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Collar lemma for Hitchin representations

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There is a classical result known as the collar lemma for hyperbolic surfaces. A consequence of the collar lemma is that if two closed curves A and B on a closed orientable hyperbolizable surface intersect each other, then there is an explicit lower bound for the length of A in terms of the length of B, which holds for every hyperbolic structure on the surface. In this article, we prove an analog of the classical collar lemma in the setting of Hitchin representations.

57M50; 30F60, 32G15

1 Introduction

Let S be a closed connected oriented topological surface of genus $g \geq 2$, and let Γ be its fundamental group. The Teichmüller space of S, which we denote by $\mathcal{T}(S)$, is the space of hyperbolic structures on S, ie the space of isotopy classes of hyperbolic metrics on S. Via the holonomy representation, $\mathcal{T}(S)$ can be identified with a component of the space of conjugacy classes of representations from Γ to $\mathrm{PSL}(2,\mathbb{R})$. One advantage of doing so is that it allows us to generalize $\mathcal{T}(S)$ in the following way. It is a standard fact in representation theory that for any $n \geq 2$, there is a unique (up to conjugation) irreducible representation $\iota_n\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$. This gives, via postcomposition, an embedding

$$\mathcal{T}(S) \hookrightarrow \mathcal{X}_n(S) := \mathrm{Hom}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))/\,\mathrm{PSL}(n,\mathbb{R}).$$

The image of this embedding is known as the *Fuchsian locus* and the component of $\mathcal{X}_n(S)$ containing the Fuchsian locus is the n^{th} *Hitchin component*, which we denote by $\mathrm{Hit}_n(S)$. By definition, $\mathrm{Hit}_2(S) = \mathcal{T}(S)$, so Hitchin representations can be thought of as generalizations of Fuchsian representations.

For the hyperbolic structures in $\mathcal{T}(S)$, there is a classical result due to Keen [14] known as the collar lemma. It gives an effective lower bound on the width of the maximal collar neighborhood of a simple closed curve in a hyperbolic surface, which grows to ∞ as the length of the simple closed curve is shrunk to 0. A consequence of the collar lemma is that if two closed curves η and γ in a hyperbolic surface have

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nonvanishing geometric intersection number and γ is simple, then there is an explicit lower bound on the length of η in terms of the length of γ. This is a powerful tool that has been used to understand surfaces. For example, it was used to study the length spectrum of Riemann surfaces; see Buser [5].

The goal of this paper is to generalize a version of the classical collar lemma to Hitchin representations. By Labourie [15], for any Hitchin representation ρ and any nonidentity element X in Γ, we know that $\rho(X)$ is diagonalizable over $\mathbb{R}$ with eigenvalues that have pairwise distinct moduli. *For the rest of this paper, we will denote by $x^+, x^- \in \partial_\infty \Gamma$ the attracting and repelling fixed points, respectively, of $X \in \Gamma \setminus \{\mathrm{id}\}$.* With this notation, we now state the main theorem of this paper.

Theorem 1.1 *Let A, B be elements in Γ such that a^+, b^+, a^-, b^- lie in $\partial_\infty \Gamma$ in that cyclic order. Also, let $\rho \in \mathrm{Hit}_n(S)$ and let $\alpha_n < \cdots < \alpha_1$ and $\beta_n < \cdots < \beta_1$ be the moduli of the eigenvalues of $\rho(A)$ and $\rho(B)$, respectively. For every $k = 0, \dots, n-2$, the following hold:*

(1)
$$\frac{\alpha_1}{\alpha_n} > \frac{\beta_{k+1}}{\beta_{k+1} - \beta_{k+2}}.$$

(2) *Let η and γ be closed curves in S corresponding to A and B, respectively, and let $i(\eta,\gamma)$ be the geometric intersection number between η and γ. If γ is simple, then*
$$\frac{\alpha_1}{\alpha_n} > \left(\frac{\beta_{k+1}}{\beta_{k+1} - \beta_{k+2}}\right)^u \cdot \left(\frac{\beta_{n-k-1}}{\beta_{n-k-1} - \beta_{n-k}}\right)^{i(\eta,\gamma)-u}$$
for some nonnegative integer $u \le i(\eta,\gamma)$ that is independent of k.

Observe that Theorem 1.1(2) does not depend on the choice of orientation on η or γ. We can also say what the constant u in Theorem 1.1(2) is. Choose orientations on η and γ and let $\hat{\imath}(\eta,\gamma)$ be the algebraic intersection number between η and γ. Then
$$u = \tfrac{1}{2}\big(i(\eta,\gamma) + |\hat{\imath}(\eta,\gamma)|\big).$$

In the setting of Hitchin representations, the width of a collar neighborhood is not well defined since Hitchin representations in general do not give a metric on S. However, for every Hitchin representation ρ, we do still have a natural notion of length for free homotopy classes of closed curves in S. Given any representation ρ in $\mathrm{Hit}_n(S)$ and any closed curve γ in S, we can define the *ρ–length* of γ to be
$$l_\rho(\gamma) = \log\left(\frac{\lambda_1}{\lambda_n}\right),$$

where λ_1 and λ_n are the largest and smallest moduli of the eigenvalues of $\rho(X)$, respectively, and $X \in \Gamma$ corresponds to the closed curve γ equipped with a choice of orientation. Observe that the ρ–length does not depend on the choice of orientation on γ or the choice of X, and is constant on each free homotopy class of closed curves in S.

If $\rho \in \mathrm{Hit}_2(S)$, then $l_\rho(\gamma)$ is exactly the hyperbolic length of the geodesic homotopic to γ, measured in the hyperbolic metric corresponding to ρ. Also, Choi and Goldman [7] proved that representations in $\mathrm{Hit}_3(S)$ are exactly holonomies of convex $\mathbb{RP}^2$ structures on S. Moreover, each such convex $\mathbb{RP}^2$ structure also induces a natural Finsler metric, known as the Hilbert metric, on S. One can then verify, in the case when $\rho \in \mathrm{Hit}_3(S)$, that $l_\rho(\gamma)$ is the length of the geodesic homotopic to γ, measured in the Hilbert metric induced by the convex $\mathbb{RP}^2$ structure corresponding to ρ.

With this notion of ρ–length, we have the following corollary of Theorem 1.1, which one can think of as a generalization of the collar lemma.

Corollary 1.2 *Let S be a surface of genus $g \geq 2$, and let η and γ be two essential closed curves in S. Then, for any $n \geq 2$ and any $\rho \in \mathrm{Hit}_n(S)$, the following hold:*

(1) *If $i(\eta,\gamma) \neq 0$, then*

$$\frac{1}{\exp(l_\rho(\eta))} < 1 - \frac{1}{\exp(l_\rho(\gamma)/(n-1))}.$$

(2) *If $i(\eta,\gamma) \neq 0$ and γ is simple, then there are nonnegative integers u and v with $u \geq v$ and $u+v = i(\eta,\gamma)$ such that*

$$\frac{1}{\exp(l_\rho(\eta))} < \left(1 - \frac{1}{\exp(l_\rho(\gamma)/(n-1))}\right)^u \left(1 - \frac{1}{\exp(l_\rho(\gamma))}\right)^v.$$

(3) *Let $\delta_n > 0$ be the unique real solution to the equation $e^{-x} + e^{-x/(n-1)} = 1$. If η is a nonsimple closed curve, then*

$$l_\rho(\eta) > \delta_n.$$

The quantity u in the above corollary is the same u as in Theorem 1.1. Observe that δ_n is an increasing unbounded sequence, and $\delta_2 = \log(2)$. Also, the expressions on the right hand side of the inequalities in parts (1) and (2) of Corollary 1.2 are maximized when $n = 2$. Hence, we can replace n by 2 in the right hand side of all three inequalities in Corollary 1.2, and they will still hold.

In the case of $\mathcal{T}(S)$, the first inequality in Corollary 1.2 can be rewritten as

$$\bigl(\exp(l_\rho(\eta)) - 1\bigr)\bigl(\exp(l_\rho(\gamma)) - 1\bigr) > 1.$$

This is weaker than a version of the classical collar lemma, which is the inequality

$$\sinh\big(\tfrac{1}{2}l_\rho(\eta)\big)\sinh\big(\tfrac{1}{2}l_\rho(\gamma)\big) > 1, \tag{1-1}$$

although in both inequalities, $l_\rho(\eta)$ grows logarithmically with $1/l_\rho(\gamma)$. In general, it is not known if the inequality (1-1) holds for all Hitchin components. However, it is a consequence of recent work of Tholozan [18] that it in fact holds for $\mathrm{Hit}_3(S)$. See Section 3.3 for more details.

Choi [6] proved an analog of the Margulis lemma for convex $\mathbb{RP}^2$ surfaces. As a consequence, he showed the existence of a collar neighborhood in the convex $\mathbb{RP}^2$ surface about a simple closed curve of sufficiently short length, and found (nonexplicit) lower bounds for the width of this collar neighborhood in terms of the length of the simple closed curve. This analog of the Margulis lemma was later extended by Cooper, Long and Tillman [8] to all convex real projective manifolds. Burger and Pozzetti [4] also recently proved a statement analogous to Theorem 1.1 for maximal representations into $\mathrm{PSp}(2k,\mathbb{R})$.

The image of the irreducible representation $\iota_n\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ lies in a conjugate of the subgroup $\mathrm{PSO}(k,k+1) \subset \mathrm{PSL}(2k+1,\mathbb{R})$ when $n = 2k+1$, and a conjugate of $\mathrm{PSp}(2k,\mathbb{R}) \subset \mathrm{PSL}(2k,\mathbb{R})$ when $n = 2k$. Hence, we can define Hitchin components in

$$\mathrm{Hom}(\Gamma,\mathrm{PSO}(k,k+1))/\,\mathrm{PSO}(k,k+1), \quad \mathrm{Hom}(\Gamma,\mathrm{PSp}(2k,\mathbb{R}))/\,\mathrm{PSp}(2k,\mathbb{R})$$

in the same way as we did for $\mathrm{PSL}(n,\mathbb{R})$. Denote these Hitchin components by $\mathrm{Hit}_n(S)'$. Since the image of ι_7 in particular lies in the exceptional Lie group $G_2 \subset \mathrm{PSO}(3,4)$, we can also define a Hitchin component $\mathrm{Hit}(S,G_2)$ in $\mathrm{Hom}(\Gamma,G_2)/G_2$. Note that $\mathrm{Hit}_n(S)'$ and $\mathrm{Hit}(S,G_2)$ can be naturally identified with a subset of $\mathrm{Hit}_n(S)$ and $\mathrm{Hit}_7(S)'$, respectively. In the case when $\rho \in \mathrm{Hit}_n(S)$ happens to be an element of $\mathrm{Hit}_n(S)'$, we can strengthen Theorem 1.1(2), which we state as the following corollary.

Corollary 1.3 *Let A and B be elements in Γ such that a^+, b^+, a^-, b^- lie in $\partial_\infty\Gamma$ in that cyclic order. Let $\rho \in \mathrm{Hit}_n(S)'$ and let $\alpha_n < \cdots < \alpha_1$ and $\beta_n < \cdots < \beta_1$ be the moduli of the eigenvalues of $\rho(A)$ and $\rho(B)$, respectively. Finally, let η and γ be closed curves on S corresponding to A and B, respectively. If γ is simple, then for every $k = 0,\dots,n-2$,*

$$\alpha_1^2 > \left(\frac{\beta_{k+1}}{\beta_{k+1}-\beta_{k+2}}\right)^{i(\eta,\gamma)}.$$

Hitchin representations into $\mathrm{PSp}(2k,\mathbb{R})$ are special examples of maximal representations. In this case, the inequality in the above corollary is stronger than the one given by Burger and Pozzetti [4].

The proof of our results relies heavily on the seminal work of Labourie [15], who showed that every Hitchin representation into $\mathrm{PSL}(n,\mathbb{R})$ (and hence into $\mathrm{PSp}(2k,\mathbb{R})$, $\mathrm{PSO}(k,k+1)$ and G_2) naturally comes with an equivariant Frenet curve; see Theorem 2.5. While Hitchin representations can be defined for any split real group, properties of the limit curve of these Hitchin representations are still poorly understood in general. As such, we are unable to generalize our techniques to prove an analog of Theorem 1.1 for Hitchin representations into split real groups other than $\mathrm{PSL}(n,\mathbb{R})$, $\mathrm{PSp}(2k,\mathbb{R})$, $\mathrm{PSO}(k,k+1)$ and G_2.

Unfortunately, for $\rho \in \mathrm{Hit}_n(S)$ when $n \geq 4$, it is not known whether there exists a metric on S that induces l_ρ as its length function. However, we can still interpret Corollary 1.2 geometrically by considering the $\mathrm{PSL}(n,\mathbb{R})$ symmetric space $\widetilde{M}$. Normalize the Riemannian metric on $\widetilde{M}$ so that for any $Z \in \mathrm{PSL}(n,\mathbb{R})$ with real eigenvalues,

$$\inf\{d_{\widetilde{M}}(o, Z\cdot o) : o \in \widetilde{M}\} = \sqrt{2\sum_{i=1}^{n}(\log\lambda_i)^2},$$

where $\lambda_1,\dots,\lambda_n$ are the moduli of the eigenvalues of Z and $d_{\widetilde{M}}$ is the distance function on $\widetilde{M}$ induced by the normalized Riemannian metric. Let $M := \rho(\Gamma)\backslash\widetilde{M}$, and for any closed curve ω in M, let $l_M(\omega)$ be the length of ω measured in the Riemannian metric on M induced by the normalized Riemannian metric on $\widetilde{M}$. Then the following corollary holds.

Corollary 1.4 *Let η and γ be two essential closed curves in S and let X and Y be elements in Γ corresponding to η and γ, respectively. For any $\rho \in \mathrm{Hit}_n(S)$, let η' and γ' be two closed curves in M that correspond to $X, Y \in \Gamma$, respectively. Then the statements in Corollary 1.2 hold, with $l_\rho(\eta)$ and $l_\rho(\gamma)$ replaced by $l_M(\eta')$ and $l_M(\gamma')$, respectively.*

It is an important remark that this corollary (and hence Corollary 1.2) is not simply a quantitative version of the Margulis lemma on $\mathrm{PSL}(n,\mathbb{R})$ because the closed curves η' and γ' do not need to intersect, even when $i(\eta,\gamma) \neq 0$.

Theorem 1.1 is a property that is special to Hitchin representations. In fact, for any pair of simple closed curves in S, one can find a sequence of quasi-Fuchsian representations

$$\rho_i\colon \Gamma \to \mathrm{PSO}(3,1)^+ \subset \mathrm{PSL}(4,\mathbb{R})$$

such that the lengths of the geodesics in $\rho_i(\Gamma)\backslash\widetilde{M}$ corresponding to both of these two simple closed curves converge to 0 along this sequence. In particular, Corollary 1.2 does not hold on the space of quasi-Fuchsian representations. This is explained in greater detail in Section 3.2.

Theorem 1.1 can also be generalized to the setting where we allow S to be compact but not necessarily closed; see Corollary 3.4.

As a final consequence of Theorem 1.1, we have the following properness result.

Corollary 1.5 *Let $\mathcal{C} := \{\gamma_1, \dots, \gamma_k\}$ be a collection of closed curves in S that contains a pants decomposition, such that the complement of $\mathcal{C}$ in S is a union of discs. Then the map*

$$\mathrm{Hit}_n(S) \to \mathbb{R}^k, \quad \rho \mapsto (l_\rho(\gamma_1), \dots, l_\rho(\gamma_k)),$$

is proper.

In other words, in order for a sequence $\{\rho_i\}_{i=1}^{\infty}$ in $\mathrm{Hit}_n(S)$ to escape, the ρ_i–length of some curve in $\mathcal{C}$ must grow to ∞. We will give the proof of this corollary in the Appendix because it uses some technical results from Zhang [19]. Refer to Section 3.1 for more corollaries of Theorem 1.1.

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2 Proof of Theorem 1.1

We start this section by discussing some useful topological properties of Γ and its boundary in Section 2.1. Then for the sake of demonstrating the proof without too many technical details, we prove Theorem 1.1(1) for the special case $\mathrm{Hit}_3(S)$ in Section 2.2. Next, we develop the technical tools that we need in Section 2.3, and apply them in Section 2.4 to prove Theorem 1.1 in its full generality.

2.1 Properties of the boundary of the group

It is well known that Γ is Gromov hyperbolic, so the Cayley graph of Γ has a natural boundary, which we denote by $\partial_\infty\Gamma$, and the action of Γ on its Cayley graph extends to an action on $\partial_\infty\Gamma$. Moreover, if we choose $\rho \in \mathcal{T}(S)$, ie a hyperbolic structure on S, we get a ρ–equivariant identification of $\partial_\infty\Gamma$ with the boundary $\partial\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$.

For any hyperbolic element $A \in \mathrm{PSL}(2,\mathbb{R})$, the *axis* of A, which we denote by L_A, is the unique geodesic in $\mathbb{H}^2$ whose endpoints are the repelling and attracting fixed points of A in $\partial\mathbb{H}^2$. The proof of the main theorem relies crucially on an important property of the action of Γ on $\partial_\infty\Gamma$, which we state as Lemma 2.2. These are well-known facts about surface groups, but for lack of a good reference, we will give the proof here.

Lemma 2.1 *Let B and B' be noncommuting elements in $\mathrm{PSL}(2,\mathbb{R})$ that generate a subgroup consisting only of hyperbolic isometries. If the translation lengths of B and B' are the same and $L_{B'} \cap L_B = \varnothing$, then $(B\cdot L_{B'}) \cap L_{B'} = \varnothing$.*

Proof Since B and B' do not commute, $L_B \neq L_{B'}$. Since the commutator $[B, B']$ is not parabolic, B and B' cannot share a fixed point. Hence, by changing coordinates and replacing B and B' with their inverses if necessary, we can assume that L_B and $L_{B'}$ are as in Figure 1, and that B and B' translate along their axes in the directions drawn.

Let L be the geodesic in $\mathbb{H}^2$ that is perpendicular to both $L_{B'}$ and L_B, and let R be the reflection about L. There is a unique geodesic K that is perpendicular to L_B and whose distances to L and $B\cdot L$ are equal. Let S be the reflection about K, and note that $B = SR$. Also, observe that the distance between K and L is realized only by the points $K\cap L_B$ and $L\cap L_B$, and is half the translation length of B, which we denote by T. Furthermore, $(B\cdot L_{B'})\cap L_{B'} = (SR\cdot L_{B'})\cap L_{B'} = (S\cdot L_{B'})\cap L_{B'}$ is empty if and only if $K\cap L_{B'}$ is empty.

Thus, it is sufficient to show that $K\cap L_{B'}$ is empty. Suppose for contradiction that it is not. As before, there is a unique geodesic K' such that $B' = S'R$, where S' is the reflection about K'. Since the translation lengths of B and B' are the same, the symmetry between B and B' ensures that $K'\cap L_B$ is also nonempty.

Now, note that $K'\cap L_{B'}$ lies between $K\cap L_{B'}$ and $L\cap L_{B'}$ because

$$d(K\cap L_{B'}, L\cap L_{B'}) > d(K\cap L_B, L\cap L_B) = \tfrac{1}{2}T = d(K'\cap L_{B'}, L\cap L_{B'}).$$

Similarly, $K\cap L_B$ lies between $K'\cap L_B$ and $L\cap L_B$. This implies that K and K' have a common point of intersection, p; see Figure 1. Observe that $B'B^{-1} = S'RR^{-1}S^{-1} = S'S$ fixes p, but that is impossible because $B'B^{-1}$ is not elliptic. □

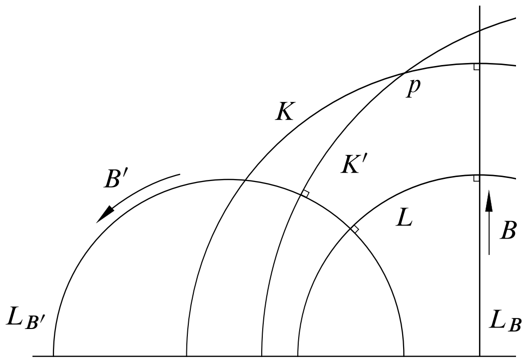

Figure 1: An impossible configuration of K and K' in Lemma 2.1

Lemma 2.2 *Let A, B and B' be pairwise noncommuting elements in Γ such that B and B' are conjugate. If*

$$a^+,\ b'^+,\ b^+,\ a^-,\ b^-,\ b'^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order, then

$$a^+,\ b'^+,\ B\cdot a^+,\ b^+,\ a^-,\ b^-,\ B^{-1}\cdot a^+,\ b'^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order; see Figure 2.

Proof Let s_0 be the open subsegment of $\partial_\infty\Gamma$ with endpoints b'^- and b^+ that does not contain b^-, and let s_1 be the open subsegment of $\partial_\infty\Gamma$ with endpoints b'^+ and b^+ that does not contain b^-. Observe that $B\cdot b'^-$ lies in s_0 and $B\cdot b'^+$ lies in s_1.

Choose a hyperbolic metric on S. This identifies $\partial_\infty\Gamma$ with $\partial\mathbb{H}^2$ and Γ with a discrete, torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Since

$$a^+,\ b'^+,\ b^+,\ a^-,\ b^-,\ b'^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order, L_B and $L_{B'}$ have to be disjoint. Moreover, B and B' have the same translation lengths and do not commute. Hence, we can apply Lemma 2.1 to conclude that $B\cdot L_{B'}$ and $L_{B'}$ are disjoint. This implies that both $B\cdot b'^-$ and $B\cdot b'^+$ have to lie in s_1. Since a^+ lies in s_0 between b'^- and b'^+, we have that $B\cdot a^+$ must lie in s_1 between $B\cdot b'^-$ and $B\cdot b'^+$. In particular,

$$a^+,\ b'^+,\ B\cdot a^+,\ b^+,\ a^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order; see Figure 2.

A similar argument, using B^{-1} instead of B, shows that

$$a^-,\ b^-,\ B^{-1}\cdot a^+,\ b'^-,\ a^+$$

lie in $\partial_\infty\Gamma$ in that cyclic order. This proves the lemma. $\square$

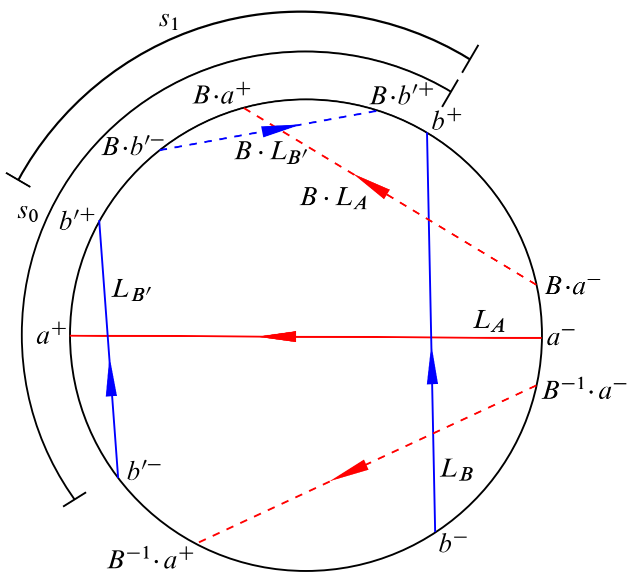

Figure 2: The cyclic order of the attracting and repelling fixed points of A, B, B' and BAB^{-1} along $\partial_\infty \Gamma$ in Lemma 2.2

2.2 Proof in the $\mathrm{PSL}(3,\mathbb{R})$ case

In order to demonstrate the main ideas of the proof without involving too many technicalities, we will first prove Theorem 1.1(1) in the special case when $n = 3$, ie $\rho\colon \Gamma \to \mathrm{PSL}(3,\mathbb{R}) = \mathrm{SL}(3,\mathbb{R})$ is a Hitchin representation.

By Choi and Goldman [7], we know that in this case, ρ is the holonomy of a convex $\mathbb{RP}^2$ structure on S. In other words, there is a strictly convex domain Ω_ρ in $\mathbb{RP}^2$ which is preserved by the Γ–action on $\mathbb{RP}^2$ induced by ρ, and on which the Γ–action is properly discontinuous and cocompact. Moreover, $\rho(X)$ is diagonalizable with positive pairwise distinct eigenvalues for any nonidentity element $X \in \Gamma$ (see Goldman [11, Theorem 3.2]), so $\rho(X)$ has an attracting and repelling fixed point in $\partial\Omega_\rho$. Since the Hilbert metric in Ω_ρ is invariant under projective transformations and the geodesics of the Hilbert metric are lines, one can use the Švarc–Milnor lemma [3, Proposition 8.19] to construct a continuous map

$$\xi^{(1)}\colon \partial_\infty \Gamma \to \partial\Omega_\rho$$

which identifies the attracting fixed point of any $X \in \Gamma \backslash \{\mathrm{id}\}$ to the attracting fixed point of $\rho(X)$.

Pick any four projective lines in $\mathbb{RP}^2$ that intersect at a common point, such that no three of the four agree. There is a classical projective invariant of these four projective

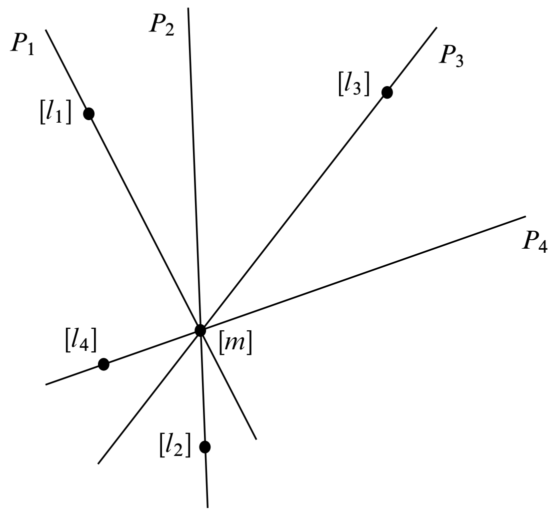

Figure 3: A choice of vectors l_i to compute the cross ratio (P_1, P_2, P_3, P_4)

lines, called the *cross ratio*, which can be defined as follows. Let the four projective lines be P_1, P_2, P_3, P_4 and let m be a vector in $\mathbb{R}^3$ such that $[m]$, the projective point corresponding to the $\mathbb{R}$–span of m, is the common point of intersection of the P_i. For each i, choose a vector $l_i \in \mathbb{R}^3$ so that $[l_i] \neq [m]$ and $[l_i]$ lies in P_i; see Figure 3. By choosing a linear identification

$$f\colon \bigwedge^3 \mathbb{R}^3 \to \mathbb{R},$$

we can evaluate the expression

$$(P_1, P_2, P_3, P_4) := \frac{m \wedge l_1 \wedge l_3}{m \wedge l_1 \wedge l_2} \cdot \frac{m \wedge l_4 \wedge l_2}{m \wedge l_4 \wedge l_3}$$

as an extended real number. One can then verify that the cross ratio (P_1, P_2, P_3, P_4) does not depend on the choice of m, l_1, l_2, l_3, l_4 or the choice of identification f.

This definition of the cross ratio agrees with the classical notion of the cross ratio of four points on a line in the following way. By taking the dual, the four lines $P_1, \dots, P_4$ become four points $p_1, \dots, p_4 \in (\mathbb{RP}^2)^*$, and they lie in the projective line in $(\mathbb{RP}^2)^*$ that is dual to the point $[m]$ in $\mathbb{RP}^2$. One can then check that (P_1, P_2, P_3, P_4) is exactly the cross ratio of the four collinear points $p_1, \dots, p_4$.

Proof of Theorem 1.1(1) when $n = 3$ Observe that

$$a^+,\ A \cdot b^+,\ b^+,\ a^-,\ b^-,\ A \cdot b^-$$

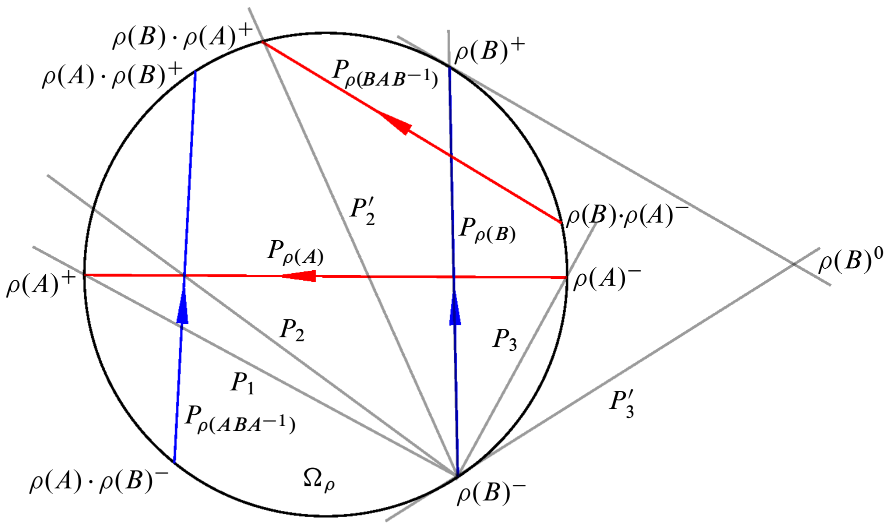

Figure 4: A schematic for the comparison between the cross ratios $(P_1, P_2, P_{\rho(B)}, P_3)$ and $(P_1, P'_2, P_{\rho(B)}, P'_3)$

lie in $\partial_\infty\Gamma$ in that cyclic order. By Lemma 2.2, we see that

$$a^+,\ A\cdot b^+,\ B\cdot a^+,\ b^+,\ a^-,\ b^-,\ B^{-1}\cdot a^+,\ A\cdot b^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order, because $A\cdot b^+$ and $A\cdot b^-$ are the attracting and repelling fixed points of ABA^{-1}, respectively.

Choose any $\rho\in\mathrm{Hit}_3(S)$. For any nonidentity element $X\in\Gamma$, let $\rho(X)^+$, $\rho(X)^0$ and $\rho(X)^-$ be the three fixed points for $\rho(X)$, where $\rho(X)^+$ is attracting and $\rho(X)^-$ is repelling. Denote by $P_{\rho(X)}$ the line segment in Ω_ρ with endpoints $\rho(X)^+$ and $\rho(X)^-$.

Now, let

- P_1 be the line through $\rho(B)^-$ and $\rho(A)^+$,
- P_2 be the line through $\rho(B)^-$ and $P_{\rho(A)}\cap P_{\rho(ABA^{-1})}$,
- P_3 be the line through $\rho(B)^-$ and $\rho(A)^-$,
- P'_2 be the line through $\rho(B)^-$ and $\rho(B)\cdot\rho(A)^+$,
- P'_3 be the line through $\rho(B)^-$ and $\rho(B)^0$.

By using $\xi^{(1)}$ to identify $\partial_\infty\Gamma$ with $\partial\Omega_\rho$, we have that

$$\rho(A)^+,\ \rho(A)\cdot\rho(B)^+,\ \rho(B)\cdot\rho(A)^+,\ \rho(B)^+,\ \rho(A)^-,\ \rho(B)^-$$

lie in $\partial\Omega_\rho$ in that cyclic order; see Figure 4. It is a classically known property of the cross ratio (see Proposition 2.10) that

$$(P_1, P_2, P_{\rho(B)}, P_3) > (P_1, P_2', P_{\rho(B)}, P_3').$$

It is an easy cross ratio computation (see Lemmas 2.8 and 2.9) that

$$(P_1, P_2', P_{\rho(B)}, P_3') = \frac{\beta_1}{\beta_1 - \beta_2} \quad \text{and} \quad (P_1, P_2, P_{\rho(B)}, P_3) = \frac{\alpha_1}{\alpha_3}.$$

Hence, we have

$$\frac{\alpha_1}{\alpha_3} > \frac{\beta_1}{\beta_1 - \beta_2}.$$

Similarly, by reversing the roles of $\rho(B)^-$ and $\rho(B)^+$, and using $\rho(B)^{-1}$ in place of $\rho(B)$, we can also show that

$$\frac{\alpha_1}{\alpha_3} > \frac{\beta_2}{\beta_2 - \beta_3}.$$

This proves Theorem 1.1(1) in the case when $n = 3$. □

2.3 Properties of Frenet curves of Hitchin representations

Next, we want to generalize the proof given in Section 2.2 to any Hitchin representation. We will devote this section to developing the tools needed to do so. In the rest of the paper, we use the same notation for points in $\mathbb{RP}^{n-1}$ and for lines in $\mathbb{R}^n$. It should be clear to which we are referring from the context.

Denote by $\mathcal{F}(\mathbb{R}^n)$ the space of complete flags in $\mathbb{R}^n$. Labourie [15] and Guichard [12] gave a beautiful characterization of representations in $\mathrm{Hit}_n(S)$ as representations that admit an equivariant Frenet curve $\partial_\infty\Gamma \to \mathcal{F}(\mathbb{R}^n)$. When $n = 3$, the Frenet curve, postcomposed with the projection from $\mathcal{F}(\mathbb{R}^3)$ to $\mathbb{RP}^2$, is exactly the map $\xi^{(1)}\colon \partial_\infty\Gamma \to \partial\Omega_\rho$ described in Section 2.2. This characterization will be the main tool we use to extend our proof in Section 2.2 to the general case.

We will start by first defining the Frenet property.

Notation 2.3 Let $\xi\colon S^1 \to \mathcal{F}(\mathbb{R}^n)$ be a continuous closed curve and denote the Grassmannian of k–dimensional subspaces of $\mathbb{R}^n$ by $\mathrm{Gr}(k,n)$. For any $k = 1, \dots, n-1$ and any point $x \in S^1$, let $\xi(x)^{(k)} := \pi_k(\xi(x))$, where $\pi_k\colon \mathcal{F}(\mathbb{R}^n) \to \mathrm{Gr}(k,n)$ is the obvious projection.

Definition 2.4 A closed curve $\xi\colon S^1 \to \mathcal{F}(\mathbb{R}^n)$ is *Frenet* if for every set of distinct points $x_1, \dots, x_k$ in S^1, for every $x \in S^1$, and for all positive integers $n_1, \dots, n_k$ such that $m := \sum_{i=1}^k n_i \le n$,

$$\dim \sum_{i=1}^{k} \xi(x_i)^{(n_i)} = m \quad \text{and} \quad \lim_{\substack{x_i \to x, \forall i \\ x_i \neq x_j, \forall i \neq j}} \sum_{i=1}^{k} \xi(x_i)^{(n_i)} = \xi(x)^{(m)}.$$

The Frenet property ensures ξ has good continuity properties and is "maximally transverse". Combining the work of Labourie [15, Theorem 1.4] and Guichard [12, théorème 1], one can characterize the representations in the $\mathrm{Hit}_n(S)$ as those that preserve an equivariant Frenet curve.

Theorem 2.5 (Guichard, Labourie) *A representation ρ in*

$$\mathrm{Hom}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))/\,\mathrm{PSL}(n,\mathbb{R})$$

lies in $\mathrm{Hit}_n(S)$ if and only if there exists a ρ–equivariant Frenet curve $\xi\colon \partial_\infty\Gamma \to \mathcal{F}(\mathbb{R}^n)$. If ξ exists, then it is unique.

We will now prove several properties of these Frenet curves that will be needed. These are special cases of more general properties that appear in Section 2 of Zhang [19]. However, for the sake of completeness, we will reproduce the proofs.

Lemma 2.6 *Let a, m_0, b, m_1, m_2 and m_3 be distinct points on $\partial_\infty\Gamma$ in that cyclic order, and let $\rho \in \mathrm{Hit}_n(S)$ with corresponding Frenet curve ξ. Also, let $P := \mathbb{P}\big(\xi(a)^{(1)} + \xi(b)^{(1)}\big)$. Then the following hold:*

(1) *Let k_0, k_1, k_2 and k_3 be nonnegative integers that sum to $n-2$, and let $M := \sum_{i=0}^3 \xi(m_i)^{(k_i)}$. The map*

$$f_M\colon \partial_\infty\Gamma \to P$$

given by

$$f_M\colon x \mapsto \begin{cases} \mathbb{P}\big(\xi(x)^{(1)} + \sum_{i=0}^3 \xi(m_i)^{(k_i)}\big) \cap P & \text{if } x \neq m_j, \\ \mathbb{P}\big(\xi(m_j)^{(k_j+1)} + \sum_{i\neq j} \xi(m_i)^{(k_i)}\big) \cap P & \text{if } x = m_j, \end{cases}$$

is a homeomorphism with $f_M(a) = \xi(a)^{(1)}$ and $f_M(b) = \xi(b)^{(1)}$.

(2) *Let k_0, k_1 and k_2 be nonnegative integers that sum to $n-1$, and let s be the closed subsegment of $\partial_\infty\Gamma$ with endpoints a and b that does not contain m_0.*

Also, let $M := \xi(m_0)^{(k_0)}$*. Then there is some closed subsegment* ω *of* P *with endpoints* $\xi(a)^{(1)}$ *and* $\xi(b)^{(1)}$ *such that the map*

$$g_M\colon s \to \omega$$

given by

$$g_M\colon x \mapsto \begin{cases} \mathbb{P}\big(\xi(x)^{(k_2)} + \xi(m_1)^{(k_1)} + \xi(m_0)^{(k_0)}\big) \cap P & \text{if } x \neq m_1, \\ \mathbb{P}\big(\xi(m_1)^{(k_1+k_2)} + \xi(m_0)^{(k_0)}\big) \cap P & \text{if } x = m_1, \end{cases}$$

is a homeomorphism with $g_M(a) = \xi(a)^{(1)}$ *and* $g_M(b) = \xi(b)^{(1)}$*.*

Proof Before we start the proof, observe that for any nonnegative integers $t_0, \dots, t_4$ such that $\sum_{i=0}^4 t_i = n-1$, the intersection $\mathbb{P}\big(\xi(x)^{(t_4)} + \sum_{i=0}^3 \xi(m_i)^{(t_i)}\big) \cap P$ is a single point; otherwise, $\mathbb{P}\big(\xi(a)^{(1)} + \xi(b)^{(1)}\big) \subset \mathbb{P}\big(\xi(x)^{(t_4)} + \sum_{i=0}^3 \xi(m_i)^{(t_i)}\big)$, which contradicts the Frenet property of ξ.

(1) Since ξ is Frenet, f_M is continuous. Moreover, because the domain and target of f_M are both topologically a circle, it is sufficient to show that f_M is injective. Suppose for contradiction that there exist $x \neq x'$ such that $f_M(x) = f_M(x')$. We will assume that $x, x' \neq m_i$ for all $i = 0, 1, 2, 3$ as the other cases are similar. Then

$$\begin{aligned} \sum_{i=0}^{3} \xi(m_i)^{(k_i)} + \xi(x)^{(1)} &= \sum_{i=0}^{3} \xi(m_i)^{(k_i)} + f_M(x) \\ &= \sum_{i=0}^{3} \xi(m_i)^{(k_i)} + f_M(x') \\ &= \sum_{i=0}^{3} \xi(m_i)^{(k_i)} + \xi(x')^{(1)}, \end{aligned}$$

which is impossible because ξ is Frenet. The fact that $f_M(a) = \xi(a)^{(1)}$ and $f_M(b) = \xi(b)^{(1)}$ is easily verified.

(2) First, observe that g_M viewed as a map from s to P is continuous. Also, for any x in s, we have that $g_M(x) = \xi(a)^{(1)}$ if and only if $x = a$ and $g_M(x) = \xi(b)^{(1)}$ if and only if $x = b$. This proves that the image of g_M is a subsegment ω of P with endpoints $\xi(a)^{(1)}$ and $\xi(b)^{(1)}$.

To finish the proof, we only need to show that g_M is injective. Choose x and x' in the interior of s with $x \neq x'$, and assume without loss of generality that a, x', x and b lie along s in that order. Again, we assume that $x, x' \neq m_1$ as the other cases are similar. For any positive integer $i \leq k_2$, let

$$M_i := \xi(x)^{(i-1)} + \xi(x')^{(k_2-i)} + \xi(m_1)^{(k_1)} + \xi(m_0)^{(k_0)}.$$

By (1), we know that $f_{M_i}(x)$ lies on ω strictly between $f_{M_i}(x')$ and $f_{M_i}(b) = \xi(b)^{(1)}$. Also, observe that $f_{M_i}(x) = f_{M_{i+1}}(x')$. This implies that $f_{M_{k_2}}(x)$ lies on ω strictly between $f_{M_1}(x')$ and $\xi(b)^{(1)}$. In particular, $g_M(x) = f_{M_{k_2}}(x) \neq f_{M_1}(x') = g_M(x')$, so g_M is injective. □

In the proof of the $n = 3$ case given in Section 2.2, the classical cross ratio in $\mathbb{RP}^2$ was the main computational tool used to obtain our estimates. We will now define a generalization of the cross ratio for $\mathbb{RP}^{n-1}$.

Definition 2.7 Let $P_1, \dots, P_4$ be four hyperplanes in $\mathbb{R}^n$ that intersect along a $(n-2)$–dimensional subspace $M = \mathrm{Span}\{m_1, \dots, m_{n-2}\} \subset \mathbb{R}^n$, such that no three of the four P_i agree. For $i = 1, \dots, 4$, let $L_i = [l_i]$ be a line through the origin in P_i that does not lie in M. Define the *cross ratio* by

$$(P_1, P_2, P_3, P_4) := \frac{m_1 \wedge \cdots \wedge m_{n-2} \wedge l_1 \wedge l_3}{m_1 \wedge \cdots \wedge m_{n-2} \wedge l_1 \wedge l_2} \cdot \frac{m_1 \wedge \cdots \wedge m_{n-2} \wedge l_4 \wedge l_2}{m_1 \wedge \cdots \wedge m_{n-2} \wedge l_4 \wedge l_3}.$$

In the above definition, choose an identification between $\bigwedge^n(\mathbb{R}^n)$ and $\mathbb{R}$ to evaluate the fraction on the right as a real number. One can check that this number does not depend on the identification chosen, the choice of basis $\{m_1, \dots, m_{n-2}\}$ for M, the choice of L_i in P_i, or the choice of representatives l_i for L_i. When convenient, we sometimes use the notation

$$(L_1, L_2, L_3, L_4)_M := (P_1, P_2, P_3, P_4).$$

Also, at times, in our notation for the cross ratio, we replace the subspaces L_i, P_i and M of $\mathbb{R}^n$ with their projectivizations. As with the $n = 3$ case, this definition of the cross ratio agrees with the classical cross ratio of four points along a projective line in $(\mathbb{RP}^{n-1})^*$.

The following two lemmas summarize some basic properties of this cross ratio.

Lemma 2.8 *Let $L_1, \dots, L_5$ be pairwise distinct lines in $\mathbb{R}^n$ through 0 and let M and M' be $(n-2)$–dimensional subspaces of $\mathbb{R}^n$ not containing L_i for any $i = 1, \dots, 5$, such that no three of the five $M + L_i$ agree and no three of the five $M' + L_i$ agree.*

(1) $(X \cdot L_1, \dots, X \cdot L_4)_{X \cdot M} = (L_1, \dots, L_4)_M$ *for any* $X \in \mathrm{PSL}(n, \mathbb{R})$.

(2) *If* L_1, L_2, L_3, L_4 *lie in a plane, then* $(L_1, L_2, L_3, L_4)_M = (L_1, L_2, L_3, L_4)_{M'}$.

(3) $(L_1, L_2, L_3, L_4)_M = (L_4, L_3, L_2, L_1)_M$.

(4) $(L_1, L_2, L_3, L_5)_M \cdot (L_1, L_3, L_4, L_5)_M = (L_1, L_2, L_4, L_5)_M$.

(5) $(L_1, L_2, L_3, L_4)_M \cdot (L_1, L_3, L_2, L_4)_M = 1$.

(6) $(L_1, L_2, L_3, L_4)_M = 1 - (L_1, L_2, L_4, L_3)_M$.

Proof (1), (3), (4) and (5) follow immediately from the definition of the cross ratio. To prove (2), observe that there is a projective transformation X that fixes L_1, L_2 and L_3, and maps M to M'. Since L_4 lies in the plane containing L_1, L_2 and L_3, X must also fix L_4. This allows us to use (1) to get (2).

To prove (6), assume that $M + L_1, \dots, M + L_4$ are distinct; the other cases are similar. Choose a basis $e_1, \dots, e_n$ for $\mathbb{R}^n$ so that

$$M = \operatorname{Span}\{e_1, \dots, e_{n-2}\}, \quad L_1 = [e_{n-1}], \quad L_4 = [e_n], \quad L_2 = \left[\sum_{i=1}^{n} e_i\right], \quad L_3 = \left[\sum_{i=1}^{n} \alpha_i e_i\right]$$

for some real numbers $\alpha_1, \dots, \alpha_n$. The assumption that $M + L_1, \dots, M + L_4$ are pairwise distinct implies that α_{n-1} and α_n are nonzero real numbers. One can then easily compute that

$$(L_1, L_2, L_3, L_4)_M = \frac{\alpha_n}{\alpha_{n-1}} \quad \text{and} \quad (L_1, L_2, L_4, L_3)_M = \frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}}. \qquad \square$$

In view of Lemma 2.8(2), we will denote $(L_1, L_2, L_3, L_4)_M$ by (L_1, L_2, L_3, L_4) in the case when L_1, L_2, L_3 and L_4 lie in the same plane.

Lemma 2.9 *Let $X \in \mathrm{PSL}(n, \mathbb{R})$ be diagonalizable with n real eigenvalues $\lambda_1, \dots, \lambda_n$ (these are only well defined up to sign) of pairwise distinct moduli, such that $|\lambda_n| < \dots < |\lambda_1|$. Let L_i and L_j be fixed lines through the origin in $\mathbb{R}^n$ corresponding to the eigenvalues λ_i and λ_j, respectively, with $i < j$, and let L be a line through the origin in the plane $L_i + L_j$ such that $L_i \neq L \neq L_j$. Then*

$$(L_j, L, X \cdot L, L_i) = \frac{\lambda_i}{\lambda_j}.$$

Proof Choose a basis $e_1, \dots, e_n$ for $\mathbb{R}^n$ so that $[e_k]$ is a fixed line through the origin of $\rho(X)$ corresponding to the eigenvalue λ_k. In this basis, $\rho(X)$ is the diagonal matrix $[x_{u,v}]$, where

$$x_{u,v} = \begin{cases} 0 & \text{if } u \neq v, \\ \lambda_u & \text{if } u = v. \end{cases}$$

Let M be the $(n-2)$–dimensional subspace $\operatorname{Span}\{e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n\}$ of $\mathbb{R}^n$. Via a projective transformation that fixes $e_1, \dots, e_n$, we can assume $L = [e_i + e_j]$. The lemma follows from an easy computation using the cross ratio definition. $\square$

The next task is to understand how the cross ratio interacts with Frenet curves.

Proposition 2.10 *Let $\rho \in \mathrm{Hit}_n(S)$, and let ξ be the corresponding Frenet curve. Also, let a, b, c, m_0, d and m_1 be distinct points along $\partial_\infty \Gamma$ in that cyclic order, and let k_0 and k_1 be nonnegative integers that sum to $n-2$. For any $x \in \partial_\infty \Gamma$, define*

$$P_x = \begin{cases} \xi(x)^{(1)} + \xi(m_0)^{(k_0)} + \xi(m_1)^{(k_1)} & \text{if } x \neq m_0, m_1, \\ \xi(m_i)^{(k_i+1)} + \xi(m_{1-i})^{(k_{1-i})} & \text{if } x = m_i. \end{cases}$$

Then the following hold:

(1) $(P_a, P_b, P_{m_0}, P_d) > (P_a, P_b, P_{m_0}, P_{m_1})$.

(2) $(P_a, P_b, P_{m_0}, P_d) > (P_a, P_c, P_{m_0}, P_d)$.

Proof We will only show the proof of (1); the same proof together with Lemma 2.8 gives (2). Let

$$\begin{aligned} L_{m_0} &= P_{m_0} \cap \big(\xi(a)^{(1)} + \xi(b)^{(1)}\big), \\ L_{m_1} &= P_{m_1} \cap \big(\xi(a)^{(1)} + \xi(b)^{(1)}\big), \\ L_d &= P_d \cap \big(\xi(a)^{(1)} + \xi(b)^{(1)}\big). \end{aligned}$$

Choose vectors $l_{m_0}, l_{m_1}, l_a, l_b, l_d \in \mathbb{R}^n$ such that

$$[l_{m_0}] = L_{m_0}, \quad [l_{m_1}] = L_{m_1}, \quad [l_a] = \xi(a)^{(1)}, \quad [l_b] = \xi(b)^{(1)}, \quad [l_d] = L_d.$$

By Lemma 2.6(1), we can ensure, by replacing each l_i with $-l_i$ if necessary, that

$$\begin{aligned} l_{m_0} &= \alpha l_a + (1-\alpha) l_b, \\ l_d &= \beta l_a + (1-\beta) l_b, \\ l_{m_1} &= \gamma l_a + (1-\gamma) l_b \end{aligned}$$

for $0 < \alpha < \beta < \gamma < 1$. Then we can compute

$$\begin{aligned} (P_a, P_b, P_{m_0}, P_d) &= \frac{1-\alpha}{1-\alpha/\beta} \\ &> \frac{1-\alpha}{1-\alpha/\gamma} \\ &= (P_a, P_b, P_{m_0}, P_{m_1}). \end{aligned}$$

□

2.4 Proof in the PSL$(n, \mathbb{R})$ case

We will now use the technical facts established in Section 2.3 to prove Theorem 1.1. For the rest of this section, fix $\rho \in \mathrm{Hit}_n(S)$ and let ξ be its corresponding Frenet curve. The next lemma is the main computation in the proof of Theorem 1.1.

Lemma 2.11 *Let B be a nonidentity element in Γ. Pick $k=0,\dots,n-2$, and for any $x\in\partial_\infty\Gamma$, define*

$$P_x=P_x^{(k)}:=\begin{cases}\xi(x)^{(1)}+\xi(b^+)^{(k)}+\xi(b^-)^{(n-k-2)} & \text{if } x\neq b^+,b^-,\\ \xi(b^+)^{(k+1)}+\xi(b^-)^{(n-k-2)} & \text{if } x=b^+,\\ \xi(b^+)^{(k)}+\xi(b^-)^{(n-k-1)} & \text{if } x=b^-.\end{cases}$$

Suppose that x_1, x_2 and x_3 are points in $\partial_\infty\Gamma$ such that

$$x_1,\ x_2,\ B\cdot x_1,\ b^+,\ x_3,\ b^-$$

lie on $\partial_\infty\Gamma$, in that cyclic order. Then

$$(P_{x_1},P_{x_2},P_{b^+},P_{x_3})>\frac{\beta_{k+1}}{\beta_{k+1}-\beta_{k+2}},$$

where $0<\beta_n<\dots<\beta_1$ are the eigenvalues of $\rho(B)$.

Proof By Proposition 2.10 and parts (5) and (6) of Lemma 2.8, we have

$$\text{(2-1)}\qquad (P_{x_1},P_{x_2},P_{b^+},P_{x_3})>(P_{x_1},P_{B\cdot x_1},P_{b^+},P_{b^-})=\frac{1}{(P_{x_1},P_{b^+},P_{B\cdot x_1},P_{b^-})}=\frac{1}{1-(P_{b^+},P_{x_1},P_{B\cdot x_1},P_{b^-})}.$$

Note that for all $j=1,\dots,n$,

$$L_j:=\xi(b^+)^{(j)}\cap\xi(b^-)^{(n-j+1)}$$

is the fixed line through the origin in $\mathbb{R}^n$ of $\rho(B)$ corresponding to the eigenvalue β_j. Also, observe that P_{b^+} and P_{b^-} intersect the plane $\xi(b^+)^{(k+2)}\cap\xi(b^-)^{(n-k)}$ at L_{k+1} and L_{k+2}, respectively. Let

$$L:=P_{x_1}\cap\bigl(\xi(b^+)^{(k+2)}\cap\xi(b^-)^{(n-k)}\bigr),$$

and it is clear that $P_{B\cdot x_1}\cap\bigl(\xi(b^+)^{(k+2)}\cap\xi(b^-)^{(n-k)}\bigr)=\rho(B)\cdot L$. Thus, we can use Lemma 2.9, to conclude that

$$(P_{b^+},P_{x_1},P_{B\cdot x_1},P_{b^-})=(L_{k+1},L,\rho(B)\cdot L,L_{k+2})=\frac{\beta_{k+2}}{\beta_{k+1}}.$$

Combining this with inequality (2-1) proves the lemma. □

Applying Lemma 2.11 to our setting, we can now prove Theorem 1.1.

Proof of Theorem 1.1 (1) Let ω be the subsegment of $\mathbb{P}\big(\xi(a^+)^{(1)}+\xi(a^-)^{(1)}\big)$ with endpoints $\xi(a^+)^{(1)}$ and $\xi(a^-)^{(1)}$ that has nonempty intersection with $\mathbb{P}\big(\xi(b^+)^{(k+1)}+\xi(b^-)^{(n-k-2)}\big)$. Define

$$p := \mathbb{P}\big(\xi(b^+)^{(k+1)}+\xi(b^-)^{(n-k-2)}\big)\cap\omega,$$
$$q := \mathbb{P}\big(\xi(A\cdot b^+)^{(1)}+\xi(b^+)^{(k)}+\xi(b^-)^{(n-k-2)}\big)\cap\omega,$$

and note that q exists by Lemma 2.6(1). Also, observe that

$$\rho(A)\cdot p = \mathbb{P}\big(\xi(A\cdot b^+)^{(k+1)}+\xi(A\cdot b^-)^{(n-k-2)}\big)\cap\omega,$$

so Lemma 2.6(2) implies that $\rho(A)\cdot p$ lies between $\xi(a^+)^{(1)}$ and q in ω. Lemma 2.8, Lemma 2.9 and Proposition 2.10 together then allow us to conclude that

$$\begin{aligned}\frac{\alpha_1}{\alpha_n} &= \big(\xi(a^+)^{(1)},\rho(A)\cdot p, p, \xi(a^-)^{(1)}\big)\\ &> \big(\xi(a^+)^{(1)}, q, p, \xi(a^-)^{(1)}\big)\\ &= (P_{a^+}, P_{A\cdot b^+}, P_{b^+}, P_{a^-}),\end{aligned}$$

where

$$P_x = P_x^{(k)} := \begin{cases}\xi(x)^{(1)}+\xi(b^+)^{(k)}+\xi(b^-)^{(n-k-2)} & \text{if } x\neq b^+, b^-,\\ \xi(b^+)^{(k+1)}+\xi(b^-)^{(n-k-2)} & \text{if } x=b^+,\\ \xi(b^+)^{(k)}+\xi(b^-)^{(n-k-1)} & \text{if } x=b^-.\end{cases}$$

By Lemma 2.2, we know that a^+, $A\cdot b^+$, $B\cdot a^+$, b^+, a^-, b^- lie along $\partial_\infty\Gamma$ in that cyclic order. This allows us to apply Lemma 2.11 with x_1, x_2 and x_3 as a^+, $A\cdot b^+$ and a^-, respectively, to obtain the desired inequality.

(2) Let r^- and r^+ be the closed subsegments of $\partial_\infty\Gamma$ with endpoints a^- and a^+ such that b^- lies in r^-, while b^+ lies in r^+. Orient both r^- and r^+ from a^- to a^+. Define $\mathcal{B}$ to be the set of unordered pairs $\{b'^+, b'^-\}$ in the Γ–orbit of $\{b^+, b^-\}$ such that b'^+ lies in r^+ between b^+ and $A\cdot b^+$, while b'^- lies in r^- between b^- and $A\cdot b^-$.

Every pair in $\mathcal{B}$ is the set of attracting and repelling fixed points for some B' in Γ that is conjugate to B. Since γ is simple, for every $\{b'^+, b'^-\}$ and $\{b''^+, b''^-\}$ in $\mathcal{B}$, we know that b'^+ precedes b''^+ (in the orientation on r^+) if and only if b'^- precedes b''^- (in the orientation of r^-). The orientations on r^- and r^+ thus induce an ordering on $\mathcal{B}$. Also, observe that $|\mathcal{B}| = i(\eta,\gamma)+1$, so we can label the pairs in $\mathcal{B}$ according to the order; ie

$$\mathcal{B} = \big\{\{b_1^+, b_1^-\},\dots,\{b_{m+1}^+, b_{m+1}^-\}\big\},$$

where $b_1^+ = b^+$, $b_1^- = b^-$, $b_{m+1}^+ = A\cdot b^+$, $b_{m+1}^- = A\cdot b^-$ and $m = i(\eta,\gamma)$.

For each i, let B_i be the element in Γ that is conjugate to either B or B^{-1} such that its attracting and repelling fixed points are b_i^+ and b_i^-, respectively. By Lemma 2.2, a^+, b_{i+1}^+, $B_i\cdot a^+$, b_i^+, a^-, b_i^- lie along $\partial_\infty\Gamma$ in that cyclic order, so we can apply Lemma 2.11 with x_1, x_2 and x_3 as a^+, b_{i+1}^+ and a^-, respectively, to conclude that

$$(P_{a^+,i}, P_{b_{i+1}^+,i}, P_{b_i^+,i}, P_{a^-,i}) > \frac{\beta_{k+1}}{\beta_{k+1}-\beta_{k+2}} \tag{2-2}$$

if B_i is conjugate to B, and

$$(P_{a^+,i}, P_{b_{i+1}^+,i}, P_{b_i^+,i}, P_{a^-,i}) > \frac{\beta_{n-k-1}}{\beta_{n-k-1}-\beta_{n-k}} \tag{2-3}$$

if B_i is conjugate to B^{-1}, where

$$P_{x,i} = P_{x,i}^{(k)} := \begin{cases} \xi(x)^{(1)} + \xi(b_i^+)^{(k)} + \xi(b_i^-)^{(n-k-2)} & \text{if } x \neq b_i^+, b_i^-, \\ \xi(b_i^+)^{(k+1)} + \xi(b_i^-)^{(n-k-2)} & \text{if } x = b_i^+, \\ \xi(b_i^+)^{(k)} + \xi(b_i^-)^{(n-k-1)} & \text{if } x = b_i^-. \end{cases}$$

Fix any $k=0,\dots,n-2$, and let ω be the subsegment of $\mathbb{P}\big(\xi(a^+)^{(1)}+\xi(a^-)^{(1)}\big)$ with endpoints $\xi(a^+)^{(1)}$, $\xi(a^-)^{(1)}$ whose intersection with $\mathbb{P}\big(\xi(b_i^+)^{(k+1)}+\xi(b_i^-)^{(n-k-2)}\big)$ is nonempty. For $i=1,\dots,m+1$, define

$$p_i := \mathbb{P}\big(\xi(b_i^+)^{(k+1)} + \xi(b_i^-)^{(n-k-2)}\big) \cap \omega,$$

and for $i=1,\dots,m$, define

$$q_i := \mathbb{P}\big(\xi(b_{i+1}^+)^{(1)} + \xi(b_i^+)^{(k)} + \xi(b_i^-)^{(n-k-2)}\big) \cap \omega.$$

Observe that Lemma 2.6(2) implies that p_i and q_i are well defined, and that $\xi(a^-)^{(1)}$, p_1, q_1, p_2, q_2, $\dots$, p_m, q_m, p_{m+1}, $\xi(a^+)^{(1)}$ lie in ω in that order. Hence, by similar arguments as those used in the proof of (1), we have

$$\begin{aligned}\big(\xi(a^+)^{(1)}, p_{i+1}, p_i, \xi(a^-)^{(1)}\big) &> \big(\xi(a^+)^{(1)}, q_i, p_i, \xi(a^-)^{(1)}\big) \\ &= (P_{a^+,i}, P_{b_{i+1}^+,i}, P_{b_i^+,i}, P_{a^-,i}).\end{aligned}$$

We can then use Lemmas 2.9 and 2.8 to obtain

$$\begin{aligned}\frac{\alpha_1}{\alpha_n} &= \big(\xi(a^+)^{(1)}, p_{m+1}, p_1, \xi(a^-)^{(1)}\big) \\ &= \prod_{i=1}^{m} \big(\xi(a^+)^{(1)}, p_{i+1}, p_i, \xi(a^-)^{(1)}\big) \\ &> \prod_{i=1}^{m} (P_{a^+,i}, P_{b_{i+1}^+,i}, P_{b_i^+,i}, P_{a^-,i}).\end{aligned} \tag{2-4}$$

Let $\mathcal{B}_+ := \{i : B_i \text{ is conjugate to } B\}$ and $\mathcal{B}_- := \{i : B_i \text{ is conjugate to } B^{-1}\}$, and let $u := |\mathcal{B}_+|$. Then combining the inequalities (2-2), (2-3) and (2-4) yields

$$\begin{aligned}\frac{\alpha_1}{\alpha_n} &> \prod_{i\in\mathcal{B}_+}(P_{a^+,i}, P_{b^+_{i+1},i}, P_{b^+_i,i}, P_{a^-,i}) \cdot \prod_{i\in\mathcal{B}_-}(P_{a^+,i}, P_{b^+_{i+1},i}, P_{b^+_i,i}, P_{a^-,i})\\ &> \left(\frac{\beta_{k+1}}{\beta_{k+1}-\beta_{k+2}}\right)^u \cdot \left(\frac{\beta_{n-k-1}}{\beta_{n-k-1}-\beta_{n-k}}\right)^{i(\eta,\gamma)-u}. \qquad \square\end{aligned}$$

3 Further remarks

In this section, we prove some corollaries of Theorem 1.1, show that it does not hold for quasi-Fuchsian representations, and perform a comparison with the classical collar lemma.

3.1 Corollaries

Theorem 1.1 has some interesting consequences. The first is an analog of the classical collar lemma for Hitchin representations.

Corollary 1.2 *Let S be a surface of genus $g \geq 2$, and let η and γ be two essential closed curves in S. Denote the geometric intersection number between η and γ by $i(\eta,\gamma)$. Then for any $n \geq 2$ and any $\rho \in \mathrm{Hit}_n(S)$, the following hold:*

(1) *If $i(\eta,\gamma) \neq 0$, then*

$$\frac{1}{\exp(l_\rho(\eta))} < 1 - \frac{1}{\exp(l_\rho(\gamma)/(n-1))}.$$

(2) *If $i(\eta,\gamma) \neq 0$ and γ is simple, then there are nonnegative integers u and v with $u \geq v$ and $u+v = i(\eta,\gamma)$ such that*

$$\frac{1}{\exp(l_\rho(\eta))} < \left(1 - \frac{1}{\exp(l_\rho(\gamma)/(n-1))}\right)^u \left(1 - \frac{1}{\exp(l_\rho(\gamma))}\right)^v.$$

(3) *Let $\delta_n > 0$ be the unique real solution to the equation $e^{-x} + e^{-x/(n-1)} = 1$. If η is a nonsimple closed curve, then*

$$l_\rho(\eta) > \delta_n.$$

Proof In this proof, we will use the same notation as we used in Theorem 1.1.

(1) Choose orientations on η and γ. The hypothesis on η and γ imply that there are group elements A and B in Γ corresponding to η and γ, respectively, such that

$$a^+,\ b^+,\ a^-,\ b^-$$

lie along $\partial_\infty \Gamma$ in that cyclic order. Let $0 < \alpha_n < \cdots < \alpha_1$ and $0 < \beta_n < \cdots < \beta_1$ be the eigenvalues of $\rho(A)$ and $\rho(B)$, respectively. By Theorem 1.1(1), we know that for all $k = 0, \dots, n-2$,

$$\frac{\alpha_1}{\alpha_n} > \frac{\beta_{k+1}}{\beta_{k+1} - \beta_{k+2}},$$

which implies that

$$\frac{\beta_{k+2}}{\beta_{k+1}} < 1 - \frac{\alpha_n}{\alpha_1}.$$

Taking the product over all $k = 0, \dots, n-2$, we get

$$\frac{\alpha_n}{\alpha_1} + \left(\frac{\beta_n}{\beta_1}\right)^{1/(n-1)} < 1.$$

Since $l_\rho(\eta) = \log(\alpha_1/\alpha_n)$ and $l_\rho(\gamma) = \log(\beta_1/\beta_n)$, the above inequality gives us (1).

(2) By Theorem 1.1(2), we know that there is some nonnegative integer $u \le i(\eta, \gamma)$ such that for any $k = 0, \dots, n-2$, we have

$$\frac{\alpha_n}{\alpha_1} < \left(1 - \frac{\beta_{k+2}}{\beta_{k+1}}\right)^u \left(1 - \frac{\beta_{n-k}}{\beta_{n-k-1}}\right)^{i(\eta,\gamma)-u}.$$

In particular, we also have that for any $k = 0, \dots, n-2$,

$$\frac{\alpha_n}{\alpha_1} < \left(1 - \frac{\beta_{k+2}}{\beta_{k+1}}\right)^{i(\eta,\gamma)-u} \left(1 - \frac{\beta_{n-k}}{\beta_{n-k-1}}\right)^{u},$$

so we can assume that

$$\frac{\alpha_n}{\alpha_1} < \left(1 - \frac{\beta_{k+2}}{\beta_{k+1}}\right)^u \left(1 - \frac{\beta_{n-k}}{\beta_{n-k-1}}\right)^{v}$$

for some nonnegative integers u and v such that $u \ge v$ and $u + v = i(\eta, \gamma)$. This implies that

$$\frac{\alpha_n}{\alpha_1} < \left(1 - \frac{\beta_{k+2}}{\beta_{k+1}}\right)^u \left(1 - \frac{\beta_n}{\beta_1}\right)^{v},$$

which we can rewrite as

$$\frac{\beta_{k+2}}{\beta_{k+1}} < 1 - \frac{(\alpha_n/\alpha_1)^{1/u}}{(1 - \beta_n/\beta_1)^{v/u}}.$$

By taking the product of the above inequality over $k = 0, \dots, n-2$, we have

$$\left(\frac{\beta_n}{\beta_1}\right)^{1/(n-1)} < 1 - \frac{(\alpha_n/\alpha_1)^{1/u}}{(1 - \beta_n/\beta_1)^{v/u}},$$

which can be rewritten as

$$\frac{\alpha_n}{\alpha_1} < \left(1-\left(\frac{\beta_n}{\beta_1}\right)^{1/(n-1)}\right)^u \left(1-\frac{\beta_n}{\beta_1}\right)^v,$$

from which (2) follows.

(3) Choose an orientation on η. Since η is nonsimple, there are group elements A and B corresponding to η such that

$$a^+,\ b^+,\ a^-,\ b^-$$

lie along $\partial_\infty \Gamma$ in that cyclic order. Let $0 < \alpha_n < \cdots < \alpha_1$ and $0 < \beta_n < \cdots < \beta_1$ be the eigenvalues of $\rho(A)$ and $\rho(B)$, respectively. Note that $\rho(B)$ is either conjugate to $\rho(A)$ or $\rho(A)^{-1}$, so $\beta_n/\beta_1 = \alpha_n/\alpha_1$. Hence, the same computation as the proof of (1) then yields the inequality

$$\frac{\alpha_n}{\alpha_1} + \left(\frac{\alpha_n}{\alpha_1}\right)^{1/(n-1)} < 1,$$

which is equivalent to

$$\left(1-\frac{\alpha_n}{\alpha_1}\right)^{n-1} - \frac{\alpha_n}{\alpha_1} > 0. \tag{3-1}$$

Consider the polynomial $P_n(x) = x^{n-1} + x - 1$. Note that for $n \geq 2$, we have that $P_n(x)$ is strictly increasing on the interval $[0,1]$, $P_n(0) = -1$ and $P_n(1) = 1$. Hence, P_n has a unique zero in the interval $(0,1)$, which we denote by x_n. It then follows that

$$\{x \in [0,1] : P_n(x) > 0\} = (x_n, 1].$$

Also, observe that

$$P_n\left(1-\frac{\alpha_n}{\alpha_1}\right) = \left(1-\frac{\alpha_n}{\alpha_1}\right)^{n-1} - \frac{\alpha_n}{\alpha_1}$$

and $0 < 1-\alpha_n/\alpha_1 < 1$. Since α_n/α_1 satisfies the inequality (3-1), we have

$$x_n < 1 - \frac{\alpha_n}{\alpha_1} < 1.$$

This implies that

$$l_\rho(\eta) = \log\left(\frac{\alpha_1}{\alpha_n}\right) > \delta_n := -\log(1-x_n). \qquad \square$$

An easy consequence of Corollary 1.2 is the following.

Corollary 3.1 *For any $n \geq 2$ and any $\rho \in \mathrm{Hit}_n(S)$, there are at most $3g-3$ closed curves in S of ρ–length at most δ_n.*

In the case of $\mathcal{T}(S)$, one can replace the number $\delta_2 = \log(2)$ with $4 \cdot \sinh^{-1}(1)$; see Buser [5, Theorem 4.2.2].

Proof By Corollary 1.2(1), if η and γ are closed curves in S such that $i(\eta, \gamma) \neq 0$, then $l_\rho(\eta)$ and $l_\rho(\gamma)$ cannot both be smaller than δ_n. Moreover, Corollary 1.2(3) tells us that any closed curve of ρ–length less than δ_n has to be simple. Thus, the set of closed curves of ρ–length less than δ_n has to be a pairwise disjoint collection of simple closed curves, so the size of this collection is at most $3g-3$. □

Let $\widetilde{M}$ be the $\mathrm{PSL}(n,\mathbb{R})$ symmetric space, and let $d_{\widetilde{M}}$ be the distance function given by the Riemannian metric on $\widetilde{M}$. It is well known that for any $Z \in \Gamma$, the translation length of $\rho(Z)$, namely $\inf\{d_{\widetilde{M}}(o, \rho(Z)\cdot o) : o \in \widetilde{M}\}$, is

$$c_n \sqrt{\sum_{i=1}^{n} (\log \lambda_i)^2}$$

for some positive constant c_n depending only on n. Here, $0 < \lambda_n < \cdots < \lambda_1$ are the eigenvalues of $\rho(Z)$. (See Chapter II.10 of Bridson and Haefliger [3].) For our purposes, we normalize the metric on $\widetilde{M}$ so that $c_n = \sqrt{2}$, ie so that the image of the totally geodesic embedding of $\mathbb{H}^2$ in $\widetilde{M}$ induced by ι_n: $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ has sectional curvature $-6/(n(n-1)(n+1))$. Then for any discrete, faithful representation ρ: $\Gamma \to \mathrm{PSL}(n,\mathbb{R})$, and for any rectifiable closed curve ω in $M := \rho(\Gamma)\backslash\widetilde{M}$, let $l_M(\omega)$ be the length of ω in the Riemannian metric on M.

In the case when $\rho \in \mathrm{Hit}_n(S)$, we can use Corollary 1.2, to obtain a relationship between the lengths of curves in the quotient locally symmetric space M.

Corollary 1.4 *Let η and γ be two essential closed curves in S and let X and Y be elements in Γ corresponding to η and γ, respectively. For any $\rho \in \mathrm{Hit}_n(S)$, let η' and γ' be two closed curves in $\rho(\Gamma)\backslash\widetilde{M} =: M$ that correspond to $X, Y \in \Gamma = \pi_1(M)$, respectively. Then the statements in Corollary 1.2 hold, with $l_\rho(\eta)$ and $l_\rho(\gamma)$ replaced by $l_M(\eta')$ and $l_M(\gamma')$, respectively.*

Proof Pick any $Z \in \Gamma \setminus \{\mathrm{id}\}$, and let ω in S and ω' in M be closed curves corresponding to Z. Observe then that the translation length of $\rho(Z)$ in $\widetilde{M}$ is a lower bound for $l_M(\omega')$.

Also, since

$$2\sum_{i=1}^{n} x_i^2 - (x_1 - x_n)^2 = (x_1 + x_n)^2 + 2(x_2^2 + \cdots + x_{n-1}^2) \geq 0,$$

we have that

$$(x_1 - x_n)^2 \leq 2 \sum_{i=1}^{n} x_i^2.$$

This allows us to compute

$$l_\rho(\omega) = \log\Big(\frac{\lambda_1}{\lambda_n}\Big) \leq \sqrt{2 \sum_{i=1}^{n} (\log \lambda_i)^2} \leq l_M(\omega'),$$

where $0 < \lambda_n < \cdots < \lambda_1$ are the eigenvalues of $\rho(Z)$. □

Let $f\colon S \to M := \rho(\Gamma)\backslash\widetilde{M}$ be a π_1–injective map such that $f(\eta)$ and $f(\gamma)$ are rectifiable curves in the Riemannian metric on M. It then follows from Corollary 1.4 that the statements in Corollary 1.2 hold, with $l_\rho(\eta)$ and $l_\rho(\gamma)$ replaced with $l_M(f(\eta))$ and $l_M(f(\gamma))$, respectively. In particular, we have a collar lemma for the image of the harmonic maps corresponding to Hitchin representations that were given by Corlette [9].

Corollary 1.2 also allow us to deduce consequences that are similar to Corollary 1.4, but with the Hilbert metric on the symmetric space instead of the Riemannian one. The symmetric space $\widetilde{M}$ can be given a Hilbert metric in the following way. Let $S(n,\mathbb{R})$ be the space of symmetric $n \times n$ matrices with real entries and let $P(n,\mathbb{R})$ be the set of positive-definite matrices in $S(n,\mathbb{R})$. Let $\mathbb{P}(P)$ and $\mathbb{P}(S)$ be the projectivizations of $P(n,\mathbb{R})$ and $S(n,\mathbb{R})$, respectively, and observe that $\mathbb{P}(P)$ is a properly convex domain in $\mathbb{P}(S) \simeq \mathbb{RP}^{N-1}$, where $N = \frac{1}{2}(n(n+1))$. This allows us to equip $\mathbb{P}(P)$ with a Hilbert metric.

Moreover, we can define a $\mathrm{PSL}(n,\mathbb{R})$–action on $\mathbb{P}(S)$ by $g \cdot A := gAg^T$ for any $g \in \mathrm{PSL}(n,\mathbb{R})$ and any $A \in \mathbb{P}(S)$. Note that this action preserves the projective structure on $\mathbb{P}(S)$, and also preserves $\mathbb{P}(P)$. In fact, $\mathrm{PSL}(n,\mathbb{R})$ acts transitively on $\mathbb{P}(P)$, and the stabilizer of the projective class of the identity matrix in $\mathbb{P}(P)$ is $\mathrm{PSO}(n)$, so the symmetric space $\widetilde{M}$ can be identified with $\mathbb{P}(P)$. This equips $\widetilde{M}$ with a Hilbert metric. Denote $\widetilde{M}$ equipped with the Hilbert metric by $\widetilde{M}'$, and for any discrete, faithful representation $\rho\colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$, let $l_{M'}$ be the length function on $M' := \rho(\Gamma)\backslash\widetilde{M}'$ induced by the Hilbert metric. Corollary 1.2 then also implies the following corollary.

Corollary 3.2 *Let η and γ be two essential closed curves in S and let X and Y be elements in Γ corresponding to η and γ, respectively. For any $\rho \in \mathrm{Hit}_n(S)$, let η' and γ' be two closed curves in M' that correspond to $X, Y \in \Gamma = \pi_1(M')$, respectively. Then the statements in Corollary 1.2 hold, with $l_\rho(\eta)$ and $l_\rho(\gamma)$ replaced with $\frac{1}{2}l_{M'}(\eta')$ and $\frac{1}{2}l_{M'}(\gamma')$, respectively.*

Proof For any $Z \in \Gamma \backslash \{\mathrm{id}\}$, let $0 < \lambda_n < \cdots < \lambda_1$ be the eigenvalues of $\rho(Z)$. We can assume without loss of generality that $\rho(Z)$ is a diagonal. Let E_{ij} be the $n \times n$ matrix with 1 at position (i, j) and zero everywhere else, and let $B_{ij} = E_{ij} + E_{ji}$. Obviously, $\{B_{ij}\}_{i \le j}$ is a basis of $S(n, \mathbb{R}) = \mathbb{R}^N$, and it is easy to verify that $\rho(Z) \cdot B_{ij} = \lambda_i \lambda_j B_{ij}$. That means B_{ij} is an eigenvector of $\rho(Z)$ with eigenvalue $\lambda_i \lambda_j$. Consequently, the translation length of $\rho(Z)$ is

$$\log\left(\frac{\lambda_1^2}{\lambda_n^2}\right) = 2\log\left(\frac{\lambda_1}{\lambda_n}\right);$$

see Cooper, Long and Tillmann [8, Proposition 2.1]. The corollary follows easily. □

As mentioned in the introduction, if we restrict to Hitchin representations that lie in $\mathrm{Hit}_n(S)' \subset \mathrm{Hit}_n(S)$, then we can strengthen Theorem 1.1(2).

Corollary 1.3 *Let A and B be elements in Γ such that a^+, b^+, a^-, b^- lie in $\partial_\infty \Gamma$ in that cyclic order. Let $\rho \in \mathrm{Hit}_n(S)'$ and let $\alpha_n < \cdots < \alpha_1$ and $\beta_n < \cdots < \beta_1$ be the moduli of the eigenvalues of $\rho(A)$ and $\rho(B)$, respectively. Finally, let η and γ be closed curves on S corresponding to A and B, respectively. If γ is a simple closed curve in S, then for every $k = 0, \dots, n-2$,*

$$\alpha_1^2 > \left(\frac{\beta_{k+1}}{\beta_{k+1} - \beta_{k+2}}\right)^{i(\eta,\gamma)}.$$

Proof Since $\rho(B)$ is a diagonalizable element in $\mathrm{PSO}(k, k+1)$ or $\mathrm{PSp}(2k, \mathbb{R})$, we see that $\beta_{k+1} = 1/\beta_{n-k}$ for $k = 0, \dots, n-1$, and $\alpha_n = 1/\alpha_1$. Apply this to Theorem 1.1(2). □

From this corollary, the same proof as Corollary 1.2(2) allows us to obtain the following stronger inequality in the case when $\rho \in \mathrm{Hit}_n(S)'$.

Corollary 3.3 *Let η and γ be two essential closed curves in S such that γ is simple and $i(\eta, \gamma) \neq 0$. Then for any $\rho \in \mathrm{Hit}_n(S)'$,*

$$\frac{1}{\exp(l_\rho(\eta))} < \left(1 - \frac{1}{\exp(l_\rho(\gamma)/(n-1))}\right)^{i(\eta,\gamma)}.$$

Our results can be generalized to surfaces with boundaries in the following way. Let S' be a connected, oriented, topological surface with boundary, such that the double of S' is S, a closed connected, oriented topological surface of genus $g \ge 2$. Let Γ' be the fundamental group of S', and note that by choosing appropriate basepoints in the universal covers of S and S', the inclusion $S' \subset S$ induces an inclusion $\Gamma' \subset \Gamma$, which in turn induces an inclusion $\partial\Gamma' \subset \partial\Gamma$. In particular, $\partial\Gamma'$ inherits a natural cyclic order from $\partial\Gamma$.

The inclusion $\Gamma' \subset \Gamma$ also allows us to define the restriction map

$$\text{res: } \text{Hit}_n(S) \to \mathcal{X}_n(S') := \text{Hom}(\Gamma', \text{PSL}(n,\mathbb{R}))/\,\text{PSL}(n,\mathbb{R})$$

by res: $[\rho] \mapsto [\rho|_{\Gamma'}]$. Using this, define the n^{th} *Hitchin component of* S' to be

$$\text{Hit}_n(S') := \text{res}(\text{Hit}_n(S)).$$

(See the introduction of Labourie and McShane [16] for an alternative definition.) While $\text{Hit}_n(S')$ is still topologically a cell, it is no longer a connected component of $\mathcal{X}_n(S')$, so it is properly contained in its closure $\overline{\text{Hit}_n(S')}$ in $\mathcal{X}_n(S')$.

Corollary 3.4 *Theorem 1.1 holds with* Γ *replaced with* Γ', $\text{Hit}_n(S)$ *replaced with* $\overline{\text{Hit}_n(S')}$, *and the strict inequalities* $>$ *replaced with weak inequalities* $\geq$.

Proof For any closed curve γ in S, let $X \in \Gamma$ be a corresponding group element. First, note that the moduli of the eigenvalues of $\rho(X)$ and $\text{res}(\rho)(X)$ agree, so Theorem 1.1 clearly holds for $\rho \in \text{Hit}_n(S')$.

Since $\alpha_n < \cdots < \alpha_1$ and $\beta_n < \cdots < \beta_1$ on $\text{Hit}_n(S')$, these moduli of eigenvalues are still well defined on $\overline{\text{Hit}_n(S')}$, and satisfy the weak inequalities $\alpha_n \leq \cdots \leq \alpha_1$ and $\beta_n \leq \cdots \leq \beta_1$. Furthermore, as functions on $\overline{\text{Hit}_n(S')}$, they are continuous. As such, the inequalities in Theorem 1.1 hold on $\overline{\text{Hit}_n(S')}$, with $>$ replaced with $\geq$. □

3.2 Counterexample for non-Hitchin representations

Note that in our proof, we used very strongly that the representations we consider are in $\text{Hit}_n(S)$ because we used properties of the Frenet curve to obtain our estimates. In fact, the collar lemma is special to Hitchin representations, and does not hold even on the space of discrete and faithful representations from Γ to $\text{PSL}(n,\mathbb{R})$.

To see this, consider the space of conjugacy classes of quasi-Fuchsian representations from Γ to $\text{PSL}(2,\mathbb{C}) = \text{PSO}(3,1)^+ \subset \text{PSL}(4,\mathbb{R})$, which is the group of orientation-preserving isometries of $\mathbb{H}^3$. These are discrete and faithful representations whose limit set in the Riemann sphere $\partial\mathbb{H}^3$ is a Jordan curve. It is well known that each quasi-Fuchsian representation ρ induces a convex cocompact hyperbolic structure on the three-manifold $S \times I$. Also, for any nonidentity element X in Γ, the closed geodesic γ in $S\times I$ (equipped with the hyperbolic metric induced by ρ) corresponding to X has ρ–length

$$l_\rho(\gamma)=\log\Big(\frac{\lambda_1}{\lambda_4}\Big),$$

where λ_1 and λ_4 are the moduli of eigenvalues of $\rho(X)$ with largest and smallest modulus, respectively.

It is a theorem of Bers [1, Theorem 1] that the space of quasi-Fuchsian representations can be naturally identified with $\mathcal{T}(S)\times\mathcal{T}(\overline{S})$, where $\overline{S}$ is S with the opposite orientation. For any quasi-Fuchsian representation ρ let (ρ^+,ρ^-) denote the pair of Fuchsian representations that corresponds to ρ, such that $\rho^+\in\mathcal{T}(S)$ and $\rho^-\in\mathcal{T}(\overline{S})$. Then for any essential closed curve γ in S, let γ_ρ be the geodesic representative of γ in the hyperbolic metric on $S\times I$ corresponding to ρ, and let γ_{ρ^+} and γ_{ρ^-} be the geodesic representatives of γ in the hyperbolic metrics on S and $\overline{S}$ corresponding to ρ^+ and ρ^-, respectively. By Epstein, Marden and Markovic [10, Theorem 3.1], we know that

$$l_\rho(\gamma_\rho)\le\min\{2\cdot l_{\rho^+}(\gamma_{\rho^+}),\ 2\cdot l_{\rho^-}(\gamma_{\rho^-})\}.$$

For any pair of simple closed curves η and γ, and for any $\epsilon>0$, let ρ be a quasi-Fuchsian representation such that

$$l_{\rho^+}(\eta_{\rho^+})<\tfrac{1}{2}\epsilon\quad\text{and}\quad l_{\rho^-}(\gamma_{\rho^-})<\tfrac{1}{2}\epsilon.$$

Hence, $l_\rho(\eta_\rho)$ and $l_\rho(\gamma_\rho)$ are both smaller than ϵ. This implies that the analog of Corollary 1.2 does not hold on the space of discrete and faithful, or even Anosov, representations from Γ to $\mathrm{PSL}(4,\mathbb{R})$. (See Guichard and Wienhard [13] for more background on Anosov representations.)

3.3 Comparison with the classical collar lemma

Let ρ be a representation in the Fuchsian locus of $\mathrm{Hit}_n(S)$ and let h be the corresponding Fuchsian representation in $\mathcal{T}(S)$. Also, let X be a nonidentity element in Γ and let γ be a curve in S corresponding to X. If λ^{-1} and λ are the two eigenvalues of $h(X)$, then $\lambda^{-n+1},\lambda^{-n+3},\dots,\lambda^{n-3},\lambda^{n-1}$ are the n eigenvalues of $\rho(X)$. Hence we can get the lengths

$$l_h(\gamma)=2\log(\lambda)\quad\text{and}\quad l_\rho(\gamma)=2(n-1)\log(\lambda).$$

Since $h\in\mathcal{T}(S)$, the classical collar lemma holds. In other words, for any pair of curves η and γ in S such that γ is simple and $i(\eta,\gamma)>0$, we have

$$I_{\eta,\gamma}(h):=\sinh\big(\tfrac{1}{2}l_h(\eta)\big)\sinh\big(\tfrac{1}{2}l_h(\gamma)\big)>1; \tag{3-2}$$

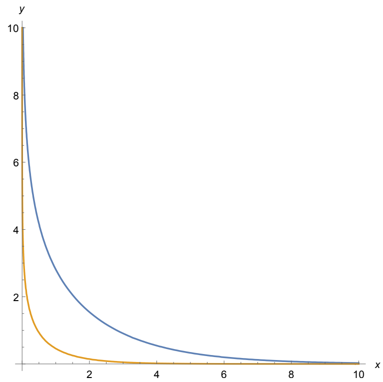

Figure 5: The upper curve $\sinh\bigl(\frac{1}{2}x\bigr)\sinh\bigl(\frac{1}{2}y\bigr) = 1$ and the lower curve $(e^x - 1)(e^y - 1) = 1$

see Buser [5, Corollary 4.1.2]. This inequality is sharp, in the sense that for any S, there are simple curves η and γ in S and a sequence of Fuchsian representations $\{h_i\}$ such that $I_{\eta,\gamma}(h_i)$ converges to 1. For more details, refer to Section 6 of Matelski [17].

On the other hand, Corollary 1.2(1), specialized to the $n = 2$ case, is the inequality

$$(e^{l_h(\eta)} - 1)(e^{l_h(\gamma)} - 1) > 1.$$

This is weaker than the inequality (3-2) because

$$e^x - 1 > \tfrac{1}{2}e^{-x/2}(e^x - 1) = \sinh\bigl(\tfrac{1}{2}x\bigr)$$

for every $x > 0$; see Figure 5. Moreover, we are unable to show that the inequality (3-2) fails in $\mathrm{Hit}_n(S)$ for any $n > 2$. This led us to conjecture in an earlier version of this paper, that for any ρ in $\mathrm{Hit}_n(S)$, there is some representation ρ' in the Fuchsian locus of $\mathrm{Hit}_n(S)$ such that

$$l_\rho(\gamma) \geq l_{\rho'}(\gamma) \quad \text{for any } \gamma \in \Gamma.$$

This conjecture implies that

$$\sinh\left(\frac{l_\rho(\eta)}{2(n-1)}\right)\sinh\left(\frac{l_\rho(\gamma)}{2(n-1)}\right) > 1$$

for any $\rho \in \mathrm{Hit}_n(S)$, which is sharp on every $\mathrm{Hit}_n(S)$ because it is sharp when restricted to the Fuchsian locus.

Recently, Tholozan proved (Section 0.4 of [18]) that the conjecture holds in the case when $n = 3$. Furthermore, Labourie disproved our conjecture in the case when $n \geq 4$. We will give his argument here.

Proposition 3.5 (Labourie) *When $n \geq 4$, there is some $\rho \in \mathrm{Hit}_n(S)$ such that for any ρ' in the Fuchsian locus of $\mathrm{Hit}_n(S)$, there is some closed curve γ in S such that $l_\rho(\gamma) < l_{\rho'}(\gamma)$.*

Proof For any closed curve γ in S, let $L_\gamma\colon \mathrm{Hit}_n(S) \to \mathbb{R}$ denote the map given by $L_\gamma(\rho) = l_\rho(\gamma)$. As before, let $\mathrm{Hit}_n(S)'$ be the $\mathrm{PSp}(2k,\mathbb{R})$ or $\mathrm{PSO}(k,k+1)$ Hitchin components when $n = 2k$ or $n = 2k+1$, respectively, and recall that $l_\rho(\gamma) = 2\log\lambda_1(\rho(X))$ for all $\rho \in \mathrm{Hit}_n(S)'$, where $X \in \Gamma$ is a group element corresponding to γ and $\lambda_1(\rho(X))$ is the modulus of eigenvalue of $\rho(X)$ with largest modulus. Proposition 10.3 of Bridgeman, Canary, Labourie and Sambarino [2] then implies that for any $\rho \in \mathrm{Hit}_n(S)'$, the set of differentials $\{dL_\gamma : \gamma \text{ a closed curve in } S\}$ generates the entire cotangent space of $\mathrm{Hit}_n(S)'$ at ρ.

Observe that if $n \geq 4$, then $\mathrm{Hit}_n(S)' \subset \mathrm{Hit}_n(S)$ properly contains the Fuchsian locus. Thus, it is sufficient to prove the proposition on $\mathrm{Hit}_n(S)'$. Suppose for contradiction that the proposition is false on $\mathrm{Hit}_n(S)'$. Choose a point ρ_0 in the Fuchsian locus, and take a smooth path ρ_t for $t \in (-\epsilon,\epsilon)$ with $\epsilon > 0$, whose nonzero tangent vector $U \in T_{\rho_0}\mathrm{Hit}_n(S)'$ is not tangential to the Fuchsian locus. Along the path, choose a sequence of representations $\{\rho_{t_i}\}_{i=1}^\infty$ which converges to ρ_0 as $i \to \infty$ so that $t_i > 0$ for all i.

Since the proposition is false on $\mathrm{Hit}_n(S)'$, there exists the corresponding sequence of Fuchsian representations ρ'_{t_i} such that $L_\gamma(\rho_{t_i}) \geq L_\gamma(\rho'_{t_i})$ for any closed curve γ in S. Also, since ρ_{t_i} converges to ρ_0, we see that $L_\gamma(\rho'_{t_i})$ is bounded for all γ, so the sequence $\{\rho'_{t_i}\}_{i=1}^\infty$ lie in a compact subset of the Fuchsian locus. By picking subsequence, we can assume without loss of generality that $\{\rho'_{t_i}\}_{i=1}^\infty$ converges to some ρ'_0 in the Fuchsian locus. The continuity of L_γ then implies that $L_\gamma(\rho_0) \geq L_\gamma(\rho'_0)$ for all γ, so $\rho_0 = \rho'_0$ because both ρ_0 and ρ'_0 lie in the Fuchsian locus.

Thus, the sequence $\{\rho'_{t_i}\}_{i=1}^\infty$ converges to ρ_0 as well. Choose a Riemannian metric on a neighborhood of ρ_0 in $\mathrm{Hit}_n(S)'$. By taking a further subsequence of $\{\rho_{t_i}\}_{i=1}^\infty$, we can also assume that either one of the following cases hold:

(i) $\rho'_{t_i} = \rho_0$ for all i;

(ii) the unit vectors at ρ_0 that are tangential to the geodesic between ρ'_{t_i} and ρ_0 converge to some unit vector $V \neq 0$ in $T_{\rho_0}\mathrm{Hit}_n(S)'$ that is tangential to the Fuchsian locus.

If (i) holds, then we have that $dL_\gamma(U) \geq 0$ for all γ. On the other hand, if (ii) holds, then for all closed curves γ in S,

$$
\begin{aligned}
dL_\gamma(U) &= \frac{d}{dt} L_\gamma(\rho_t)|_{t=0} \\
&= \lim_{i\to\infty} \frac{L_\gamma(\rho_{t_i}) - L_\gamma(\rho_0)}{t_i} \\
&\geq \lim_{i\to\infty} \frac{L_\gamma(\rho'_{t_i}) - L_\gamma(\rho'_0)}{t_i} \\
&= \Bigl(\lim_{i\to\infty} s_i\Bigr) \cdot dL_\gamma(V)
\end{aligned}
$$

for some sequence of positive numbers $\{s_i\}_{i=1}^{\infty}$. More precisely, if V_i denotes the tangent vector whose exponential is ρ'_{t_i} and $\|V_i\|$ is the norm of V_i with respect to the chosen Riemannian metric, then $s_i = \|V_i\|/t_i$.

Note that if $\lim_{i\to\infty} s_i = \infty$, then $dL_\gamma(V) \leq 0$ for all γ, which is impossible since $V \neq 0$ is tangential to the Fuchsian locus. Hence, $dL_\gamma(U + sV) \geq 0$ for some $s \leq 0$.

In either case, there is some vector $W \in T_{\rho_0} \operatorname{Hit}_n(S)'$ (possibly the zero vector) that is tangential to the Fuchsian locus such that $dL_\gamma(U + W) \geq 0$ for all γ. Furthermore, since $U + W \neq 0$, the fact that the differentials dL_γ generate the cotangent space of $\operatorname{Hit}_n(S)'$ at ρ_0 implies that $dL_\gamma(U + W) > 0$ for some γ. By a similar argument, we can also show that there is some vector $W' \in T_{\rho_0} \operatorname{Hit}_n(S)'$ that is tangential to the Fuchsian locus such that $dL_\gamma(-U + W') \geq 0$ for all γ, and this inequality holds strictly for some γ.

Adding these two inequalities together gives $dL_\gamma(W + W') \geq 0$ for all γ, and $dL_\gamma(W + W') > 0$ for some γ. However, this is impossible since $W + W'$ is tangential to the Fuchsian locus. □

Note that Labourie's argument to disprove our conjecture relied very heavily on the fact that $\operatorname{Hit}_n(S)' \subset \operatorname{Hit}_n(S)$ properly contains the Fuchsian locus. There is thus still hope that the following modified conjecture might be true.

Conjecture 3.6 *Let ρ be a representation in $\operatorname{Hit}_n(S)$. Then there is some representation ρ' in $\operatorname{Hit}_n(S)'$ such that*

$$l_\rho(\gamma) \geq l_{\rho'}(\gamma)$$

for any closed curve γ in S.

Appendix: Proof of Corollary 1.5

In this appendix, we will prove the properness result stated as Corollary 1.5. We begin by recalling some results from Zhang [19] that we will need.

Let $\mathcal{P} := \{\gamma_1, \dots, \gamma_{3g-3}\}$ be an oriented pants decomposition of S, ie a maximal collection of pairwise nonintersecting, pairwise nonhomotopic, homotopically nontrivial, oriented simple closed curves on S. These curves cut S into $2g-2$ pairs of pants, which we label by $P_1, \dots, P_{2g-2}$, and also gives us a real analytic diffeomorphism

$$\mathrm{Hit}_n(S) \to (\mathbb{R}^+)^{(3g-3)(n-1)} \times \mathbb{R}^{(3g-3)(n-1)} \times \mathbb{R}^{(2g-2)(n-1)(n-2)},$$

which one should think of as a generalization of the Fenchel–Nielsen coordinates on the Teichmüller space $\mathcal{T}(S)$; see [19, Proposition 3.5].

The first $(3g-3)(n-1)$ positive numbers are called the *boundary invariants*. For any $\rho \in \mathrm{Hit}_n(S)$, these are the numbers

$$\beta_{\gamma_j,k} := \log\left(\frac{\lambda_k(\rho(X_j))}{\lambda_{k+1}(\rho(X_j))}\right),$$

where $k = 1, \dots, n-1$ and $j = 1, \dots, 3g-3$. Here, $X_j \in \Gamma$ is a group element that corresponds to γ_j, and $\lambda_1(\rho(X_j)), \dots, \lambda_n(\rho(X_j))$ are the moduli of eigenvalues of $\rho(X_j)$ arranged in decreasing order. Note that each of the $3g-3$ curves in $\mathcal{P}$ are associated $n-1$ of these numbers. They capture the eigenvalue data of the holonomy about each of the curves in $\mathcal{P}$, and are a generalization of the Fenchel–Nielsen length coordinates.

The next $(3g-3)(n-1)$ real numbers are called the *gluing parameters*, and these are also associated to the curves in $\mathcal{P}$. Informally, the $n-1$ gluing parameters associated to each curve in $\mathcal{P}$ is the data specifying how one should "glue" the representations on adjacent pair of pants together along a common boundary component. Hence, these generalize the Fenchel–Nielsen twist coordinates. Just like the Fenchel–Nielsen twist coordinates, to specify these gluing parameters formally, we need to make additional topological choices to define what is "zero gluing". In this case, this additional topological choice we make is a pair of distinct points $a_j, b_j \in \partial_\infty \Gamma$ so that x_j^-, a_j, x_j^+, b_j lie in $\partial_\infty \Gamma$ in that cyclic order.

For simplicity, we will fix such a choice once and for all in the following way. Let P_1 and P_2 be the two pairs of pants that have γ_j as a common boundary component (it is possible for $P_1 = P_2$). For $i = 1, 2$, choose A_i, B_i and C_i to be elements in Γ corresponding the boundary components of P_i so that $C_i \cdot B_i \cdot A_i = \mathrm{id}$ and $A_1 = A_2^{-1} = X_j$. Let a_j be the repelling fixed point of B_1 and b_j be the repelling fixed point of C_2. The gluing parameters are then

$$g_{\gamma_j,k} := \log\bigl(-(P_{k,1}, P_{k,2}, P_{k,4}, P_{k,3})\bigr)$$

for $k = 1, \dots, n-1$, where $\xi\colon \partial_\infty \Gamma \to \mathcal{F}(\mathbb{R}^n)$ is the Frenet curve corresponding to ρ, and

$$\begin{aligned}
P_{k,1} &:= \xi(x_j^+)^{(k)} + \xi(x_j^-)^{(n-k-1)},\\
P_{k,2} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)} + \xi(a_j)^{(1)},\\
P_{k,3} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k)},\\
P_{k,4} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)} + \xi(b_j)^{(1)}
\end{aligned}$$

are four hyperplanes in $\mathbb{R}^n$ that intersect at $M_k := \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)}$.

Finally, the remaining $(2g-2)(n-1)(n-2)$ real numbers are called the *internal parameters*, and are associated to the pairs of pants $P_1, \dots, P_{2g-2}$. To each P_j, we associate $(n-1)(n-2)$ internal parameters, and they parametrize the Hitchin representations on a pair of pants after the boundary invariants are fixed. These are defined in great detail in Section 3 of [19]. For our purposes though, we do not need to know what these parameters are, but only the following proposition.

Proposition A.1 *Fix a pair of pants P_{j_0} given by $\mathcal{P}$. Let $\{\rho_i\}$ be a sequence in $\mathrm{Hit}_n(S)$ such that*

- *the boundary invariants corresponding to ∂P_{j_0} remain bounded away from 0 and ∞ along $\{\rho_i\}$, and*
- *some internal parameter corresponding to P_{j_0} grows to ∞ or $-\infty$ along $\{\rho_i\}$.*

Let γ be a closed curve in S with the property that any closed curve homotopic to γ has nonempty intersection with P_{j_0}. Then $\lim_{i\to\infty} l_{\rho_i}(\gamma) = \infty$.

Proof The proof of this proposition is a slight modification of the proof of the main theorem given in Section 5.1 of [19]. In Section 3.2 of [19], there is a description of a particular way to cut each P_j into two ideal triangles that share all three edges. Doing this over all P_j gives us $6g-6$ edges. Here, we view each of these edges $e = [a, b]$ as a Γ–orbit of a pair of distinct points $a, b \in \partial_\infty \Gamma$.

Let $\rho \in \mathrm{Hit}_n(S)$ and ξ the corresponding Frenet curve. As was done in Section 4.4 of [19], one can associate a particular positive number $K[a, b]$ to each of these $6g-6$ edges $[a, b]$. Using this, define

$$K(\rho, j_0) := \min\{K[a, b] : [a, b] \subset P_{j_0}\}.$$

The same argument as given in Section 5.1 of [19] proves that

$$\lim_{i\to\infty} K(\rho_i, j_0) = \infty.$$

Let $X \in \Gamma$ be a group element corresponding to γ. Let $e = [a,b]$ be an edge in P_{j_0} such that there is a lift $\widetilde{e} = \{a,b\}$ with the property that x^-, a, x^+, b lie in $\partial_\infty\Gamma$ in that cyclic order. Such an edge exists by the hypothesis we imposed on γ. For any $p = 0,\dots,n-1$, one can define subsegments $c_p(\widetilde{e})$ of the projective line $\mathbb{P}\big(\xi(x^-)^{(1)} + \xi(x^+)^{(1)}\big) \subset \mathbb{RP}^{n-1}$ associated to each lift $\widetilde{e} = \{a,b\}$ of $e = [a,b]$. These are called the *crossing* (p)*–subsegments*; see Definition 4.7 of [19]. Using the cross ratio, we can define a notion of length for these subsegments, which we denote by $l(c_p(\widetilde{e}))$; see Definition 4.8 of [19].

By the proof of [19, Proposition 4.16], we see that

$$\frac{1}{n}\sum_{p=0}^{n-1} l(c_p(\widetilde{e})) \geq K(\rho, j_0).$$

Furthermore, by Lemmas 4.9 and 4.10 of [19], we have

$$l_\rho(\gamma) \geq l(c_p(\widetilde{e}))$$

for all $p = 0,\dots,n-1$, which allows us to conclude that

$$l_\rho(\gamma) \geq K(\rho, j_0).$$

Combining this with the fact that $\lim_{i\to\infty} K(\rho_i, j_0) = \infty$ gives the proposition. □

With the above proposition, we are ready to prove Corollary 1.5. Let $\{\rho_i\}$ be a sequence in $\mathrm{Hit}_n(S)$, let $\mathcal{C} := \{\gamma_1,\dots,\gamma_k\}$ satisfy the hypothesis of Corollary 1.5 and let $\mathcal{P} := \{\gamma_1,\dots,\gamma_{3g-3}\} \subset \mathcal{C}$ be a pants decomposition. Observe that the hypothesis on $\mathcal{C}$ ensures the following:

- For any $\gamma \in \mathcal{P}$, there is some $\gamma' \in \mathcal{C}$ that intersects γ transversely.
- For each pair of pants P given by $\mathcal{P}$, there is some $\gamma \in \mathcal{C}$ such that any closed curve homotopic to γ has nonempty intersection with P.

The pants decomposition $\mathcal{P}$ then gives us a parametrization of $\mathrm{Hit}_n(S)$ as described above. We will prove Corollary 1.5 in the following steps.

(1) If there is some boundary invariant $\beta_{\gamma_j,k}$ such that $\lim_{i\to\infty} \beta_{\gamma_j,k}(\rho_i) = \infty$, then $\lim_{i\to\infty} l_{\rho_i}(\gamma_j) = \infty$.

(2) If there is some boundary invariant $\beta_{\gamma_j,k}$ such that $\lim_{i\to\infty} \beta_{\gamma_j,k}(\rho_i) = 0$, then $\lim_{i\to\infty} l_{\rho_i}(\gamma) = \infty$ for any closed curve γ that intersects γ_j transversely.

(3) If all the boundary invariants remain bounded away from 0 and ∞ and some internal parameter associated to a pair of pants P grows to $\pm\infty$, then $\lim_{i\to\infty} l_{\rho_i}(\gamma) = \infty$ for any closed curve γ with the property that any closed curve homotopic to γ has nonempty intersection with P.

(4) If all the boundary invariants remain bounded away from 0 and ∞ and there is some gluing parameter $g_{\gamma_j,k}$ such that $\lim_{i\to\infty} g_{\gamma_j,k}(\rho_i) = \pm\infty$, then $\lim_{i\to\infty} l_{\rho_i}(\gamma) = \infty$ for any γ that intersects γ_j transversely.

Note that together, the four statements above prove Corollary 1.5. Statement (1) is obvious because

$$l_\rho(\gamma_j) = \sum_{k=1}^{n-1} \beta_{\gamma_j,k}(\rho),$$

and all the boundary invariants are positive. Also, statement (3) is a restatement of Proposition A.1, and statement (2) is an immediate consequence of Theorem 1.1(1), which is a main result in this paper. The rest of this appendix will be the proof of statement (4).

Let $X_j, X \in \Gamma$ correspond to γ_j and γ, respectively, such that x_j^-, x^-, x_j^+, x^+ lie in $\partial_\infty\Gamma$ in that cyclic order. We previously chose a pair of points $a_j, b_j \in \partial_\infty\Gamma$ so that x_j^-, a_j, x_j^+, b_j lie in $\partial_\infty\Gamma$ in that cyclic order in order to define the gluing parameters $g_{\gamma_j,k}$ associated to γ_j. If we choose $X_j^l \cdot a_j$ and $X_j^m \cdot b_j$ in place of a_j and b_j, we get another collection of gluing parameters, which we denote by $g^{l,m}_{\gamma_j,k}$. The next lemma explains the relationship between $g_{\gamma_j,k} = g^{0,0}_{\gamma_j,k}$ and $g^{l,m}_{\gamma_j,k}$. Its proof is an easy computation which we omit.

Lemma A.2 *Let $\rho \in \mathrm{Hit}_n(S)$, and let $\lambda_1, \dots, \lambda_n$ be the moduli of eigenvalues of $\rho(X_j)$ arranged in decreasing order. For any integers l and m, we have*

$$g^{l,m}_{\gamma_j,k}(\rho) = (l-m)\log\Bigl(\frac{\lambda_k}{\lambda_{k+1}}\Bigr) + g_{\gamma_j,k}(\rho).$$

In particular, when the boundary invariants corresponding to γ_j are bounded away from 0 and ∞ along a sequence of representations $\{\rho_i\}$ in $\mathrm{Hit}_n(S)$, then we have $\lim_{i\to\infty} g_{\gamma_j,k}(\rho_i) = \pm\infty$ if and only if $\lim_{i\to\infty} g^{l,m}_{\gamma_j,k}(\rho_i) = \pm\infty$. Statement (4) then follows immediately from this observation and the following proposition.

Proposition A.3 *Let $\rho \in \mathrm{Hit}_n(S)$ and let γ_j, γ, x_j^-, x_j^+, x^-, x^+, a_j, b_j be as above. Let l and m be integers such that x_j^-, $X_j^{l-1}\cdot a_j$, x^-, $X_j^l \cdot a_j$, x_j^+, $X_j^{m+1}\cdot b_j$, x^+, $X_j^m \cdot b_j$ lie in $\partial_\infty\Gamma$ in that cyclic order. Then*

$$3l_\rho(\gamma) \ge g^{l-1,m+1}_{\gamma_j,k}(\rho) \quad \textit{and} \quad 3l_\rho(\gamma) \ge -g^{l,m}_{\gamma_j,k}(\rho)$$

for all $k = 1, \dots, n-1$.

Proof The technique used in this proof is the same as that used in the proof of [19, Lemma 4.18]. For any $k=1,\dots,n-1$, let

$$\begin{aligned}
P_{k,0} &:= \xi(x_j^+)^{(k)} + \xi(x_j^-)^{(n-k-1)},\\
P_{k,1} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)} + \xi(x^-)^{(1)},\\
P_{k,2} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)} + \xi(X_j^{l-1}\cdot a_j)^{(1)},\\
P'_{k,2} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)} + \xi(X_j^{l}\cdot a_j)^{(1)},\\
P_{k,3} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k)},\\
P_{k,4} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)} + \xi(X_j^{m}\cdot b_j)^{(1)},\\
P'_{k,4} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)} + \xi(X_j^{m+1}\cdot b_j)^{(1)},\\
P_{k,5} &:= \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)} + \xi(x^+)^{(1)}.
\end{aligned}$$

Also, for all i, let

$$\begin{aligned}
L'_{k,i} &:= P'_{k,i} \cap \big(\xi(x^-)^{(1)} + \xi(x^+)^{(1)}\big),\\
L_{k,i} &:= P_{k,i} \cap \big(\xi(x^-)^{(1)} + \xi(x^+)^{(1)}\big),
\end{aligned}$$

and let

$$\begin{aligned}
L_{k,a_j} &:= \big(\xi(X_j^{l}\cdot a_j)^{(k-1)} + \xi(X_j^{l-1}\cdot a_j)^{(n-k)}\big) \cap \big(\xi(x^-)^{(1)} + \xi(x^+)^{(1)}\big),\\
L_{k,b_j} &:= \big(\xi(X_j^{m+1}\cdot b_j)^{(k-1)} + \xi(X_j^{m}\cdot b_j)^{(n-k)}\big) \cap \big(\xi(x^-)^{(1)} + \xi(x^+)^{(1)}\big).
\end{aligned}$$

It follows from [19, Lemma 2.5] that

$$\xi(x^-)^{(1)},\ L_{k,a_j},\ L_{k,2},\ L_{k,3},\ L_{k,4},\ L_{k,b_j},\ \xi(x^+)^{(1)}$$

lie in the projective line $\xi(x^-)^{(1)} + \xi(x^+)^{(1)}$ in that cyclic order. Also, by [19, Lemma 4.11], we know

$$3l_\rho(\gamma) \ge \log\big(\xi(x^-)^{(1)}, L_{k,a_j}, L_{k,b_j}, \xi(x^+)^{(1)}\big),$$

which implies that

$$\begin{aligned}
3l_\rho(\gamma) &\ge \log\big(\xi(x^-)^{(1)}, L_{k,2}, L_{k,3}, \xi(x^+)^{(1)}\big),\\
3l_\rho(\gamma) &\ge \log\big(\xi(x^-)^{(1)}, L_{k,3}, L_{k,4}, \xi(x^+)^{(1)}\big)
\end{aligned}$$

by Lemma 2.9. Using Lemmas 2.8 and 2.6, we can also deduce that

$$\begin{aligned}\big(\xi(x^-)^{(1)}, L_{k,2}, L_{k,3}, \xi(x^+)^{(1)}\big) &= \big(\xi(x^-)^{(1)}, L_{k,2}, L_{k,3}, \xi(x^+)^{(1)}\big)_{M_k} \\ &= (P_{k,1}, P_{k,2}, P_{k,3}, P_{k,5}) \\ &\geq (P_{k,0}, P_{k,2}, P_{k,3}, P'_{k,4}) \\ &= 1-(P_{k,0}, P_{k,2}, P'_{k,4}, P_{k,3}) \\ &= 1+e^{g^{l-1,m+1}_{\gamma_j,k}} \\ &\geq e^{g^{l-1,m+1}_{\gamma_j,k}}\end{aligned}$$

and

$$\begin{aligned}\big(\xi(x^-)^{(1)}, L_{k,3}, L_{k,4}, \xi(x^+)^{(1)}\big) &= \big(\xi(x^-)^{(1)}, L_{k,3}, L_{k,4}, \xi(x^+)^{(1)}\big)_{M_k} \\ &= (P_{k,1}, P_{k,3}, P_{k,4}, P_{k,5}) \\ &\geq (P'_{k,2}, P_{k,3}, P_{k,4}, P_{k,0}) \\ &= 1-\frac{1}{(P_{k,0}, P'_{k,2}, P_{k,4}, P_{k,3})} \\ &= 1+e^{-g^{l,m}_{\gamma_j,k}} \\ &\geq e^{-g^{l,m}_{\gamma_j,k}},\end{aligned}$$

where $M_k := \xi(x_j^+)^{(k-1)} + \xi(x_j^-)^{(n-k-1)}$. □

References

[1] **L Bers**, *Simultaneous uniformization*, Bull. Amer. Math. Soc. 66 (1960) 94–97 MR

[2] **M Bridgeman**, **R Canary**, **F Labourie**, **A Sambarino**, *The pressure metric for Anosov representations*, Geom. Funct. Anal. 25 (2015) 1089–1179 MR

[3] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer, New York (1999) MR

[4] **M Burger**, **M B Pozzetti**, *Maximal representations, non Archimedean Siegel spaces, and buildings*, preprint (2015) arXiv

[5] **P Buser**, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics 106, Birkhäuser, Boston (1992) MR

[6] **S Choi**, *The Margulis lemma and the thick and thin decomposition for convex real projective surfaces*, Adv. Math. 122 (1996) 150–191 MR

[7] **S Choi**, **W M Goldman**, *Convex real projective structures on closed surfaces are closed*, Proc. Amer. Math. Soc. 118 (1993) 657–661 MR

[8] **D Cooper**, **D D Long**, **S Tillmann**, *On convex projective manifolds and cusps*, Adv. Math. 277 (2015) 181–251 MR

[9] **K Corlette**, *Flat G–bundles with canonical metrics*, J. Differential Geom. 28 (1988) 361–382 MR

[10] **D B A Epstein**, **A Marden**, **V Markovic**, *Quasiconformal homeomorphisms and the convex hull boundary*, Ann. of Math. 159 (2004) 305–336 MR

[11] **W M Goldman**, *Convex real projective structures on compact surfaces*, J. Differential Geom. 31 (1990) 791–845 MR

[12] **O Guichard**, *Composantes de Hitchin et représentations hyperconvexes de groupes de surface*, J. Differential Geom. 80 (2008) 391–431 MR

[13] **O Guichard**, **A Wienhard**, *Anosov representations: domains of discontinuity and applications*, Invent. Math. 190 (2012) 357–438 MR

[14] **L Keen**, *Collars on Riemann surfaces*, from "Discontinuous groups and Riemann surfaces" (L Greenberg, editor), Ann. of Math. Studies 79, Princeton Univ. Press (1974) 263–268 MR

[15] **F Labourie**, *Anosov flows, surface groups and curves in projective space*, Invent. Math. 165 (2006) 51–114 MR

[16] **F Labourie**, **G McShane**, *Cross ratios and identities for higher Teichmüller–Thurston theory*, Duke Math. J. 149 (2009) 279–345 MR

[17] **J P Matelski**, *A compactness theorem for Fuchsian groups of the second kind*, Duke Math. J. 43 (1976) 829–840 MR

[18] **N Tholozan**, *Entropy of Hilbert metrics and length spectrum of Hitchin representations in* $\mathrm{PSL}(3,\mathbb{R})$, preprint (2015) arXiv

[19] **T Zhang**, *Degeneration of Hitchin representations along internal sequences*, Geom. Funct. Anal. 25 (2015) 1588–1645 MR

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