ and π_Ω. Then $S_{\Delta,\Omega}$ is finite, and if v is a vertex in Δ, then $\pi_\Delta^{-1}(v)$ is a graph of groups decomposition of G_v corresponding to its action on $\pi_T^{-1}(\widetilde{v})$. Similarly, if m is a midpoint of an edge of Δ then $\pi_\Delta^{-1}(m)$ is a graph of groups decomposition of G_e. The situation is the same for vertex and edge groups of Ω. See Figure 4.

The complex $S_{\Delta,\Omega}$ should be thought of as a complex of groups which interpolates between Δ and Ω, it is used extensively in [9] in the construction of the slender JSJ decomposition of a finitely presented group.

Lemma 6.2 *Let G be a finitely generated group, let $\mathcal{Y}_G$ be a finite hierarchy of G over finitely generated edge groups, and let Δ_G be a graph of groups decomposition of G with finitely generated edge groups. Then for each vertex group G_v of Δ_G, there is a finite hierarchy $\mathcal{X}_{G_v}$, of the same height as $\mathcal{Y}_G$, such that vertex and edge groups at level n in $\mathcal{X}_{G_v}$ are subgroups of vertex and edge groups at level n of $\mathcal{Y}_G$.*

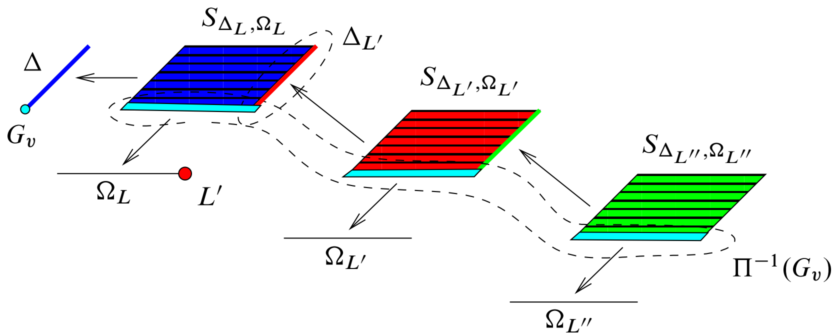

Figure 5: A piece of the hierarchy of square complexes associated to $\mathcal{Y}_G$ and Δ

Proof Let Ω_L be the decomposition of L for some $L \in \mathcal{Y}_G$. For each vertex group L' of Ω_L, define inductively $\Delta_{L'} = \pi_\Omega^{-1}(L') \subset \mathcal{S}_{\Delta_L,\Omega_L}$. See Figure 5. Consider a (nonterminal) vertex group L' of Ω_L and the projection

$$S_{\Delta_{L'},\Omega_{L'}} \to \Delta_{L'} = \pi_\Omega^{-1}(L') \subset S_{\Delta_L,\Omega_L}.$$

There is then a natural map Π from this hierarchy of square complexes to Δ, and if G_v is a vertex group of Δ, then $\mathcal{X}_{G_v} = \Pi^{-1}(v)$ is a hierarchy of G_v with the desired properties. □

7 $\mathcal{H}$–structures

There is no natural way to construct an $\mathcal{H}$–complex for each group L in a hierarchy $\mathcal{H}$ without losing control over the number of orbits of triangles. To get around this difficulty, we define an $\mathcal{H}$–structure, which is a combination of a hierarchy $\mathcal{X}$ (distinct from $\mathcal{H}$!) and a collection of $\mathcal{H}$–complexes (recall Definition 3.1) for terminal groups in $\mathcal{X}$. We associate, to each group L in a slender hyperbolic hierarchy $\mathcal{H}$, an $\mathcal{H}$–structure $\mathcal{X}_L$ and show in Section 8 that for groups sufficiently far down the hierarchy, the $\mathcal{H}$–structures may be taken to have terminal vertex groups which are $\mathcal{H}$–elliptic or slender. This will complete the proof of Theorem 2.5.

Definition 7.1 Let $\mathcal{H}$ be a hierarchy of a group H. An *$\mathcal{H}$–structure* on a group $L < H$ is a finite hierarchy over slender or small edge groups equipped with an action, for each terminal group V of $\mathcal{X}_L$, of V on an $\mathcal{H}$–complex X_V. If X_V is not a point then V is *nondegenerate*, and if X_V is a point then V is *degenerate*. The complexity $\operatorname{covol}(\mathcal{X}_L)$ is the total number of orbits of triangles over all X_V under their respective actions. See Figure 6.

An $\mathcal{H}$–structure for L with slender edge groups will be denoted by $\mathcal{X}_L$, and if an $\mathcal{H}$–structure for L possibly has small edge groups, then it is denoted by $\mathcal{Y}_L$. If $L \in \mathcal{H}$

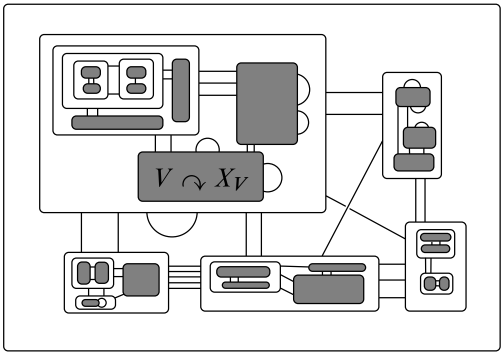

Figure 6: Schematic picture of an $\mathcal{H}$–structure. The outer box represents the top of the $\mathcal{H}$–structure, and the nesting indicates the hierarchy. Lines connecting rounded boxes are edges of the graph of groups decomposition at that level. Shaded boxes are terminal groups in the structure, and are either slender or are equipped with an action of their associated groups on an $\mathcal{H}$–complex.

has an $\mathcal{H}$–structure $\mathcal{Y}_L$, then we require that all nonslender small edge groups in $\mathcal{Y}_L$ act parabolically in the Bass–Serre tree T_{Δ_L}.

The *height* of an $\mathcal{H}$–structure on L is the number of levels in $\mathcal{X}_L$, and it is denoted by $\mathrm{height}(\mathcal{X}_L)$. We denote graphs of groups decompositions in $\mathcal{H}$–structures by Ω; ie if $L' \in \mathcal{X}_L$, then the graph of groups decomposition of L' will be denoted by $\Omega_{L'}$.

Resolving the action of G on T

In this section, $\mathcal{H}$ is assumed to be a slender hyperbolic hierarchy of a finitely generated group. Let X be a triangular complex with a G action, and let T_X be the cutpoint tree. Collapse edges with nonsmall stabilizers to obtain D_X.

Lemma 7.2 *Let $\mathcal{X}_G$ be an $\mathcal{H}$–structure for $G \in \mathcal{H}$. There are $\mathcal{H}$–structures $\mathcal{X}_{G_v}$, for $v \in \Delta_G$, such that*

$$\sum_{v\in\Delta_G} \mathrm{covol}(\mathcal{X}_{G_v}) \le \mathrm{covol}(\mathcal{X}_G),$$

and if X_B is a point for each terminal group B of $\mathcal{X}_G$, then

$$\mathrm{height}(\mathcal{X}_{G_v}) \le \mathrm{height}(\mathcal{X}_G).$$

Proof If there are no nondegenerate terminal vertex groups, set $\mathcal{Y}_G = \mathcal{X}_G$. Each terminal vertex group in the resulting decomposition is either $\mathcal{H}$–elliptic or slender. Note that $\mathrm{height}(\mathcal{Y}_G) \le \mathrm{height}(\mathcal{X}_G)$.

Let $\mathcal{X}_G$ be an $\mathcal{H}$–structure on G, and suppose G acts on a slender G–tree T with quotient Δ_G. Suppose $\mathcal{X}_G$ has a nondegenerate terminal group B acting on an $\mathcal{H}$–complex X_B. Let X' be the complex associated to B and T provided by Lemma 3.9. Let Ω_B be the graphs of groups decomposition $D_{X'}/B$, and for each vertex w of Ω_B, let X_w be the subcomplex of X' stabilized by B_w. There is a natural B_w–map $X_w \to \widehat{T}$, obtained by restriction, with X_w^* connected. Let Y_w be the $\mathcal{H}$–complex $(X_w)_T$. Now let $\Omega_{B_w} = D_{Y_w}/B_w$, and for each vertex z of D_{Y_w}/B_w, let $(B_w)_z$ act on the subcomplex of Y_w corresponding to a lift of z. Repeat over all nondegenerate terminal groups B of G to obtain an $\mathcal{H}$–structure $\mathcal{Y}_G$.

Let G_v be a vertex group of Δ_G, and let $\mathcal{X}_{G_v}$ be the hierarchy of G_v provided by Lemma 6.2 applied to $\mathcal{Y}_G$ and Δ_G. Let W be a terminal vertex group of $\mathcal{Y}_G$. By construction, W is elliptic in Δ_G, and Δ_W is a finite tree representing the trivial graph of groups decomposition of W. Suppose first that W is nondegenerate. For each vertex group V of Δ_W in $\mathcal{X}_{G_v}$, if V is slender, let X_V be a point, and if $W \geq V = W$, let $X_V = X_W$, the $\mathcal{H}$–complex associated to V. If W is degenerate, then each vertex group W of Δ_W is either $\mathcal{H}$–elliptic or slender, and in these cases, let X_W be a point. □

Hierarchy of $\mathcal{H}$–structures

Let H be a finitely generated group and $\mathcal{H}$ a slender hyperbolic hierarchy for H, and suppose H is $\mathcal{H}$–almost finitely presented. Let $\mathcal{X}_H$ be the trivial $\mathcal{H}$–structure with trivial graph of groups decomposition, and let X_H be any $\mathcal{H}$–complex for H.

Suppose that $\mathcal{X}_L$ has been defined for $L \in \mathcal{H}$. For Δ_L and a vertex group Z of Δ_L, let $\mathcal{X}_Z$ be the $\mathcal{H}$–structure on Z constructed in the previous subsection. Denote by $B_{L,1}, \dots, B_{L,n_L}$ the terminal vertex groups acting on nondegenerate $\mathcal{H}$–complexes $X_{B_{L,i}}$. Let $L_1, \dots, L_k$ be the descendants of L. Then each $X_{B_{L_j,k}}$ is obtained from some $X_{B_{L,i(k)}}$ by resolving the action of $B_{L,i(k)}$ on T_{Δ_L}. We call the collection of $B_{L_j,i(k)}$ such that $i(k) = i$ the *descendants* of $B_{L,i}$. Since X_H has finitely many triangles, for all but finitely many L, each $B_{L,i}$ has exactly one descendant $B_{L_j,i}$ and $\operatorname{covol}(X_{B_{L,i}}) = \operatorname{covol}(X_{B_{L_j,i}})$. We have:

Lemma 7.3 (cf [5, page 627])

$$\operatorname{covol}(X_{B_{L,i}}) \geq \sum_{\{k \mid i(k)=i\}} \operatorname{covol}(X_{B_{L_j,k}}),$$

and for all but finitely many $L \in \mathcal{H}$, the sum on the right is over one element and the inequality is an equality. There is some N_{tri} so that for $i \geq N_{\mathrm{tri}}$, this is the case.

Henceforth $i \geq N_{\mathrm{tri}}$.

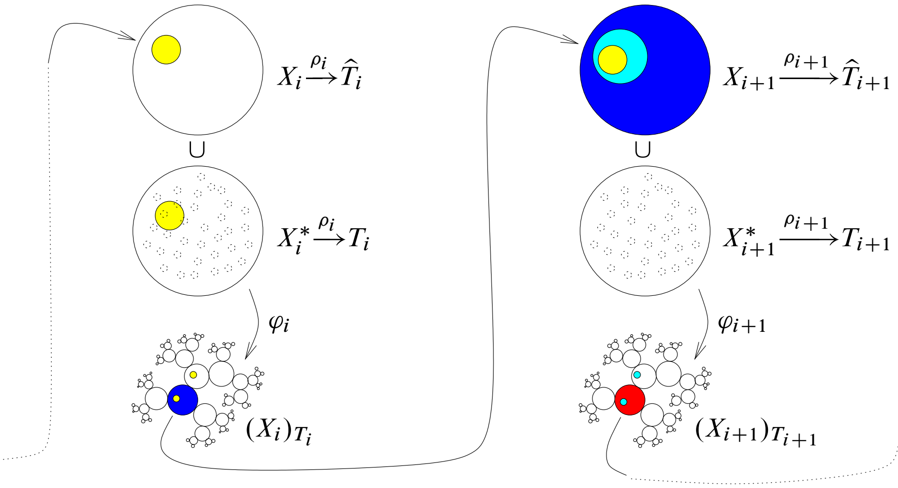

Figure 7: Sufficiently far down $\mathcal{H}$, the descendant X_{i+1} of X_i is a non-$\mathcal{H}$–elliptic cutpoint-free component of $(X_i)_{T_i}$ constructed from a Type 1 resolution.

8 Nondegenerate complexes converge to trees

The aim of this section is to replace, for groups sufficiently far down the hierarchy $\mathcal{H}$, each $\mathcal{H}$–structure $\mathcal{X}_L$ by an $\mathcal{H}$–structure with no triangles. This, along with the fact that the depth of the $\mathcal{H}$–structures is nonincreasing in this case (Lemma 7.2), will complete the proof of Theorem 2.5.

Consider the finite collection of infinite sequences of terminal vertex groups

$$\{G^p_{N_{\mathrm{tri}}} > G^p_{N_{\mathrm{tri}}+1} > G^p_{N_{\mathrm{tri}}+2} > \cdots\}$$

such that $G_i^p \in \mathcal{X}_{L(p,i)}$ ($L(p,i) \in \mathcal{H}^i$) is terminal, acts on a nondegenerate $\mathcal{H}$–complex $X_{G_i^p}^p$, and is the only descendant of G_{i-1}^p with

$$\mathrm{covol}(X^p_{G_i^p}) = \mathrm{covol}(X^p_{G^p_{i+1}}).$$

To simplify notation, we drop the p and denote G_i^p by G_i and $X^p_{G_i^p}$ by X_i. See Figure 7.

Note that if v is a vertex in X_i with non-$\mathcal{H}$–elliptic stabilizer, then the stabilizer of v is slender; hence all stabilizers of connected components of the link of v are slender, and following the steps in the construction of resolving complexes, v is not a cutpoint of X_i. Hence the link l of v has exactly one connected component, and $\mathrm{Stab}(l) = \mathrm{Stab}(v)$.

Let Δ_i be the decomposition G_i inherits from $\Delta_{L(p,i)}$, and let T_i be the associated tree. We now argue that for sufficiently large i, we can replace X_i by a graph of groups such that each vertex group acts on a tree with $\mathcal{H}$–elliptic or slender vertex stabilizers.

Let $\mathcal{L}_i$ be the collection of orbits of connected components of links of vertices of X_i, and denote the orbit of a link l by $[l]$. Say that $[l] \in \mathcal{L}_i$ *dies* in X_{i+1} if $\mathrm{Stab}(l)$ acts hyperbolically in T_i; otherwise, $[l]$ *survives*. Let $\mathcal{L}_i^s$ be the collection of orbits of connected components of links of vertices which survive, and let $\mathcal{I}_i$ be the collection of orbits of links of vertices which survive forever. There is a natural map $\iota_i\colon \mathcal{I}_i \to \mathcal{I}_{i+1}$, and since $\mathrm{covol}(X_{i+1}) \le \mathrm{covol}(X_i) \le \mathrm{covol}(X_H)$, eventually ι_i is bijective. The links which survive forever have $\mathcal{H}$–elliptic stabilizers, and if $\mathrm{Stab}(l)$ is $\mathcal{H}$–elliptic, then l is *$\mathcal{H}$–elliptic*.

Let $\mathcal{V}_i$ be the collection of orbits of vertices v such that v has an $\mathcal{H}$–elliptic component in its link. Since the number of orbits of connected components of links which survive forever is constant, $|\mathcal{V}_i|$ is nondecreasing in i, and is eventually constant. Furthermore, if a component of the link of v is not slender, then *all* components of the link of v have nonslender stabilizer, and if the link of v has an $\mathcal{H}$–elliptic component l and $\mathrm{Stab}(l)$ is slender, then l is the only component of $\mathrm{lk}(v)$.

Let l be an $\mathcal{H}$–elliptic component of a link. Then

$$|l/\,\mathrm{Stab}(l)| \ge |l'/\,\mathrm{Stab}(l')|, \quad l' \in \iota_i([l]).$$

For sufficiently large i, this number stabilizes as well, giving a bijection $\mathcal{V}_i \to \mathcal{V}_{i+1}$ and, for each vertex v with an $\mathcal{H}$–elliptic link component, a $\mathrm{Stab}(v)$–equivariant isomorphism of links $\mathrm{lk}_{X_i}(v) \to \mathrm{lk}_{X_{i+1}}(\varphi_i(v))$ (we illustrate φ_i in Figure 7).

We assume below that $N_{\mathrm{link}} \ge N_{\mathrm{tri}}$ has been chosen large enough to arrange all of the above, over all sequences $\{G_i^p\}$, for $i \ge N_{\mathrm{link}}$.

For $i \ge N_{\mathrm{link}}$, we have that X_i^* is connected and $\varphi_i\colon X_i^* \to (X_i)_{T_i}$ induces bijections on orbits of triangles and stars of vertices with $\mathcal{H}$–elliptic components in their links.

Finding $\mathcal{H}$–elliptic subgroups

Let $\nabla(X_i)$ be the set of triangles in X_i, and let $\tau_i\colon \nabla(X_i) \to \nabla((X_i)_{T_i})$ be the induced map. A pair of triangles is an unordered pair of triangles (t, t') where $t, t' \in \nabla(X_i)$ overlap in an edge. Denote the collection of orbits of pairs of triangles in X_i by $\boxslash(X_i)$.

The map φ_i *separates* a class of pairs $P = [(t,t')] \in \boxslash(X_i)/G_i$ if $\tau_i(t)$ and $\tau_i(t')$ lie in different cutpoint-free components of $(X_i)_{T_i}$. See Figure 8. If φ_i doesn't separate P, then it descends to an element

$$\tau_i(P) = [(\tau_i(t), \tau_i(t'))] \in \boxslash(X_{i+1}).$$

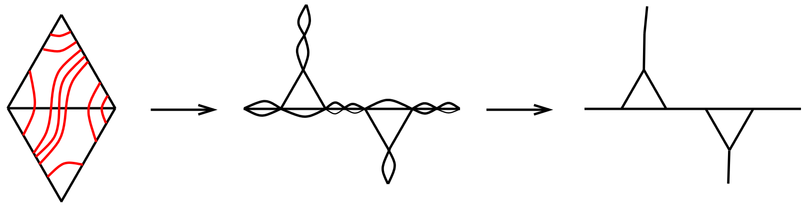

Figure 8: An adjacent pair of triangles in X_i separated by φ_i under a Type 1 resolution

Similarly, φ_{i+1} *doesn't separate* P if φ_i doesn't separate P and φ_{i+1} doesn't separate $\tau_i(P)$, and likewise for φ_k for $k = i+2, \dots$.

Definition 8.1 (stable pairs of triangles) Let $\mathrm{s}\boxslash(X_i)$ be the collection of equivalence classes of pairs of triangles which are not eventually separated by any φ_j, where $j \geq i$. Elements of $\mathrm{s}\boxslash(X_i)$ are called *stable pairs*.

There are induced (injective) maps

$$\sigma_{i,j}\colon \mathrm{s}\boxslash(X_i) \to \mathrm{s}\boxslash(X_j).$$

The purpose of this section is to show that the sequence

$$\cdots \to \mathrm{s}\boxslash(X_i) \xrightarrow{\sigma_{i,i+1}} \mathrm{s}\boxslash(X_{i+1}) \to \cdots \tag{2}$$

eventually stabilizes.

Let $\sim_i$ be the equivalence relation on $\nabla(X_i)$ generated by $s \sim_i t$ if $[(s,t)] \in \mathrm{s}\boxslash(X_i)$. Let $\{P_\alpha\}$ be the collection of subcomplexes, each of which is the union of elements in a $\sim_i$ equivalence class, and let $P_i^1, \dots, P_i^{n_i}$ be a set of representatives of orbits under the action of G_i. Then $\bigcup_j (G_i / \operatorname{Stab}(P_i^j)) P_i^j$ contains all triangles in X_i, and if gP_i^j and $hP_i^{j'}$ overlap in a triangle then $j = j'$ and $h^{-1}g \in P_i^j$.

Each $P_i^j \subset X_i$ pushes forward under φ_i to a subcomplex $\varphi_i(P_i^j)$ of $(X_{i+1})_{T_i}$, and there exists an element $h_{i,j} \in G_i$ such that $h_{i,j}\varphi_i(P_i^{j'}) \subset P_{i+1}^j$. Abusing notation, we will suppress mentioning the elements $h_{i,j}$ and simply say that P_i^j pushes forward to a subcomplex of $P_{i+1}^{j'}$. Similarly for the stabilizers of the P_i^j: we have $h_{i,j} \operatorname{Stab}(P_i^j) h_{i,j}^{-1} \leq \operatorname{Stab}(P_{i+1}^{j'})$, but we will drop the $h_{i,j}$ and simply say that $\operatorname{Stab}(P_i^j) \leq \operatorname{Stab}(P_{i+1}^{j'})$.

Since every triangle in X_{i+1} is contained in some $P_{i+1}^{j'}$, and $n_i \geq n_{i+1} \geq 1$, we can assume from now on that i is chosen sufficiently large so that $n_i = n_{i+1}$, and that P_i^j pushes forward to a subcomplex of P_{i+1}^j. Let E_i^j be the number of orbits of edges in P_i^j under the action of $\operatorname{Stab}(P_i^j)$. Then $E_i^j \geq E_{i+1}^j$. Since the number of orbits of edges is bounded from above by $3\,\mathrm{covol}(X_i)$ this quantity is nonincreasing as well. Choose $N_{\mathrm{edges}} \geq N_{\mathrm{link}}$ sufficiently large so that $E_i^j = E_{i+1}^j$ for $i \geq N_{\mathrm{edges}}$.

Lemma 8.2 *If $\sigma_{i,i+1}$ in (2) is not bijective for $i > N_{\mathrm{edges}}$, then there is j and an edge e in P_i^j such that*

$$\mathrm{Stab}^+_{P_i^j}(e) \lneq \mathrm{Stab}^+_{P_{i+1}^j}(e).$$

Furthermore, $\mathrm{Stab}^+_{P_i^j}(e)$ *is conjugate into a non-$\mathcal{H}$–elliptic edge group of T_i/G_i.*

Proof If $\mathrm{s}\boxslash(X_i) \hookrightarrow \mathrm{s}\boxslash(X_{i+1})$ is not surjective, there are triangles $t \subset P_i^j$ and $t' \subset g P_i^{j'}$, and edges $e \subset t$ and $g \cdot e \subset t'$, with $g \in G_i \setminus \mathrm{Stab}(P_i^j)$ such that $[(t,t')] \notin \mathrm{s}\boxslash(X_i)$ but $[\tau_i(t), \tau_i(t')] \in \mathrm{s}\boxslash(X_{i+1})$. Since g does not stabilize P_i^j, clearly $\mathrm{Stab}^+_{P_i^j}(e) \lneq \mathrm{Stab}^+_{P_{i+1}^j}(e) \ni g$.

Recall the construction of the pattern Λ given in the Type 1 case on page 1813. Since t and t' don't form a stable pair but their push-forwards do, there is a component λ of Λ that meets both t and t' in the edges e and $g{\cdot}e$, respectively. Then $\mathrm{Stab}^+_{P_i^j}(e) \leq \mathrm{Stab}(\lambda)$, and $\mathrm{Stab}(\lambda)$ is conjugate into an edge group of Δ_i. Since $i \geq N_{\mathrm{link}}$, no component of the link of the vertex of X_{i+1} corresponding to λ is $\mathcal{H}$–elliptic; otherwise, a new equivalence class of $\mathcal{H}$–elliptic link stabilizers would have to have appeared, contradicting the fact that the map $\iota_i\colon \mathcal{I}_i \to \mathcal{I}_{i+1}$ is a bijection for $i > N_{\mathrm{link}}$. □

Since $\mathcal{H}$ satisfies the acc (Definition 2.3) on $\mathcal{C}_{\mathcal{H}}$, there is some first index $M_p \geq N_{\mathrm{edges}}$ (recall we are working in the branch $G_i^p = G_i$) such that for every edge $e \subset P_i^j$, $\mathrm{Stab}^+_{P_i^j}(e) = \mathrm{Stab}^+_{P_{i+1}^j}(e)$ for $i \geq M_p$; hence $\sim_i = \sim_{i+1}$ for $i \geq M_p$ by Lemma 8.2.

Proof of Theorem 2.5

Definition 8.3 An *unstable* edge is an edge e such that there are triangles t and t' with $t \cap t' = e$ but $[(t,t')] \notin \mathrm{s}\boxslash(X_i)$. Let $W \subset X_i$ be the union of unstable edges in X_i.

A *cone* C is a triangulated disk with exactly one interior vertex. A cone in a triangular complex X is a combinatorial map $\gamma\colon C \to X$ which maps triangles to triangles. A cone $\gamma\colon C \to X$ is *simple* if the associated path in the link of the image of the cone point is simple.

Let $C \to X$ be a cone in X, and let C^* be the space obtained by removing vertices of C which are mapped to $\rho^{-1}(\partial T)$. Let Λ denote also the preimage of Λ in C^*. The map $C^* \to X^*/\Lambda$ induces maps $C^*/\Lambda \to X^*/\Lambda = X_T$, where $C*/\Lambda$ is the space

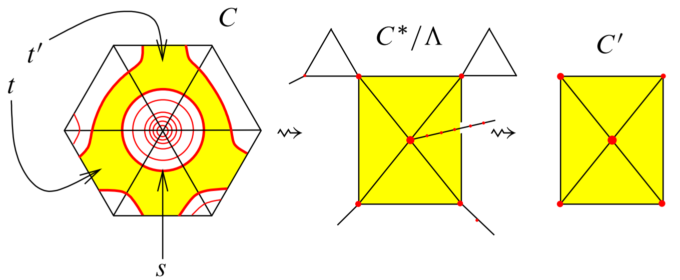

Figure 9: Constructing the push-forward C' of C. In this example, the triangles t and t' are not adjacent, but they have adjacent push-forwards.

obtained by collapsing each connected component of Λ in C to a point, followed by collapsing bigons to edges, ie reducing. Let c be the cone point in C, and let s be the outermost component of Λ encircling c if there is one; otherwise, let $s = c$. The *push-forward* C' of C to X_T is the cone obtained from C^*/Λ by taking all triangles in C^*/Λ containing the image of s. See Figure 9.

Lemma 8.4 *Suppose that $C_i \to X_i$ is a simple cone and that there are two triangles t and t' in the image of C_i such that $t \not\sim_i t'$. Then $i < M_p$.*

Proof Suppose $i \geq M_p$. Let $C_j \to X_j$ be the push-forward of C_i to X_j. Since there are triangles t and t' such that $t \not\sim_i t'$, then for some $j > i$, we have $|C_j| < |C_i|$; otherwise, each pair of adjacent triangles in C_i is a stable pair. Let j be the first index such that $|C_j| = |C_{j'}|$ for $j' \geq j$. Then all triangles in the image of C_j are $\sim_j$ equivalent. Let $t_1, \dots, t_n$ be the triangles in the image of C_j, indexed so that $[(t_k, t_{k+1})] \in \mathrm{s}\boxslash(X_j)/G_j$. Let $\tilde{t}_k$ be the triangle in the image of C_i in X_i corresponding to t_k. Then since $i \geq M_p$, there are edges $\tilde{e}_k$ in X_i such that $\tilde{t}_k \cap \tilde{t}_{k+1} = \tilde{e}_k$, hence $[(\tilde{t}_k, \tilde{t}_{k+1})] \in \mathrm{s}\boxslash(X_i)/G_i$, but this implies that the cone $C_i \to X_i$ was not simple. $\square$

Lemma 8.5 *Suppose that $i > M_p$, that $t, t' \in \nabla(X_i)$ intersect in an edge e, and that $[(t, t')] \notin \mathrm{s}\boxslash(X_i)$. Then e separates X_i, with $t \setminus e$ and $t' \setminus e$ lying in different components of $X_i \setminus e$.*

Note that there may be edges which are not unstable, but which still separate X_i.

Proof of Lemma 8.5. Let a and b be the vertices of t and t', respectively, distinct from the endpoints v and w of e. Suppose that e doesn't separate X_i into at least two components, with $t \setminus e$ lying in one and $t' \setminus e$ lying in another. Then there is an edge path $q\colon I \to X_i$ of a subdivided interval such that $q(0) = a$, $q(1) = b$ and $q^{-1}(e) = \varnothing$. Let f and g be the oriented edges of t and t' connecting a to v and v

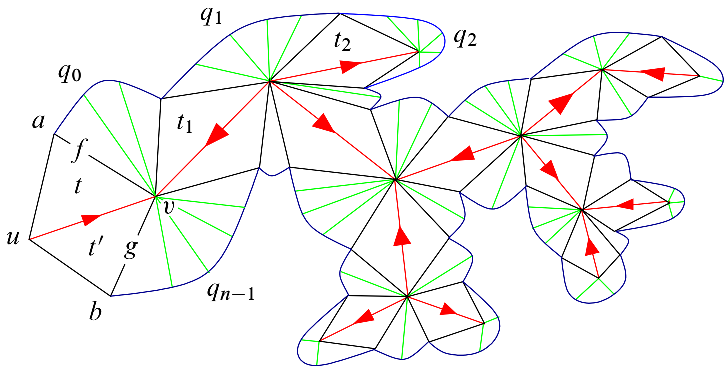

Figure 10: The homology h from Lemma 8.5, which we may assume is a disk. The edges with arrows are mapped to e.

to b, respectively. Let $h'\colon D' \to X_i$ be a combinatorial map of a triangulated surface D' representing a homology between the edge paths gf and q, and let $h\colon D \to X_i$ be the combinatorial map of a surface obtained by attaching two triangles representing $t \cup_e t'$ to D'. See Figure 10.

Without loss, by perhaps changing q and h, we may assume that the union of edges of D which are mapped to e does not separate D, and that D is a disk, as illustrated in Figure 10. The path q may be divided into subpaths $q_0, \dots, q_{n-1}$ such that q_j connects the apex of a triangle t_j to the apex of a triangle t_{j+1}, and such that the side of t_j is mapped to e by h. Furthermore, by identifying the edges labeled e in the sequence of triangles determined by t_j, q_j and t_{j+1}, we obtain a cone in the link of one of v or w. Then either $t_j = t_{j+1}$ or there is a simple cone $C \to X_i$ containing t_j and t_{j+1}; hence by Lemma 8.4, $t_j \sim_i t_{j+1}$ for all j. Therefore, $t \sim_i t'$, contrary to hypothesis. □

Proof of Theorem 2.5 Fix some $N_{\mathrm{equiv}} > \max_p\{M_p\}$, and let $\{U_\alpha^p\}$ be the collection of maximal connected subcomplexes of $X^p_{N_{\mathrm{equiv}}}$ which aren't eventually separated by any φ_j^p where $j > N_{\mathrm{equiv}}$. Each U_α^p is a union of $\sim_{N_{\mathrm{equiv}}}$ equivalence classes which are either disjoint or meet in a vertex with nonslender $\mathcal{H}$–elliptic link component stabilizers. Clearly $\mathrm{Stab}(U_\alpha^p)$ is $\mathcal{H}$–elliptic.

Let T^p be the bipartite graph whose vertex set is the collection of U_α^p and unstable edges, and whose edge set is the set of pairs (U_α^p, e), where $e \subset U_\alpha^p$. Then clearly, T^p is connected, and since the endpoints of unstable edges are not cutpoints, Lemma 8.5 implies that T^p is a tree. Vertex stabilizers correspond to stabilizers of U_α^p and unstable edges, hence are $\mathcal{H}$–elliptic or slender, and edge stabilizers are stabilizers of pairs (U_α^p, e), hence are slender. For each p, replace the $G^p_{N_{\mathrm{equiv}}}$–complex $X^p_{N_{\mathrm{equiv}}}$ by the graph of groups decomposition $T^p/G_{N_{\mathrm{equiv}}}$ given above. □

9 Strong accessibility

Almost finitely presented groups

As it is rather long and technical, we will not restate the definition of the JSJ decomposition of a finitely presented group over slender edge groups here, and instead refer the reader to [9, Theorem 5.13] and [8]. We need the following from [9].

Theorem 9.1 [9, Theorem 5.15] *Let G be a finitely presented group and Γ a graph decomposition we obtain in* [9, Theorem 5.13]. (Note: Γ is the slender JSJ.) *Let $G = A *_C B$, $A*_C$ be a splitting along a slender group C, and T_C its Bass–Serre tree.*

(1) *If the group C is elliptic with respect to any minimal splitting of G along a slender group, then all vertex groups of Γ are elliptic on T_C.*

(2) *Suppose the group C is hyperbolic with respect to some minimal splitting of G along a slender group. Then*
   (a) *All nonenclosing vertex groups of Γ are elliptic on T_C.*
   (b) *For each enclosing vertex group V of Γ, there is a graph of groups decomposition of V, $\mathcal{V}$, whose edge groups are in conjugates of C, which we can substitute for V in Γ such that if we substitute for all enclosing vertex groups of Γ, then all vertex groups of the resulting refinement of Γ are elliptic on T_C.*

In other words, the nonenclosing vertex groups of the slender JSJ decomposition Γ are elliptic in every slender splitting of G. A few remarks are in order:

- Let $T_{C_1}, \dots, T_{C_n}, \dots$ be a collection of Bass–Serre trees associated to splittings of a *finitely generated* group G along slender edge groups. Then there is a graph of groups decomposition Γ_n satisfying the bullets of Theorem 9.1 for the trees $T_{C_1}, \dots, T_{C_n}$. The decomposition Γ_m is a refinement of Γ_{m+1}.
- If $E_1, \dots, E_k < G$ is a family of subgroups such that each E_i acts elliptically in T_{C_j} for all j, then we may assume that each E_i is elliptic in Γ_m for all m.
- If G is accessible relative to $\{E_i\}$ over the family of slender subgroups, ie there is a constant bounding the number of vertices in a reduced graph of groups decomposition relative to $\{E_i\}$ of G over slender edge groups, then there is a *slender* JSJ *decomposition* of G relative to $\{E_i\}$, ie a graph of groups decomposition Γ satisfying the conclusion of Theorem 9.1, where T_C is only allowed to vary over all slender G–trees in which the E_i are elliptic.

Let H be almost finitely presented relative to $\mathcal{E}$, and suppose that H doesn't contain any slender subgroups outside $\mathcal{E}$ which have an infinite dihedral quotient. Let Δ_H be the slender JSJ decomposition of H and let X_H be an acyclic simplicial complex that H acts on with cell stabilizers which are either slender or in $\mathcal{E}$. Let $\mathcal{X}_H$ be the trivial hierarchy of H, where Ω_H is just a point. For each nonslender vertex group L of Δ_H, let $\mathcal{X}_L$ be the hierarchy obtained by resolving the action of G on the tree associated to Δ_H. Then, by the above, L is accessible relative to $\mathcal{E}$, hence has a slender JSJ decomposition relative to $\mathcal{E}$. Repeat to construct a hierarchy $\mathcal{H}$ of G. We call $\mathcal{H}$ the slender JSJ hierarchy of H relative to $\mathcal{E}$. By construction, $\mathcal{X}_L$ is an $\mathcal{H}$–structure for L. Corollary 2.6 claims that $\mathcal{H}$ is finite:

Proof of Corollary 2.6 Let $L \in \mathcal{H}^n$ for some $n > N$, where N is as in Theorem 2.5. Groups in $\mathcal{H}$ are either slender-by-orbifold, hence have JSJ decompositions which are graphs of slender groups over slender edge groups, or are elliptic in the top level of the $\mathcal{H}$–structure of their parents. Hence if $K < K'$ is a vertex group of the JSJ of K' and is not a graph of slender groups, then we may assume $\operatorname{height}(\mathcal{X}_K) < \operatorname{height}(\mathcal{X}'_K)$. Hence $\mathcal{H}_L$ is finite and has terminal leaves which are either slender or are the nonslender terminal leaves of $\mathcal{X}_L$. □

Relatively hyperbolic groups

In this section, we prove Corollary 2.7. Since relatively hyperbolic groups are finitely presented relative to their peripheral subgroups, it suffices to show that relatively hyperbolic groups satisfy the acc on $\mathcal{C}_{\mathcal{H}}$.

Lemma 9.2 *Let G and $\mathcal{H}$ be as in Corollary 2.7. Then G satisfies the acc on $\mathcal{C}_{\mathcal{H}}$.*

We have chosen to use a definition of relative hyperbolicity (first introduced in [10]) which will facilitate the proof of Lemma 9.4: it is easily seen to be equivalent to the standard definitions. See, for instance, [13, Definition 3.3]. If Z is a δ–hyperbolic metric space, then it has a Gromov boundary ∂Z. Horofunctions are defined in [13, Section 2]; if $h\colon Z \to \mathbb{R}$ is a horofunction centered at some $\xi \in \partial Z$, we denote by $B(n) = \{x \in X \mid h(x) \geq n\}$ the *depth-n horoball.*

Definition 9.3 (see [13, Definition 3.3]) A group G is *hyperbolic relative to peripheral subgroups* $P_1, \dots, P_r$ if it acts properly on a δ–hyperbolic graph Z such that

- each peripheral subgroup P_i fixes a point $p_i \in \partial Z$,
- centered at each p_i, there is a horofunction h_i so that if $B_i(0)$ is corresponding depth 0–horoball, the G–translates of the $B_i(0)$ are all disjoint, and
- G acts cocompactly on $Z \setminus U$, where U is the union of the translates of these horoballs.

The points in ∂Z that are translates of the p_i are called *parabolic limit points*. We denote this set by $\Pi \subset \partial Z$.

Lemma 9.4 *Let G be a relatively hyperbolic group and let $Q < G$ be a finite subgroup. There is a constant K such that if*

- *H is a two-ended nonperipheral subgroup of G and $Q < H$, or*
- *Q is contained in two distinct conjugates of peripheral subgroups of G,*

then $|Q| \leq K$

The next lemma (needed for Lemma 9.4) though not explicitly stated in [3], is easily extracted from their proof that there are only finitely many conjugacy classes of finite subgroups of a hyperbolic group.

Lemma 9.5 (see [3]) *Let Q be a finite group of isometries of a δ–hyperbolic metric space Z. Then there is a point $x_Q \in Z$ which is displaced by at most 3δ by each element of Q.*

Proof of Lemma 9.4. Recall that every two-ended group H either has an infinite dihedral quotient or splits as a semidirect product

$$H \approx Q \rtimes \langle t \rangle. \tag{3}$$

Since G is relatively hyperbolic, it acts freely (but not necessarily cocompactly) on a proper δ–hyperbolic graph Z (see [11]) such that the stabilizers of parabolic limit points p in the Gromov boundary Π of Z are precisely the peripheral subgroups of G. Furthermore, there is a G–equivariant collection of disjoint horoballs $B(p)$ centered at parabolic limit points $p \in \Pi$ whose stabilizers G_p are the peripheral subgroups of G, and G permutes this collection and maps horospheres to horospheres of the same depth. See [13].

Recall that Bowditch's characterization of relatively hyperbolic groups as those groups that act cocompactly on *fine* hyperbolic graphs (see [4, Definition 2] or [13, Section 3.3]) immediately implies that the intersections of any two distinct conjugates of peripheral subgroups of G have orders bounded by some constant $K_1 = K_1(G)$. Otherwise, one could construct arbitrarily many circuits of some bounded length passing through a fixed edge.

By Lemma 9.5, if F is a finite group of isometries of a δ–hyperbolic metric space Z, there is a point x_F which is displaced at most 3δ by each element of F. Let Q and t be as in (3), and suppose first that x_Q is at least 3δ–deep in a horoball $B(p)$. Then

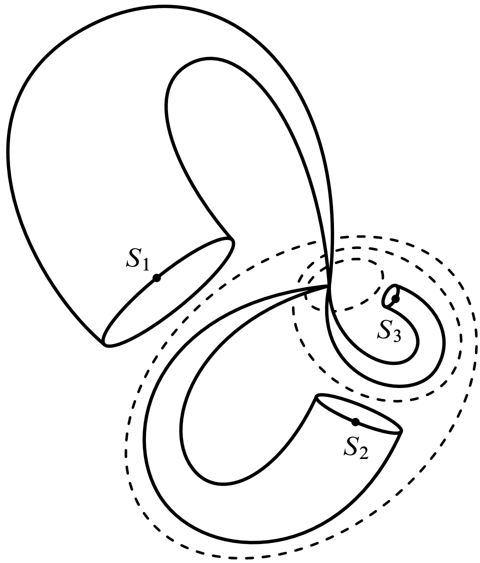

Figure 11: A chain S_i of $\mathcal{H}$–elliptic finite subgroups of non-$\mathcal{H}$–elliptic edge groups. In this case, a peripheral subgroup is represented by the dot at the center of the picture. Each S_i is peripheral, but contained in two distinct conjugates of the peripheral subgroup, hence have orders uniformly bounded above.

$Q \cdot x_Q \subset B(p)$; hence $Q \cdot B(p) = B(p)$ and $Q \leq G_p$. On the other hand, tQt^{-1} must fix the point $tp \in \Pi$. Since H is not parabolic, $tp \neq p$, but since $\langle t \rangle$ normalizes Q, we must have $Q \leq G_p \cap tG_pt^{-1}$; hence $|Q| \leq K_1$.

Otherwise, x_Q lies in the neutered space $W = Z \setminus B(3\delta)$ obtained by removing all 3δ–deep horoballs. Since G acts freely and cocompactly on W, the number of vertices in a ball of radius 3δ in W is bounded by some $K_2 = K_2(G, W)$; thus $|Q| = |Q \cdot x_Q| \leq K_2$. Set $K = \max\{K_1, K_2\}$. □

We are now finally prepared to prove Lemma 9.2.

Proof of Lemma 9.2 Let K be the constant from Lemma 9.4. If $S_1 < S_2 < \cdots$ is an ascending chain in $\mathcal{C}_{\mathcal{H}}$, then $S_i < H_i$, where H_i is a non-$\mathcal{H}$–elliptic edge group. If H_i is nonperipheral in G, then $|S_i| < K$. If H_i is peripheral in G then H_i is two-ended and hyperbolic in the slender JSJ decomposition of some lower level L in $\mathcal{H}$, and S_i is contained in an edge group E of Δ_L (since S_i is finite and H_i fixes an axis). (See Figure 11.) Either E is peripheral, in which case S_i is contained in two distinct conjugates of peripheral subgroups of G, or E is not peripheral. In either case, $|Q| \leq K$. Since the S_i are of uniformly bounded orders, G satisfies the acc on $\mathcal{C}_{\mathcal{H}}$. □

The following may also be of interest.

Corollary 9.6 *Let G be a relatively hyperbolic group. There are only finitely many isomorphism classes of nonperipheral two-ended subgroups.*

Proof If $H \leq G$ is a nonperipheral two-ended subgroup, then if it maps surjectively onto $\mathbb{Z}$, it splits as in (3), and the bound on the order of Q given by Lemma 9.4 bounds the number of isomorphism classes. Otherwise, G surjects onto $\mathbb{Z}_2 * \mathbb{Z}_2$ and therefore has an index-2 subgroup of the form (3). □

Hierarchies over elementary families

In this section, we prove Theorem 2.9. The proof is formally identical to the proof of Theorem 2.5, however, since we are allowed to construct a hierarchy by hand, we don't need to use $\mathcal{H}$–structures.

Proof of Theorem 2.9 We define a hierarchy $\mathcal{H}$ of G inductively.

Case 1 Let X be a G–complex with stabilizers in an elementary family $\mathcal{C}$. By collapsing cells with infinite stabilizers in $\mathcal{C}$, we may assume that edge and face stabilizers in X are finite. Let T_X be the cutpoint tree. Edge stabilizers in T_X are either in $\mathcal{C}$ or are finite. If G acts on T_X with global fixed point, there is a cutpoint-free component Y of X stabilized by G, and we go to case 2. Otherwise, let the descendants of G be the vertex groups of T_X/G. The vertex groups of T_X/G are either in $\mathcal{C}$, are finite, or are stabilizers of one of $Y_1, \dots, Y_n$, where each Y_i is a representative of an orbit of cutpoint-free components of X. Note that $\sum_i \operatorname{covol}(Y_i) = \operatorname{covol}(X)$.

Case 2 Suppose that G and X are as above, and that X has no cutpoints. Suppose that X has a separating edge. Let S_X be the cut-edge tree. If G acts on S_X with global fixed point, there is a maximal connected subcomplex Y of X which is stabilized by G and doesn't have a separating edge; then we replace X by Y and go to case 3. If G doesn't have a global fixed point, let the descendants of G be the vertex groups of S_X/G. They are either finite or conjugate to a stabilizer of one of $Y_1, \dots, Y_n$, where each Y_i is a representative of a maximal cut-edge-free component of X. Note that $\sum_i \operatorname{covol}(Y_i) = \operatorname{covol}(X)$.

Case 3 In the remaining case, G has an action on a cutpoint and cut-edge-free G–complex X. Suppose that G has a nontrivial graph of groups decomposition over elements of $\mathcal{C}$; then G has one in which every element of $\mathcal{C}$ acts parabolically, elliptically, or hyperbolically [5, Lemma 1.4]. Let T be the associated tree, and let ρ: $X \to \widehat{T}$ be the resolving map. Then X^* is connected, and G acts on X_T. Now we are again in the first case, and G doesn't act on T_{X_T} with global fixed point.

Since $\sum_i \mathrm{covol}(Y_i) \le \mathrm{covol}(X)$, there are at most finitely many infinite branches

$$G_1^p > G_2^p > \cdots$$

in $\mathcal{H}$, where each G_i^p is the sole nonelementary descendant of G_{i-1}^p. (This is the same principle as in Lemma 7.3.) As before, we drop the p and let X_i be the G_i complex produced above. Again, there exist $N_{\mathrm{link}} \le N_{\mathrm{edges}}$ so that $\mathcal{H}$–elliptic vertex stabilizers in X_i stabilize, and the number of orbits of edges in $\sim_i$ equivalence classes stabilizes. The ascending chain condition on finite subgroups in $\mathcal{C}$ immediately implies the analogue of Lemma 8.2. Now argue, as in the proof of Theorem 2.5, that $G_{N_{\mathrm{equiv}}}$ acts on a tree T_p with $\mathcal{H}$–elliptic or finite edge stabilizers for some N_{equiv}. □

Acknowledgements We would like to thank Thomas Delzant and Leonid Potyagailo for some interesting conversations. Thanks to Daniel Groves for help with an earlier version of the paper, as well as to Peter Haïssinsky and Matias Carrasco Piaggio for encouragement, and to Thierry Coulbois for bringing our attention to [12]. We also thank Lee Mosher, Misha Kapovich, and Ian Agol for input on finite subgroups of relatively hyperbolic groups. We are also grateful to the numerous referees whose comments improved the paper.

The first author was supported in part by EPSRC grant EP/D073626/2 and NSF grant DMS-0602191. The second author was supported by an NSERC PDF, the ANR grant ANR-2010-BLAN-116-01 GGAA and a Fields Postdoctoral Fellowship.

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Received: 22 September 2015
Accepted: 5 April 2016