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Building Anosov flows on 3–manifolds

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We prove we can build (transitive or nontransitive) Anosov flows on closed three-dimensional manifolds by gluing together filtrating neighborhoods of hyperbolic sets. We give several applications of this result; for example:

(1) We build a closed three-dimensional manifold supporting both a transitive Anosov vector field and a nontransitive Anosov vector field.
(2) For any n, we build a closed three-dimensional manifold M supporting at least n pairwise different Anosov vector fields.
(3) We build transitive hyperbolic attractors with prescribed entrance foliation; in particular, we construct some incoherent transitive hyperbolic attractors.
(4) We build a transitive Anosov vector field admitting infinitely many pairwise nonisotopic transverse tori.

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Dedicated to the memory of Marco Brunella
(Varese, Italy, 1964 – Rio de Janeiro, Brazil, 2012)

1 Introduction

1.1 General setting and aim of this paper

Anosov vector fields are the paradigm of hyperbolic chaotic dynamics. They are nonsingular vector fields on compact manifolds for which the whole manifold is a hyperbolic set: if X is an Anosov vector field, the tangent bundle to the manifold M splits in a direct sum $E^{\mathrm{s}} \oplus \mathbb{R}X \oplus E^{\mathrm{u}}$, where E^{s} and E^{u} are continuous subbundles invariant under the flow of X, and the vectors in E^{s} and E^{u} are respectively uniformly contracted and uniformly expanded by the derivative of this flow. As with all hyperbolic dynamics, Anosov vector fields are structurally stable: any vector field C^1–close to X is *topologically equivalent* (or *orbit equivalent*) to X. Therefore, there is hope for a topological classification of Anosov vector fields up to topological equivalence, and many works started this kind of classification in dimension 3 (see for instance Ghys [19; 20], Barbot [1; 3], Fenley [13; 14] and Barbot and Fenley [5]).

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However, we are still very far from proposing a classification, even in dimension 3. For instance we still do not know which manifolds support Anosov vector fields, or how many Anosov vector fields may be carried by the same manifold.

The simplest examples of Anosov vector fields on three-dimensional manifolds are the suspension of an Anosov–Thom linear automorphism of the torus T^2 and the geodesic flow of a hyperbolic Riemann surface. These two (classes of) examples share some strong rigidity properties:

- Plante [27] has proved that every Anosov vector field on a torus bundle over the circle is topologically equivalent to the suspension of an Anosov–Thom automorphism.
- Ghys [19] has proved that, up to finite covering, every Anosov vector field on a circle bundle is topologically equivalent to the geodesic flow of a hyperbolic surface.
- Ghys [20] has proved that, up to finite cover, every Anosov vector field on a three-manifold whose stable and unstable foliations are C^∞ is topologically equivalent to a suspension or a geodesic flow.

However, a lot of nonalgebraic examples of Anosov vector fields on closed three-manifolds have been built by different authors (including Handel and Thurston [22], Goodman [21], Fried [17], Fenley [13], Barbot [4] and Foulon and Hasselblatt [15]). Let us mention in particular Franks and Williams [16], who have built a nontransitive Anosov vector field, and Bonatti and Langevin [9], who have built an Anosov vector field admitting a closed transverse cross-section (diffeomorphic to a torus) which does not cut every orbit.

Both Franks and Williams' and Bonatti and Langevin's examples were obtained by gluing filtrating neighborhoods of hyperbolic sets along their boundaries. The aim of the present paper is to develop a general theory of this class of examples. We first describe some elementary bricks (called *hyperbolic plugs*). Then we prove a quite general result allowing us to build Anosov vector fields (on closed three-manifolds) by gluing together such elementary bricks. We also provide a simple criterion allowing us to decide whether an Anosov flow built in this manner is topologically transitive or not. Finally we illustrate our "construction game" by several examples.

1.2 Hyperbolic plugs

In order to present our main results, we need to state some definitions. We call a *plug* any pair (V, X) where V is a compact 3–manifold with boundary and X is a vector field on V, transverse to the boundary of V (in particular, X is assumed to be nonsingular on ∂V). Given such a plug (V, X), we can decompose ∂V as the disjoint union of an *entrance boundary* $\partial^{\mathrm{in}} V$ (the part of ∂V where X is pointing inwards) and an *exit boundary* $\partial^{\mathrm{out}} V$ (the part of ∂V where X is pointing outwards). The plug (V, X) will be called an *attracting plug* if $\partial^{\mathrm{out}} V = \varnothing$, and a *repelling plug* if $\partial^{\mathrm{in}} V = \varnothing$. The plug (V, X) will be called a *hyperbolic plug* if its maximal invariant set $\Lambda = \bigcap_{t\in\mathbb{R}} X^t(V)$ is nonsingular and hyperbolic with 1–dimensional stable and unstable bundles. Here X^t is the flow induced by X on V.

If (V, X) is a hyperbolic plug, the stable lamination $W^{\mathrm{s}}(\Lambda)$ (resp. the unstable lamination $W^{\mathrm{u}}(\Lambda)$) of the maximal invariant set $\Lambda = \bigcap_{t\in\mathbb{R}} X^t(V)$ intersects transversally the entrance boundary $\partial^{\mathrm{in}} V$ (resp. the exit boundary $\partial^{\mathrm{out}} V$) and is disjoint from $\partial^{\mathrm{out}} V$ (resp. $\partial^{\mathrm{in}} V$). By transversality, $\mathcal{L}^{\mathrm{s}}_X := W^{\mathrm{s}}(\Lambda) \cap \partial^{\mathrm{in}} V$ and $\mathcal{L}^{\mathrm{u}}_X := W^{\mathrm{u}}(\Lambda) \cap \partial^{\mathrm{out}} V$ are one-dimensional laminations; we call them the *entrance lamination* and the *exit lamination* of V. We will see that the laminations $\mathcal{L}^{\mathrm{s}}_X$ and $\mathcal{L}^{\mathrm{u}}_X$ satisfy the following properties (see Proposition 3.8):

(i) They contain finitely many compact leaves.

(ii) Every half noncompact leaf is asymptotic to a compact leaf.

(iii) Each compact leaf may be oriented such that its holonomy is a contraction. This orientation will be called the *contracting orientation*.

A lamination satisfying this three properties will be called a *Morse–Smale lamination*, or an *MS lamination* for short. If the lamination is indeed a foliation, we will call it an *MS foliation*.

If (V, X) is a hyperbolic attracting plug, then $\mathcal{L}^{\mathrm{s}}_X$ is a foliation on $\partial^{\mathrm{in}} V$. In particular, every connected component of $\partial V = \partial^{\mathrm{in}} V$ is a two-torus (or a Klein bottle if we allow V to be nonorientable).

Let us first state an elementary result, which is nevertheless a fundamental tool for our “construction game”:

Proposition 1.1 *Let (U, X) and (V, Y) be two hyperbolic plugs. Let T^{out} be a union of connected components of $\partial^{\mathrm{out}} U$ and T^{in} be a union of connected components of $\partial^{\mathrm{in}} V$. Assume that there exists a diffeomorphism $\varphi\colon T^{\mathrm{out}} \to T^{\mathrm{in}}$ such that $\varphi_*(\mathcal{L}^{\mathrm{u}}_X)$ is transverse to the foliation $\mathcal{L}^{\mathrm{s}}_Y$. Let Z be the vector field induced by X and Y on the manifold $W := U \sqcup V/\varphi$. Then (W, Z) is a hyperbolic plug.*[1]

[1] This entails that there is a differentiable structure on W (compatible with the differentiable structures of U and V by restriction) such that Z is a differentiable vector field.

This proposition will be proven in Section 3.1. A classical example of the use of Proposition 1.1 is the famous Franks–Williams example of a nontransitive Anosov vector field. It corresponds to the case where (U,X) is an attracting plug, (V,Y) is a repelling plug, $T^{\mathrm{out}} = \partial^{\mathrm{out}} U$ and $T^{\mathrm{in}} = \partial^{\mathrm{in}} V$. In that case, W is boundaryless and Z is Anosov and nontransitive.

If (V,X) is a hyperbolic plug such that V is embedded in a closed three-dimensional manifold M and X is the restriction of an Anosov vector field $\overline{X}$ on M, then the stable (resp. unstable) lamination of the maximal invariant set of V is embedded in the stable (resp. unstable) foliation of the Anosov vector field $\overline{X}$. This leads to some restrictions on the entrance and exit laminations of V, and motivates the following definition.

A lamination $\mathcal{L}$ on a compact surface S is a *filling MS lamination* if it satisfies properties (i), (ii) and (iii) above and if every connected component C of $S \setminus \mathcal{L}$ is "a strip whose width tends to 0 at both ends": more precisely, C is simply connected, and the accessible boundary of C consists of two distinct noncompact leaves L_1 and L_2 which are asymptotic to each other at both ends.

Any filling MS lamination can be embedded in a $\mathcal{C}^{0,1}$ foliation.[2] As a consequence, a closed surface carrying a filling MS lamination is either a torus or a Klein bottle. We will prove that the entrance lamination $\mathcal{L}^{\mathrm{s}}$ of a hyperbolic plug is a filling MS lamination if and only if this is also the case for the exit lamination $\mathcal{L}^{\mathrm{u}}$ (see Lemma 3.21). Therefore we will speak of *hyperbolic plugs with filling MS laminations*. Theorem 1.12 shows that every hyperbolic plug with filling MS laminations can be embedded in an Anosov vector field.

Example 1.2 Consider a transitive Anosov vector field X on a closed 3–dimensional manifold M, and a finite collection $\alpha_1,\dots,\alpha_m$, $\beta_1,\dots\beta_n$ of (pairwise different) orbits of Y. Let Y be the vector field obtained from X by performing some DA bifurcations (see Section 8.1) in the stable direction on the orbits $\alpha_1,\dots,\alpha_m$, and some DA bifurcations in the unstable direction on the orbits $\beta_1,\dots\beta_n$. Let U be the compact manifold with boundary obtained by cutting out from M some pairwise disjoint tubular neighborhoods of $\alpha_1,\dots,\alpha_m$, $\beta_1,\dots\beta_n$ whose boundaries are transverse to Y. Then (U,Y) is a hyperbolic plug with filling MS laminations. More precisely, on each connected component of $\partial^{\mathrm{in}} V$ (resp. $\partial^{\mathrm{out}} V$) the entrance lamination $\mathcal{L}^{\mathrm{s}}_X$ (resp. the exit lamination $\mathcal{L}^{\mathrm{u}}_X$) consists of one or two Reeb components (see Figure 14 as an illustration of the case of two Reeb components).

[2] Notice that the converse is not true; see Remark 3.20.

We finish this section by stating an addendum to Proposition 1.1 which allows us to build more and more complicated hyperbolic plugs with filling MS laminations. Two filling MS laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ are called *strongly transverse* if they are transverse and if every connected component C of $S \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$ is a topological disc whose boundary ∂C consists of exactly four segments a_1, a_2, b_1 and b_2, where a_1 and b_1 lie on leaves of $\mathcal{L}_1$, and a_2 and b_2 lie on leaves of $\mathcal{L}_2$.

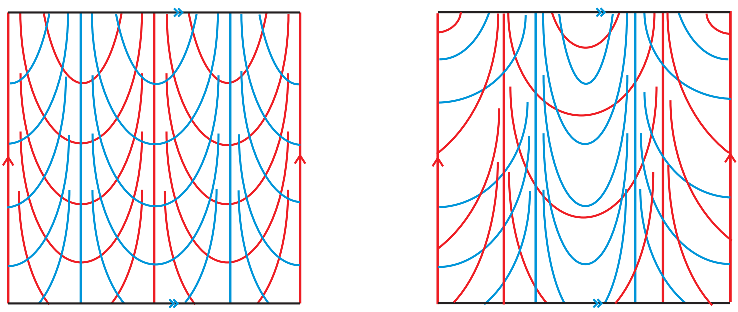

Figure 1: An example of pairs of strongly transverse filling MS laminations on the torus $\mathbb{T}^2$ (left) and an example of pairs of transverse filling MS laminations on the torus $\mathbb{T}^2$ which are not strongly transverse (right)

Proposition 1.3 *In Proposition 1.1, assume furthermore that the plugs (U, X) and (V, Y) have filling MS laminations and that $\varphi_*(\mathcal{L}^{\mathrm{u}}_X)$ is strongly transverse to $\mathcal{L}^{\mathrm{s}}_Y$. Then the plug (W, Z) has filling MS laminations.*

1.3 Building transitive Anosov flows

In this section, we consider a hyperbolic plug with filling MS laminations (U, X), and a diffeomorphism $\varphi\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$ such that the laminations $\mathcal{L}^{\mathrm{u}}_X$ and $\mathcal{L}^{\mathrm{s}}_X$ are strongly transverse; we say that φ is *a strongly transverse gluing diffeomorphism*. We denote by Z the vector field induced by X on the closed manifold U/φ. Note that there is a differentiable structure on U/φ such that Z is a differentiable vector field.

In general the vector field Z is not hyperbolic,[3] and the dynamics of Z depends on the gluing diffeomorphism φ. It could even happen (in principle) that two strongly transverse gluing diffeomorphisms φ_0 and φ_1, isotopic through strongly transverse gluing diffeomorphisms, lead to vector fields Z_0 and Z_1 which are not topologically equivalent. It is therefore necessary to choose carefully the gluing diffeomorphism φ.

[3] Starting from any example, one can perturb the gluing map φ in such a way that the new vector field Z exhibits a bunch of parallel periodic orbits filling a solid torus. This clearly prevents this vector field Z from being hyperbolic.

Question 1.4 *For a hyperbolic plug (U,X) with filling MS laminations and a strongly transverse gluing diffeomorphism $\varphi_0\colon \partial^{\mathrm{out}}U\to\partial^{\mathrm{in}}U$, does there exist a strongly transverse gluing diffeomorphism φ_1, isotopic to φ_0 through strongly transverse gluing diffeomorphisms, such that the vector field Z_1 induced by X on U/φ_1 is Anosov?*

We are not able to answer this question in general. Nevertheless we can give a positive answer in the particular case where the maximal invariant set Λ of X in U admits an affine Markov partition (ie a Markov partition such that the first return map on the rectangles is affine for some coordinates). If Λ does not contain any attractor nor repeller, then a slight perturbation of X leads to a vector field Y, topologically equivalent to X, whose maximal invariant set Λ_Y admits such an affine Markov partition (see Lemma 5.3). We will therefore allow ourselves to use such perturbations.

Let (U,Y) be another hyperbolic plug with filling MS laminations and $\psi\colon \partial^{\mathrm{out}}U\to\partial^{\mathrm{in}}U$ be a strongly transverse gluing diffeomorphism. We say that (U,X,φ) and (U,Y,ψ) are *strongly isotopic* if there is a continuous path (U,X_t,φ_t) of hyperbolic plugs with filling MS laminations and strongly transverse gluing diffeomorphisms such that $(U,X_0,\varphi_0)=(U,X,\varphi)$ and $(U,X_1,\varphi_1)=(U,Y,\psi)$. Notice that this implies that (U,X) and (U,Y) are topologically equivalent.

We will prove the following result:

Theorem 1.5 *Let (U,X) be a hyperbolic plug with filling MS laminations such that the maximal invariant set of X contains neither attractors nor repellers, and let $\varphi\colon \partial^{\mathrm{out}}U\to\partial^{\mathrm{in}}U$ be a strongly transverse gluing diffeomorphism. Then there exist a hyperbolic plug (U,Y) with filling MS laminations and a strongly transverse gluing diffeomorphism $\psi\colon \partial^{\mathrm{out}}U\to\partial^{\mathrm{in}}U$ such that (U,X,φ) and (U,Y,ψ) are strongly isotopic, and such that the vector field Z induced by Y on U/ψ is Anosov.*

Theorem 1.5 allows us to build an Anosov vector field Z by gluing the entrance and the exit boundary of a hyperbolic plug. We will now state a result providing a simple criterion to decide whether this Anosov vector field Z is transitive or not.

Given a hyperbolic plug (U,X) with filling MS laminations and a strongly transverse gluing map $\varphi\colon \partial^{\mathrm{out}}U\to\partial^{\mathrm{in}}U$, consider the oriented graph $\mathcal{P}$ defined as follows:

- The vertices of $\mathcal{P}$ are the basic pieces $\Lambda_1,\dots,\Lambda_k$ of X.
- There is an edge going from Λ_i to Λ_j if $W^{\mathrm{u}}_X(\Lambda_i)$ intersects $W^{\mathrm{s}}_X(\Lambda_j)$, or $\varphi_*(W^{\mathrm{u}}_X(\Lambda_i)\cap\partial^{\mathrm{out}}U)$ intersects $W^{\mathrm{u}}_X(\Lambda_j)\cap\partial^{\mathrm{in}}U$.

We say that (U,X,φ) is *combinatorially transitive* if $\mathcal{P}$ is *strongly connected*, ie if any two edges of $\mathcal{P}$ can be joined by an oriented path.

Proposition 1.6 *Under the hypotheses of Theorem 1.5, if (U, X, φ) is combinatorially transitive, then the Anosov vector field Z is transitive.*

In Theorem 1.5, starting with a hyperbolic plug without attractors nor repellers and a strongly transverse gluing map φ, we build another vector field Y orbit equivalent to X and a gluing map ψ isotopic to φ through strongly transverse gluing such that the vector field induced by the gluing is Anosov. It is natural to ask if the resulting Anosov flow is independent of the choices (of Y and ψ), up to orbit equivalence.

Question 1.7 *Let (V, Y_1, ψ_1) and (V, Y_2, ψ_2) be two hyperbolic plugs with filling MS laminations endowed with strongly transverse gluing diffeomorphisms. Moreover, suppose (V, Y_1, ψ_1) and (V, Y_2, ψ_2) are strongly isotopic. Let Z_1 and Z_2 be the vector fields induced by Y_1 and Y_2 on V/ψ_1 and V/ψ_2. Assume that Z_1 and Z_2 both are Anosov. Are Z_1 and Z_2 topologically equivalent?*

In a forthcoming paper [8], we will provide a positive answer to this question, at least in the case where the Anosov flows Z_1 and Z_2 are topologically transitive.

1.4 Examples

To illustrate the power of Theorem 1.5, we will use it to build various types of examples of Anosov vector fields. We like to think of hyperbolic plugs as elementary bricks of a "construction game". Proposition 1.1 and Theorem 1.5 allow us to glue these elementary bricks together in order to build more complicated hyperbolic plugs, hyperbolic attractors, transitive or nontransitive Anosov vector fields. We hope that the statements below will convince the reader of the interest and of the versatility of this "construction game".

1.4.1 The "blow-up, excise and glue surgery" As a first application of Theorem 1.5, we shall prove the following result:

Theorem 1.8 *Given any transitive Anosov vector field X on a closed (orientable) three-manifold M, there exists a transitive Anosov vector field Z on a closed (orientable) three-manifold N such that "the dynamics of Z is richer than the dynamics of X". More precisely, there exists a compact set $\Lambda \subset N$ invariant under the flow of Z, and a continuous onto map $\pi\colon \Lambda \to M$ such that $\pi \circ X^t = Z^t \circ \pi$ for every $t \in \mathbb{R}$.*

The proof of Theorem 1.8 relies on what we call the *blow-up, excise and glue surgery*. Let us briefly describe this surgery (details will be given in Section 8). We start with a transitive Anosov vector field X on a closed three-manifold M.

Step 1 (blow-up) We blow-up two periodic orbits of X. More precisely, we pick two periodic orbits (with positive eigenvalues) O and O' of X; we perform an attracting DA (derived from Anosov) bifurcation on O and a repelling DA bifurcation on O'. This gives rise to a new vector field $\overline{X}$ on M which has three basic pieces: a saddle hyperbolic set Λ, an attracting periodic orbit O and a repelling periodic orbit O'.

Step 2 (excise) Then we excise two solid tori: we consider the manifold with boundary $U = M \setminus (T \sqcup T')$, where T and T' are small tubular neighborhoods of O and O'. Under some mild assumptions, $(U, \overline{X}|_U)$ is a hyperbolic plug.

Step 3 (glue) Finally, we glue the exit boundary to the entrance boundary of U: we consider the manifold $N := U/\varphi$, where $\varphi\colon \partial^{\mathrm{out}}U \to \partial^{\mathrm{in}}U$ is an appropriate gluing map, and the vector field Z induced by $\overline{X}$ on M. Theorem 1.5 ensures that (up to perturbing $\overline{X}$ within its topological equivalence class) φ can be chosen such that Z is Anosov, and Proposition 1.6 implies that Z is transitive. The hyperbolic set Λ can be seen as a compact subset of N which is invariant under the flow of Z. Well-known facts on DA bifurcations show that there exists a continuous map $\pi\colon \Lambda \to M$ inducing a semiconjugacy between the flow of X and Z, as stated by Theorem 1.8.

Starting with a given transitive Anosov vector field on a closed three-manifold, Theorem 1.8 can be applied inductively, giving birth to an infinite sequence of transitive Anosov vector fields which are "more and more complicated". Moreover, the "blow-up, excise and glue surgery" admits many variants, allowing to construct myriads of examples of Anosov vector fields. For example, instead of starting with a single Anosov vector field X, we could have started with n transitive Anosov vector fields $X_1, \dots, X_n$ in order to get a single transitive Anosov vector field Z which "contains" the dynamics of $X_1, \dots, X_n$. We could also have started with a nontransitive Anosov vector field X; in this case, we get an Anosov vector field Z which might or might not be transitive depending on the choice of the periodic orbits we use for the DA bifurcations. As an application, we will obtain the following result, which answers a question that A Katok asked us:

Theorem 1.9 *There exists a closed orientable three-manifold supporting both a transitive Anosov vector field and a nontransitive Anosov vector field.*

1.4.2 Hyperbolic attractors As already mentioned above, the entrance foliation $\mathcal{L}^s(U, X)$ of an attracting hyperbolic plug (U, X) is always an *MS foliation* (it has finitely many compact leaves, every half leaf is asymptotic to a compact leaf and every compact leaf can be oriented so that its holonomy is a contraction). Using Theorem 1.5, we shall prove a converse statement: every MS foliation can be realized as the entrance foliation of a transitive attracting hyperbolic plug. More precisely:

Theorem 1.10 *For every MS foliation $\mathcal{F}$ on a closed orientable surface S, there exists an orientable transitive attracting hyperbolic plug (U, X) and a homeomorphism $h\colon \partial^{\mathrm{in}} U \to S$ such that $h_*(\mathcal{L}^{\mathrm{s}}(U, X)) = \mathcal{F}$.*

Let (U, X) be an attracting hyperbolic plug, and Λ be the maximal invariant set of (U, X). Each compact leaf γ of the foliation $\mathcal{L}^{\mathrm{s}}(U, X)$ can be endowed with the contracting orientation. Recall that the contracting orientation of γ is defined so that its holonomy is a contraction. The attractor Λ is said to be *incoherent* if one can find two compact leaves γ_1 and γ_2 of $\mathcal{L}^{\mathrm{s}}(U, X)$ in the same connected component of $\partial^{\mathrm{in}} U$ such that γ_1 and γ_2 equipped with their contracting orientations are not freely homotopic. The notion of incoherent hyperbolic attractors was introduced by J Christy in his PhD thesis [11]. Christy studied the existence of Birkhoff sections for hyperbolic attractors of vector fields on three-manifolds. He proved that a transitive hyperbolic attractor admits a Birkhoff section if and only if it is coherent, and he announced that he could build incoherent transitive hyperbolic attractors. He did publish an example of incoherent hyperbolic attractor in Christy [12], but it is not clear whether this example is transitive or not. The existence of incoherent transitive hyperbolic attractors is an immediate consequence of Theorem 1.10:

Corollary 1.11 *There exist incoherent transitive hyperbolic attractors (on orientable manifolds).*

1.4.3 Embedding hyperbolic plugs in Anosov flows Using Theorem 1.10, we will be able to prove that every hyperbolic plug with filling MS laminations can be embedded in an Anosov flow. More precisely:

Theorem 1.12 *Consider a hyperbolic plug with filling MS laminations (U_0, X_0).*

- *Up to changing (U_0, X_0) by a topological equivalence, we can find an Anosov vector field X on a closed orientable three-manifold M such that there exists an embedding $\theta\colon U_0 \hookrightarrow M$ with $\theta_* X_0 = X$.*
- *Moreover, if the maximal invariant set of (U_0, X_0) contains neither attractors nor repellers, the construction can be done in such a way that the Anosov vector field X is transitive.*

In other words, the first item of Theorem 1.12 states that, for every hyperbolic plug with filling MS laminations (U_0, X_0), we can find a closed three-manifold M, an Anosov vector field X on M, and a closed submanifold with boundary U of M, such that X is transverse to ∂U and such that (U_0, X_0) is topologically equivalent to $(U, X|_U)$. The manifold M and the Anosov vector field X will be constructed by gluing appropriate attracting and repelling hyperbolic plugs on (U_0, X_0). These attracting and repelling hyperbolic plugs will be provided by Theorem 1.10.

In order to get the second item of Theorem 1.12, we will modify the Anosov vector field provided by the first item, using the "blow-up, excise and glue procedure" discussed above.

1.4.4 Manifolds with several Anosov vector fields Our techniques also allow to construct examples of three-manifolds supporting several Anosov flows. Barbot [4] constructed an infinite family of three-manifolds, each of which supports at least two (topologically nonequivalent) Anosov flows. We shall prove the following result:

Theorem 1.13 *For any $n \geq 1$, there is a closed orientable three-manifold M supporting at least n transitive Anosov vector fields $Z_1, \dots, Z_n$ which are pairwise topologically nonequivalent.*

Remark 1.14 The manifold M that we will construct to prove Theorem 1.13 has a JSJ decomposition with three pieces: two hyperbolic pieces and one Seifert piece. These examples also positively answer two questions asked by Barbot and Fenley in the final section of their recent paper [5].[4] The vector fields $Z_1, \dots, Z_n$ that we will construct are pairwise homotopic in the space of nonsingular vector fields on M.

Theorem 1.13 admits many variants. For example, we claim that the manifold M that we will construct also supports at least n nontransitive Anosov flows. We also claim that, for every $n \geq 1$, there exists a graph manifold supporting at least n transitive Anosov vector fields. We will not prove those claims to avoid increasing too much the length of the paper; we leave them as exercises for the reader.

1.4.5 Transitive Anosov vector fields with infinitely many transverse tori By Theorem 1.5 we can build transitive Anosov vector fields by gluing hyperbolic plug along their boundary. Conversely, one may try to decompose a transitive Anosov vector field X on a closed three-manifold M into hyperbolic plugs by cutting M along a finite family of pairwise disjoint tori that are transverse to X. Ideally, one would like to find a canonical (maximal) finite family of pairwise disjoint tori embedded in M and transverse to X such that, by cutting M along these tori, one gets a canonical (maximal) decomposition of (M, X) into hyperbolic plugs. This raises the following

[4] They asked the following questions: Do there exist examples of manifolds, with a JSJ decomposition containing more than one hyperbolic piece, supporting transitive Anosov or pseudo-Anosov flows? Do there exist examples of manifolds, with both hyperbolic and Seifert pieces in their JSJ decomposition, supporting transitive Anosov or pseudo-Anosov flows?

question: given an Anosov vector field X on a closed three-manifold M, are there only finitely many tori (up to isotopy) embedded in M and transverse to X? Unfortunately, we shall prove that this is not the case:

Theorem 1.15 *There exists a transitive Anosov vector field Z on a closed orientable three-manifold M such that there exist infinitely many pairwise nonisotopic tori embedded in M and transverse to Z.*

Roughly speaking, Theorem 1.15 implies that there is no possibility of finding a "fully canonical" maximal decomposition of any transitive Anosov vector field into hyperbolic plugs. In another paper [7], we prove that one can find a maximal decomposition of any transitive Anosov vector field X into hyperbolic plugs, such that the maximal invariant sets of the plugs are canonically associated with X. This is what we call the *spectral-like decomposition* for transitive Anosov vector fields.

Homage This work started in January 2012, as we were reading in a working group the beautiful paper [10] of Marco Brunella. It was during the same month that we learned the sad news of Marco's death. We dedicate this paper to his memory.

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Part I Proof of the gluing theorem

2 A brief outline of the proof

In this section, we will give a brief outline of the proof of the gluing theorem, Theorem 1.5.

We consider a hyperbolic plug (V, X) with filling MS laminations and without attractors and repellers, and a strongly transverse gluing diffeomorphism $\varphi_0\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$. We want to construct a hyperbolic plug (V, Y) and a strongly transverse gluing diffeomorphism $\psi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$ such that (V, X, φ) and (V, Y, ψ) are strongly isotopic, and such that the vector field Z induced by Y on V/ψ is Anosov.

By classical arguments, to show that a vector field is Anosov, it is sufficient to consider the return map of the orbits of Z on some "section", and prove that this return map is hyperbolic. In our situation, it is sufficient and convenient to consider the return map Θ_ψ of the orbit of Z on $\partial^{\mathrm{in}} V$. Note that this return map Θ_ψ is the product of the "crossing map" $\Gamma\colon \partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}_Y \to \partial^{\mathrm{out}} V \setminus \mathcal{L}^{\mathrm{u}}_Y$, defined by the flowlines of Y, and the gluing map $\psi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$. We will see that the hyperbolicity of the vector field Z can be deduced from the existence of two cone fields $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ on $\partial^{\mathrm{in}} V$, satisfying some invariance and expansion/contraction properties under the return map Θ_ψ (Lemma 6.1).

So we are left to construct the vector field Y, the gluing map ψ and the cone fields $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$. We begin by a series of perturbations of (V, X) and φ_0, which will yield some nice properties:

- First, we replace the vector field X by a vector field Y, arbitrarily close to X in C^1–topology, such that the maximal invariant set Λ_Y of (V, Y) admits an *affine* Markov partition (ie a Markov partition such that the return map on the rectangles is affine in some coordinates). This yields a very useful property: the holonomy of every compact leaf of the entrance (resp. exit) lamination $\mathcal{L}^{\mathrm{s}}_Y$ (resp. $\mathcal{L}^{\mathrm{u}}_Y$) is a homothety. More details can be found in Lemma 5.3.

- Then we extend the laminations $W^{\mathrm{s}}(\Lambda_Y)$ and $W^{\mathrm{u}}(\Lambda_Y)$ to foliations $\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ on an invariant neighborhood $\mathcal{U}_0$ of Λ_Y such that the holonomy of every compact leaf in $\mathcal{W}^{\mathrm{s}}_{\mathrm{in}} = \mathcal{F}^{\mathrm{s}} \cap \partial^{\mathrm{in}} V$ and $\mathcal{W}^{\mathrm{u}}_{\mathrm{out}} = \mathcal{F}^{\mathrm{u}} \cap \partial^{\mathrm{out}} V$ is a homothety (Lemma 5.6).

- As Y can be chosen arbitrarily C^1–close to X, the laminations $\varphi_{0,*}(\mathcal{L}^{\mathrm{u}}_Y)$ and $\mathcal{L}^{\mathrm{s}}_Y$ are still strongly transverse. In Proposition 5.7, we perturb the gluing map $\varphi_0\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$ to $\varphi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$ such that, in some invariant neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of Λ_Y, the following properties hold:
  1. the laminations $\varphi_*(\mathcal{L}^{\mathrm{u}})$ and $\mathcal{L}^{\mathrm{s}}$ are strongly transverse;
  2. the foliations $\varphi_*(\mathcal{W}^{\mathrm{s}}_{\mathrm{out}})$ and $\varphi_*(\mathcal{W}^{\mathrm{u}}_{\mathrm{out}})$ coincide with $\mathcal{W}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{W}^{\mathrm{u}}_{\mathrm{in}}$, respectively, on $\varphi(\mathcal{U} \cap \partial^{\mathrm{out}} V) \cap (\mathcal{U} \cap \partial^{\mathrm{in}} V)$, where $\mathcal{W}^{\mathrm{s}}_{\mathrm{out}} = \mathcal{F}^{\mathrm{s}} \cap \partial^{\mathrm{out}} V$ and $\mathcal{W}^{\mathrm{u}}_{\mathrm{in}} = \mathcal{F}^{\mathrm{s}} \cap \partial^{\mathrm{in}} V$;
  3. φ is isotopic to φ_0 among strongly transverse gluing maps.

- Finally, we can extend the foliations $\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ on $\mathcal{U}_0$ to two foliations $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{u}}$ on V, and modify the gluing map $\varphi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$ to $\varphi_1\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$ to get a global transversality property (see Lemmas 5.13 and 5.15).

The outcome of this series of perturbations is summarized in Proposition 5.2.

Then comes the core of the proof: we perturb the gluing map φ_1 into a gluing map ψ in order to get the hyperbolicity properties. For any $\lambda > 1$ and $\varepsilon > 0$, we construct $\psi^{\mathrm{in}} = \psi^{\mathrm{in}}_{\lambda,\varepsilon}\colon \partial^{\mathrm{in}} V \to \partial^{\mathrm{in}} V$ (see Proposition 6.2) such that

(1) ψ^{in} coincides with the identity map on a neighborhood of the lamination $\mathcal{L}^{\mathrm{s}}$;

(2) ψ^{in} preserves each leaf of the foliation $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$;

(3) the foliation $(\psi^{\mathrm{in}})^{-1}_{*}(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}})$ is ε–C^1–close to the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$;

(4) the derivative of $\Gamma\circ\psi^{\mathrm{in}}$ expands vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than λ: for any vector u tangent to a leaf of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$, one has $\|(\Gamma\circ\psi^{\mathrm{in}})_*(u)\| > \lambda\|u\|$.

A key ingredient in the construction of ψ^{in} is the following: If x is a point of $\partial^{\mathrm{in}}V\setminus\mathcal{L}^{\mathrm{s}}$ which is very close to the lamination $\mathcal{L}^{\mathrm{s}}$, then the forward orbit of x will spend a very long time near Λ. As a consequence, the derivative of the crossing map Γ at x will expand a vector tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a very large factor. Therefore, property (4) above is automatically satisfied on a sufficiently small neighborhood of $\partial^{\mathrm{in}}V\setminus\mathcal{L}^{\mathrm{s}}$, no matter what ψ^{in} might be.

Analogously, there exists a diffeomorphism $\psi^{\mathrm{out}}=\psi^{\mathrm{out}}_{\lambda,\varepsilon}\colon \partial^{\mathrm{out}}V\to\partial^{\mathrm{out}}V$. Then we obtain $\psi=\psi_{\lambda,\varepsilon}:=\psi^{\mathrm{in}}_{\lambda,\varepsilon}\circ\varphi_1\circ\psi^{\mathrm{out}}_{\lambda,\varepsilon}$ and $\Theta_{\lambda,\varepsilon}:=\psi_{\lambda,\varepsilon}\circ\Gamma$. The properties of $\psi^{\mathrm{in}}_{\lambda,\varepsilon}$ and $\psi^{\mathrm{out}}_{\lambda,\varepsilon}$ ensure that we can find some cone fields $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ satisfying the desired contraction/expansion properties for the return map $\Theta_\psi=\psi\circ\Gamma$.

3 Definitions and elementary properties

In this paper we consider nonsingular vector fields on compact three-dimensional manifolds (with or without boundary).

3.1 Hyperbolic plugs

Definitions 3.1 Throughout this paper, a *plug* is a pair (V,X) where V is a (not necessarily connected) compact three-dimensional manifold with boundary, and X is a nonsingular C^1 vector field on V transverse to the boundary ∂V. Given such a plug (V,X), we decompose ∂V as a disjoint union

$$\partial V=\partial^{\mathrm{in}}V\sqcup\partial^{\mathrm{out}}V,$$

where X points inward (resp. outward) V along $\partial^{\mathrm{in}}V$ (resp. $\partial^{\mathrm{out}}V$). We call $\partial^{\mathrm{in}}V$ the *entrance boundary* of V, and $\partial^{\mathrm{out}}V$ the *exit boundary* of V. If $\partial^{\mathrm{out}}V=\varnothing$, we say that (V,X) is an *attracting plug*. If $\partial^{\mathrm{in}}V=\varnothing$, we say that (V,X) is a *repelling plug*.

If ∂V is nonempty, the flow of X is not complete. Every orbit of X is defined for a closed time interval of $\mathbb{R}$. The *maximal invariant set* Λ of X in V is the set of points $x\in V$ whose forward and backward orbits are defined forever; equivalently, Λ is the set of points whose orbit is disjoint from ∂V. The *stable set* $W^{\mathrm{s}}(\Lambda)$ is the set of points whose forward orbit is defined forever. Equivalently, $W^{\mathrm{s}}(\Lambda)$ is the set of points whose forward orbit accumulates on Λ, and is the set of points whose positive orbit is disjoint from $\partial^{\mathrm{out}} V$. Analogously the *unstable set* $W^{\mathrm{u}}(\Lambda)$ is the set of points whose backward orbit is defined forever. Equivalently, it is the set of points whose negative orbit is disjoint from $\partial^{\mathrm{in}} V$.

Definition 3.2 Throughout the paper, a *hyperbolic plug* is a plug (V, X) whose maximal invariant set Λ is hyperbolic with one-dimensional strong stable and strong unstable bundles: for $x \in \Lambda$, there is a splitting

$$T_x V = E^{\mathrm{s}}(x) \oplus \mathbb{R}X(x) \oplus E^{\mathrm{u}}(x)$$

which depends continuously on x and is invariant under the derivative of the flow of X, and there is a Riemannian metric on V such that the differential of the time one map of the flow of X contracts uniformly the vectors in E^{s} and expands uniformly the vectors in E^{u}.

We cannot recall here the whole hyperbolic theory. Let us just recall some elementary properties from the classical theory of hyperbolic dynamical systems that we will use in the other sections:

- For every x, the *strong stable manifold*

$$W^{\mathrm{ss}}(x) = \{y \in V \mid d(X^t(y), X^t(x)) \to 0 \text{ as } t \to +\infty\}$$

is a C^1 curve through x tangent at x to $E^{\mathrm{s}}(x)$. The strong unstable manifold $W^{\mathrm{uu}}(x)$ of x is the strong stable manifold of x for $-X$.

- The weak stable manifold $W^{\mathrm{s}}(x)$ (resp. weak unstable manifold $W^{\mathrm{u}}(x)$) of x is the union of the stable manifolds (resp. unstable manifolds) of the points in the orbit of x. The weak stable and weak unstable manifolds of points $x \in \Lambda$ are C^1 injectively immersed surfaces which depend continuously on x. The weak stable (resp. unstable) manifold of x is invariant under the positive (resp. negative) flow, and by the negative (resp. positive) flow for the times it is defined.

- There exist two 2–dimensional laminations $W^{\mathrm{s}}(\Lambda)$ and $W^{\mathrm{u}}(\Lambda)$ whose leaves are the weak stable and unstable manifolds, respectively, of the points of Λ. These laminations are of class $C^{0,1}$, ie the leaves are C^1–immersed manifolds tangent to a continuous plane field. Moreover, these laminations are of class C^1 when the vector field X is of class C^2 (see Hasselblatt [23, Corollary 2.3.4], for example).

- The 2–dimensional laminations $W^{\mathrm{s}}(\Lambda)$ and $W^{\mathrm{u}}(\Lambda)$ are everywhere transverse and the intersection $W^{\mathrm{s}}(\Lambda) \cap W^{\mathrm{u}}(\Lambda)$ is precisely Λ.

- The nonwandering set $\Omega(X)$ is contained in Λ. It is the union of finitely many transitive hyperbolic sets, called *basic pieces of* X. Every point in Λ belongs to the intersection of the stable manifold of one basic piece with the unstable manifold of a basic piece.

- There is a C^1–neighborhood $\mathcal{U}$ of X such that, for every $Y \in \mathcal{U}$, the pair (V, Y) is a hyperbolic plug, and it is topologically (orbitally) equivalent to X: there is a homeomorphism $\varphi_Y\colon V \to V$ mapping the oriented orbits of Y on the orbits of X. Furthermore, for Y C^1–close to X, the homeomorphism φ_Y can be chosen C^0–close to the identity.

3.2 Separatrices, free separatrices and boundary leaves

We will now introduce some notions which are useful for describing more precisely the geometry of the stable and unstable laminations of hyperbolic sets of vectors fields on three-manifolds. These notions were introduced by the first two authors in [6].

Let (V, X) be a hyperbolic plug, and Λ be its maximal invariant set. By gluing some manifolds with boundary R and A, respectively, on $\partial^{\text{in}} V$ and $\partial^{\text{out}} V$, one can embed V in a closed three-manifold $M = R \cup V \cup A$ and extend X to a smooth vector field on M, such that $\partial^{\text{in}} V$ is the exit boundary of a repelling region and $\partial^{\text{out}} V$ is the entrance boundary of an attracting region for $\overline{X}$ for the new vector field. After this operation, V appears as a *filtrating neighborhood*, ie the intersection of a repelling and an attracting region, for a vector field on a closed three-manifold. This allows us to use some notions that were defined in [6] for hyperbolic maximal invariant sets of filtrating neighborhoods of flows on closed three-manifolds.

Definitions 3.3 A *stable separatrix* of a periodic orbit $O \subset \Lambda$ is a connected component of $W^{\text{s}}(O) \setminus O$. A stable separatrix of a periodic orbit $O \subset \Lambda$ is called a *free separatrix* if it is disjoint from Λ.

Remark 3.4 For $x \in \Lambda$, the weak stable manifold $W^{\text{s}}(x)$ is an injectively immersed manifold.

- If $W^{\text{s}}(x)$ does not contain periodic orbit, then $W^{\text{s}}(x)$ is diffeomorphic to the plane $\mathbb{R}^2$, and the foliation of $W^{\text{s}}(x)$ by the orbits of flow of X is topologically equivalent to the trivial foliation of $\mathbb{R}^2$ by horizontal lines.

- If $W^{\text{s}}(x)$ contains a periodic orbit O, then O is unique, and every orbit of X on $W^{\text{s}}(x)$ is asymptotic to O in the future. If the multipliers[5] of O are positive,

[5] That is, the eigenvalues of the derivative at p of the first return map of the orbits of X on a local section intersecting O at a single point p.

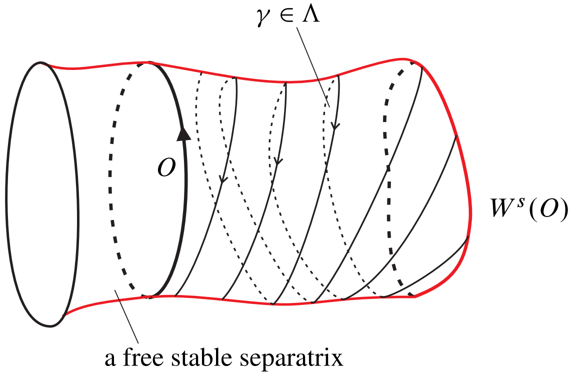

Figure 2

then $W^s(x)$ is diffeomorphic to a cylinder, and O has two stable separatrices. If the multipliers of O are negative, then $W^s(x)$ is diffeomorphic to a Möbius band, and O has only one stable separatrix. In any case, each stable separatrix of O is diffeomorphic to a cylinder $S^1 \times \mathbb{R}$, and the foliation of this separatrix by the orbits of X is topologically equivalent to the trivial foliation of $S^1 \times \mathbb{R}$ by vertical lines.

Remarks 3.5 Let x be a point in Λ.

(1) Each orbit of X on $W^s(x)$ cuts $\partial^{in} V$ in at most one point, and this point depends continuously on the orbit. Thus, the connected components of $W^s(x) \cap \partial^{in} V$ are in one-to-one correspondence with the connected components of $W^s(x) \setminus \Lambda$.

(2) Due to the dynamics inside each leaf $W^s(x)$, one easily shows that, for each connected component C of $W^s(x) \setminus \Lambda$, one has one of the two possible situations below:

- Either there is an orbit $O \in \Lambda \cap W^s(x)$ such that C is a connected component of $W^s(x) \setminus O$. In that case, Lemma 1.6 of [6] proves that O is a periodic orbit, and C is a free stable separatrix of O. In particular, C is diffeomorphic to a cylinder, and the foliation of C by the orbits of X is topologically equivalent to the trivial foliation of $S^1 \times \mathbb{R}$ by vertical lines. Since each orbit X of C cuts $\partial^{in} V$ at exactly one point, which depends continuously on the orbit, one deduces than $C \cap \partial^{in} V$ is diffeomorphic to a circle (see Figure 3, left).
- Or there are two orbits $O_1, O_2 \in \Lambda \cap W^s(x)$ such that C is a connected component of $W^s(x) \setminus (O_1 \cup O_2)$; in other words, C is a strip bounded by O_1 and O_2 and the foliation of C by the orbits of X is topologically equivalent to the trivial foliation of $\mathbb{R}^2$ by horizontal lines. Since each orbit X of C cuts $\partial^{in} V$ at exactly one point, which depends continuously on the orbit, one deduces that $C \cap \partial^{in} V$ is diffeomorphic to a line (see Figure 3, right, or Figure 8 from another point of view).

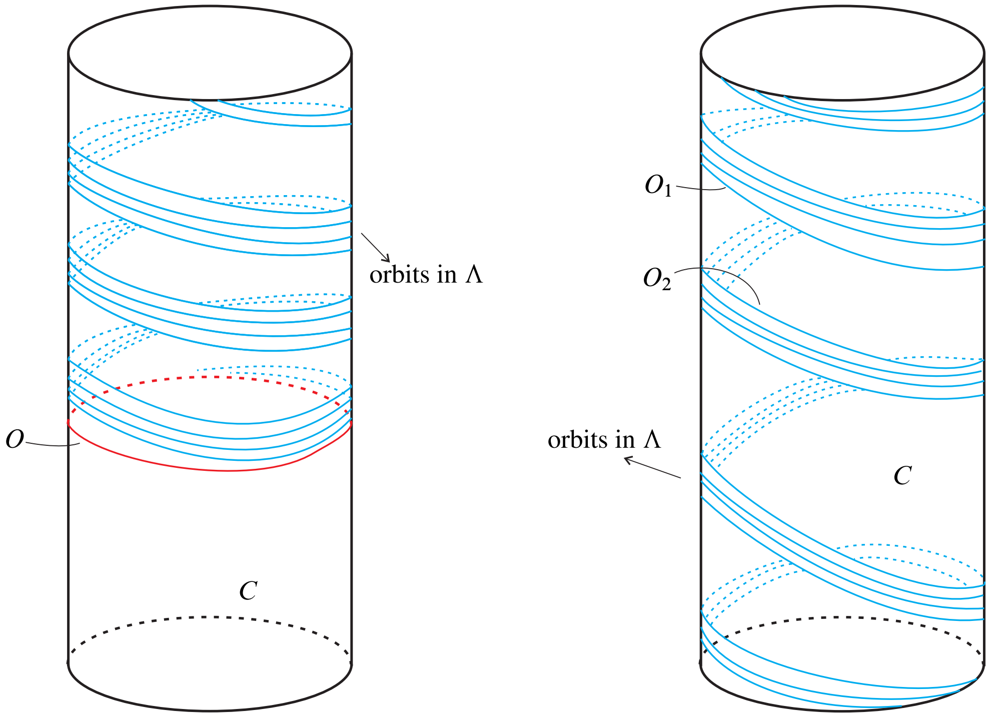

Figure 3: The two cases for a connected components C of $W^s(x) \setminus \Lambda$ (Remark 3.5(2))

Definition 3.6 An unstable manifold $W^u(x)$ is called *an unstable boundary leaf* if there is an open path I cutting $W^u(x)$ transversely at a point y such that one connected component of $I \setminus \{y\}$ is disjoint from $W^u(\Lambda)$.

Remark 3.7 Lemma 1.6 of [6] shows that the unstable boundary leaves are precisely the unstable manifolds of the periodic orbits having a free stable separatrix, and, moreover, that there are only finitely many periodic orbits in Λ having a free stable separatrix. As an immediate consequence, there are only finitely many boundary leaves in $W^u(\Lambda)$.

One defines, similarly, free unstable separatrices and the stable boundary leaves.

3.3 Entrance and exit laminations

Let (V, X) be a hyperbolic plug, Λ be its maximal invariant set, and $W^s(\Lambda)$ and $W^u(\Lambda)$ be the 2–dimensional stable and unstable laminations of Λ, respectively.

The vector field X is tangent to the laminations $W^s(\Lambda)$ and $W^u(\Lambda)$, and is transverse to ∂V. This implies that the laminations $W^s(\Lambda)$ and $W^u(\Lambda)$ are transverse to ∂V. As a consequence, each leaf of these 2–dimensional laminations cuts ∂V along C^1–curves, and the laminations $W^s(\Lambda)$ and $W^u(\Lambda)$ cut ∂V along 1–dimensional laminations. Thus

$$\mathcal{L}^s = W^s(\Lambda) \cap \partial V = W^s(\Lambda) \cap \partial^{in} V \quad \text{and} \quad \mathcal{L}^u = W^u(\Lambda) \cap \partial V = W^u(\Lambda) \cap \partial^{out} V$$

are 1–dimensional. The aim of this section is to describe elementary properties of the laminations $\mathcal{L}^s$ of $\partial^{in} V$ and $\mathcal{L}^u$ of $\partial^{out} V$. More precisely:

Proposition 3.8 *The laminations $\mathcal{L}^s$ and $\mathcal{L}^u$ satisfy the following properties:*

(1) *The laminations contains finitely many compact leaves.*

(2) *Every half leaf is asymptotic to a compact leaf.*

(3) *Each compact leaf may be oriented so that its holonomy is a contraction.*

Sketch of proof Item (1) is a direct consequence of Remarks 3.5 and 3.7: the compact leaves of $\mathcal{L}^s$ are in one-to-one correspondence with the free stable separatrices of periodic orbits, and there are only finitely many such separatrices.

Consider a noncompact leaf γ of $\mathcal{L}^s$. According to Remarks 3.5, it corresponds to a connected component of some $W^s(x) \setminus \Lambda$ which is a strip B bounded by two orbits O_1 and O_2 in $\Lambda \cap W^s(x)$. Consider the unstable manifold $W^u(O_i)$. These unstable leaves are boundary leaves of $W^u(\Lambda)$. According to Remark 3.7, it follows that O_1 and O_2 belong to the unstable manifolds of periodic orbits having a free separatrix. Therefore the λ–lemma (see [23]) implies that B accumulates these free separatrices, and γ accumulates on the corresponding compact leaves of $\mathcal{L}^s$. Lemma 1.8 of [6] makes this argument rigorously for proving item (2).

Let us now explain item (3). Let γ_0 be a compact leaf of $\mathcal{L}^s$. Then γ_0 is the intersection of $\partial^{in} V$ with a free stable separatrix of a periodic orbit $O_0 \subset \Lambda$. The leaves of $\mathcal{L}^s$, in the neighborhood of γ_0, are the transverse intersection of the weak stable manifolds of orbits which belong to $W^u(O_0)$. Notice that $W^u(O_0)$ and $\partial^{in} V$ are both transverse to the lamination $W^s(\Lambda)$. Furthermore, γ_0 and O_0 are contained in the same leaf $W^s(O_0)$, which is either a cylinder or a Möbius band.

- If $W^s(O_0)$ is a cylinder then γ_0 and O_0 are homotopic in the leaf $W^s(O_0)$. Therefore the holonomy of $\mathcal{L}^s$ along γ_0 is conjugated to the holonomy of the lamination induced by $W^s(\Lambda)$ on $W^u(O_0)$. This holonomy is a contraction if one endows the orbit O_0 with the orientation induced by the vector field $-X$.

- If $W^{\mathrm{s}}(O_0)$ is a Möbius band then γ_0 is homotopic (in $W^{\mathrm{s}}(O_0)$) to $2 \cdot O_0$. Therefore the holonomy of $\mathcal{L}^{\mathrm{s}}$ along γ_0 is conjugated to the square of the holonomy of the lamination induced by $W^{\mathrm{s}}(\Lambda)$ on $W^{\mathrm{u}}(O_0)$; once again, this holonomy is a contraction if one endows the orbit O_0 with the orientation induced by the vector field $-X$. □

Definition 3.9 A lamination (resp. a foliation) on a closed surface is called an *MS lamination*[6] (resp. an *MS foliation*) if it satisfies the following properties:

(1) It has only finitely many compact leaves.

(2) Every half leaf is asymptotic to a compact leaf.

(3) each compact leaf may be oriented such that its holonomy is a contraction.

Proposition 3.8 states that the entrance/exit laminations of a hyperbolic plug are MS laminations.

Remark 3.10 If (V, X) is a hyperbolic attracting plug, then $\mathcal{L}^{\mathrm{s}}$ is a foliation on $\partial^{\mathrm{in}} V$ (and $\partial^{\mathrm{out}} V$ is empty). In particular, ∂V consists of some tori (and possibly some Klein bottles if V is not orientable).

3.4 Connected component of the complement of the laminations

Let us start with a very general observation:

Definition 3.11 Let S be a closed surface, $\mathcal{L}$ be a 1–dimensional lamination with finitely many compact leaves, and C be a connected component of $S \setminus \mathcal{L}$. We call C a *strip* if it satisfies the two following properties:

- C is homeomorphic to $\mathbb{R}^2$.
- The accessible boundary[7] of C consists of exactly two noncompact leaves of $\mathcal{L}$ which are asymptotic to each other at both ends.

Otherwise we say that C is an *exceptional component* of $S \setminus \mathcal{L}$. See Figure 3 as an illustration.

Lemma 3.12 *Let S be a closed surface and $\mathcal{L}$ be a 1–dimensional lamination with finitely many compact leaves. Then there are only finitely many exceptional components in $S \setminus \mathcal{L}$.*

[6] "MS" stands for "Morse–Smale"

[7] That is, the points in the boundary which are an extremal point of a segment whose interior is contained in C.

Proof There is a smooth Morse–Smale vector field Z on S transverse to the lamination $\mathcal{L}$ — one easily builds a continuous vector field transverse to the lamination; transversality is an open property; hence (see Palis and de Melo [25, Chapter 4]) one may perturb this vector field to turn it into a smooth Morse–Smale vector field. Now, at most finitely many connected components or $S \setminus \mathcal{L}$ may contain a singular point of Z or a whole periodic orbit of Z.

If C is a component which does not contain any singular point or any periodic orbit, then the dynamics of Z restricted to C does not contain any nonwandering point. Furthermore, by transversality of Z with $\mathcal{L}$ (hence of the accessible boundary or C) any orbit reaching a small neighborhood of ∂C goes out of C in a finite time. As a direct consequence, there is $T > 0$ such that no orbit of $Z|_C$ is defined for a time interval of length larger than T. This implies that either C is an annulus bounded by two compact leaves of $\mathcal{L}$, or C is a strip (as Definition 3.11). □

3.5 The crossing map

We consider a hyperbolic plug (V, X). As usual, we denote by Λ the maximal invariant set of (V, X), by $\mathcal{L}^{\mathrm{s}} \subset \partial^{\mathrm{in}} V$ the entrance lamination of (V, X) and by $\mathcal{L}^{\mathrm{u}} \subset \partial^{\mathrm{out}} V$ the exit lamination of (V, X).

Definition 3.13 The positive orbit of any point $x \in \partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$ reaches $\partial^{\mathrm{out}} V$ in a finite time at a point $\Gamma(x) \in \partial^{\mathrm{out}} V \setminus \mathcal{L}^{\mathrm{u}}$. The map $x \mapsto \Gamma(x)$ will be called *the crossing map* of the plug (V, X). Using the fact that the orbits of X are transverse to ∂V, one easily sees that Γ defines a diffeomorphism from $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$ to $\partial^{\mathrm{out}} V \setminus \mathcal{L}^{\mathrm{u}}$. See Figure 4.

Lemma 3.14 *A connected component C of $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$ is a strip if and only if $\Gamma(C)$ is a strip in $\partial^{\mathrm{out}} V \setminus \mathcal{L}^{\mathrm{u}}$.*

Proof Using the fact that Γ is a diffeomorphism, up to reversing the flow of X, one is reduced to proving the following assertion: if C is a strip then $\Gamma(C)$ is a strip.

Assume that C is a strip. First notice that $\Gamma(C)$ is homeomorphic to C, hence to $\mathbb{R}^2$. It remains to check that the accessible boundary consists of two leaves. For that purpose, one considers a smooth vector field Z on $\partial^{\mathrm{in}} V$ transverse to $\mathcal{L}^{\mathrm{s}}$ and without singular point in C. Then the orbits of Z restricted to the union of C with its accessible boundary induces a trivial foliation by segments. Consider now the image of this foliation by Γ. This is a foliation of $\Gamma(C)$. However, as Z is transverse to the boundary of C which is contained in the stable manifold of the hyperbolic set Λ, the λ–lemma (or a cone field argument) implies that $\Gamma_*(Z)$ tends uniformly to the tangent direction to the unstable lamination $\mathcal{L}^{\mathrm{u}}$ when $\Gamma(x)$ tends to the boundary of $\Gamma(C)$.

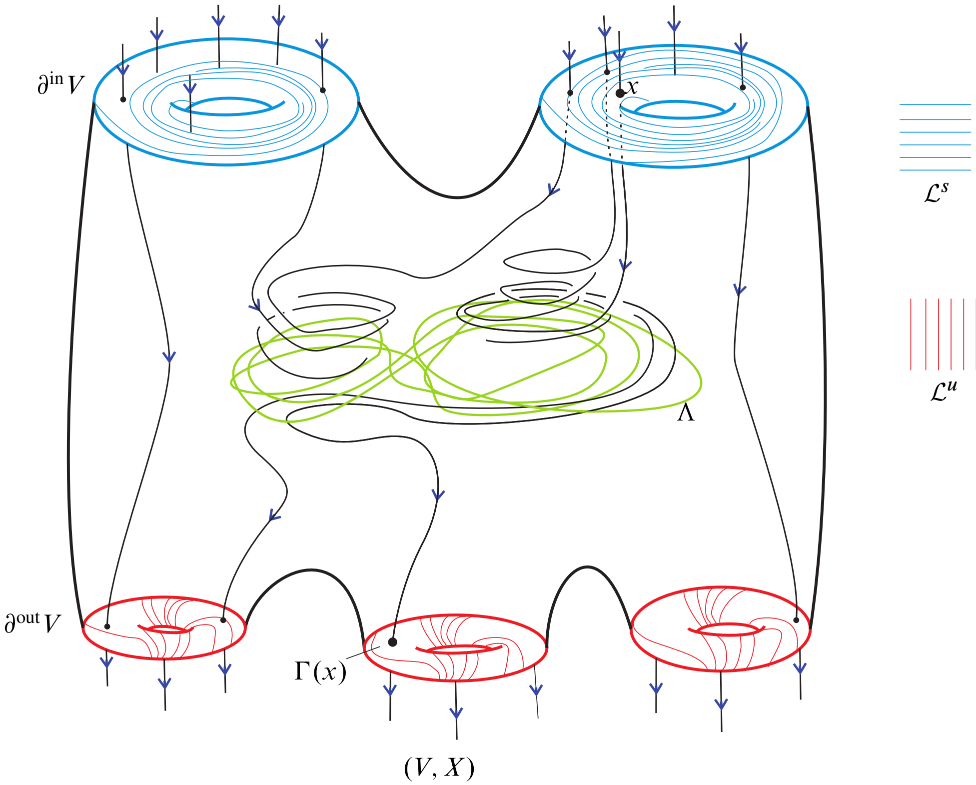

Figure 4: A hyperbolic plug (V, X), its maximal invariant set Λ (in green) and its crossing map Γ

Thus one gets a foliation on the closure of $\Gamma(C)$; this implies that $\Gamma(C)$ is a strip, which concludes the proof. $\square$

A very similar proof allows us to prove:

Lemma 3.15 *A connected component C of $\partial^{\text{in}} V \setminus \mathcal{L}^{\text{s}}$ is an annulus bounded by two compact leaves (resp. a Möbius band bounded by one compact leaf) of $\mathcal{L}^{\text{s}}$ if and only if $\Gamma(C)$ is an annulus bounded by two compact leaves (resp. a Möbius band bounded by one compact leaf) of $\mathcal{L}^{\text{u}}$.*

3.6 Filling MS lamination and prefoliation

If a hyperbolic plug can be embedded in an Anosov flow, the stable and unstable manifolds of its maximal invariant set are sublaminations of the stable and unstable foliations, respectively. This leads to restrictions on its stable and unstable laminations.

Definition 3.16 We say that a lamination $\mathcal{L}$ of a closed surface S is a *prefoliation* if it can be completed as a foliation of S. Notice that this implies that every connected component of S is either the torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}$.

As a direct consequence of Proposition 3.8 and Lemmas 3.12 and 3.14, one obtains:

Lemma 3.17 *Let (V, X) be an hyperbolic plug. The lamination $\mathcal{L}^s$ is a prefoliation if and only if every exceptional component of $\partial^{in} V \setminus \mathcal{L}^s$ is either an annulus or a Möbius band bounded by compact leaves of $\mathcal{L}^s$. Furthermore,*

$$\mathcal{L}^s \text{ is a prefoliation} \iff \mathcal{L}^u \text{ is a prefoliation}.$$

The components of $\partial^{in} V \setminus \mathcal{L}^s$ and $\partial^{out} V \setminus \mathcal{L}^u$ which are annuli or Möbius bands will sometimes lead to specific difficulties. For this reason we introduce a more restrictive notion:

Definition 3.18 A lamination $\mathcal{L}$ on a closed surface S is called a *filling MS lamination* if

- it is an MS lamination (see Definition 3.9);
- $S \setminus \mathcal{L}$ has no exceptional component (in other words, every connected component of $S \setminus \mathcal{L}$ is a strip; see Definition 3.11 and Figure 5).

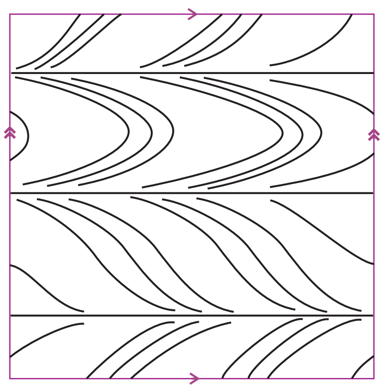

Figure 5: A filling MS lamination

Lemma 3.19 *Let $\mathcal{L}$ be a filling MS lamination of a closed surface S. Then $\mathcal{L}$ is a prefoliation. In particular, every component of S is either a torus $\mathbb{T}^2$ or a Klein bottle $\mathbb{K}$. Furthermore, if $\mathcal{F}$ is a foliation containing $\mathcal{L}$ as a sublamination, then $\mathcal{F}$ is an MS foliation. Finally, two foliations containing $\mathcal{L}$ as a sublamination are topologically conjugated.*

Idea for the proof One just need to foliate each connected component of the complement of $\mathcal{L}$. Such a component is a strip bounded by two asymptotic leaves. There is a unique way to foliate such a strip up to topological conjugacy. □

Remark 3.20 The converse of the first statement of Lemma 3.19 is not true, ie there exist prefoliations which are not filling MS laminations. An easy example is given by a lamination on $\mathbb{T}^2$ consisting of a single closed leaf not homotopic to 0.

Once again, as a consequence of Proposition 3.8 and Lemmas 3.12 and 3.14, one easily shows:

Lemma 3.21 *Let (V, X) be a hyperbolic plug. The stable lamination $\mathcal{L}^s$ is a filling MS lamination if and only if $\partial^{in} V \setminus \mathcal{L}^s$ has no exceptional component. The entrance lamination $\mathcal{L}^s$ is a filling MS lamination if and only if $\mathcal{L}^u$ is a filling MS lamination.*

If $\mathcal{L}^s$ and $\mathcal{L}^u$ are filling MS laminations, we will say that (V, X) is a *hyperbolic plug with filling MS laminations*.

3.7 Hyperbolic plugs with prefoliations and invariant foliation

Let (V, X) be a hyperbolic plug such that the entrance lamination $\mathcal{L}^s_X$ and the exit lamination $\mathcal{L}^u_X$ are prefoliations. The following lemma shows that every foliation on $\partial^{out} V$ transverse to $\mathcal{L}^u$ extends to an X–invariant foliation of V containing $W^s(\Lambda)$ as a sublamination.

Lemma 3.22 *Let F^s be a foliation on $\partial^{out} V$ which is transverse to $\mathcal{L}^u$. Then F^s extends on V to an invariant foliation $\mathcal{F}^s$ with two-dimensional leaves containing $W^s(\Lambda)$ as a sublamination. In particular, $\mathcal{F}^s \cap \partial^{in} V$ is a foliation which extends $\mathcal{L}^s$.*

Proof First notice that $V \setminus W^s(\Lambda)$ is the (backwards) X–orbit of the set $\partial^{out} V$. The X–orbits of the leaves of F^s are the leaves of a foliation $\mathcal{F}^s_0$ of $V \setminus W^s(\Lambda)$. The foliation $\mathcal{F}^s_0$ is tangent to X and therefore transverse to $\partial^{in} V$. It induces a 1–foliation on $\partial^{in} V \setminus \mathcal{L}^s$.

Thus, it is enough to check that the leaves of $\mathcal{F}^s_0$ tend to the leaves of $W^s(\Lambda)$. Notice that a point in $V \setminus W^s(\Lambda)$ in a very small neighborhood of $W^s(\Lambda)$ has its positive orbit which meets $\partial^{out} V$ at a point very close to $\mathcal{L}^u$. Thus, to prove that $\mathcal{F}^s_0$ extends by continuity to $W^s(\Lambda)$, it is enough to consider the negative orbits through small segments of leaves of F^s centered at points of $\mathcal{L}^u$.

Thus we fix $\varepsilon > 0$ and we consider the family of segments of size ε of leaves of F^{s} centered at the points of $\mathcal{L}^{\mathrm{u}}$. It is a C^1–continuous family of segments parametrized by a compact set. We consider the negative orbits by the flow of X of these segments. We need to prove that these orbits tend to the stable leaves of Λ as time tends to $-\infty$. This is the classical λ–lemma (see [23]). □

Lemma 3.23 *Every X–invariant $\mathcal{C}^{0,1}$ foliation $\mathcal{F}$ containing $W^{\mathrm{s}}(\Lambda)$ as a sublamination induces on $\partial^{\mathrm{out}} V$ a one-dimensional foliation $\mathcal{L}$ transverse to $\mathcal{L}^{\mathrm{u}}$.*

Proof As $\mathcal{F}$ is X–invariant, it is transverse to $\partial^{\mathrm{out}} V$. Thus it induces a one-dimensional foliation $\mathcal{L}$ on $\partial^{\mathrm{out}} V$. Furthermore, as $\mathcal{F}$ is $\mathcal{C}^{0,1}$ and contains $W^{\mathrm{s}}(\Lambda)$, which is transverse to $W^{\mathrm{u}}(\Lambda)$ along Λ, one gets that $\mathcal{F}$ is transverse to $W^{\mathrm{u}}(\Lambda)$ in a neighborhood $\mathcal{O}$ of Λ. Now every point x of $\mathcal{L}^{\mathrm{u}}$ is on the orbit of X of a point in $\mathcal{O}$. The X–invariance of both $W^{\mathrm{s}}(\Lambda)$ and $\mathcal{F}$ implies that $\mathcal{F}$ and $W^{\mathrm{u}}(\Lambda)$ are transverse at x. One deduces that $\mathcal{L}$ is transverse to $\mathcal{L}^{\mathrm{u}}$. □

Thus, the 2–dimensional X–invariant foliations on V containing $W^{\mathrm{s}}(\Lambda)$ as a sublamination are in one-to-one correspondence with the 1–dimensional foliation on $\partial^{\mathrm{out}} V$ transverse to $\mathcal{L}^{\mathrm{u}}$.

This shows in particular that V admits many invariant foliations. These foliations will help us to recover the hyperbolic structure when we will glue the exit with the entrance boundaries. For that we need to control the expansion/contraction properties of the crossing map along the directions tangent to these foliations. This is the aim of the next lemma:

Lemma 3.24 *Let $\mathcal{F}^{\mathrm{s}}$ be an invariant $\mathcal{C}^{0,1}$–foliation on V containing $W^{\mathrm{s}}(\Lambda)$ and let $F^{\mathrm{s}}_{\mathrm{in}}$ denote the intersection $\mathcal{F}^{\mathrm{s}} \cap \partial^{\mathrm{in}} V$. Then the derivative Γ_* of the crossing map contracts arbitrarily uniformly the vectors tangent to $F^{\mathrm{s}}_{\mathrm{in}}$ in small neighborhoods of $\mathcal{L}^{\mathrm{s}}$. More precisely, for every $\varepsilon > 0$ there is $\delta > 0$ such that, given any $x \in \partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$ with $d(x, \mathcal{L}^{\mathrm{s}}) < \delta$ and any vector $u \in T_x \partial^{\mathrm{in}} V$ tangent to the leaf of $F^{\mathrm{s}}_{\mathrm{in}}$, one has*

$$\|\Gamma_*(u)\| < \varepsilon \|u\|.$$

Proof The maximal invariant set Λ admits arbitrarily small filtrating neighborhoods.[8] Recall that Λ is hyperbolic and such that the area on the center stable space $E^{\mathrm{cs}}(x) =$

[8] Recall that a *filtrating neighborhood* of a compact set is a neighborhood which is a *filtrating set* where a filtrating set is the intersection of an attracting region with a repelling region. Here an attracting region is a compact region with boundary transverse to the flow such that every orbit enters into the region at each point of the boundary, and a repelling region is an attracting region for the reverse flow. As an important consequence, an orbit which exits a filtrating neighborhood never comes back. More basic properties of filtrating neighborhoods can be found in Section 3.2.

$E^{ss}(x) \oplus \mathbb{R}X(x)$ is uniformly contracted along the orbit of $x \in \Lambda$. Let $E^{cs}(x)$ for $x \in V$ denote the tangent plane to $\mathcal{F}^s$. One denotes by

$$J_t^s(x) = \left|\mathrm{Det}((X_t)_*|_{E^{cs}(x)})\right|$$

the determinant of the restriction to $E^{cs}(x)$ of the derivative of the flow of X at time t.

One deduces that there is a filtrating neighborhood $\mathcal{U}$ of Λ and $0 < \lambda < 1$ such that

$$x \in \mathcal{U} \quad \text{and} \quad t \geq 1 \quad \Longrightarrow \quad J_t^s(x) < \lambda^t < 1.$$

The proof consists now in noting that, for $x \in \partial^{in} V$ close to $\mathcal{L}^s$, the orbit segment joining x to $\Gamma(x)$ consists of a segment contained in $\mathcal{U}$ whose length tends to infinity as x tends to $\mathcal{L}^s$ and two bounded segments: one joining $x \in \partial^{in} V$ to $\mathcal{U}$ and one joining $\mathcal{U}$ to $\Gamma(x) \in \partial^{out} V$.

The first and the last segments have a bounded effect on J_t^s, as their lengths are uniformly bounded. One deduces that

$$J_{\tau(x)}^s(x) \to 0 \quad \text{as } x \to \Lambda^s,$$

where $\tau(x)$ is the crossing time of x (that is, $\Gamma(x) = X_{\tau(x)}(x)$).

Let u be a unit tangent vector to F_{in}^s at $x \in F_{in}^s$. The difficulty is that the vector $\Gamma_*(u)$ is not the image $(X_{\tau(x)})_*(u)$; the vector $\Gamma_*(u)$ is the projection along $X(y)$ on $T_y \partial^{out} V$ of $(X_{\tau(x)})_*(u)$, where $y = \Gamma(x) = X_{\tau(x)}(x)$. In order to simplify the calculation let us choose a metric on V such that X is orthogonal to ∂V and $\|X\| = 1$. With these notations, one gets

$$\|\Gamma_*(u)\| = J_{\tau(x)}^s(x).$$

Thus, $\|\Gamma_*(u)\| / \|u\|$ tends to 0 as $x \to \mathcal{L}^s$. This completes the proof. □

3.8 Strongly transverse lamination

Consider an Anosov flow X on a closed 3–manifold M, and assume that two plugs (V_1, X_1) and (V_2, X_2) are embedded in (M, X) such that a component S of $\partial^{out} V_1$ is also a component of $\partial^{in} V_2$. Then the laminations $\mathcal{L}_1^u$ and $\mathcal{L}_2^s$ are not only transverse, they extend on S as two transverse foliations (the unstable and stable foliation of X); not every two transverse filling MS laminations may extend as two transverse foliations. This difficulty leads to the following definition:

Definition 3.25 Let S be a compact surface and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two laminations on S. We say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *strongly transverse* if

- $\mathcal{L}_1$ and $\mathcal{L}_2$ are transverse at every point of $\mathcal{L}_1 \cap \mathcal{L}_2$;
- every connected component of $S \setminus \mathcal{L}_1 \cap \mathcal{L}_2$ is a disc whose closure is the image of a square $[0,1]^2$ by an immersion which is a diffeomorphism on $(0,1)^2$, and such that $[0,1] \times \{0,1\}$ is mapped in leaves of $\mathcal{L}_1$ and $\{0,1\} \times [0,1]$ is mapped in leaves of $\mathcal{L}_2$.

One can easily show the following lemma:

Lemma 3.26 *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are strongly transverse, they extend in transverse foliations. In particular, $\mathcal{L}_1$ and $\mathcal{L}_2$ are prefoliations.*

3.9 Gluing vector fields

Let V_1 and V_2 be manifolds with boundary, S_1 and S_2 be unions of boundary components of ∂V_1 and ∂V_2, and X_1 and X_2 be vector fields on V_1 and V_2, transverse to S_1 and S_2 such that X_1 goes out V_1 through S_1 and X_2 goes in V_2 through S_2. Let $\varphi\colon S_1 \to S_2$ be a diffeomorphism. Let V be the quotient $(V_1 \amalg V_2)/x \simeq \varphi(x)$.

Lemma 3.27 *With the notation above there is a differential structure on V such that V is a smooth manifold and there is a C^1 vector field on V such that the restriction of X to V_i is X_i for $i = 1, 2$.*

Proof Just notice that the flow box theorem implies that there is a tubular neighborhood U_i of S_i in V_i and some coordinates $(x,t)\colon U_1 \to S_i \times (-\varepsilon, 0]$ (resp. $(x,t)\colon U_2 \to S_2 \times [0,\varepsilon)$) such that, in these coordinates X_i is the trivial vector field $\frac{\partial}{\partial t}$. □

Definition 3.28 Let X_0 and X_1 be two vector fields on the same compact 3–manifold with boundary V such that (V, X_0) and (V, X_1) both are hyperbolic plugs with filling MS laminations. Let $\varphi_0\colon \partial^{\text{out}}_{X_0} V \to \partial^{\text{in}}_{X_0} V$ and $\varphi_1\colon \partial^{\text{out}}_{X_1} V \to \partial^{\text{in}}_{X_1} V$ be strongly transverse gluing maps for (V, X_0) and (V, X_1), respectively. We say that (V, X_0, φ_0) and (V, X_1, φ_1) are *strongly isotopic* if there exists a continuous path $(U, X_t, \varphi_t)_{t\in[0,1]}$ of hyperbolic plugs with filling MS laminations and strongly transverse gluing maps.

Remark 3.29 If (V, X_0, φ_0) and (V, X_1, φ_1) are strongly isotopic, the structural stability of hyperbolic plugs implies that (V, X_0) and (V, X_1) are topologically equivalent.

4 Gluing hyperbolic plugs without cycles

In this section, we consider two hyperbolic plugs (U, X) and (V, Y). We also consider a union T^{out} of connected components of $\partial^{\text{out}} U$, a union T^{in} of connected components of $\partial^{\text{in}} V$ and a gluing map $\varphi\colon T^{\text{out}} \to T^{\text{in}}$ such that the lamination $\varphi_*(\mathcal{L}^{\text{u}}_X)$ is transverse to the lamination $\mathcal{L}^{\mathrm{s}}_Y$. Then we consider the manifold with boundary $W := (U \sqcup V)/\varphi$ and the vector field Z induced by X and Y on W.

The aim of the section is to prove that (W, Z) is a hyperbolic plug (Proposition 1.1) and that (W, Z) has filling MS laminations provided that this is the case for (U, X) and (V, Y) and provided that φ is a strongly transverse gluing map (Proposition 1.3).

4.1 Hyperbolicity of the new plug: proof of Proposition 1.1

Proof of Proposition 1.1 The vector field Z is transverse to the boundary of W; hence (W, Z) is a plug. So we only need to check that the maximal invariant set of (W, Z) is a hyperbolic set. Let Λ_X, Λ_Y and Λ_Z be the maximal invariant sets of (U, X), (V, Y) and (W, Z), respectively. Then Λ_Z is the union of Λ_X, Λ_Y and the Z–orbit of the set $\varphi_*(\mathcal{L}^{\mathrm{u}}_X) \cap \mathcal{L}^{\mathrm{s}}_Y$. The orbit of $\varphi_*(\mathcal{L}^{\mathrm{u}}_X) \cap \mathcal{L}^{\mathrm{s}}_Y$ inherits a hyperbolic structure for Z: for $x \in \varphi_*(\mathcal{L}^{\mathrm{u}}_X) \cap \mathcal{L}^{\mathrm{s}}_Y$, the stable (resp. unstable) bundle at x is the direct sum of the line $\mathbb{R} \cdot Z(x)$ and the line tangent to $\mathcal{L}^{\mathrm{s}}_Y$ (resp. $\varphi_*(\mathcal{L}^{\mathrm{u}}_X)$) at x. Moreover, the so-called λ–lemma (see [26, page 155], for example) implies that this hyperbolic structure on the orbit of $\varphi_*(\mathcal{L}^{\mathrm{u}}_X) \cap \mathcal{L}^{\mathrm{s}}_Y$ extends continuously to the (previously existing) hyperbolic structures on Λ_X and Λ_Y. This provides the desired hyperbolic structure on the maximal invariant set Λ_Z. □

The following simple observation (whose proof is left to the reader) will be used many times in the remainder of the paper:

Proposition 4.1 *The exit boundary of the plug (W, Z) is*

$$\partial^{\mathrm{out}} W = (\partial^{\mathrm{out}} U \setminus T^{\mathrm{out}}) \cup \partial^{\mathrm{out}} V.$$

Furthermore, the lamination $\mathcal{L}^{\mathrm{u}}_Z$ coincides with

$$\begin{cases} \mathcal{L}^{\mathrm{u}}_X & \text{on } \partial^{\mathrm{out}} U \setminus T^{\mathrm{out}}, \\ \mathcal{L}^{\mathrm{u}}_Y \sqcup (\Gamma_Y)_*(\varphi_*(\mathcal{L}^{\mathrm{u}}_X) \setminus \mathcal{L}^{\mathrm{s}}_Y) & \text{on } \partial^{\mathrm{out}} V, \end{cases}$$

where Γ_Y is the crossing map of the plug (V, Y).

4.2 Filling MS laminations: proof of Proposition 1.3

In this subsection, we assume that the hyperbolic plugs (U, X) and (V, Y) have filling MS laminations and that φ is a strongly transverse gluing map.

Proof of Proposition 1.3 According to Lemma 3.21 it is enough to prove that $\mathcal{L}^{\mathrm{u}}_Z$ is a filling MS lamination. For that we consider a connected component C of $\partial^{\mathrm{out}} W \setminus \mathcal{L}^{\mathrm{u}}_Z$.

If $C \subset \partial^{\mathrm{out}} U \setminus T^{\mathrm{out}}$, then C is a connected component of $\partial^{\mathrm{out}} U \setminus \mathcal{L}^{\mathrm{u}}_X$ and therefore is a strip bounded by two asymptotic leaves, ending the proof in this case.

According to Proposition 4.1, we can now assume that $C \subset \partial^{\mathrm{out}} V$. Furthermore, according to Proposition 4.1, C is disjoint from $\mathcal{L}^{\mathrm{u}}_Y$, so one can consider $\Gamma_Y^{-1}(C)$:

If $\Gamma_Y^{-1}(C)$ is contained in $\partial^{\mathrm{in}} V \setminus T^{\mathrm{in}}$, then $\Gamma_Y^{-1}(C)$ is a connected component of $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}_Y$ such that C itself is a connected component of $\partial^{\mathrm{out}} V \setminus \mathcal{L}^{\mathrm{u}}_Y$. This is a strip bounded by two asymptotic leaves, ending the proof in this case.

So assume that $\Gamma_Y^{-1}(C)$ is contained in T^{in}.

Lemma 4.2 *Assume that $\Gamma_Y^{-1}(C)$ is contained in $T^{\mathrm{in}} \subset \partial^{\mathrm{in}} V$. Then $\Gamma_Y^{-1}(C)$ is a connected component of $T^{\mathrm{in}} \setminus (\mathcal{L}^{\mathrm{s}}_Y \cup \varphi_*(\mathcal{L}^{\mathrm{u}}_X))$.*

Proof The set $\Gamma_Y^{-1}(C)$ is disjoint from $\mathcal{L}^{\mathrm{s}}(Y)$ because the range of Γ_Y^{-1} is $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}_Y$. Proposition 4.1 implies that it is also disjoint from $\varphi_*(\mathcal{L}^{\mathrm{u}}_X)$, as C is disjoint from $\Gamma(\varphi_*(\mathcal{L}^{\mathrm{u}}_X)) \subset \mathcal{L}^{\mathrm{u}}_Z$. Therefore $\Gamma_Y^{-1}(C)$ is contained in a connected component C_1 of $T^{\mathrm{in}} \setminus (\mathcal{L}^{\mathrm{s}}_Y \cup \varphi_*(\mathcal{L}^{\mathrm{u}}_X))$. Now $\Gamma_Y(C_1)$ is disjoint from $\mathcal{L}^{\mathrm{u}}_Z$ (Proposition 4.1) so one deduces the other inclusion: $\Gamma_Y(C_1) \subset C$. □

Let $C_1 := \Gamma_Y^{-1}(C)$. By definition of strongly transverse lamination, C_1 is an immersed square $[0,1]^2$ bounded by two segments γ_1 and γ_2 (images of the horizontal segments $[0,1] \times \{0,1\}$) in $\mathcal{L}^{\mathrm{s}}_Y$ and two segments σ_1 and σ_2 in $\varphi_*(\mathcal{L}^{\mathrm{u}}_X)$ (images of the vertical segments $\{0,1\} \times [0,1]$). Notice that $L_i = \Gamma_Y(\sigma_i)$ for $i = 1, 2$ is a leaf of $\mathcal{L}^{\mathrm{u}}_Z$ contained in the accessible boundary of C. We will see that L_1 and L_2 are asymptotic on both sides. More precisely:

Lemma 4.3 *With the notation above, there is a foliation on $C \cup L_1 \cup L_2$ whose leaves are segments with one endpoint in L_1 and the other endpoint on L_2. Furthermore, the length of the leaves tends to 0 when one of their endpoints tends to one end of L_i.*

Proof According to Lemma 3.22 applied to (V, Y), there is a Y–invariant foliation $\mathcal{F}^{\mathrm{s}}$ on V containing $W^{\mathrm{s}}(\Lambda_Y)$ as a sublamination and such that the foliation $\mathcal{F}^s_{\mathrm{out}}$ induced by $\mathcal{F}^{\mathrm{s}}$ on $\partial^{\mathrm{out}} V$ is transverse to $\mathcal{L}^{\mathrm{u}}_Y$. We denote by $\mathcal{F}^s_{\mathrm{in}}$ the foliation induced by $\mathcal{F}^{\mathrm{s}}$ on $\partial^{\mathrm{in}} V$. The lamination $\mathcal{L}^{\mathrm{s}}_Y$ is a sublamination of $\mathcal{F}^s_{\mathrm{in}}$. In particular, the sides γ_1 and γ_2 are leaf segments of $\mathcal{F}^s_{\mathrm{in}}$. Unfortunately, the foliation $\mathcal{F}^s_{\mathrm{in}}$ may fail to be transverse to the segments σ_1 and σ_2. However, since the transversality is an open property, there is a neighborhood of $\gamma_1 \cup \gamma_2$ in C_1 which is foliated by segments of leaves of $\mathcal{F}^s_{\mathrm{in}}$ with one endpoint on σ_1 and the other endpoint on σ_2. As a consequence, there is a foliation $\mathcal{G}_1$ of C_1, which coincides with $\mathcal{F}^s_{\mathrm{in}} \cap C_1$ on a neighborhood of $\gamma_1 \cup \gamma_2$, such that the leaves of $\mathcal{G}_1$ are segments with one endpoint on σ_1 and the other endpoint on σ_2. The announced foliation is $\Gamma_Y(\mathcal{G}_1)$. Lemma 3.24 implies that the length of the leaves of this foliation tends to 0 when one of their endpoints tends to one of the ends of $\Gamma_Y(\sigma_i) = L_i$. □

Lemma 4.3 implies that C is a strip whose accessible boundary consists of two leaves of $\mathcal{L}^{\mathrm{u}}_Z$ which are asymptotic to each other at both ends. This concludes the proof of Proposition 1.3. □

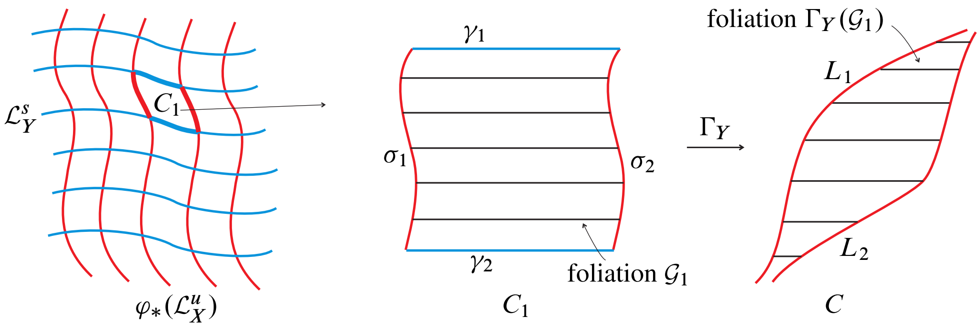

Figure 6: The crossing map $\Gamma_Y\colon (C_1, \mathcal{G}_1) \to (C, \Gamma_Y(\mathcal{G}_1))$

5 Normal form

Given a hyperbolic plug (V, X) with filling MS laminations, and a strongly transverse gluing map $\varphi_0\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$, the main purpose of this section is to perturb (V, X) (within its topological equivalence class) and the map φ_0 (with its isotopy class of strongly transverse gluing map) in order to get some foliations on $\partial^{\mathrm{in}} V$ and $\partial^{\mathrm{out}} V$ satisfying some nice properties. Our perturbation uses Markov partition by disjoint rectangles. Such a Markov partition exists if and only if the hyperbolic set does not contain any attractor or repeller. For this reason, we will need to assume that the maximal invariant set of (V, X) does not contain attractors or repellers.

Definition 5.1 A hyperbolic plug (V, X) is called a *saddle hyperbolic plug* if the maximal invariant set of (V, X) does not contain attractors or repellers.

More precisely, we will prove the following proposition:

Proposition 5.2 *Let (V, X) be a saddle hyperbolic plug with filling MS laminations, endowed with a strongly transverse gluing map $\varphi_0\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$. Then there exists a vector field Y on V arbitrarily C^1–close to X and a map $\varphi_1\colon \partial^{\mathrm{out}}_X V \to \partial^{\mathrm{in}}_X V$ with the following properties:*

- *(V,Y) is a hyperbolic plug, φ_1 is a strongly transverse gluing map for (V,Y), and (V,Y,φ_1) is strongly isotopic to (V,X,φ_0) (Definition 3.28). We denote by Λ_Y the maximal invariant set of (V,Y).*
- *There exist smooth Y–invariant foliations $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{u}}$ on V, such that $W^{\mathrm{s}}(\Lambda_Y)$ is a sublamination of $\mathcal{G}^{\mathrm{s}}$ and $W^{\mathrm{u}}(\Lambda_Y)$ is a sublamination of $\mathcal{G}^{\mathrm{u}}$. We denote by $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$, $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$, $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ the intersections of $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{u}}$ with $\partial^{\mathrm{in}}V$ and $\partial^{\mathrm{out}}V$.*
- *For each compact leaf of $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ and of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$, the holonomy is conjugated to a homothety.*
- *The image of $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ by φ_1 is transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$.*

Apart from proving this proposition, we will also establish some bounds on the rate of contraction/expansion of the crossing map (see Lemma 5.14) and build specific invariant neighborhoods of the maximal invariant set of (V,X), called adapted neighborhoods.

5.1 Linear model for the vector field X

If Λ is a locally maximal hyperbolic set without attractors and repellers for a vector field in dimension 3, then Λ admits a local transverse cross-section Σ (see [6]). Moreover, the first return map on Σ admits a Markov partition by disjoint rectangles contained in Σ such that $\Sigma\cap\Lambda$ is the maximal invariant set of the union of the rectangles (see [6], for instance). One can refine such a Markov partition by considering intersections of the rectangles with their (positive or negative) images under the first return map. Iterating the process, the diameters of the rectangles can be made arbitrarily small. When the diameters are small enough the restrictions of the first return map to the rectangles are almost affine. Hence, one can perform a C^1–small perturbation[9] of the vector field such that the first return map becomes affine on each rectangles, preserving the vertical and horizontal foliations. This proves the following lemma:

Lemma 5.3 *Let (V,X) be a saddle hyperbolic plug and Λ_X its maximal invariant set. There is an arbitrarily small C^1 perturbation Y of X, topologically equivalent to X, admitting an* affine Markov partition, *which is a Markov partition consisting of smooth disjoint rectangles such that*

[9] A complete argument is somewhat more delicate. Consider a Markov partition $\mathcal{R}$ in Σ and denote by f the return map on Σ. The diameter of the rectangles of the refined Markov partition $\mathcal{R}^{(n)} := f^{-n}(\mathcal{R})\wedge\cdots\wedge\mathcal{R}\wedge\cdots\wedge f^{n}(\mathcal{R})$ goes to 0 as n goes to infinity. Unfortunately, the distance between two rectangles could a priori become much smaller than the size of the rectangles: in this case, the linearization on the rectangles requires a perturbation whose C^1 norm is large outside the rectangles. However, by performing a first arbitrary C^1–small perturbation, we can always assume that the flow is C^2. And when the flow is C^2, the distance between two rectangles of the Markov partition $\mathcal{R}^{(n)}$ remains proportional to the size of the rectangles as n goes to infinity. Then the argument become rigorous: the linearization can be performed by a arbitrary C^1–small perturbation.

- *the boundary of each rectangle is disjoint from Λ_Y ;*
- *every orbit of Λ_Y meets the union of the interior of the rectangles;*
- *there are coordinates on the rectangles such that the first return map preserves the horizontal and vertical foliations defined by these coordinates and is affine on each rectangle.*

Lemma 5.3 motivates the following definition:

Definition 5.4 An *affine plug* is a saddle hyperbolic plug admitting an affine Markov partition by disjoint rectangles.

Remark 5.5 For an affine plug, the holonomy of each compact leaf of the entrance (resp. exit) lamination $\mathcal{L}^{\mathrm{s}}$ (resp. $\mathcal{L}^{\mathrm{u}}$) is an affine contraction, ie an homothety.

As Y can be chosen arbitrarily C^1–close to X, the laminations $(\varphi_0)_*(\mathcal{L}^{\mathrm{u}}_Y)$ and $\mathcal{L}^{\mathrm{s}}_Y$ are still strongly transverse. Thus (V, Y, φ_0) is still a saddle hyperbolic plug endowed with a strongly transverse gluing diffeomorphism.

By pushing along the flow the vertical and horizontal foliations of the rectangle of an affine Markov partition, one gets a pair of invariant foliations which extend the stable and unstable laminations of the maximal invariant set. More precisely:

Lemma 5.6 *Let (V, Y) be an affine plug with maximal invariant set Λ. There is an invariant neighborhood $\mathcal{U}_0$ of Λ endowed with two smooth invariant 2–dimensional foliations $\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ such that*

- *$\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ are transverse to each other;*
- *$\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ are both transverse to $\partial V = \partial^{\mathrm{in}} V \cup \partial^{\mathrm{out}} V$;*
- *the leaves of the laminations $W^{\mathrm{s}}(\Lambda)$ and $W^{\mathrm{u}}(\Lambda)$ are leaves of $\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$, respectively.*

Let $\mathcal{U}_0^{\mathrm{in}}$ and $\mathcal{U}_0^{\mathrm{out}}$ denote the intersections of $\mathcal{U}$ with $\partial^{\mathrm{in}} V$ and $\partial^{\mathrm{out}} V$, respectively. By transversality, $\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ induce

- *two smooth transverse 1–dimensional foliations $\mathcal{W}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{W}^{\mathrm{u}}_{\mathrm{in}}$ on $\mathcal{U}_0^{\mathrm{in}}$ such that $\mathcal{L}^{\mathrm{s}}$ is a sublamination of $\mathcal{W}^{\mathrm{s}}_{\mathrm{in}}$;*
- *two smooth transverse 1–dimensional foliations $\mathcal{W}^{\mathrm{s}}_{\mathrm{out}}$ and $\mathcal{W}^{\mathrm{u}}_{\mathrm{out}}$ on $\mathcal{U}_0^{\mathrm{out}}$ such that $\mathcal{L}^{\mathrm{u}}$ is a sublamination of $\mathcal{W}^{\mathrm{u}}_{\mathrm{out}}$.*

Furthermore, one can choose the foliations $\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ so that the only compact leaves of $\mathcal{W}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{W}^{\mathrm{u}}_{\mathrm{out}}$ are those of the laminations $\mathcal{L}^{\mathrm{s}}$ and $\mathcal{L}^{\mathrm{u}}$, and their holonomies are homotheties.

Proof The neighborhood $\mathcal{U}_0$ is just the union of the orbits of Y intersecting the rectangles of an affine Markov partition of Λ. The foliation $\mathcal{F}^s$ and $\mathcal{F}^u$ are obtained by saturating by the flow of Y the vertical and horizontal foliations of the rectangles. All the desired properties are immediate, except maybe for the fact that the only compact leaves of $\mathcal{W}^s_{in}$ and $\mathcal{W}^u_{out}$ are those of the laminations $\mathcal{L}^s$ and $\mathcal{L}^u$. Let us prove this. On the one hand, by the Poincaré–Bendixson theorem, a compact leaf in $\mathcal{W}^s_{in}$ (resp. $\mathcal{W}^u_{out}$) cannot be induced by a planar leaf in $\mathcal{F}^s$ (resp. $\mathcal{F}^u$). On the other hand, since the return map on the rectangles of the Markov partition is affine, every leaf of $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) which is not planar contains a periodic orbit of Y, hence belongs to the lamination $W^s(\Lambda)$ (resp. $W^u(\Lambda)$). It follows that the only compact leaves of $\mathcal{W}^s_{in}$ and $\mathcal{W}^u_{out}$ are those of the laminations $\mathcal{L}^s$ and $\mathcal{L}^u$. □

5.2 Local linearization of the gluing map

The aim of this section is to prove the following result, which provides a kind of "normal form" for the gluing map of an affine plug in a neighborhood of the intersection of the laminations of the boundary.

Proposition 5.7 *Let (V, X) be an affine plug, with maximal invariant set Λ, such that the entrance lamination $\mathcal{L}^s$ and the exit lamination $\mathcal{L}^u$ of (V, X) are prefoliations. Let $\varphi_0\colon \partial^{out}V \to \partial^{in}V$ be a diffeomorphism such that $\varphi_{0,*}(\mathcal{L}^u)$ and $\mathcal{L}^s$ are strongly transverse. Let $\mathcal{U}_0$ be an invariant neighborhood of Λ endowed with two smooth foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ as given by Lemma 5.6. Let $\mathcal{U}_0^{in}$ and $\mathcal{U}_0^{out}$ be the intersections of $\mathcal{U}_0$ with $\partial^{in}V$ and $\partial^{out}V$. Observe that these are neighborhoods of the laminations $\mathcal{L}^s$ and $\mathcal{L}^u$ in $\partial^{in}V$ and $\partial^{out}V$, respectively. Let $\mathcal{W}^s_{in}$, $\mathcal{W}^u_{in}$, $\mathcal{W}^s_{out}$ and $\mathcal{W}^u_{out}$ be the foliations induced by $\mathcal{F}^s$ and $\mathcal{F}^u$ on $\mathcal{U}_0^{in}$ and $\mathcal{U}_0^{out}$.*

Then there exists a diffeomorphism $\varphi\colon \partial^{out}V \to \partial^{in}V$ and an invariant neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of Λ such that

- *$\varphi_*(\mathcal{L}^u)$ and $\mathcal{L}^s$ are strongly transverse;*
- *the foliations $\varphi_*(\mathcal{W}^s_{out})$ and $\varphi_*(\mathcal{W}^u_{out})$ coincide with $\mathcal{W}^s_{in}$ and $\mathcal{W}^u_{in}$ on the intersection $\varphi(\mathcal{U}^{out}) \cap \mathcal{U}^{in}$;*
- *φ is isotopic to φ_0 among strongly transverse gluing maps.*

The proof of the proposition uses the following technical lemma:

Lemma 5.8 *Let D be a compact disc of dimension 2 endowed with three smooth foliations $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. Assume that $\mathcal{F}$ is transverse to both $\mathcal{G}$ and $\mathcal{H}$. Let $\mathcal{K}$ and $\mathcal{L}$*

be sublaminations of $\mathcal{F}$ and $\mathcal{H}$, respectively. We assume that $\mathcal{K}$ and $\mathcal{L}$ have empty interior and that $\mathcal{K} \cap \mathcal{L}$ is disjoint from the boundary ∂D.

Then there is a smooth isotopy $(\psi_t)_{t \in [0,1]}$ of diffeomorphisms of D with the following properties:

- $\psi_0 = \mathrm{id}$.
- *ψ_t coincides with the identity map in a neighborhood of ∂D for every t.*
- *ψ_t preserves each leaf of $\mathcal{F}$ for every t.*
- *$\psi_1(\mathcal{H})$ coincides with $\mathcal{G}$ in a neighborhood of $\mathcal{K} \cap \psi_1(\mathcal{L})$.*

Proof First observe that any diffeomorphism coinciding with the identity close to ∂D and preserving every leaf of $\mathcal{F}$ is isotopic to the identity inside the set of diffeomorphisms preserving each leaf of $\mathcal{F}$: to prove this fact, one just need consider a barycentric isotopy along the leaves. Therefore, we only need to build the diffeomorphism ψ_1.

Using the fact that $\mathcal{K}$ and $\mathcal{L}$ have empty interior, the intersection of every leaf of $\mathcal{K}$ with $\mathcal{L}$ is totally discontinuous. As $\mathcal{K}$ has empty interior, one deduces that one can cover $\mathcal{K} \cap \mathcal{L}$ by finitely many pairwise disjoint rectangles such that the vertical segments of these rectangles are segments of leaves of $\mathcal{F}$, the horizontal segments of these rectangles are segments of leaves of $\mathcal{G}$ and the boundaries of these rectangles are disjoint from $\mathcal{K} \cap \mathcal{L}$. Fix such a rectangle R.

Let σ be a connected component of $\mathcal{K} \cap R$ (note that σ is a vertical segment of R). One can find an arbitrarily thin vertical subrectangle R_σ of R containing σ such that the vertical sides of R_σ are disjoint from $\mathcal{K}$. Any connected component of the intersection of a leaf of $\mathcal{L}$ with R_σ is disjoint from the horizontal boundary of R_σ, and "crosses R_σ horizontally" intersecting every vertical segment of R_σ once. If R_σ is thin enough, then the same is true for a connected component of the intersection of a leaf of $\mathcal{H}$ with R_σ in a neighborhood of $\mathcal{L} \cap R$ (because $\mathcal{H}$ is transverse to σ and the horizontal boundary of R_σ is disjoint from $\mathcal{L}$). Therefore one can find a diffeomorphism ψ_σ supported in the interior of R_σ, preserving the vertical segments of R (ie the leaves of $\mathcal{F}$) and such that any connected component of R_σ with a leaf $\mathcal{H}$ is mapped on an horizontal segment of R_σ (ie in a leaf of $\mathcal{G}$) in a neighborhood of $L \cap R_\sigma$. Now one can cover $\mathcal{L} \cap R$ by finitely many disjoint such rectangles R_{σ_i} and the announced diffeomorphism ψ_1 is the product of the ψ_{σ_i}. □

We are now ready to prove the proposition.

Proof of Proposition 5.7 We consider the foliations $\mathcal{W}^{\mathrm{s}}_{\mathrm{in}}$, $\mathcal{W}^{\mathrm{u}}_{\mathrm{in}}$, $\varphi_0(\mathcal{W}^{\mathrm{s}}_{\mathrm{out}})$ and $\varphi_0(\mathcal{W}^{\mathrm{u}}_{\mathrm{out}})$ defined by Lemma 5.6 on some neighborhoods $\mathcal{U}^{\mathrm{in}}_0 \subset \partial^{\mathrm{in}} V$ and $\mathcal{U}^{\mathrm{out}}_0 \subset \partial^{\mathrm{out}} V$ of $\mathcal{L}^{\mathrm{s}}$ and $\mathcal{L}^{\mathrm{u}}$, respectively.

Since (V, X) is a saddle hyperbolic plug, the laminations $\mathcal{L}^s$ and $\mathcal{L}^u$ have empty interior. By assumption, the laminations $\varphi_0(\mathcal{L}^u)$ and $\mathcal{L}^s$ are (strongly) transverse to each other. One deduces that the intersection $\varphi_0(\mathcal{L}^u) \cap \mathcal{L}^s$ is a totally discontinuous compact subset of the surface $\partial^{in} V$. Therefore $\varphi_0(\mathcal{L}^u) \cap \mathcal{L}^s$ can be covered by the interior of a finite union of disjoint arbitrarily small compact discs D_i.

As the laminations $\varphi_0(\mathcal{L}^u)$ and $\mathcal{L}^s$ are (strongly) transverse, there is a neighborhood $\mathcal{O}$ of $\varphi_0(\mathcal{L}^u) \cap \mathcal{L}^s$ on which the foliations $\varphi_0(\mathcal{W}^u_{out})$ and $\mathcal{W}^s_{in}$ are transverse. By shrinking $\mathcal{O}$ if necessary, one may assume that $\mathcal{O} \subset \varphi_0(\mathcal{U}^{out}_0) \cap \mathcal{U}^{in}_0$. We choose the discs D_i small enough so that they are contained in $\mathcal{O}$.

According to Lemma 5.8 for each of the disc D_i there is a diffeomorphism φ_i supported in the interior of D_i, preserving each leaf of $\mathcal{W}^s_{in}$ (and isotopic to the identity through diffeomorphisms preserving the leaves of $\mathcal{W}^s_{in}$) and such that $\psi_i(\varphi_0(\mathcal{W}^u_{out}))$ coincides with $\mathcal{W}^u_{in}$ in the neighborhood of $\psi_i(\varphi_0(\mathcal{L}^u) \cap D_i) \cap \mathcal{L}^s$. Let ψ^{in} be the diffeomorphism of $\partial^{in} V$ coinciding with ψ_i on each D_i and with the identity out of the D_i. Let $\varphi_1 := \psi^{in} \circ \varphi_0$. Note that $\varphi_1(\mathcal{W}^u_{out})$ coincides with $\mathcal{W}^u_{in}$ in the neighborhood of $\varphi_1(\mathcal{L}^u) \cap \mathcal{L}^s$. So we got half of the conclusion. Now, what is left is to push $\varphi_0(\mathcal{W}^s_{out})$ on $\mathcal{W}^s_{in}$ without destroying what has been done.

For that purpose, we consider the foliations $\mathcal{W}^s_{out}$, $\mathcal{W}^u_{out}$, $\varphi_1^{-1}(\mathcal{W}^s_{in})$ and $\varphi_1^{-1}(\mathcal{W}^u_{in})$. Notice that $\mathcal{W}^u_{out}$ and $\varphi_1^{-1}(\mathcal{W}^u_{in})$ coincide in a neighborhood $\mathcal{O}_1$ of $\mathcal{L}^u \cap \varphi_1^{-1}(\mathcal{L}^s)$, where the foliations $\mathcal{W}^s_{out}$ and $\varphi_1^{-1}(\mathcal{W}^s_{in})$ are transverse. We cover $\mathcal{L}^u \cap \varphi_1^{-1}(\mathcal{L}^s)$ by a family of disjoint discs Δ_j contained in $\mathcal{O}_1$ and with boundary disjoint from $\mathcal{L}^u \cap \varphi_1^{-1}(\mathcal{L}^s)$. We apply Lemma 5.8 with $\mathcal{F} = \mathcal{W}^u_{out} = \varphi_1^{-1}(\mathcal{W}^u_{in})$, $\mathcal{G} = \mathcal{W}^s_{out}$ and $\mathcal{H} = \varphi_1^{-1}(\mathcal{W}^s_{in})$. This provides a diffeomorphism, denoted by $(\psi^{out})^{-1}$, supported in the union of the interiors of the discs Δ_j keeping invariant each leaf of $\varphi_1^{-1}(\mathcal{W}^u_{in})$ and sending $\varphi_1^{-1}(\mathcal{W}^s_{in})$ on $\mathcal{W}^s_{out}$, in a small neighborhood of $(\psi^{out})^{-1}(\varphi_1^{-1})(\mathcal{L}^s) \cap \mathcal{L}^u$. Notice that $(\psi^{out})^{-1}(\varphi_1^{-1}(\mathcal{W}^s_{in})) = \varphi_1^{-1}(\mathcal{W}^s_{in}) = \mathcal{W}^u_{out}$. The desired diffeomorphism is $\varphi := \varphi_1 \circ \psi^{out} = \psi^{in} \circ \varphi_0 \circ \psi^{out}$. □

5.3 Adapted neighborhoods

In the remainder of the section, we consider an affine saddle plug (V, X), a strongly transverse gluing map $\varphi\colon \partial^{out} V \to \partial^{in} V$, an invariant neighborhood $\mathcal{U}$ of the maximal invariant set Λ of (V, X) and some transverse foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ on $\mathcal{U}$ containing respectively $W^s(\Lambda)$ and $W^u(\Lambda)$ as sublaminations. We set $\mathcal{U}^{in} := \mathcal{U} \cap \partial^{in} V$ and $\mathcal{U}^{out} := \mathcal{U} \cap \partial^{out} V$. We denote by $\mathcal{W}^s_{in}$, $\mathcal{W}^u_{in}$, $\mathcal{W}^s_{out}$ and $\mathcal{W}^u_{out}$ the foliations induced by $\mathcal{F}^s$ and $\mathcal{F}^u$ on $\mathcal{U}^{in}$ and $\mathcal{U}^{out}$. We assume that these objects satisfy the conclusion of Proposition 5.7, ie we assume that the foliations $\varphi_*(\mathcal{W}^s_{out})$ and $\varphi_*(\mathcal{W}^u_{out})$ coincide with $\mathcal{W}^s_{in}$ and $\mathcal{W}^u_{in}$ on $\varphi(\mathcal{U}^{out}) \cap \mathcal{U}^{in}$. Recall that $\mathcal{U}^{in} := \mathcal{U} \cap \partial^{in} V$ is a neighborhood of $\mathcal{L}^s$. Hence $\mathcal{U}^{in}$ contains all but finitely many elements of the connected components of $\partial^{in} V \setminus \mathcal{L}^s$.

Definitions 5.9 An *in-square* is a disc $C \subset \partial^{in} V$ with the following properties:

- The boundary ∂C is contained in $\mathcal{U}^{in}$.
- ∂C consists of exactly four segments: two segments of leaves of $\mathcal{W}^s_{in}$ and two segments of leaves of $\mathcal{W}^u_{in}$.
- There is a diffeomorphism from C to $[0,1]^2$ such that, on a neighborhood of the boundary of C, this diffeomorphism maps $\mathcal{W}^s_{in}$ and $\mathcal{W}^u_{in}$ on the horizontal and vertical foliations of $[0,1]^2$.

A compact neighborhood $\mathcal{U}^{in}_1 \subset \mathcal{U}^{in}$ of $\mathcal{L}^s$ will be called an *adapted neighborhood* of $\mathcal{L}^s$ if, for any connected component C of the complement of $\mathcal{L}^s$, the complement $C \setminus \mathcal{U}^{in}_1$ is either empty or is the interior of an in-square.

As $\mathcal{L}^s$ is a filling MS lamination one easily proves:

Lemma 5.10 *The lamination $\mathcal{L}^s$ admits a basis of adapted neighborhoods: every neighborhood of $\mathcal{L}^s$ in $\partial^{in} V$ contains an adapted neighborhood.*

One defines analogously adapted neighborhoods of $\mathcal{L}^u$, and proves that $\mathcal{L}^u$ admits a basis of adapted neighborhoods.

5.4 The crossing map

Recall that, for every point x in $\partial^{in} V \setminus \mathcal{L}^s$, the positive orbit of x exits V at a point $\Gamma(x)$ of $\partial^{out} V \setminus \mathcal{L}^u$. The map $\Gamma\colon \partial^{in} V \setminus \mathcal{L}^s \to \partial^{out} V \setminus \mathcal{L}^u$ is a diffeomorphism called the *crossing map* of (V, X) (see Section 3.5). The invariance (under the flow of X) of the neighborhood $\mathcal{U}$ and the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ imply that

$$\Gamma(\mathcal{U}^{in} \setminus \mathcal{L}^s) = \mathcal{U}^{out} \setminus \mathcal{L}^u, \quad \Gamma_*(\mathcal{W}^s_{in}) = \mathcal{W}^s_{out}, \quad \Gamma_*(\mathcal{W}^u_{in}) = \mathcal{W}^u_{out}.$$

One easily deduces the next two lemmas:

Lemma 5.11 *If $\mathcal{V}^{in} \subset \mathcal{U}^{in}$ is an adapted neighborhood of $\mathcal{L}^s$, then $\Gamma(\mathcal{V}^{in}) \cup \mathcal{L}^u$ is an adapted neighborhood of $\mathcal{L}^u$.*

Definition 5.12 An invariant neighborhood $\mathcal{V}$ of Λ will be called *an adapted neighborhood of* Λ if both $\mathcal{V}^{in} = \mathcal{V} \cap \partial^{in} V$ and $\mathcal{V}^{out} = \mathcal{V} \cap \partial^{out} V$ are adapted neighborhoods of $\mathcal{L}^s$ and $\mathcal{L}^u$, respectively.

Lemma 5.13 *Let $\mathcal{G}^{s}_{in}$ and $\mathcal{G}^{u}_{in}$ be smooth transverse foliations on $\partial^{in}V$, which coincide respectively with $\mathcal{W}^{s}_{in}$ and $\mathcal{W}^{u}_{in}$ on an adapted neighborhood of $\mathcal{L}^{s}$. Then $\mathcal{G}^{s}_{in}$ and $\mathcal{G}^{u}_{in}$ extend to smooth transverse X–invariant foliations $\mathcal{G}^{s}$ and $\mathcal{G}^{u}$ in V, which coincide with $\mathcal{F}^{s}$ and $\mathcal{F}^{u}$, respectively, on a neighborhood of Λ. As a consequence, $\Gamma_*(\mathcal{G}^{s}_{in})$ and $\Gamma_*(\mathcal{G}^{u}_{in})$ extend on $\partial^{out}V$ to smooth transverse foliations $\mathcal{G}^{s}_{out}$ and $\mathcal{G}^{u}_{out}$ which coincide with $\mathcal{W}^{s}_{out}$ and $\mathcal{W}^{u}_{out}$, respectively, on an adapted neighborhood of $\mathcal{L}^{u}$.*

The following lemma shows that the crossing map is "as strongly hyperbolic as we want" in small neighborhoods of $\mathcal{L}^{s}$:

Lemma 5.14 *Given any $\lambda > 1$, there is an adapted neighborhood $\mathcal{U}^{in}_{\lambda}$ of $\mathcal{L}^{s}$ such that, for $x \in \mathcal{U}^{in}_{\lambda} \setminus \mathcal{L}^{s}$, the crossing map Γ expands vectors tangent to the leaves of $\mathcal{W}^{u}_{in}$ by more than λ, and contracts vectors tangent to the leaves of $\mathcal{W}^{s}_{in}$ by a factor smaller than λ^{-1}.*

Proof It is a direct consequence of Lemma 3.24. Let us just recall the idea. For every point $x \in \partial^{in}U \setminus \mathcal{L}^{s}$ close enough to $\mathcal{L}^{s}$, the positive orbit goes in finite time in a small neighborhood of the hyperbolic set Λ_X, then spends an arbitrarily large interval of time close to Λ_X, and then reaches $\partial^{out}U$ in a finite time. Therefore it is enough to choose the adapted neighborhood $\mathcal{U}^{in}_{\lambda}$ small enough for getting the desired strength of hyperbolicity for the crossing map. □

5.5 Modifying the gluing map to get some transversality

Let $\mathcal{G}^{s}$ and $\mathcal{G}^{u}$ be a choice of smooth transverse X–invariant foliations given by Lemma 5.13. We denote by $\mathcal{G}^{s}_{in}$, $\mathcal{G}^{u}_{in}$, $\mathcal{G}^{s}_{out}$ and $\mathcal{G}^{u}_{out}$ the one-dimensional foliations induced by $\mathcal{G}^{s}$ and $\mathcal{G}^{u}$ on $\partial^{in}V$ and $\partial^{out}V$, respectively. According to Proposition 5.7, we can (and we do) assume that there exists an invariant neighborhood $\mathcal{U}$ of Λ such that $\varphi_*(\mathcal{G}^{s}_{out})$ and $\varphi_*(\mathcal{G}^{u}_{out})$ coincide with $\mathcal{G}^{s}_{in}$ and $\mathcal{G}^{u}_{in}$ on $\varphi(\mathcal{U}^{out}) \cap \mathcal{U}^{in}$. Up to shrinking $\mathcal{U}$, we may (and we do) assume that $\mathcal{U}$ is an adapted neighborhood.

Lemma 5.15 *There is a map $\varphi_1\colon \partial^{out}V \to \partial^{in}V$, isotopic to φ and coinciding with φ on an adapted neighborhood of the exit lamination $\mathcal{L}^{u}$, such that $(\varphi_1)_*(\mathcal{G}^{u}_{out})$ is transverse to $\mathcal{G}^{s}_{in}$.*

Remark 5.16 The map φ_1 is a strongly transverse gluing map since it coincides with φ on an adapted neighborhood of $\mathcal{L}^{u}$. Moreover, φ_1 is isotopic to φ inside the set of strongly transverse gluing maps.

We start the proof of Lemma 5.15 with a very general lemma:

Lemma 5.17 *Let $\mathcal{F}$ and $\mathcal{G}$ be smooth foliations of the square $C=[0,1]^2$ such that $[0,1]\times\{0,1\}$ consists of leaves of $\mathcal{F}$ and $\{0,1\}\times[0,1]$ consists of leaves of $\mathcal{G}$, and $\mathcal{F}$ and $\mathcal{G}$ are transverse in a neighborhood of the boundary ∂C. Then there is a smooth diffeomorphisms ψ of C equal to the identity map in a neighborhood of ∂C such that $\psi(\mathcal{G})$ is transverse to $\mathcal{F}$.*

Proof The hypothesis imply that each of the foliations $\mathcal{F}$ and $\mathcal{G}$ are smoothly conjugated to trivial foliations, so we may assume that $\mathcal{F}$ is the horizontal foliation $\{[0,1]\times\{t\}\}_{t\in[0,1]}$. Now there is a foliation $\mathcal{H}$ on C, transverse to the leaves of $\mathcal{F}$, coinciding with $\mathcal{G}$ in a neighborhood of ∂C, and having the same holonomy from $\{0\}\times[0,1]\to\{1\}\times[0,1]$ as $\mathcal{G}$.

The foliations $\mathcal{G}$ and $\mathcal{H}$ are smoothly conjugated by a diffeomorphism ψ which coincides with the identity close to ∂C, completing the proof. □

Proof of Lemma 5.15 The proof consists of three steps:

Claim 5.18 *There is a diffeomorphism $\psi^{\mathrm{in}}\colon \partial^{\mathrm{in}}V\to\partial^{\mathrm{in}}V$ which coincides with the identity map on a neighborhood of $\mathcal{L}^{\mathrm{s}}$ and such that $(\psi^{\mathrm{in}}\circ\varphi)_*(\mathcal{L}^{\mathrm{u}})$ is transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$.*

Proof We have assumed that there exists an adapted neighborhood $\mathcal{U}$ of Λ such that $\varphi_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})$ and $\varphi_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ coincide with $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ on $\varphi(\mathcal{U}^{\mathrm{out}})\cap\mathcal{U}^{\mathrm{in}}$. Hence, $\varphi_*(\mathcal{L}^{\mathrm{u}})$ is already transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ on $\mathcal{U}^{\mathrm{in}}$. So it is enough to consider a connected component R of $\partial^{\mathrm{in}}V\setminus\mathcal{U}^{\mathrm{in}}$. Since $\mathcal{U}^{\mathrm{in}}$ is an adapted neighborhood of $\mathcal{L}^{\mathrm{s}}$, R is an in-rectangle: the restrictions to R of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ are the trivial horizontal and vertical foliations of R. Moreover, $\varphi_*(\mathcal{L}^{\mathrm{u}})\cap R$ is a lamination coinciding with $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ in a neighborhood of ∂R, and each leaf of $\varphi_*(\mathcal{L}^{\mathrm{u}})\cap R$ is a segment joining the bottom horizontal segment to the top horizontal segment of ∂R. A similar proof to the one of Lemma 5.17 proves the existence of a diffeomorphism ψ^{in}_R equal to the identity in a neighborhood of ∂R and such that $(\psi^{\mathrm{in}}_R\circ\varphi)_*(\mathcal{L}^{\mathrm{u}})\cap R$ is transverse in R to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$. The announced diffeomorphism ψ^{in} is the product of the diffeomorphisms $\psi^{\mathrm{in}}_{R_1},\dots,\psi^{\mathrm{in}}_{R_n}$ associated with the connected components $R_1,\dots,R_n$ of $\partial^{\mathrm{in}}V\setminus\mathcal{U}^{\mathrm{in}}$. □

Let $\varphi_1:=\psi^{\mathrm{in}}\circ\varphi$. Notice that φ_1 is isotopic to φ through strongly transverse gluing diffeomorphisms. Moreover, there is an adapted neighborhood $\mathcal{V}\subset\mathcal{U}$ of Λ such that $(\varphi_1)_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})$ and $(\varphi_1)_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ coincide with $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ on $\varphi_1(\mathcal{V}^{\mathrm{out}})\cap\mathcal{V}^{\mathrm{in}}$.

Claim 5.19 *There is $\psi^{\mathrm{out}}\colon\partial^{\mathrm{out}}V\to\partial^{\mathrm{out}}V$ which is the identity map in a neighborhood of $\mathcal{L}^{\mathrm{u}}$ and such that $(\varphi_1\circ\psi^{\mathrm{out}})^{-1}_*(\mathcal{L}^{\mathrm{s}})$ is transverse to $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$.*

Proof The proof is identical to the one of the first claim, reversing the flow of X. □

Now we set $\varphi_2 := \varphi_1 \circ \psi^{\mathrm{out}}$. Then φ_2 is isotopic to φ through strongly transverse diffeomorphisms, $(\varphi_2)_*(\mathcal{L}^{\mathrm{u}}) = (\varphi_1)_*(\mathcal{L}^{\mathrm{u}})$ is transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$, and $(\varphi_2)_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ is transverse to $\mathcal{L}^{\mathrm{s}}_{\mathrm{in}}$. So $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $\varphi_2(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ may fail to be transverse only in the interior of a connected component of $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}} \cup (\varphi_2)_*(\mathcal{L}^{\mathrm{u}})$. As $\mathcal{L}^{\mathrm{s}}$ and $(\psi_2)_*(\mathcal{L}^{\mathrm{u}})$ are strongly transverse, the closure of each component of $\partial^{\mathrm{in}} V \setminus (\mathcal{L}^{\mathrm{s}} \cup (\psi_2)_*(\mathcal{L}^{\mathrm{u}}))$ is a square having two sides on leaves of $\mathcal{L}^{\mathrm{s}}$ and two sides on leaves of $(\psi_2)_*(\mathcal{L}^{\mathrm{u}})$. One concludes by applying Lemma 5.17 in each of these squares with $\mathcal{F} = \mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{G} = (\varphi_2)_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$, which completes the proof. □

Proof of Proposition 5.2 It suffices to put together Lemmas 5.3 and 5.6, Proposition 5.7 and Lemma 5.15. Lemma 5.3 explains how to obtain the plug (V, Y). Lemma 5.6 implies the existence of the desired foliations $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{u}}$. Proposition 5.7 and Lemma 5.15 ensure the existence of a map $\varphi_1\colon \partial^{\mathrm{out}}_X V \to \partial^{\mathrm{in}}_X V$ satisfying the required properties. □

Definition 5.20 The *return map* $\Theta\colon \partial^{\mathrm{in}} V \to \partial^{\mathrm{out}} V$ associated with (V, X, φ_1) is obtained by composing the crossing map Γ and the gluing map φ_1:

$$\Theta := \varphi_1 \circ \Gamma.$$

Note that if φ_1 satisfies the conclusion of Lemma 5.15 then the foliation $\Theta_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{in}})$ and $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ are transverse:

$$\Theta_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}) \pitchfork \mathcal{G}^{\mathrm{s}}_{\mathrm{in}}.$$

6 Perturbation of the return map and proof of Theorem 1.5

The aim of this section is to prove Theorem 1.5 and Proposition 1.6.

In all of this section, we consider a saddle hyperbolic plug (V, X) with filling MS laminations, and a strongly transverse gluing map $\varphi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$. We denote by Λ the maximal invariant set of (V, X). Since (V, X) is a saddle hyperbolic plug, Λ does not contain attractors nor repellers. We denote by $\Gamma\colon \partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}} \to \partial^{\mathrm{out}} V \setminus \mathcal{L}^{\mathrm{u}}$ the crossing map of (V, X), and by $\Theta := \varphi \circ \Gamma$ the return map of X on $\partial^{\mathrm{in}} V$.

According to Proposition 5.2, we may (and we do) assume that V is endowed with a pair of two-dimensional smooth X–invariant foliations $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{u}}$, transverse to ∂V and transverse to each other, containing respectively $W^{\mathrm{s}}(\Lambda)$ and $W^{\mathrm{u}}(\Lambda)$ as sublaminations. We denote by $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$, $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$, $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ the one-dimensional foliations induced by $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{u}}$ on $\partial^{\mathrm{in}} V$ and $\partial^{\mathrm{out}} V$, respectively. Recall that the entrance lamination $\mathcal{L}^{\mathrm{s}}$ and the exit lamination $\mathcal{L}^{\mathrm{u}}$ are sublaminations of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$, respectively. Again by Proposition 5.2, we may (and we do) assume that the holonomy of each compact leaf of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ is conjugated to a homothety, and that $\varphi_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ is transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$.

6.1 Reduction of Theorem 1.5 to a perturbation of the return map Θ

Theorem 1.5 states that there exists a strongly transverse gluing diffeomorphism $\psi\colon \partial^{\mathrm{out}}V \to \partial^{\mathrm{in}}V$, which is isotopic to φ through strongly transverse gluing diffeomorphisms and such that the vector field induced by X on the closed manifold V/ψ is Anosov. As stated by the following lemma, proving that X_ψ is Anosov amounts to proving that its first return map $\Theta_\psi := \psi \circ \Gamma$ is hyperbolic:

Lemma 6.1 *Consider a strongly transverse gluing map $\psi\colon \partial^{\mathrm{out}}V \to \partial^{\mathrm{in}}V$. Denote by Z_ψ the vector field induced by X on the closed manifold V/ψ, and by $\Theta_\psi := \psi \circ \Gamma$ the return map of Z_ψ on $\partial^{\mathrm{in}}V$. Assume that there exist two continuous cone fields $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ on $\partial^{\mathrm{in}}V$ such that:*

- *The cone fields $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ are invariant under $d\Theta_\psi$ and $d\Theta_\psi^{-1}$, respectively, and the vectors in $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ are uniformly expanded by $d\Theta_\psi$ and $d\Theta_\psi^{-1}$, respectively (for some Riemannian metric).*
- *$\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ contains the direction tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ and the direction tangent to $\psi_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$, but it contains neither the direction tangent to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ nor the direction tangent to $\psi_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})$.*
- *$\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ contains the direction tangent to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and the direction tangent to $\psi_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})$, but it contains neither the direction tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ nor the direction tangent to $\psi_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$.*

Then the induced vector field induced Z_ψ is Anosov.

Proof By assumption, the maximal invariant set Λ of (V, X) does not contain attractors or repellers. Therefore, Λ is transversally totally discontinuous, and we may consider a local section Σ of Λ. By this, we mean that Σ is a collection of closed topological discs, Σ is transverse to X (or, equivalently, Z_ψ), the boundary of Σ is disjoint from Λ, and the interior of Σ intersects every orbit of X in Λ. We denote by f the first return map of the orbit of X on Σ. We denote by $\mathcal{G}^{\mathrm{s}}_\Sigma$ and $\mathcal{G}^{\mathrm{u}}_\Sigma$ the 1–dimensional foliations induced by $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{u}}$ on Σ. Note that Σ can be chosen so that it is contained in an arbitrarily small neighborhood of Λ.

Every orbit of Z_ψ either is contained in Λ or intersects $\partial^{\mathrm{in}}V$. Therefore, the interior of $\Sigma \cup \partial^{\mathrm{in}}V$ intersects every orbit of Z_ψ. We denote by f_ψ the first return map of the vector field Z_ψ on $\Sigma \cup \partial^{\mathrm{in}}V$. By classical elementary arguments, proving that the vector field Z_ψ is hyperbolic (ie Anosov) amounts to proving that f_ψ is hyperbolic. In order to prove that f_ψ is indeed hyperbolic, we will construct some cone fields $\mathcal{C}^{\mathrm{s}}$ and $\mathcal{C}^{\mathrm{u}}$ on $\mathrm{int}(\Sigma) \cup \partial^{\mathrm{in}}V$, prove that $\mathcal{C}^{\mathrm{u}}$ and $\mathcal{C}^{\mathrm{s}}$ are invariant under df_ψ and df_ψ^{-1}, respectively, and that the vectors in $\mathcal{C}^{\mathrm{u}}$ and $\mathcal{C}^{\mathrm{s}}$ are uniformly expanded by df_ψ and df_ψ^{-1}, respectively.

By assumption, we already have some cone fields $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ on $\partial^{\mathrm{in}} V$. Moreover, Λ is a hyperbolic set for X; hence there exist some cone fields $\mathcal{C}^{\mathrm{u}}_{\Sigma}$ and $\mathcal{C}^{\mathrm{s}}_{\Sigma}$ on $\mathrm{int}(\Sigma)$ which are invariant under df and df^{-1}, respectively, and such that the vectors in $\mathcal{C}^{\mathrm{u}}_{\Sigma}$ and $\mathcal{C}^{\mathrm{s}}_{\Sigma}$ and uniformly expanded by df and df^{-1}, respectively. We may assume that these cone fields $\mathcal{C}^{\mathrm{u}}_{\Sigma}$ and $\mathcal{C}^{\mathrm{s}}_{\Sigma}$, respectively, contain the directions tangent to the foliations induced by $\mathcal{G}^{\mathrm{u}}$ and $\mathcal{G}^{\mathrm{s}}$ on Σ. We consider the cone fields $\mathcal{C}^{\mathrm{u}}$ and $\mathcal{C}^{\mathrm{s}}$ on $\mathrm{int}(\Sigma) \cup \partial^{\mathrm{in}} V$, which coincide with $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ on $\partial^{\mathrm{in}} V$ and coincide with $\mathcal{C}^{\mathrm{u}}_{\Sigma}$ and $\mathcal{C}^{\mathrm{s}}_{\Sigma}$ on $\mathrm{int}(\Sigma)$. In order to check that these cone fields satisfy the desired properties, we will decompose the first return map f_ψ into four parts. Namely, we consider the restrictions of f_ψ to $\Sigma \cap f_\psi^{-1}(\Sigma)$, $\partial^{\mathrm{in}} V \cap f_\psi^{-1}(\partial^{\mathrm{in}} V)$, $\partial^{\mathrm{in}} V \cap f_\psi^{-1}(\Sigma)$ and $\partial^{\mathrm{in}} V \cap f_\psi^{-1}(\Sigma)$. We denote these restrictions by $f_{\psi,1}$, $f_{\psi,2}$, $f_{\psi,3}$ and $f_{\psi,4}$, respectively.

- The map $f_{\psi,1}\colon \Sigma \to \Sigma$ is nothing but the first return map f of the orbits of X on Σ (because a segment of orbit of Z_ψ which does not cross $\partial^{\mathrm{in}} V$ is a segment of orbit of Σ). Hence, $\mathcal{C}^{\mathrm{u}}_{\Sigma}$ and $\mathcal{C}^{\mathrm{s}}_{\Sigma}$ are invariant under $df_{\psi,1}$ and $df_{\psi,1}^{-1}$, respectively, and the vectors in $\mathcal{C}^{\mathrm{u}}_{\Sigma}$ and $\mathcal{C}^{\mathrm{s}}_{\Sigma}$ are uniformly expanded by $df_{\psi,1}$ and $df_{\psi,1}^{-1}$, respectively.

- The map $f_{\psi,2}\colon \partial^{\mathrm{in}} V \to \partial^{\mathrm{in}} V$ is a restriction of the return map Θ_ψ (namely, the restriction to the set of points x such that the forward Z_ψ–orbit of x intersects $\partial^{\mathrm{in}} V$ before intersecting Σ). Hence our assumption ensures that $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ are invariant under $df_{\psi,2}$ and $df_{\psi,2}^{-1}$, respectively, and that the vectors in $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ are uniformly expanded by $df_{\psi,2}$ and $df_{\psi,2}^{-1}$, respectively.

- Consider $\mathcal{U}$ a neighborhood of Λ. By taking $\mathcal{U}$ small enough, we can assume that there is uniform expansion (resp. contraction) of the vectors tangent to $\mathcal{G}^{\mathrm{u}}$ (resp. $\mathcal{G}^{\mathrm{s}}$) along the orbits of X while they stay in $\mathcal{U}$. Now let $\mathcal{U}_0$ be a much smaller neighborhood of Λ. By taking Σ small enough, one can assume that Σ is contained in $\mathcal{U}_0$. Hence, any orbit of X that enters $\mathcal{U}$ and hits Σ will have to spend a long time in $\mathcal{U}$ before hitting Σ. Therefore, up to choosing Σ small enough, one can apply arguments similar to those of the proof of Lemma 5.14, and get that, for any segment of an orbit of X that starts at $\partial^{\mathrm{in}} V$ and ends in Σ, the vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ (resp. $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$) will be expanded (resp. contracted) by a very large factor along that segment. According to our assumptions, the cone fields $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ contains the direction tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ but does not contain the direction tangent to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$. Therefore, provided that the section Σ is contained in a small enough neighborhood of Λ, the derivative of $f_{\psi,3}\colon \partial^{\mathrm{in}} V \to \Sigma$ will map the cone field $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ to an arbitrarily thin cone field around $\mathcal{G}^{\mathrm{u}}_{\Sigma}$ (in particular, the image of $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ will be contained in $\mathcal{C}^{\mathrm{u}}_{\Sigma}$) and will expand uniformly the vectors in $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$. Similarly, the derivative of $f_{\psi,3}^{-1}$ will map $\mathcal{C}^{\mathrm{s}}_{\Sigma}$ inside $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$, and will expand uniformly the vectors in $\mathcal{C}^{\mathrm{s}}_{\Sigma}$.

- Similar arguments show that, provided that the section Σ is contained in a small enough neighborhood of Λ, the cone fields satisfy the desired properties with respect to the map $f_{\psi,4}\colon \Sigma \to \partial^{\mathrm{in}} V$ (here we use the fact that $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ respectively contain the directions tangent to $\psi(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ and $\psi(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})$, but do not contain the directions tangent to $\psi(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})$ and $\psi(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$).

The four points above show that, if Σ is contained in a small enough neighborhood of Λ, then the cone fields $\mathcal{C}^{\mathrm{u}}$ and $\mathcal{C}^{\mathrm{s}}$ are invariant under df_ψ and df_ψ^{-1}, respectively, and such that the vectors in $\mathcal{C}^{\mathrm{u}}$ and $\mathcal{C}^{\mathrm{s}}$ and uniformly expanded by df_ψ and df_ψ^{-1}, respectively. In other words, the first return map f is hyperbolic provided that Σ is small enough. Hence the vector field Z_ψ is Anosov. □

The gluing map $\psi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$ (whose existence is claimed by Theorem 1.5) will be obtained as a composition $\psi = \psi^{\mathrm{in}} \circ \varphi \circ \psi^{\mathrm{out}}$, where φ is the original gluing map, ψ^{in} is a self-diffeomorphism of the entrance boundary $\partial^{\mathrm{in}} V$ and ψ^{out} is a self-diffeomorphism of the exit boundary $\partial^{\mathrm{out}} V$. The diffeomorphisms ψ^{in} and ψ^{out} will be provided by the following proposition:

Proposition 6.2 *Given any $\lambda > 1$ and $\varepsilon > 0$, there is a diffeomorphism $\psi^{\mathrm{in}}\colon \partial^{\mathrm{in}} V \to \partial^{\mathrm{in}} V$ with the following properties:*

- *ψ^{in} coincides with the identity map on a neighborhood of the lamination $\mathcal{L}^{\mathrm{s}}$.*
- *ψ^{in} preserves each leaf of the foliation $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$.*
- *the foliation $(\psi^{\mathrm{in}})^{-1}_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}})$ is ε–C^1–close to the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$.*
- *the derivative of $\Gamma \circ \psi^{\mathrm{in}}$ expands vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than λ: for any vector u tangent to a leaf of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$, one has $\|(\Gamma \circ \psi^{\mathrm{in}})_*(u)\| > \lambda \|u\|$.*

Analogously, there exists a diffeomorphism $\psi^{\mathrm{out}}\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{out}} V$ such that:

- *ψ^{out} coincide with the identity map on a neighborhood of the lamination $\mathcal{L}^{\mathrm{u}}$.*
- *ψ^{out} preserves each leaf of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$.*
- *The foliation $\psi^{\mathrm{out}}_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ is ε–C^1–close to the foliation $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$.*
- *The derivative of $(\Gamma \circ \psi^{\mathrm{in}})^{-1}$ expands vectors tangent to $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ by a factor larger than λ: for any unit vector u tangent to a leaf of $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$, one has $\|(\Gamma\circ\psi^{\mathrm{out}})^{-1}_*(u)\| > \lambda \cdot \|u\|$.*

Proof of Theorem 1.5 assuming Proposition 6.2 Given $\lambda > 1$ and $\varepsilon > 0$, we consider the diffeomorphisms $\psi^{\mathrm{in}}_{\lambda,\varepsilon}$ and $\psi^{\mathrm{out}}_{\lambda,\varepsilon}$ associated with λ, ε by Proposition 6.2. Then we consider the gluing map

$$\psi_{\lambda,\varepsilon} := \psi^{\mathrm{in}}_{\lambda,\varepsilon} \circ \varphi \circ \psi^{\mathrm{out}}_{\lambda,\varepsilon}$$

and the vector field $Z_{\lambda,\varepsilon}$ induced by X on the closed manifold $V/\psi_{\lambda,\varepsilon}$. Observe that, since ψ^{in} and ψ^{out} coincide with the identity on neighborhoods of $\mathcal{L}^{\mathrm{s}}$ and $\mathcal{L}^{\mathrm{u}}$, respectively, $\psi^{\mathrm{in}}_{\lambda,\varepsilon}$ is a strongly transverse gluing map which is isotopic to φ inside the set of strongly transverse gluing maps. We want to prove that the vector field $Z_{\lambda,\varepsilon}$ is Anosov provided that ε is small enough and λ is large enough. So we are left to proving that the return map

$$\Theta_{\lambda,\varepsilon} := \psi_{\lambda,\varepsilon} \circ \Gamma$$

satisfies the hypotheses of Lemma 6.1 provided that ε is small enough and λ is large enough. For technical reasons, it is convenient to introduce the map

$$\widehat{\Theta}_{\lambda,\varepsilon} := (\psi^{\mathrm{in}})^{-1} \circ \Theta_{\lambda,\varepsilon} \circ \psi^{\mathrm{in}} = \varphi \circ \psi^{\mathrm{out}} \circ \Gamma \circ \psi^{\mathrm{in}},$$

and the foliations

$$\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon} := (\psi^{\mathrm{in}}_{\lambda,\varepsilon})^{-1}_{*}(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}) \quad \text{and} \quad \mathcal{G}^{\mathrm{u}}_{\mathrm{out},\lambda,\varepsilon} = (\psi^{\mathrm{out}}_{\lambda,\varepsilon})_{*}(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}).$$

Note that

$$(\psi^{\mathrm{in}}_{\lambda,\varepsilon})^{-1}_{*}(\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}) = \mathcal{G}^{\mathrm{u}}_{\mathrm{in}} \quad \text{and} \quad (\psi^{\mathrm{out}}_{\lambda,\varepsilon})_{*}(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}) = \mathcal{G}^{\mathrm{s}}_{\mathrm{out}},$$

since $\psi^{\mathrm{in}}_{\lambda,\varepsilon}$ preserves the foliation $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ and $\psi^{\mathrm{out}}_{\lambda,\varepsilon}$ preserves the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ by assumption.

Since the diffeomorphism $\psi^{\mathrm{out}}_{\lambda,\varepsilon}$ preserves each leaf of $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$, we can define $h^{\mathrm{s}}_{\mathrm{out},\lambda,\varepsilon}$ as the holonomy of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ between x and $\psi^{\mathrm{out}}_{\lambda,\varepsilon}(x)$.

Claim 6.3 *The length of the segment of leaf of $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ joining a point x to $\psi^{\mathrm{out}}_{\lambda,\varepsilon}(x)$ is bounded by a constant independent of ε, λ and x. As a consequence, the action of the holonomy $h^{\mathrm{s}}_{\mathrm{out},\lambda,\varepsilon}$ on vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ is uniformly bounded independently of ε and λ.*

Proof Recall that $\mathcal{L}^{\mathrm{u}}$ is a filling MS lamination, $\mathcal{L}^{\mathrm{u}}$ is a sublamination of the foliation $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ and $\psi^{\mathrm{out}}_{\lambda,\varepsilon}$ is the identity map in a neighborhood $\mathcal{L}^{\mathrm{u}}$. Therefore, if x is not a fixed point of $\psi^{\mathrm{out}}_{\lambda,\varepsilon}$, then the points x and $\psi^{\mathrm{out}}_{\lambda,\varepsilon}(x)$ belong to the same segment of leaf of $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}} \setminus \mathcal{L}^{\mathrm{u}}$. As $\mathcal{L}^{\mathrm{u}}$ is a filling MS lamination, these segments have a uniformly bounded length, proving the first assertion. The second assertion is a direct consequence of the first one and the fact that $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ is a smooth foliation. □

Claim 6.4 *The perturbed foliations $\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{out},\lambda,\varepsilon}$ tend to the nonperturbed foliations $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ when $\varepsilon \to 0$ for the C^1–topology. As a consequence, for ε small enough, the foliation $(\widehat{\Theta}_{\lambda,\varepsilon})_{*}(\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}) = \varphi_{*}(\mathcal{G}^{\mathrm{u}}_{\mathrm{out},\lambda,\varepsilon})$ is uniformly (in λ, ε) transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon}$.*

Proof The first assertion follows immediately from the definition of the foliations $\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon}$ and $\mathcal{G}^{\mathrm{u}}_{\mathrm{out},\lambda,\varepsilon}$, and from the properties of the maps ψ^{in} and ψ^{out}. The second assertion is a direct consequence of the first one. □

Claim 6.5 *There exist some constants $C > 0$ and $\varepsilon_0 > 0$ such that, for any $\lambda > 1$ and $0 < \varepsilon < \varepsilon_0$, the return map $\widehat{\Theta}_{\lambda,\varepsilon}$ expands uniformly the vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than $C\lambda$, and its inverse $(\widehat{\Theta}_{\lambda,\varepsilon})^{-1}$ expands uniformly the vectors tangent to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon}$ by a factor larger than $C\lambda$.*

Proof Let u be a vector tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}} = \mathcal{G}^{\mathrm{u}}_{\mathrm{in},\lambda,\varepsilon}$. One writes

$$\widehat{\Theta}_{\lambda,\varepsilon,*}(u) = \varphi_* \circ (\psi^{\mathrm{out}}_{\lambda,\varepsilon})_* \circ \Gamma_* \circ (\psi^{\mathrm{in}}_{\lambda,\varepsilon})_*(u).$$

See Figure 7. By definition of the map $\psi^{\mathrm{in}}_{\lambda,\varepsilon,*}$, one has

$$\|\Gamma_* \circ (\psi^{\mathrm{in}}_{\lambda,\varepsilon})_*(u)\| > \lambda \cdot \|u\|.$$

Furthermore, $\Gamma_* \circ (\psi^{\mathrm{in}}_{\lambda,\varepsilon})_*(u)$ is tangent to $\mathcal{G}^{u}_{\mathrm{out}}$. Thus we just need to see that the action of $\psi^{\mathrm{out}}_{\lambda,\varepsilon}$ on the vectors tangent to $\mathcal{G}^{u}_{\mathrm{out}}$ is bounded, independently of λ and ε (for ε smaller than some ε_0).

Recall that $h^{\mathrm{s}}_{\mathrm{out},\lambda,\varepsilon}$ is the holonomy of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ between x and $\psi^{\mathrm{out}}_{\lambda,\varepsilon}(x)$. The foliation $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ is transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$, and $\psi^{\mathrm{out}}_{\lambda,\varepsilon}(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ is ε–C^1–close to $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$. In particular, there exists ε_0 such that, for $0 < \varepsilon < \varepsilon_0$, the foliation $(\psi^{\mathrm{out}}_{\lambda,\varepsilon})_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out}})$ is uniformly (in λ and ε) transverse to the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$. Therefore, for v tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$, the ratio between $\|(h^{\mathrm{s}}_{\mathrm{out},\lambda,\varepsilon})_*(v)\|$ and $\|(\psi^{\mathrm{out}}_{\lambda,\varepsilon})_*(v)\|$ is bounded independently of λ, ε and v. Using Claim 6.3, we conclude that the map $\widehat{\Theta}_{\lambda,\varepsilon}$ expands uniformly the vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than $C\lambda$. See Figure 7. The arguments are similar for the action of the map $(\widehat{\Theta}_{\lambda,\varepsilon})^{-1}$ on the vectors tangent to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon}$. □

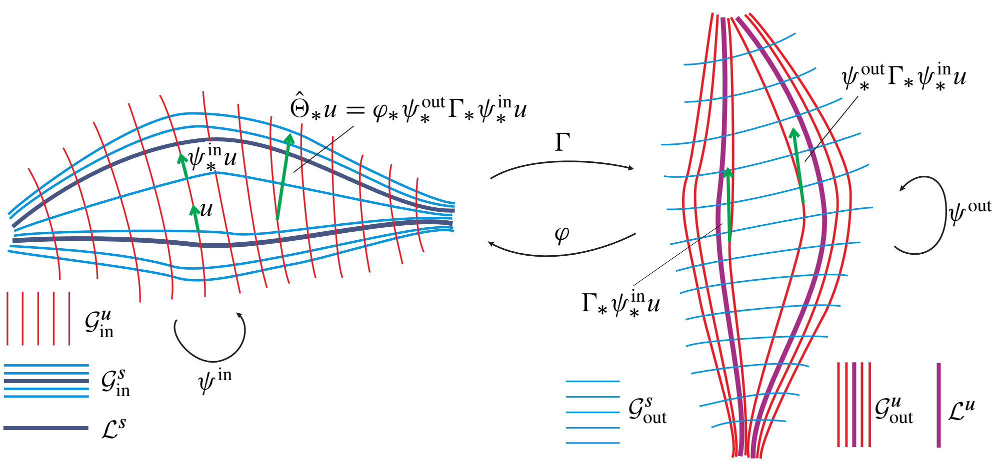

Figure 7: The action of $\widehat{\Theta}_{\lambda,\varepsilon}$ on a vector u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$

During the proof of Claim 6.5, we have chosen ε_0 such that the foliations $\varphi_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out},\lambda,\varepsilon})$ are uniformly transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon}$ for $\varepsilon\in[0,\varepsilon_0]$ and $\lambda>1$. Moreover, the foliation $\varphi_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})$ is transverse to the foliation $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$. This allows us to choose a continuous cone field $\widehat{\mathcal{C}}^{\mathrm{u}}_{\mathrm{in}}$ on $\partial^{\mathrm{in}}V$ such that $\widehat{\mathcal{C}}^{\mathrm{u}}_{\mathrm{in}}$ contains the direction tangent to the foliation $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ and the direction tangent to the foliation $\varphi_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out},\lambda,\varepsilon})$ for $\varepsilon\in[0,\varepsilon_0]$ and $\lambda>1$, and such that $\widehat{\mathcal{C}}^{\mathrm{u}}_{\mathrm{in}}$ contains neither the direction tangent to the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon}$ for any $\varepsilon\in[0,\varepsilon_0]$ and $\lambda>1$ nor the direction tangent to the foliation $\varphi_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})$. As a direct consequence of Claim 6.5 one gets:

Fact 6.6 *For every $\varepsilon\in[0,\varepsilon_0]$, when $\lambda\to\infty$ the cone field $\widehat{\mathcal{C}}^{\mathrm{u}}_{\mathrm{in}}$ is mapped by $\Gamma_{\lambda,\varepsilon}$ in an arbitrarily small cone field around $\mathcal{G}^{\mathrm{u}}_{\mathrm{out},\lambda,\varepsilon}$ and the vectors in $\widehat{\mathcal{C}}^{\mathrm{u}}_{\mathrm{in}}$ are expanded by an arbitrarily large factor. As a consequence, there exists λ_0 such that, for every $\varepsilon\in[0,\varepsilon_0]$ and $\lambda\geq\lambda_0$, the cone field $\widehat{\mathcal{C}}^{\mathrm{u}}_{\mathrm{in}}$ is strictly invariant by $\widehat{\Theta}_{\lambda,\varepsilon}$ and the vectors in that cone field are uniformly expanded by $\widehat{\Theta}_{\lambda,\varepsilon}$.*

From now on, we fix $\varepsilon\in[0,\varepsilon_0]$ and $\lambda\geq\lambda_0$. We consider the cone field $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}:=\psi^{\mathrm{in}}_{\lambda,\varepsilon}(\widehat{\mathcal{C}}^{\mathrm{u}}_{\mathrm{in}})$. This cone field contains the direction tangent to the foliations

$$(\psi^{\mathrm{in}}_{\lambda,\varepsilon})^{-1}_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{in}})=\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}\quad\text{and}\quad(\psi^{\mathrm{in}}_{\lambda,\varepsilon})^{-1}_*\varphi_*(\mathcal{G}^{\mathrm{u}}_{\mathrm{out},\lambda,\varepsilon})=(\psi_{\lambda,\varepsilon})_*(\mathcal{G}^{u}_{\mathrm{out}}),$$

and it does not contain the direction tangent to the foliations

$$(\psi^{\mathrm{in}}_{\lambda,\varepsilon})^{-1}_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{in},\lambda,\varepsilon})=\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}\quad\text{or}\quad(\psi^{\mathrm{in}}_{\lambda,\varepsilon})^{-1}_*\circ\varphi_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{out}})=(\psi_{\lambda,\varepsilon})_*(\mathcal{G}^{s}_{\mathrm{out}}).$$

Moreover, Fact 6.6 implies that the cone field $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ is strictly invariant by $\Theta_{\lambda,\varepsilon}$ and that the vectors in $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ are uniformly expanded by $\Theta_{\lambda,\varepsilon}$ (for the norm associated with the pullback under $\psi^{\mathrm{in}}_{\lambda,\varepsilon}$ of the initial Riemannian metric). In other words, the cone field $\mathcal{C}^{\mathrm{u}}_{\mathrm{in}}$ satisfies all the hypotheses of Lemma 6.1 (for the return map $\Theta_{\lambda,\varepsilon}$). The construction of a cone field $\mathcal{C}^{\mathrm{s}}_{\mathrm{in}}$ is completely similar. So we can apply Lemma 6.1, which shows that the vector field $Z_{\lambda,\varepsilon}$ is Anosov. □

Remark 6.7 There is a version of Theorem 1.5 where one does not glue the whole exit boundary of a hyperbolic plug on the whole entrance boundary. More precisely, let (V,X) be a saddle hyperbolic plug with filling MS laminations (V,X). Let T^{in} and T^{out} be unions of connected components of $\partial^{\mathrm{in}}V$ and $\partial^{\mathrm{out}}V$, respectively. Let $\varphi\colon T^{\mathrm{out}}\to T^{\mathrm{in}}$ be a map such that $\varphi_*(\mathcal{L}^{\mathrm{u}}\cap T^{\mathrm{out}})$ is strongly transverse to $\mathcal{L}^{\mathrm{s}}\cap T^{\mathrm{in}}$. Exactly the same arguments as above allow to prove that there is a vector field Y on V which is C^1–close to X and map $\psi\colon T^{\mathrm{out}}\to T^{\mathrm{in}}$ such that

- (V,X,φ) and (V,Y,ψ) are strongly isotopic;
- if Z_ψ is the vector field induced by X on V/ψ, then $(V/\psi,Z_\psi)$ is a hyperbolic plug.

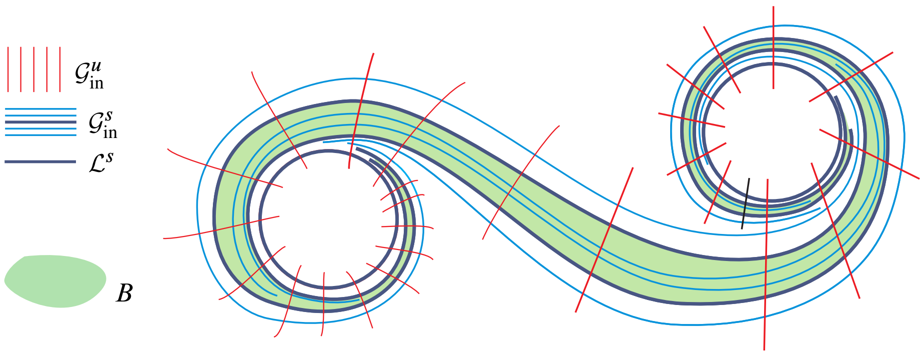

Figure 8: A connected component of $\partial^{\text{in}} V \setminus \mathcal{L}^{\text{s}}$

6.2 Perturbation of the return map Θ: proof of Proposition 6.2

This section is devoted to the proof of Proposition 6.2. We will only deal with the diffeomorphism $\psi^{\text{in}}\colon \partial^{\text{in}} V \to \partial^{\text{in}} V$. The construction of the diffeomorphism ψ^{out} is analogous, up to reversing the flow.

Let us briefly present the construction of ψ^{in}. Fix $\varepsilon > 0$ and $\lambda > 1$. According to Lemma 5.14, the crossing map Γ expands the vectors tangent to $\mathcal{G}^{\text{u}}_{\text{in}}$ by a factor of at least λ on some neighborhood of $\mathcal{L}^{\text{s}}$. The image by Γ of such a neighborhood is a neighborhood of $\mathcal{L}^{\text{u}}$. This neighborhood contains an adapted neighborhood, whose complement consists of finitely many in-squares (see Lemma 5.10). Our proof will consist in building the diffeomorphism ψ^{in} in one of these in-squares and extending it on the whole $\partial^{\text{in}} V$ by gluing it with the identity map by a bump function, using the fact that the expansion of vectors tangent to $\mathcal{G}^{\text{u}}_{\text{in}}$ is arbitrarily large out of these in-squares.

As $\mathcal{L}^{\text{s}}$ is a filling MS lamination, every connected component B of $\partial^{\text{in}} V \setminus \mathcal{L}^{\text{s}}$ is a strip whose accessible boundary consists of two noncompact leaves of $\mathcal{L}^{\text{s}}$, which are asymptotic to each other at both ends. Each end of B spirals around a compact leaf of $\mathcal{L}^{\text{s}}$,[10] with contracting linear holonomy (see Figure 8). Our construction will be divided in two steps:

- We will first build a diffeomorphism ψ_h of B, defined as the product of a diffeomorphism h of a segment I^{u} of a $\mathcal{G}^{\text{u}}_{\text{in}}$ leaf by the identity map in the direction of the leaves of $\mathcal{G}^{\text{s}}_{\text{in}}$. The diffeomorphism ψ_h will have all the announced properties, except that it will not coincide with the identity close to the boundary of S, so it cannot be extended to the whole $\partial^{\text{in}} V$.

[10] This fact is a direct consequence of the definition of a filling MS lamination; see Definitions 3.9 and 3.18.

- Then, we will "slow down" the diffeomorphism ψ_h close to the ends of B, in order to be able to extend ψ_h continuously on $\partial^{\mathrm{in}} V$ (in a way that the extension of ψ_h will coincide with the identity on the complement of B).

The main difficulty is to manage to "slow down" ψ_h without destroying the hyperbolicity. A key ingredient to do that will be the uniform control of distortion of the holonomies of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ (this is the reason why we need the holonomy of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ along a compact leaf to be conjugated to a homothety).

6.2.1 Distortion control of the holonomies

Lemma 6.8 *Let $\mathcal{F}$ be an MS foliation of a compact surface S such that the holonomy of each compact leaf is conjugated to a homothety. Let $\mathcal{L}$ be a filling MS sublamination of $\mathcal{F}$. Let $\mathcal{G}$ be a smooth foliation transverse to $\mathcal{F}$.*

Then there is $C > 1$ with the following property: Let I and J be two segments of $\mathcal{G}$–leaves whose interiors are contained in a connected component B of $S \setminus \mathcal{L}$ and whose endpoints are on the boundary of B. Let $H_{I,J}$ be the holonomy of the foliation $\mathcal{F}$ from J to I. Then, for every $x, y \in J$, one has

$$C^{-1} < \frac{DH_{I,J}(x)}{DH_{I,J}(y)} < C.$$

An important point is that the constant C depends neither on the connected component B of $S \setminus \mathcal{L}$ nor on the segments I or J.

Proof First notice that the existence of such an announced constant C does not depend on the metric on the surface S (only the value of C will depend on the metric). Therefore we may choose a metric on S such that the holonomy of every compact leaf of $\mathcal{F}$ is a homothety. More precisely, denote by $\gamma_1, \dots, \gamma_p$ the compact leaves of $\mathcal{F}$. We choose a metric on S such that, for every $i \in \{1, \dots, p\}$, there is a tubular neighborhood T_i of the compact leaf γ_i such that the fibers of the tubular neighborhood are segments of leaves of $\mathcal{G}$, and such that the holonomy map from any fiber to any other fiber is a homothety.

Since $\mathcal{L}$ is a filling MS lamination, every half-leaf of $\mathcal{F}$ spirals around some compact leaf. It follows that the length of a segment of leaf of $\mathcal{F}$ which is disjoint from $T_1 \cup \dots \cup T_p$ is uniformly bounded. As a consequence, there exists a constant ℓ with the following property: Given a connected component B of $S \setminus \mathcal{L}$, and two segments I and J of $\mathcal{G}$–leaves as in the statement of Lemma 6.8, the holonomy map $H_{I,J}$ can be decomposed as

$$H_{I,J} = H_{I,I_1} \circ H_{I_1,J_1} \circ H_{J_1,J},$$

where I_1 and J_1 are segments of $\mathcal{G}$–leaves in B of $S \setminus \mathcal{L}$, and with endpoints on the boundary of B such that H_{I,I_1} and $H_{J_1,J}$ are homotheties, and the holonomy H_{I_1,J_1} is along $\mathcal{F}$–leaf segments of length bounded by ℓ. Notice that I_1 and J_1 may be equal to I and J, respectively.

On the one hand, the distortion of $H_{I,J}$ coincides with the distortion of H_{I_1,J_1} (since H_{I,I_1} and $H_{J_1,J}$ are homotheties). On the other hand, the distortion of H_{I_1,J_1} is uniformly bounded (because the holonomies of the foliation $\mathcal{F}$ along segments of leaves of length bounded by ℓ have uniformly bounded derivative). Hence, the distortion of $H_{I,J}$ is uniformly bounded. □

6.2.2 Building ψ^{in} on a large square

Definition 6.9 Let I be a compact segment of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–leaf contained in a connected component B of $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$. Given a diffeomorphism h of I such that the endpoints of I are flat fixed points for h, we denote by ψ_h the unique diffeomorphism of B such that:

- ψ_h is the identity out of the $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$–saturation of I.
- ψ_h preserves (globally) the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$.
- ψ_h preserves each leaf segment of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$.
- The restriction of ψ_h to I is h.

The aim of this subsection is to prove the following result:

Proposition 6.10 *Given any $\lambda > 1$ and any component B of $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$, there is a segment I of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–leaf contained in B and a diffeomorphism $h\colon I \to I$ such that the endpoints of I are flat fixed points for h and such that, for any vector u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some point $x \in B$, one has*

$$\|(\Gamma \circ \psi_h)_*(u)\| > \lambda \cdot \|u\|.$$

Proposition 6.10 announces a control of the expansion on unit vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at any point x of a connected component B of $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$. We start by getting such a control along one segment of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–leaf crossing B:

Lemma 6.11 *Let B be a component of $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$, and σ be a leaf of the restriction of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ to B. Fix any constant $A > 1$. Then there is a diffeomorphism $h\colon \sigma \to \sigma$, equal to the identity map outside of some compact part of σ, such that for every vector u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some point $x \in \sigma$ one has*

$$\|(\Gamma \circ \psi_h)_*(u)\| > A \cdot \|u\|.$$

Proof Observe that σ is a interval of bounded length, and $\Gamma(\sigma)$ is an entire leaf of $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ hence isometric to $\mathbb{R}$. Furthermore, according to Lemma 5.14, the rate of expansion of the crossing map Γ for vectors tangent to σ tends to infinity close to the ends of σ. Therefore Lemma 6.11 is a direct consequence of the following general lemma (whose proof is left to the reader).

Lemma 6.12 *Let ϕ: $]0,1[\to\mathbb{R}$ be a diffeomorphism whose derivative tends to $+\infty$ when t tends to 0 or 1. For any $A>1$, there is a diffeomorphism $\widetilde{\phi}$ which coincides with ϕ in a neighborhood of 0 and of 1 and whose derivative is everywhere larger than A.*

This completes the proof of Lemma 6.11 □

The following lemma allows us to compare the rate of expansion of the map $\Gamma\circ\psi_h$ for vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at different points of $\partial^{\mathrm{in}}V\setminus\mathcal{L}^{\mathrm{s}}$.

Lemma 6.13 *Let B be a connected component of $\partial^{\mathrm{in}}V\setminus\mathcal{L}^{\mathrm{s}}$ and σ be a leaf of the restriction of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ to B. There exists a constant $\alpha_\sigma>0$ with the following property: for every diffeomorphism h: $\sigma\to\sigma$ supported in a compact segment $I\subset\sigma$ and every vectors u and v tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some points $x,y\in B$ such that x and y belong to the same leaf of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ and $y\in\sigma$, one has*

$$\frac{\|(\Gamma\circ\psi_h)_*(u)\|}{\|u\|}>\alpha_\sigma\frac{\|(\Gamma\circ\psi_h)_*(v)\|}{\|v\|}.$$

Proof Denote by σ_x the leaf of the restriction of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ to B containing x. Observe that $\Gamma(B)$ is a connected component of $\partial^{\mathrm{out}}V\setminus\mathcal{L}^{\mathrm{u}}$, and $\Sigma:=\Gamma(\sigma)$ and $\Sigma_x:=\Gamma(\sigma_x)$ are two leaves of $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ contained in $\Gamma(B)$. We denote by $H_{\sigma_x\to\sigma}$: $\sigma_x\to\sigma$ the holonomy of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ from σ_x to σ. We denote by $H_{\Sigma\to\Sigma_x}$: $\Sigma\to\Sigma_x$ the holonomy of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ from Σ to Σ_x.

By construction, the restriction of ψ_h to σ_x is conjugated to h by $H_{\sigma_x\to\sigma}$. One deduces that the restriction of $\Gamma\circ\psi_h$ to σ_x can be written as

$$(\Gamma\circ\psi_h)|_{\sigma_x}=H_{\Sigma\to\Sigma_x}\circ(\Gamma\circ\psi_h)|_\sigma\circ H_{\sigma_x\to\sigma}. \tag{1}$$

The following lemma gives a uniform upper bound for the derivative of $H_{\Sigma\to\Sigma_x}$:

Lemma 6.14 *There exists a constant $\beta>1$ such that the holonomy of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ between two leaves of $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ in the same connected component $\partial^{\mathrm{out}}V\setminus\mathcal{L}^{\mathrm{u}}$ has a derivative which is bounded by β.*

Proof Just notice that $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$ is a smooth foliation, and that the segment of leaves of $\mathcal{G}^{\mathrm{u}}_{\mathrm{out}}$ contained in $\partial^{\mathrm{out}}V\setminus\mathcal{L}^{\mathrm{u}}$ have uniformly bounded length. □

The following lemma gives a uniform lower bound for the derivative of $H_{\Sigma\to\Sigma_x}$:

Lemma 6.15 *There is a constant $\beta_\sigma > 0$ such that, for every $x \in B$ and any vector u tangent to σ_x, one has*

$$\|(H_{\sigma_x\to\sigma})_*(u)\| > \beta_\sigma \cdot \|u\|.$$

Proof The component B is a strip whose ends converge to compact leaves whose holonomies are conjugated to homotheties. Lemma 6.8 asserts that the holonomy of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ between two leaves of the restriction to B of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ have uniformly bounded distortion C. As a consequence, for every $x \in B$, the derivative on the holonomy $H_{\sigma_x,\sigma}$ is larger than $\ell(\sigma)/(C\cdot\ell(\sigma_x))$, where ℓ is the length. One concludes by noticing that the length $\ell(\sigma_x)$ is uniformly bounded, with

$$\inf_{x\in B}\frac{\ell(\sigma)}{C\cdot\ell(\sigma_x)} > 0. \qquad \square$$

Putting together equality (1) and Lemmas 6.14 and 6.15, one easily sees that the constant $\alpha_\sigma := \beta_\sigma\cdot\beta^{-1}$ satisfies the properties announced in Lemma 6.13. This completes the proof of Lemma 6.13. □

Proof of Proposition 6.10 One just needs to combine Lemmas 6.11 and 6.13, with a constant A larger than $\alpha_\sigma\cdot\lambda$. □

6.2.3 Estimates for the derivative of ψ_h

Corollary 6.16 *Let B be a connected component of $\partial^{\mathrm{in}}V\setminus\mathcal{L}^{\mathrm{s}}$ and I be a segment of a $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–leaf contained in B, and let h be a diffeomorphism of I such that the endpoints of I are flat fixed points for h. We consider the diffeomorphism ψ_h of B associated with h* (see Section 6.2.2)*. Let u be a vector tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some point $y \in B$. Then*

$$C^{-1}\inf_{x\in I}|Dh(x)|\cdot\|u\| \le \|D\psi_h(u)\| \le C\sup_{x\in I}|Dh(x)|\cdot\|u\|,$$

where C is the bound on the distortion of the holonomy of foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ given by Lemma 6.8.

Proof Let σ_y be the leaf through y of the restriction of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ to B. Notice that the restriction of ψ_h to σ_y is the conjugation of h by the holonomy of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$. By Lemma 6.8, the distortion of this holonomy is bounded by C. This yields the desired estimates. □

6.2.4 "Slowing down" the diffeomorphisms ψ_h close to the ends of the strip

Proposition 6.10 built a diffeomorphism ψ_h of a connected component B of $\partial^{\mathrm{in}}V\setminus\mathcal{L}^{\mathrm{s}}$. Recall that B is a strip bounded by two noncompact leaves of $\mathcal{L}^{\mathrm{s}}$ which are asymptotic to each other at both ends. Each end of the strip B spirals around a compact leaf of $\mathcal{L}^{\mathrm{s}}$.

The diffeomorphism ψ_h coincides with the identity outside of the $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$–saturation of some compact interval $I \subset B$. Nevertheless, ψ_h does not tend to the identity close to the ends of B. This is the reason why we need to "slow down" ψ_h close to the ends of B.

We consider a compact leaf c of $\mathcal{L}^{\mathrm{s}}$ (or equivalently of $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$) contained in the closure of B (ie there is one end of B spiraling around c). We orient c so that its holonomy is a linear contraction. Recall that $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ is a smooth foliation transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$. So, one can choose a smooth tubular neighborhood O of c such that

- the boundary ∂O is transverse to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$,
- the fibers of O are segments of leaves of $\mathcal{G}^{\mathrm{u}}$.

We choose a parametrization of c by $S^1 = \mathbb{R}/\mathbb{Z}$ such that the universal cover of O can be identified with $\mathbb{R}\times[-1,1]$, where the lifts of the leaves of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ are the segments $\{t\}\times[-1,1]$. For every $\theta \in S^1$ and every $t \in \mathbb{R}$, we will denote by $H_{\theta,t}$ the holonomy of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ from the fiber $\sigma^{\mathrm{u}}_{\theta} = \{\theta\}\times[-1,1]$ to the fiber $\sigma^{\mathrm{u}}_{\theta+t} = \{\theta+t\}\times[-1,1]$. More precisely, we choose a lift $\bar\theta$ of θ and consider the holonomy of the lifted foliation $\bar{\mathcal{G}}^{\mathrm{s}}_{\mathrm{in}}$ from the fiber $\{\bar\theta\}\times[-1,1]$ to the fiber $\{\bar\theta+t\}\times[-1,1]$; the projection of this holonomy does not depend on the lift $\bar\theta$. Notice that, for every $t>0$ and every θ, the holonomy $H_{\theta,t}$ is defined on the whole fiber, and is a contraction.

Lemma 6.17 *Let C be the constant given by Lemma 6.8. Let I be a segment of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–leaf contained in $\sigma^{\mathrm{u}}_{\theta} \cap B$. We denote by I_t the image of I by the holonomy $H_{\theta,t}$. Let h be a diffeomorphism of I such that the endpoints of I are flat fixed points of h. For every $\varepsilon>0$, there is a diffeomorphism ψ^+ of B, with the following properties:*

- *ψ^+ preserves each leaf σ^{u} of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$.*
- *ψ^+ is equal to h on I.*
- *ψ^+ is the identity out of the $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$–saturation of I.*
- *ψ^+ coincides with ψ_h outside O, and also coincides with $\psi_h = H_{\theta,-t} h H^{-1}_{\theta,-t}$ on I_{-t} for every $t>0$.*
- *For $t>0$ large enough, ψ_+ is the identity map on I_t.*
- *$\psi^+(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}})$ is ε–C^1–close to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$.*
- *The action of ψ^+ on vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ is controlled by the derivative of h; more precisely, for every vector u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$,*

$$C^{-1}\inf_{x\in I}|Dh(x)|\cdot\|u\| \le \|(\psi^+)_*(u)\| \le C\cdot\sup_{x\in I}|Dh(x)|\cdot\|u\|. \tag{2}$$

Proof We consider the isotopy $(h_t)_{t\in[0,1]}$ joining h to the identity by convex sum, ie $h_t(x)-x=t(h(x)-x)$. Hence $Dh_t(x)-1=t\cdot(Dh(x)-1)$. We consider a smooth decreasing map $\tau\colon \mathbb{R}\to[0,1[$ such that $\tau(t)=1$ for $t<0$ and $\tau(t)=0$ for t large enough. We define ψ^+ as follows:

- $\psi^+=\psi_h$ outside O.
- $\psi^+=\mathrm{Id}$ outside the $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$–saturation of I.
- $\psi^+=H_{\theta,t}h_{\tau(t)}H_{\theta,t}^{-1}$ on I_t.

One easily checks that this definition is coherent. The diffeomorphism ψ^+ trivially satisfies all the desired properties, except for the two last ones (the control of distance between the foliations $\psi^+(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}})$ and $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$, and the control of the action of the derivative of ψ^+ on the vectors tangent to the $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–leaves).

To obtain proximity between the foliations $\psi^+(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}})$ and $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$, one just needs to notice that

- the C^1–distance between $\psi^+(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}})$ and $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ tends to 0 when $\sup_{\mathbb{R}}|D\tau(t)|$ tends to 0;
- we can choose the function τ so that $\sup_{\mathbb{R}}|D\tau(t)|$ is arbitrarily small.

So we are left to check the last property. For this purpose, we consider a vector u tangent to a $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–leaf at some point $y\in B$. Assume that the point y belongs to O. Hence, we have

$$\begin{aligned}\|(\psi^+)_*(u)\| &= \|(H_{\theta,t}\circ h_{\tau(t)}\circ H_{\theta,t}^{-1})_*(u)\|\\ &= \frac{\|DH_{\theta,t}(z_2)\|}{\|DH_{\theta,t}(z_1)\|}\cdot\|1+\tau(t)\cdot(Dh(z_1)-1)\|\cdot\|u\|,\end{aligned}$$

where $z_1=H_{\theta,t}^{-1}(y)$ and $z_2=h(z_1)$. Using Lemma 6.8 and the fact that $|\tau(t)|$ is less than 1, this yields the desired inequality (2). If the point y is not in O, the inequality follows from Corollary 6.16 (since $\psi^+=\psi_h$ outside O). □

6.2.5 End of the proof of Proposition 6.2

Proof of Proposition 6.2 Fix $\lambda>1$ and $\varepsilon>0$. According to Lemma 5.14, there is an adapted neighborhood of $\mathcal{U}^{\mathrm{in}}_{\lambda}$ of $\mathcal{L}^{\mathrm{s}}$ in $\partial^{\mathrm{in}}V$, such that

(3) $\|(\Gamma)_*(u)\|\ge\lambda\cdot\|u\|$ for every vector u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some point $x\in\mathcal{U}^{\mathrm{in}}_{\lambda}\setminus\mathcal{L}^{\mathrm{s}}$.

The set $\partial^{\mathrm{in}}V\setminus\mathcal{U}^{\mathrm{in}}_{\lambda}$ is contained in finitely many connected components $B_1,\dots,B_m$ of $\partial^{\mathrm{in}}V\setminus\mathcal{L}^{\mathrm{s}}$. Recall that each B_i is a strip bounded by two noncompact leaves of $\mathcal{L}^{\mathrm{s}}$ which are spiraling (at both ends) around some compact leaves of $\mathcal{L}^{\mathrm{s}}$.

For $i = 1, \dots, m$, we consider a homeomorphism h_i associated with λ and B_i by Proposition 6.10. The map $\Gamma \circ \psi_{h_i}$ expands vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than λ:

(4) $\|(\Gamma \circ \psi_{h_i})_*(u)\| \geq \lambda \cdot \|u\|$ for every u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some point $x \in B_i$.

The only trouble is that ψ_{h_i} cannot be extended as a diffeomorphism on the closure of B_i. To overcome this problem, we will modify ψ_{h_i} on the ends of the strip B_i using Lemma 6.17.

Let m be a lower bound for the derivatives of all the h_i. According to Corollary 6.16, there exists a constant C such that, for every i,

(5) $\|(\psi_{h_i})_*(u)\| \geq C^{-1} m \cdot \|u\|$ for every u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some point $x \in B_i$.

Now, we use Lemma 5.10 and again Lemma 5.14 to get an adapted neighborhood $\mathcal{U}^{\mathrm{in}} \subset \mathcal{U}^{\mathrm{in}}_{\lambda}$ of $\mathcal{L}^{\mathrm{s}}$ such that

(6) $\|(\Gamma)_*(u)\| \geq (\lambda \cdot C^2 m^{-1}) \cdot \|u\|$
for every vector u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some point $x \in \partial^{\mathrm{in}} V \setminus \mathcal{U}^{\mathrm{in}}$.

By definition of adapted neighborhood, the complement of $\mathcal{U}^{\mathrm{in}}$ consists of finitely many in-rectangles, and each connected component of $\partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}$ contains at most one of these in-rectangles. We denote by R_i the in-rectangle contained in the strip B_i. The set $\partial^{\mathrm{in}} V \setminus \mathcal{U}^{\mathrm{in}}_{\lambda}$ is contained in the interior of the union of the R_i. Up to fattening the in-rectangles R_i (that is, up to shrinking the adapted neighborhood $\mathcal{U}^{\mathrm{in}}$) one may assume that the $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–sides of the in-rectangle R_i are contained in the tubular neighborhoods of the compact leaves of $\mathcal{L}^{\mathrm{s}}$ in the closure of B_i.

Applying Lemma 6.17 to both the $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$–sides of the rectangle R_i, one gets a diffeomorphism ψ_i of the strip B_i such that:

- ψ_i preserves every leaf of the restriction of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ to B_i.
- The restriction of ψ_i to the in-rectangle R_i is ψ_{h_i}.
- ψ_i coincides with the identity map out of a compact subset of B_i.
- ψ_i expands vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than $C^{-2} m$:

(7) $\|(\psi_i)_*(u)\| \geq C^{-2} m \cdot \|u\|$ for every u tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ at some point $x \in B_i$.

- The C^1–distance between the foliations $(\psi_i)_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}})$ and $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$ is smaller than ε.

We consider the diffeomorphism ψ^{in} of $\partial^{\mathrm{in}} V$ which coincides with ψ_i on the strip B_i and coincides with the identity map out of the union of the B_i. Let us check that ψ^{in} satisfies all the announced properties: it is the identity map on a neighborhood of $\mathcal{L}^{\mathrm{s}}$, preserves every leaf of $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$, and $(\psi^{\mathrm{in}})_*(\mathcal{G}^{\mathrm{s}}_{\mathrm{in}})$ is ε–C^1–close to $\mathcal{G}^{\mathrm{s}}_{\mathrm{in}}$. It remains to control the action of derivative of $\Gamma \circ \psi^{\mathrm{in}}$ on vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$.

- On $\partial^{\mathrm{in}} V \setminus \bigcup_i B_i$, the diffeomorphism ψ^{in} coincides with the identity. Therefore, (3) and the inclusion of $\partial^{\mathrm{in}} V \setminus \bigcup_i B_i$ in $\mathcal{U}^{\mathrm{in}}_{\lambda}$ imply that $\Gamma \circ \psi^{\mathrm{in}}$ expands vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than λ.
- On $B_i \setminus \mathcal{U}^{\mathrm{in}} = R_i$, the diffeomorphism ψ^{in} coincides with ψ_{h_i}. Therefore, (4) implies that $\Gamma \circ \psi^{\mathrm{in}}$ expands vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than λ.
- On $B_i \cap \mathcal{U}^{\mathrm{in}}$, the diffeomorphism ψ^{in} coincides with ψ_i. Therefore (6) and (7) imply that $\Gamma \circ \psi^{\mathrm{in}}$ expands vectors tangent to $\mathcal{G}^{\mathrm{u}}_{\mathrm{in}}$ by a factor larger than λ.

This completes the proof of Proposition 6.2 (and therefore also of Theorem 1.5). □

6.3 Transitivity

The aim of this subsection is to prove Proposition 1.6.

Lemma 6.18 *Every orbit of the Anosov flow given by Theorem 1.5 which is not contained in V has its stable and unstable manifold cutting $\mathcal{L}^{\mathrm{u}}$ and $\mathcal{L}^{\mathrm{s}}$, respectively.*

Proof This orbit cuts $\partial^{\mathrm{out}} V$ such that its stable manifold contains a leaf of the foliation $\mathcal{G}^{\mathrm{s}}_{\mathrm{out}}$, all of whose leaves cut $\mathcal{L}^{\mathrm{u}}$. □

Proof of Proposition 1.6 Lemma 6.18 implies that every orbit γ of the resulting Anosov flow has its stable (resp. unstable) manifold cutting transversely the unstable (resp. stable) manifold of a basic piece of the maximal invariant set Λ in V. The combinatorial transitivity means that, after gluing, all the basic pieces of Λ are related by a cycle, and hence belong to the same basic piece of the Anosov flow. Now the stable and unstable manifolds of γ cut the unstable and stable manifold of this basic piece of the Anosov flow, so that γ belongs to this basic piece. One deduces that the whole manifold is a unique basic piece, which means that the flow is transitive. □

Part II Applications of the gluing theorem

7 MS foliations and filling MS laminations

The purpose of this section is to investigate the geometry of filling MS laminations on closed orientable surfaces. We will define the *combinatorial types* of a filling MS lamination: these are simple combinatorial objects encoding the orientations of the compact leaves of the lamination. Then we will focus on the particular case of MS foliations, and prove that every MS foliation is characterized up to topological equivalence by any of its combinatorial types. This result will play a crucial role in the proofs of Theorems 1.8, 1.10, 1.12 and 1.13.

All along this section, we will consider filling MS laminations on the torus $\mathbb{T}^2$. Indeed, up to diffeomorphism, $\mathbb{T}^2$ is the only closed, connected, orientable surface which carries filling MS laminations. From now on, we assume that an orientation of $\mathbb{T}^2$ is fixed.

7.1 Combinatorial type of a filling MS lamination

Lemma 7.1 *Let $\mathcal{L}$ be a filling MS lamination on $\mathbb{T}^2$. The compact leaves of $\mathcal{L}$, regarded as nonoriented closed curves on $\mathbb{T}^2$, are noncontractible and pairwise freely homotopic.*

Proof According to Lemma 3.17, $\mathcal{L}$ can be completed to an MS foliation $\mathcal{F}$. Lemma 7.1 is a consequence of the Poincaré–Hopf theorem applied to the foliation $\mathcal{F}$. □

Lemma 7.1 allows us to compare the orientations of two compact leaves of a filling MS lamination:

Definition 7.2 (coherently orientated compact leaves) Let $\mathcal{L}$ be a filling MS lamination on $\mathbb{T}^2$, and γ and γ' be some compact leaves of $\mathcal{L}$, endowed with their contracting orientations. We say that γ and γ' are *coherently oriented* if they are freely homotopic, when regarded as oriented closed curves.

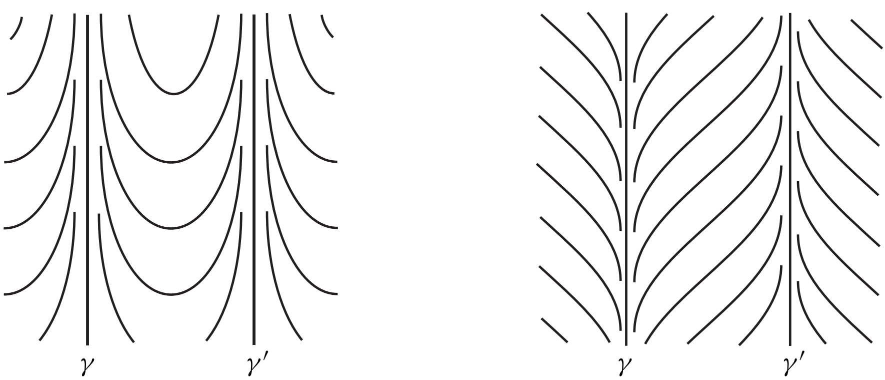

Figure 9: Compact leaves with coherent contracting orientations (left) and compact leaves with incoherent contracting orientations (right)

Lemma 7.1 implies that the compact leaves of a filling MS lamination are "cyclically ordered"; let us formalize this:

Definition 7.3 (geometrical enumeration) Let $\mathcal{L}$ be a filling MS lamination of $\mathbb{T}^2$, with n compact leaves. An enumeration $\gamma_0, \dots, \gamma_{n-1}$ of the compact leaves of $\mathcal{L}$ is called a *geometrical enumeration* if it satisfies the following properties:

- For $i = 0, \dots, n-1$, the leaves γ_i and $\gamma_{(i+1) \bmod n}$ bound a connected component A_i of $\mathbb{T}^2 \setminus \bigcup_k \gamma_k$.
- A_1 is on the right-hand side[11] of γ_0, with respect to the contracting orientation of γ_0.

Definition 7.4 (combinatorial type) Let $\mathcal{L}$ be a filling MS lamination on $\mathbb{T}^2$, and $\gamma_0, \dots, \gamma_{n-1}$ be a geometrical enumeration of the compact leaves of $\mathcal{L}$. The *combinatorial type* of the lamination $\mathcal{L}$ (associated with the enumeration $\gamma_0, \dots, \gamma_{n-1}$) is the map

$$\sigma\colon \{0, \dots, n-1\} \to \{-, +\}$$

defined as follows: $\sigma(i) = +$ if and only if the contracting orientations of γ_i and γ_0 are coherent.[12]

Remark 7.5 Let $\mathcal{L}$ be a filling MS lamination on $\mathbb{T}^2$, with n compact leaves. There are n possible geometrical enumeration of the compact leaves of $\mathcal{L}$. To each geometrical enumeration is associated a combinatorial type of $\mathcal{L}$. These combinatorial types can easily be deduced from one another.

7.2 MS foliations are characterized by their combinatorial types

Definition 7.6 (topological equivalence on oriented surfaces) Let $\mathcal{L}$ and $\mathcal{L}'$ be laminations on oriented surfaces S and S'. We will say that $\mathcal{L}$ and $\mathcal{L}'$ are *topologically equivalent* if there exists an orientation-preserving homeomorphism $h\colon S \to S'$ such that $h_*(\mathcal{L}) = \mathcal{L}'$

Keep in mind that we only consider topological equivalences induced by *orientation-preserving* homeomorphism. Now, let us focus our attention on MS foliations.

Proposition 7.7 *An MS foliation on $\mathbb{T}^2$ is characterized up to topological equivalence by any of its combinatorial types.*

Proof Consider two MS foliations $\mathcal{F}^1$ and $\mathcal{F}^2$ on $\mathbb{T}^2$. Fix some geometrical enumerations $\gamma_0^1, \dots, \gamma_{n-1}^1$ and $\gamma_0^2, \dots, \gamma_{n-1}^2$ of the compact leaves of $\mathcal{F}^1$ and $\mathcal{F}^2$, and denote by σ^1 and σ^2 the corresponding combinatorial types of $\mathcal{F}^1$ and $\mathcal{F}^2$. Assume that $\sigma^1 = \sigma^2$. We will prove that $\mathcal{F}^1$ and $\mathcal{F}^2$ are topologically equivalent.

[11] The orientation of $\mathbb{T}^2$ provides an notion of *local right-hand side of an oriented closed curve*. To include the particular case where $\mathcal{L}$ has a single compact leaf, we allow A_0 to be on both sides of γ_0.

[12] In particular, $\sigma(0)$ is always equal to $+$.

If we endow the compact leaf γ_i^j with its contracting orientation, then the holonomy of $\mathcal{F}^i$ along γ_i^j is a contraction. Therefore, we can find an arbitrarily small tubular neighborhood U_i^j of γ_i^j such that $\mathcal{F}^j$ is transverse to ∂U_i^j. For $j=1,2$, we can assume that the neighborhoods $U_1^j,\dots,U_n^j$ are pairwise disjoint. We denote by A_i^j the connected component of $\mathbb{T}^2 \setminus \bigcup_i \operatorname{int}(U_i^j)$ which lies between U_i^j and U_{i+1}^j. We denote by $\partial^r U_i^j$ (resp. $\partial^\ell U_i^j$) the boundary component of U_i^j which is also a boundary component of A_i^j (resp. A_{i-1}^j). The contracting orientation of the leaf γ_1^j induces an orientation of the closed curves $\partial^\ell U_i^j$ and $\partial^r U_i^j$. Note that $\partial A_i^j = \partial^r U_i^j \cup \partial^\ell U_{i+1}^j$.

Claim 7.8 *For each i, there is an orientation-preserving homeomorphism $\psi_i\colon A_i^1 \to A_i^2$ which maps $\partial^r U_i^1$ and $\partial^\ell U_{i+1}^1$ onto $\partial^r U_i^2$ and $\partial^\ell U_{i+1}^2$, respectively, and is such that $(\psi_i)_*(\mathcal{F}^1)=\mathcal{F}^2$.*

Proof Indeed, A_i^j is a compact annulus, disjoint from the compact leaves of $\mathcal{F}^j$, whose boundary is transverse to $\mathcal{F}^j$. Since every half-leaf of $\mathcal{F}^j$ accumulates on a compact leaf, this implies that the restriction of $\mathcal{F}^j$ to A_i^j is topologically conjugate to the vertical foliation on the annulus $(\mathbb{R}/\mathbb{Z})\times[-1,1]$. The claim follows. □

Claim 7.9 *For each i, there is an orientation-preserving homeomorphism $\phi_i\colon U_i^1 \to U_i^2$ which coincides with ψ_i on $\partial^r U_i^1$ and ψ_{i-1} on $\partial^\ell U_i^1$ and is such that $\phi_*(\mathcal{F}^1)=\mathcal{F}^2$.*

Proof We endow γ_i^1 (resp. γ_i^2) with the orientation that is coherent with the contracting orientation of γ_1^1 (resp. γ_1^2). Since $\sigma^1(i)=\sigma^2(i)$, there are two possibilities: either both the holonomies of γ_i^1 and γ_i^2 are contractions, or both the holonomies of γ_i^1 and γ_i^2 are dilations. Assume, for example, that they both are contractions.

Choose an oriented arc α_i^j in U_i^j, transverse to $\mathcal{F}^j$ and going from $\partial^\ell U_i^j$ to $\partial^r U_i^j$, such that:

- $\psi_i(e(\alpha_i^1)) = e(\alpha_i^2)$ and $\psi_i(s(\alpha_{i+1}^1)) = s(\alpha_{i+1}^2)$ (mod n), where $e(\alpha_i^j)$ and $s(\alpha_i^j)$ are the endpoints of the arc α_i^j.
- $e(\alpha_i^1)$ and $s(\alpha_{i+1}^1)$ are in the same leaf of the restriction of $\mathcal{F}^1$ to A_i^1 (mod n).

The arc α_i^j is a cross-section for the restriction of $\mathcal{F}^j$ to U_i^j. Denote by f_i^j the first return map of the leaves of $\mathcal{F}^j$ on α^j. The maps f_i^1 and f_i^2 are contractions. Hence they are topologically conjugate by an orientation-preserving homeomorphism $h_i\colon \alpha_i^1 \to \alpha_i^2$. One deduces automatically that there exists an orientation-preserving homeomorphism $\phi_i\colon U_i^1 \to U_i^2$ which maps $\partial^\ell U_i^1$ and $\partial^r U_i^1$ onto $\partial^\ell U_i^2$ and $\partial^r U_i^2$, respectively, with $\phi_{i,*}(\mathcal{F}^1)=\mathcal{F}^2$. Note that there is some freedom for the choice of the conjugating homeomorphism h_i: the restriction of h_i to a fundamental domain of the contraction f_i^j can be chosen arbitrarily. As a consequence, ϕ_i can be chosen so that it coincides with ψ_i (resp. ψ_{i-1}) on $\partial^r U_i^1$ (resp. $\partial^\ell U_i^1$). The claim is proved. □

The homeomorphisms $\phi_1, \psi_1, \dots, \phi_n, \psi_n$ provided by the two claims can be glued together to obtain a global orientation-preserving homeomorphism $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $\phi_*(\mathcal{F}^1) = \mathcal{F}^2$. This completes the proof of Proposition 7.7. □

Remark 7.10 Proposition 7.7 is false for filling MS laminations: by removing noncompact leaves to a given filling MS lamination, one can easily find infinitely many filling MS laminations with the same combinatorial type which are pairwise not topologically equivalent. Nevertheless, every filling MS lamination $\mathcal{L}$ can be embedded in an MS foliation $\mathcal{F}$ with the same compact leaves as $\mathcal{L}$, the combinatorial types of $\mathcal{F}$ are the same as those of $\mathcal{L}$, and $\mathcal{F}$ is characterized up to topological equivalence by these combinatorial types.

Definition 7.11 (zipped Reeb lamination/foliation) A *zipped Reeb lamination* (resp. *foliation*) is a filling MS lamination (resp. foliation) on $\mathbb{T}^2$ with a single compact leaf. See Figure 10.

Example 7.12 Consider the vector field X on $\mathbb{R}^2$ defined by

$$X(x,y) := \sin(\pi x)\frac{\partial}{\partial x} + \cos(\pi x)\frac{\partial}{\partial y}.$$

The orbits of this vector field define a foliation on $\mathbb{R}^2$. This foliation is invariant under the standard action of $\mathbb{Z}^2$. Therefore, it induces a foliation on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. One easily checks that this a zipped Reeb foliation.

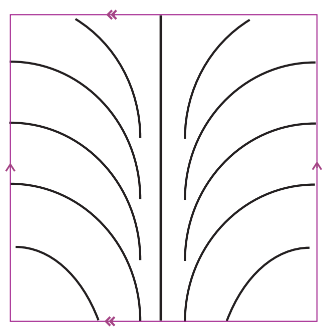

Figure 10: A zipped Reeb lamination

Definition 7.13 Let $\mathcal{F}$ and $\mathcal{F}'$ be MS foliations on $\mathbb{T}^2$. Suppose that $\mathcal{F}'$ has one more compact leaf than $\mathcal{F}$, and suppose that there is a combinatorial type $\sigma\colon \{1,\dots,n\} \to \{+,-\}$ of $\mathcal{F}$ and a combinatorial type $\sigma'\colon \{1,\dots,n+1\} \to \{+,-\}$ of $\mathcal{F}'$ such that $\sigma'|_{\{1,\dots,n\}} = \sigma$. We say that the foliation $\mathcal{F}'$ is obtained by *adding a compact leaf* to $\mathcal{F}$.

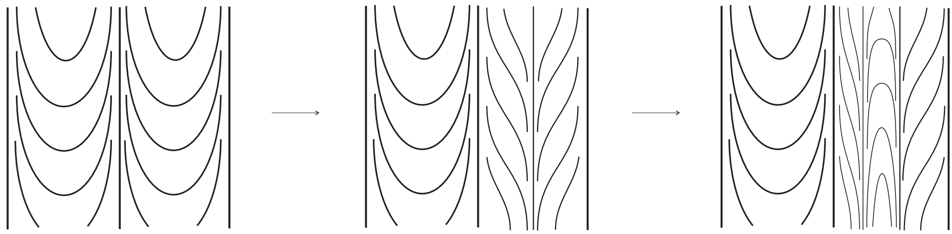

Figure 11: Adding compact leaves to MS foliations

The two following statements are immediate consequences of Proposition 7.7:

Corollary 7.14 *All zipped Reeb foliations are topologically equivalent.*

Corollary 7.15 *Up to topological equivalence, every simple foliation on $\mathbb{T}^2$ can be obtained by adding inductively a finite number of compact leaves to a zipped Reeb foliation.*

7.3 Contracting orientation versus dynamical orientation

Consider a hyperbolic plug (U, X). The compact leaves of the laminations $\mathcal{L}^{\mathrm{s}}_X$ and $\mathcal{L}^{\mathrm{u}}_X$ can be equipped with their contracting orientation. We will define another natural orientation for these compact leaves.

Definition 7.16 (dynamical orientation) Let γ be a compact leaf of the lamination $\mathcal{L}^{\mathrm{u}}_X$. According to (the proof of) Proposition 3.8, there exists a periodic orbit O of X such that γ is a connected component of $W^{\mathrm{u}}(O) \cap \partial^{\mathrm{out}} U$. The orbit O has a natural orientation defined by the vector field X.

- If O has positive multipliers, then $W^{\mathrm{u}}(O)$ is a cylinder, and both γ and O are noncontractible closed curves on this cylinder. The *dynamical orientation* of γ is the orientation for which γ is freely homotopic to the orbit O endowed with its natural orientation, in the cylinder $W^{\mathrm{u}}(O)$. See Figure 12.
- If O has negative multipliers, then $W^{\mathrm{u}}(O)$ is a Möbius band. The *dynamical orientation* of γ is the orientation for which γ is freely homotopic to two times the orbit O endowed with its natural orientation, in the cylinder $W^{\mathrm{u}}(O)$.

We define similarly the dynamical orientation of a compact leaf of $\mathcal{L}^{\mathrm{s}}_X$.

Proposition 7.17 *Let (U, X) be a hyperbolic plug. If γ is a compact leaf of $\mathcal{L}^{\mathrm{u}}_X$, the contracting orientation and the dynamical orientation of γ coincide. If γ is a compact leaf of $\mathcal{L}^{\mathrm{s}}_X$, the contracting orientation and the dynamical orientation of γ are opposite.*

Proof Let Λ be the maximal invariant set of (U, X). Recall that $\mathcal{L}_X^{\mathrm{u}} = W^{\mathrm{u}}(\Lambda) \cap \partial^{\mathrm{out}} U$ and $\mathcal{L}_X^{\mathrm{s}} = W^{\mathrm{s}}(\Lambda) \cap \partial^{\mathrm{in}} U$. The proposition is a consequence of the definitions together with the following fact: if O is a periodic orbit of X, the holonomy of the two-dimensional lamination $W^{\mathrm{u}}(\Lambda)$ along O (where the orbit O is equipped with its natural orientation induced by X) is a contraction, and the holonomy of the two-dimensional lamination $W^{\mathrm{s}}(\Lambda)$ along O is a dilation. □

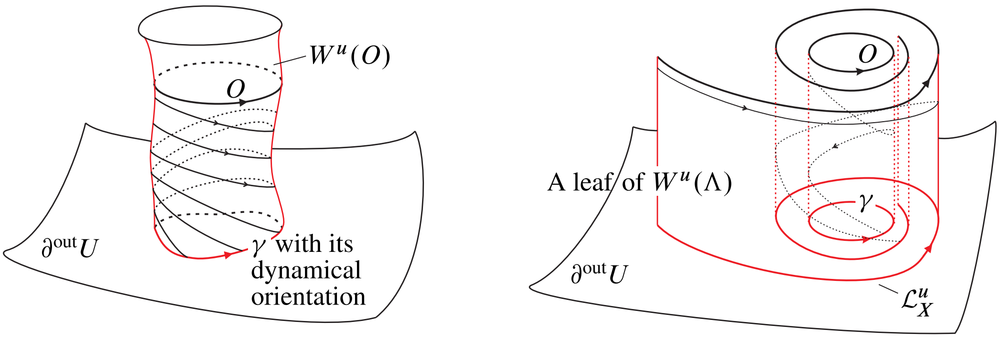

Figure 12: Dynamical orientation (left) and proof of Proposition 7.17 (right)

7.4 Simplification of an MS foliation

The following elementary proposition provides a kind of "normal form" for a filling MS lamination of $\mathbb{T}^2$.

Proposition 7.18 *Let $\mathcal{L}$ be a filling MS lamination of class C^1 on the torus $\mathbb{T}^2$, $\gamma_0, \dots, \gamma_{n-1}$ a geometrical enumeration of its compact leaves and $\sigma\colon \{\gamma_0, \dots, \gamma_{n-1}\} \to \{+,-\}$ a combinatorial type. Write $\sigma_i := \sigma(\gamma_i) \in \{+,-\}$. We endow $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with its standard euclidean coordinates, and we assume that γ_0 is isotopic to $\{0\} \times \mathbb{S}^1$. Then there is a diffeomorphism $\varphi\colon \mathbb{T}^2 \to \mathbb{T}^2$ isotopic to the identity map such that the lamination $\varphi_*(\mathcal{L})$ has the following properties:*

- $\varphi(\gamma_i) = \left\{\frac{i}{n}\right\} \times \mathbb{S}^1$.
- *On the annulus $\left(\frac{i}{n}, \frac{i+1}{n}\right) \times \mathbb{S}^1$, the leaves of the lamination $\varphi_*(\mathcal{L})$ are the graphs of C^1 functions from $\left(\frac{i}{n}, \frac{i+1}{n}\right)$ to $\mathbb{S}^1$; moreover, the derivatives of these functions are*
  1. *positive on the whole interval $\left(\frac{i}{n}, \frac{i+1}{n}\right)$ if $\sigma_i < 0$ and $\sigma_{i+1} > 0$;*
  2. *negative on the whole interval $\left(\frac{i}{n}, \frac{i+1}{n}\right)$ if $\sigma_i > 0$ and $\sigma_{i+1} < 0$;*
  3. *positive on $\left(\frac{i}{n}, \frac{i}{n} + \frac{1}{2}\right)$ and negative on $\left(\frac{i}{n} + \frac{1}{2}, \frac{i+1}{n}\right)$ if $\sigma_i < 0$ and $\sigma_{i+1} < 0$;*
  4. *negative on $\left(\frac{i}{n}, \frac{i}{n} + \frac{1}{2}\right)$ and positive on $\left(\frac{i}{n} + \frac{1}{2}, \frac{i+1}{n}\right)$ if $\sigma_i > 0$ and $\sigma_{i+1} > 0$.*

Proposition 7.18 can be thought of as a kind of "differentiable version" of Proposition 7.7 (indeed, Proposition 7.7 shows that an MS foliation is *fully* characterized *up to topological equivalence* by its combinatorial types, whereas Proposition 7.18 shows that a C^1 filling MS lamination is *partially* characterized *up to differentiable equivalence* by its combinatorial types).

Idea of the proof The proof of Proposition 7.18 roughly follows the same scheme as that of Proposition 7.7. One first embeds $\mathcal{L}$ in an MS foliation $\mathcal{F}$, using Lemma 3.19. Then one chooses a diffeomorphism mapping the compact leaf γ_i on $\{\frac{i}{n}\} \times \mathbb{S}^1$ for every i. To get the normal form on a small tubular neighborhood U_i of the compact leaf γ_i, one uses the fact that a foliation of a surface is C^1–equivalent on a neighborhood of a compact leaf to the suspension of the holonomy of this compact leaf. To conclude, it remains to get the announced normal form on a compact annulus A_i lying between the tubular neighborhoods U_i and U_{i+1}; this is an easy task since the restriction of $\mathcal{F}$ to the compact annulus A_i is a trivial foliation by segments joining one boundary component of A_i to the other one. We leave the details to the reader. □

8 The "blow-up, excise and glue surgery"

The purpose of this section is to describe the "blow-up, excise and glue surgery" which was sketched in the introduction. As immediate applications, we will prove Theorems 1.8 and 1.9.

8.1 DA bifurcations

In his seminal paper [28], S Smale constructed one of the first examples of surface diffeomorphism displaying a one-dimensional hyperbolic attractor. This diffeomorphism was obtained by bifurcating a linear Anosov diffeomorphism of $\mathbb{T}^2$. Smale's construction is known as a *DA bifurcation*.[13] Since then DA bifurcations have been generalized to various contexts, including Axiom A vector fields in dimension 3 (a good reference for this purpose is Ghrist, Holmes and Sullivan [18, Section 2.2.2]).

Given some hyperbolic plug (U, X), one can build another hyperbolic plug (U', X') by performing a DA bifurcation on a periodic orbit of X and excising a small tubular neighborhood of this orbit. We shall describe this operation in detail.

8.1.1 Attracting DA bifurcation on a periodic orbit with positive multipliers We consider a hyperbolic plug[14] with filling MS laminations (U, X), and a periodic orbit

[13] "DA" stands for "derived from Anosov".

[14] Note that the entrance boundary $\partial^{\text{in}} U$ and/or the exit boundary $\partial^{\text{out}} U$ can be empty.

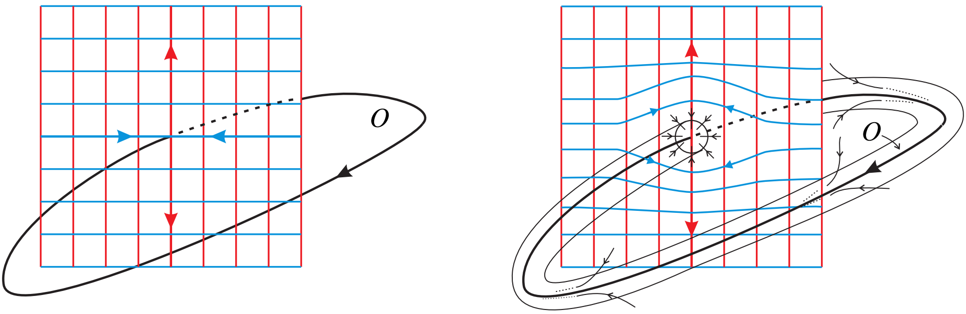

Figure 13: An attracting DA bifurcation

O of the vector field X. We assume that O has positive multipliers. To avoid dealing with some particular cases, we assume moreover that O has no free separatrix.[15] We denote by Λ the maximal invariant set of (U, X).

The vector field X is structurally stable. Therefore, up to perturbing X within its topological equivalence class, we can assume that X is C^1–linearizable on a neighborhood of the periodic orbit O. This means that there exists a coordinate system $(x, y, \theta)\colon V \to [-1,1]\times[-1,1]\times\mathbb{R}/\mathbb{Z}$, defined on a neighborhood $V \subset \mathrm{int}(U)$ of the orbit O, such that

$$X(x,y,\theta) = \lambda x\frac{\partial}{\partial x} + \mu y\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}$$

for some constants $\lambda < 0 < \mu$. For $0 < \eta < 1$, we consider the vector field X which vanishes on $U \setminus V$, and which is defined on V by

$$Y_\eta(x,y,\theta) = -2\mu y\phi\Big(\frac{x}{\eta}\Big)\phi\Big(\frac{y}{\eta}\Big)\frac{\partial}{\partial y},$$

where $\phi\colon [-1,1] \to \mathbb{R}^+$ is the bump function defined by $\phi(t) = (1-t^2)^2\mathbf{1}_{[-1,1]}(t)$. Then we consider the vector field

$$X' := X + Y_\eta.$$

A straightforward computation shows that O is an attracting hyperbolic periodic orbit for the vector field X'. We say that *X' is derived from X by an attracting DA bifurcation on the orbit O*.

[15]Recall that a *stable separatrix* of O is a connected component of $W^s_X(O)\setminus O$. Since O has positive multipliers, $W^s_X(O)$ is a cylinder and O has two stable separatrices. A stable separatrix is said to be *free* if it is disjoint from the maximal invariant set of (U, X). *Free unstable separatrices* are defined in the same way. Note that the assumption "O has no free separatrix" is not very restrictive since it is satisfied by all but finitely many periodic orbits. See the proof of Proposition 3.8.

Now we pick a (small) real number $\epsilon > 0$, and we consider the solid torus $T \subset V$ defined by

$$T := \{(x, y, \theta) \mid x^2 + y^2 < \epsilon^2\}.$$

Obviously, T is a tubular neighborhood of the periodic orbit O. We assume that ϵ is small enough that the two following properties hold: T is included in the basin of attraction (for X') of O, and X' is transverse to ∂T. We consider the manifold with boundary

$$U' := U \setminus T$$

endowed with the vector field X'. Note that X' is transverse to $\partial U'$ (since X' is transverse to ∂T and $X' = X$ on $\partial U' \setminus \partial T = \partial U$). Hence (U', X') is a plug. The following proposition summarizes the relationships between the plugs (U, X) and (U', X'):

Proposition 8.1 *The plug (U', X') satisfies the following properties:*

(1) $U' = U \setminus T$*, where T is a tubular neighborhood of a periodic orbit of X.*

(2) $\partial^{\text{in}} U' = \partial^{\text{in}} U$ *and* $\partial^{\text{out}} U' = \partial^{\text{out}} U \cup \partial T$*.*

Moreover, if η is small enough:

(3) (U', X') *is a hyperbolic plug: the maximal invariant set Λ' of (U', X') is a hyperbolic set.*

(4) $(X')|_{\Lambda'}$ *is a topological extension of $X|_\Lambda$: there exists a continuous onto map $\pi\colon \Lambda' \to \Lambda$ inducing a semiconjugacy between a reparametrization of the flow of X' and the flow of X. Moreover, π is "almost one-to-one": the set $\pi^{-1}(x)$ is a single point for every $x \in \Lambda \setminus W^{\text{s}}_X(O)$.*

(5) *If (U, X) is a transitive plug, then so is (U', X').*

(6) $\mathcal{L}^{\text{s}}_{X'}$ *is a filling MS lamination, with the same combinatorial types as $\mathcal{L}^{\text{s}}_X$.*

(7) $\mathcal{L}^{\text{u}}_{X'} \cap \partial^{\text{out}} U$ *is a filling MS lamination, topologically equivalent to $\mathcal{L}^{\text{u}}_X$.*

(8) $\mathcal{L}^{\text{u}}_{X'} \cap \partial T$ *is a filling MS lamination with two coherently oriented compact leaves (Figure 14).*

Proof Let us start by setting some notations. Recall that the periodic orbit O corresponds to the circle $(x = y = 0)$ in the (x, y, θ) coordinate system. Recall that O is a saddle hyperbolic orbit for the vector field X, and an attracting hyperbolic orbit for the vector field X'. We denote by $B := W^{\text{s}}_{X',V}(O)$ the basin of attraction of O for the vector field X'. We denote by $O_\pm$ the circle $(x = 0, y = \pm\delta)$, where

δ is the unique positive solution of the equation $\phi(y) = \frac{1}{2}$. Straightforward computations show that O_- and O_+ are saddle hyperbolic periodic orbits for the vector field X'. One can easily check that, in the (x, y, θ) coordinate system, the local stable manifolds $W^{\mathrm{s}}_{X',V}(O^-)$ and $W^{\mathrm{s}}_{X',V}(O^+)$ are "horizontal" graphs, and the local basin $W^{\mathrm{s}}_{X',V}(O)$ is the open band between the graphs $W^{\mathrm{s}}_{X',V}(O^-)$ and $W^{\mathrm{s}}_{X',V}(O^+)$. It follows that the accessible boundary of B is precisely $W^{\mathrm{s}}_{X'}(O_-) \cup W^{\mathrm{s}}_{X'}(O_+)$. We set $\widehat{B} := B \cup W^{\mathrm{s}}_{X'}(O_-) \cup W^{\mathrm{s}}_{X'}(O_+)$

Items (1) and (2) follow immediately from the construction of U' and X'.

Items (3) and (4) are consequences of well-known properties of DA bifurcations. Let us give more details. Using classical techniques of hyperbolic theory, one can prove that, for η small enough, the maximal invariant set Λ' of (U', X') is a saddle hyperbolic set. The very rough idea is the following. Denote by W the support of the vector field $Y_\eta = X' - X$, and observe that W is contained in a solid torus which gets thinner and thinner when η goes to 0. Therefore, when η is very small, every orbit spends a long time outside W between two visits of W. Therefore, the possible loss of hyperbolicity in W is counterbalanced by the hyperbolicity outside W. See [18, Section 2.2.2] for a detailed proof. Moreover, one can prove that the vector field X' is a topological extension of the vector field X: there exists a continuous onto map $\pi\colon U \to U$, inducing a semiconjugacy between a reparametrization of the flow of X' and the flow of X. Moreover, the map π admits a concrete description: it "squashes $\widehat{B}$ onto $W^{\mathrm{s}}_X(O)$". More precisely,

- π maps $\widehat{B}$ on $W^{\mathrm{s}}_X(O)$, and maps $U \setminus \widehat{B}$ on $U \setminus W^{\mathrm{s}}_X(O)$;
- $\pi\colon U \setminus \widehat{B} \to U \setminus W^{\mathrm{s}}_X(O)$ is a homeomorphism;
- for $x \in W^{\mathrm{s}}_X(O)$, the set $\pi^{-1}(x)$ is an arc crossing $\widehat{B}$ from $W^{\mathrm{s}}_{X'}(O_-)$ to $W^{\mathrm{s}}_{X'}(O_-)$.

In particular, the restriction $\pi\colon \Lambda' \to \Lambda$ is onto, and $\pi\colon \Lambda' \setminus \widehat{B} \to \Lambda \setminus W^{\mathrm{s}}_X(O)$ is a homeomorphism. See again [18, Section 2.2.2] for a detailed proof. Items (3) and (4) follow.

Let us prove (5). Assume that (U, X) is a transitive plug. By definition, this means that Λ is a transitive hyperbolic set for X. Note that Λ is not a single orbit since we have assumed that the orbit O has no free separatrix. Hence, we can find an orbit Q of X such that $Q \subset \Lambda \setminus W^{\mathrm{s}}_X(O)$ and such that Q is dense in Λ. We have seen above that $\pi\colon \Lambda' \setminus \widehat{B} \to \Lambda \setminus W^{\mathrm{s}}_X(O)$ is an homeomorphism. It follows that the set $\Lambda' \setminus \widehat{B}$ is topologically transitive for the vector field X'. On the other hand, since O has no free unstable separatrix, $W^{\mathrm{s}}_X(O)$ is accumulated on both sides by leaves of $W^{\mathrm{s}}_X(\Lambda) \setminus W^{\mathrm{s}}_X(O)$. Using the properties of the map π, it follows that neither $W^{\mathrm{s}}_{X'}(O_-)$ nor $W^{\mathrm{s}}_{X'}(O_+)$ is isolated in $W^s_{X'}(\Lambda')$. In other words, $\Lambda'\setminus\widehat{B}=\Lambda'\setminus(W^s_{X'}(O_-)\cup W^s_{X'}(O_+))$ is dense in Λ'. Hence, Λ' is transitive (for the vector field X'). By definition, this means that (U',X') is a transitive plug.

Let us turn to (6). Recall that the orbit O has no free separatrix. According to the proof of Proposition 3.8, this implies that $W^s_X(O)$ does not contain any compact leaf of the lamination $\mathcal{L}^s_X$. The map π induces a homeomorphism from $\mathcal{L}^s_{X'}\setminus\widehat{B}$ to $\mathcal{L}^s_X\setminus W^s_X(O)$. Moreover, if γ is a (noncompact) leaf of $\mathcal{L}^s_X\cap W^s_X(O)$, then $\pi^{-1}(\gamma)$ is a strip, bounded by two (noncompact) leaves of $\mathcal{L}^s_{X'}$, whose interior is contained in the basin B (hence disjoint from $\mathcal{L}^s_{X'}$). Item (6) follows.

Now we prove (7). The surface $\partial^{\mathrm{out}}U$ is disjoint from the basin B since every orbit of X' in B must accumulate on O in the future, and therefore must exit from U' by crossing ∂T. The surface $\partial^{\mathrm{out}}U$ is also disjoint from the stable manifolds $W^s_{X'}(O_-)$ and $W^s_{X'}(O_+)$ (since every orbit in $W^s_{X'}(O_-)$ and $W^s_{X'}(O_+)$ accumulates on O_+ and O_- in the future, and therefore remains in U' forever). Hence, $\partial^{\mathrm{out}}U$ is disjoint from $\widehat{B}=B\sqcup W^s_{X'}(O_-)\sqcup W^s_{X'}(O_-)$. But we know that π is a homeomorphism on the complement of $\widehat{B}$. Hence, π induces a topological equivalence between the laminations $\mathcal{L}^u_{X'}\cap\partial^{\mathrm{out}}U$ and $\mathcal{L}^u_X\cap\partial^{\mathrm{out}}U=\mathcal{L}^u_X$.

We are left to prove (8). Let $\gamma_\pm$ be the circle $\{x=0,\ y=\pm\epsilon\}$ in the (x,y,θ) coordinate system. Let W_+ be the cylinder $\{x=0,\ y>0\}$ and W_- be the cylinder $\{x=0,\ y>0\}$. It is easy to check that the cylinder $W_\pm$ is contained in $W^u_{X'}(O_\pm)$. It follows that the circles γ_+ and γ_- are compact leaves of the lamination $\mathcal{L}^s_{X'}\cap\partial T$. On the other hand, let γ be a compact leaf of $\mathcal{L}^s_{X'}\cap\partial T$. According to the proof of Proposition 3.8, $\gamma=W\cap\partial T$, where W is a free unstable separatrix of a periodic orbit $P\subset\Lambda'$. Since ∂T is contained in the basin B, the separatrix W must be contained in B, and the orbit P must be contained in the accessible boundary of B. But O_+ and O_- are the only periodic orbits in the accessible boundary of B. Hence, the separatrix W must be equal to either W_+ or W_-. As a further consequence, the compact leaf γ must be equal to either γ_+ or γ_-. So we have proved that the circles γ_+ and γ_- are the only compact leaves of the lamination $\mathcal{L}^u_{X'}\cap\partial T$. For later use, note that these compact leaves are not homotopic to 0 in the torus ∂T.

By assumption, the periodic orbit O has no free stable separatrix. Hence $W^u_X(O)$ is accumulated on both sides by leaves of $W^u_X(\Lambda)$. Using the properties of the map π, it follows that $W^u_{X'}(O_\pm)$ is accumulated on both sides by leaves of $W^u_{X'}(\Lambda')$. As a further consequence, the compact leaf $\gamma_\pm$ is accumulated on both sides by noncompact leaves of $\mathcal{L}^s_{X'}\cap\partial T$.

The surface ∂T is a torus, no compact leaf of $\mathcal{L}^s_{X'}\cap\partial T$ is homotopic to 0, and every compact leaf of $\mathcal{L}^s_{X'}\cap\partial T$ is accumulated on both sides by noncompact leaves of

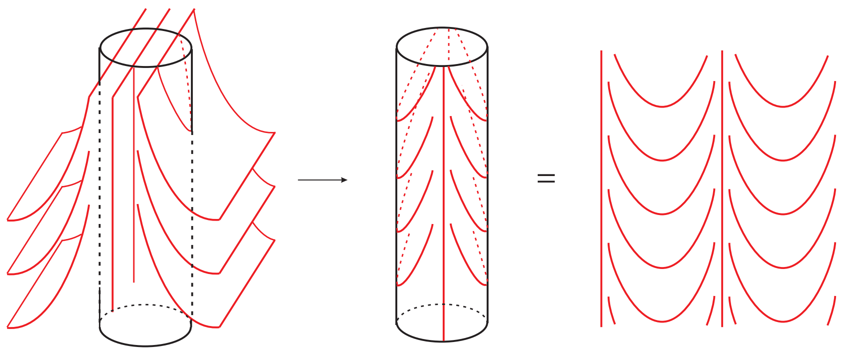

Figure 14: The exit lamination $\mathcal{L}^{\mathrm{u}}_{X'} \cap \partial T$

$\mathcal{L}^{\mathrm{s}}_{X'} \cap \partial T$. It follows that every connected component of $\partial T \setminus \mathcal{L}^{\mathrm{s}}_{X'}$ is a strip bounded by two leaves of $\mathcal{L}^{\mathrm{s}}_{X'}$ which are asymptotic to each other at both ends. Hence, $\mathcal{L}^{\mathrm{s}}_{X'} \cap \partial T$ is a filling MS lamination.

Using explicit formula for the vector field X', one easily checks that the dynamical orientation of the compact leaf $\gamma_{\pm}$ coincides with the orientation induced by the vector field $\frac{\partial}{\partial\theta}$. According to Proposition 7.17, the attracting orientation of the leaf $\gamma_{\pm}$ coincides with its dynamical orientation. It follows that the attracting orientation of γ_- and γ_+ are coherent. The proof is complete. □

Remark 8.2 For later use, we note that, in the (x, y, θ) coordinate system, the two compact leaves of the lamination $\mathcal{L}^{\mathrm{u}}_{X'} \cap \partial T$ are the circles $\{x = 0,\ y = \pm\epsilon\}$. Moreover, the attracting orientation of these leaves is the orientation induced by the vector field $\frac{\partial}{\partial\theta}$ (see the proof of Proposition 8.1).

8.1.2 Attracting DA bifurcation on orbits with negative multipliers

In the preceding subsection, the orbit O was assumed to have positive multipliers. Actually, we can also make an attracting DA bifurcation on a periodic orbit O with *negative multipliers*. In this case, the stable manifold $W^{\mathrm{s}}_X(O)$ is a Möbius band, and Proposition 8.1 must be replaced by the following statement:

Proposition 8.3 *The same as Proposition 8.1, except for (8), which is replaced by:*

(8′) $\mathcal{L}^{\mathrm{u}}_{X'} \cap \partial T$ *is a filling MS lamination with a single compact leaf, ie a zipped Reeb lamination.*

8.1.3 Repelling DA bifurcations

Instead of an attracting DA bifurcation, it is also possible to make *repelling DA bifurcation* on a periodic orbit O. This bifurcation creates a repelling periodic orbit, instead of an attracting one. As in Section 8.1.1, one can excise a tubular neighborhood of this repelling periodic orbit, and get a hyperbolic plug (U', X'). The properties of this hyperbolic plug are analogous to those listed in Propositions 8.1 and 8.3, after having exchanged the roles of the stable and the unstable directions, and the roles of the entrance and exit boundaries. Remark 8.2 must replaced by the following statement:

Remark 8.4 In the (x, y, θ) coordinate system, the compact leaves of the lamination $\mathcal{L}^s_{X'} \cap \partial T$ are the circles $\{x = \pm\epsilon,\ y = 0\}$. Moreover, the attracting orientation of these leaves coincides with the orientation induced by the vector field $-\frac{\partial}{\partial\theta}$.

8.2 The "blow-up, excise and glue surgery"

We will now explain what we call the "blow-up, excise and glue surgery", and prove Theorem 1.8. Recall that this theorem states that, for a given transitive Anosov vector field X on a closed orientable 3–manifold M, there exists a transitive Anosov vector field Z on a closed orientable 3–manifold N such that "the dynamics of Z is richer than those of X". By such we mean that there exists a proper compact invariant set $\Lambda \subsetneq N$ and a continuous surjective map $\pi\colon \Lambda \to M$ such that $\pi \circ X^t = Z^t \circ \pi$ for every $t \in \mathbb{R}$.

We shall need the following lemma:

Lemma 8.5 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be filling MS laminations on $\mathbb{T}^2$ with the same number of compact leaves. Assume that all the compact leaves of $\mathcal{L}_1$ and $\mathcal{L}_2$ are coherently oriented (see Definition 7.2). Then there exists an orientation-preserving diffeomorphism $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $\phi_*(\mathcal{L}_1)$ is strongly transverse to $\mathcal{L}_2$. If $\mathcal{L}_1 = \mathcal{L}_2$, then ϕ can be chosen isotopic to the identity.*

Proof Let n be the number of compact leaves of $\mathcal{L}_1$ and $\mathcal{L}_2$. By Proposition 7.18, there exist two diffeomorphisms $\phi_1, \phi_2\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that, for $i = 1, 2$:

- The compact leaves of the lamination $(\phi_i)_*(\mathcal{L}_i)$ are the vertical circles $\left\{\frac{i}{n}\right\} \times \mathbb{S}^1$ for $i = 0, \dots, n-1$.
- In the open annulus $\left(\frac{i}{n}, \frac{i+1}{n}\right) \times \mathbb{S}^1$, the leaves of the lamination $(\phi_i)_*(\mathcal{L}_i)$ are the graphs of C^1 functions from $\left(\frac{i}{n}, \frac{i+1}{n}\right)$ to $\mathbb{S}^1$; moreover, the derivatives of these functions are strictly positive on $\left(\frac{i}{n}, \frac{i}{n} + \frac{1}{2}\right)$, strictly negative on $\left(\frac{i}{n} + \frac{1}{2}, \frac{i+1}{n}\right)$ and vanish precisely on $\left\{\frac{i}{n} + \frac{1}{2}\right\}$.

Let ϕ be the diffeomorphism of $\mathbb{T}^2$ given by $\psi_0(x, y) = \left(x + \frac{1}{2n}, y\right)$. Let $\phi := (\phi_2)^{-1} \circ \psi \circ \phi_1$. One easily checks that $\phi_*(\mathcal{L}_1)$ is strongly transverse to $\mathcal{L}_2$. If $\mathcal{L}_1 = \mathcal{L}_2$, then one may take $\phi_1 = \phi_2$, which implies that ϕ is isotopic to the identity. □

We begin with a transitive Anosov vector field X on a closed three-manifold M. We consider two distinct periodic orbits O and O' of X, both of which have positive multipliers.[16] We proceed as follows:

Step 1 (blow-up) We consider a vector field $X' = X'_\eta$ on M, derived from X by an attracting DA bifurcation on the orbit O (see Section 8.1). Note that O is an attracting hyperbolic periodic orbit for this new vector field X'.

Step 2 (excise) As in Section 8.1, we consider a tubular neighborhood T of the attracting orbit of O' such that T is included in the basin of attraction of O', and such that X' is transverse to ∂T. We set $U' := M \setminus T$. Clearly, (U', X') is a repelling hyperbolic plug with $\partial^{\mathrm{out}} U' = \partial T$. According to Proposition 8.1(8), $\mathcal{L}^{\mathrm{u}}_{X'}$ is a filling MS lamination with two coherently oriented compact leaves. We denote by Λ' the maximal invariant set of X'. According to Proposition 8.1(4), there is a continuous onto map $\pi\colon \Lambda' \to M$ inducing a semiconjugacy between a reparametrization of the flow of X' and the flow of X. Since $O' \not\subseteq W^{\mathrm{s}}_X(O)$, the preimage $\pi^{-1}(O')$ is a periodic orbit of X'. In other words, we can (and we will) regard O' as an orbit of X'.

Step 1′ (blow-up) Now, we consider a vector field X'' on U', derived from X' by a repelling DA bifurcation on the orbit O'. Note that O' is a repelling hyperbolic periodic orbit for X''.

Step 2′ (excise) We consider a tubular neighborhood T' of the repelling orbit of O' such that T' is included in the basin of repulsion of O', and such that X'' is transverse to $\partial T'$. We set $U'' := U' \setminus T'$. Then (U'', X'') is a hyperbolic plug with $\partial^{\mathrm{out}} U'' = \partial^{\mathrm{out}} U' = \partial T$ and $\partial^{\mathrm{in}} U'' = \partial T'$. We denote by Λ'' the maximal invariant set of X'. According to Proposition 8.1(6), the lamination $\mathcal{L}^{\mathrm{u}}_{X''}$ is a filling MS lamination with the same combinatorial type as $\mathcal{L}^{\mathrm{u}}_{X'}$, ie $\mathcal{L}^{\mathrm{u}}_{X''}$ has two coherently oriented compact leaves. According to Proposition 8.1(8), the lamination $\mathcal{L}^{\mathrm{s}}_{X''}$ is also a filling MS lamination with two coherently oriented compact leaves. According to Proposition 8.1(4), there is a continuous onto map $\pi'\colon \Lambda'' \to \Lambda'$ inducing a semiconjugacy between a reparametrization of the flow of X'' and the flow of X'.

Step 3 (glue) The laminations $\mathcal{L}^{\mathrm{s}}_{X''}$ and $\mathcal{L}^{\mathrm{u}}_{X''}$ satisfy the hypothesis of Lemma 8.5. Hence we can find an orientation-preserving diffeomorphism of $\phi\colon \partial^{\mathrm{out}} U'' \to \partial^{\mathrm{in}} U''$ such that $\phi_*(\mathcal{L}^{\mathrm{u}}_{X''})$ is strongly transverse to $\mathcal{L}^{\mathrm{s}}_{X''}$. By Proposition 8.1, (U'', X'') is a hyperbolic plug. Moreover, since X is transitive, the maximal invariant set of (U'', X'') is also transitive, and, therefore, (U'', X'') is a saddle hyperbolic plug. We consider the closed manifold $N := U''/\phi$ and the vector field Z induced by X'' on N. According to Theorem 1.5, up to modifying X'' by a topological equivalence and ϕ by a strongly transverse isotopy, Z is Anosov.

[16] The surgery can also be made if O and O' both have negative multipliers, but we will not need it.

We say that the Anosov vector field (N, Z) are derived from the Anosov vector field (M, X) by a *"blow-up, excise and glue surgery"*.

Lemma 8.6 *If X is transitive, then so is Z.*

Proof Assume that X is transitive. According to Proposition 8.1(5), it follows that $X''|_{\Lambda''}$ is also transitive. Now, using Proposition 1.6, we deduce that Z is transitive. □

Lemma 8.7 *The dynamics of the new vector field Z is "richer" than the dynamics of the initial vector field X. More precisely, there exists a compact subset Λ'' of N, which is invariant under the flow of Z, and a continuous onto map $\pi' \circ \pi \colon \Lambda'' \to M$ inducing a semiconjugacy between some reparametrization of the flow of $Z|_{\Lambda''}$ on the flow of X.*

Proof The set Λ'' and the maps π' and π were defined above. Observe that Λ'' can indeed be seen as a subset of N, since $\Lambda'' \subset \mathrm{int}(U'') \subset N$. Moreover, the vector field Z coincides with X'' on Λ''. The lemma follows from the properties of the maps π and π'. □

Proof of Theorem 1.8 The theorem immediately follows from the construction above and Lemmas 8.6 and 8.7. □

8.3 A transitive and a nontransitive Anosov vector field on the same manifold

The "blow-up, excise and glue" surgery described in the previous paragraph admits many variants. We shall use one of these variants to prove Theorem 1.9, ie to construct a closed three-manifold N supporting both a nontransitive Anosov vector field Y and a transitive vector field Z.

Proof of Theorem 1.9 We start with a transitive Anosov vector field X on a closed manifold M. We pick two periodic orbits O and O' of X with positive multipliers. Then we consider four vector fields $X_1, \dots, X_4$ on M which are derived from X by DA bifurcations on O and O'. More precisely:

- X_1 is obtained by an attracting DA bifurcation on O and an attracting DA bifurcation on O'.
- X_2 is obtained by a repelling DA bifurcation on O and a repelling DA bifurcation on O'.
- X_3 is obtained by an attracting DA bifurcation on O and a repelling DA bifurcation on O'.
- X_4 is obtained by a repelling DA bifurcation on O and a attracting DA bifurcation on O'.

Observe that O is an attracting orbit for X_1 and X_3 and a repelling periodic orbit for X_2 and X_4, whereas O' is an attracting orbit for X_1 and X_4 and a repelling orbit for X_2 and X_3. We can find some tubular neighborhoods T and T' of O and O', respectively, such that T and T' are contained in the basins[17] of O and O', respectively, and such that the four vector fields $X_1, \dots, X_4$ are transverse to ∂T and $\partial T'$. We consider the manifold with boundary $U := M \setminus (\mathrm{int}(T) \cup \mathrm{int}(T'))$. Note that (U, X_1), (U, X_2), (U, X_3) and (U, X_4) are hyperbolic plugs (Proposition 8.1(3)).

We construct a nontransitive Anosov vector field Y by gluing the hyperbolic plugs (U, X_1) and (U, X_2). The periodic orbits O and O' are attracting for X_1. Hence (U, X_1) is a repelling hyperbolic plug: $\partial^{\mathrm{out}}_{X_1} U = \partial U = \partial T \cup \partial T'$. On the other hand, the periodic orbits O and O' are repelling for X_2. Hence, (U, X_2) is an attracting hyperbolic plug: $\partial^{\mathrm{in}}_{X_2} U = \partial U = \partial T \cup \partial T'$. According to Proposition 8.1(8), $\mathcal{L}^{\mathrm{s}}_{X_1} \cap \partial T$, $\mathcal{L}^{\mathrm{s}}_{X_1} \cap \partial T'$, $\mathcal{L}^{\mathrm{u}}_{X_2} \cap T$ and $\mathcal{L}^{\mathrm{u}}_{X_2} \cap T'$ are MS foliations with two coherently oriented compact leaves. Lemma 8.5 provides an orientation-preserving diffeomorphism

$$\phi\colon \partial^{\mathrm{out}}_{X_2} U = \partial U \to \partial^{\mathrm{in}}_{X_1} U = \partial U$$

such that $\phi_*(\mathcal{L}^{\mathrm{u}}_{X_1})$ is transverse to $\mathcal{L}^{\mathrm{s}}_{X_2}$. We consider the closed manifold $N_\phi := (U \sqcup U)/\phi$. The vector fields X_1 and X_2 induce a vector field Y on N_ϕ. According to Proposition 1.1, (N_ϕ, Y) is a hyperbolic plug. Since $\partial N_\phi = \varnothing$, this means that Y is an Anosov vector field. Note that Y is not transitive, since (N_ϕ, Y) was constructed by gluing an attracting plug and a repelling plug.

Now, we construct a transitive Anosov vector field Z by gluing the hyperbolic plugs (U, X_3) and (U, X_4). Recall that O is an attracting orbit for X_3 and a repelling orbit for X_4, whereas O' is a repelling orbit for X_3 and an attracting orbit for X_4. Therefore, $\partial^{\mathrm{in}}_{X_3} U = \partial^{\mathrm{out}}_{X_4} U = \partial T'$ and $\partial^{\mathrm{out}}_{X_3} U = \partial^{\mathrm{in}}_{X_4} U = \partial T$. According to Proposition 8.1(8), $\mathcal{L}^{\mathrm{s}}_{X_3}$, $\mathcal{L}^{\mathrm{u}}_{X_3}$, $\mathcal{L}^{\mathrm{s}}_{X_4}$ and $\mathcal{L}^{\mathrm{u}}_{X_4}$) are filling MS laminations with two coherently oriented compact leaves. Lemma 8.5 provides an orientation-preserving diffeomorphism

$$\psi\colon (\partial^{\mathrm{out}}_{X_3} U \sqcup \partial^{\mathrm{out}}_{X_4} U) = \partial U \to (\partial^{\mathrm{in}}_{X_3} U \sqcup \partial^{\mathrm{in}}_{X_4} U) = \partial U$$

such that $\psi_*(\mathcal{L}^{\mathrm{u}}_{X_3})$ is strongly transverse to $\mathcal{L}^{\mathrm{s}}_{X_4}$ on ∂T, and $\psi_*(\mathcal{L}^{\mathrm{u}}_{X_4})$ is strongly transverse to $\mathcal{L}^{\mathrm{s}}_{X_3}$ on $\partial T'$. We consider the closed manifold $N_\psi := (U \sqcup U)/\psi$. The vector fields X_3 and X_4 induce a vector field Z on N_ψ. According to Theorem 1.5, up to perturbing X_3 and X_4 by a small topological equivalence and ψ by a strongly transverse isotopy, we can assume that Z is an Anosov vector field.

Let us prove that the vector field Z is transitive. Since the initial Anosov vector field X is transitive, Proposition 8.1(5) ensures that the maximal invariant set Λ_3

[17] With respect to each of the four vector fields $X_1, \dots, X_4$.

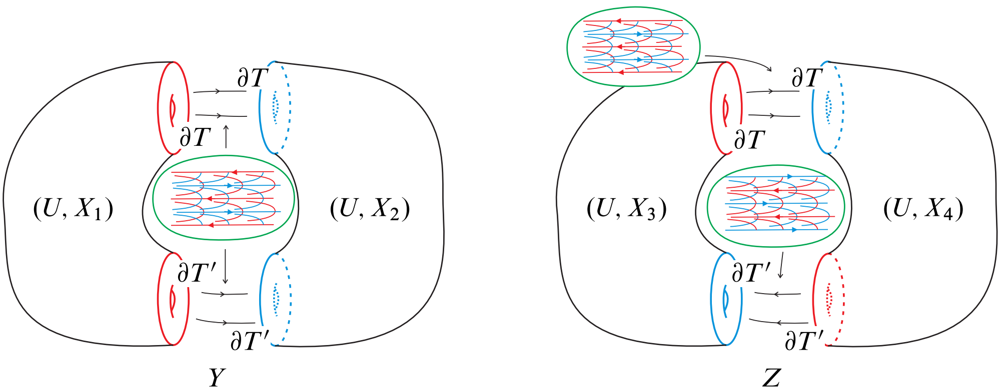

Figure 15: Construction of the Anosov vector fields Y and Z

of (U, X_3) and the maximal invariant set Λ_4 of (U, X_4) are both transitive. Moreover, $\psi_*(\mathcal{L}^{\mathrm{u}}_{X_4}) = \psi_*(W^{\mathrm{u}}_{X_4}(\Lambda_4)) \cap \partial^{\mathrm{out}}_{X_4} U$ intersects $\mathcal{L}^{\mathrm{s}}_{X_3} = W^{\mathrm{s}}_{X_3}(\Lambda_3) \cap \partial^{\mathrm{in}}_{X_3} U$, and $\psi_*(\mathcal{L}^{\mathrm{u}}_{X_3}) = \psi_*(W^{\mathrm{u}}_{X_3}(\Lambda_3)) \cap \partial^{\mathrm{out}}_{X_3} U$ intersects $\mathcal{L}^{\mathrm{s}}_{X_4} = W^{\mathrm{s}}_{X_4}(\Lambda_4) \cap \partial^{\mathrm{in}}_{X_4} U$. Hence, Proposition 1.6 guarantees that Z is transitive.

The proof of Lemma 8.5 together with Remarks 8.2 and 8.4 imply that the maps ϕ and ψ can be chosen isotopic to $-\mathrm{Id}$ on each of the two connected components of $\partial U \simeq \mathbb{T}^2 \sqcup \mathbb{T}^2$. As a consequence, the gluing maps ϕ and ψ can be chosen in such a way that the manifolds N_ϕ and N_ψ are diffeomorphic. As a further consequence, Y and Z can be regarded as vector fields on the same manifold N. The proof is complete. □

9 Attractors with prescribed entrance foliation

Let (V, X) be an oriented plug (ie a hyperbolic plug such that V is oriented). Then the entrance boundary $\partial^{\mathrm{in}} V$ inherits a canonical orientation, characterized by the following property: if (e_1, e_2) is a basis of the tangent space $T_p \partial^{\mathrm{in}} V$ of $\partial^{\mathrm{in}} V$ at some point p, then (e_1, e_2) is a direct basis of $T_p \partial^{\mathrm{in}} V$ if and only if $(e_1, e_2, X(p))$ is a direct basis of $T_p V$. The canonical orientation of the exit boundary $\partial^{\mathrm{out}} V$ is defined similarly.

Definition 9.1 Let $\mathcal{F}$ be an MS foliation on a closed oriented surface S and (U, X) be an oriented attracting hyperbolic plug. If the entrance foliation $\mathcal{L}^{\mathrm{s}}_X$ is topologically equivalent to $\mathcal{F}$, then we say that $\mathcal{F}$ is *realized* by the plug (U, X).

The purpose of the present section is to prove Theorem 1.10, which states that every MS foliation (on a closed oriented surface) can be realized by a transitive attracting hyperbolic plug. For pedagogical results, we will first prove a weaker result (Proposition 9.2).

9.1 Nontransitive attracting plugs with prescribed entrance foliation

As a first step towards Theorem 1.10, we will prove the following proposition:

Proposition 9.2 *Assume that some orientation of $\mathbb{T}^2$ has been fixed. Any MS foliation on $\mathbb{T}^2$ can be realized by a (not necessarily transitive) attracting hyperbolic plug.*

The proof of Proposition 9.2 relies on the results of Section 7. In particular, we will use the fact that every MS foliation on $\mathbb{T}^2$ can be obtained by adding compact leaves to a zipped Reeb foliation (Corollary 7.15).

Lemma 9.3 *There exists an oriented (transitive) attracting hyperbolic plug (U, X) whose entrance boundary $\partial^{\text{in}} U$ is connected and whose entrance foliation $\mathcal{L}^{\text{s}}_X$ is a zipped Reeb foliation* (see Definition 7.11).

Proof Let X_0 be a transitive Anosov vector field on a closed oriented three-manifold M such that X_0 has some periodic orbits with negative multipliers. For example, X_0 can be the suspension of the linear Anosov automorphism $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by $A(x, y) = (-2x - y, -x - y)$. Choose a periodic orbit O of X_0 such that O has negative multiplier. Make a repelling DA bifurcation on O (see Section 8.1). This gives rise to a vector field X on M, for which O is a repelling hyperbolic periodic orbit. As in Section 8.1, consider a tubular neighborhood T of O, such that T is contained in the basin of O, and such that X is transverse to ∂T. Set $U := M \setminus T$. Proposition 8.3 shows that (U, X) satisfies the required properties. □

Remark 9.4 More generally, for every $p \geq 1$, there exists an oriented transitive attracting hyperbolic plug (U, X), whose entrance boundary $\partial^{\text{in}} U$ has p connected components, and such that the restriction of the entrance foliation $\mathcal{L}^{\text{s}}_X$ to each connected component of $\partial^{\text{in}} U$ is a zipped Reeb foliation. The construction of the plug (U, X) is similar to those of the proof of Lemma 9.3. The only difference is that we need to make a DA bifurcation on p periodic orbits instead of a single one.

The core of the proof of Proposition 9.2 is the following lemma:

Lemma 9.5 *Let $\mathcal{F}$ and $\mathcal{F}'$ be MS foliations on $\mathbb{T}^2$. Suppose that:*

- *$\mathcal{F}'$ can be obtained by adding a compact leaf to $\mathcal{F}$* (in the sense of Definition 7.13).
- *$\mathcal{F}$ is realized by an attracting hyperbolic plug.*

Then $\mathcal{F}'$ can also be realized by an attracting hyperbolic plug.

During the proof of the Lemma 9.5, we will need the existence of a hyperbolic plug as provided by the following lemma:

Lemma 9.6 *There exists an oriented hyperbolic plug (V, Y) with the following properties:*

- *V is a Seifert bundle over a 2–sphere minus two discs.*
- *The maximal invariant set $\bigcap_{t\in\mathbb{R}} Y^t(V)$ is a saddle hyperbolic periodic orbit O with negative multipliers.*
- *The entrance boundary $\partial^{\mathrm{in}} V$ is a torus, and the entrance lamination $\mathcal{L}_Y^{\mathrm{s}} = W_Y^{\mathrm{s}}(O) \cap \partial^{\mathrm{in}} V$ is made of a single (closed) leaf γ^{s}, which is essential in $\partial^{\mathrm{in}} V$.*
- *The exit boundary $\partial^{\mathrm{out}} V$ is a torus, and the exit lamination $\mathcal{L}_Y^{\mathrm{u}} = W_Y^{\mathrm{u}}(O) \cap \partial^{\mathrm{out}} V$ is made of a single (closed) curve γ^{u}, which is essential in $\partial^{\mathrm{in}} V$.*

Proof Consider a gradient-like diffeomorphism $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that the nonwandering set of f consists of a repelling hyperbolic fixed point r, a saddle hyperbolic fixed point (with negative eigenvalues) o and an attracting hyperbolic periodic orbit of period two. Denote by (N, Y) the suspension of $(\mathbb{S}^2, f)$. The nonwandering set of Y is made of a repelling hyperbolic periodic orbit R, a saddle hyperbolic periodic orbit O and an attracting hyperbolic periodic orbit A. Let V be the manifold with boundary obtained by excising from N some small tubular neighborhoods of the periodic orbits A and R whose boundary are transverse to Y. Then (V, Y) is a hyperbolic plug, and one can easily check that it satisfies the desired properties. □

Proof of Lemma 9.5 By assumption, we can find a connected attracting hyperbolic plug (U, X) realizing the foliation $\mathcal{F}$. Let (V, Y) be the hyperbolic plug provided by Lemma 9.6. We have to construct a plug (U', X') realizing the foliation $\mathcal{F}'$. This plug will be obtained by gluing together the plugs (U, X) and (V, Y). We proceed to the construction.

By assumption, $\mathcal{F}'$ can be obtained by adding a compact leaf to $\mathcal{F}$. Since $\mathcal{L}_X^{\mathrm{s}}$ is topologically equivalent to $\mathcal{F}$, the foliation $\mathcal{F}'$ can also be obtained by adding a compact leaf to $\mathcal{L}_X^{\mathrm{s}}$. By definition, this means that, one can find a geometrical enumeration $\gamma_1, \dots, \gamma_n$ of the compact leaves of $\mathcal{L}_X^{\mathrm{s}}$ and a geometrical enumeration $\gamma_1', \dots, \gamma_{n+1}'$ of the compact leaves of $\mathcal{F}'$ such that the corresponding combinatorial types $\sigma\colon \{1, \dots, n\} \to \{+, -\}$ and $\sigma'\colon \{1, \dots, n+1\} \to \{+, -\}$ of $\mathcal{F}'$ satisfy $\sigma'|_{\{1,\dots,n\}} = \sigma$.

Since $\gamma_1, \dots, \gamma_n$ is a geometrical enumeration of the compact leaves of $\mathcal{L}_X^{\mathrm{s}}$, the compact leaves γ_n and γ_1 bound an open annulus $A \subset \partial^{\mathrm{in}} U$ which is disjoint from the compact leaves of $\mathcal{L}_X^{\mathrm{s}}$ (in the particular case $n = 2$, the leaves γ_n and γ_1 bound two annuli; we denote by A the annulus which is on the left-hand side of γ_1 with respect to the contracting orientation of γ_1).

According to Lemma 9.6(4), the surface $\partial^{\mathrm{out}} V$ is a torus and the lamination $\mathcal{L}^{\mathrm{u}}_Y$ is made of a single (closed) leaf γ^{u}. We consider a diffeomorphism $\phi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} U$ with the following properties:

(i) The curve $\phi_*(\gamma^{\mathrm{u}})$ is contained in the annulus A.

(ii) The curve $\phi_*(\gamma^{\mathrm{u}})$ is transverse to the foliation $\mathcal{L}^{\mathrm{s}}_X$.

Such a diffeomorphism ϕ does exist: indeed, the restriction of the foliation $\mathcal{L}^{\mathrm{s}}_X$ to the annulus A is topologically equivalent to the foliation by vertical lines of the annulus $\mathbb{S}^1 \times \mathbb{R}$. Now we glue the plugs (U, X) and (V, Y), using ϕ as a gluing map. In other words, we consider the manifold with boundary $U' := (U \sqcup V)/\phi$, and the vector field X' on U' induced by X and Y. Proposition 1.1 ensures that (U', X') is a connected attracting hyperbolic plug.

We want to prove that the foliation $\mathcal{F}'$ is realized by (U', X'), ie that the foliations $\mathcal{L}^{\mathrm{s}}_{X'}$ and $\mathcal{F}'$ are topologically equivalent. For this purpose, we will use the crossing map

$$\Gamma_V\colon \partial^{\mathrm{in}} V \setminus \gamma^{\mathrm{s}} \to \partial^{\mathrm{out}} V \setminus \gamma^{\mathrm{u}}.$$

Recall that Γ_V maps a point $x \in \partial^{\mathrm{in}} V \setminus \gamma^{\mathrm{s}}$ to the unique point of intersection of the orbit of x (for the flow of the vector field Y) with the surface $\partial^{\mathrm{out}} V$. As stated in Proposition 4.1,

$$\mathcal{L}^{\mathrm{s}}_{X'} = \gamma^{\mathrm{s}} \sqcup (\Gamma_V^{-1})_*(\phi^{-1})_*((\mathcal{L}^{\mathrm{s}}_X) \setminus \gamma^{\mathrm{u}}). \tag{8}$$

The foliation $\mathcal{L}^{\mathrm{s}}_X$ has n compact leaves $\gamma_1, \dots, \gamma_n$. By definition of the gluing map ϕ, the closed curve $\phi_*(\gamma^{\mathrm{u}})$ is disjoint from $\gamma_1, \dots, \gamma_n$. Hence, the foliation $\mathcal{L}^{\mathrm{s}}_{X'}$ has $n+1$ compact leaves $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{n+1}$, where we let

$$\widehat{\gamma}_i := (\Gamma_V^{-1} \circ \phi^{-1})_*(\gamma_1) \quad \text{for } i = 1 \dots n \qquad \text{and} \qquad \widehat{\gamma}_{n+1} := \gamma^{\mathrm{s}}.$$

Let $\widehat{A} := (\Gamma_V^{-1} \circ \phi^{-1})(A)$. Since $\phi_*(\gamma^{\mathrm{u}})$ is contained in A, the map $\Gamma_V^{-1} \circ \phi^{-1}$ is defined on $\partial^{\mathrm{in}} U \setminus A$ and (8) shows that

$$\mathcal{L}^{\mathrm{s}}_{X'} \cap (\partial^{\mathrm{in}} U' \setminus \widehat{A}) \text{ is topologically equivalent to } \mathcal{L}^{\mathrm{s}}_X \cap (\partial^{\mathrm{in}} U \setminus A).$$

Since the annulus $\partial^{\mathrm{in}} U' \setminus \widehat{A}$ contains the compact leaves $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{n+1}$, this proves that:

- $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{n+1}$ is a geometrical enumeration of the compact leaves of $\mathcal{L}^{\mathrm{s}}_{X'}$.
- The combinatorial type $\widehat{\sigma}\colon \{1, \dots, n+1\} \to \{+, -\}$ of $\mathcal{L}^{\mathrm{s}}_{X'}$ associated with this geometrical enumeration satisfies $\widehat{\sigma}|_{\{1,\dots,n\}} = \sigma = \sigma'|_{\{1,\dots,n\}}$.

We are left to prove that $\widehat{\sigma}(n+1) = \sigma'(n+1)$. Actually, we will modify the gluing map ϕ in order to adjust the value of $\widehat{\sigma}(n+1)$. Let $\tau\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{out}} V$ be an orientation-

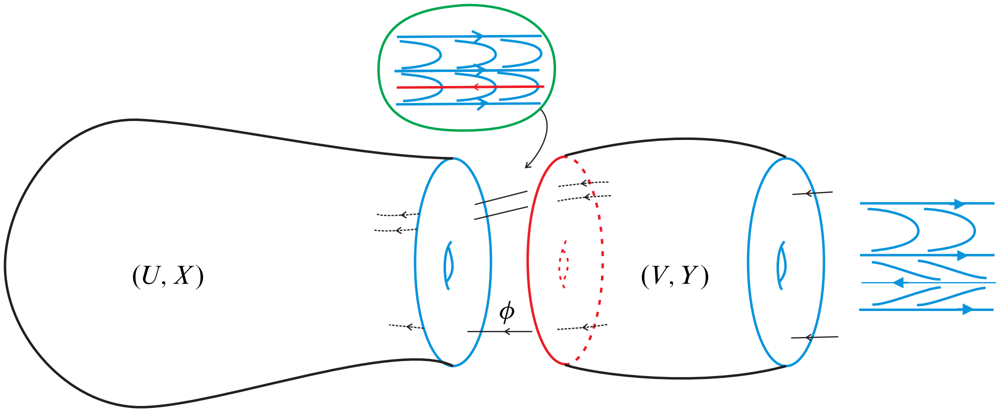

Figure 16: Proof of Lemma 9.5

preserving homeomorphism which maps the closed curve γ^{u} on the same curve with the opposite orientation. Observe that $\phi \circ \tau$ still satisfies properties (i) and (ii), so we may replace ϕ by $\phi \circ \tau$ in our construction. Replacing ϕ by $\phi \circ \tau$ has the following effect:

- It changes the contracting orientation of $\hat{\gamma}_1$. Indeed, the contracting orientation of $\hat{\gamma}_1$ is the image under $(\Gamma_V^{-1} \circ \phi^{-1})_*$ of the contracting orientation of γ_1, and τ_* reverses the orientation of $(\phi^{-1})_*(\gamma_1)$ since $\phi^{-1}(\gamma_1)$ is in the same free homotopy class as γ^{u}.
- It does not change the contracting orientation of $\hat{\gamma}^{n+1} = \gamma^{\mathrm{s}}$. Indeed, Proposition 7.17 ensures that the contracting orientation of γ^{s} as a leaf of $\mathcal{L}^{\mathrm{s}}_{X'}$ is opposite to the dynamical orientation of γ^{s}. And the dynamical orientation of γ^{s} does not depend on the gluing map.

Therefore, up to replacing ϕ by $\tau \circ \phi$, we can decide whether the contracting orientation of the compact leaves $\hat{\gamma}_1$ and $\hat{\gamma}_{n+1} = \gamma^{\mathrm{s}}$ are coherent or not. In other words, up to replacing ϕ by $\tau \circ \phi$, we may assume that $\hat{\sigma}(n+1) = \sigma'(n+1)$. We have proved that the foliations $\mathcal{L}^{\mathrm{s}}_{X'}$ and $\mathcal{F}'$ have the same combinatorial type. According to Proposition 7.7, this implies that these foliations are topologically equivalent. The proof is complete. □

Remark 9.7 The attracting plug (U', X') constructed in the proof above is never transitive.

Proof of Proposition 9.2 The proposition is obtained by combining Corollary 7.15 and Lemmas 9.3 and 9.5. □

9.2 Transitive attracting plug with prescribed entrance foliations

We are now ready to prove Theorem 1.10. For this purpose, we need an improved version of Lemma 9.5:

Lemma 9.8 *Let $\mathcal{F}$ and $\mathcal{F}'$ be MS foliations on $\mathbb{T}^2$. Suppose that*

- *$\mathcal{F}'$ can be obtained by adding a compact leaf to $\mathcal{F}$;*
- *$\mathcal{F}$ is realized by a transitive attracting hyperbolic plug (U, X) which has infinitely many periodic orbits with negative multipliers.*

Then $\mathcal{F}'$ can be realized by a transitive attracting hyperbolic plug (U', X') which has infinitely many periodic orbits with negative multipliers.

The proof of Lemma 9.8 follows the same strategy as those of Lemma 9.5, but is slightly more complicated. Instead of using the plug (V, Y) provided by Lemma 9.6, we will use a plug (W, Z) with the following characteristics:

Lemma 9.9 *There exists a connected oriented hyperbolic plug (W, Z) such that:*

- *W is diffeomorphic to the product of a pair of pants by a circle.*
- *The maximal invariant set $\bigcap_{t\in\mathbb{R}} Z^t(W)$ is an isolated saddle hyperbolic periodic orbit (with positive multipliers).*
- *The entrance boundary $\partial^{\text{in}} W$ is made of two tori $\partial^{\text{in}}_1 W$ and $\partial^{\text{in}}_2 W$; the entrance lamination $\mathcal{L}^{\text{s}}_Z$ is made of two isolated compact leaves γ^{s}_1 and γ^{s}_2; more precisely, γ^{s}_1 is an essential simple closed curve in $\partial^{\text{in}}_1 W$ and γ^{s}_2 is an essential simple closed curve in $\partial^{\text{in}}_2 W$.*
- *The exit boundary $\partial^{\text{out}} W$ is a torus, and the exit lamination $\mathcal{L}^{\text{u}}_Z$ is made of two closed leaves γ^u_1 and γ^u_2, which are essential in the torus $\partial^{\text{out}} W$; moreover, the dynamical orientations of γ^u_1 and γ^u_2 coincide.*

Proof Consider a gradient-like diffeomorphism $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that the nonwandering set of f consists of two repelling fixed points r and r', one saddle hyperbolic fixed point o and one attracting fixed point a. Denote by (N, Z) the suspension of $(\mathbb{S}^2, f)$. The nonwandering set of Z is made of two repelling periodic orbits R and R', one saddle hyperbolic periodic orbit O and one attracting periodic orbit A. Let W be the manifold with boundary obtained by excising from N some small tubular neighborhoods of the periodic orbits R, R' and A. Then (W, Z) is a hyperbolic plug, and one can easily check that it satisfies the desired properties. □

Proof of Lemma 9.8 By assumption, the foliation $\mathcal{F}$ is realized by a transitive attracting hyperbolic plug (U, X), which has infinitely many periodic orbits with negative multipliers. The foliation $\mathcal{F}'$ can be obtained by adding a compact leaf to $\mathcal{L}^{\text{s}}_X$. This means that we can find a geometrical enumeration $\gamma_1, \dots, \gamma_n$ of the compact leaves of $\mathcal{L}^{\text{s}}_X$ and a geometrical enumeration $\gamma'_1, \dots, \gamma'_{n+1}$ of the compact leaves of $\mathcal{F}'$ such that the corresponding combinatorial types σ and σ' satisfy $\sigma'|_{\{1,\dots,n\}} = \sigma$.

We denote by A the connected component of $\partial^{\mathrm{in}} U \setminus \bigcup_i \gamma_i$ which is bounded by the compact leaves γ_n and γ_1, and which is on the left-hand side of γ_1.

Let (W, Z) be the plug provided by Lemma 9.6, and $\Gamma_W\colon \partial^{\mathrm{in}} W \setminus \mathcal{L}^{\mathrm{s}}_Z \to \partial^{\mathrm{out}} W \setminus \mathcal{L}^{\mathrm{u}}_Z$ be the crossing map of this plug. Observe that $\partial^{\mathrm{in}} W \setminus \mathcal{L}^{\mathrm{s}}_Z$ is the disjoint union of the open annuli $A^{\mathrm{s}}_1 := \partial^{\mathrm{in}}_1 W \setminus \gamma^{\mathrm{s}}_1$ and $A^{\mathrm{s}}_2 := \partial^{\mathrm{in}}_2 W \setminus \gamma^{\mathrm{s}}_2$. Therefore, $\partial^{\mathrm{out}} W \setminus \mathcal{L}^{\mathrm{u}}_Z$ is the disjoint union of the open annuli $A^{\mathrm{u}}_1 := \Gamma_W(A^{\mathrm{s}}_1)$ and $A^{\mathrm{u}}_2 := \Gamma_W(A^{\mathrm{s}}_2)$. Both A^{u}_1 and A^{u}_2 are bounded by the compact leaves γ^{u}_1 and γ^{u}_2.

We consider a gluing diffeomorphism $\phi\colon \partial^{\mathrm{out}} W \to \partial^{\mathrm{in}} U$ satisfying the two following properties:

(1) $\phi(\overline{A^{\mathrm{u}}_2})$ is contained in A (equivalently, $\partial^{\mathrm{in}} U \setminus A$ is contained in $\phi(A^{\mathrm{u}}_1)$).

(2) The compact leaves $\phi_*(\gamma^{\mathrm{u}}_1)$ and $\phi_*(\gamma^{\mathrm{u}}_2)$ are transverse to the foliation $\mathcal{L}^{\mathrm{s}}_X$.

We consider the attracting plug (U', X') obtained by gluing (W, Z) and (U, X) thanks to the gluing map ϕ. Proposition 1.1 implies that (U', X') is an hyperbolic plug. Note that $\partial^{\mathrm{in}} U' = \partial^{\mathrm{in}} W = \partial^{\mathrm{in}}_1 W \sqcup \partial^{\mathrm{in}}_2 W$. Property (1) above implies that the preimage under $\Gamma_W \circ \phi$ of the n compact leaves of $\mathcal{L}^{\mathrm{s}}_X$ are in $\partial^{\mathrm{in}}_1 W$. This remark and the same arguments as in the proof of Lemma 9.5 show that the foliation $\mathcal{L}^{\mathrm{s}}_{X'} \cap \partial^{\mathrm{in}}_1 W$ has the same combinatorial type as $\mathcal{F}'$. It also shows that γ^{s}_2 is the only compact leaf of the foliation $\mathcal{L}^{\mathrm{s}}_{X'} \cap \partial^{\mathrm{in}}_2 W$; in other words, $\mathcal{L}^{\mathrm{s}}_{X'} \cap \partial^{\mathrm{in}}_2 W$ is a zipped Reeb foliation.

Let us summarize. We have constructed an attracting hyperbolic plug (U', X') with the following properties:

- The entrance boundary $\partial^{\mathrm{in}} U'$ has two connected components $\partial^{\mathrm{in}}_1 W$ and $\partial^{\mathrm{in}}_2 W$.
- On the first component $\partial^{\mathrm{in}}_1 W$, the entrance foliation $\mathcal{L}^{\mathrm{s}}_{X'}$ is topologically equivalent to the foliation $\mathcal{F}'$.
- On the other connected component $\partial^{\mathrm{in}}_2 W$, the entrance foliation $\mathcal{L}^{\mathrm{s}}_{X'}$ is a zipped Reeb foliation.

The plug (U', X') is not transitive. We will use the "blow-up, excise and glue surgery" of Section 8 to turn (U', X') into a transitive plug.

On the one hand, the maximal invariant set of (U, X) is a transitive hyperbolic attractor; let us denote it by Λ. On the other hand, the maximal invariant set of (W, Z) is a single hyperbolic periodic orbit O. Hence (U', X') has two basic pieces: Λ and O. Observe that $W^{\mathrm{u}}(O)$ intersects $W^{\mathrm{s}}(\Lambda)$ in U', since $\phi(W^{\mathrm{u}}(O) \cap \partial^{\mathrm{out}} W) = \phi(\gamma^{\mathrm{s}}_1 \cup \gamma^{\mathrm{s}}_2)$ intersects $W^{\mathrm{s}}(\Lambda) \cap \partial^{\mathrm{in}} U$. By assumption, (U, X) contains some periodic orbits with negative multipliers. Choose such a periodic orbit Ω, and make an attracting DA bifurcation on Ω. This gives rise to a new vector field X'' on U' which is a topological extension of X', and has three basic pieces: a saddle hyperbolic periodic orbit O, a

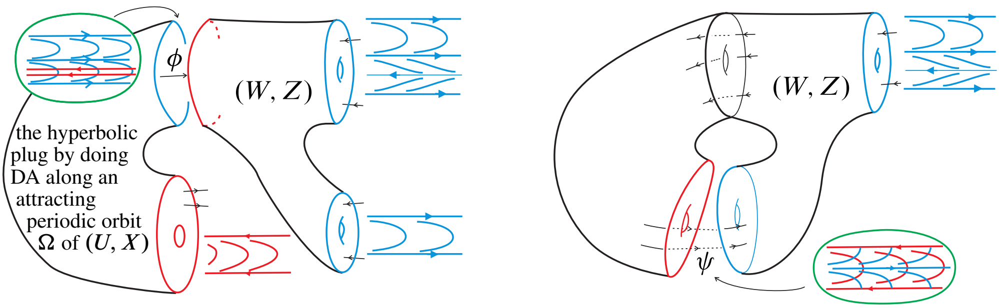

Figure 17: Proof of Lemma 9.8

nontrivial saddle basic piece Λ' and an attracting periodic orbit Ω (see Section 8.1 for more details). Let U'' be obtained by excising from U' a small tubular neighborhood of the attracting orbit Ω. According to Proposition 8.1, (U'',X'') is a hyperbolic plug with the following properties:

- The exit boundary $\partial^{\mathrm{out}}U''$ is a torus.
- The exit lamination $\mathcal{L}^{\mathrm{u}}_{X''}$ is a zipped Reeb lamination.
- The entrance boundary $\partial^{\mathrm{in}}U''$ coincides with $\partial^{\mathrm{in}}U'$,
- The entrance lamination $\mathcal{L}^{\mathrm{s}}_{X''}$ has the same combinatorial type as $\mathcal{L}^{\mathrm{s}}_{X'}$ (hence, $\mathcal{L}^{\mathrm{s}}_{X''}\cap\partial^{\mathrm{in}}_1 W$ has the same combinatorial type as $\mathcal{F}'$ and $\mathcal{L}^{\mathrm{s}}_{X''}\cap\partial^{\mathrm{in}}_2 W$ is a zipped Reeb lamination).

The laminations $\mathcal{L}^{\mathrm{u}}_{X''}$ and $\mathcal{L}^{\mathrm{s}}_{X''}\cap\partial^{\mathrm{in}}_2 W$ both are zipped Reeb laminations. So, by Lemma 8.5, we can find a strongly transverse gluing map $\psi\colon \partial^{\mathrm{out}}U''\to\partial^{\mathrm{in}}_2 U''$. We consider the manifold with boundary $U''':=U''/\psi$ and the vector field X''' induced by X'' on U'''. Clearly, (U''',X''') is an attracting plug and $\partial^{\mathrm{in}}U'''=\partial^{\mathrm{in}}_1 W$. Recall that (U'',X'') is a saddle hyperbolic plug, so, according to Theorem 1.5 and Remark 6.7, up to modifying X'' by a topological equivalence and ψ by a strongly transverse isotopy we may assume that (U''',X''') is a hyperbolic plug.

The interior of U'' is embedded in U''', and X''' coincides with X'' on its restriction to $\mathrm{int}(U'')$. It follows that $\mathcal{L}^{\mathrm{s}}_{X''}\cap\partial^{\mathrm{in}}_1 W$ must be a sublamination of the foliation $\mathcal{L}^{\mathrm{s}}_{X'''}$. Since $\mathcal{L}^{\mathrm{s}}_{X''}\cap\partial^{\mathrm{in}}_1 U''$ is a filling MS lamination, this implies that $\mathcal{L}^{\mathrm{s}}_{X'''}$ has the same combinatorial type as $\mathcal{L}^{\mathrm{s}}_{X''}\cap\partial^{\mathrm{in}}_1 W$. As a further consequence, $\mathcal{L}^{\mathrm{s}}_{X'''}$ has the same combinatorial type as $\mathcal{F}'$. According to Proposition 7.7, this implies that $\mathcal{F}'$ is realized by the attracting hyperbolic plug (U''',X''').

Clearly, (U''',X''') has infinitely many periodic orbits with negative multipliers. It remains to check that the maximal invariant set of (U''',X''') is transitive. The hyperbolic plug (U'',X'') has two basic pieces: the transitive attractor Λ' and the periodic orbit O. On the one hand, the unstable manifold of O intersects the stable manifold of Λ'. On the other hand, $\psi_*(W^{\mathrm{u}}_{X''}(\Lambda')\cap\partial^{\mathrm{out}}W)$ intersects $W^{\mathrm{s}}_{X''}(O)\cap\partial^{\mathrm{in}}_2W$, since the unique compact leaf of $\mathcal{L}^{\mathrm{s}}_{X''}\cap\partial^{\mathrm{in}}_2W$ is $\gamma_2^{\mathrm{s}}\subset W^{\mathrm{s}}_{X''}(O)$. Therefore, the system (U'',X'',ψ) is combinatorially transitive. According to Proposition 1.6, this implies that (U''',X''') is a transitive plug. The proof is complete. □

Remark 9.10 The plug (U,X) provided by the proof of Lemma 9.3 has infinitely many periodic orbits with negative multipliers.

Proof of Theorem 1.10 Let $\mathcal{F}$ be an MS foliation on a closed oriented surface S. We have to prove that $\mathcal{F}$ is realized by a transitive attracting plug. If S is connected (ie is a torus), this immediately follows from Corollary 7.15, Lemma 9.3, Remark 9.10 and Lemma 9.8. If S has several connected components, the proof is roughly the same, except for the fact that we have to use Remark 9.4 instead of Lemma 9.3. □

Remark 9.11 If (U,X) is an oriented transitive attracting hyperbolic plug, then $(U,-X)$ is an oriented transitive repelling hyperbolic plug, and $\mathcal{L}^{\mathrm{u}}(U,-X)=\mathcal{L}^{\mathrm{s}}_X$. Using this observation, we deduce from Theorem 1.10 that every MS foliation can be realized as the exit foliation of a transitive repelling hyperbolic plug.

10 Embedding hyperbolic plugs in Anosov flows

The purpose of the present section is to prove Theorem 1.12 which states that every hyperbolic plug with filling MS laminations can be embedded in an Anosov flow. We shall need the following lemma:

Lemma 10.1 *For every MS foliation $\mathcal{F}$ on a surface S, there exists an MS foliation $\mathcal{G}$ on S which is transverse to $\mathcal{F}$.*

Proof Choose a riemannian metric on S, and consider the orthogonal $\mathcal{F}^\perp$ of $\mathcal{F}$ for this metric. Obviously, $\mathcal{F}^\perp$ is a foliation on S which is transverse to $\mathcal{F}$. In general, $\mathcal{F}^\perp$ is not an MS foliation. Nevertheless, a generic C^1–small perturbation of $\mathcal{F}^\perp$ is an MS foliation which is still transverse to $\mathcal{F}$. □

Proof of Theorem 1.12 Let us prove the first item. We consider a hyperbolic plug with filling MS laminations (U_0,X_0). We have to build a closed three-manifold M and a (not necessarily transitive) Anosov vector field X on M such that there is a compact submanifold U of M such that X is transverse to ∂U and $(U,X|_U)$ is topologically equivalent to (U_0,X_0).

According to Lemma 3.19, the lamination $\mathcal{L}^{\mathrm{s}}_{X_0}$ can be completed to an MS foliation $\mathcal{F}^{\mathrm{s}}$. According to Lemma 10.1, we can find an MS foliation $\mathcal{G}$ on $\partial^{\mathrm{in}}U_0$ which is transverse to $\mathcal{F}^{\mathrm{s}}$. Theorem 1.10 provides a transitive repelling hyperbolic plug (U_R, X_R) and a homeomorphism $\phi\colon \partial^{\mathrm{out}} U_1 \to \partial^{\mathrm{in}} U_0$ such that $\phi_*(\mathcal{L}^{\mathrm{u}}_{X_R}) = \mathcal{G}$. In particular, the foliation $\phi_*(\mathcal{L}^{\mathrm{u}}_{X_R})$ is strongly transverse to the lamination $\mathcal{L}^{\mathrm{s}}_{X_0}$. We consider the manifold $U_1 := (U_R \sqcup U_0)/\phi$, endowed with the vector field X_1 induced by X_R and X_0. According to Proposition 1.1, (U_1, X_1) is a repelling hyperbolic plug.

Now we consider the exit foliation $\mathcal{L}^{\mathrm{u}}_{X_1}$. According to Lemma 10.1, we can find an MS foliation $\mathcal{H}$ on $\partial^{\mathrm{out}} U_1 = \partial^{\mathrm{out}} U_0$ which is transverse to $\mathcal{L}^{\mathrm{u}}_{X_1}$. Theorem 1.10 provides a transitive attracting hyperbolic plug (U_A, X_A) and a homeomorphism $\psi\colon \partial^{\mathrm{in}} U_A \to \partial^{\mathrm{out}} U_1$ such that $\psi_*(\mathcal{L}^{\mathrm{s}}_{X_A}) = \mathcal{H}$. In particular, the foliation $\psi_*(\mathcal{L}^{\mathrm{s}}_{X_A})$ is transverse to the foliation $\mathcal{L}^{\mathrm{u}}_{X_1}$. We consider the closed manifold $M := (U_1 \sqcup U_A)/\psi$, endowed with and the vector field X induced by X_1 and X_A. According to Proposition 1.1, X is a (nontransitive) Anosov vector field.

The plug (M, X) was constructed by gluing together the plugs (U_R, X_R), (U_0, X_0) and (U_A, X_A). Therefore, U_0 can be regarded as a submanifold with boundary of M, and X_0 can be regarded as the restriction of X to U_0. This completes the proof of the first item of Theorem 1.12.

Now we prove the second item. We assume that the maximal invariant set of (U_0, X_0) contains neither attractors nor repellers. We will use the blow-up, excise and glue surgery to "turn X into a transitive vector field".

Recall that (M, X) was obtained by gluing together the hyperbolic plugs (U_R, X_R), (U_0, X_0) and (U_A, X_A). Therefore, U_R, U_0 and U_A can be regarded as compact submanifolds with boundary of M. We pick two periodic orbits O_R and O_A of X, both with positive multipliers, contained respectively in U_R and U_A. We make a repelling DA bifurcation on O_R, and an attracting DA bifurcation on O_A (see Section 8.1). This gives rise to a new vector field $\overline{X}$ on M, which has a repelling periodic orbit $O_R \subset U_R$ and an attracting periodic orbit $O_A \subset U_A$. As in Section 8.1, we consider some open tubular neighborhoods T_R and T_A of O_A and O_R, respectively, such that $\overline{X}$ is transverse to ∂T_R and ∂T_A. We assume that T_A and T_R are contained respectively in U_A and U_R (therefore T_A and T_R are disjoint from U_0). We excise these tubular neighborhoods from M, ie we consider the compact manifold with boundary $U := M \setminus (T_A \sqcup T_R)$. As explained in Section 8.2, $(U, \overline{X})$ is a hyperbolic plug, and we can find a strongly transverse gluing diffeomorphism $\chi\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$. Notice moreover that the maximal invariant set of $(U, \overline{X})$ contains neither attractors nor repellers, since we have made a DA bifurcation on a periodic orbit of the unique attractor of X (turning this attractor into a saddle basic piece) and a DA bifurcation on a periodic orbit of the unique repeller of X (turning this repeller into a saddle basic piece). We consider the closed manifold $M' := U/\chi$, and the vector field X'

induced by $\overline{X}$ on M'. According to Theorem 1.5 (up to perturbing $\overline{X}$ by topological equivalence, and χ by a strongly transverse isotopy), the vector field X' is Anosov. Observe that U_0 can be regarded as a compact submanifold with boundary of M' (since the solid tori T_A and T_R were assumed to be disjoint from U_0), and X_0 can be regarded as the restriction of X' to U_0 (indeed, X' coincides with $\overline{X}$ on $\mathrm{int}(\overline{M}) \supset U_0$, and $\overline{X}$ coincides with X outside a small neighborhood of the orbits O and O').

We are left to prove that X' is transitive. For this purpose, we will use the criterion provided by Proposition 1.6. Let Λ_A (resp. Λ_R) be the maximal invariant set of the plug (U_A, X_A) (resp. (U_R, X_R)). Let $\Lambda_1, \dots, \Lambda_n$ be the collection of the basic pieces of the hyperbolic plug (U, X). Recall that (M, X) was obtained by gluing the hyperbolic plugs (U_R, X_R), (U_0, X_0) and (U_A, X_A), without creating any new recurrent orbit. Therefore, the basic pieces of (M, X) are $\Lambda_R, \Lambda_1, \dots, \Lambda_n, \Lambda_A$. For each $i = 1, \dots, n$, $W^{\mathrm{s}}_X(\Lambda_i)$ must intersect $W^{\mathrm{u}}_X(\Lambda_R)$, because Λ_R is the only repelling basic piece for X. Similarly, $W^{\mathrm{u}}_X(\Lambda_i)$ must intersect $W^{\mathrm{s}}_X(\Lambda_A)$. Proposition 8.1(4) implies that the basic pieces of $\overline{X}|_U$ are in one-to-one correspondence with the basic pieces in X. We denote by $\overline{\Lambda}_R, \overline{\Lambda}_1, \dots, \overline{\Lambda}_n, \overline{\Lambda}_A$ the basic pieces of $\overline{X}|_U$ (using obvious notations). For $i = 1, \dots, n$, $W^{\mathrm{s}}_{\overline{X}}(\overline{\Lambda}_i)$ intersects $W^{\mathrm{u}}_{\overline{X}}(\overline{\Lambda}_R)$, and $W^{\mathrm{u}}_{\overline{X}}(\overline{\Lambda}_i)$ intersects $W^{\mathrm{s}}_{\overline{X}}(\overline{\Lambda}_A)$. Moreover, Proposition 8.1(4) implies that $W^{\mathrm{u}}_{\overline{X}}(\overline{\Lambda}_A) \cap \partial^{\mathrm{out}} U$ is dense in $\mathcal{L}^{\mathrm{u}}_{\overline{X}}$ and $W^{\mathrm{s}}_{\overline{X}}(\overline{\Lambda}_R) \cap \partial^{\mathrm{in}} U$ is dense in $\mathcal{L}^{\mathrm{s}}_{\overline{X}}$. Hence, the image under χ_* of $W^{\mathrm{u}}_{\overline{X}}(\overline{\Lambda}_A) \cap \partial^{\mathrm{out}} U$ intersects $W^{\mathrm{s}}_{\overline{X}}(\overline{\Lambda}_R) \cap \partial^{\mathrm{in}} U$. As a consequence, the system $(U, \overline{X}, \chi)$ is combinatorially transitive. According to Proposition 1.6, this implies that the Anosov vector field X' is transitive. This completes the proof of the second item of Theorem 1.12. □

11 A manifold supporting n transitive Anosov flows

The purpose of this section is to prove Theorem 1.13. We fix an integer $n \geq 1$. In Section 11.1, we construct a manifold M supporting n transitive Anosov vector $Z_1, \dots, Z_n$. In Section 11.2, we prove that these vector fields are pairwise topologically nonequivalent.

11.1 Construction of a manifold M supporting n transitive Anosov flows

Lemma 11.1 *There exists a transitive hyperbolic plug with filling MS laminations (U, X) such that:*

1. *$\mathrm{int}(U)$ is a hyperbolic manifold.*
2. *$\partial^{\mathrm{in}} U$ is connected (ie is a torus) and the lamination $\mathcal{L}^{\mathrm{s}}_X$ has $2n+2$ compact leaves, all of them being coherently oriented.*
3. *For each connected component T of $\partial^{\mathrm{out}} U$, all the compact leaves of the lamination $\mathcal{L}^{\mathrm{u}}_X \cap T$ are coherently oriented.*

Proof Consider a pseudo-Anosov diffeomorphism f of a closed surface Σ such that f has at least two singularities and such that one of the singularities of f has exactly $2n+2$ prongs. The existence of such a pseudo-Anosov diffeomorphism follows for example from Masur and Smillie [24, Theorem 2]. After possibly replacing f by a power, we can assume that all the prongs of all the singularities of f are fixed by f. We denote these singularities by $p_1, \dots, p_m$, where p_1 has $2n+2$ prongs. Then we make a repelling DPA (derived from pseudo-Anosov) bifurcation at p_1, and some attracting DPA bifurcations at $p_2, \dots, p_m$. This yields an Axiom A diffeomorphism g of Σ, whose nonwandering set is composed of a nontrivial saddle basic piece, a repelling fixed point p_1 and some attracting fixed points $p_2, \dots, p_m$. Then we consider the suspension (N, X) of this diffeomorphism: N is a closed three-manifold, and X is a nonsingular Axiom A vector field on N whose nonwandering set is made of a nontrivial saddle basic piece Λ, a repelling periodic orbit γ_1 and some periodic attracting orbits $\gamma_2, \dots, \gamma_m$. We set $U := M \setminus (T_1 \cup \cdots \cup T_m)$, where $T_1, \dots, T_m$ are "small" open tubular neighborhoods of the periodic orbits $\gamma_1, \gamma_2, \dots, \gamma_m$. More precisely, we choose $T_1, \dots, T_m$ to be included in the basins of the orbits $\gamma_1, \gamma_2, \dots, \gamma_m$, respectively, and such that their boundaries are transverse to X (just as in Section 8.1). By construction, (U, X) is a plug, $\partial^{\text{in}} U = \partial T_1$ (in particular $\partial^{\text{in}} U$ is connected as announced) and $\partial^{\text{out}} U = \partial T_2 \cup \cdots \cup \partial T_m$. Moreover, the same arguments as in the proof of Proposition 8.1 show that:

- The maximal invariant set of (U, X) is transitive and hyperbolic.
- $\mathcal{L}^{\text{s}}_X$ is a filling MS lamination with $2n+2$ compact leaves, all of them being coherently oriented.
- $\mathcal{L}^{\text{u}}_X \cap \partial T_k$ is a filling MS lamination with s_k compact leaves, where s_k is the number of prongs of the singularity p_k, all of them being coherently oriented.

Finally, by a well-known theorem of Thurston [29], the interior of U is hyperbolic. □

We will use the hyperbolic plug (V, Y) provided by Lemma 9.6 to prove the following:

Lemma 11.2 *There exists a hyperbolic plug with filling MS laminations (W, Z) such that:*

(1) *(W, Z) is obtained by gluing the hyperbolic plugs (U, X) and (V, Y), provided respectively by Lemmas 11.1 and 9.6, along $\partial^{\text{in}} U$ and $\partial^{\text{out}} V$; in particular, $\partial^{\text{in}} W = \partial^{\text{in}} V$ is connected.*

(2) *The lamination $\mathcal{L}^{\text{s}}_Z$ has $2n+3$ compact leaves; exactly $2n+2$ of these $2n+3$ compact leaves are coherently oriented.*

(3) *On each connected components of $\partial^{\text{out}} W$, all the compact leaves of the lamination $\mathcal{L}^{\text{u}}_Z \cap T$ are coherently oriented.*

Proof According to Lemma 11.1, $\partial^{\mathrm{in}} U$ is a torus, and $\mathcal{L}^{\mathrm{s}}_X$ is a filling MS lamination with $2n+2$ compact leaves. Let $\gamma_1, \dots, \gamma_{2n+2}$ be a geometrical enumeration of the compact leaves of $\mathcal{L}^{\mathrm{s}}_X$ (see Definition 7.3). Let A_1 be the connected component of $\partial^{\mathrm{in}} U \setminus \bigcup_i \gamma_i$ bounded by the compact leaves γ_1 and γ_2. Recall that $\partial^{\mathrm{out}} V$ is a torus, and that the lamination $\mathcal{L}^{\mathrm{u}}_Y$ consists of a single (compact) leaf γ^{u}. Then we choose a diffeomorphism $\psi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} U$ such that the closed leaf $\psi(\gamma^{\mathrm{u}})$ is contained in the interior of the annulus A_1, and transverse to the lamination $\mathcal{L}^{\mathrm{s}}_X$ (in particular, the compact leaf $\psi(\gamma^{\mathrm{u}})$ does intersect the lamination $\mathcal{L}^{\mathrm{s}}(U, X)$). We glue the plugs (U, X) and (V, Y) thanks to the diffeomorphism ψ. More precisely, we consider the manifold with boundary $W := (U \sqcup V)/\psi$ and the vector field Z induced by X and Y on W. Proposition 1.1 asserts that (W, Z) is a hyperbolic plug.

Let us describe the lamination $\mathcal{L}^{\mathrm{u}}_Z$. According to Proposition 4.1, we have

$$\mathcal{L}^{\mathrm{u}}(W, Z) = \mathcal{L}^{\mathrm{u}}_X \sqcup (\Gamma_U)_*(\psi(\gamma^{\mathrm{u}}) \setminus \mathcal{L}^{\mathrm{s}}_X),$$

where $\Gamma_U\colon \partial^{\mathrm{in}} U \setminus \mathcal{L}^{\mathrm{s}}_X \to \partial^{\mathrm{out}} U \setminus \mathcal{L}^{\mathrm{u}}_X$ is the crossing map associated with the plug (U, X). Now observe that $\psi(\gamma^{\mathrm{u}}) \setminus \mathcal{L}^{\mathrm{s}}_X$ does not contain any compact leaf of $\mathcal{L}^{\mathrm{u}}_Z$ (recall that the compact leaf $\psi(\gamma^{\mathrm{u}})$ does intersect the lamination $\mathcal{L}^{\mathrm{s}}_X$). Therefore the compact leaves of $\mathcal{L}^{\mathrm{u}}_Z$ are exactly the same as the compact leaves of $\mathcal{L}^{\mathrm{u}}_X$. In particular, for each connected component T of $\partial^{\mathrm{out}} W = \partial^{\mathrm{out}} U$, all the compact leaves of $\mathcal{L}^{\mathrm{u}}_Z \cap T$ are coherently oriented.

Now we describe the entrance lamination $\mathcal{L}^{\mathrm{s}}_Z$. The arguments are very similar to those of the proof of Lemma 9.5. We have

$$\mathcal{L}^{\mathrm{s}}_Z = \mathcal{L}^{\mathrm{s}}_Y \sqcup (\Gamma_V^{-1})_*(\psi_*^{-1}(\mathcal{L}^{\mathrm{s}}_X) \setminus \gamma^{\mathrm{u}}),$$

where $\Gamma_V\colon \partial^{\mathrm{in}} V \setminus \mathcal{L}^{\mathrm{s}}_Y \to \partial^{\mathrm{out}} V \setminus \mathcal{L}^{\mathrm{u}}_Y$ is the crossing map associated with the plug (V, Y). The lamination $(\psi^{-1})_*(\mathcal{L}^{\mathrm{s}}_X)$ has $2n+2$ compact leaves, and we have chosen ψ so that these leaves are disjoint from γ^{u}. Therefore $(\Gamma_V^{-1})_*((\psi^{-1})_*(\mathcal{L}^{\mathrm{s}}_X) \setminus \mathcal{L}^{\mathrm{u}}_Y)$ has $2n+2$ compact leaves. The lamination $\mathcal{L}^{\mathrm{s}}_Y$ consists of a single isolated compact leaf γ^{s}. This proves that the lamination $\mathcal{L}^{\mathrm{s}}(W, Z)$ has exactly $2n+3$ compact leaves. Let us examine the contracting orientations of the leaves. The compact leaves of $\mathcal{L}^{\mathrm{s}}_X$ are coherently oriented. Moreover, these compact leaves are contained in the annulus $\partial^{\mathrm{in}} U \setminus A_1$, and the map $\Gamma_V^{-1} \circ \psi^{-1}$ is well-defined on $\partial^{\mathrm{in}} U \setminus A_1$ (since $\psi_*(\gamma^{\mathrm{u}})$ is contained in A_1). It follows that the $2n+2$ compact leaves of $\mathcal{L}^{\mathrm{s}}_Z$ contained in $(\Gamma_V^{-1})_*((\psi^{-1})_*(\mathcal{L}^{\mathrm{s}}_X))$ are coherently oriented. It remains to check that the orientation of the last compact leaf of $\mathcal{L}^{\mathrm{s}}_Z$ is not coherent with the orientations of the $2n+2$ other compact leaves. Actually, we can use exactly the same trick as in the proof of Lemma 9.5: we consider an orientation-preserving homeomorphism $\tau\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{out}} V$ which reverses the orientation of the compact leaf γ^{u}. Exactly the same arguments as in the proof of Lemma 9.5 show that either ψ or $\psi\circ\tau$ lead to a plug (W, Z) satisfying the desired property. □

Now we consider two copies W_- and W_+ of the manifold with boundary W provided by Lemma 11.2. We endow W_+ with the vector field $Z_+ := Z$, and we endow W_- with the vector field $Z_- := -Z$. There are some natural identifications:

- $\partial^{\mathrm{in}} W_+ \simeq \partial^{\mathrm{out}} W_- \simeq \partial^{\mathrm{in}} W$ and $\partial^{\mathrm{out}} W_+ \simeq \partial^{\mathrm{in}} W_- \simeq \partial^{\mathrm{out}} W$;
- $\mathcal{L}^{\mathrm{s}}_{Z_+} \simeq \mathcal{L}^{\mathrm{u}}_{Z_-} \simeq \mathcal{L}^{\mathrm{s}}_{Z}$ and $\mathcal{L}^{\mathrm{u}}_{Z_+} \simeq \mathcal{L}^{\mathrm{s}}_{Z_-} \simeq \mathcal{L}^{\mathrm{u}}_{Z}$.

By Lemma 11.2, there is one (and only one) compact leaf c of the lamination $\mathcal{L}^{\mathrm{s}}_{Z}$ such that the contracting orientation of c is incoherent with the contracting orientations of the other compact leaves of $\mathcal{L}^{\mathrm{s}}_{Z}$. We denote by c_+ (resp. c_-) the corresponding compact leaf of $\mathcal{L}^{\mathrm{s}}_{Z_+}$ (resp. $\mathcal{L}^{\mathrm{u}}_{Z_-}$).

Lemma 11.3 *There exists a diffeomorphism $\phi\colon \partial^{\mathrm{out}} W_+ \to \partial^{\mathrm{in}} W_-$ such that:*

1. *The filling MS laminations $\phi_*(\mathcal{L}^{\mathrm{u}}_{Z_+})$ and $\mathcal{L}^{\mathrm{s}}_{Z_-}$ are strongly transverse.*
2. *If we view ϕ as a self-homeomorphism of $\partial^{\mathrm{out}} W$ (using the natural identifications of $\partial^{\mathrm{out}} W_+$ and $\partial^{\mathrm{in}} W_-$ with $\partial^{\mathrm{out}} W$), then ϕ is isotopic to the identity.*

Proof This follows immediately from Lemmas 8.5 and 11.2(3). □

Lemma 11.4 *For every $k\in\{1,\dots,n\}$, there exists a diffeomorphism $\phi_k\colon \partial^{\mathrm{out}} W_- \to \partial^{\mathrm{in}} W_+$ with the following properties:*

1. *$(\phi_k)_*(\mathcal{L}^{\mathrm{u}}_{Z_-})$ and $\mathcal{L}^{\mathrm{s}}_{Z_+}$ are strongly transverse.*
2. *The compact leaves $(\phi_k)_*(c^-)$ and c^+ bound two open annuli[18] in the torus $\partial^{\mathrm{in}} W_+$, which contain respectively k and $2n+2-k$ compacts leaves of the lamination $\mathcal{L}^{\mathrm{s}}_{Z_+}$.*
3. *If we view ϕ as a self-homeomorphism of $\partial^{\mathrm{out}} W$, then ϕ_k is isotopic to the identity.*

Remark 11.5 Lemma 11.4(2) implies in particular that $\mathcal{L}^{\mathrm{s}}_{Z_+}\cup(\phi_i)_*(\mathcal{L}^{\mathrm{u}}_{Z_-})$ is homeomorphic to $\mathcal{L}^{\mathrm{s}}_{Z_+}\cup(\phi_j)_*(\mathcal{L}^{\mathrm{u}}_{Z_-})$ only if $i=j$.

Proof of Lemma 11.4 See Figure 18. Proposition 7.18 provides a diffeomorphism $\psi^0_-\colon \partial^{\mathrm{out}} W_- \to \mathbb{T}^2$ such that:

[18] In order to avoid unnecessary complications, we will not try to distinguish these two annuli (although this can be done using the orientations of the leaves $(\phi_k)_*(c^-)$ and c^+).

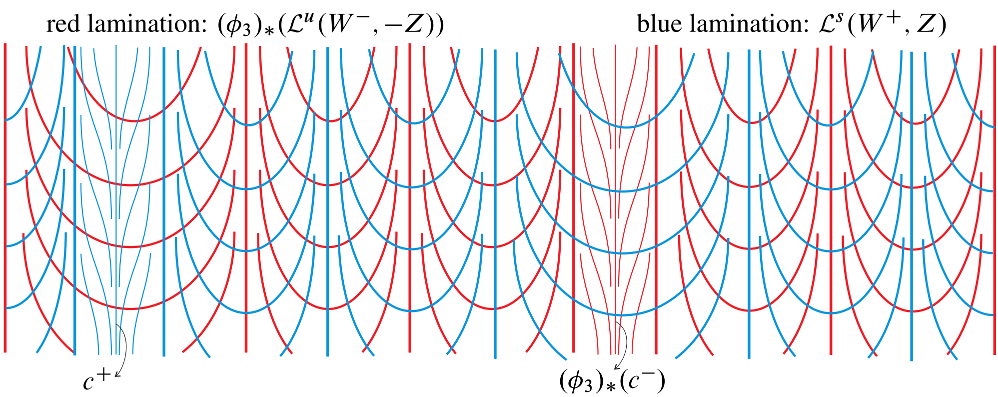

Figure 18: The diffeomorphism ϕ_k in the case $n=2$ and $k=3$

- The $2n+3$ compact leaves of the lamination $(\psi_-)_*(\mathcal{L}^{\mathrm{u}}_{Z_-})$ are the vertical circles $\{\frac{i}{2n+3}\}\times\mathbb{S}^1$ for $i=0,\dots,2n+2$, the leaf $(\psi_-)_*(c^-)$ being the circle $\{\frac{1}{2n+3}\}\times\mathbb{S}^1$.
- In the open annulus $\left(\frac{i}{2n+3},\frac{i+1}{2n+3}\right)\times\mathbb{S}^1$, the leaves of the lamination $(\psi_-^0)_*(\mathcal{L}^{\mathrm{u}}_{Z_-})$ are graphs of C^1 functions from $\left(\frac{i}{2n+3},\frac{i+1}{2n+3}\right)$ to $\mathbb{S}^1$.
- The derivatives of these functions are positive on $\left(0,\frac{1}{2n+3}\right)$ and negative on $\left(\frac{1}{2n+3},\frac{2}{2n+3}\right)$.
- For $i=2,\dots,2n+2$, the derivatives of these functions are positive on the interval $\left(\frac{i}{2n+3},\frac{i}{2n+3}+\frac{1}{2}\right)$ and negative on $\left(\frac{i}{2n+3}+\frac{1}{2},\frac{i+1}{2n+3}\right)$.

Similarly, one gets a diffeomorphism $\psi_+\colon \partial^{\mathrm{in}}W_+\to\mathbb{T}^2$ such that the lamination $(\psi_+)_*(\mathcal{L}^{\mathrm{s}}_{Z_+})$ satisfies similar properties (with the leaf $(\psi_+)_*(c^+)$ instead of the leaf $(\psi_-)_*(c^-)$). Now pick $\epsilon\ll 1$ and consider the diffeomorphism $\Theta_\epsilon\colon\mathbb{T}^2\to\mathbb{T}^2$ defined by

$$\Theta_\epsilon(x,y)=(\theta_\epsilon(x),y),$$

where $\theta_\epsilon\colon[0,1]\to[0,1]$ is the function which maps affinely $\left[0,\frac{2}{2n+3}\right]$ on $[0,\epsilon]$ and maps affinely $\left[\frac{2}{2n+3},1\right]$ on $[\epsilon,1]$ (in other words, Θ_k shrinks the annulus $\left(0,\frac{2}{2n+3}\right)\times\mathbb{S}^1$ to a very thin annulus). For $k\in\{1,\dots,n\}$, also consider the diffeomorphism $\xi_k\colon\mathbb{T}^2\to\mathbb{T}^2$ defined by

$$\xi_k(x,y)=\left(x+\frac{k+\frac{1}{2}}{2n+1},y\right).$$

One easily checks that the diffeomorphism $(\Theta_\epsilon\circ\psi_+)^{-1}\circ\xi_k\circ(\Theta_\epsilon\circ\psi_-)$ satisfies the desired properties provided that ϵ is small enough. □

Definition 11.6 (the vector fields $Z_1,\dots,Z_n$) Consider the hyperbolic plug

$$(W_+,Z_+)\sqcup(W_-,Z_-).$$

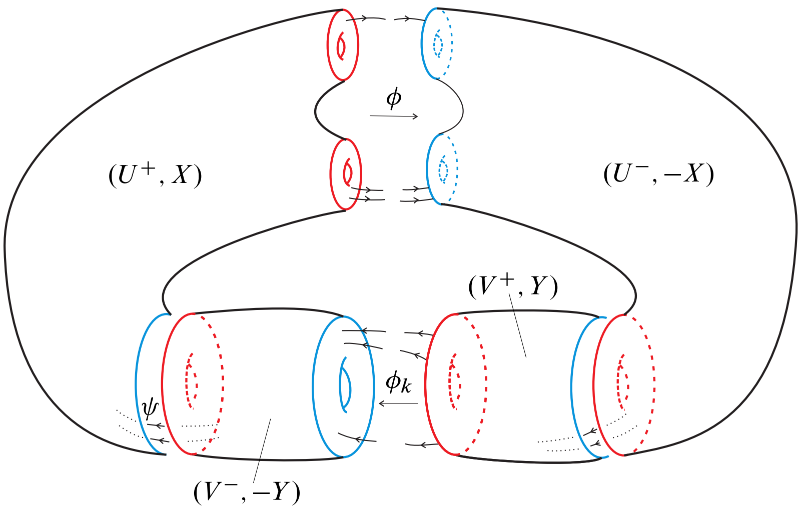

Figure 19: The hyperbolic plugs used to build the manifold M and the Anosov vector field Z_k

For $k = 1 \dots n$, consider the diffeomorphism

$$\Phi_k \colon \partial^{\text{out}} W_+ \sqcup \partial^{\text{out}} W_- \to \partial^{\text{in}} W_+ \sqcup \partial^{\text{in}} W_-$$

defined by $\Phi_k := \phi$ on $\partial^{\text{out}} W_+$ and $\Phi_k := \phi_k$ on $\partial^{\text{out}} W_-$. According to Lemmas 11.3 and 11.4, Φ_k is a strongly transverse gluing diffeomorphism. Now consider the closed manifold $M_k := (W_+ \sqcup W_-)/\Phi_k$ and the vector field Z_k on M_k induced by the vector fields Z and $-Z$ (more precisely, $Z_k := Z$ on W_+ and $Z_k := -Z$ on W_-). According to Theorem 1.5, up to modifying Z by a topological equivalence and Φ_k by a strongly transverse isotopy, Z_k is an Anosov vector field.

Since the gluing map Φ_k is isotopic to the identity for every k, the manifolds $M_1, \dots, M_n$ are pairwise diffeomorphic. From now on, we identify the manifolds $M_1, \dots, M_n$ with a single manifold M, and view $Z_1, \dots, Z_n$ as vector fields on M.

The Anosov vector field (M, Z_k) was obtained by gluing cyclically four hyperbolic plugs (U_-, X_-), (V_-, Y_-), (V_+, Y_+) and (U_+, X_+), where U_+ and U_- are two copies of U, V_+ and V_- are two copies of V, $X_+ = X$, $X_- = -X$, $Y_+ = Y$ and $Y_- = -Y$. See Figure 19.

Proposition 11.7 *The JSJ decomposition of the manifold M has three pieces: two hyperbolic pieces U_- and U_+, and one Seifert piece $S := V_- \cup V_+$.*

Proof As explained above, M was obtained by gluing two copies U_- and U_+ of U and two copies V_- and V_+ of V. The interior of U is hyperbolic; therefore, U_- and

U_+ must be hyperbolic pieces of the JSJ decomposition of M. The manifold V is a Seifert fiber bundle. During the construction of the manifold M, the two copies V_- and V_+ of V were glued together using the map ψ_k. The map ψ_k is isotopic to the identity, and therefore maps the regular fibers of V_- on the regular fibers of V_+ (up to free homotopy). Therefore $S := V_- \cup V_+$ is a Seifert bundle, and corresponds to a single piece in the JSJ decomposition of M. □

Remark 11.8 The vector fields $Z_1, \dots, Z_n$ are pairwise homotopic through nonzero vector fields on M; this follows easily from the construction.

Proposition 11.9 *For every $k \in \{1, \dots, n\}$, the Anosov vector field Z_k is transitive.*

Proof As explained above, (M_k, Z_k) was obtained by gluing the four hyperbolic plugs (U_-, X_-), (V_-, Y_-), (V_+, Y_+) and (U_+, X_+). These four plugs form a cycle, as shown in Figure 19. The choice of the gluing maps ensures that the unstable manifold of the maximal invariant set of any of these four plugs intersects the stable manifold of the maximal invariant set of the next plug in the cycle. Moreover, each of the four hyperbolic plugs is transitive. Therefore, the graph associated with the gluing procedure has four vertices, and these four vertices belong to an oriented cycle; in particular, the gluing procedure is combinatorially transitive. By Proposition 1.6, it follows that Z_k is topologically transitive. □

11.2 The vector fields $Z_1, \dots, Z_n$ are not topologically equivalent

The strategy to prove the vector fields $Z_1, \dots, Z_n$ are pairwise topologically nonequivalent is the following. First, we prove that a topological equivalence between Z_i and Z_j must leave invariant the submanifolds W_- and W_+. Then we use Remark 11.5 to conclude that such a topological equivalence cannot exist, unless $i = j$.

We will use the following result, which was proved by Barbot [2, théorème A], elaborating on some arguments of Brunella [10]:

Lemma 11.10 (Barbot) *Let Z be an Anosov vector field on a closed three-manifold M and let T and T' be some tori embedded in M and transverse to Z. If T is homotopic to T', then T is isotopic to T' along the orbits of Z.*

The phrase "T is isotopic to T' along the orbits of Z" means that there exists a continuous function $u\colon T \to \mathbb{R}$ such that $x \mapsto Z^{u(x)}(x)$ maps T onto T'. This implies that there is a homeomorphism $g\colon M \to M$ preserving each orbit of Z and mapping T onto T'.

Lemma 11.11 *Assume that the vector fields Z_i and Z_j are topologically equivalent. Then one can choose the topological equivalence so that it preserves W_- and W_+.*

Proof By assumption, Z_i and Z_j are topologically equivalent: there exists a homeomorphism $h\colon M \to M$ mapping the oriented orbits of Z_i onto the oriented orbits of Z_j.

Recall that the JSJ decomposition of M comprises two hyperbolic pieces U_-, U_+ and one Seifert piece $S = V_- \cup V_+$ (see Proposition 11.7). By the well-known fact that the JSJ decomposition of a given closed 3–manifold is unique up to isotopy, the homeomorphism h permutes the three JSJ pieces up to isotopy, mapping a hyperbolic piece on a hyperbolic piece, and a Seifert piece on a Seifert piece. Moreover, h cannot map (even up to isotopy) U_- on U_+, because h preserves the orientation of the orbits, and the orbits of Z_i and Z_j go from U_+ to U_- (see Figure 19). Therefore, h must leave invariant the three JSJ pieces U^-, U^+ and S up to isotopy.

Now, recall that the boundaries of U_-, U_+ and S are transverse to the vector fields Z_i and Z_j (see the proof of Proposition 11.7). Hence, using Lemma 11.10, we can modify h so that it leaves invariant U_- and U_+, S in the set-theoretic sense (not "up to isotopy"). And since $\partial^{\mathrm{out}} W_+ = \partial^{\mathrm{in}} W_- = \partial^{\mathrm{out}} U_+ = \partial^{\mathrm{in}} U_-$ (see Figure 19), it follows that h preserves the surface $\partial^{\mathrm{out}} W_+ = \partial^{\mathrm{in}} W_-$.

It remains to prove that h also preserves the surface $\partial^{\mathrm{out}} W_- = \partial^{\mathrm{in}} W_+$. Recall that $\partial^{\mathrm{out}} W_- = \partial^{\mathrm{in}} W_+ = \partial^{\mathrm{out}} V_- = \partial^{\mathrm{in}} V_+$ is an incompressible torus in the interior of the Seifert piece $S = V_- \cup V_+$ (see again Figure 19). But the topology of S is quite simple. Indeed, V_- and V_+ are Seifert bundles over the projective plane minus two discs. It follows that, up to homotopy, there are only three incompressible tori in the Seifert piece S: the two connected components $\partial^{\mathrm{in}} V_-$ and $\partial^{\mathrm{out}} V_+$ of the boundary of S, and the torus $\partial^{\mathrm{out}} W_- = \partial^{\mathrm{in}} W_+ = \partial^{\mathrm{out}} V_- = \partial^{\mathrm{in}} V_+$. As a consequence, h must preserve the torus $\partial^{\mathrm{out}} W_- = \partial^{\mathrm{in}} W_+$ up to isotopy. And, using once again Lemma 11.10, we can modify h so that it leaves invariant $\partial^{\mathrm{out}} W_- = \partial^{\mathrm{in}} W_+$ in the set-theoretic sense.

Now h leaves invariant $\partial^{\mathrm{out}} W_+ = \partial^{\mathrm{in}} W_-$ and $\partial^{\mathrm{out}} W_- = \partial^{\mathrm{in}} W_+$. Thus, it must leave invariant W_- and W_+. □

Proposition 11.12 *The vector fields Z_i and Z_j are not topologically equivalent unless $i = j$.*

Proof Consider two integers $i, j \in \{1, \dots, n\}$ and assume that the vector fields Z_i and Z_j are topologically equivalent. According to Lemma 11.11, there exists a homeomorphism $h\colon M \to M$ mapping the orbits of Z_i to the oriented orbits of Z_j and leaving invariant the two submanifolds W^- and W^+. This homeomorphism h maps the laminations $\mathcal{L}^{\mathrm{s}}_{Z^-}$ and $(\phi_i)_*(\mathcal{L}^{\mathrm{u}}_{Z_+})$ onto the laminations $\mathcal{L}^{\mathrm{s}}_{Z^-}$ and $(\phi_j)_*(\mathcal{L}^{\mathrm{u}}_{Z_+})$, respectively. According to Remark 11.5, this is possible only if $i = j$. □

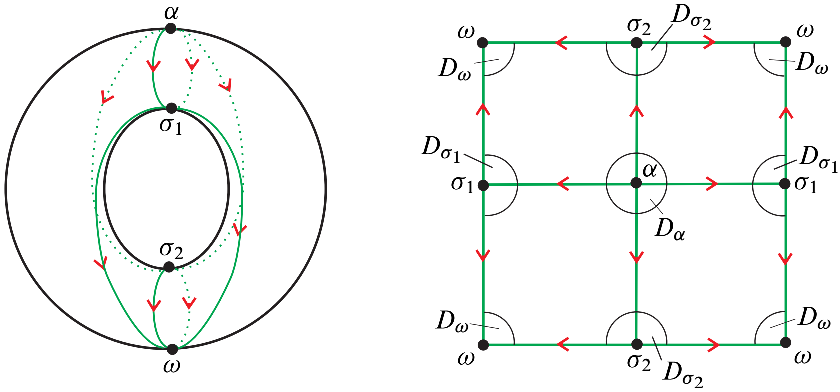

Figure 20: The gradient-like vector field X_0

12 An Anosov flow with infinitely many transverse tori

The purpose of this last section is to prove Theorem 1.15, ie to build a transitive Anosov vector field Z on a closed three-manifold M such that there exist infinitely many pairwise nonisotopic tori embedded in M which are transverse to Z.

We first consider the vector field X_0 on the torus $\mathbb{T}^2$ defined by

$$X_0(x,y) = \sin(2\pi x)\frac{\partial}{\partial x} + \sin(2\pi y)\frac{\partial}{\partial y}.$$

One can easily check that the nonwandering set of X_0 consists of four hyperbolic singularities: a source $\alpha := (0,0)$, two saddles $\sigma_1 := \left(\frac{1}{2}, 0\right)$ and $\sigma_2 := \left(0, \frac{1}{2}\right)$, and a sink $\omega := \left(\frac{1}{2}, \frac{1}{2}\right)$. Moreover, the invariant manifolds of σ_1 are disjoint from the invariant manifold of σ_2. See Figure 20.

Now, we consider some pairwise disjoint (small) open discs D_α, D_{σ_1}, D_{σ_2} and D_ω centered at α, σ_1, σ_2 and ω, respectively, such that the vector field X_0 is transverse to the boundaries of D_α and D_ω. We consider a smooth function $\varphi\colon \mathbb{T}^2 \to \mathbb{R}$ such that $\varphi > 0$ on D_{σ_1}, $\varphi < 0$ on D_{σ_2} and $\varphi = 0$ on $S = \mathbb{T}^2 \setminus (D_{\sigma_1} \cup D_{\sigma_2})$ (in particular, $\varphi = 0$ on $D_\alpha \cup D_\omega$). Then we consider the vector field X on $\mathbb{T}^2 \times \mathbb{S}^1$ defined by

$$X(x,y,t) = X_0(x,y) + \varphi(x,y)\frac{\partial}{\partial t}.$$

We consider the compact three-manifold with boundary $U := (\mathbb{T}^2 \setminus (D_\alpha \cup D_\omega)) \times \mathbb{S}^1$.

Lemma 12.1 *The pair (U, X) is a hyperbolic plug with the following characteristics:*

(1) *The maximal invariant set of (U, X) consists of two saddle hyperbolic periodic orbits.*

(2) *Both the entrance and exit boundaries $\partial^{\mathrm{in}} U$ and $\partial^{\mathrm{out}} U$ of (U, X) are tori.*

(3) *The lamination* $\mathcal{L}^{\mathrm{s}}_X$ *(resp.* $\mathcal{L}^{\mathrm{u}}_X$*) consists of four closed leaves. These leaves are parallel essential curves in* $\partial^{\mathrm{in}}U$ *(resp.* $\partial^{\mathrm{out}}U$*). Moreover, the dynamical orientations* (see Definition 7.16) *of these leaves are "alternating": the dynamical orientations of two adjacent leaves of* $\mathcal{L}^{\mathrm{s}}_X$ *(resp.* $\mathcal{L}^{\mathrm{u}}_X$*) are always incoherent.*

Proof The vector field X_0 is transverse to $\partial D_\alpha \cup \partial D_\omega$, and the function φ vanishes on $D_\alpha \cup D_\omega$. Hence, the vector field X is transverse to $\partial U = (\partial D_\alpha \times \mathbb{S}^1) \sqcup (\partial D_\omega \times \mathbb{S}^1)$. In other words, (U, X) is a plug. Moreover, since α is a source and ω is a sink, the vector field X_0 is pointing out of D_α and into D_ω. It follows that $\partial^{\mathrm{in}}U = \partial D_\alpha \times \mathbb{S}^1$ and $\partial^{\mathrm{out}}U = \partial D_\omega \times \mathbb{S}^1$. In particular, both $\partial^{\mathrm{in}}U$ and $\partial^{\mathrm{out}}U$ are tori.

The definitions of X_0 and S imply that maximal invariant set of (S, X_0) is made of the two saddles σ_1 and σ_2. It follows that the maximal invariant set of (U, X) consists of the saddle hyperbolic periodic orbits $\sigma_1 \times \mathbb{S}^1$ and $\sigma_2 \times \mathbb{S}^1$. In particular, (U, X) is a hyperbolic plug.

The intersection of $W^{\mathrm{s}}(\sigma_1) \cup W^{\mathrm{s}}(\sigma_2)$ with ∂D_α consists of four points. By definition of the vector field X, the lamination $\mathcal{L}^{\mathrm{s}}_X$ is the product by $\mathbb{S}^1$ of $(W^{\mathrm{s}}(\sigma_1) \cup W^{\mathrm{s}}(\sigma_2)) \cap \partial D_\alpha$. This shows that $\mathcal{L}^{\mathrm{s}}_X$ consists of four closed leaves, which are parallel essential curves in $\partial^{\mathrm{in}}U = \partial D_\alpha \times \mathbb{S}^1$.

Let γ_1 and γ_2 be two adjacent leaves in $\mathcal{L}^{\mathrm{s}}(U, X)$. Note that the points of $W^{\mathrm{s}}(\sigma_1) \cap \partial D_\alpha$ and $W^{\mathrm{s}}(\sigma_2) \cap \partial D_\alpha$ are alternating with respect to the cyclic order of ∂D_α. Therefore, up to exchanging the names, γ_1 and γ_2 belong respectively to $W^{\mathrm{s}}(\sigma_1 \times \mathbb{S}^1)$ and $W^{\mathrm{s}}(\sigma_2 \times \mathbb{S}^1)$. And since the function φ is positive on D_{σ_1} and negative on D_{σ_2}, it follows that the dynamical orientations of γ_1 and γ_2 are incoherent. □

Now we consider two copies (U_1, X_1) and (U_2, X_2) of the plug (U, X). We choose a diffeomorphism $\psi\colon \partial^{\mathrm{out}}U_1 \to \partial^{\mathrm{in}}U_2$ such that each of the four compact leaves of $\psi_*(\mathcal{L}^{\mathrm{u}}_{X_1})$ intersects transversally each of the four leaves of $\mathcal{L}^{\mathrm{s}}_{X_2}$ at exactly one point (see Figure 21, left). We consider the manifold with boundary $V := (U_1 \sqcup U_2)/\psi$. We denote by Y the vector field on V induced by X_1 and X_2. According to Proposition 1.1, (V, Y) is a hyperbolic plug. By construction, $\partial^{\mathrm{in}}V = \partial^{\mathrm{in}}U_1$ and $\partial^{\mathrm{out}}V = \partial^{\mathrm{out}}U_2$. In particular, both $\partial^{\mathrm{in}}V$ and $\partial^{\mathrm{out}}V$ are tori.

Remark 12.2 Each connected component A^{s} of $\partial^{\mathrm{in}}U_2 \setminus \mathcal{L}^{\mathrm{s}}_{X_2}$ is an annulus bounded by two (compact) leaves of $\mathcal{L}^{\mathrm{s}}_{X_2}$. The assumptions on the gluing map ψ imply that $\psi_*(\mathcal{L}^{\mathrm{u}}_{X_2}) \cap A^{\mathrm{s}}$ consists of four open arcs, each of which is "crossing" the annulus A^{s} (ie going from one end of A^{s} to the other).

Lemma 12.3 *$\mathcal{L}^{\mathrm{u}}_Y$ and $\mathcal{L}^{\mathrm{s}}_Y$ are filling MS laminations. Each has four compact leaves. These leaves have "alternating contracting orientations": the contracting orientations of two adjacent leaves are always incoherent.*

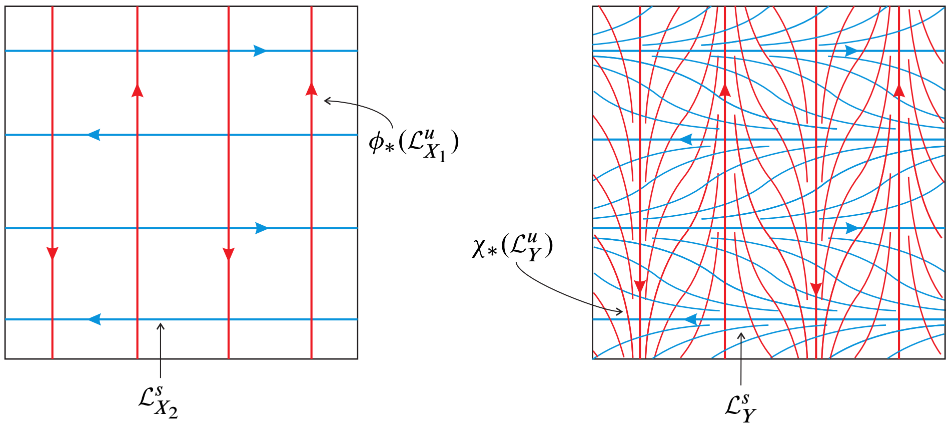

Figure 21: The gluing maps ψ (left) and χ (right)

Proof As usual, we use Proposition 4.1 to write $\mathcal{L}^{\mathrm{s}}_Y$ as a disjoint union

$$\mathcal{L}^{\mathrm{u}}_Y = \mathcal{L}^{\mathrm{u}}_{X_2} \sqcup (\Gamma_2)_*(\phi_*(\mathcal{L}^{\mathrm{u}}_{X_1}) \setminus \mathcal{L}^{\mathrm{s}}_{X_2}),$$

where $\Gamma_2\colon \partial^{\mathrm{in}} U_2 \setminus \mathcal{L}^{\mathrm{s}}_{X_2} \to \partial^{\mathrm{out}} U_2 \setminus \mathcal{L}^{\mathrm{u}}_{X_2}$ is the crossing map of the plug (U_1, X_1).

Each of the four leaves of $\phi_*(\mathcal{L}^{\mathrm{u}}_{X_1})$ intersects $\mathcal{L}^{\mathrm{s}}_{X_2}$. Therefore, $\Gamma_*(\phi_*(\mathcal{L}^{\mathrm{u}}_{X_1}) \setminus \mathcal{L}^{\mathrm{s}}_{X_2})$ does not contain any compact leaf. As a further consequence, the compact leaves of the lamination $\mathcal{L}^{\mathrm{u}}_Y$ are exactly those of the lamination $\mathcal{L}^{\mathrm{u}}_{X_2}$. Hence, the lamination $\mathcal{L}^{\mathrm{u}}_Y$ has four compact leaves (which are parallel essential curves in the torus $\partial^{\mathrm{out}} V = \partial^{\mathrm{out}} U_2$), and the dynamical orientation of two adjacent compact leaves are incoherent. By Proposition 7.17, this is equivalent to the analogous statement with the contracting orientation instead of the dynamical orientations.

We are left to prove that $\mathcal{L}^{\mathrm{u}}_Y$ is a filling MS lamination. We already know that this is an MS lamination thanks to Proposition 3.8. So we are left to prove that every connected component of $\partial^{\mathrm{out}} V \setminus \mathcal{L}^{\mathrm{u}}_Y$ is a strip in the sense of Definition 3.11. For this purpose, we consider a connected component A^{u} of $\partial^{\mathrm{out}} U_2 \setminus \mathcal{L}^{\mathrm{u}}_{X_2}$; this is an open annulus bounded by two compact leaves γ^{u}_1 and γ^{u}_2 of $\mathcal{L}^{\mathrm{u}}_{X_2}$. The set $A^{\mathrm{s}} := \Gamma^{-1}(A^{\mathrm{u}})$ is a connected component of $\partial^{\mathrm{in}} U_2 \setminus \mathcal{L}^{\mathrm{s}}_{X_2}$. The leaves of $\mathcal{L}^{\mathrm{u}}_Y$ contained in the annulus A^{u} are exactly the images under Γ_* of the connected components of $\psi_*(\mathcal{L}^{\mathrm{u}}_{X_1}) \cap A^{\mathrm{s}}$. Together with Remark 12.2, this implies that there are exactly four leaves of $\mathcal{L}^{\mathrm{u}}_Y$ in the annulus A^{u}, and that each of these four leaves is "crossing" the annulus A^{u}, ie is accumulating on both γ^{u}_1 and γ^{u}_2. As a further consequence, every connected component of $A^{\mathrm{u}} \setminus \mathcal{L}^{\mathrm{u}}_Y$ is a strip bounded by two compact leaves of $\mathcal{L}^{\mathrm{u}}_Y$ which are asymptotic at both ends. In other words, $\mathcal{L}^{\mathrm{u}}_Y$ is a filling MS lamination. □

Lemma 12.4 *There exists a diffeomorphism $\chi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$ such that:*

- *The laminations $\chi_*(\mathcal{L}^{\mathrm{u}}_Y)$ and $\mathcal{L}^{\mathrm{s}}_Y$ are strongly transverse.*
- *Every leaf of $\chi_*(\mathcal{L}^{\mathrm{u}}_Y)$ intersects every leaf of $\mathcal{L}^{\mathrm{s}}_Y$* (see Figure 21, right).

Proof See Figure 21. Proposition 7.18 provides a diffeomorphism $\psi^{\mathrm{out}}\colon \partial^{\mathrm{out}} V \to \mathbb{T}^2$ such that:

- The compact leaves of the lamination $(\psi^{\mathrm{out}})_*(\mathcal{L}^{\mathrm{u}}_Y)$ are the vertical circles $\{\frac{i}{4}\} \times \mathbb{S}^1$ for $i = 0, \dots, 3$.
- In the open annulus $\left(\frac{i}{4}, \frac{i+1}{4}\right) \times \mathbb{S}^1$, the leaves of the lamination $(\psi^{\mathrm{out}})_*(\mathcal{L}^{\mathrm{u}}_Y)$ are graphs of C^1 functions from $\left(\frac{i}{4}, \frac{i+1}{4}\right)$ to $\mathbb{S}^1$.
- The derivatives of these functions are positive for $i = 0$ and 2, and negative for $i = 1$ and 3. Given any constant A, an elementary modification of the proof of Proposition 7.18 allows us to assume that the derivatives of these functions are larger than A for $i = 0$ and 2, and smaller than $-A$ for $i = 1$ and 3.

There is a diffeomorphism $\psi^{\mathrm{in}}\colon \partial^{\mathrm{in}} V \to \mathbb{T}^2$ such that the lamination $(\psi^{\mathrm{in}})_*(\mathcal{L}^{\mathrm{s}}_Y)$ satisfies analogous properties. Now, let $\chi_0\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the orientation-preserving diffeomorphism defined by $\chi_0(x, y) = (-y, x)$. A straightforward computation shows that the diffeomorphism $\chi\colon \partial^{\mathrm{out}} V \to \partial^{\mathrm{in}} V$ defined by $\chi := (\psi^{\mathrm{out}})^{-1} \circ \chi_0 \circ \psi^{\mathrm{in}}$ satisfies the desired properties. □

We consider the closed manifold $M := V/\chi$. We denote by Z the vector field induced by Y on M. The first item of Lemma 12.4 and Theorem 1.5 imply that the vector field Z is Anosov (up to perturbing Y within its topological equivalence class and modifying χ by a strongly transverse isotopy). The second item of Lemma 12.4 and Proposition 1.6 imply that the vector field Z is topologically transitive.

Proposition 12.5 *There exist infinitely many pairwise nonisotopic tori embedded in M which are transverse to the vector field Z.*

Proof Let c_x and c_y be the closed curves on $\mathbb{T}^2$ defined respectively by the equations $x = \frac{1}{4}$ and $y = \frac{1}{4}$. We endow c_x and c_y with the orientations defined by the vector fields $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$, respectively. One can easily check that the vector field X_0 is transverse to c_x and c_y.

Let $q \in \mathbb{N} \setminus \{0\}$. Using classical and elementary desingularization process, one can easily find an oriented simple closed curve c_q on $\mathbb{T}^2$, freely homotopic to $c_x + q \cdot c_y$, transverse to the vector field X_0 and disjoint from the discs D_α, D_{σ_1}, D_{σ_2} and D_ω (see Figure 22). The torus $T_q := c_q \times \mathbb{S}^1$ is embedded in $U = (\mathbb{T}^2 \setminus (D_\alpha \cup D_\beta)) \times \mathbb{S}^1$ and transverse to the vector field X (because c_q is transverse to X_0 and since $X(x, y, t) = X_0(x, y)$ for $(x, y) \in \mathbb{T}^2 \setminus (D_{\sigma_1}, D_{\sigma_2})$). Now recall that (M, Z) has been obtained by gluing

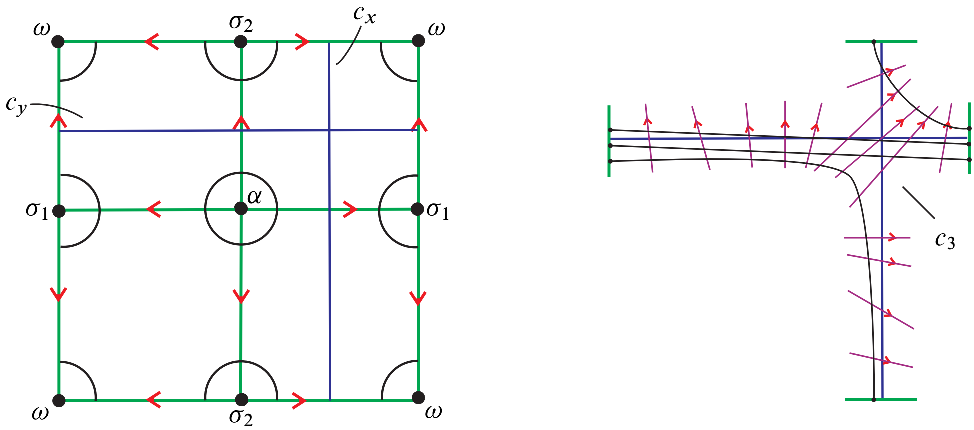

Figure 22: The curves c_x, c_y and c_3

together two copies of the plug (U, X). Therefore the torus T_q can be seen as a torus embedded in M transverse to the vector field Z (with the additional property that T_q is contained in one of the two copies of the plug (U, X) which were glued together to obtain the manifold M).

Now consider two different positive integers q and q'. Fix any $\theta_0 \in \mathbb{S}^1$ and consider the simple closed curve $\widehat{c}_q := c_q \times \{\theta_0\} \subset T_q$. Obviously the algebraic intersection number of the curve $\widehat{c}_q$ and the torus $T_{q'}$ is equal to $\pm|q-q'|$, which is nonzero. An easy cohomological argument shows that T_q is not isotopic to $T_{q'}$ in M.

So we have found infinitely many pairwise nonisotopic tori embedded in M and transverse to Z. The proof of Proposition 12.5 and Theorem 1.15 is complete. □

Remark 12.6 The construction above does not seem to be optimal. Indeed, it should be possible to find a gluing map $\theta\colon \partial^{\text{out}}U \to \partial^{\text{in}}U$ such that the vector field Z_θ induced by X on the closed manifold U/θ is Anosov. Nevertheless, the existence of such a gluing map does not follow from Theorem 1.5, since the entrance/exit laminations of the plug (U, X) are not filling MS laminations.

Remark 12.7 The manifold M constructed above is a graph manifold (it was obtained by gluing together two copies of $S \times \mathbb{S}^1$, where S is the torus minus two discs). Nevertheless, the construction can easily be modified in order to get a manifold M which has hyperbolic pieces in its JSJ decomposition.

Proposition 12.5 motivates us to ask the following question, to which we do not know the answer:

Question 12.8 *For every two different positive integers q and q', is there a homeomorphism of M respecting the orbits of Z and sending T_q to $T_{q'}$?*

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