Geometry & Topology 21 (2017) 2049–2092

The Eynard–Orantin recursion and equivariant mirror symmetry for the projective line

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We study the equivariantly perturbed mirror Landau–Ginzburg model of $\mathbb{P}^1$. We show that the Eynard–Orantin recursion on this model encodes all-genus, all-descendants equivariant Gromov–Witten invariants of $\mathbb{P}^1$. The nonequivariant limit of this result is the Norbury–Scott conjecture, while by taking large radius limit we recover the Bouchard–Mariño conjecture on simple Hurwitz numbers.

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1 Introduction

The equivariant Gromov–Witten theory of $\mathbb{P}^1$ has been studied extensively. Okounkov and Pandharipande [27; 28] completely solved the equivariant Gromov–Witten theory of the projective line and established a correspondence between the stationary sector of Gromov–Witten theory of $\mathbb{P}^1$ and Hurwitz theory. Givental [20] derived a quantization formula for the all-genus descendant potential of the equivariant Gromov–Witten theory of $\mathbb{P}^1$ (and more generally, $\mathbb{P}^n$). In the nonequivariant limit, these results imply the Virasoro conjecture of $\mathbb{P}^1$.

The Norbury–Scott conjecture [26] relates (nonequivariant) Gromov–Witten invariants of $\mathbb{P}^1$ to Eynard–Orantin invariants [10] of the affine plane curve

$$\left\{x = Y + \frac{1}{Y} \;\middle|\; (x, Y) \in \mathbb{C} \times \mathbb{C}^*\right\}.$$

P Dunin-Barkowski, N Orantin, S Shadrin and L Spitz [5] relate the Eynard–Orantin topological recursion to the Givental formula for the ancestor formal Gromov–Witten potential, and prove the Norbury–Scott conjecture using their main result and Givental's quantization formula for the all-genus descendant potential of the (nonequivariant) Gromov–Witten theory of $\mathbb{P}^1$. It is natural to ask if the Norbury–Scott conjecture can be extended to the equivariant setting, in a way that the original conjecture can be recovered in the nonequivariant limit.

Published: 19 May 2017 DOI: 10.2140/gt.2017.21.2049

1.1 Main results

Our first main result (Theorem A in Section 3.7) relates equivariant Gromov–Witten invariants of $\mathbb{P}^1$ to the Eynard–Orantin invariants [10] of the affine curve

$$\left\{x = t^0 + Y + \frac{Qe^{t^1}}{Y} + \mathsf{w}_1 \log Y + \mathsf{w}_2 \log \frac{Qe^{t^1}}{Y} \;\middle|\; (x, Y) \in \mathbb{C} \times \mathbb{C}^*\right\},$$

where t^0 and t^1 are complex parameters, w_1 and w_2 are equivariant parameters of the torus $T = (\mathbb{C}^*)^2$ acting on $\mathbb{P}^1$, and Q is the Novikov variable encoding the degree of the stable maps to $\mathbb{P}^1$ (see Section 2.2). The superpotential of the T–equivariant Landau–Ginzburg mirror of the projective line is given by

$$W_t^{\mathsf{w}}\colon \mathbb{C}^* \to \mathbb{C}, \quad W_t^{\mathsf{w}}(Y) = t_0 + Y + \frac{Qe^{t^1}}{Y} + \mathsf{w}_1 \log Y + \mathsf{w}_2 \log \frac{Qe^{t^1}}{Y},$$

so Theorem A can be viewed as a version of all-genus equivariant mirror symmetry for $\mathbb{P}^1$. We prove Theorem A using the main result in [5] and a suitable version of Givental's formula [20] for all-genus *equivariant* descendant Gromov–Witten potential of $\mathbb{P}^n$ (see also Lee and Pandharipande [24]).

Our second main result (Theorem B in Section 3.7) gives a precise correspondence between genus-g, n–point descendant equivariant Gromov–Witten invariants of $\mathbb{P}^1$ and Laplace transforms of the Eynard–Orantin invariant $\omega_{g,n}$ along Lefschetz thimbles. This result generalizes the known relation between the A–model, genus-0, 1–point descendant Gromov–Witten invariants and the B–model oscillatory integrals.

1.2 Nonequivariant limit and the Norbury–Scott conjecture

Taking the nonequivariant limit $\mathsf{w}_1 = \mathsf{w}_2 = 0$, we obtain

$$W_t(Y) = t^0 + Y + \frac{Qe^{t^1}}{Y},$$

which is the superpotential of the (nonequivariant) Landau–Ginzburg mirror for the projective line. We obtain all-genus (nonequivariant) mirror symmetry for the projective line.

In the stationary phase $t^0 = t^1 = 0$ and $Q = 1$, the curve becomes

$$\{x = Y + Y^{-1} : (x, Y) \in \mathbb{C} \times \mathbb{C}^*\},$$

and Theorem A specializes to the Norbury–Scott conjecture [26]. (See Section 4.2 for details.)

1.3 Large radius limit and the Bouchard–Mariño conjecture

Let $\mathsf{w}_2 = 0$, $t_0 = 0$ and $q = Qe^{t^1}$; we obtain

$$x = Y + \frac{q}{Y} + \mathsf{w}_1 \log Y,$$

which reduces to

$$x = Y + \mathsf{w}_1 \log Y$$

in the large radius limit $q \to 0$. The $\mathbb{C}^*$–equivariant mirror of the affine line $\mathbb{C}$ is given by

$$W\colon \mathbb{C}^* \to \mathbb{C}, \quad W(Y) = Y + \mathsf{w}_1 \log Y.$$

In the large radius limit, we obtain a version of all-genus $\mathbb{C}^*$–equivariant mirror symmetry of the affine line $\mathbb{C}$.

In particular, letting $\mathsf{w}_1 = -1$ and $X = e^{-x}$, we obtain the Lambert curve

$$X = Ye^{-Y}.$$

In this limit, Theorem A specializes to the Bouchard–Mariño conjecture [2] relating simple Hurwitz numbers (related to linear Hodge integrals by the ELSV formula of Ekedahl, Lando, Shapiro and Vainshtein [6] and Graber and Vakil [21]) to Eynard–Orantin invariants of the Lambert curve. (See Section 5 for details.)

Borot, Eynard, Mulase and Safnuk [1] introduced a new matrix model representation for the generating function of simple Hurwitz numbers, and proved the Bouchard–Mariño conjecture. Eynard, Mulase and Safnuk [9] provided another proof of the Bouchard–Mariño conjecture using the cut-and-joint equation of simple Hurwitz numbers. Recently, new proofs of the ELSV formula and the Bouchard–Mariño conjecture have been given by Dunin-Barkowski, Kazarian, Orantin, Shadrin and Spitz [4].

Acknowledgment We thank P Dunin-Barkowski, B Eynard, M Mulase, P Norbury and N Orantin for helpful conversations. The research of the authors is partially supported by NSF DMS-1206667 and NSF DMS-1159416.

2 A–model

Let $T = (\mathbb{C}^*)^2$ act on $\mathbb{P}^1$ by

$$(t_1, t_2) \cdot [z_1, z_2] = [t_1^{-1} z_1, t_2^{-1} z_2].$$

Let $\mathbb{C}[\mathsf{w}] := \mathbb{C}[\mathsf{w}_1, \mathsf{w}_2] = H_T^*(\text{point}; \mathbb{C})$ be the T–equivariant cohomology of a point.

2.1 Equivariant cohomology of $\mathbb{P}^1$

The T–equivariant cohomology of $\mathbb{P}^1$ is given by

$$H_T^*(\mathbb{P}^1;\mathbb{C}) = \mathbb{C}[H,\mathsf{w}]/\langle (H-\mathsf{w}_1)(H-\mathsf{w}_2)\rangle,$$

where $\deg H = \deg \mathsf{w}_i = 2$. Let $p_1 = [1,0]$ and $p_2 = [0,1]$ be the T fixed points. Then $H|_{p_i} = \mathsf{w}_i$. The T–equivariant Poincaré dual of p_1 and p_2 are $H-\mathsf{w}_2$ and $H-\mathsf{w}_1$, respectively. Let

$$\phi_1 := \frac{H-\mathsf{w}_2}{\mathsf{w}_1-\mathsf{w}_2},\ \phi_2 := \frac{H-\mathsf{w}_1}{\mathsf{w}_2-\mathsf{w}_1} \in H_T^*(\mathbb{P}^1;\mathbb{C}) \otimes_{\mathbb{C}[\mathsf{w}]} \mathbb{C}\Big[\mathsf{w}, \frac{1}{\mathsf{w}_1-\mathsf{w}_2}\Big]$$

Then $\deg \phi_\alpha = 0$, and

$$\phi_\alpha \cup \phi_\beta = \delta_{\alpha\beta}\phi_\alpha,$$

So $\{\phi_1,\phi_2\}$ is a canonical basis of the semisimple algebra

$$H_T^*(\mathbb{P}^1;\mathbb{C}) \otimes_{\mathbb{C}[\mathsf{w}]} \mathbb{C}\Big[\mathsf{w}, \frac{1}{\mathsf{w}_1-\mathsf{w}_2}\Big].$$

We have

$$\phi_1+\phi_2 = 1,$$
$$(\phi_\alpha,\phi_\beta) := \int_{\mathbb{P}^1} \phi_\alpha \cup \phi_\beta = \delta_{\alpha\beta}\int_{\mathbb{P}^1}\phi_\alpha = \frac{\delta_{\alpha\beta}}{\Delta^\alpha}, \quad \alpha,\beta \in \{1,2\},$$

where

$$\Delta^1 = \mathsf{w}_1-\mathsf{w}_2, \quad \Delta^2 = \mathsf{w}_2-\mathsf{w}_1.$$

Cup product with the hyperplane class is given by

$$H \cup \phi_\alpha = \mathsf{w}_\alpha\phi_\alpha, \quad \alpha = 1,2.$$

2.2 Equivariant Gromov–Witten invariants of $\mathbb{P}^1$

Suppose that $d>0$ or $2g-2+n>0$, so that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)$ is nonempty. Given $\gamma_1,\dots,\gamma_n \in H_T^*(\mathbb{P}^1,\mathbb{C})$ and $a_1,\dots,a_n \in \mathbb{Z}_{\geq 0}$, we define genus-$g$, degree-$d$, T–equivariant descendant Gromov–Witten invariants of $\mathbb{P}^1$:

$$\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\rangle_{g,n,d}^{\mathbb{P}^1,T} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{\mathrm{vir}}} \prod_{j=1}^n \psi_j^{a_j}\,\mathrm{ev}_j^*(\gamma_j) \in \mathbb{C}[\mathsf{w}],$$

where $\mathrm{ev}_j\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d) \to \mathbb{P}^1$ is the evaluation at the j^{th} marked point, which is a T–equivariant map. We define genus-g, degree-d primary Gromov–Witten invariants:

$$\langle \gamma_1,\dots,\gamma_n\rangle_{g,n,d}^{\mathbb{P}^1,T} := \langle \tau_0(\gamma_1)\cdots\tau_0(\gamma_n)\rangle_{g,n,d}^{\mathbb{P}^1,T}.$$

Let $\boldsymbol{t} = t^0 1 + t^1 H$. If $2g-2+n>0$, define

$$\langle\langle \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n) \rangle\rangle_{g,n}^{\mathbb{P}^1,T} := \sum_{d\geq 0} Q^d \sum_{l=0}^{\infty} \frac{1}{l!} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \underbrace{\tau_0(\boldsymbol{t}) \cdots \tau_0(\boldsymbol{t})}_{l \text{ times}} \rangle_{g,n+l,d}^{\mathbb{P}^1,T}.$$

Suppose that $2g-2+n+m>0$. Given $\gamma_1,\dots,\gamma_{n+m} \in H_T^*(\mathbb{P}^1)$, we define

$$\begin{aligned}\Big\langle \frac{\gamma_1}{z_1-\psi_1}, \dots, \frac{\gamma_n}{z_n-\psi_n}, \gamma_{n+1}, \dots, \gamma_{n+m} \Big\rangle_{g,n+m,d}^{\mathbb{P}^1,T} \\ := \sum_{a_1,\dots,a_n\geq 0} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_0(\gamma_{n+1}) \cdots \tau_0(\gamma_{n+m}) \rangle_{g,n+m,d}^{\mathbb{P}^1,T} \prod_{j=1}^{n} z_j^{-a_j-1}.\end{aligned}$$

In particular, if $n+m \geq 3$ then

$$\begin{aligned}(1)\quad \Big\langle \frac{\gamma_1}{z_1-\psi_1}, \dots, \frac{\gamma_n}{z_n-\psi_n}, \gamma_{n+1}, \dots, \gamma_{n+m} \Big\rangle_{0,n+m,0}^{\mathbb{P}^1,T} \\ = \frac{1}{z_1\cdots z_n} \Big(\frac{1}{z_1} + \cdots + \frac{1}{z_n} \Big)^{n+m-3} \int_{\mathbb{P}^1} \gamma_1 \cup \cdots \cup \gamma_{n+m},\end{aligned}$$

where we use the fact $\overline{\mathcal{M}}_{0,n+m}(\mathbb{P}^1,0) = \overline{\mathcal{M}}_{0,m+n} \times \mathbb{P}^1$, and the identity

$$\int_{\overline{\mathcal{M}}_{0,k}} \psi_1^{a_1} \cdots \psi_k^{a_k} = \begin{cases} \dfrac{(k-3)!}{\prod_{j=1}^k a_j!} & \text{if } a_1+\cdots+a_k = k-3, \\ 0 & \text{otherwise.} \end{cases}$$

We use the second line of (1) to extend the definition of the correlator in the first line of (1) to the unstable cases $(n,m) = (1,0)$, $(1,1)$, $(2,0)$:

$$\begin{aligned}\Big\langle \frac{\gamma_1}{z_1-\psi_1} \Big\rangle_{0,1,0}^{\mathbb{P}^1,T} &:= z_1 \int_{\mathbb{P}^1} \gamma_1, \\ \Big\langle \frac{\gamma_1}{z_1-\psi_1}, \gamma_2 \Big\rangle_{0,2,0}^{\mathbb{P}^1,T} &:= \int_{\mathbb{P}^1} \gamma_1 \cup \gamma_2, \\ \Big\langle \frac{\gamma_1}{z_1-\psi_1}, \frac{\gamma_2}{z_2-\psi_2} \Big\rangle_{0,2,0}^{\mathbb{P}^1,T} &:= \frac{1}{z_1+z_2} \int_{\mathbb{P}^1} \gamma_1 \cup \gamma_2.\end{aligned}$$

Suppose that $2g-2+n+m>0$ or $n>0$. Given $\gamma_1,\dots,\gamma_{n+m} \in H_T^*(\mathbb{P}^1)$, we define

$$\left\langle\!\!\left\langle \frac{\gamma_1}{z_1-\psi_1},\dots,\frac{\gamma_n}{z_n-\psi_n},\gamma_{n+1},\dots,\gamma_{n+m}\right\rangle\!\!\right\rangle_{g,n+m}^{\mathbb{P}^1,T}$$
$$:=\sum_{d\geq 0}\sum_{l\geq 0}\frac{Q^d}{l!}\left\langle \frac{\gamma_1}{z_1-\psi_1},\dots,\frac{\gamma_n}{z_n-\psi_n},\gamma_{n+1},\dots,\gamma_{n+m},\underbrace{\boldsymbol{t},\dots,\boldsymbol{t}}_{l\text{ times}}\right\rangle_{g,n+m+l,d}^{\mathbb{P}^1,T}.$$

Let $q=Qe^{t^1}$. Then, for $m\geq 3$,

$$\langle\!\langle \gamma_1,\dots,\gamma_m\rangle\!\rangle_{0,m}^{\mathbb{P}^1,T}=\sum_{d\geq 0}q^d\langle\gamma_1,\dots,\gamma_m\rangle_{0,m,d}^{\mathbb{P}^1,T}=\delta_{m,3}\int_{\mathbb{P}^1}\gamma_1\cup\dots\cup\gamma_m+q\prod_{i=1}^m\int_{\mathbb{P}^1}\gamma_i.$$

2.3 Equivariant quantum cohomology of $\mathbb{P}^1$

As a $\mathbb{C}[\mathsf{w}]$–module, $QH_T^*(\mathbb{P}^1;\mathbb{C})=H_T^*(\mathbb{P}^1;\mathbb{C})$. The ring structure is given by the quantum product $*$ defined by

$$(\gamma_1\star\gamma_2,\gamma_3)=\langle\!\langle\gamma_1,\gamma_2,\gamma_3\rangle\!\rangle_{0,3}^{\mathbb{P}^1,T},$$

or equivalently,

$$\gamma_1\star\gamma_2=\gamma_1\cup\gamma_2+q\left(\int_{\mathbb{P}^1}\gamma_1\right)\left(\int_{\mathbb{P}^1}\gamma_2\right),$$

where $\cup$ is the product in $H_T^*(\mathbb{P}^1)$ and $q=Qe^{t^1}$. In particular,

$$H\star H=(\mathsf{w}_1+\mathsf{w}_2)H-\mathsf{w}_1\mathsf{w}_2+q.$$

The T–equivariant quantum cohomology of $\mathbb{P}^1$ is

$$QH_T^*(\mathbb{P}^1;\mathbb{C})=\mathbb{C}[H,\mathsf{w},q]/\langle(H-\mathsf{w}_1)\star(H-\mathsf{w}_2)-q\rangle,$$

where $\deg H=\deg\mathsf{w}_i=2$ and $\deg q=4$.

The (nonequivariant) quantum cohomology of $\mathbb{P}^1$ is

$$\mathbb{C}[H,q]/\langle H\star H-q\rangle.$$

Let

$$\phi_1(q)=\frac{1}{2}+\frac{H-\frac{1}{2}(\mathsf{w}_1+\mathsf{w}_2)}{(\mathsf{w}_1-\mathsf{w}_2)\sqrt{1+4q/(\mathsf{w}_1-\mathsf{w}_2)^2}},$$
$$\phi_2(q)=\frac{1}{2}+\frac{H-\frac{1}{2}(\mathsf{w}_1+\mathsf{w}_2)}{(\mathsf{w}_2-\mathsf{w}_1)\sqrt{1+4q/(\mathsf{w}_1-\mathsf{w}_2)^2}}.$$

Then

$$\phi_\alpha(q)\star\phi_\beta(q)=\delta_{\alpha\beta}\phi_\alpha(q),$$

so $\{\phi_1(q),\phi_2(q)\}$ is a canonical basis of the semisimple algebra

$$QH_T^*(\mathbb{P}^1;\mathbb{C})\otimes\mathbb{C}\Big[\mathsf{w},\frac{1}{\Delta^1(q)}\Big],$$

where $\Delta^1(q)$ is defined by (2). We also have

$$\begin{aligned}(\phi_\alpha(q),\phi_\beta(q)) &= (1\star\phi_\alpha(q),\phi_\beta(q)) = (1,\phi_\alpha(q)\star\phi_\beta(q))\\ &= \delta_{\alpha\beta}(1,\phi_\alpha(q)) = \delta_{\alpha\beta}\int_{\mathbb{P}^1}\phi_\alpha(q) = \frac{\delta_{\alpha\beta}}{\Delta^\alpha(q)},\end{aligned}$$

where

$$\begin{aligned}\Delta^1(q) &= (\mathsf{w}_1-\mathsf{w}_2)\sqrt{1+\frac{4q}{(\mathsf{w}_1-\mathsf{w}_2)^2}},\\ \Delta^2(q) &= (\mathsf{w}_2-\mathsf{w}_1)\sqrt{1+\frac{4q}{(\mathsf{w}_1-\mathsf{w}_2)^2}} = -\Delta^1(q).\end{aligned} \tag{2}$$

Quantum multiplication by the hyperplane class is given by

$$H\star\phi_\alpha = \frac{\mathsf{w}_1+\mathsf{w}_2+\Delta^\alpha(q)}{2}\phi_\alpha,\quad \alpha=1,2.$$

Finally, we take the nonequivariant limit $\mathsf{w}_2=0$, $\mathsf{w}_1\to 0^+$. We obtain:

$$\begin{aligned}\phi_1(q) &= \frac{1}{2}+\frac{H}{2\sqrt{q}}, & \phi_2(q) &= \frac{1}{2}-\frac{H}{2\sqrt{q}},\\ \Delta^1(q) &= 2\sqrt{q}, & \Delta^2(q) &= -2\sqrt{q},\\ H\star\phi_1(q) &= \sqrt{q}\phi_1(q), & H\star\phi_2(q) &= -\sqrt{q}\phi_2(q).\end{aligned}$$

These nonequivariant limits coincide with the results in [29, Section 2].

2.4 The A–model canonical coordinates and the Ψ–matrix

Let $\{t^0,t^1\}$ be the flat coordinates with respect to the basis $\{\mathbf{1},H\}$, and let $\{u^1,u^2\}$ be the canonical coordinates with respect to the basis $\{\phi_1(q),\phi_2(q)\}$. Then

$$\begin{aligned}\frac{\partial}{\partial u^1} &= \frac{1}{2}\Big(1-\frac{\mathsf{w}_1+\mathsf{w}_2}{\Delta^1(q)}\Big)\frac{\partial}{\partial t^0}+\frac{1}{\Delta^1(q)}\frac{\partial}{\partial t^1},\\ \frac{\partial}{\partial u^2} &= \frac{1}{2}\Big(1-\frac{\mathsf{w}_1+\mathsf{w}_2}{\Delta^2(q)}\Big)\frac{\partial}{\partial t^0}+\frac{1}{\Delta^2(q)}\frac{\partial}{\partial t^1},\\ du^1 &= dt^0+\frac{1}{2}(\Delta^1(q)+\mathsf{w}_1+\mathsf{w}_2)dt^1,\\ du^2 &= dt^0+\frac{1}{2}(\Delta^2(q)+\mathsf{w}_1+\mathsf{w}_2)dt^1.\end{aligned}$$

The above equations determine the canonical coordinates u^1 and u^2 up to a constant in $\mathbb{C}[\mathsf{w}_1, \mathsf{w}_2, 1/(\mathsf{w}_1-\mathsf{w}_2)]$. Givental's A–model canonical coordinates (u^1, u^2) are characterized by their large radius limits

$$\lim_{q\to 0}(u^1 - t^0 - \mathsf{w}_1 t^1) = 0, \quad \lim_{q\to 0}(u^2 - t^0 - \mathsf{w}_2 t^1) = 0. \tag{3}$$

For $\alpha \in \{1,2\}$ and $i \in \{0,1\}$, define ${\Psi_i}^\alpha$ by

$$\frac{du^\alpha}{\sqrt{\Delta^\alpha(q)}} = \sum_{i=0}^{1} dt^i {\Psi_i}^\alpha,$$

and define the Ψ–matrix to be

$$\Psi := \begin{bmatrix} {\Psi_0}^1 & {\Psi_0}^2 \\ {\Psi_1}^1 & {\Psi_1}^2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} \frac{du^1}{\sqrt{\Delta^1(q)}} & \frac{du^2}{\sqrt{\Delta^2(q)}} \end{bmatrix} = [\, dt^0 \;\; dt^1 \,]\Psi,$$

$${\Psi_0}^\alpha = \frac{1}{\sqrt{\Delta^\alpha(q)}}, \quad {\Psi_1}^\alpha = \frac{\Delta^\alpha(q)+\mathsf{w}_1+\mathsf{w}_2}{2\sqrt{\Delta^\alpha(q)}}.$$

Let

$$\Psi^{-1} = \begin{bmatrix} {(\Psi^{-1})_1}^0 & {(\Psi^{-1})_1}^1 \\ {(\Psi^{-1})_2}^0 & {(\Psi^{-1})_2}^1 \end{bmatrix}$$

be the inverse matrix of Ψ, so that

$$\sum_{i=0}^{1} {(\Psi^{-1})_\alpha}^i {\Psi_i}^\beta = {\delta_\alpha}^\beta.$$

Then

$${(\Psi^{-1})_\alpha}^0 = \frac{\Delta^\alpha(q)-\mathsf{w}_1-\mathsf{w}_2}{2\sqrt{\Delta^\alpha(q)}}, \quad {(\Psi^{-1})_\alpha}^1 = \frac{1}{\sqrt{\Delta^\alpha(q)}}.$$

Let $Q=1$, ie $q=e^{t^1}$. We take the nonequivariant limit $\mathsf{w}_2=0$, $\mathsf{w}_1\to 0^+$:

$$u^1 = t^0 + 2\sqrt{q}, \quad u^2 = t^0 - 2\sqrt{q},$$

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{1}{4}t^1} & -\sqrt{-1}e^{-\frac{1}{4}t^1} \\ e^{\frac{1}{4}t^1} & \sqrt{-1}e^{\frac{1}{4}t^1} \end{pmatrix},$$

$$\Psi^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{1}{4}t^1} & e^{-\frac{1}{4}t^1} \\ \sqrt{-1}e^{\frac{1}{4}t^1} & -\sqrt{-1}e^{-\frac{1}{4}t^1} \end{pmatrix}.$$

These nonequivariant limits agree with the results in [29, Section 2].

2.5 The $\mathcal{S}$–operator

The $\mathcal{S}$–operator is defined as follows: for any cohomology classes $a, b \in H^*_T(\mathbb{P}^1;\mathbb{C})$,

$$(a, \mathcal{S}(b)) = \left\langle\!\!\left\langle a, \frac{b}{z-\psi} \right\rangle\!\!\right\rangle_{0,2}^{\mathbb{P}^1,T}.$$

The T–equivariant J–function is characterized by

$$(J, a) = (1, \mathcal{S}(a))$$

for any $a \in H^*_T(\mathbb{P}^1)$.

Let

$$\chi^1 = \mathsf{w}_1 - \mathsf{w}_2, \quad \chi^2 = \mathsf{w}_2 - \mathsf{w}_1.$$

We consider several different (flat) bases for $H^*_T(\mathbb{P}^1;\mathbb{C})$:

- The canonical basis: $\phi_1 = (H-\mathsf{w}_2)/(\mathsf{w}_1-\mathsf{w}_2)$ and $\phi_2 = (H-\mathsf{w}_1)/(\mathsf{w}_2-\mathsf{w}_1)$.
- The basis dual to the canonical basis with respect to the T–equivariant Poincaré pairing: $\phi^1 = \chi^1\phi_1$ and $\phi^2 = \chi^2\phi_2$.
- The normalized canonical basis $\hat{\phi}_1 = \sqrt{\chi^1}\phi_1$ and $\hat{\phi}_2 = \sqrt{\chi^2}\phi_2$, which is self-dual: $\hat{\phi}^1 = \hat{\phi}_1$ and $\hat{\phi}^2 = \hat{\phi}_2$.
- The natural basis: $T_0 = 1$ and $T_1 = H$.
- The basis dual to the natural basis: $T^0 = H - \mathsf{w}_1 - \mathsf{w}_2$ and $T^1 = 1$.

For $\alpha, \beta \in \{1,2\}$, define

$$S^\alpha{}_\beta(z) := (\phi^\alpha, \mathcal{S}(\phi_\beta)).$$

Then $S(z) = (S^\alpha{}_\beta(z))$ is the matrix[1] of the $\mathcal{S}$–operator with respect to the ordered basis (ϕ_1, ϕ_2):

$$\mathcal{S}(\phi_\beta) = \sum_{\alpha=1}^{2} \phi_\alpha S^\alpha{}_\beta(z). \tag{4}$$

For $i \in \{0,1\}$ and $\alpha \in \{1,2\}$, define

$$S_i{}^{\hat{\alpha}}(z) := (T_i, \mathcal{S}(\hat{\phi}^\alpha)).$$

Then $(S_i{}^{\hat{\alpha}})$ is the matrix of the $\mathcal{S}$–operator with respect to the ordered bases $(\hat{\phi}^1, \hat{\phi}^2)$ and (T^0, T^1):

$$\mathcal{S}(\hat{\phi}^\alpha) = \sum_{i=0}^{1} T^i S_i{}^{\hat{\alpha}}(z). \tag{5}$$

[1] We use the convention that the *left* superscript/subscript is the *row* number and the *right* superscript/subscript is the *column* number.

We have

$$z\frac{\partial J}{\partial t^i} = \sum_{\alpha=1}^{2} S_i^{\,\widehat{\alpha}}(z)\widehat{\phi}_\alpha.$$

By [17; 25], the equivariant J–function is

$$J = e^{(t^0+t^1 H)/z}\biggl(1+\sum_{d=1}^{\infty}\frac{q^d}{\prod_{m=1}^{d}(H-\mathsf{w}_1+mz)\prod_{m=1}^{d}(H-\mathsf{w}_2+mz)}\biggr).$$

For $\alpha = 1, 2$, define

$$J^\alpha := J|_{p_\alpha} = e^{(t^0+t^1\mathsf{w}^\alpha)/z}\sum_{d=0}^{\infty}\frac{q^d}{d!z^d}\frac{1}{\prod_{m=1}^{d}(\chi^\alpha+mz)}.$$

Then

$$z\frac{\partial J}{\partial t^0} = J = \sum_{\alpha=1}^{2} J^\alpha\phi_\alpha,\quad z\frac{\partial J}{\partial t^1} = z\sum_{\alpha=1}^{2}\frac{\partial J^\alpha}{\partial t^1}\phi_\alpha,$$

so

$$S_i^{\,\widehat{\alpha}}(z) = \frac{z}{\sqrt{\chi^\alpha}}\cdot\frac{\partial J^\alpha}{\partial t^i}.$$

Following Givental, we define

$$\widetilde{S}_i^{\,\widehat{\alpha}}(z) := S_i^{\,\widehat{\alpha}}(z)\exp\biggl(-\sum_{n=1}^{\infty}\frac{B_{2n}}{2n(2n-1)}\Bigl(\frac{z}{\chi^\alpha}\Bigr)^{2n-1}\biggr).$$

Then

$$\widetilde{S}_0^{\,\widehat{\alpha}}(z) = \frac{1}{\sqrt{\chi^\alpha}}\exp\biggl(\frac{t^0+t^1\mathsf{w}_\alpha}{z}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2n(2n-1)}\Bigl(\frac{z}{\chi^\alpha}\Bigr)^{2n-1}\biggr)\cdot\biggl(\sum_{d=0}^{\infty}\frac{q^d}{d!z^d}\frac{1}{\prod_{m=1}^{d}(\chi^\alpha+mz)}\biggr),$$

$$\widetilde{S}_1^{\,\widehat{\alpha}}(z) = \frac{1}{\sqrt{\chi^\alpha}}\exp\biggl(\frac{t^0+t^1\mathsf{w}_\alpha}{z}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2n(2n-1)}\Bigl(\frac{z}{\chi^\alpha}\Bigr)^{2n-1}\biggr)\cdot\biggl(\mathsf{w}_\alpha\sum_{d=0}^{\infty}\frac{q^d}{d!z^d}\frac{1}{\prod_{m=1}^{d}(\chi^\alpha+mz)}+\sum_{d=1}^{\infty}\frac{q^d}{(d-1)!z^d}\frac{1}{\prod_{m=1}^{d}(\chi^\alpha+mz)}\biggr).$$

2.6 The A–model R–matrix

By Givental [20], the matrix $(\widetilde{S}_i{}^{\widehat{\beta}})(z)$ is of the form

$$\widetilde{S}_i{}^{\widehat{\beta}}(z) = \sum_{\alpha=1}^{2} \Psi_i{}^{\alpha} R_\alpha{}^{\beta}(z) e^{u^\beta/z} = (\Psi R(z))_i{}^{\beta} e^{u^\beta/z},$$

where $R(z) = (R_\alpha{}^\beta(z)) = I + \sum_{k=1}^{\infty} R_k z^k$ and is unitary, and

$$\lim_{q\to 0} R_\alpha{}^\beta(z) = \delta_{\alpha\beta} \exp\biggl(-\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)}\Bigl(\frac{z}{\chi^\beta}\Bigr)^{2n-1}\biggr).$$

2.7 Gromov–Witten potentials

Introducing formal variables

$$\boldsymbol{u} = \sum_{a\ge 0} u_a z^a, \quad \text{where } u_a = \sum_{\alpha=1}^{2} u_a^\alpha \phi_\alpha(q),$$

we define

$$\begin{aligned} F_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t}) &:= \sum_{\substack{a_1,\dots,a_n\\ a_i\in\mathbb{Z}_{\ge 0}}} \frac{1}{n!}\langle\langle \tau_{a_1}(u_{a_1})\cdots\tau_{a_n}(u_{a_n})\rangle\rangle_{g,n}^{\mathbb{P}^1,T} \\ &= \sum_{\substack{a_1,\dots,a_n\\ a_i\in\mathbb{Z}_{\ge 0}}} \sum_{m=0}^{\infty}\sum_{d=0}^{\infty} \frac{Q^d}{n!m!}\int_{[\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}^1,d)]^{\mathrm{vir}}} \prod_{j=1}^{n} \mathrm{ev}_j^*(u_{a_j})\psi_j^{a_j} \prod_{i=1}^{m} \mathrm{ev}_{i+n}^*(\boldsymbol{t}). \end{aligned}$$

We define the total descendent potential of $\mathbb{P}^1$ to be

$$D^{\mathbb{P}^1,T}(\boldsymbol{u}) = \exp\biggl(\sum_{n,g} \hbar^{g-1} F_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u},0)\biggr).$$

Consider the map $\pi\colon \overline{\mathcal{M}}_{g,n+m}(\mathbb{P}^1,d)\to\overline{\mathcal{M}}_{g,n}$ which forgets the map to the target and the last m marked points. Let $\overline{\psi}_i := \pi^*(\psi_i)$ be the pull-backs of the classes ψ_i for $i=1,\dots,n$ from $\overline{\mathcal{M}}_{g,n}$. Then we can define

$$\overline{F}_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t}) := \sum_{\substack{a_1,\dots,a_n\\ a_i\in\mathbb{Z}_{\ge 0}}} \sum_{m=0}^{\infty}\sum_{d=0}^{\infty} \frac{Q^d}{n!m!}\int_{[\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}^1,d)]^{\mathrm{vir}}} \prod_{j=1}^{n} \mathrm{ev}_j^*(u_{a_j})\overline{\psi}_j^{a_j} \prod_{i=1}^{m} \mathrm{ev}_{i+n}^*(\boldsymbol{t}).$$

Let the ancestor potential of $\mathbb{P}^1$ be

$$A^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t}) = \exp\biggl(\sum_{n,g} \hbar^{g-1} \overline{F}_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t})\biggr).$$

2.8 Givental's formula for equivariant Gromov–Witten potential and the A–model graph sum

The quantization of the $\mathcal{S}$–operator relates the ancestor potential and the descendent potential of $\mathbb{P}^1$ via Givental's formula. Concretely, we have (see [19])

$$D^{\mathbb{P}^1,T}(\boldsymbol{u}) = \exp(F_1^{\mathbb{P}^1,T})\widehat{\mathcal{S}}^{-1} A^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t}),$$

where $F_1^{\mathbb{P}^1,T}$ denotes $\sum_n F_{1,n}^{\mathbb{P}^1,T}(\boldsymbol{u},0)$ at $u_0 = u$ and $u_1 = u_2 = \cdots = 0$, and $\widehat{\mathcal{S}}$ is the quantization [19] of $\mathcal{S}$. For our purpose, we need to describe a formula for a slightly different potential: $F_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t})$ — the descendent potential with arbitrary primary insertions.

Now we first describe a graph sum formula for the ancestor potential $A^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t})$. Given a connected graph Γ, we introduce the following notation:

- $V(\Gamma)$ is the set of vertices in Γ.
- $E(\Gamma)$ is the set of edges in Γ.
- $H(\Gamma)$ is the set of half-edges in Γ.
- $L^o(\Gamma)$ is the set of ordinary leaves in Γ.
- $L^1(\Gamma)$ is the set of dilaton leaves in Γ.

With the above notation, we introduce the following labels:

- **Genus** $g\colon V(\Gamma) \to \mathbb{Z}_{\geq 0}$.
- **Marking** $\beta\colon V(\Gamma) \to \{1,2\}$. This induces $\beta\colon L(\Gamma) = L^o(\Gamma) \cup L^1(\Gamma) \to \{1,2\}$, as follows: if $l \in L(\Gamma)$ is a leaf attached to a vertex $v \in V(\Gamma)$, define $\beta(l) = \beta(v)$.
- **Height** $k\colon H(\Gamma) \to \mathbb{Z}_{\geq 0}$.

Given an edge e, let $h_1(e)$ and $h_2(e)$ be the two half-edges associated to e. The order of the two half-edges does not affect the graph sum formula in this paper. Given a vertex $v \in V(\Gamma)$, let $H(v)$ denote the set of half-edges emanating from v. The valency of the vertex v is equal to the cardinality of the set $H(v)$, written $\mathrm{val}(v) = |H(v)|$. A labeled graph $\vec{\Gamma} = (\Gamma, g, \beta, k)$ is *stable* if

$$2g(v) - 2 + \mathrm{val}(v) > 0$$

for all $v \in V(\Gamma)$.

Let $\mathbf{\Gamma}(\mathbb{P}^1)$ denote the set of all stable labeled graphs $\vec{\Gamma} = (\Gamma, g, \beta, k)$. The genus of a stable labeled graph $\vec{\Gamma}$ is defined to be

$$g(\vec{\Gamma}) := \sum_{v \in V(\Gamma)} g(v) + |E(\Gamma)| - |V(\Gamma)| + 1 = \sum_{v \in V(\Gamma)} (g(v) - 1) + \Big(\sum_{e \in E(\Gamma)} 1\Big) + 1.$$

Define

$$\mathbf{\Gamma}_{g,n}(\mathbb{P}^1) = \{\vec{\Gamma} = (\Gamma, g, \beta, k) \in \mathbf{\Gamma}(\mathbb{P}^1) : g(\vec{\Gamma}) = g,\ |L^o(\Gamma)| = n\}.$$

Given $\alpha \in \{1,2\}$, define

$$\boldsymbol{u}^\alpha(z) = \sum_{a \geq 0} u^\alpha_a z^a.$$

We assign weights to leaves, edges, and vertices of a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}(\mathbb{P}^1)$ as follows:

(1) **Ordinary leaves** To each ordinary leaf $l \in L^o(\Gamma)$ with $\beta(l) = \beta \in \{1,2\}$ and $k(l) = k \in \mathbb{Z}_{\geq 0}$, we assign

$$(\mathcal{L}^{\boldsymbol{u}})^\beta_k(l) = [z^k]\biggl(\sum_{\alpha=1,2} \frac{\boldsymbol{u}^\alpha(z)}{\sqrt{\Delta^\alpha(q)}} R_\alpha{}^\beta(-z)\biggr).$$

(2) **Dilaton leaves** To each dilaton leaf $l \in L^1(\Gamma)$ with $\beta(l) = \beta \in \{1,2\}$ and $2 \leq k(l) = k \in \mathbb{Z}_{\geq 0}$, we assign

$$(\mathcal{L}^1)^\beta_k(l) = [z^{k-1}]\biggl(-\sum_{\alpha=1,2} \frac{1}{\sqrt{\Delta^\alpha(q)}} R_\alpha{}^\beta(-z)\biggr).$$

(3) **Edges** To an edge connecting a vertex marked by $\alpha \in \{1,2\}$ to a vertex marked by $\beta \in \{1,2\}$ and with heights k and l at the corresponding half-edges, we assign

$$\mathcal{E}^{\alpha,\beta}_{k,l}(e) = [z^k w^l]\biggl(\frac{1}{z+w}\biggl(\delta_{\alpha,\beta} - \sum_{\gamma=1,2} R_\gamma{}^\alpha(-z) R_\gamma{}^\beta(-w)\biggr)\biggr).$$

(4) **Vertices** To a vertex v with genus $g(v) = g \in \mathbb{Z}_{\geq 0}$ and marking $\beta(v) = \beta$, with n ordinary leaves and half-edges attached to it with heights $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ and m more dilaton leaves with heights $k_{n+1}, \ldots, k_{n+m} \in \mathbb{Z}_{\geq 0}$, we assign

$$(\sqrt{\Delta^\beta(q)})^{2g-2+n+m} \int_{\overline{\mathcal{M}}_{g,n+m}} \psi_1^{k_1} \cdots \psi_{n+m}^{k_{n+m}}.$$

We define the weight of a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}(\mathbb{P}^1)$ to be

$$w(\vec{\Gamma}) = \prod_{v \in V(\Gamma)} (\sqrt{\Delta^{\beta(v)}(q)})^{2g(v)-2+\mathrm{val}(v)} \biggl\langle \prod_{h \in H(v)} \tau_{k(h)} \biggr\rangle_{g(v)} \prod_{e \in E(\Gamma)} \mathcal{E}^{\beta(v_1(e)),\beta(v_2(e))}_{k(h_1(e)),k(h_2(e))}(e) \cdot \prod_{l \in L^o(\Gamma)} (\mathcal{L}^{\boldsymbol{u}})^{\beta(l)}_{k(l)}(l) \prod_{l \in L^1(\Gamma)} (\mathcal{L}^1)^{\beta(l)}_{k(l)}(l).$$

Then

$$\log(A^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t})) = \sum_{\vec{\Gamma}\in\boldsymbol{\Gamma}(\mathbb{P}^1)} \frac{\hbar^{g(\vec{\Gamma})-1}w(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|} = \sum_{g\geq 0}\hbar^{g-1}\sum_{n\geq 0}\sum_{\vec{\Gamma}\in\boldsymbol{\Gamma}_{g,n}(\mathbb{P}^1)} \frac{w(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

Now we describe a graph sum formula for $F_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t})$ — the descendant potential with arbitrary primary insertions. For $\alpha = 1, 2$, let

$$\widehat{\phi}_\alpha(q) := \sqrt{\Delta^\alpha(q)}\phi_\alpha(q).$$

Then $\widehat{\phi}_1(q)$, $\widehat{\phi}_2(q)$ is the normalized canonical basis of $QH_T^*(\mathbb{P}^1;\mathbb{C})$, the T–equivariant quantum cohomology of $\mathbb{P}^1$. Define

$$S^{\underline{\widehat{\alpha}}}_{\underline{\widehat{\beta}}}(z) := \big(\widehat{\phi}_\alpha(q), \mathcal{S}(\widehat{\phi}_\beta(q))\big).$$

Then this is the matrix of the $\mathcal{S}$–operator with respect to the ordered basis $(\widehat{\phi}_1(q), \widehat{\phi}_2(q))$:

$$\mathcal{S}(\widehat{\phi}_\beta(q)) = \sum_{\alpha=1}^{2}\widehat{\phi}_\alpha(q)S^{\underline{\widehat{\alpha}}}_{\underline{\widehat{\beta}}}(z). \tag{6}$$

We define a new weight of the ordinary leaves:

(1′) **Ordinary leaves** To each ordinary leaf $l\in L^o(\Gamma)$ with $\beta(l)=\beta\in\{1,2\}$ and $k(l)=k\in\mathbb{Z}_{\geq 0}$, we assign

$$(\mathring{\mathcal{L}}^{\boldsymbol{u}})^\beta_k(l) = [z^k]\bigg(\sum_{\alpha,\gamma=1,2}\frac{\boldsymbol{u}^\alpha(z)}{\sqrt{\Delta^\alpha(q)}}S^{\underline{\widehat{\gamma}}}_{\underline{\widehat{\alpha}}}(z)R(-z)_\gamma{}^\beta\bigg).$$

We define a new weight of a labeled graph $\vec{\Gamma}\in\boldsymbol{\Gamma}(\mathbb{P}^1)$ to be

$$\mathring{w}(\vec{\Gamma}) = \prod_{v\in V(\Gamma)}\big(\sqrt{\Delta^{\beta(v)}(q)}\big)^{2g(v)-2+\mathrm{val}(v)}\Big\langle\prod_{h\in H(v)}\tau_{k(h)}\Big\rangle_{g(v)}\prod_{e\in E(\Gamma)}\mathcal{E}^{\beta(v_1(e)),\beta(v_2(e))}_{k(h_1(e)),k(h_2(e))}(e) \cdot\prod_{l\in L^o(\Gamma)}(\mathring{\mathcal{L}}^{\boldsymbol{u}})^{\beta(l)}_{k(l)}(l)\prod_{l\in L^1(\Gamma)}(\mathcal{L}^1)^{\beta(l)}_{k(l)}(l).$$

Then

$$\sum_{g\geq 0}\hbar^{g-1}\sum_{n\geq 0}F_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u},\boldsymbol{t}) = \sum_{\vec{\Gamma}\in\boldsymbol{\Gamma}(\mathbb{P}^1)}\frac{\hbar^{g(\vec{\Gamma})-1}\mathring{w}(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|} = \sum_{g\geq 0}\hbar^{g-1}\sum_{n\geq 0}\sum_{\vec{\Gamma}\in\boldsymbol{\Gamma}_{g,n}(\mathbb{P}^1)}\frac{\mathring{w}(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

We can slightly generalize this graph sum formula to the case where we have n *ordered* variables $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$ and n *ordered* ordinary leaves. Let

$$\boldsymbol{u}_j = \sum_{a\geq 0}(u_j)_a z^a$$

and let

$$F_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n,\boldsymbol{t}) := \sum_{\substack{a_1,\dots,a_n\\ a_i\in\mathbb{Z}_{\geq 0}}} \sum_{m=0}^{\infty}\sum_{d=0}^{\infty} \frac{1}{m!}\int_{[\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}^1,d)]^{\mathrm{vir}}} \prod_{j=1}^{n} \mathrm{ev}_j^*((u_j)_{a_j})\psi_j^{a_j} \cdot \prod_{i=1}^{m} \mathrm{ev}_{i+n}^*(\boldsymbol{t}).$$

Define the set of graphs $\widetilde{\boldsymbol{\Gamma}}_{g,n}(\mathbb{P}^1)$ as the definition of $\boldsymbol{\Gamma}_{g,n}(\mathbb{P}^1)$ except that the n ordinary leaves are *ordered*. Let $\{l_1,\dots,l_n\}$ be the ordinary leaves in $\Gamma\in\widetilde{\boldsymbol{\Gamma}}_{g,n}(\mathbb{P}^1)$ and for $j=1,\dots,n$ let

$$(\mathring{\mathcal{L}}^{\boldsymbol{u}_j})_k^{\beta}(l_j) = [z^k]\biggl(\sum_{\alpha,\gamma=1,2} \frac{\boldsymbol{u}_j^{\alpha}(z)}{\sqrt{\Delta^{\alpha}(q)}} S^{\hat{\gamma}}_{\;\underline{\hat{\alpha}}}(z) R(-z)_{\gamma}^{\;\beta}\biggr).$$

Define the weight

$$\mathring{w}(\vec{\Gamma}) = \prod_{v\in V(\Gamma)} \bigl(\sqrt{\Delta^{\beta(v)}(q)}\bigr)^{2g(v)-2+\mathrm{val}(v)} \biggl\langle \prod_{h\in H(v)} \tau_{k(h)} \biggr\rangle_{g(v)} \prod_{e\in E(\Gamma)} \mathcal{E}^{\beta(v_1(e)),\beta(v_2(e))}_{k(h_1(e)),k(h_2(e))}(e) \cdot \prod_{j=1}^{n} (\mathring{\mathcal{L}}^{\boldsymbol{u}_j})^{\beta(l_j)}_{k(l_j)}(l_j) \prod_{l\in L^1(\Gamma)} (\mathcal{L}^1)^{\beta(l)}_{k(l)}(l).$$

Then

$$\sum_{g\geq 0}\hbar^{g-1}\sum_{n\geq 0} F_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_n,\boldsymbol{t}) = \sum_{\vec{\Gamma}\in\widetilde{\boldsymbol{\Gamma}}(\mathbb{P}^1)} \frac{\hbar^{g(\vec{\Gamma})-1}\mathring{w}(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|} = \sum_{g\geq 0}\hbar^{g-1}\sum_{n\geq 0}\sum_{\vec{\Gamma}\in\widetilde{\boldsymbol{\Gamma}}_{g,n}(\mathbb{P}^1)} \frac{\mathring{w}(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

3 B–model

3.1 The equivariant superpotential and the Frobenius structure of the Jacobian ring

Let Y be coordinates on $\mathbb{C}^*$. The T–equivariant superpotential $W_t^{\mathsf{w}}\colon \mathbb{C}^*\to\mathbb{C}$ is given by

$$W_t^{\mathsf{w}}(Y) = Y + t_0 + \frac{q}{Y} + \mathsf{w}_1\log Y + \mathsf{w}_2\log\frac{q}{Y},$$

where $q = Qe^{t_1}$ and $Y = e^y$. In this section, we assume $\mathsf{w}_1-\mathsf{w}_2$ is a positive real number. The Jacobian ring of W_t^{w} is

$$\mathrm{Jac}(W_t^{\mathsf{w}}) \cong C[Y,Y^{-1},q,\mathsf{w}]\Big/\Big\langle\frac{\partial W_t^{\mathsf{w}}}{\partial y}\Big\rangle = \mathbb{C}[Y,Y^{-1},q,\mathsf{w}]\Big/\Big\langle Y - \frac{q}{Y} + \mathsf{w}_1 - \mathsf{w}_2\Big\rangle.$$

Let

$$B := q\frac{\partial W_t^{\mathsf{w}}}{\partial q} = \frac{q}{Y} + \mathsf{w}_2.$$

The Jacobian ring is isomorphic to $QH_T^*(\mathbb{P}^1;\mathbb{C})$ if one identifies B with H:

$$\mathrm{Jac}(W_t^{\mathsf{w}}) \cong \mathbb{C}[B,q,\mathsf{w}]/\langle (B-\mathsf{w}_1)(B-\mathsf{w}_2)-q\rangle.$$

The critical points of W_t^{w} are P_1 and P_2, where

$$P_\alpha = \frac{\mathsf{w}_2-\mathsf{w}_1+\Delta^\alpha(q)}{2}, \quad \alpha=1,2.$$

Endow a metric on $\mathrm{Jac}(W_q^{\mathsf{w}})$ by the residue pairing

$$(f,g) = \sum_{\alpha=1}^{2} \mathrm{Res}_{Y=P_\alpha} \frac{f(Y)g(Y)}{\partial W_t^{\mathsf{w}}/\partial y}\frac{dY}{Y}.$$

By direct calculation, we have

$$(B,B)=\mathsf{w}_1+\mathsf{w}_2, \quad (B,\mathbf{1})=(\mathbf{1},B)=1, \quad (\mathbf{1},\mathbf{1})=0.$$

We let $b_0=\mathbf{1}$, $b_1=B$ and define b^i by $(b^i,b_j)=\delta^i_j$. These calculations show the following well-known fact:

Proposition 3.1 *There is an isomorphism of Frobenius manifolds*

$$QH_T^*(\mathbb{P}^1;\mathbb{C})\otimes_{\mathbb{C}[\mathsf{w}]}\mathbb{C}\Big[\mathsf{w},\frac{1}{\mathsf{w}_1-\mathsf{w}_2}\Big] \cong \mathrm{Jac}(W_t^{\mathsf{w}})\otimes_{\mathbb{C}[\mathsf{w}]}\mathbb{C}\Big[\mathsf{w},\frac{1}{\mathsf{w}_1-\mathsf{w}_2}\Big].$$

We denote $\mathrm{Jac}(W_t^{\mathsf{w}})\otimes_{\mathbb{C}[\mathsf{w}]}\mathbb{C}[\mathsf{w},1/(\mathsf{w}_1-\mathsf{w}_2)]$ by H_B. The Dubrovin–Givental connection is denoted by $\nabla^B_v = z\partial_v + v\bullet$ on $\mathcal{H}_B := H_B((z))$.

3.2 The B–model canonical coordinates

The isomorphism of Frobenius structures automatically ensures their canonical coordinates are the same up to a permutation and constants. We fix the B–model canonical coordinates in this subsection by the critical values of the superpotential W_t^{w}, and find the constant difference to the A–model coordinates that we set up in earlier sections.

Let $C_t^{\mathsf{w}} = \{(x,y)\in\mathbb{C}^2 : x = W_t^{\mathsf{w}}(e^y)\}$ be the graph of the equivariant superpotential. It is a covering of $\mathbb{C}^*$, given by $y\mapsto e^y$. Let $\overline{\Sigma}\cong\mathbb{P}^1$ be the compactification of $\mathbb{C}^*$ with $Y\in\mathbb{C}^*\subset\mathbb{P}^1$ as its coordinate. At each branch point $Y=P_\alpha$, we have the expansions

$$x = \check{u}^\alpha - \zeta_\alpha^2, \quad y = \check{v}^\alpha - \sum_{k=1}^{\infty} h_k^\alpha(q)\zeta_\alpha^k,$$

where $h_1^\alpha(q) = \sqrt{2/\Delta^\alpha(q)}$. Note that we define ζ_α by $\zeta_\alpha^2 = \check{u}^\alpha - x$, which differs from the definition of ζ in [7; 11] by a factor of $\sqrt{-1}$.

The critical values are

$$\check{u}^\alpha = t^0 + \mathsf{w}_\alpha t^1 + \Delta^\alpha(q) - \chi^\alpha \log \frac{\chi^\alpha + \Delta^\alpha(q)}{2}.$$

Since

$$\frac{\partial \check{u}^\alpha}{\partial t^0} = 1, \quad \frac{\partial \check{u}^\alpha}{\partial t^1} = \frac{q}{P_\alpha} + \mathsf{w}_2 = \frac{\mathsf{w}_1 + \mathsf{w}_2 + \Delta^\alpha(q)}{2},$$

we have

$$d\check{u}^\alpha = du^\alpha, \quad \alpha = 1, 2. \tag{7}$$

Recall that $\lim_{q\to 0} \Delta^1(q) = \mathsf{w}_1 - \mathsf{w}_2$, so in the large radius limit $q \to 0$ we have

$$\lim_{q\to 0} (\check{u}^\alpha - t^0 - \mathsf{w}_\alpha t^1) = \chi^\alpha - \chi^\alpha \log \chi^\alpha. \tag{8}$$

From (7), (8) and (3), we conclude that

$$\check{u}^\alpha = u^\alpha + a_\alpha, \quad \alpha = 1, 2,$$

where

$$a_\alpha = \chi^\alpha - \chi^\alpha \log \chi^\alpha.$$

3.3 The Liouville form and Bergman kernel

On C_t^{w}, let

$$\lambda = x\, dy$$

be the Liouville form on $\mathbb{C}^2 = T^*\mathbb{C}$. Then $d\lambda = dx \wedge dy$. Let

$$\Phi := \lambda|_{C_t^{\mathsf{w}}} = W_t^{\mathsf{w}}(e^y)\, dy = (e^y + t_0 + qe^{-y} + (\mathsf{w}_1 - \mathsf{w}_2) y + \mathsf{w}_2 \log q)\, dy.$$

Then Φ is a holomorphic 1–form on $\mathbb{C}$. Recall that $q = Qe^{t^1}$ and $Y = e^y$. Define

$$\Phi_0 := \frac{\partial \Phi}{\partial t^0} = \frac{dY}{Y},$$
$$\Phi_1 := \frac{\partial \Phi}{\partial t^1} = \Big(\frac{q}{Y} + \mathsf{w}_2\Big)\frac{dY}{Y}.$$

Then Φ_0 and Φ_1 descend to holomorphic 1–forms on $\mathbb{C}^*$ which extends to meromorphic 1–forms on $\mathbb{P}^1$. We have:

- Φ_0 has simple poles at $Y = 0$ and $Y = \infty$, and

$$\mathrm{Res}_{Y\to 0}\, \Phi_0 = 1, \quad \mathrm{Res}_{Y\to\infty}\, \Phi_1 = -1.$$

- $\Phi_1 - \mathsf{w}_2 \Phi_0 = -q\, d(Y^{-1})$ is an exact 1–form.

Let $B(p_1, p_2)$ be the fundamental normalized differential of the second kind on $\overline{\Sigma}$ (see eg [16]). It is also called the Bergman kernel in [10; 11]. In this simple case with $\overline{\Sigma} \cong \mathbb{P}^1$, we have

$$B(Y_1, Y_2) = \frac{dY_1 \otimes dY_2}{(Y_1 - Y_2)^2}.$$

3.4 Differentials of the second kind

Following [7; 11], given $\alpha = 1, 2$ and $d \in \mathbb{Z}_{\geq 0}$, define

$$d\xi_{\alpha,d}(p) := (2d-1)!!\, 2^{-d}\, \mathrm{Res}_{p' \to P_\alpha} B(p, p')(\sqrt{-1}\zeta_\alpha)^{-2d-1}.$$

Then $d\xi_{\alpha,d}$ satisfies the following properties:

- $d\xi_{\alpha,d}$ is a meromorphic 1–form on $\mathbb{P}^1$ with a single pole of order $2d+2$ at P_α.
- In the local coordinate ζ_α near P_α,

$$d\xi_{\alpha,d} = \left(\frac{-(2d+1)!!}{2^d \sqrt{-1}^{2d+1} \zeta_\alpha^{2d+2}} + f(\zeta_\alpha) \right) d\zeta_\alpha,$$

  where $f(\zeta_\alpha)$ is analytic around P_α. The residue of $d\xi_{\alpha,d}$ at P_α is zero, so $d\xi_{\alpha,d}$ is a differential of the second kind.

The meromorphic 1–form $d\xi_{\alpha,d}$ is characterized by the above properties; $d\xi_{\alpha,d}$ can be viewed as a section in $H^0\big(\mathbb{P}^1, \omega_{\mathbb{P}^1}((2d+2)P_\alpha)\big)$. In particular, $d\xi_{\alpha,0}$ is

$$d\xi_{\alpha,0} = \sqrt{\frac{-2}{\Delta^\alpha(q)}}\, d\left(\frac{P_\alpha}{Y - P_\alpha} \right).$$

Then we have

$$\begin{aligned}
d\left(\frac{\Phi_0}{dW}\right) &= d\left(\frac{Y}{(Y-P_1)(Y-P_2)} \right) = \frac{1}{P_1 - P_2} d\left(\frac{P_1}{Y-P_1} - \frac{P_2}{Y-P_2} \right) \\
&= \frac{1}{\sqrt{-1}} \frac{1}{\sqrt{2\Delta^1(q)}} d\xi_{1,0} + \frac{1}{\sqrt{-1}} \frac{1}{\sqrt{2\Delta^2(q)}} d\xi_{2,0} \\
&= \frac{1}{\sqrt{-2}} \sum_{\alpha=1}^{2} {\Psi_0}^{\alpha} d\xi_{\alpha,0}, \\
d\left(\frac{\Phi_1}{dW}\right) &= d\left(\frac{q + \mathsf{w}_2 Y}{(Y-P_1)(Y-P_2)} \right) \\
&= \frac{1}{P_1 - P_2} d\left(\frac{q + P_1 \mathsf{w}_2}{Y - P_1} - \frac{q + P_2 \mathsf{w}_2}{Y - P_2} \right) \\
&= \frac{1}{\sqrt{-1}} \frac{1}{\Delta^1(q)} \left(\sqrt{\frac{\Delta^1(q)}{2}} \left(\frac{q}{P_1} + \mathsf{w}_2 \right) d\xi_{1,0} - \sqrt{\frac{\Delta^2(q)}{2}} \left(\frac{q}{P_2} + \mathsf{w}_2 \right) d\xi_{2,0} \right)
\end{aligned}$$

$$= \frac{1}{2\sqrt{-2}}\left(\left(\sqrt{\Delta^1(q)}+\frac{\mathsf{w}_1+\mathsf{w}_2}{\sqrt{\Delta^1(q)}}\right)d\xi_{1,0}+\left(\sqrt{\Delta^2(q)}+\frac{\mathsf{w}_1+\mathsf{w}_2}{\sqrt{\Delta^2(q)}}\right)d\xi_{2,0}\right)$$

$$= \frac{1}{\sqrt{-2}}\sum_{\alpha=1}^{2}\Psi_1{}^{\alpha}\,d\xi_{\alpha,0},$$

so

$$(9)\qquad \begin{pmatrix} d\left(\frac{\Phi_0}{dW}\right) \\ d\left(\frac{\Phi_1}{dW}\right)\end{pmatrix} = \frac{1}{\sqrt{-2}}\Psi\begin{pmatrix} d\xi_{1,0} \\ d\xi_{2,0}\end{pmatrix}, \quad \sqrt{-2}\Psi^{-1}\begin{pmatrix} d\left(\frac{\Phi_0}{dW}\right) \\ d\left(\frac{\Phi_1}{dW}\right)\end{pmatrix} = \begin{pmatrix} d\xi_{1,0} \\ d\xi_{2,0}\end{pmatrix}.$$

3.5 Oscillating integrals and the B–model R–matrix

For $\alpha, \beta \in \{1,2\}$, $i \in \{0,1\}$ and $z > 0$, define

$$\check{S}_i{}^{\boldsymbol{\alpha}}(z) := \int_{y\in\gamma_\alpha} e^{W_q^{\mathsf{w}}(Y)/z}\Phi_i = -z\int_{y\in\gamma_\alpha} e^{W_q^{\mathsf{w}}(Y)/z}\, d\left(\frac{\Phi_i}{dW}\right),$$

where γ_α is the Lefschetz thimble going through P_α such that $W_q^{\mathsf{w}}(Y) \to -\infty$ near its ends. It is straightforward to check that $\sum_{i=0}^{1} b^i \check{S}_i{}^{\boldsymbol{\alpha}}$ is a solution to the quantum differential equation $\nabla^B f = 0$ for $\alpha = 1, 2$. We quote the following theorem:

Theorem 3.2 [3; 18; 20] *Near a semisimple point on a Frobenius manifold of dimension n, there is a fundamental solution S to the quantum differential equation satisfying the following properties:*

(1) *S has the form*

$$S = \Psi R(z) e^{U/z},$$

where $R(z)$ is a matrix of formal power series in z and $U = \mathrm{diag}(u^1, \dots, u^n)$ is a matrix formed by canonical coordinates.

(2) *If S is unitary under the pairing of the Frobenius structure, then $R(z)$ is unique up to a right multiplication of $e^{\sum_{i=1}^{\infty} A_{2i-1} z^{2i-1}}$, where the A_k are constant diagonal matrices.*

Remark 3.3 For equivariant Gromov–Witten theory of $\mathbb{P}^1$, the fundamental solution S in Theorem 3.2 is viewed as a matrix with entries in $\mathbb{C}[\mathsf{w}, 1/(\mathsf{w}_1-\mathsf{w}_2)]((z))[\![q, t^0, t^1]\!]$. We choose the canonical coordinates $\{u^\alpha(t)\}$ such that there is no constant term by (3). Then, if we fix the powers of q, t^0 and t^1, only finitely many terms in the expansion of $e^{U/z}$ contribute. So the multiplication $\Psi R(z) e^{U/z}$ is well-defined and the result matrix indeed has entries in $\mathbb{C}[\mathsf{w}, 1/(\mathsf{w}_1 - \mathsf{w}_2)]((z))[\![q, t^0, t^1]\!]$.

Remark 3.4 For a general abstract semisimple Frobenius manifold defined over a ring A, the expression $S = \Psi R(z)e^{U/z}$ in Theorem 3.2 can be understood in the following way. We consider the free module $M = \langle e^{u^1/z}\rangle \oplus \cdots \oplus \langle e^{u^n/z}\rangle$ over the ring $A((z))[\![t^1, \cdots, t^n]\!]$, where $t^1, \ldots, t^n$ are the flat coordinates of the Frobenius manifold. We formally define the differential $de^{u^i/z} = e^{u^i/z}\, du^i/z$ and we extend the differential to M by the product rule. Then we have a map $d\colon M \to M\, dt^1 \oplus \cdots \oplus M\, dt^n$. We consider the fundamental solution $S = \Psi R(z)e^{U/z}$ as a matrix with entries in M. The meaning that S satisfies the quantum differential equation is understood by the above formal differential.

In our case, the multiplication in the A–model fundamental solution $S = \Psi R(z)e^{U/z}$ is formal in z, as in Remark 3.3. On the B–model side, we use the stationary phase expansion to obtain a product of the form $\Psi R(z)e^{U/z}$. The multiplications $\Psi R(z)e^{U/z}$ on both the A–model and B–model can be viewed as matrices with entries in M, and their differentials are obviously the same with the formal differential above.

We repeat the argument in Givental [19] and state it as the following fact:

Proposition 3.5 *The fundamental solution matrix $\{\check{S}_i{}^{\boldsymbol{\alpha}}/\sqrt{-2\pi z}\}$ has the asymptotic expansion, where $\check{R}(z)$ is a formal power series in z,*

$$\frac{\check{S}_i{}^{\boldsymbol{\alpha}}(z)}{\sqrt{-2\pi z}} \sim \sum_{\gamma=1}^{2} \Psi_i{}^{\gamma}\, \check{R}_{\gamma}^{\ \alpha}(z) e^{\check{u}^{\alpha}/z}.$$

Proof By the stationary phase expansion,

$$\check{S}_i{}^{\alpha}(z) \sim \sqrt{2\pi z}\, e^{\check{u}^{\alpha}/z}(1 + a_{i,1}^{\alpha} z + a_{i,2}^{\alpha} z^2 + \cdots),$$

it follows that $\{\check{S}_i{}^{\alpha}\}$ can be asymptotically expanded in the desired form (notice that Ψ is a matrix in z–degree 0). In particular, by (9),

$$\check{R}_{\beta}^{\ \alpha}(z) \sim \frac{\sqrt{z}\, e^{-\check{u}^{\alpha}/z}}{2\sqrt{\pi}} \int_{\gamma_\alpha} e^{W_t^{\mathrm{w}}/z}\, d\xi_{\beta,0}.$$

The above B–model R–matrix $\check{R}_{\beta}^{\ \alpha}(z)$ is related to $f_{\beta}^{\alpha}(u)$ in Eynard [8] by

$$f_{\beta}^{\alpha}(u) = \check{R}_{\beta}^{\ \alpha}\Big(-\frac{1}{u}\Big). \tag{10}$$

Following Eynard [8], define the Laplace transform of the Bergman kernel

$$\check{B}^{\alpha,\beta}(u,v,q) := \frac{uv}{u+v}\delta_{\alpha,\beta} + \frac{\sqrt{uv}}{2\pi} e^{u\check{u}^{\alpha} + v\check{u}^{\beta}} \int_{p_1 \in \gamma_\alpha} \int_{p_2 \in \gamma_\beta} B(p_1, p_2) e^{-ux(p_1) - vx(p_2)},$$

where $\alpha, \beta \in \{1,2\}$. By [8, Equation (B.9)] and (10),

$$\check{B}^{\alpha,\beta}(u,v,q) = \frac{uv}{u+v}\Big(\delta_{\alpha,\beta} - \sum_{\gamma=1}^{2} \check{R}_{\gamma}^{\ \alpha}\Big(-\frac{1}{u}\Big)\check{R}_{\gamma}^{\ \beta}\Big(-\frac{1}{v}\Big)\Big). \tag{11}$$

Setting $u = -v$, we conclude that

$$\Big(\check{R}^*\Big(\frac{1}{u}\Big)\check{R}\Big(-\frac{1}{u}\Big)\Big)^{\alpha\beta} = \Big\{\sum_{\gamma=1}^{2} \check{R}_{\gamma}^{\ \alpha}\Big(\frac{1}{u}\Big)\check{R}_{\gamma}^{\ \beta}\Big(-\frac{1}{u}\Big)\Big\} = \delta^{\alpha\beta}.$$

This shows $\check{R}$ is unitary. □

Following Iritani [22] (with slight modification), we introduce the following definition:

Definition 3.6 (equivariant K–theoretic framing) We define

$$\widetilde{\mathrm{ch}}_z\colon K_T(\mathbb{P}^1) \to H_T^*(\mathbb{P}^1;\mathbb{Q})\Big[\Big[\frac{\mathsf{w}_1-\mathsf{w}_2}{z}\Big]\Big]$$

by the following two properties, which uniquely characterize it:

(a) $\widetilde{\mathrm{ch}}_z$ is a homomorphism of additive groups:

$$\widetilde{\mathrm{ch}}_z(\mathcal{E}_1 \oplus \mathcal{E}_2) = \widetilde{\mathrm{ch}}_z(\mathcal{E}_1) + \widetilde{\mathrm{ch}}_z(\mathcal{E}_2).$$

(b) If $\mathcal{L}$ is a T–equivariant line bundle on $\mathbb{P}^1$ then

$$\widetilde{\mathrm{ch}}_z(\mathcal{L}) = \exp\Big(-\frac{2\pi\sqrt{-1}(c_1)_T(\mathcal{L})}{z}\Big).$$

For any $\mathcal{E} \in K_T(\mathbb{P}^1)$, we define the *K–theoretic framing* of $\mathcal{E}$ by

$$\kappa(\mathcal{E}) := (-z)^{1-(c_1)_T(T\mathbb{P}^1)/z}\,\Gamma\Big(1 - \frac{(c_1)_T(T\mathbb{P}^1)}{z}\Big)\widetilde{\mathrm{ch}}_z(\mathcal{E}),$$

where $(c_1)_T(T\mathbb{P}^1) = 2H - \mathsf{w}_1 - \mathsf{w}_2$.

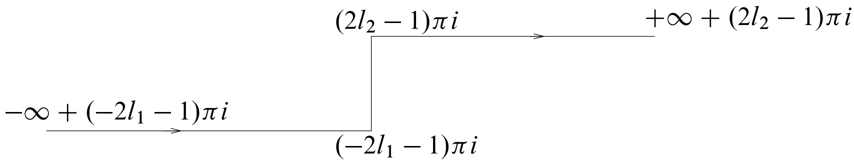
By localization, property (b) in the above definition is characterized by

$$\iota_{p_\alpha}^*\kappa(\mathcal{O}_{\mathbb{P}^1}(l_1p_1 + l_2p_2)) = (-z)^{1-\chi^\alpha/z}\,\Gamma\Big(1 - \frac{\chi^\alpha}{z}\Big)e^{-2l_\alpha\pi\sqrt{-1}\chi^\alpha/z}, \quad \alpha = 1,2,$$

where $\iota_{p_\alpha}\colon p_\alpha \to \mathbb{P}^1$ is the inclusion map.

The following definition is motivated by [12; 14]:

Definition 3.7 (equivariant SYZ T–dual) Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(l_1 p_1 + l_2 p_2)$ be an equivariant ample line bundle on $\mathbb{P}^1$, where l_1 and l_2 are integers such that $l_1 + l_2 > 0$. We define the equivariant SYZ T–dual $\mathrm{SYZ}(\mathcal{L})$ of $\mathcal{L}$ to be the oriented graph



in $\mathbb{C}$. We extend the definition additively to the equivariant K–theory group $K_T(\mathbb{P}^1)$.

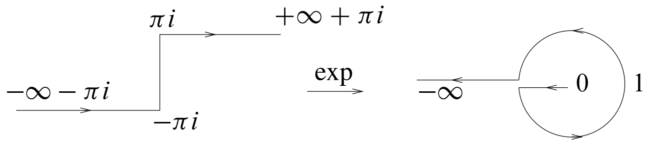


Figure 1: The equivariant SYZ T–dual of $\mathcal{O}_{\mathbb{P}^1}(p_2)$ in $\mathbb{C}$ and the (nonequivariant) SYZ T–dual of $\mathcal{O}_{\mathbb{P}^1}(1)$ in $\mathbb{C}^*$

The following theorem gives a precise correspondence between the B–model oscillatory integrals and the A–model 1–point descendant invariants.

Theorem 3.8 *Suppose that $z, q, \mathsf{w}_1 - \mathsf{w}_2 \in (0,\infty)$. Then, for any $\mathcal{L} \in K_T(\mathbb{P}^1)$,*

$$\int_{y \in \mathrm{SYZ}(\mathcal{L})} e^{W_t^{\mathsf{w}}/z}\, dy = \left\langle\!\!\left\langle 1, \frac{\kappa(\mathcal{L})}{z-\psi} \right\rangle\!\!\right\rangle_{0,2}^{\mathbb{P}^1,T}, \tag{12}$$

$$\int_{y \in \mathrm{SYZ}(\mathcal{L})} e^{W_t^{\mathsf{w}}/z}\, y\, dx = -\left\langle\!\!\left\langle \frac{\kappa(\mathcal{L})}{z-\psi} \right\rangle\!\!\right\rangle_{0,1}^{\mathbb{P}^1,T}. \tag{13}$$

Here $dx = d(W_t^{\mathsf{w}}(y))$.

Proof The left-hand side of (12) is

$$\int_{y \in \mathrm{SYZ}(\mathcal{L})} e^{W_t^{\mathsf{w}}/z}\, dy = -\frac{1}{z} \int_{y \in \mathrm{SYZ}(\mathcal{L})} e^{W_t^{\mathsf{w}}/z}\, y\, d(W_t^{\mathsf{w}}).$$

By the string equation, the right-hand side of (12) is

$$\left\langle\!\!\left\langle 1, \frac{\kappa(\mathcal{L})}{z-\psi} \right\rangle\!\!\right\rangle_{0,2}^{\mathbb{P}^1,T} = \left\langle\!\!\left\langle \frac{\kappa(\mathcal{L})}{z(z-\psi)} \right\rangle\!\!\right\rangle_{0,1}^{\mathbb{P}^1,T}.$$

So (12) is equivalent to (13).

It remains to prove (12) for $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(l_1 p_1 + l_2 p_1)$, where $l_1 + l_2 \geq 0$. We will express both sides of (12) in terms of (modified) Bessel functions. A brief review of Bessel functions is given in Appendix A. The equivariant quantum differential equation of $\mathbb{P}^1$ is related to the modified Bessel differential equation by a simple transform (see Appendix B).

Let γ_{l_1,l_2} be defined as in Appendix A. Then

$$
\begin{aligned}
\int_{\mathrm{SYZ}(\mathcal{L})} e^{W_t^{\mathsf{w}}/z}\,dy &= \int_{\mathrm{SYZ}(\mathcal{L})} \exp\Big(\frac{e^y + t^0 + qe^{-y} + \mathsf{w}_1 y + \mathsf{w}_2(t^1 - y)}{z}\Big)\,dy \\
&= e^{(t^0 + \mathsf{w}_2 t^1)/z} \int_{\gamma_{l_1,l_2}} \exp\Big(\frac{e^{y-i\pi} + qe^{i\pi - y} + (\mathsf{w}_1 - \mathsf{w}_2)(y - \pi i)}{z}\Big)\,dy \\
&= (-1)^{(\mathsf{w}_1 - w_2)/z} \exp\Big(\frac{t^0}{z} + \frac{\mathsf{w}_1 + \mathsf{w}_2}{2z} t^1\Big) \\
&\qquad \times \int_{\gamma_{l_1,l_2}} \exp\Big(-\frac{2\sqrt{q}}{z}\cosh\Big(y - \frac{t^1}{2}\Big) + \frac{\mathsf{w}_1 - \mathsf{w}_2}{z}\Big(y - \frac{t^1}{2}\Big)\Big)\,dy \\
&= (-1)^{(\mathsf{w}_1 - w_2)/z} \exp\Big(\frac{t^0}{z} + \frac{\mathsf{w}_1 + \mathsf{w}_2}{2z} t^1\Big) \\
&\qquad \times \int_{\gamma_{l_1,l_2}} \exp\Big(-\frac{2\sqrt{q}}{z}\cosh(y) + \frac{\mathsf{w}_1 - \mathsf{w}_2}{z} y\Big)\,dy.
\end{aligned}
$$

By Lemma A.1,

$$
\begin{aligned}
\int_{\gamma_{l_1,l_2}} &\exp\Big(-\frac{2\sqrt{q}}{z}\cosh(y) + \frac{\mathsf{w}_1 - \mathsf{w}_2}{z} y\Big)\,dy \\
&= \frac{\pi}{\sin\big(((\mathsf{w}_2 - \mathsf{w}_1)/z)\pi\big)} \Big(e^{-2\pi i l_1(\mathsf{w}_1 - \mathsf{w}_2)/z} I_{(\mathsf{w}_1 - \mathsf{w}_2)/z}\Big(\frac{2\sqrt{q}}{z}\Big) \\
&\qquad\qquad - e^{-2\pi i l_2(\mathsf{w}_2 - \mathsf{w}_1)/z} I_{(\mathsf{w}_2 - \mathsf{w}_1)/z}\Big(\frac{2\sqrt{q}}{z}\Big)\Big) \\
&= -\sum_{\alpha=1}^{2} e^{-2\pi i l_\alpha \chi_\alpha/z} \frac{\pi}{\sin((\chi^\alpha/z)\pi)} I_{\chi^\alpha/z}\Big(\frac{2\sqrt{q}}{z}\Big).
\end{aligned}
$$

Therefore, the left-hand side of (12) is

$$
\begin{aligned}
\int_{\mathrm{SYZ}(\mathcal{L})} & e^{W_t^{\mathsf{w}}/z}\,dy \\
&= -\exp\Big(\frac{t^0}{z} + \frac{\mathsf{w}_1 + \mathsf{w}_2}{2z} t^1\Big) \sum_{\alpha=1}^{2} e^{-(2l_\alpha - 1)\pi i \chi_\alpha/z} \frac{\pi}{\sin((\chi^\alpha/z)\pi)} I_{\chi^\alpha/z}\Big(\frac{2\sqrt{q}}{z}\Big).
\end{aligned}
$$

Recall from Section 2.5 that

$$J^\alpha = \left\langle\!\!\left\langle 1, \frac{\phi^\alpha}{z-\psi} \right\rangle\!\!\right\rangle_{0,2}^{\mathbb{P}^1,T} = \chi^\alpha \left\langle\!\!\left\langle 1, \frac{\phi_\alpha}{z-\psi} \right\rangle\!\!\right\rangle_{0,2}^{\mathbb{P}^1,T}.$$

We have

$$\begin{aligned} J^\alpha &= e^{(t^0+t^1 \mathsf{w}_\alpha)/z} \sum_{d=0}^\infty \frac{q^d}{d! z^d} \frac{1}{\prod_{m=1}^d (\chi^\alpha + mz)} \\ &= e^{(t^0+t^1 \mathsf{w}_\alpha)/z} \sum_{m=0}^\infty \left(\frac{2\sqrt{q}}{z}\right)^{2m} \frac{\Gamma(\chi^\alpha/z+1)}{m!\,\Gamma(\chi^\alpha/z+m+1)} \\ &= \exp\left(\frac{t^0}{z} + \frac{\mathsf{w}^1+\mathsf{w}^2}{2z} t^1\right) z^{\chi^\alpha/z} \Gamma\left(\frac{\chi^\alpha}{z}+1\right) I_{\chi^\alpha/z}\left(\frac{2\sqrt{q}}{z}\right), \\ \kappa(\mathcal{L}) &= \sum_{\alpha=1}^2 (-z)^{\chi^\alpha/(-z)+1} \Gamma\left(1-\frac{\chi^\alpha}{z}\right) e^{-2l_\alpha \pi \sqrt{-1}\chi^\alpha/z} \phi_\alpha. \end{aligned}$$

So the right-hand side of (12) is

$$\begin{aligned} \left\langle\!\!\left\langle 1, \frac{\kappa(\mathcal{L})}{z-\psi} \right\rangle\!\!\right\rangle_{0,2}^{\mathbb{P}^1,T} &= \sum_{\alpha=1}^2 (-z)^{\chi^\alpha/(-z)+1} \Gamma\left(1-\frac{\chi^\alpha}{z}\right) e^{-2\pi i l_\alpha \chi^\alpha/z} \frac{J^\alpha}{\chi^\alpha} \\ &= -\exp\left(\frac{t^0}{z} + \frac{\mathsf{w}^1+\mathsf{w}^2}{2z} t^1\right) \\ &\qquad \times \sum_{\alpha=1}^2 (-1)^{\chi^\alpha/(-z)} e^{-2\pi i l_\alpha \chi^\alpha/z} \frac{\pi}{\sin((\chi^\alpha/z)\pi)} I_{\chi^\alpha/z}\left(\frac{2\sqrt{q}}{z}\right) \\ &= -\exp\left(\frac{t^0}{z} + \frac{\mathsf{w}^1+\mathsf{w}^2}{2z} t^1\right) \\ &\qquad \times \sum_{\alpha=1}^2 e^{-(2l_\alpha-1)\pi i \chi^\alpha/z} \frac{\pi}{\sin((\chi^\alpha/z)\pi)} I_{\chi^\alpha/z}\left(\frac{2\sqrt{q}}{z}\right). \qquad \square \end{aligned}$$

Remark 3.9 Definition 3.6 (equivariant K–theoretic framing) and Definition 3.7 (equivariant SYZ T–dual) can be extended to any projective toric manifold. In [13], the first author uses the mirror theorem [17; 25] and results in [22] to extend Theorem 3.8 to any semi-Fano projective toric manifold. The left-hand side of (12) is known as the central charge of the Lagrangian brane $\mathrm{SYZ}(\mathcal{L})$.

Proposition 3.10 *The A– and B–model R–matrices are equal:*

$$R_\beta^{\ \alpha}(z) = \check{R}_\beta^{\ \alpha}(z).$$

Proof By the asymptotic decomposition theorem of the S–matrix (Theorem 3.2),we only have to compare at the limit $q=0,\ t_0=0$ since both $\widetilde{S}$ and $\check{S}$ are unitary. Notice that Ψ has a nondegenerate limit at $q=0$, so it suffices to show that

$$\widetilde{S}_i^{\ \hat{\alpha}} e^{-u^\alpha/z}\big|_{q=0,\,t_0=0} \sim \frac{1}{\sqrt{-2\pi z}} \check{S}_i^{\ \boldsymbol{\alpha}} e^{-\check{u}^\alpha/z}\Big|_{q=0,\,t_0=0}.$$

The Lefschetz thimble γ_2 is $\{Y \mid Y \in (-\infty,0)\}$. While the Lefschetz thimble γ_1 could not be explicitly depicted, we could alternatively consider the thimble $\gamma_1' = \{Y \mid Y \in (0,\infty)\}$ for $z<0$ of the oscillating integral $\int e^{W_t^{\mathrm{w}}/z}\,dy$. The integral yields the same asymptotic answer once we analytically continue $z<0$ to $z>0$, since the stationary phase expansion only depends on the local behavior (higher-order derivatives) of W_t^{w} at the critical points.

So, letting $Y=-Tz$ for $\alpha=2$, or $Y=-q/(Tz)$ for $\alpha=1$,

$$e^{-\check{u}^\alpha/z}\check{S}_0^{\ \boldsymbol{\alpha}} = e^{-\Delta^\alpha(q)/z}\left(\frac{\chi^\alpha+\Delta^\alpha(q)}{2}\right)^{\frac{\chi^\alpha}{z}}(-z)^{-\chi^\alpha/z}\int_0^\infty e^{-T}e^{-q/(Tz^2)}T^{\chi_\alpha/z-1}\,dT.$$

Taking the limit $q\to 0$,

$$\begin{aligned}\frac{1}{\sqrt{-2\pi z}}e^{-\check{u}^\alpha/z}\check{S}_0^{\ \boldsymbol{\alpha}}\Big|_{q=0} &= \frac{1}{\sqrt{-2\pi z}}e^{-\chi^\alpha/z}\left(\frac{-\chi^\alpha}{z}\right)^{\frac{\chi^\alpha}{z}}\Gamma\left(\frac{-\chi^\alpha}{z}\right)\\ &\sim \sqrt{\frac{1}{\chi^\alpha}}\exp\left(-\sum_{n=1}^\infty \frac{B_{2n}}{2n(2n-1)}\left(\frac{z}{\chi^\alpha}\right)^{2n-1}\right)\\ &\sim \widetilde{S}_0^{\ \hat{\alpha}}e^{-u^\alpha/z}\big|_{q=0}.\end{aligned}$$

Here we use the Stirling formula

$$\log\Gamma(z)\sim\frac12\log(2\pi)+\left(z-\frac12\right)\log z - z+\sum_{n=1}^\infty\frac{B_{2n}}{2n(2n-1)}z^{1-2n}.$$

Notice that

$$\check{S}_1^{\ \boldsymbol{\alpha}} = z\frac{\partial}{\partial t_1}\check{S}_0^{\ \alpha} = z\int_{\gamma_\alpha} e^{W_t^{\mathrm{w}}/z}\left(\frac{q}{Y}+\mathrm{w}_2\right)\frac{dY}{Y},$$

and similar calculation shows (letting $Y=-Tz$ if $\alpha=2$ and $Y=-q/(Tz)$ if $\alpha=1$)

$$\begin{aligned}\frac{1}{\sqrt{-2\pi z}}e^{-\check{u}^\alpha/z}\check{S}_1^{\ \boldsymbol{\alpha}}\Big|_{q=0} &\sim \mathrm{w}^\alpha\sqrt{\frac{1}{\chi^\alpha}}\exp\left(-\sum_{n=1}^\infty\frac{B_{2n}}{2n(2n-1)}\left(\frac{z}{\chi^\alpha}\right)^{2n-1}\right)\\ &\sim \widetilde{S}_1^{\ \hat{\alpha}}e^{-u^\alpha/z}\big|_{q=0}.\end{aligned}$$

This concludes the proof. □

Notice that the matrix $\check{R}$ is given by the asymptotic expansion. This theorem does not imply $\widetilde{S}_i{}^{\widehat{\alpha}} e^{-u^\alpha/z} = \check{S}_i{}^{\boldsymbol{\alpha}} e^{-\check{u}^\alpha/z}/\sqrt{-2\pi z}$, which are unequal.

3.6 The Eynard–Orantin topological recursion and the B–model graph sum

Let $\omega_{g,n}$ be defined recursively by the Eynard–Orantin topological recursion [10]:

$$\omega_{0,1} = 0, \quad \omega_{0,2} = B(Y_1, Y_2) = \frac{dY_1 \otimes dY_2}{(Y_1 - Y_2)^2}.$$

When $2g - 2 + n > 0$,

$$\begin{aligned}\omega_{g,n}(Y_1, \dots, Y_n) = \sum_{\alpha=1}^{2} \operatorname{Res}_{Y \to P_\alpha} & \frac{-\int_{\xi=Y}^{\widehat{Y}} B(Y_n, \xi)}{2(\log(Y) - \log(\widehat{Y}))dW} \\ & \times \Big(\omega_{g-1,n+1}(Y, \widehat{Y}, Y_1, \dots, Y_{n-1}) + \\ & \sum_{g_1+g_2=g} \sum_{\substack{I \cup J = \{1,\dots,n-1\} \\ I \cap J = \varnothing}} \omega_{g_1,|I|+1}(Y, Y_I)\omega_{g_2,|J|+1}(\widehat{Y}, Y_J) \Big),\end{aligned}$$

where $Y \neq P_\alpha$ is in a small neighborhood of P_α and $\widehat{Y} \neq Y$ is the other point in the neighborhood such that $W_q^{\mathsf{w}}(\widehat{Y}) = W_q^{\mathsf{w}}(Y)$.

The B–model invariants $\omega_{g,n}$ can be expressed as graph sums [23; 7; 8; 5]. We will use the formula stated in [5, Theorem 3.7], which is equivalent to the formula in [7, Theorem 5.1]. Given a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}_{g,n}(\mathbb{P}^1)$ with $L^o(\Gamma) = \{l_1, \dots, l_n\}$, we define its weight to be

$$\begin{aligned} w(\vec{\Gamma}) = (-1)^{g(\vec{\Gamma})-1+n} \prod_{v \in V(\Gamma)} \Big(\frac{h_1^\alpha}{\sqrt{2}}\Big)^{2-2g-\mathrm{val}(v)} \Big\langle \prod_{h \in H(v)} \tau_{k(h)} \Big\rangle_{g(v)} \prod_{e \in E(\Gamma)} \check{B}^{\alpha(v_1(e)),\alpha(v_2(e))}_{k(e),l(e)} \\ \cdot \prod_{j=1}^{n} \frac{1}{\sqrt{-2}} d\xi^{\alpha(l_j)}_{k(l_j)}(Y_j) \prod_{l \in \mathcal{L}^1(\Gamma)} \Big(-\frac{1}{\sqrt{-2}}\Big) \check{h}^{\alpha(l)}_{k(l)}. \end{aligned}$$

Here,

$$\check{h}^\alpha_k = -\frac{2(2k-1)!!\, h^\alpha_{2k-1}}{\sqrt{-1}^{2k-1}}, \quad \check{B}^{\alpha,\beta}_{k,l} = [u^{-k} v^{-l}] \check{B}^{\alpha,\beta}(u, v, q).$$

Note that the definitions of $\check{B}^{\alpha,\beta}_{k,l}$, $\check{h}^\alpha_k$ and $d\xi^\alpha_k$ in this paper are slightly different from those in [5]; for example, the definition of $\check{B}^{\alpha,\beta}_{k,l}$ in this paper differs from [5, Equation (3.11)] by a factor of 2^{-k-l-1}. In our notation, [5, Theorem 3.7] is equivalent to:

Theorem 3.11 *For* $2g-2+n>0$,

$$\omega_{g,n}=\sum_{\Gamma\in\mathbf{\Gamma}_{g,n}(\mathbb{P}^1)}\frac{w(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

3.7 All-genus mirror symmetry

Given a meromorphic function $f(Y)$ on $\mathbb{P}^1$ which is holomorphic on $\mathbb{P}^1\setminus\{P_1,P_2\}$, define

$$\theta(f)=\frac{df}{dW}=\frac{Y^2}{(Y-P_1)(Y-P_2)}\frac{df}{dY}.$$

Then $\theta(f)$ is also a meromorphic function which is holomorphic on $\mathbb{P}^1\setminus\{P_1,P_2\}$. For $\alpha\in\{1,2\}$, let

$$\xi_{\alpha,0}=\frac{1}{\sqrt{-1}}\sqrt{\frac{2}{\Delta^\alpha(q)}}\frac{P_\alpha}{Y-P_\alpha}.$$

Then $\xi_{\alpha,0}$ is a meromorphic function on $\mathbb{P}^1$ with a simple pole at $Y=P_\alpha$ and holomorphic elsewhere. Moreover, the differential of $\xi_{\alpha,0}$ is $d\xi_{\alpha,0}$. For $k>0$, define

$$W_k^\alpha:=d((-1)^k\theta^k(\xi_{\alpha,0})).$$

Define

$$\check{S}^{\boldsymbol{\alpha}}_{\underline{\widehat{\beta}}}(z)=-z\int_{y\in\gamma_\alpha}e^{x/z}\frac{d\xi_{\beta,0}}{\sqrt{-2}},\quad \check{S}^{\kappa(\mathcal{L})}_{\underline{\widehat{\beta}}}(z)=-z\int_{y\in\mathrm{SYZ}(\mathcal{L})}e^{x/z}\frac{d\xi_{\beta,0}}{\sqrt{-2}}. \tag{14}$$

Then

$$\check{S}^{\boldsymbol{\alpha}}_{\underline{\widehat{\beta}}}(z)=-z^{k+1}\int_{y\in\gamma_\alpha}e^{W(y)/z}\frac{W_k^\beta}{\sqrt{-2}},\quad \check{S}^{\kappa(\mathcal{L})}_{\underline{\widehat{\beta}}}(z)=-z^{k+1}\int_{y\in\mathrm{SYZ}(\mathcal{L})}e^{W(y)/z}\frac{W_k^\beta}{\sqrt{-2}}.$$

Therefore,

$$\int_{y\in\mathrm{SYZ}(\mathcal{L})}e^{W(y)/z}\frac{W_k^\beta}{\sqrt{-2}}=-z^{-k-1}\check{S}^{\kappa(\mathcal{L})}_{\underline{\widehat{\beta}}}(z)=-z^{-k-1}\left\langle\!\left\langle\widehat{\phi}_\alpha(q),\frac{\kappa(\mathcal{L})}{z-\psi}\right\rangle\!\right\rangle_{0,2}^{\mathbb{P}^1,T}, \tag{15}$$

where the last equality follows from Theorem 3.8.

For $\alpha=1,2$ and $j=1,\dots,n$, let

$$\widetilde{\boldsymbol{u}}_j^\alpha(z)=\sum_{\beta=1}^2 S^{\underline{\widehat{\alpha}}}_{\underline{\widehat{\beta}}}(z)\frac{\boldsymbol{u}_j^\beta(z)}{\sqrt{\Delta^\beta(q)}}. \tag{16}$$

Theorem A (all-genus equivariant mirror symmetry for $\mathbb{P}^1$) *When* $n>0$ *and* $2g-2+n>0$, *we have*

$$\omega_{g,n}\big|_{W_k^\alpha(Y_j)/\sqrt{-2}=(\widetilde{u}_j)_k^\alpha}=(-1)^{g-1+n}F_{g,n}^{\mathbb{P}^1,T}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n,\boldsymbol{t}). \tag{17}$$

Proof We will prove this theorem by comparing the A–model graph sum in the end of Section 2.8 and the B–model graph sum in Section 3.6.

- **Vertex** By Section 3.2, we have $h_1^\alpha(q) = \sqrt{2/\Delta^\alpha(q)}$. So, in the B–model vertex, $h_1^\alpha/\sqrt{2} = \sqrt{1/\Delta^\alpha(q)}$. Therefore the B–model vertex matches the A–model vertex.

- **Edge** By (11), we know that

$$\begin{aligned}\check{B}^{\alpha,\beta}_{k,l} &= [u^{-k}v^{-l}]\Big(\frac{uv}{u+v}\Big(\delta_{\alpha,\beta} - \sum_{\gamma=1,2} \check{R}_\gamma{}^\alpha\Big(-\frac{1}{u}\Big)\check{R}_\gamma{}^\beta\Big(-\frac{1}{v}\Big)\Big)\Big)\\ &= [z^k w^l]\Big(\frac{1}{z+w}\Big(\delta_{\alpha,\beta} - \sum_{\gamma=1,2} \check{R}_\gamma{}^\alpha(-z)\check{R}_\gamma{}^\beta(-w)\Big)\Big).\end{aligned}$$

By definition,

$$\mathcal{E}^{\alpha,\beta}_{k,l} = [z^k w^l]\Big(\frac{1}{z+w}\Big(\delta_{\alpha,\beta} - \sum_{\gamma=1,2} R_\gamma{}^\alpha(-z)R_\gamma{}^\beta(-w)\Big)\Big).$$

By Proposition 3.10, $\check{R}_\beta{}^\alpha(z) = R_\beta{}^\alpha(z)$, so

$$\check{B}^{\alpha,\beta}_{k,l} = \mathcal{E}^{\alpha,\beta}_{k,l}.$$

- **Ordinary leaf** We have the following expression for $d\xi_k^\alpha$ (see [15]):

$$d\xi_k^\alpha = W_k^\alpha - \sum_{i=0}^{k-1}\sum_\beta \check{B}^{\alpha,\beta}_{k-1-i,0} W_i^\beta.$$

By the calculation for edge above, for $k, l \in \mathbb{Z}_{\geq 0}$,

$$\check{B}^{\alpha,\beta}_{k,l} = [z^k w^l]\Big(\frac{1}{z+w}\Big(\delta_{\alpha,\beta} - \sum_{\gamma=1,2} R_\gamma{}^\alpha(-z)R_\gamma{}^\beta(-w)\Big)\Big).$$

We also have

$$[z^0](R_\beta{}^\alpha(-z)) = \delta_{\alpha,\beta}.$$

Therefore,

$$d\xi_k^\alpha = \sum_{i=0}^{k}\sum_{\beta=1}^{2}([z^{k-i}]R_\beta{}^\alpha(-z))W_i^\beta,$$

so under the identification

$$\frac{1}{\sqrt{-2}}W_k^\alpha(Y_j) = (\tilde{u}_j)_k^\alpha,$$

the B–model ordinary leaf matches the A–model ordinary leaf.

- **Dilaton leaf** We have the following relation between $\check{h}_k^{\alpha}$ and $f_{\beta}^{\alpha}(u,q)$ (see [15]):

$$\check{h}_k^{\alpha} = [u^{1-k}] \sum_{\beta} \sqrt{-1} h_1^{\beta} f_{\beta}^{\alpha}(u,q).$$

By the relation

$$R_{\beta}{}^{\alpha}(z) = f_{\beta}^{\alpha}\Big(\frac{-1}{z}\Big)$$

and the fact $h_1^{\beta}(q) = \sqrt{2/\Delta^{\beta}(q)}$, it is easy to see that the B–model dilaton leaf matches the A–model dilaton leaf. □

Taking Laplace transforms at appropriate cycles to Theorem A produces a theorem concerning descendant potential.

Theorem B (all-genus full descendant equivariant mirror symmetry for $\mathbb{P}^1$) *Suppose that $n > 0$ and $2g-2+n > 0$. For any $\mathcal{L}_1, \dots, \mathcal{L}_n \in K_T(\mathbb{P}^1)$, there is a formal power series identity*

$$\int_{y_1 \in \mathrm{SYZ}(\mathcal{L}_1)} \cdots \int_{y_n \in \mathrm{SYZ}(\mathcal{L}_n)} e^{W(y_1)/z_1 + \cdots + W(y_n)/z_n} \omega_{g,n} = (-1)^{g-1} \Big\langle\!\Big\langle \frac{\kappa(\mathcal{L}_1)}{z_1 - \psi_1}, \dots, \frac{\kappa(\mathcal{L}_n)}{z_n - \psi_n} \Big\rangle\!\Big\rangle_{g,n}. \tag{18}$$

Remark 3.12 By Theorem 3.8,

$$\int_{y_1 \in \mathrm{SYZ}(\mathcal{L})} e^{W(y_1)/z_1} y\, dx = -\Big\langle\!\Big\langle \frac{\kappa(\mathcal{L}_1)}{z_1 - \psi_1} \Big\rangle\!\Big\rangle_{0,1}^{\mathbb{P}^1, T}, \tag{19}$$

which is the analogue of (18) in the unstable case $(g,n) = (0,1)$.

Proof of Theorem B By (16),

$$\tilde{\boldsymbol{u}}_j^{\alpha}(z) = \sum_{\beta=1}^{2} \sqrt{\Delta^{\alpha}(q)} \Big\langle\!\Big\langle \phi_{\alpha}(q), \frac{\phi_{\beta}(q)}{z - \psi} \Big\rangle\!\Big\rangle_{0,2}^{\mathbb{P}^1, T} \boldsymbol{u}_j^{\beta}(z).$$

Define the flat coordinates $\bar{\boldsymbol{u}}_j^{\alpha}$ by

$$\sum_{\alpha=1}^{2} \boldsymbol{u}_j^{\alpha}(z) \phi_{\alpha}(q) = \sum_{\alpha=1}^{2} \bar{\boldsymbol{u}}_j^{\alpha}(z) \phi_{\alpha}(0),$$

and a power series in $\frac{1}{z}$,

$$S^{\hat{\alpha}}{}_{\beta}(z) = \Big\langle\!\Big\langle \hat{\phi}_{\alpha}(q), \frac{\phi_{\beta}(0)}{z - \psi} \Big\rangle\!\Big\rangle_{0,2}.$$

Then

$$\widetilde{\boldsymbol{u}}_j^\alpha(z) = \sum_{\beta=1}^{2} \left(\left\langle\!\!\left\langle \widehat{\phi}_\alpha(q), \frac{\phi_\beta(0)}{z-\psi} \right\rangle\!\!\right\rangle \overline{\boldsymbol{u}}_j^\beta(z) \right)_+ = \sum_{\beta=1}^{2} (S^{\widehat{\underline{\alpha}}}_{\ \beta}(z)\overline{\boldsymbol{u}}_j^\beta(z))_+ .$$

Notice that $(S^{\widehat{\underline{\alpha}}}_{\ \beta})$ is unitary, ie $\sum_\gamma S^{\widehat{\underline{\gamma}}}_{\ \alpha}(z) S^{\widehat{\underline{\gamma}}}_{\ \beta}(-z) = \delta/\chi^\beta_{\alpha\beta}$. We have

$$\sum_{\alpha=1}^{2} (S^{\widehat{\underline{\alpha}}}_{\ \gamma}(-z)\widetilde{\boldsymbol{u}}_j^\alpha(z))_+ = \sum_{\alpha=1}^{2} \left(\sum_{\beta=1}^{2} S^{\widehat{\underline{\alpha}}}_{\ \beta}(z) S^{\widehat{\underline{\alpha}}}_{\ \gamma}(-z) \overline{\boldsymbol{u}}_j^\beta(z) \right) = \frac{\overline{\boldsymbol{u}}_j^\gamma(z)}{\chi^\gamma}.$$

Taking the Laplace transform of $\omega_{g,n}$,

$$\begin{aligned}
&\int_{y_1\in \mathrm{SYZ}(\mathcal{L}_1)} \cdots \int_{y_n \in \mathrm{SYZ}(\mathcal{L}_n)} e^{W(y_1)/z_1+\cdots+W(y_n)/z_n} \omega_{g,n} \\
&= \int_{y_1\in \mathrm{SYZ}(\mathcal{L}_1)} \cdots \int_{y_n \in \mathrm{SYZ}(\mathcal{L}_n)} e^{\sum_{i=1}^n W(y_i)/z_i} (-1)^{g-1+n} \\
&\qquad\qquad \cdot \left(\sum_{\beta_i,a_i} \left\langle\!\!\left\langle \prod_{i=1}^n \tau_{a_i}(\phi_{\beta_i}(0)) \right\rangle\!\!\right\rangle_{g,n} \prod_{i=1}^n (\overline{u}_i)^{\beta_i}_{a_i} \right) \Bigg|_{(\widetilde{u}_j)^\beta_k = W^\beta_k(y_j)/\sqrt{-2}} \\
&= \int_{y_1\in \mathrm{SYZ}(\mathcal{L}_1)} \cdots \int_{y_n \in \mathrm{SYZ}(\mathcal{L}_n)} e^{\sum_{i=1}^n W(y_i)/z_i} (-1)^{g-1+n} \\
&\quad \cdot \left(\sum_{\beta_i,a_i} \left\langle\!\!\left\langle \prod_{i=1}^n \tau_{a_i}(\phi_{\beta_i}(0)) \right\rangle\!\!\right\rangle_{g,n} \prod_{i=1}^n \left(\chi^{\beta_i} \sum_{\alpha=1}^{2} \sum_{k\in\mathbb{Z}_{\geq 0}} [z_i^{a_i-k}] S^{\widehat{\underline{\alpha}}}_{\ \beta_i}(-z_i) \frac{W^\alpha_k(y_i)}{\sqrt{-2}} \right) \right).
\end{aligned}$$

Using (15),

$$\begin{aligned}
&\int_{y_1\in \mathrm{SYZ}(\mathcal{L}_1)} \cdots \int_{y_n \in \mathrm{SYZ}(\mathcal{L}_n)} e^{W(y_1)/z_1+\cdots+W(y_n)/z_n} \omega_{g,n} \\
&= (-1)^{g-1+n} \left(\sum_{\beta_i,a_i} \left\langle\!\!\left\langle \prod_{i=1}^n \tau_{a_i}(\phi_{\beta_i}(0)) \right\rangle\!\!\right\rangle_{g,n} \right. \\
&\qquad\qquad \left. \cdot \prod_{i=1}^n \left(\chi^{\beta_i} \sum_{\alpha=1}^{2} \sum_{k\in\mathbb{Z}_{\geq 0}} ([z_i^{a_i-k}] S^{\widehat{\underline{\alpha}}}_{\ \beta_i}(-z_i)) S_{\widehat{\underline{\alpha}}}^{\ \kappa(\mathcal{L}_i)}(z_i)(-z_i^{-k-1}) \right) \right) \\
&= (-1)^{g-1} \sum_{\beta_i,a_i} \left\langle\!\!\left\langle \prod_{i=1}^n \tau_{a_i}(\phi_{\beta_i}(0)) \right\rangle\!\!\right\rangle_{g,n} \prod_{i=1}^n \chi^{\beta_i}(\phi_{\beta_i}(0), \kappa(\mathcal{L}_i)) z_i^{-a_i-1} \\
&= (-1)^{g-1} \left\langle\!\!\left\langle \frac{\kappa(\mathcal{L}_1)}{z_1-\psi_1}, \ldots, \frac{\kappa(\mathcal{L}_n)}{z_n-\psi_n} \right\rangle\!\!\right\rangle_{g,n}.
\end{aligned}$$

□

4 The nonequivariant limit and the Norbury–Scott conjecture

In this section, we consider the nonequivariant limit $\mathsf{w}_1 = \mathsf{w}_2 = 0$.

4.1 The nonequivariant R–matrix

By [20, Section 1.3], $R(z) = I + \sum_{n=1}^{\infty} R_n z^n$ is uniquely determined by:

(1) The recursive relation $(d + \Psi^{-1} d\Psi) R_n = [dU, R_{n+1}]$.

(2) The homogeneity of $R(z)$: $R_n q^{n/2}$ is a constant matrix.

The unique solution $R(z)$ satisfying the above conditions was computed explicitly in [29]:

Lemma 4.1 [29, Lemma 3.1] *We have*

$$R_n = q^{-n/2} \frac{(2n-1)!!\,(2n-3)!!}{n!\,2^{4n}} \begin{pmatrix} -1 & 2n\sqrt{-1}(-1)^{n+1} \\ 2n\sqrt{-1} & (-1)^{n+1} \end{pmatrix}.$$

By Proposition 3.10 , $R(z) = \check{R}(z)$. In this subsection, we recover the above lemma by computing the stationary phase expansion of $\check{S}$.

We assume $z, q \in (0, \infty)$, where $q = Qe^{t^1}$. Then

$$\begin{aligned} \check{S}_0{}^2 &= \int_{y=-\infty}^{y=+\infty} e^{(t^0 + e^{y-i\pi} + qe^{-(y-i\pi)})/z}\, dy \\ &= e^{t^0/z} \int_{y=-\infty}^{y=+\infty} e^{-2\sqrt{q}\cosh(y - t^1/2)/z}\, dy \\ &= e^{t^0/z} \int_{y=-\infty}^{y=+\infty} e^{-2\sqrt{q}\cosh(y)/z}\, dy \\ &= 2e^{(t^0 - 2\sqrt{q})/z} \int_{y=0}^{y=+\infty} e^{-2\sqrt{q}(\cosh(y)-1)/z}\, dy. \end{aligned}$$

Let $T = 2\sqrt{q}(\cosh(y) - 1)/z$; then

$$y = \cosh^{-1}\Big(1 + \frac{zT}{2\sqrt{q}}\Big), \quad dy = \frac{1}{2} q^{-\frac{1}{4}} T^{-\frac{1}{2}} \sqrt{\frac{z}{1 + zT/(4\sqrt{q})}},$$

$$\begin{aligned}
\check{S}_0{}^2 &= e^{(t^0-2\sqrt{q})/z}\sum_{n=0}^{\infty}\Big(\frac{z}{\sqrt{q}}\Big)^{n+\frac{1}{2}}\binom{-1/2}{n}2^{-2n}\int_{T=0}^{T=+\infty}e^{-T}T^{n-\frac{1}{2}}\,dT\\
&= e^{(t^0-2\sqrt{q})/z}\sum_{n=0}^{\infty}\Big(\frac{z}{\sqrt{q}}\Big)^{n+\frac{1}{2}}\frac{(-1)^n(2n-1)!!}{n!2^{3n}}\Gamma\Big(n+\frac{1}{2}\Big)\\
&= \sqrt{\pi}e^{(t^0-2\sqrt{q})/z}\sum_{n=0}^{\infty}\Big(\frac{z}{\sqrt{q}}\Big)^{n+\frac{1}{2}}\frac{(-1)^n((2n-1)!!)^2}{n!2^{4n}},\\
\check{S}_1{}^2 &= z\frac{\partial}{\partial t_1}\check{S}_0{}^2\\
&= \sqrt{\pi}ze^{(t^0-2\sqrt{q})/z}\sum_{n=0}^{\infty}\Big(\frac{z}{\sqrt{q}}\Big)^{n-\frac{1}{2}}\Big(1+\Big(\frac{1}{4}+\frac{n}{2}\Big)\frac{z}{\sqrt{q}}\Big)\frac{(-1)^{n+1}((2n-1)!!)^2}{n!2^{4n}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\check{S}_0{}^1 &= \sqrt{-\pi}e^{(t^0+2\sqrt{q})/z}\sum_{n=0}^{\infty}\Big(\frac{z}{\sqrt{q}}\Big)^{n+\frac{1}{2}}\frac{((2n-1)!!)^2}{n!2^{4n}};\\
\check{S}_1{}^1 &= \sqrt{-\pi}ze^{(t^0+2\sqrt{q})/z}\sum_{n=0}^{\infty}\Big(\frac{z}{\sqrt{q}}\Big)^{n-\frac{1}{2}}\Big(1-\Big(\frac{1}{4}+\frac{n}{2}\Big)\frac{z}{\sqrt{q}}\Big)\frac{((2n-1)!!)^2}{n!2^{4n}}.
\end{aligned}$$

Therefore,

$$\widetilde{S}(z)=\frac{1}{\sqrt{-2\pi z}}\check{S}(z),\quad [z^n](\widetilde{S}(z)e^{-U/z})=\begin{pmatrix}A & B\\ C & D\end{pmatrix},$$

where

$$\begin{aligned}
A &= \frac{((2n-1)!!)^2}{\sqrt{2}n!2^{4n}q^{\frac{1}{2}n+\frac{1}{4}}},\quad B=\frac{\sqrt{-1}(-1)^{n+1}((2n-1)!!)^2}{\sqrt{2}n!2^{4n}q^{\frac{1}{2}n+\frac{1}{4}}},\\
C &= \frac{((2n-1)!!)^2}{\sqrt{2}n!2^{4n}q^{\frac{1}{2}n-\frac{1}{4}}}-\Big(\frac{n}{2}-\frac{1}{4}\Big)\frac{((2n-3)!!)^2}{\sqrt{2}(n-1)!2^{4n-4}q^{\frac{1}{2}n-\frac{1}{4}}},\\
D &= \frac{\sqrt{-1}(-1)^n((2n-1)!!)^2}{\sqrt{2}n!2^{4n}q^{\frac{1}{2}n-\frac{1}{4}}}+\Big(\frac{n}{2}-\frac{1}{4}\Big)\frac{\sqrt{-1}(-1)^{n+1}((2n-3)!!)^2}{\sqrt{2}(n-1)!2^{4n-4}q^{\frac{1}{2}n-\frac{1}{4}}},
\end{aligned}$$

and

$$\begin{aligned}
R_n &= \begin{pmatrix}-\frac{(2n-1)!!\,(2n-3)!!}{n!2^{4n}} & \frac{\sqrt{-1}(-1)^{n+1}(2n-1)!!\,(2n-3)!!}{(n-1)!2^{4n-1}}\\ \frac{\sqrt{-1}(2n-1)!!\,(2n-3)!!}{(n-1)!2^{4n-1}} & \frac{(-1)^{n+1}(2n-1)!!\,(2n-3)!!}{n!2^{4n}}\end{pmatrix}q^{-\frac{1}{2}n}\\
&= q^{-\frac{1}{2}n}\frac{(2n-1)!!\,(2n-3)!!}{n!2^{4n}}\begin{pmatrix}-1 & 2n\sqrt{-1}(-1)^{n+1}\\ 2n\sqrt{-1} & (-1)^{n+1}\end{pmatrix}.
\end{aligned}$$

4.2 The Norbury–Scott conjecture

In this subsection, we assume $\mathsf{w}_1 = \mathsf{w}_2 = t^0 = 0$. Then

$$\langle\!\langle \tau_{a_1}(H) \cdots \tau_{a_n}(H) \rangle\!\rangle_{g,n}^{\mathbb{P}^1} = q^{\frac{1}{2}(\sum_{i=1}^n a_i)+1-g} \langle \tau_{a_1}(H) \cdots \tau_{a_n}(H) \rangle_{g,n}^{\mathbb{P}^1}.$$

Note that when $\frac{1}{2}\big(\sum_{i=1}^n a_i\big) + 1 - g$ is not an nonnegative integer, both sides are zero.

When $2g - 2 + n > 0$, the symmetric n–form $\omega_{g,n}$ is holomorphic near $Y = 0$, and one may expand it in the local holomorphic coordinate $\tilde{x} = x^{-1} = (Y + q/Y)^{-1}$.

Theorem 4.2 *Suppose that $2g-2+n > 0$. Then, near $Y = 0$, the symmetric n–form $\omega_{g,n}$ has the expansion*

$$\omega_{g,n} = (-1)^{g-1+n} \sum_{\substack{a_1,\ldots,a_n \\ a_i \in \mathbb{Z}_{\geq 0}}} \langle\!\langle \tau_{a_1}(H) \cdots \tau_{a_n}(H) \rangle\!\rangle_{g,n}^{\mathbb{P}^1} \prod_{j=1}^n \frac{(a_j+1)!}{x^{a_j+2}} dx_j.$$

The Norbury–Scott conjecture corresponds to the specialization $q = 1$, ie $t^1 = 0$ and $Q = 1$.

Proof Define $\widetilde{W}_k^\alpha$ by

$$\frac{1}{\sqrt{-2}} \widetilde{W}_k^\alpha = \tilde{\boldsymbol{u}}_k^\alpha \big|_{t_a^0 = 0,\, t_a^1 = (a+1)! x^{-a-2} dx}.$$

By Theorem A, it suffices to show that $\widetilde{W}_k^\alpha$ agrees with the expansion of W_k^α near $Y = 0$ in $\tilde{x} = x^{-1}$.

We now compute $\widetilde{W}_k^\alpha$ explicitly:

$$\begin{aligned}
J &= e^{(t^0 + t^1 H)/z} \bigg(1 + \sum_{d=1}^\infty \frac{q^d}{\prod_{m=1}^d (H + mz)^2} \bigg) \\
&= e^{t^0/z} \Big(1 + t^1 \frac{H}{z} \Big) \bigg(1 + \sum_{d=1}^\infty \frac{q^d}{z^{2d}(d!)^2} - 2 \bigg(\sum_{d=1}^\infty \frac{q^d}{z^{2d}(d!)^2} \sum_{m=1}^d \frac{1}{m} \bigg) \frac{H}{z} \bigg) \\
&= e^{t^0/z} \bigg(1 + \sum_{d=1}^\infty \frac{q^d}{z^{2d}(d!)^2} \bigg) \\
&\qquad + e^{t^0/z} \bigg(t^1 \bigg(1 + \sum_{d=1}^\infty \frac{q^d}{z^{2d+1}(d!)^2} \bigg) - 2 \sum_{d=1}^\infty \frac{q^d}{z^{2d+1}(d!)^2} \sum_{m=1}^d \frac{1}{m} \bigg) H,
\end{aligned}$$

$$\begin{aligned}
z\frac{\partial J}{\partial t^1} &= e^{t^0/z}\Big(\sum_{d=1}^{\infty}\frac{dq^d}{z^{2d-1}(d!)^2}\Big) \\
&\quad + e^{t^0/z}\Big(t^1\Big(\sum_{d=1}^{\infty}\frac{dq^d}{z^{2d}(d!)^2}\Big)+1+\sum_{d=1}^{\infty}\frac{q^d}{z^{2d}(d!)^2}\Big(1-2d\sum_{m=1}^{d}\frac{1}{m}\Big)\Big)H, \\
S^0{}_0(z) &= (H,\mathcal{S}(1)) = \Big(1, z\frac{\partial J}{\partial t^1}\Big) \\
&= e^{t^0/z}\Big(t^1\Big(\sum_{d=1}^{\infty}\frac{dq^d}{z^{2d}(d!)^2}\Big)+1+\sum_{d=1}^{\infty}\frac{q^d}{z^{2d}(d!)^2}\Big(1-2d\sum_{m=1}^{d}\frac{1}{m}\Big)\Big), \\
S^1{}_0(z) &= (1,\mathcal{S}(1)) = (1, J) \\
&= e^{t^0/z}\Big(t^1\Big(1+\sum_{d=1}^{\infty}\frac{q^d}{z^{2d+1}(d!)^2}\Big)-2\sum_{d=1}^{\infty}\frac{q^d}{z^{2d+1}(d!)^2}\sum_{m=1}^{d}\frac{1}{m}\Big), \\
S^0{}_1(z) &= (H,\mathcal{S}(H)) = (H, z\frac{\partial J}{\partial t^1}) = e^{t^0/z}\Big(\sum_{d=0}^{\infty}\frac{q^{d+1}}{z^{2d+1}d!(d+1)!}\Big), \\
S^1{}_1(z) &= (1,\mathcal{S}(H)) = (H, J) = e^{t^0/z}\Big(1+\sum_{d=1}^{\infty}\frac{q^d}{z^{2d}(d!)^2}\Big)
\end{aligned}$$

$$S^{\underline{\hat{\alpha}}}{}_j(z) = \sum_{i=0}^{1}\Psi_i{}^{\alpha}S^i{}_j(z),$$

$$S^{\underline{\hat{1}}}{}_1(z) = \frac{1}{\sqrt{2}}e^{t^0/z}\sum_{n=0}^{\infty}\frac{(\sqrt{q})^{n+\frac12}}{z^n}\frac{1}{\lfloor n/2\rfloor!\,\lceil n/2\rceil!},$$

$$S^{\underline{\hat{2}}}{}_1(z) = \frac{1}{\sqrt{2}}e^{t^0/z}\sum_{n=0}^{\infty}\frac{(-\sqrt{q})^{n+\frac12}}{z^n}\frac{1}{\lfloor n/2\rfloor!\,\lceil n/2\rceil!},$$

$$\tilde{\boldsymbol{u}}^{\alpha}(z) = \sum_{i=0}^{1}S^{\underline{\hat{\alpha}}}{}_i(z)t^i(z),$$

$$\tilde{u}^1_k\big|_{t^0_a=0} = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(\sqrt{q})^{n+\frac12}}{\lfloor n/2\rfloor!\,\lceil n/2\rceil!}t^1_{k+n},$$

$$\tilde{u}^2_k\big|_{t^0_a=0} = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(-\sqrt{q})^{n+\frac12}}{\lfloor n/2\rfloor!\,\lceil n/2\rceil!}t^1_{k+n}.$$

For $\alpha = 1, 2$,

$$\widetilde{W}^{\alpha}_k = \sqrt{-2}\,\tilde{u}^{\alpha}_k\big|_{t^0_a=0,\, t^1_a=(a+1)!x^{-a-2}dx} = d\Big(\Big(-\frac{d}{dx}\Big)^k\tilde{\xi}_{\alpha,0}\Big), \tag{20}$$

where

$$\widetilde{\xi}_{1,0} := -\frac{1}{\sqrt{-1}}\sum_{n=0}^{\infty}(\sqrt{q})^{n+\frac{1}{2}}\binom{n}{\lfloor n/2\rfloor}x^{-n-1}, \tag{21}$$

$$\widetilde{\xi}_{2,0} := -\frac{1}{\sqrt{-1}}\sum_{n=0}^{\infty}(-\sqrt{q})^{n+\frac{1}{2}}\binom{n}{\lfloor n/2\rfloor}x^{-n-1}. \tag{22}$$

Recall that

$$W_k^{\alpha} = d\Big(\Big(-\frac{d}{dx}\Big)^k \xi_{\alpha,0}\Big). \tag{23}$$

By (20) and (23), to complete the proof it remains to show that $\widetilde{\xi}_{\alpha,0}$ agrees with the expansion of $\xi_{\alpha,0}$ near $Y=0$ in $\widetilde{x} = x^{-1} = \big(Y+\frac{q}{Y}\big)^{-1}$.

Assume that $q \in (0,\infty)$. We have

$$P_1 = \sqrt{q}, \qquad \Delta^1 = 2\sqrt{q}, \qquad \xi_{1,0} = \frac{1}{\sqrt{-1}}\frac{q^{\frac{1}{4}}}{Y-\sqrt{q}},$$

$$P_2 = -\sqrt{q}, \quad \Delta^2 = -2\sqrt{q}, \quad \xi_{2,0} = \frac{q^{\frac{1}{4}}}{Y+\sqrt{q}}.$$

The n^{th} coefficient in the expansion of $\widetilde{x} = \big(Y+\frac{q}{Y}\big)^{-1}$ at $Y=0$ is given by the residue

$$\begin{aligned}
\operatorname{Res}_{Y=0}\widetilde{x}^{-n-1}\xi_{1,0}\,d\widetilde{x} &= -\frac{1}{\sqrt{-1}}q^{\frac{1}{4}}\operatorname{Res}_{Y=0}\Big(Y+\frac{q}{Y}\Big)^{n-1}\Big(1-\frac{q}{Y^2}\Big)\frac{dY}{Y-\sqrt{q}} \\
&= -\frac{1}{\sqrt{-1}}q^{\frac{1}{4}}\operatorname{Res}_{Y=0}\frac{(Y^2+q)^{n-1}(Y+\sqrt{q})}{Y^{n+1}}dY \\
&= -\frac{1}{\sqrt{-1}}(\sqrt{q})^{n-\frac{1}{2}}\binom{n-1}{\lfloor n/2\rfloor},
\end{aligned}$$

where

$$\begin{aligned}
\xi_{1,0} &= -\frac{1}{\sqrt{-1}}\sum_{n=1}^{\infty}(\sqrt{q})^{n-\frac{1}{2}}\binom{n-1}{\lfloor n/2\rfloor}\widetilde{x}^{n} = -\frac{1}{\sqrt{-1}}\sum_{n=0}^{\infty}(\sqrt{q})^{n+\frac{1}{2}}\binom{n}{\lfloor (n+1)/2\rfloor}\widetilde{x}^{n+1} \\
&= -\frac{1}{\sqrt{-1}}\sum_{n=0}^{\infty}(\sqrt{q})^{n+\frac{1}{2}}\binom{n}{\lfloor n/2\rfloor}x^{-n-1},
\end{aligned}$$

which agrees with $\widetilde{\xi}_{1,0}$, defined in (21), and

$$\operatorname{Res}_{Y=0}\widetilde{x}^{-n-1}\xi_{2,0}\,d\widetilde{x} = -\,q^{\frac{1}{4}}\operatorname{Res}_{Y=0}\Big(Y+\frac{q}{Y}\Big)^{n-1}\Big(1-\frac{q}{Y^2}\Big)\frac{dY}{Y+\sqrt{q}}$$

$$= -q^{\frac{1}{4}}\mathrm{Res}_{Y=0}\frac{(Y^2+q)^{n-1}(Y-\sqrt{q})}{Y^{n+1}}dY$$
$$= -\frac{1}{\sqrt{-1}}(-\sqrt{q})^{n-\frac{1}{2}}\binom{n-1}{\lfloor n/2\rfloor},$$

where

$$\xi_{2,0} = -\frac{1}{\sqrt{-1}}\sum_{n=0}^{\infty}(-\sqrt{q})^{n+\frac{1}{2}}\binom{n}{\lfloor n/2\rfloor}x^{-n-1},$$

which agrees with $\widetilde{\xi}_{2,0}$, defined in (22). □

5 The large radius limit and the Bouchard–Mariño conjecture

In this section, we will specialize Theorem A to the large radius limit case. In this case, Theorem A relates the invariant $\omega_{g,n}$ of the limit curve to the equivariant descendent theory of $\mathbb{C}$. After expanding $\xi_{\alpha,0}$ in suitable coordinates, we can relate the corresponding expansion of $\omega_{g,n}$ to the generation function of Hurwitz numbers and therefore reprove the Bouchard–Mariño conjecture [2] on Hurwitz numbers.

Let $\mathsf{w}_2 = 0$ and $t_0 = 0$, and take the large radius limit $q \to 0$. Then our mirror curve becomes

$$x = Y + \mathsf{w}_1 \log Y.$$

When $\mathsf{w}_1 = -1$, this is just the Lambert curve. Recall that the two critical points P_1 and P_2 of $W_t^{\mathsf{w}}(Y)$ are

$$P_\alpha = \frac{\mathsf{w}_2 - \mathsf{w}_1 + \Delta^\alpha(q)}{2}.$$

Since $\Delta^1(0) = \mathsf{w}_1 - \mathsf{w}_2$, we have $P_1 \to 0$ under the limit $q \to 0$. In other words, P_1 goes out of the curve under the limit $q \to 0$ and $\xi_{1,0} = \sqrt{2/\Delta^\alpha(q)}\,P_1/(Y-P_1) \to 0$. As a result, $W_k^1 = d(\theta^k(\xi_{1,0}))$ also tends to zero under the large radius limit.

Under the identification $W_k^\alpha(Y_j)/\sqrt{-2} = (\widetilde{u}_j)_k^\alpha$ in Theorem A, we have $(\widetilde{u}_j)_k^1 \to 0$ when $q \to 0$. On the A–model side, since $q = 0$, the $S-$matrix $(\mathring{S}^\alpha{}_\beta(z))$ is diagonal. Therefore, we also have $(u_j)_k^1 \to 0$ when $q \to 0$ under the identification in Theorem A. This means that in the localization graph of the equivariant GW invariants of $\mathbb{P}^1$, we can only have a constant map to $p_2 \in \mathbb{P}^1$. Since $H|_{p_2} = \mathsf{w}_2 = 0$ and $t^0 = 0$, we cannot have any primary insertions. Therefore, in the large radius limit, we get

$$F_{g,n}^{\mathbb{P}^1,\mathbb{C}^*}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_n;\boldsymbol{t}) = \sum_{a_1,\dots,a_n\in\mathbb{Z}_{\geq 0}} \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,0)]^{\mathrm{vir}}} \prod_{j=1}^{n} \mathrm{ev}_j^*((u_j)^2_{a_j}\phi_2(0))\psi_j^{a_j}$$
$$= \sum_{a_1,\dots,a_n\in\mathbb{Z}_{\geq 0}} \frac{1}{-\mathsf{w}_1}\int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^{n} (u_j)^2_{a_j}\psi_j^{a_j}\Lambda_g^\vee(-\mathsf{w}_1),$$

where

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g\lambda_g$$

and $\lambda_j = c_j(\mathbb{E})$ is the j^{th} Chern class of the Hodge bundle. At the same time, we also have $\mathring{S}^2{}_2 = (\hat{\phi}_2(0),\hat{\phi}_2(0)) = 1$, so $(u_j)^2_k/\sqrt{-\mathsf{w}_1} = (\tilde{u}_j)^2_k$. Therefore Theorem A specializes to

$$\omega_{g,n}\big|_{W_k^2(Y_j)/\sqrt{-2}=(u_j)^2_k/\sqrt{-\mathsf{w}_1}} = (-1)^{g-1+n}\sum_{\substack{a_1,\dots,a_n\\ a_i\in\mathbb{Z}_{\geq 0}}} \frac{1}{-\mathsf{w}_1}\int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^{n} (u_j)^2_{a_j}\psi_j^{a_j}\Lambda_g^\vee(-\mathsf{w}_1).$$

Now we study the expansion of $\xi_{2,0}$ near the point $Y=0$ in the coordinate $Z=e^{x/\mathsf{w}_1}$. We have

$$\xi_{2,0} = \frac{1}{\sqrt{-1}}\sqrt{\frac{2}{-\mathsf{w}_1}}\frac{-\mathsf{w}_1}{Y+\mathsf{w}_1}.$$

Since $Z = Ye^{Y/\mathsf{w}_1}$, by taking the differential we have

$$\frac{dZ}{Z} = \frac{Y+\mathsf{w}_1}{Y\mathsf{w}_1}dY.$$

Therefore,

$$\xi_{2,0} = -\frac{1}{\sqrt{-1}}\sqrt{\frac{2}{-\mathsf{w}_1}}\frac{dY}{dZ/Z}\frac{1}{Y}.$$

Let

$$\xi_{2,0} = \sum_{\mu=0}^{\infty} C_\mu Z^\mu$$

near the point $Y=0$. Then we have

$$C_\mu = \mathrm{Res}_{Y\to 0}\,\xi_{2,0}Z^{-\mu}\frac{dZ}{Z} = -\frac{1}{\sqrt{-1}}\sqrt{\frac{2}{-\mathsf{w}_1}}\,\mathrm{Res}_{Y\to 0}\, e^{-\mu Y/\mathsf{w}_1}\frac{dY}{Y^{\mu+1}}$$
$$= -\frac{1}{\sqrt{-1}}\sqrt{\frac{2}{-\mathsf{w}_1}}\frac{(-\mu/\mathsf{w}_1)^\mu}{\mu!}.$$

Therefore,

$$W_k^2 = -\frac{1}{\sqrt{-1}}\sqrt{\frac{2}{-\mathsf{w}_1}}\mathsf{w}_1 \sum_{\mu=0}^{\infty} \frac{(-\mu/\mathsf{w}_1)^{\mu}}{\mu!}\Big(-\frac{\mu}{\mathsf{w}_1}\Big)^{k+1} Z^{\mu-1} dZ.$$

On the A–model side, let

$$(u_j)^2_{a_j} = \sum_{\mu_j=0}^{\infty} \frac{(-\mu_j/\mathsf{w}_1)^{\mu_j}}{\mu_j!}\Big(\frac{\mu_j}{\mathsf{w}_1}\Big)^{a_j} Z_j^{\mu_j}.$$

Then

$$\begin{aligned}
&F_{g,n}^{\mathbb{C},\mathbb{C}^*}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_n)\\
&= \sum_{\substack{a_1,\dots,a_n\\ a_i\in\mathbb{Z}_{\geq 0}}} \frac{1}{-\mathsf{w}_1}\int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^n \psi_j^{a_j}\Lambda_g^{\vee}(-\mathsf{w}_1)\prod_{j=1}^n\Big(\sum_{\mu_j=0}^{\infty}\frac{(-\mu_j/\mathsf{w}_1)^{\mu_j}}{\mu_j!}\Big(-\frac{\mu_j}{\mathsf{w}_1}\Big)^{a_j} Z_j^{\mu_j}\Big)\\
&= \sum_{\substack{a_1,\dots,a_n\\ a_i\in\mathbb{Z}_{\geq 0}}} \frac{1}{-\mathsf{w}_1}\int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^n \Big(-\frac{\mu_j\psi_j}{\mathsf{w}_1}\Big)^{a_j}\Lambda_g^{\vee}(-\mathsf{w}_1)\prod_{j=1}^n\Big(\sum_{\mu_j=0}^{\infty}\frac{(-\mu_j/\mathsf{w}_1)^{\mu_j}}{\mu_j!} Z_j^{\mu_j}\Big).
\end{aligned}$$

By the ELSV formula [6; 21],

$$\begin{aligned}
H_{g,\mu} &= \frac{(2g-2+|\mu|+n)!}{|\mathrm{Aut}(\mu)|}\prod_{j=1}^n\frac{\mu_j^{\mu_j}}{\mu_j!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda_g^{\vee}(1)}{\prod_{j=1}^n(1-\mu_j)}\\
&= \frac{(2g-2+|\mu|+n)!}{|\mathrm{Aut}(\mu)|}\prod_{j=1}^n\frac{\mu_j^{\mu_j}}{\mu_j!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda_g^{\vee}(-\mathsf{w}_1)(-\mathsf{w}_1)^{2g-3+2n}}{\prod_{j=1}^n(-\mathsf{w}_1-\mu_j)},
\end{aligned}$$

so

$$F^{\mathbb{C},\mathbb{C}^*} = \sum_{l(\mu)=n}\frac{|\mathrm{Aut}(\mu)|}{(2g-2+|\mu|+n)!\,(-\mathsf{w}_1)^{2g-2+|\mu|+n}}H_{g,\mu}\sum_{\sigma\in S_n}\prod_{j=1}^n Z_{\sigma(j)}^{\mu_j}.$$

When $\mathsf{w}_1 = -1$, this is just the generating function of the Hurwitz numbers.

Let $W_{g,n}(Z_1,\dots,Z_n)$ be the expansion of $\omega_{g,n}(Y_1,\dots,Y_n)$ in the coordinate Z near $Y=0$. Then we have:

Corollary 5.1 (Bouchard–Mariño conjecture) *For $n>0$ and $2g-2+n>0$, the invariant $W_{g,n}(Z_1,\cdots,Z_n)$ for the curve $x = Y+\mathsf{w}_1\log Y$ satisfies*

$$
\begin{aligned}
&\int_0^{Z_1}\cdots\int_0^{Z_n} W_{g,n}(Z_1,\cdots,Z_n)\\
&\quad=(-1)^{g-1+n}\sum_{\substack{a_1,\dots,a_n\\ a_i\in\mathbb{Z}_{\geq 0}}}\frac{1}{-\mathsf{w}_1}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{j=1}^n\psi_j^{a_j}\Lambda_g^\vee(-\mathsf{w}_1)\\
&\qquad\qquad\cdot\prod_{j=1}^n\Bigl(\sum_{\mu_j=0}^\infty\frac{(-\mu_j/\mathsf{w}_1)^{\mu_j+a_j}}{\mu_j!}Z_j^{\mu_j}\Bigr)\\
&\quad=(-1)^{g-1+n}\sum_{l(\mu)=n}\frac{|\mathrm{Aut}(\mu)|H_{g,\mu}}{(2g-2+|\mu|+n)!\,(-\mathsf{w}_1)^{2g-2+|\mu|+n}}\sum_{\sigma\in S_n}\prod_{j=1}^n Z_{\sigma(j)}^{\mu_j}.
\end{aligned}
$$

In particular, when $\mathsf{w}_1=-1$*, the right-hand side is the generating function of the Hurwitz numbers and the Bouchard–Mariño conjecture is recovered.*

Appendix A: Bessel functions

In this section, we give a brief review of Bessel functions.

The Bessel differential equation is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0. \tag{24}$$

The *Bessel function of the first kind* is defined by

$$J_\alpha(x)=\sum_{m=0}^\infty\frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\Bigl(\frac{x}{2}\Bigr)^{2m+\alpha}.$$

The *Bessel function of the second kind* is defined by

$$Y_\alpha(x)=\frac{J_\alpha(x)\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

When n is an integer, $Y_n(x):=\lim_{\alpha\to n}Y_\alpha(x)$.

$J_\alpha(x)$ and $Y_\alpha(x)$ form a basis of the 2–dimensional space of solutions to the Bessel differential equation (24).

Replacing x by ix in (24), one obtains the modified Bessel differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+\alpha^2)y=0. \tag{25}$$

The *modified Bessel function of the first kind* is defined by

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

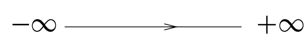
The *modified Bessel function of the second kind* is defined by

$$K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}.$$

The following integral formulas are valid when $\Re(x) > 0$:

$$\begin{aligned} I_\alpha(x) &= \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos(\alpha\theta)\,d\theta - \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-x\cosh t - \alpha t}\,dt, \\ K_\alpha(x) &= \int_0^\infty e^{-x\cosh t}\cosh(\alpha t)\,dt = \frac{1}{2}\int_{t\in\gamma_{0,0}} e^{-x\cosh t-\alpha t}\,dt, \end{aligned}$$

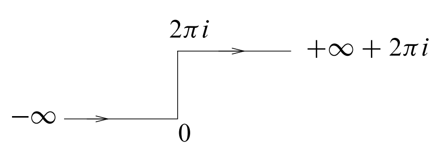
where $\gamma_{0,0}$ is the real line with the standard orientation:



We have

$$\begin{aligned} &e^{\alpha\pi i} K_\alpha(x) + i\pi I_\alpha(x) \\ &\quad= \frac{\pi}{2}\frac{e^{\alpha\pi i} I_{-\alpha}(x) - e^{-\alpha\pi i} I_\alpha(x)}{\sin(\alpha\pi)} \\ &\quad= \frac{e^{\alpha\pi i}}{2}\int_{-\infty}^0 e^{-x\cosh t-\alpha t}\,dt + \frac{e^{\alpha\pi i}}{2}\int_0^{2\pi} e^{-x\cos(i\theta)-\alpha(i\theta)}\,d(i\theta) \\ &\qquad\qquad + \frac{e^{-\alpha\pi i}}{2}\int_0^\infty e^{-x\cosh t-\alpha t}\,dt \\ &\quad= \frac{e^{\alpha\pi i}}{2}\int_{\gamma_{0,1}} e^{-x\cosh t-\alpha t}\,dt, \end{aligned}$$

where $\gamma_{0,1}$ is the following contour:

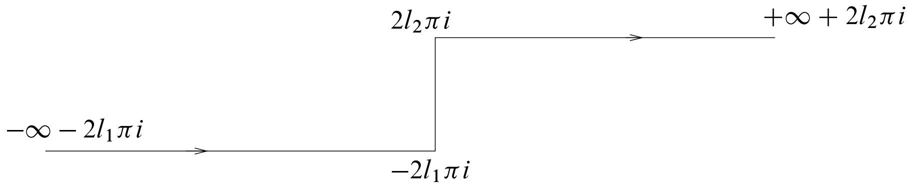


Therefore,

$$\int_{\gamma_{0,0}} e^{-x\cosh t-\alpha t}\,dt=\frac{\pi}{\sin(\alpha\pi)}(I_{-\alpha}(x)-I_{\alpha}(x)),\tag{26}$$

$$\int_{\gamma_{0,1}} e^{-x\cosh t-\alpha t}\,dt=\frac{\pi}{\sin(\alpha\pi)}(I_{-\alpha}(x)-e^{-2\alpha\pi i}I_{\alpha}(x)).\tag{27}$$

For any integers l_1 and l_2 with $l_1+l_2\geq 0$, let γ_{l_1,l_2} be the following contour:



Lemma A.1 *For any $l_1, l_2\in\mathbb{Z}$ such that $l_1+l_2\geq 0$, we have*

$$\int_{\gamma_{l_1,l_2}} e^{-x\cosh t-\alpha t}\,dt=\frac{\pi}{\sin(\alpha\pi)}(e^{2l_1\alpha\pi i}I_{-\alpha}(x)-e^{-2l_2\alpha\pi i}I_{\alpha}(x)).\tag{28}$$

Proof We observe that

$$\int_{\gamma_{l_1-k,l_2+k}} e^{-x\cosh t-\alpha t}\,dt=e^{-2k\alpha\pi i}\int_{\gamma_{l_1,l_2}} e^{-x\cosh t-\alpha t}\,dt.\tag{29}$$

In particular,

$$\begin{aligned}\int_{\gamma_{l_1,-l_1}} e^{-x\cosh t-\alpha t}\,dt&=e^{2l_1\alpha\pi i}\int_{\gamma_{0,0}} e^{-x\cosh t-\alpha t}\,dt\\&=\frac{\pi}{\sin(\alpha\pi)}\big(e^{-2l_1\alpha\pi i}I_{-\alpha}(x)-e^{2l_1\alpha\pi i}I_{\alpha}(x)\big).\end{aligned}$$

This proves (28) in the case $l_1+l_2=0$. If $l_1+l_2>0$ then

$$\gamma_{l_1,l_2}=\sum_{k=-l_1}^{l_2-1}\gamma_{1-k,k}-\sum_{k=1-l_1}^{l_2-1}\gamma_{-k,k}.\tag{30}$$

Equations (29) and (30) imply

$$\begin{aligned}&\int_{\gamma_{l_1,l_2}} e^{-x\cosh t-\alpha t}\,dt\\&=\Big(\sum_{k=-l_1}^{l_2-1}e^{-2k\alpha\pi i}\Big)\int_{\gamma_{0,1}} e^{-x\cosh t-\alpha t}\,dt-\Big(\sum_{k=1-l_1}^{l_2-1}e^{-2k\alpha\pi i}\Big)\int_{\gamma_{0,0}} e^{-x\cosh t-\alpha t}\,dt.\end{aligned}$$

Equation (28) follows from the above equation and (26)–(27). □

Appendix B: The equivariant quantum differential equation for $\mathbb{P}^1$

The equivariant quantum differential equation of $\mathbb{P}^1$ is the vector equation

$$zq\frac{d}{dq}\vec{I} = \begin{pmatrix} 0 & q - \mathsf{w}_1\mathsf{w}_2 \\ 1 & \mathsf{w}_1 + \mathsf{w}_2 \end{pmatrix}\vec{I},$$

which is equivalent to the scalar equation

$$\Big(zq\frac{d}{dq} - \mathsf{w}_1\Big)\Big(zq\frac{d}{dq} - \mathsf{w}_2\Big)I = qI. \tag{31}$$

Let

$$I = \exp\bigg(\frac{\mathsf{w}_1 + \mathsf{w}_2}{2z}\log q\bigg)y, \quad x = \frac{2\sqrt{q}}{z}.$$

Then (31) is equivalent to

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \bigg(x^2 + \Big(\frac{\mathsf{w}_1 - \mathsf{w}_2}{2z}\Big)^2\bigg)y = 0,$$

which is the modified Bessel differential equation (25) with $\alpha = (\mathsf{w}_1 - \mathsf{w}_2)/(2z)$. When $\mathsf{w}_1 - \mathsf{w}_2 \neq 0$, any solution to (31) is of the form

$$I = \exp\bigg(\frac{\mathsf{w}_1 + \mathsf{w}_2}{2z}\log q\bigg)\bigg(c_1 I_{\chi^1/z}\Big(\frac{2\sqrt{q}}{z}\Big) + c_2 I_{\chi^2/z}\Big(\frac{2\sqrt{q}}{z}\Big)\bigg),$$

where $\chi^1 = \mathsf{w}_1 - \mathsf{w}_2 = -\chi^2$, and c_1 and c_2 are functions of w_1, w_2 and z.

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