Geometry & Topology 21 (2017) 2419–2460

Hodge modules on complex tori and generic vanishing for compact Kähler manifolds

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We extend the results of generic vanishing theory to polarizable real Hodge modules on compact complex tori, and from there to arbitrary compact Kähler manifolds. As applications, we obtain a bimeromorphic characterization of compact complex tori (among compact Kähler manifolds), semipositivity results and a description of the Leray filtration for maps to tori.

14C30; 14F17

A Introduction

The term "generic vanishing" refers to a collection of theorems about the cohomology of line bundles with trivial first Chern class. The first results of this type were obtained by Green and Lazarsfeld in the late 1980s [13; 14]; they were proved using classical Hodge theory and are therefore valid on arbitrary compact Kähler manifolds. About ten years ago, Hacon [15] found a more algebraic approach, using vanishing theorems and the Fourier–Mukai transform, that has led to many additional results in the projective case; see also Chen and Jiang [9], Pareschi and Popa [23] and Popa and Schnell [26]. The purpose of this paper is to show that the newer results are in fact also valid on arbitrary compact Kähler manifolds.

Besides Hacon [15], our motivation also comes from a paper by Chen and Jiang [9], in which they prove, roughly speaking, that the direct image of the canonical bundle under a generically finite morphism to an abelian variety is semiample. Before we can state more precise results, recall the following definitions (see Section 13 for more details).

Definition Given a coherent $\mathscr{O}_T$–module $\mathscr{F}$ on a compact complex torus T, define

$$S^i(T,\mathscr{F}) = \{L \in \operatorname{Pic}^0(T) \mid H^i(T,\mathscr{F}\otimes L) \neq 0\}.$$

We say that $\mathscr{F}$ is a *GV-sheaf* if $\operatorname{codim} S^i(T,\mathscr{F}) \geq i$ for every $i \geq 0$; we say that $\mathscr{F}$ is *M–regular* if $\operatorname{codim} S^i(T,\mathscr{F}) \geq i+1$ for every $i \geq 1$.

Published: 19 May 2017 DOI: 10.2140/gt.2017.21.2419

Hacon [15, Section 4] showed that if $f\colon X \to A$ is a morphism from a smooth projective variety to an abelian variety, then the higher direct image sheaves $R^j f_* \omega_X$ are GV-sheaves on A; in the special case where f is generically finite over its image, Chen and Jiang [9, Theorem 1.2] proved the much stronger result that $f_*\omega_X$ is, up to tensoring by line bundles in $\mathrm{Pic}^0(A)$, the direct sum of pullbacks of M–regular sheaves from quotients of A. Since GV-sheaves are nef, whereas M–regular sheaves are ample, one should think of this as saying that $f_*\omega_X$ is not only nef but actually semiample. One of our main results is the following generalization of this fact:

Theorem A *Let $f\colon X \to T$ be a holomorphic mapping from a compact Kähler manifold to a compact complex torus. Then, for $j \geq 0$, one has a decomposition*

$$R^j f_* \omega_X \simeq \bigoplus_{k=1}^{n} (q_k^* \mathscr{F}_k \otimes L_k),$$

where each $\mathscr{F}_k$ is an M–regular (hence ample) coherent sheaf with projective support on the compact complex torus T_k, each $q_k\colon T \to T_k$ is a surjective morphism with connected fibers, and each $L_k \in \mathrm{Pic}^0(T)$ has finite order. In particular, $R^j f_* \omega_X$ is a GV-sheaf on T.

This leads to strong positivity properties for higher direct images of canonical bundles under maps to tori. For instance, if f is a surjective map that is submersive away from a divisor with simple normal crossings, then $R^j f_* \omega_X$ is a semipositive vector bundle on T. See Section 20 for more on this circle of ideas.

One application of Theorem A is the following effective criterion for a compact Kähler manifold to be bimeromorphically equivalent to a torus; this generalizes a well-known theorem of Chen and Hacon in the projective case [6].

Theorem B *A compact Kähler manifold X is bimeromorphic to a compact complex torus if and only if $\dim H^1(X, \mathbb{C}) = 2 \dim X$ and $P_1(X) = P_2(X) = 1$.*

The proof is inspired by the approach to the Chen–Hacon theorem given by Pareschi [20]; even in the projective case, however, the result in Corollary 16.2 greatly simplifies the existing proof. In Theorem 19.1, we deduce that the Albanese map of a compact Kähler manifold with $P_1(X) = P_2(X) = 1$ is surjective with connected fibers; in the projective case, this was first proved by Jiang [16], as an effective version of Kawamata's theorem about projective varieties of Kodaira dimension zero. It is likely that the present methods can also be applied to the classification of compact Kähler manifolds with $\dim H^1(X, \mathbb{C}) = 2 \dim X$ and small plurigenera; for the projective case, see for instance Chen and Hacon [8].

In a different direction, Theorem A combined with results of Lazarsfeld, Popa and Schnell [18] leads to a concrete description of the Leray filtration on the cohomology of ω_X, associated with a holomorphic mapping $f\colon X \to T$ as above. Recall that, for each $k \geq 0$, the Leray filtration is a decreasing filtration $L^\bullet H^k(X, \omega_X)$ with the property that

$$\mathrm{gr}_L^i H^k(X, \omega_X) = H^i(T, R^{k-i} f_* \omega_X).$$

One can also define a natural decreasing filtration $F^\bullet H^k(X, \omega_X)$ induced by the cup product action of $H^1(T, \mathscr{O}_T)$, namely

$$F^i H^k(X, \omega_X) = \mathrm{Im}\big(\textstyle\bigwedge^i H^1(T, \mathscr{O}_T) \otimes H^{k-i}(X, \omega_X) \to H^k(X, \omega_X)\big).$$

Theorem C *The filtrations $L^\bullet H^k(X, \omega_X)$ and $F^\bullet H^k(X, \omega_X)$ coincide.*

We give a dual description of the filtration on global holomorphic forms in Corollary 21.3. Despite the elementary nature of the statement, we do not know how to prove Theorem C using only methods from classical Hodge theory; finding a more elementary proof is an interesting problem.

Our approach to Theorem A is to address generic vanishing for a larger class of objects of Hodge-theoretic origin, namely polarizable real Hodge modules on compact complex tori. This is not just a matter of higher generality; we do not know how to prove Theorem A using methods of classical Hodge theory in the spirit of Green and Lazarsfeld [13]. This is precisely due to the lack of an a priori description of the Leray filtration on $H^k(X, \omega_X)$ as in Theorem C.

The starting point for our proof of Theorem A is a result by Saito [29], which says that the coherent $\mathscr{O}_T$–module $R^j f_* \omega_X$ is part of a polarizable real Hodge module $M = (\mathcal{M}, F_\bullet \mathcal{M}, M_\mathbb{R}) \in \mathrm{HM}_\mathbb{R}(T, \dim X + j)$ on the torus T; more precisely,

$$R^j f_* \omega_X \simeq \omega_T \otimes F_{p(M)} \mathcal{M}$$

is the first nontrivial piece in the Hodge filtration $F_\bullet \mathcal{M}$ of the underlying regular holonomic $\mathscr{D}$–module $\mathcal{M}$. (Please see Section 1 for some background on Hodge modules.) Note that M is supported on the image $f(X)$, and that its restriction to the smooth locus of f is the polarizable variation of Hodge structure on the $(\dim f + j)^{\text{th}}$ cohomology of the fibers. The reason for working with real coefficients is that the polarization is induced by a choice of Kähler form in $H^2(X, \mathbb{R}) \cap H^{1,1}(X)$; the variation of Hodge structure itself is of course defined over $\mathbb{Z}$.

In light of the above identity, Theorem A is a consequence of the following general statement about polarizable real Hodge modules on compact complex tori:

Theorem D *Let $M = (\mathcal{M}, F_\bullet\mathcal{M}, M_\mathbb{R}) \in \mathrm{HM}_\mathbb{R}(T, w)$ be a polarizable real Hodge module on a compact complex torus T. Then, for each $k \in \mathbb{Z}$, the coherent $\mathscr{O}_T$–module $\mathrm{gr}_k^F \mathcal{M}$ decomposes as*

$$\mathrm{gr}_k^F \mathcal{M} \simeq \bigoplus_{j=1}^{n} (q_j^* \mathscr{F}_j \otimes_{\mathscr{O}_T} L_j),$$

where $q_j \colon T \to T_j$ is a surjective map with connected fibers to a complex torus, $\mathscr{F}_j$ is an M–regular coherent sheaf on T_j with projective support and $L_j \in \mathrm{Pic}^0(T)$. If M admits an integral structure, then each L_j has finite order.

Let us briefly describe the most important elements in the proof. In Popa and Schnell [26] we already exploited the relationship between generic vanishing and Hodge modules on abelian varieties, but the proofs relied on vanishing theorems. What allows us to go further is a beautiful idea by Botong Wang [41], namely that, up to taking direct summands and tensoring by unitary local systems, every polarizable real Hodge module on a complex torus actually comes from an abelian variety. (Wang showed this for Hodge modules of geometric origin.) This is a version with coefficients of Ueno's result [39] that every irreducible subvariety of T is a torus bundle over a projective variety, and is proved by combining this geometric fact with some arguments about variations of Hodge structure.

The existence of the decomposition in Theorem D is due to the fact that the regular holonomic $\mathscr{D}$–module $\mathcal{M}$ is semisimple, hence isomorphic to a direct sum of simple regular holonomic $\mathscr{D}$–modules. This follows from a theorem by Deligne and Nori (see Deligne [11]), which says that the local system underlying a polarizable real variation of Hodge structure on a Zariski-open subset of a compact Kähler manifold is semisimple. It turns out that the decomposition of $\mathcal{M}$ into simple summands is compatible with the Hodge filtration $F_\bullet\mathcal{M}$; in order to prove this, we introduce the category of "polarizable complex Hodge modules" (which are polarizable real Hodge modules together with an endomorphism whose square is minus the identity), and show that every simple summand of $\mathcal{M}$ underlies a polarizable complex Hodge module in this sense.

Note Our ad hoc definition of complex Hodge modules is good enough for the purposes of this paper, but is certainly not the final word. A more satisfactory treatment, in terms of $\mathscr{D}$–modules and distribution-valued pairings, is currently being developed by Claude Sabbah and the third author [27].

The M–regularity of the individual summands in Theorem D turns out to be closely related to the Euler characteristic of the corresponding $\mathscr{D}$–modules. The results in [26] show that when $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a polarizable complex Hodge module on an abelian variety A, the Euler characteristic satisfies $\chi(A, \mathcal{M}) \geq 0$, and each coherent $\mathscr{O}_A$–module $\mathrm{gr}_k^F \mathcal{M}$ is a GV-sheaf. The new result (in Lemma 15.1) is that each $\mathrm{gr}_k^F \mathcal{M}$ is actually M–regular provided that $\chi(A, \mathcal{M}) > 0$. That we can always get into the situation where the Euler characteristic is positive follows from some general results about simple holonomic $\mathscr{D}$–modules from Schnell [34].

Theorem D implies that each graded quotient $\mathrm{gr}_k^F \mathcal{M}$ with respect to the Hodge filtration is a GV-sheaf, the Kähler analogue of a result in [26]. However, the stronger formulation above is new even in the case of smooth projective varieties, and has further useful consequences. One such is the following: for a holomorphic mapping $f\colon X \to T$ that is generically finite onto its image, the locus

$$S^0(T, f_*\omega_X) = \{L \in \mathrm{Pic}^0(T) \mid H^i(T, f_*\omega_X \otimes_{\mathscr{O}_T} L) \neq 0\}$$

is preserved by the involution $L \mapsto L^{-1}$ on $\mathrm{Pic}^0(T)$; see Corollary 16.2. This is a crucial ingredient in the proof of Theorem B.

Going back to Wang's paper [41], its main purpose was to prove Beauville's conjecture, namely that, on a compact Kähler manifold X, every irreducible component of every $\Sigma^k(X) = \{\rho \in \mathrm{Char}(X) \mid H^k(X, \mathbb{C}_\rho) \neq 0\}$ contains characters of finite order. In the projective case, this is of course a famous theorem by Simpson [37]. Combining the structural Theorem 7.1 with known results about Hodge modules on abelian varieties (Schnell [35]) allows us to prove the following generalization of Wang's theorem (which dealt with Hodge modules of geometric origin):

Theorem E *If a polarizable real Hodge module $M \in \mathrm{HM}_{\mathbb{R}}(T, w)$ on a compact complex torus admits an integral structure, then the sets*

$$S_m^i(T, M) = \{\rho \in \mathrm{Char}(T) \mid \dim H^i(T, M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}_\rho) \geq m\}$$

are finite unions of translates of linear subvarieties by points of finite order.

The idea is to use Kronecker's theorem (about algebraic integers all of whose conjugates have absolute value one) to prove that certain characters have finite order. Roughly speaking, the characters in question are unitary because of the existence of a polarization on M, and they take values in the group of algebraic integers because of the existence of an integral structure on M.

Projectivity questions

We conclude by noting that many of the results in this paper can be placed in the broader context of the following problem: how far are natural geometric or sheaf-theoretic constructions on compact Kähler manifolds in general, and on compact complex tori in particular, from being determined by similar constructions on projective manifolds? Theorems A and D provide the answer on tori in the case of Hodge-theoretic constructions. We thank János Kollár for suggesting this point of view, and also the statements of the problems in the paragraph below.

Further structural results could provide a general machine for reducing certain questions about Kähler manifolds to the algebraic setting. For instance, by analogy with positivity conjectures in the algebraic case, one hopes for the following result in the case of varying families: if X and Y are compact Kähler manifolds and $f\colon X \to Y$ is a fiber space of maximal variation, ie such that the general fiber is bimeromorphic to at most countably many other fibers, then Y is projective. More generally, for an arbitrary such f, is there a mapping $g\colon Y \to Z$, with Z projective, such that the fibers of f are bimeromorphically isotrivial over those of Y?

A slightly more refined version in the case when $Y = T$ is a torus, which is essentially a combination of Iitaka fibrations and Ueno's conjecture, is this: there should exist a morphism $h\colon X \to Z$, where Z is a variety of general type generating an abelian quotient $g\colon T \to A$, such that the fibers of h have Kodaira dimension 0 and are bimeromorphically isotrivial over the fibers of g.

B Real and complex Hodge modules

1 Real Hodge modules

In this paper, we work with polarizable real Hodge modules on complex manifolds. This is the natural setting for studying compact Kähler manifolds, because the polarizations induced by Kähler forms are defined over $\mathbb{R}$ (but usually not over $\mathbb{Q}$, as in the projective case). Saito originally developed the theory of Hodge modules with rational coefficients, but as explained in [29], everything works just as well with real coefficients and with the following weaker assumption on the local monodromy: the eigenvalues of the monodromy operator on the nearby cycles are allowed to be arbitrary complex numbers of absolute value one, rather than just roots of unity. This has already been observed several times in the literature [33]; the point is that Saito's theory rests on certain results about polarizable variations of Hodge structure [32; 43; 5], which hold in this generality.

Let X be a complex manifold. We first recall some terminology.

Definition 1.1 We denote by $\mathrm{HM}_{\mathbb{R}}(X, w)$ the category of polarizable real Hodge modules of weight w; this is a semisimple $\mathbb{R}$–linear abelian category, endowed with a faithful functor to the category of real perverse sheaves.

Saito constructs $\mathrm{HM}_{\mathbb{R}}(X, w)$ as a full subcategory of the category of all filtered regular holonomic $\mathscr{D}$–modules with real structure, in several stages. To begin with, recall that a *filtered regular holonomic $\mathscr{D}$–module with real structure* on X consists of the following four pieces of data: (1) a regular holonomic left $\mathscr{D}_X$–module $\mathcal{M}$; (2) a good filtration $F_\bullet\mathcal{M}$ by coherent $\mathscr{O}_X$–modules; (3) a perverse sheaf $M_\mathbb{R}$ with coefficients in $\mathbb{R}$; (4) an isomorphism $M_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \simeq \mathrm{DR}(\mathcal{M})$. Although the isomorphism is part of the data, we usually suppress it from the notation and simply write $M = (\mathcal{M}, F_\bullet\mathcal{M}, M_\mathbb{R})$. The *support* $\operatorname{Supp} M$ is defined to be the support of the underlying perverse sheaf $M_\mathbb{R}$; one says that M has *strict support* if $\operatorname{Supp} M$ is irreducible and if M has no nontrivial subobjects or quotient objects that are supported on a proper subset of $\operatorname{Supp} M$.

Now M is called a *real Hodge module of weight w* if it satisfies several additional conditions that are imposed by recursion on the dimension of $\operatorname{Supp} M$. Although they are not quite stated in this way in [28], the essence of these conditions is that (1) every Hodge module decomposes into a sum of Hodge modules with strict support, and (2) every Hodge module with strict support is generically a real variation of Hodge structure, which uniquely determines the Hodge module. Given $k \in \mathbb{Z}$, set $\mathbb{R}(k) = (2\pi i)^k\mathbb{R} \subseteq \mathbb{C}$; then one has the *Tate twist*

$$M(k) = (\mathcal{M}, F_{\bullet-k}\mathcal{M}, M_\mathbb{R} \otimes_\mathbb{R} \mathbb{R}(k)) \in \mathrm{HM}_\mathbb{R}(X, w-2k).$$

Every real Hodge module of weight w has a well-defined *dual* $\boldsymbol{D}M$, which is a real Hodge module of weight $-w$ whose underlying perverse sheaf is the Verdier dual $\boldsymbol{D}M_\mathbb{R}$. A *polarization* is an isomorphism of real Hodge modules $\boldsymbol{D}M \simeq M(w)$, subject to certain conditions that are again imposed recursively; one says that M is *polarizable* if it admits at least one polarization.

Example 1.2 Every polarizable real variation of Hodge structure of weight w on X gives rise to an object of $\mathrm{HM}_\mathbb{R}(X, w + \dim X)$. If $\mathcal{H}$ is such a variation, we denote the underlying real local system by $\mathcal{H}_\mathbb{R}$, its complexification by $\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ and the corresponding flat bundle by $(\mathcal{H}, \nabla)$; then $\mathcal{H} \simeq \mathcal{H}_\mathbb{C} \otimes_\mathbb{C} \mathscr{O}_X$. The flat connection makes $\mathcal{H}$ into a regular holonomic left $\mathscr{D}$–module, filtered by $F_\bullet\mathcal{H} = F^{-\bullet}\mathcal{H}$; the real structure is given by the real perverse sheaf $\mathcal{H}_\mathbb{R}[\dim X]$.

We list a few useful properties of polarizable real Hodge modules. By definition, every object $M \in \mathrm{HM}_\mathbb{R}(X, w)$ admits a locally finite *decomposition by strict support*; when X is compact, this is a finite decomposition

$$M \simeq \bigoplus_{j=1}^n M_j,$$

where each $M_j \in \mathrm{HM}_\mathbb{R}(X, w)$ has strict support equal to an irreducible analytic subvariety $Z_j \subseteq X$. There are no nontrivial morphisms between Hodge modules with different strict support; if we assume that $Z_1, \dots, Z_n$ are distinct, the decomposition by strict support is therefore unique. Since the category $\mathrm{HM}_{\mathbb{R}}(X, w)$ is semisimple, it follows that every polarizable real Hodge module of weight w is isomorphic to a direct sum of simple objects with strict support.

One of Saito's most important results is the following structure theorem, relating polarizable real Hodge modules and polarizable real variations of Hodge structure.

Theorem 1.3 (Saito) *The category of polarizable real Hodge modules of weight w with strict support $Z \subseteq X$ is equivalent to the category of generically defined polarizable real variations of Hodge structure of weight $w - \dim Z$ on Z.*

In other words, for any $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ with strict support Z, there is a dense Zariski-open subset of the smooth locus of Z over which it restricts to a polarizable real variation of Hodge structure; conversely, every such variation extends uniquely to a Hodge module with strict support Z. The proof in [30, Theorem 3.21] carries over to the case of real coefficients; see [29] for further discussion.

Lemma 1.4 *The support of $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ lies in a submanifold $i\colon Y \hookrightarrow X$ if and only if M belongs to the image of the functor $i_*\colon \mathrm{HM}_{\mathbb{R}}(Y, w) \to \mathrm{HM}_{\mathbb{R}}(X, w)$.*

This result is often called *Kashiwara's equivalence*, because Kashiwara proved the same thing for arbitrary coherent $\mathscr{D}$–modules. In the case of Hodge modules, the point is that the coherent $\mathscr{O}_X$–modules $F_k\mathcal{M}/F_{k-1}\mathcal{M}$ are in fact $\mathscr{O}_Y$–modules.

2 Compact Kähler manifolds and semisimplicity

In this section, we prove some results about the underlying regular holonomic $\mathscr{D}$–modules of polarizable real Hodge modules on compact Kähler manifolds. Our starting point is the following semisimplicity theorem:

Theorem 2.1 (Deligne, Nori) *Let X be a compact Kähler manifold. If*

$$M = (\mathcal{M}, F_\bullet\mathcal{M}, M_{\mathbb{R}}) \in \mathrm{HM}_{\mathbb{R}}(X, w),$$

then the perverse sheaf $M_{\mathbb{R}}$ and the $\mathscr{D}$–module $\mathcal{M}$ are semisimple.

Proof Since the category $\mathrm{HM}_{\mathbb{R}}(X, w)$ is semisimple, we may assume without loss of generality that M is simple, with strict support an irreducible analytic subvariety $Z \subseteq X$. By Saito's Theorem 1.3, M restricts to a polarizable real variation of Hodge structure $\mathcal{H}$ of weight $w - \dim Z$ on a Zariski-open subset of the smooth locus of Z; note that $\mathcal{H}$ is a simple object in the category of real variations of Hodge structure. Now

$M_{\mathbb{R}}$ is the intersection complex of $\mathcal{H}_{\mathbb{R}}$, and so it suffices to prove that $\mathcal{H}_{\mathbb{R}}$ is semisimple. After resolving singularities, we can assume that $\mathcal{H}$ is defined on a Zariski-open subset of a compact Kähler manifold; in that case, Deligne and Nori have shown that $\mathcal{H}_{\mathbb{R}}$ is semisimple [11, Section 1.12]. It follows that the complexification $M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ of the perverse sheaf is semisimple as well; by the Riemann–Hilbert correspondence, the same is true for the underlying regular holonomic $\mathscr{D}$–module $\mathcal{M}$. □

A priori, there is no reason why the decomposition of the regular holonomic $\mathscr{D}$–module $\mathcal{M}$ into simple factors should lift to a decomposition in the category $\mathrm{HM}_{\mathbb{R}}(X, w)$. Nevertheless, it turns out that we can always chose the decomposition in such a way that it is compatible with the filtration $F_\bullet\mathcal{M}$.

Proposition 2.2 *Let $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ be a simple polarizable real Hodge module on a compact Kähler manifold. Then one of the following two statements is true:*

1. *The underlying perverse sheaf $M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is simple.*
2. *There is an endomorphism $J \in \mathrm{End}(M)$ with $J^2 = -\mathrm{id}$ such that*
$$(\mathcal{M}, F_\bullet\mathcal{M}, M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = \ker(J - i \cdot \mathrm{id}) \oplus \ker(J + i \cdot \mathrm{id}),$$
and the perverse sheaves underlying $\ker(J \pm i \cdot \mathrm{id})$ are simple.

We begin by proving the following key lemma:

Lemma 2.3 *Let $\mathcal{H}$ be a polarizable real variation of Hodge structure on a Zariski-open subset of a compact Kähler manifold. If $\mathcal{H}$ is simple, then*

(a) *either the underlying complex local system $\mathcal{H}_{\mathbb{C}}$ is also simple,*

(b) *or there is an endomorphism $J \in \mathrm{End}(\mathcal{H})$ with $J^2 = -\mathrm{id}$ such that*
$$\mathcal{H}_{\mathbb{C}} = \ker(J_{\mathbb{C}} - i \cdot \mathrm{id}) \oplus \ker(J_{\mathbb{C}} + i \cdot \mathrm{id})$$
is the sum of two (possibly isomorphic) simple local systems.

Proof Since X is a Zariski-open subset of a compact Kähler manifold, the theorem of the fixed part holds on X, and the local system $\mathcal{H}_{\mathbb{C}}$ is semisimple [11, Section 1.12]. Choose a base point $x_0 \in X$, and write $H_{\mathbb{R}}$ for the fiber of the local system $\mathcal{H}_{\mathbb{R}}$ at the point x_0; it carries a polarizable Hodge structure
$$H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=w} H^{p,q},$$
say of weight w. The fundamental group $\Gamma = \pi_1(X, x_0)$ acts on $H_{\mathbb{R}}$, and, as we remarked above, $H_{\mathbb{C}}$ decomposes into a sum of simple Γ–modules. The proof of

[11, Proposition 1.13] shows that there is a nontrivial simple Γ–module $V \subseteq H_{\mathbb{C}}$ compatible with the Hodge decomposition, meaning that

$$V = \bigoplus_{p+q=w} V \cap H^{p,q}.$$

Let $\overline{V} \subseteq H_{\mathbb{C}}$ denote the conjugate of V with respect to the real structure $H_{\mathbb{R}}$; it is another nontrivial simple Γ–module with

$$\overline{V} = \bigoplus_{p+q=w} \overline{V} \cap H^{p,q}.$$

The intersection $(V + \overline{V}) \cap H_{\mathbb{R}}$ is therefore a Γ–invariant real sub-Hodge structure of $H_{\mathbb{R}}$. By the theorem of the fixed part, it extends to a real subvariation of $\mathcal{H}$; since $\mathcal{H}$ is simple, this means that $H_{\mathbb{C}} = V + \overline{V}$. Now there are two possibilities:

(1) If $V = \overline{V}$, then $H_{\mathbb{C}} = V$, and $\mathcal{H}_{\mathbb{C}}$ is a simple local system.

(2) If $V \neq \overline{V}$, then $H_{\mathbb{C}} = V \oplus \overline{V}$, and $\mathcal{H}_{\mathbb{C}}$ is the sum of two (possibly isomorphic) simple local systems.

The endomorphism algebra $\operatorname{End}(\mathcal{H}_{\mathbb{R}})$ coincides with the subalgebra of Γ–invariants in $\operatorname{End}(H_{\mathbb{R}})$; by the theorem of the fixed part, it is also a real sub-Hodge structure. Let $p \in \operatorname{End}(H_{\mathbb{C}})$ and $\overline{p} \in \operatorname{End}(H_{\mathbb{C}})$ denote the projections to the two subspaces V and $\overline{V}$; both preserve the Hodge decomposition, and are therefore of type $(0,0)$. This shows that the element $J = i(p - \overline{p}) \in \operatorname{End}(H_{\mathbb{C}})$ is a real Hodge class of type $(0,0)$ with $J^2 = -\mathrm{id}$; by the theorem of the fixed part, J is the restriction to x_0 of an endomorphism of the variation of Hodge structure $\mathcal{H}$. This completes the proof because V and $\overline{V}$ are exactly the $\pm i$–eigenspaces of J. □

Proof of Proposition 2.2 Since M is simple, it has strict support equal to an irreducible analytic subvariety $Z \subseteq X$; by Theorem 1.3, M is obtained from a polarizable real variation of Hodge structure $\mathcal{H}$ of weight $w - \dim Z$ on a dense Zariski-open subset of the smooth locus of Z. Let $\mathcal{H}_{\mathbb{R}}$ denote the underlying real local system; then $M_{\mathbb{R}}$ is isomorphic to the intersection complex of $\mathcal{H}_{\mathbb{R}}$. Since we can resolve the singularities of Z by blowing up along submanifolds of X, Lemma 2.3 applies to this situation; it shows that $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ has at most two simple factors. The same is true for $M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and, by the Riemann–Hilbert correspondence, for $\mathcal{M}$.

Now we have to consider two cases. If $\mathcal{H}_{\mathbb{C}}$ is simple, then $\mathcal{M}$ is also simple, and we are done. If $\mathcal{H}_{\mathbb{C}}$ is not simple, then by Lemma 2.3 there is an endomorphism $J \in \operatorname{End}(\mathcal{H})$ with $J^2 = -\mathrm{id}$ such that the two simple factors are the $\pm i$–eigenspaces of J. By Theorem 1.3, it extends uniquely to an endomorphism of $J \in \mathrm{End}(M)$ in the category $\mathrm{HM}_{\mathbb{R}}(X, w)$; in particular, we obtain an induced endomorphism

$$J\colon \mathcal{M} \to \mathcal{M}$$

that is strictly compatible with the filtration $F_\bullet\mathcal{M}$ by [28, Proposition 5.1.14]. Now the $\pm i$–eigenspaces of J give us the desired decomposition

$$(\mathcal{M}, F_\bullet\mathcal{M}) = (\mathcal{M}', F_\bullet\mathcal{M}') \oplus (\mathcal{M}'', F_\bullet\mathcal{M}'');$$

note that the two regular holonomic $\mathscr{D}$–modules $\mathcal{M}'$ and $\mathcal{M}''$ are simple because the corresponding perverse sheaves are the intersection complexes of the simple complex local systems $\ker(J_{\mathbb{C}} \pm i \cdot \mathrm{id})$, where $J_{\mathbb{C}}$ stands for the induced endomorphism of the complexification $M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. □

3 Complex Hodge modules

In Saito's recursive definition of the category of polarizable Hodge modules, the existence of a real structure is crucial: to say that a given filtration on a complex vector space is a Hodge structure of a certain weight, or that a given bilinear form is a polarization, one needs to have complex conjugation. This explains why there is as yet no general theory of "polarizable complex Hodge modules" — although it seems likely that such a theory can be constructed within the framework of twistor $\mathscr{D}$–modules developed by Sabbah and Mochizuki. We now explain a workaround for this problem, suggested by Proposition 2.2.

Definition 3.1 A *polarizable complex Hodge module* on a complex manifold X is a pair (M, J), consisting of a polarizable real Hodge module $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ and an endomorphism $J \in \mathrm{End}(M)$ with $J^2 = -\mathrm{id}$.

The space of morphisms between two polarizable complex Hodge modules (M_1, J_1) and (M_2, J_2) is defined in the obvious way:

$$\mathrm{Hom}((M_1, J_1), (M_2, J_2)) = \{f \in \mathrm{Hom}(M_1, M_2) \mid f \circ J_1 = J_2 \circ f\}.$$

Note that composition with J_1 (or equivalently, J_2) puts a natural complex structure on this real vector space.

Definition 3.2 We denote by $\mathrm{HM}_{\mathbb{C}}(X, w)$ the category of polarizable complex Hodge modules of weight w; it is $\mathbb{C}$–linear and abelian.

From a polarizable complex Hodge module (M, J), we obtain a filtered regular holonomic $\mathscr{D}$–module as well as a complex perverse sheaf, as follows. Denote by

$$\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}'' = \ker(J - i \cdot \mathrm{id}) \oplus \ker(J + i \cdot \mathrm{id})$$

the induced decomposition of the regular holonomic $\mathscr{D}$–module underlying M, and observe that $J \in \operatorname{End}(\mathcal{M})$ is strictly compatible with the Hodge filtration $F_\bullet\mathcal{M}$. This means that we have a decomposition

$$(\mathcal{M}, F_\bullet\mathcal{M}) = (\mathcal{M}', F_\bullet\mathcal{M}') \oplus (\mathcal{M}'', F_\bullet\mathcal{M}'')$$

in the category of filtered $\mathscr{D}$–modules. Similarly, let $J_{\mathbb{C}} \in \operatorname{End}(M_{\mathbb{C}})$ denote the induced endomorphism of the complex perverse sheaf underlying M; then

$$M_{\mathbb{C}} = M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \ker(J_{\mathbb{C}} - i \cdot \mathrm{id}) \oplus \ker(J_{\mathbb{C}} + i \cdot \mathrm{id}),$$

and the two summands correspond to $\mathcal{M}'$ and $\mathcal{M}''$ under the Riemann–Hilbert correspondence. Note that they are isomorphic as *real* perverse sheaves; the only difference is in the $\mathbb{C}$–action. We obtain a functor

$$(M, J) \mapsto \ker(J_{\mathbb{C}} - i \cdot \mathrm{id})$$

from $\mathrm{HM}_{\mathbb{C}}(X, w)$ to the category of complex perverse sheaves on X; it is faithful, but depends on the choice of i.

Definition 3.3 Given $(M, J) \in \mathrm{HM}_{\mathbb{C}}(X, w)$, we call

$$\ker(J_{\mathbb{C}} - i \cdot \mathrm{id}) \subseteq M_{\mathbb{C}}$$

the *underlying complex perverse sheaf*, and

$$(\mathcal{M}', F_\bullet\mathcal{M}') = \ker(J - i \cdot \mathrm{id}) \subseteq (\mathcal{M}, F_\bullet\mathcal{M})$$

the *underlying filtered regular holonomic $\mathscr{D}$–module*.

There is also an obvious functor from polarizable real Hodge modules to polarizable complex Hodge modules: it takes $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ to the pair

$$(M \oplus M, J_M), \quad J_M(m_1, m_2) = (-m_2, m_1).$$

Not surprisingly, the underlying complex perverse sheaf is isomorphic to $M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and the underlying filtered regular holonomic $\mathscr{D}$–module to $(\mathcal{M}, F_\bullet\mathcal{M})$. The proof of the following lemma is left as an easy exercise.

Lemma 3.4 *A polarized complex Hodge module $(M, J) \in \mathrm{HM}_{\mathbb{C}}(X, w)$ belongs to the image of $\mathrm{HM}_{\mathbb{R}}(X, w)$ if and only if there exists $r \in \operatorname{End}(M)$ with*

$$r \circ J = -J \circ r \quad \text{and} \quad r^2 = \mathrm{id}.$$

In particular, (M, J) should be isomorphic to its *complex conjugate* $(M, -J)$, but this in itself does not guarantee the existence of a real structure — for example when M is simple and $\operatorname{End}(M)$ is isomorphic to the quaternions $\mathbb{H}$.

Proposition 3.5 *The category* $\mathrm{HM}_{\mathbb{C}}(X, w)$ *is semisimple, and the simple objects are of the following two types:*

(i) $(M \oplus M, J_M)$, *where* $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ *is simple and* $\mathrm{End}(M) = \mathbb{R}$.

(ii) (M, J), *where* $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ *is simple and* $\mathrm{End}(M) \in \{\mathbb{C}, \mathbb{H}\}$.

Proof Since $\mathrm{HM}_{\mathbb{R}}(X, w)$ is semisimple, every object of $\mathrm{HM}_{\mathbb{C}}(X, w)$ is isomorphic to a direct sum of polarizable complex Hodge modules of the form

$$(M^{\oplus n}, J), \tag{3.6}$$

where $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ is simple and J is an $n \times n$ matrix with entries in $\mathrm{End}(M)$ such that $J^2 = -\mathrm{id}$. By Schur's lemma and the classification of real division algebras, the endomorphism algebra of a simple polarizable real Hodge module is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If $\mathrm{End}(M) = \mathbb{R}$, elementary linear algebra shows that n must be even and that (3.6) is isomorphic to the direct sum of $\frac{n}{2}$ copies of (i). If $\mathrm{End}(M) = \mathbb{C}$, one can diagonalize the matrix J; this means that (3.6) is isomorphic to a direct sum of n objects of type (ii). If $\mathrm{End}(M) = \mathbb{H}$, it is still possible to diagonalize J, but this needs some nontrivial results about matrices with entries in the quaternions [42]. Write $J \in M_n(\mathbb{H})$ in the form $J = J_1 + J_2 j$, with $J_1, J_2 \in M_n(\mathbb{C})$, and consider the "adjoint matrix"

$$\chi_J = \begin{pmatrix} J_1 & J_2 \\ -\bar{J}_2 & \bar{J}_1 \end{pmatrix} \in M_{2n}(\mathbb{C}).$$

Since $J^2 = -\mathrm{id}$, one also has $\chi_J^2 = -\mathrm{id}$, and so the matrix J is normal by [42, Theorem 4.2]. According to [42, Corollary 6.2], this implies the existence of a unitary matrix $U \in M_n(\mathbb{H})$ such that $U^{-1} A U = i \cdot \mathrm{id}$; here unitary means that $U^{-1} = U^*$ is equal to the conjugate transpose of U. The consequence is that (3.6) is again isomorphic to a direct sum of n objects of type (ii). Since it is straightforward to prove that both types of objects are indeed simple, this concludes the proof. □

Note The three possible values for the endomorphism algebra of a simple object $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ reflect the splitting behavior of its complexification $(M \oplus M, J_M)$ in $\mathrm{HM}_{\mathbb{C}}(X, w)$: if $\mathrm{End}(M) = \mathbb{R}$, it remains irreducible; if $\mathrm{End}(M) = \mathbb{C}$, it splits into two nonisomorphic simple factors; if $\mathrm{End}(M) = \mathbb{H}$, it splits into two isomorphic simple factors. Note that the endomorphism ring of a simple polarizable complex Hodge module is always isomorphic to $\mathbb{C}$, in accordance with Schur's lemma.

Our ad hoc definition of the category $\mathrm{HM}_{\mathbb{C}}(X, w)$ has the advantage that every result about polarizable real Hodge modules that does not explicitly mention the real structure extends to polarizable complex Hodge modules. For example, each $(M, J) \in \mathrm{HM}_{\mathbb{C}}(X, w)$ admits a unique decomposition by strict support: M admits such a decomposition, and since there are no nontrivial morphisms between objects with different strict support, J is automatically compatible with the decomposition. For much the same reason, Kashiwara's equivalence (in Lemma 1.4) holds also for polarizable complex Hodge modules.

Another result that immediately carries over is Saito's direct image theorem. The strictness of the direct image complex is one of the crucial properties of polarizable Hodge modules; in the special case of the morphism from a projective variety X to a point, it is equivalent to the E_1–degeneration of the spectral sequence

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}_p^F \mathrm{DR}(\mathcal{M}')) \Rightarrow H^{p+q}(X, \mathrm{DR}(\mathcal{M}')),$$

a familiar result in classical Hodge theory when $\mathcal{M}' = \mathscr{O}_X$.

Theorem 3.7 *Let $f\colon X \to Y$ be a projective morphism between complex manifolds.*

(a) *If $(M, J) \in \mathrm{HM}_{\mathbb{C}}(X, w)$, then for each $k \in \mathbb{Z}$, the pair*

$$\mathcal{H}^k f_*(M, J) = (\mathcal{H}^k f_* M, \mathcal{H}^k f_* J) \in \mathrm{HM}_{\mathbb{C}}(Y, w+k)$$

is again a polarizable complex Hodge module.

(b) *The direct image complex $f_+(\mathcal{M}', F_\bullet\mathcal{M}')$ is strict, and $\mathcal{H}^k f_+(\mathcal{M}', F_\bullet\mathcal{M}')$ is the filtered regular holonomic $\mathscr{D}$–module underlying $\mathcal{H}^k f_*(M, J)$.*

Proof Since $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ is a polarizable real Hodge module, $\mathcal{H}^k f_* M$ is in $\mathrm{HM}_{\mathbb{R}}(Y, w+k)$ by Saito's direct image theorem [28, Théorème 5.3.1]. Now it suffices to note that $J \in \mathrm{End}(M)$ induces an endomorphism $\mathcal{H}^k f_* J \in \mathrm{End}(\mathcal{H}^k f_* M)$ whose square is equal to minus the identity. Since

$$(\mathcal{M}, F_\bullet\mathcal{M}) = (\mathcal{M}', F_\bullet\mathcal{M}') \oplus (\mathcal{M}'', F_\bullet\mathcal{M}''),$$

the strictness of the complex $f_+(\mathcal{M}', F_\bullet\mathcal{M}')$ follows from that of $f_+(\mathcal{M}, F_\bullet\mathcal{M})$, which is part of the above-cited theorem by Saito. □

On compact Kähler manifolds, the semisimplicity results from the previous section can be summarized as follows:

Proposition 3.8 *Let X be a compact Kähler manifold.*

(a) *A polarizable complex Hodge module $(M, J) \in \mathrm{HM}_{\mathbb{C}}(X, w)$ is simple if and only if the underlying complex perverse sheaf*

$$\ker\bigl(J_{\mathbb{C}} - i \cdot \mathrm{id}\colon M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}\bigr)$$

is simple.

(b) *If $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$, then every simple factor of the complex perverse sheaf $M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ underlies a polarizable complex Hodge module.*

Proof This is a restatement of Proposition 2.2. □

4 Complex variations of Hodge structure

In this section, we discuss the relation between polarizable complex Hodge modules and polarizable complex variations of Hodge structure.

Definition 4.1 A *polarizable complex variation of Hodge structure* is a pair $(\mathcal{H}, J)$, where $\mathcal{H}$ is a polarizable real variation of Hodge structure and $J \in \operatorname{End}(\mathcal{H})$ is an endomorphism with $J^2 = -\mathrm{id}$.

As before, the *complexification* of a real variation $\mathcal{H}$ is defined as

$$(\mathcal{H} \oplus \mathcal{H}, J_{\mathcal{H}}), \quad J_{\mathcal{H}}(h_1, h_2) = (-h_2, h_1),$$

and a complex variation $(\mathcal{H}, J)$ is real if and only if there is an endomorphism $r \in \operatorname{End}(\mathcal{H})$ with $r \circ J = -J \circ r$ and $r^2 = \mathrm{id}$. Note that the direct sum of $(\mathcal{H}, J)$ with its *complex conjugate* $(\mathcal{H}, -J)$ has an obvious real structure.

The definition above is convenient for our purposes; it is also not hard to show that it is equivalent to the one in [11, Section 1], up to the choice of weight. (Deligne only considers complex variations of weight zero.)

Example 4.2 Let $\rho \in \operatorname{Char}(X)$ be a unitary character of the fundamental group, and denote by $\mathbb{C}_\rho$ the resulting unitary local system. It determines a polarizable complex variation of Hodge structure in the following manner. The underlying real local system is $\mathbb{R}^2$, with monodromy acting by

$$\begin{pmatrix} \operatorname{Re}\rho & -\operatorname{Im}\rho \\ \operatorname{Im}\rho & \operatorname{Re}\rho \end{pmatrix};$$

the standard inner product on $\mathbb{R}^2$ makes this into a polarizable real variation of Hodge structure $\mathcal{H}_\rho$ of weight zero, with $J_\rho \in \operatorname{End}(\mathcal{H}_\rho)$ acting as $J_\rho(x, y) = (-y, x)$; for simplicity, we continue to denote the pair $(\mathcal{H}_\rho, J_\rho)$ by the symbol $\mathbb{C}_\rho$.

We have the following criterion for deciding whether a polarizable complex Hodge module is *smooth*, meaning induced by a complex variation of Hodge structure.

Lemma 4.3 *Given $(M, J) \in \mathrm{HM}_{\mathbb{C}}(X, w)$, let us denote by*

$$\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}'' = \ker(J - i \cdot \mathrm{id}) \oplus \ker(J + i \cdot \mathrm{id})$$

the induced decomposition of the regular holonomic $\mathscr{D}$–module underlying M. If $\mathcal{M}'$ is coherent as an $\mathscr{O}_X$–module, then M is smooth.

Proof Let $M_{\mathbb{C}} = \ker(J_{\mathbb{C}} - i \cdot \mathrm{id}) \oplus \ker(J_{\mathbb{C}} + i \cdot \mathrm{id})$ be the analogous decomposition of the underlying perverse sheaf. Since $\mathcal{M}'$ is $\mathscr{O}_X$–coherent, it is a vector bundle with flat connection; by the Riemann–Hilbert correspondence, the first factor is therefore (up to a shift in degree by $\dim X$) a complex local system. Since it is isomorphic to $M_{\mathbb{R}}$ as a real perverse sheaf, it follows that $M_{\mathbb{R}}$ is also a local system; but then M is smooth by [28, Lemme 5.1.10]. □

In general, the relationship between complex Hodge modules and complex variations of Hodge structure is governed by the following theorem; it is of course an immediate consequence of Saito's results (see Theorem 1.3).

Theorem 4.4 *The category of polarizable complex Hodge modules of weight w with strict support $Z \subseteq X$ is equivalent to the category of generically defined polarizable complex variations of Hodge structure of weight $w - \dim Z$ on Z.*

5 Integral structures on Hodge modules

By working with polarizable real (or complex) Hodge modules, we lose certain arithmetic information about the monodromy of the underlying perverse sheaves, such as the fact that the monodromy eigenvalues are roots of unity. One can recover some of this information by asking for the existence of an "integral structure" [35, Definition 1.9], which is just a constructible complex of sheaves of $\mathbb{Z}$–modules that becomes isomorphic to the perverse sheaf underlying the Hodge module after tensoring by $\mathbb{R}$.

Definition 5.1 An *integral structure* on a polarizable real Hodge module M in $\mathrm{HM}_{\mathbb{R}}(X, w)$ is a constructible complex $E \in \mathrm{D}^b_c(\mathbb{Z}_X)$ such that $M_{\mathbb{R}} \simeq E \otimes_{\mathbb{Z}} \mathbb{R}$.

As explained in [35, Section 1.2.2], the existence of an integral structure is preserved by many of the standard operations on (mixed) Hodge modules, such as direct and inverse images or duality. Note that even though it makes sense to ask whether a given (mixed) Hodge module admits an integral structure, there appears to be no good functorial theory of "polarizable integral Hodge modules".

Lemma 5.2 *If $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ admits an integral structure, then the same is true for every summand in the decomposition of M by strict support.*

Proof Consider the decomposition

$$M = \bigoplus_{j=1}^{n} M_j$$

by strict support, with $Z_1, \dots, Z_n \subseteq X$ distinct irreducible analytic subvarieties. Each M_j is a polarizable real Hodge module with strict support Z_j, and therefore comes from a polarizable real variation of Hodge structure $\mathcal{H}_j$ on a dense Zariski-open subset of Z_j. What we must prove is that each $\mathcal{H}_j$ can be defined over $\mathbb{Z}$. Let $M_{\mathbb{R}}$ be the underlying real perverse sheaf, and set $d_j = \dim Z_j$. According to [2, Proposition 2.1.17], Z_j is an irreducible component in the support of the $(-d_j)^{\text{th}}$ cohomology sheaf of $M_{\mathbb{R}}$ and $\mathcal{H}_{j,\mathbb{R}}$ is the restriction of that constructible sheaf to a Zariski-open subset of Z_j. Since $M_{\mathbb{R}} \simeq E \otimes_{\mathbb{Z}} \mathbb{R}$, it follows that $\mathcal{H}_j$ is defined over $\mathbb{Z}$. □

6 Operations on Hodge modules

In this section, we recall three useful operations for polarizable real (and complex) Hodge modules. If $\operatorname{Supp} M$ is compact, we define the *Euler characteristic* of $M = (\mathcal{M}, F_\bullet\mathcal{M}, M_{\mathbb{R}}) \in \mathrm{HM}_{\mathbb{R}}(X, w)$ by the formula

$$\chi(X, M) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i(X, M_{\mathbb{R}}) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathrm{DR}(\mathcal{M})).$$

For $(M, J) \in \mathrm{HM}_{\mathbb{C}}(X, w)$, we let $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}'' = \ker(J - i \cdot \mathrm{id}) \oplus \ker(J + i \cdot \mathrm{id})$ be the decomposition into eigenspaces, and define

$$\chi(X, M, J) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathrm{DR}(\mathcal{M}')).$$

With this definition, one has $\chi(X, M) = \chi(X, M, J) + \chi(X, M, -J)$.

Given a smooth morphism $f\colon Y \to X$ of relative dimension $\dim f = \dim Y - \dim X$, we define the *naive inverse image*

$$f^{-1}M = (f^*\mathcal{M}, f^*F_\bullet\mathcal{M}, f^{-1}M_{\mathbb{R}}).$$

One can show that $f^{-1}M \in \mathrm{HM}_{\mathbb{R}}(Y, w + \dim f)$; see [36, Section 9] for more details. The same is true for polarizable complex Hodge modules: if $(M, J) \in \mathrm{HM}_{\mathbb{C}}(X, w)$, then one obviously has

$$f^{-1}(M, J) = (f^{-1}M, f^{-1}J) \in \mathrm{HM}_{\mathbb{C}}(Y, w + \dim f).$$

One can also twist a polarizable complex Hodge module by a unitary character.

Lemma 6.1 *For any unitary character $\rho \in \operatorname{Char}(X)$, there is an object*

$$(M, J) \otimes_{\mathbb{C}} \mathbb{C}_\rho \in \mathrm{HM}_{\mathbb{C}}(X, w)$$

whose associated complex perverse sheaf is $\ker(J_{\mathbb{C}} - i \cdot \mathrm{id}) \otimes_{\mathbb{C}} \mathbb{C}_\rho$.

Proof In the notation of Example 4.2, consider the tensor product

$$M \otimes_{\mathbb{R}} \mathcal{H}_\rho \in \mathrm{HM}_{\mathbb{R}}(X, w);$$

it is again a polarizable real Hodge module of weight w because $\mathcal{H}_\rho$ is a polarizable real variation of Hodge structure of weight zero. The square of the endomorphism $J \otimes J_\rho$ is the identity, and so

$$N = \ker(J \otimes J_\rho + \mathrm{id}) \subseteq M \otimes_{\mathbb{R}} \mathcal{H}_\rho$$

is again a polarizable real Hodge module of weight w. Now $K = J \otimes \mathrm{id} \in \mathrm{End}(N)$ satisfies $K^2 = -\mathrm{id}$, which means that the pair (N, K) is a polarizable complex Hodge module of weight w. On the associated complex perverse sheaf

$$\ker(K_{\mathbb{C}} - i \cdot \mathrm{id}) \subseteq M_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{H}_{\rho,\mathbb{C}},$$

both $J_{\mathbb{C}} \otimes \mathrm{id}$ and $\mathrm{id} \otimes J_{\rho,\mathbb{C}}$ act as multiplication by i, which means that

$$\ker(K_{\mathbb{C}} - i \cdot \mathrm{id}) = \ker(J_{\mathbb{C}} - i \cdot \mathrm{id}) \otimes_{\mathbb{C}} \mathbb{C}_{\rho}.$$

The corresponding regular holonomic $\mathscr{D}$–module is obviously

$$\mathcal{N}' = \mathcal{M}' \otimes_{\mathscr{O}_X} (L, \nabla),$$

with the filtration induced by $F_{\bullet}\mathcal{M}'$; here (L, ∇) denotes the flat bundle corresponding to the complex local system $\mathbb{C}_{\rho}$, and $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ as above. □

Note The proof shows that

$$N_{\mathbb{C}} = \bigl(\ker(J_{\mathbb{C}} - i \cdot \mathrm{id}) \otimes_{\mathbb{C}} \mathbb{C}_{\rho}\bigr) \oplus \bigl(\ker(J_{\mathbb{C}} + i \cdot \mathrm{id}) \otimes_{\mathbb{C}} \mathbb{C}_{\overline{\rho}}\bigr),$$
$$\mathcal{N} = \bigl(\mathcal{M}' \otimes_{\mathscr{O}_X} (L, \nabla)\bigr) \oplus \bigl(\mathcal{M}'' \otimes_{\mathscr{O}_X} (L, \nabla)^{-1}\bigr),$$

where $\overline{\rho}$ is the complex conjugate of the character $\rho \in \mathrm{Char}(X)$.

C Hodge modules on complex tori

7 Main result

The paper [26] contains several results about Hodge modules of geometric origin on abelian varieties. In this chapter, we generalize these results to arbitrary polarizable complex Hodge modules on compact complex tori. To do so, we develop a beautiful idea due to Wang [41], namely that, up to direct sums and character twists, every such object actually comes from an abelian variety.

Theorem 7.1 *Let $(M, J) \in \mathrm{HM}_{\mathbb{C}}(T, w)$ be a polarizable complex Hodge module on a compact complex torus T. Then there is a decomposition*

$$(M, J) \simeq \bigoplus_{j=1}^{n} q_j^{-1}(N_j, J_j) \otimes_{\mathbb{C}} \mathbb{C}_{\rho_j}, \tag{7.2}$$

where $q_j \colon T \to T_j$ is a surjective morphism with connected fibers, $\rho_j \in \mathrm{Char}(T)$ is a unitary character and $(N_j, J_j) \in \mathrm{HM}_{\mathbb{C}}(T_j, w - \dim q_j)$ is a simple polarizable complex Hodge module with $\mathrm{Supp}\, N_j$ projective and $\chi(T_j, N_j, J_j) > 0$.

For Hodge modules of geometric origin, a less precise result was proved by Wang [41]. His proof makes use of the decomposition theorem, which in the setting of arbitrary compact Kähler manifolds is only known for Hodge modules of geometric origin [29]. This technical issue can be circumvented by putting everything in terms of generically defined variations of Hodge structure.

To get a result for a polarizable real Hodge module $M \in \mathrm{HM}_{\mathbb{R}}(T, w)$, we simply apply Theorem 7.1 to its complexification $(M \oplus M, J_M) \in \mathrm{HM}_{\mathbb{C}}(T, w)$. One could say more about the terms in the decomposition below, but the following version is enough for our purposes.

Corollary 7.3 *Let $M \in \mathrm{HM}_{\mathbb{R}}(T, w)$ be a polarizable real Hodge module on a compact complex torus T. Then, in the notation of Theorem 7.1, one has*

$$(M \oplus M, J_M) \simeq \bigoplus_{j=1}^{n} q_j^{-1}(N_j, J_j) \otimes_{\mathbb{C}} \mathbb{C}_{\rho_j}.$$

If M admits an integral structure, then each $\rho_j \in \mathrm{Char}(T)$ has finite order.

The proof of these results takes up the rest of the chapter.

8 Subvarieties of complex tori

This section contains a structure theorem for subvarieties of compact complex tori. The statement is contained in [41, Propositions 2.3 and 2.4], but we give a simpler argument below.

Proposition 8.1 *Let X be an irreducible analytic subvariety of a compact complex torus T. Then there is a subtorus $S \subseteq T$ with the following two properties:*

(a) *$S + X = X$ and the quotient $Y = X/S$ is projective.*

(b) *If $D \subseteq X$ is an irreducible analytic subvariety with $\dim D = \dim X - 1$, then $S + D = D$.*

In particular, every divisor on X is the preimage of a divisor on Y.

Proof It is well known that the algebraic reduction of T is an abelian variety. More precisely, there is a subtorus $S \subseteq T$ such that $A = T/S$ is an abelian variety, and every other subtorus with this property contains S; see eg [4, Chapter 2, Section 6].

Now let $X \subseteq T$ be an irreducible analytic subvariety of T; without loss of generality, we may assume that $0 \in X$ and that X is not contained in any proper subtorus of T. By a theorem of Ueno [39, Theorem 10.9], there is a subtorus $S' \subseteq T$ with $S' + X \subseteq X$

and such that $X/S' \subseteq T/S'$ is of general type. In particular, X/S' is projective; but then T/S' must also be projective, which means that $S \subseteq S'$. Setting $Y = X/S$, we get a cartesian diagram

$$\begin{array}{ccc} X & \hookrightarrow & T \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & A \end{array}$$

with Y projective. Now it remains to show that every divisor on X is the pullback of a divisor from Y.

Let $D \subseteq X$ be an irreducible analytic subvariety with $\dim D = \dim X - 1$; as before, we may assume that $0 \in D$. For dimension reasons, either $S + D = D$ or $S + D = X$; let us suppose that $S + D = X$ and see how this leads to a contradiction. Define $T_D \subseteq T$ to be the smallest subtorus of T containing D; then $S + T_D = T$. If $T_D = T$, then the same reasoning as above would show that $S + D = D$; therefore $T_D \neq T$, and $\dim(T_D \cap S) \leq \dim S - 1$. Now

$$D \cap S \subseteq T_D \cap S \subseteq S,$$

and, because $\dim(D \cap S) = \dim S - 1$, it follows that $D \cap S = T_D \cap S$ consists of a subtorus S'' and finitely many of its translates. After dividing out by S'', we may assume that $\dim S = 1$ and that $D \cap S = T_D \cap S$ is a finite set; in particular, D is finite over Y, and therefore also projective. Now consider the addition morphism

$$S \times D \to T.$$

Since $S + D = X$, its image is equal to X; because S and D are both projective, it follows that X is projective, and hence that T is projective. But this contradicts our choice of S. The conclusion is that $S + D = D$, as asserted. $\square$

Note It is possible for S to be itself an abelian variety; this is why the proof that $S + D \neq X$ requires some care.

9 Simple Hodge modules and abelian varieties

We begin by proving a structure theorem for *simple* polarizable complex Hodge modules on a compact complex torus T; this is evidently the most important case, because every polarizable complex Hodge module is isomorphic to a direct sum of simple ones. Fix a simple polarizable complex Hodge module $(M, J) \in \mathrm{HM}_{\mathbb{C}}(T, w)$. By Proposition 3.5, the polarizable real Hodge module $M \in \mathrm{HM}_{\mathbb{R}}(X, w)$ has strict support equal to an irreducible analytic subvariety; we assume in addition that $\operatorname{Supp} M$ is not contained in any proper subtorus of T.

Theorem 9.1 *There is an abelian variety A, a surjective morphism $q\colon T \to A$ with connected fibers, a simple $(N, K) \in \mathrm{HM}_{\mathbb{C}}(A, w - \dim q)$ with $\chi(A, N, K) > 0$, and a unitary character $\rho \in \mathrm{Char}(T)$, such that*

$$(M, J) \simeq q^{-1}(N, K) \otimes_{\mathbb{C}} \mathbb{C}_{\rho}. \tag{9.2}$$

In particular, $\operatorname{Supp} M = q^{-1}(\operatorname{Supp} N)$ *is covered by translates of* $\ker q$.

Let $X = \operatorname{Supp} M$. By Proposition 8.1, there is a subtorus $S \subseteq T$ such that $S + X = X$ and such that $Y = X/S$ is projective. Since Y is not contained in any proper subtorus, it follows that $A = T/S$ is an abelian variety. Let $q\colon T \to A$ be the quotient mapping, which is proper and smooth of relative dimension $\dim q = \dim S$. This will not be our final choice for Theorem 9.1, but it does have almost all the properties that we want (except for the lower bound on the Euler characteristic).

Proposition 9.3 *There is a simple $(N, K) \in \mathrm{HM}_{\mathbb{C}}(A, w - \dim q)$ with strict support Y and a unitary character $\rho \in \mathrm{Char}(T)$ for which* (9.2) *holds.*

By Theorem 4.4, (M, J) corresponds to a polarizable complex variation of Hodge structure of weight $w - \dim X$ on a dense Zariski-open subset of X. The crucial observation, due to Wang, is that we can choose this set to be of the form $q^{-1}(U)$, where U is a dense Zariski-open subset of the smooth locus of Y.

Lemma 9.4 *There is a dense Zariski-open subset $U \subseteq Y$, contained in the smooth locus of Y, and a polarizable complex variation of Hodge structure $(\mathcal{H}, J)$ of weight $w - \dim X$ on $q^{-1}(U)$ such that (M, J) is the polarizable complex Hodge module corresponding to $(\mathcal{H}, J)$ in Theorem 4.4.*

Proof Let $Z \subseteq X$ be the union of the singular locus of X and the singular locus of M. Then Z is an analytic subset of X, and according to Theorem 1.3, the restriction of M to $X \setminus Z$ is a polarizable real variation of Hodge structure $\mathcal{H}$ of weight $w - \dim X$. By Proposition 8.1, no irreducible component of Z of dimension $\dim X - 1$ dominates Y; we can therefore find a Zariski-open subset $U \subseteq Y$, contained in the smooth locus of Y, such that the intersection $q^{-1}(U) \cap Z$ has codimension ≥ 2 in $q^{-1}(U)$. Now $\mathcal{H}$ extends uniquely to a polarizable real variation of Hodge structure on the entire complex manifold $q^{-1}(U)$, see [32, Proposition 4.1]. The assertion about J follows easily. □

For any $y \in U$, the restriction of $(\mathcal{H}, J)$ to the fiber $q^{-1}(y)$ is a polarizable complex variation of Hodge structure on a translate of the compact complex torus $\ker q$. By Lemma 11.1, the restriction to $q^{-1}(y)$ of the underlying local system

$$\ker(J_{\mathbb{C}} - i \cdot \mathrm{id}\colon \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}})$$

is the direct sum of local systems of the form $\mathbb{C}_\rho$ for $\rho \in \mathrm{Char}(T)$ unitary; when M admits an integral structure, ρ has finite order in the group $\mathrm{Char}(T)$.

Proof of Proposition 9.3 Let $\rho \in \mathrm{Char}(T)$ be one of the unitary characters in question, and let $\bar\rho \in \mathrm{Char}(T)$ denote its complex conjugate. The tensor product $(\mathcal{H}, J) \otimes_{\mathbb{C}} \mathbb{C}_{\bar\rho}$ is a polarizable complex variation of Hodge structure of weight $w - \dim X$ on the open subset $q^{-1}(U)$. Since all fibers of $q\colon q^{-1}(U) \to U$ are translates of the compact complex torus $\ker q$, classical Hodge theory for compact Kähler manifolds [43, Theorem 2.9] implies that

$$q_*((\mathcal{H}, J) \otimes_{\mathbb{C}} \mathbb{C}_{\bar\rho}) \tag{9.5}$$

is a polarizable complex variation of Hodge structure of weight $w - \dim X$ on U; in particular, it is again semisimple. By our choice of ρ, the adjunction morphism

$$q^{-1}q_*((\mathcal{H}, J) \otimes_{\mathbb{C}} \mathbb{C}_{\bar\rho}) \to (\mathcal{H}, J) \otimes_{\mathbb{C}} \mathbb{C}_{\bar\rho}$$

is nontrivial. Consequently, (9.5) must have at least one simple summand $(\mathcal{H}_U, K)$ in the category of polarizable complex variations of Hodge structure of weight $w - \dim X$ for which the induced morphism $q^{-1}(\mathcal{H}_U, K) \to (\mathcal{H}, J) \otimes_{\mathbb{C}} \mathbb{C}_{\bar\rho}$ is nontrivial. Both sides being simple, the morphism is an isomorphism; consequently,

$$q^{-1}(\mathcal{H}_U, K) \otimes_{\mathbb{C}} \mathbb{C}_\rho \simeq (\mathcal{H}, J). \tag{9.6}$$

Now let $(N, K) \in \mathrm{HM}_{\mathbb{C}}(A, w - \dim q)$ be the polarizable complex Hodge module on A corresponding to $(\mathcal{H}_U, K)$; by construction, (N, K) is simple with strict support Y. Arguing as in [34, Lemma 20.2], one proves that the naive pullback $q^{-1}(N, K) \in \mathrm{HM}_{\mathbb{C}}(T, w)$ is simple with strict support X. By (9.6), this means that (M, J) is isomorphic to $q^{-1}(N, K) \otimes_{\mathbb{C}} \mathbb{C}_\rho$ in the category $\mathrm{HM}_{\mathbb{C}}(T, w)$. □

We have thus proved Theorem 9.1, except for the inequality $\chi(A, N, K) > 0$. Let $\mathcal{N}$ denote the regular holonomic $\mathscr{D}$–module underlying N; then

$$\mathcal{N} = \mathcal{N}' \oplus \mathcal{N}'' = \ker(K - i \cdot \mathrm{id}) \oplus \ker(K + i \cdot \mathrm{id}),$$

where $K \in \mathrm{End}(\mathcal{N})$ refers to the induced endomorphism. By Proposition 3.8, both $\mathcal{N}'$ and $\mathcal{N}''$ are simple with strict support Y. Since A is an abelian variety, one has, for example by [34, Section 5], that

$$\chi(A, N, K) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(A, \mathrm{DR}(\mathcal{N}')) \geq 0.$$

Now the point is that a simple holonomic $\mathscr{D}$–module with vanishing Euler characteristic is always (up to a twist by a line bundle with flat connection) the pullback from a lower-dimensional abelian variety [34, Section 20].

Proof of Theorem 9.1 Keeping the notation from Proposition 9.3, we have a surjective morphism $q\colon T \to A$ with connected fibers, a simple polarizable complex Hodge module $(N, K) \in \mathrm{HM}_{\mathbb{C}}(Y, w - \dim q)$ with strict support $Y = q(X)$, and a unitary character $\rho \in \mathrm{Char}(T)$ such that

$$(M, J) \simeq q^{-1}(N, K) \otimes_{\mathbb{C}} \mathbb{C}_\rho.$$

If (N, K) has positive Euler characteristic, we are done, so let us assume from now on that $\chi(A, N, K) = 0$. This means that $\mathcal{N}'$ is a simple regular holonomic $\mathscr{D}$–module with strict support Y and Euler characteristic zero.

By [34, Corollary 5.2], there is a surjective morphism $f\colon A \to B$ with connected fibers from A to a lower-dimensional abelian variety B, such that $\mathcal{N}'$ is (up to a twist by a line bundle with flat connection) the pullback of a simple regular holonomic $\mathscr{D}$–module with positive Euler characteristic. Setting

$$\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}'' = \ker(J - i \cdot \mathrm{id}) \oplus \ker(J + i \cdot \mathrm{id}),$$

it follows that $\mathcal{M}'$ is (again up to a twist by a line bundle with flat connection) the pullback by $f \circ q$ of a simple regular holonomic $\mathscr{D}$–module on B. Consequently, there is a dense Zariski-open subset $U \subseteq f(Y)$ such that the restriction of $\mathcal{M}'$ to $(f \circ q)^{-1}(U)$ is coherent as an $\mathscr{O}$–module. By Lemma 4.3, the restriction of (M, J) to this open set is therefore a polarizable complex variation of Hodge structure of weight $w - \dim X$. After replacing our original morphism $q\colon T \to A$ by the composition $f \circ q\colon T \to B$, we can argue as in the proof of Proposition 9.3 to show that (9.2) is still satisfied (for a different choice of $\rho \in \mathrm{Char}(T)$, perhaps).

With some additional work, one can prove that now $\chi(A, N, K) > 0$. Alternatively, the same result can be obtained by the following more indirect method: as long as $\chi(A, N, K) = 0$, we can repeat the argument above; since the dimension of A goes down each time, we must eventually get to the point where $\chi(A, N, K) > 0$. This completes the proof of Theorem 9.1. $\square$

10 Proof of the main result

As in Theorem 7.1, let $(M, J) \in \mathrm{HM}_{\mathbb{C}}(T, w)$ be a polarizable complex Hodge module on a compact complex torus T. Using the decomposition by strict support, we can assume without loss of generality that (M, J) has strict support equal to an irreducible analytic subvariety $X \subseteq T$. After translation, we may assume moreover that $0 \in X$. Let $T' \subseteq T$ be the smallest subtorus of T containing X; by Kashiwara's equivalence, we have $(M, J) = i_*(M', J')$ for some $(M', J') \in \mathrm{HM}_{\mathbb{C}}(T', w)$, where $i\colon T' \hookrightarrow T$ is the inclusion. Now Theorem 9.1 gives us a morphism $q'\colon T' \to A'$ such that (M', J') is isomorphic to the direct sum of pullbacks of polarizable complex Hodge modules twisted by unitary local systems. Since $i^{-1}\colon \mathrm{Char}(T) \to \mathrm{Char}(T')$ is surjective, the same is then true for (M, J) with respect to the quotient mapping $q\colon T \to T/\ker q'$. This proves Theorem 7.1.

Proof of Corollary 7.3 By considering the complexification

$$(M \oplus M, J_M) \in \mathrm{HM}_{\mathbb{C}}(T, w),$$

we reduce the problem to the situation of Theorem 7.1. It remains to show that all the characters in (7.2) have finite order in $\mathrm{Char}(T)$ if M admits an integral structure. By Lemma 5.2, every summand in the decomposition of M by strict support still admits an integral structure, and so we may assume without loss of generality that M has strict support equal to $X \subseteq T$ and that $0 \in X$. As before, we have $(M, J) = i_*(M', J')$, where $i\colon T' \hookrightarrow T$ is the smallest subtorus of T containing X; it is easy to see that M' again admits an integral structure. Now we apply the same argument as in the proof of Theorem 7.1 to the finitely many simple factors of (M, J), noting that the characters $\rho \in \mathrm{Char}(T)$ that come up always have finite order by Lemma 11.1 below. □

Note As in the proof of Lemma 6.1, it follows that $M \oplus M$ is isomorphic to the direct sum of the polarizable real Hodge modules

$$\ker(q_j^{-1} J_j \otimes J_{\rho_j} + \mathrm{id}) \subseteq q_j^{-1} N_j \otimes_{\mathbb{R}} \mathcal{H}_{\rho_j}. \tag{10.1}$$

Furthermore, one can show that, for each $j = 1, \dots, n$, exactly one of two things happens:

(1) Either the object in (10.1) is simple, and therefore occurs among the simple factors of M; in this case, the underlying regular holonomic $\mathscr{D}$–module $\mathcal{M}$ will contain the two simple factors

$$(q_j^* \mathcal{N}_j' \otimes_{\mathscr{O}_T} (L_j, \nabla_j)) \oplus (q_j^* \mathcal{N}_j'' \otimes_{\mathscr{O}_T} (L_j, \nabla_j)^{-1}).$$

(2) Or the object in (10.1) splits into two copies of a simple polarizable real Hodge module, which also has to occur among the simple factors of M. In this case, one can actually arrange that (N_j, J_j) is real and that the character ρ_j takes values in $\{-1, +1\}$. The simple object in question is the twist of (N_j, J_j) by the polarizable real variation of Hodge structure of rank one determined by ρ_j; moreover, $\mathcal{M}$ will contain $q_j^* \mathcal{N}_j' \otimes_{\mathscr{O}_T} (L_j, \nabla_j) \simeq q_j^* \mathcal{N}_j'' \otimes_{\mathscr{O}_T} (L_j, \nabla_j)^{-1}$ as a simple factor.

11 A lemma about variations of Hodge structure

The fundamental group of a compact complex torus is abelian, and so every polarizable complex variation of Hodge structure is a direct sum of unitary local systems of rank one; this is the content of the following elementary lemma [35, Lemma 1.8]:

Lemma 11.1 *Let $(\mathcal{H}, J)$ be a polarizable complex variation of Hodge structure on a compact complex torus T. Then the local system $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to a direct sum of unitary local systems of rank one. If $\mathcal{H}$ admits an integral structure, then each of these local systems of rank one has finite order.*

Proof According to [11, Section 1.12], the underlying local system of a polarizable complex variation of Hodge structure on a compact Kähler manifold is semisimple; in the case of a compact complex torus, it is therefore a direct sum of rank-one local systems. The existence of a polarization implies that the individual local systems are unitary [11, Proposition 1.13]. Now suppose that $\mathcal{H}$ admits an integral structure, and let $\mu\colon \pi_1(A, 0) \to \mathrm{GL}_n(\mathbb{Z})$ be the monodromy representation. We already know that the complexification of μ is a direct sum of unitary characters. Since μ is defined over $\mathbb{Z}$, the values of each character are algebraic integers of absolute value one; by Kronecker's theorem, they must be roots of unity. □

12 Integral structure and points of finite order

One can combine the decomposition in Corollary 7.3 with known results about Hodge modules on abelian varieties [35] to prove the following generalization of Wang's theorem:

Corollary 12.1 *If $M \in \mathrm{HM}_{\mathbb{R}}(T, w)$ admits an integral structure, then the sets*

$$S^i_m(T, M) = \{\rho \in \mathrm{Char}(T) \mid \dim H^i(T, M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}_\rho) \geq m\}$$

are finite unions of translates of linear subvarieties by points of finite order.

Proof The result in question is known for abelian varieties: if $M \in \mathrm{HM}_{\mathbb{R}}(A, w)$ is a polarizable real Hodge module on an abelian variety, and if M admits an integral structure, then the sets $S^i_m(A, M)$ are finite unions of "arithmetic subvarieties" (namely translates of linear subvarieties by points of finite order). This is proved in [35, Theorem 1.4] for polarizable rational Hodge modules, but the proof carries over unchanged to the case of real coefficients. The same argument shows more generally that if the underlying perverse sheaf $M_{\mathbb{C}}$ of a polarizable real Hodge module $M \in \mathrm{HM}_{\mathbb{R}}(A, w)$ is isomorphic to a direct factor in the complexification of some $E \in \mathrm{D}^b_c(\mathbb{Z}_A)$, then each $S^i_m(A, M)$ is a finite union of arithmetic subvarieties.

Now let us see how to extend this result to compact complex tori. Passing to the underlying complex perverse sheaves in Corollary 7.3, we get

$$M_{\mathbb{C}} \simeq \bigoplus_{j=1}^{n} (q_j^{-1} N_{j,\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}_{\rho_j});$$

recall that $\operatorname{Supp} N_j$ is a projective subvariety of the complex torus T_j, and that $\rho_j \in \operatorname{Char}(T)$ has finite order. In light of this decomposition and the comments above, it is therefore enough to prove that each $N_{j,\mathbb{C}}$ is isomorphic to a direct factor in the complexification of some object of $\mathrm{D}_c^b(\mathbb{Z}_{T_j})$.

Let $E \in \mathrm{D}_c^b(\mathbb{Z}_T)$ be some choice of integral structure on the real Hodge module M; obviously $M_{\mathbb{C}} \simeq E \otimes_{\mathbb{Z}} \mathbb{C}$. Let $r \geq 1$ be the order of the point $\rho_j \in \operatorname{Char}(T)$, and denote by $[r]\colon T \to T$ the finite morphism given by multiplication by r. We define

$$E' = \boldsymbol{R}[r]_*([r]^{-1} E) \in \mathrm{D}_c^b(\mathbb{Z}_T)$$

and observe that the complexification of E' is isomorphic to the direct sum of $E \otimes_{\mathbb{Z}} \mathbb{C}_\rho$, where $\rho \in \operatorname{Char}(T)$ runs over the finite set of characters whose order divides r. This set includes ρ_j^{-1}, and so $q_j^{-1} N_{j,\mathbb{C}}$ is isomorphic to a direct factor of $E' \otimes_{\mathbb{Z}} \mathbb{C}$. Because $q_j\colon T \to T_j$ has connected fibers, this implies that

$$N_{j,\mathbb{C}} \simeq \mathcal{H}^{-\dim q_j} q_{j*}(q_j^{-1} N_{j,\mathbb{C}})$$

is isomorphic to a direct factor of

$$\mathcal{H}^{-\dim q_j} q_{j*}(E' \otimes_{\mathbb{Z}} \mathbb{C}).$$

As explained in [35, Section 1.2.2], this is again the complexification of a constructible complex in $\mathrm{D}_c^b(\mathbb{Z}_{T_j})$, and so the proof is complete. □

D Generic vanishing theory

Let X be a compact Kähler manifold, and let $f\colon X \to T$ be a holomorphic mapping to a compact complex torus. The main purpose of this chapter is to show that the higher direct image sheaves $R^j f_* \omega_X$ have the same properties as in the projective case (such as being GV-sheaves). As explained in the introduction, we do not know how to obtain this using classical Hodge theory; this forces us to prove a more general result for arbitrary polarizable complex Hodge modules.

13 GV-sheaves and M–regular sheaves

We begin by reviewing a few basic definitions. Let T be a compact complex torus, $\hat{T} = \operatorname{Pic}^0(T)$ its dual, and P the normalized Poincaré bundle on the product $T \times \hat{T}$. It induces an integral transform

$$\boldsymbol{R}\Phi_P\colon \mathrm{D}^b_{\mathrm{coh}}(\mathscr{O}_T) \to \mathrm{D}^b_{\mathrm{coh}}(\mathscr{O}_{\hat{T}}), \quad \boldsymbol{R}\Phi_P(\mathscr{F}) = \boldsymbol{R} p_{2*}(p_1^* \mathscr{F} \otimes P),$$

where $\mathrm{D}^b_{\mathrm{coh}}(\mathscr{O}_T)$ is the derived category of cohomologically bounded and coherent complexes of $\mathscr{O}_T$–modules. Likewise, we have $\boldsymbol{R}\Psi_P\colon \mathrm{D}^b_{\mathrm{coh}}(\mathscr{O}_{\hat{T}}) \to \mathrm{D}^b_{\mathrm{coh}}(\mathscr{O}_T)$ going in the opposite direction. An argument analogous to Mukai's for abelian varieties shows that the Fourier–Mukai equivalence holds in this case as well [3, Theorem 2.1].

Theorem 13.1 *With the notations above, $\boldsymbol{R}\Phi_{\boldsymbol{P}}$ and $\boldsymbol{R}\Psi_{\boldsymbol{P}}$ are equivalences of derived categories. More precisely, one has*

$$\boldsymbol{R}\Psi_{\boldsymbol{P}} \circ \boldsymbol{R}\Phi_{\boldsymbol{P}} \simeq (-1)_T^*[-\dim T] \quad \textit{and} \quad \boldsymbol{R}\Phi_{\boldsymbol{P}} \circ \boldsymbol{R}\Psi_{\boldsymbol{P}} \simeq (-1)_{\hat{T}}^*[-\dim T].$$

Given a coherent $\mathscr{O}_T$–module $\mathscr{F}$ and an integer $m \geq 1$, we define

$$S^i_m(T, \mathscr{F}) = \{L \in \mathrm{Pic}^0(T) \mid \dim\ H^i(T, \mathscr{F} \otimes_{\mathscr{O}_T} L) \geq m\}.$$

It is customary to denote

$$S^i(T, \mathscr{F}) = S^i_1(T, \mathscr{F}) = \{L \in \mathrm{Pic}^0(T) \mid H^i(T, \mathscr{F} \otimes_{\mathscr{O}_T} L) \neq 0\}.$$

Recall the following definitions, from [23] and [21], respectively.

Definition 13.2 A coherent $\mathscr{O}_T$–module $\mathscr{F}$ is called a *GV-sheaf* if the inequality

$$\mathrm{codim}_{\mathrm{Pic}^0(T)} S^i(T, \mathscr{F}) \geq i$$

is satisfied for every integer $i \geq 0$. It is called *M–regular* if the inequality

$$\mathrm{codim}_{\mathrm{Pic}^0(T)} S^i(T, \mathscr{F}) \geq i + 1$$

is satisfied for every integer $i \geq 1$.

A number of local properties of integral transforms for complex manifolds, based only on commutative algebra results, were proved in [22; 25]. For instance, the following is a special case of [22, Theorem 2.2]:

Theorem 13.3 *Let $\mathscr{F}$ be a coherent sheaf on a compact complex torus T. Then the following statements are equivalent:*

(i) *$\mathscr{F}$ is a GV-sheaf.*

(ii) *$R^i\Phi_{\boldsymbol{P}}(\boldsymbol{R}\Delta\mathscr{F}) = 0$ for $i \neq \dim T$, where $\boldsymbol{R}\Delta\mathscr{F} := \boldsymbol{R}\mathcal{H}om(\mathscr{F}, \mathscr{O}_T)$.*

Note that this statement was inspired by work of Hacon [15] in the projective setting. In the course of the proof of Theorem 13.3, and also for some of the results below, the following consequence of Grothendieck duality for compact complex manifolds is needed:

$$\boldsymbol{R}\Phi_{\boldsymbol{P}}(\mathscr{F}) \simeq \boldsymbol{R}\Delta(\boldsymbol{R}\Phi_{\boldsymbol{P}^{-1}}(\boldsymbol{R}\Delta\mathscr{F})[\dim T]); \tag{13.4}$$

see the proof of [22, Theorem 2.2], and especially the references there. In particular, if $\mathscr{F}$ is a GV-sheaf, then if we let $\widehat{\mathscr{F}} := R^{\dim T}\Phi_{\boldsymbol{P}^{-1}}(\boldsymbol{R}\Delta\mathscr{F})$, Theorem 13.3 and (13.4) imply that

$$R\Phi_P(\mathscr{F}) \simeq R\mathcal{H}om(\widehat{\mathscr{F}}, \mathscr{O}_{\widehat{A}}). \tag{13.5}$$

As in [24, Proposition 2.8], $\mathscr{F}$ is M–regular if and only if $\widehat{\mathscr{F}}$ is torsion-free.

The fact that Theorems 13.1 and 13.3 and (13.5) hold for arbitrary compact complex tori allows us to deduce important properties of GV-sheaves in this setting. Besides these statements, the proofs only rely on local commutative algebra and base change, and so are completely analogous to those for abelian varieties; we will thus only indicate references for that case.

Proposition 13.6 *Let $\mathscr{F}$ be a GV-sheaf on T.*

(a) *One has $S^{\dim T}(T,\mathscr{F}) \subseteq \cdots \subseteq S^1(T,\mathscr{F}) \subseteq S^0(T,\mathscr{F}) \subseteq \hat{T}$.*

(b) *If $S^0(T,\mathscr{F})$ is empty, then $\mathscr{F} = 0$.*

(c) *If an irreducible component $Z \subseteq S^0(T,\mathscr{F})$ has codimension k in $\operatorname{Pic}^0(X)$, then $Z \subseteq S^k(T,\mathscr{F})$, and hence $\dim \operatorname{Supp} \mathscr{F} \geq k$.*

Proof For (a), see [23, Proposition 3.14]; for (b), see [20, Lemma 1.12]; for (c), see [20, Lemma 1.8]. □

14 Higher direct images of dualizing sheaves

Saito [29] and Takegoshi [38] have extended to Kähler manifolds many of the fundamental theorems on higher direct images of canonical bundles proved by Kollár for smooth projective varieties. The following theorem summarizes some of the results in [38, pages 390–391] in the special case that is needed for our purposes.

Theorem 14.1 (Takegoshi) *Let $f\colon X \to Y$ be a proper holomorphic mapping from a compact Kähler manifold to a reduced and irreducible analytic space, and let $L \in \operatorname{Pic}^0(X)$ be a holomorphic line bundle with trivial first Chern class.*

(a) *The Leray spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_*(\omega_X \otimes L)) \Rightarrow H^{p+q}(X, \omega_X \otimes L)$$

degenerates at E_2.

(b) *If f is surjective, then $R^q f_*(\omega_X \otimes L)$ is torsion-free for every $q \geq 0$; in particular, it vanishes for $q > \dim X - \dim Y$.*

Saito [29] obtained the same results in much greater generality, using the theory of Hodge modules. In fact, his method also gives the splitting of the complex $R f_*\omega_X$

in the derived category, thus extending the main result of [17] to all compact Kähler manifolds.

Theorem 14.2 (Saito) *Keeping the assumptions of the previous theorem, one has*

$$\boldsymbol{R} f_* \omega_X \simeq \bigoplus_j (R^j f_* \omega_X)[-j]$$

in the derived category $\mathrm{D}^b_{\mathrm{coh}}(\mathscr{O}_Y)$.

Proof Given [29], the proof in [31] goes through under the assumption that X is a compact Kähler manifold. □

15 Euler characteristic and M–regularity

In this section, we relate the Euler characteristic of a simple polarizable complex Hodge module on a compact complex torus T to the M–regularity of the associated graded object.

Lemma 15.1 *Let* $(M, J) \in \mathrm{HM}_{\mathbb{C}}(T, w)$ *be a simple polarizable complex Hodge module on a compact complex torus. If* $\operatorname{Supp} M$ *is projective and* $\chi(T, M, J) > 0$, *then the coherent* $\mathscr{O}_T$*–module* $\mathrm{gr}_k^F \mathcal{M}'$ *is* M*–regular for every* $k \in \mathbb{Z}$.

Proof $\operatorname{Supp} M$ is projective, hence contained in a translate of an abelian subvariety $A \subseteq T$; because Lemma 1.4 holds for polarizable complex Hodge modules, we may therefore assume without loss of generality that $T = A$ is an abelian variety.

As usual, let $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}'' = \ker(J - i \cdot \mathrm{id}) \oplus \ker(J + i \cdot \mathrm{id})$ be the decomposition into eigenspaces. The summand $\mathcal{M}'$ is a simple holonomic $\mathscr{D}$–module with positive Euler characteristic on an abelian variety, and so [34, Theorem 2.2 and Corollary 20.5] show that

$$\{\rho \in \mathrm{Char}(A) \mid H^i(A, \mathrm{DR}(\mathcal{M}') \otimes_{\mathbb{C}} \mathbb{C}_\rho) \neq 0\} \tag{15.2}$$

is equal to $\mathrm{Char}(A)$ when $i = 0$, and is equal to a finite union of translates of linear subvarieties of codimension $\geq 2i + 2$ when $i \geq 1$.

We have a one-to-one correspondence between $\mathrm{Pic}^0(A)$ and the subgroup of unitary characters in $\mathrm{Char}(A)$; it takes a unitary character $\rho \in \mathrm{Char}(A)$ to the holomorphic line bundle $L_\rho = \mathbb{C}_\rho \otimes_{\mathbb{C}} \mathscr{O}_A$. If $\rho \in \mathrm{Char}(A)$ is unitary, the twist $(M, J) \otimes_{\mathbb{C}} \mathbb{C}_\rho$ is still a polarizable complex Hodge module by Lemma 6.1, and so the complex computing its hypercohomology is strict. It follows that

$$H^i(A, \mathrm{gr}_k^F \mathrm{DR}(\mathcal{M}') \otimes_{\mathscr{O}_A} L_\rho) \quad \text{is a subquotient of} \quad H^i(A, \mathrm{DR}(\mathcal{M}') \otimes_{\mathbb{C}} \mathbb{C}_\rho).$$

If we identify $\operatorname{Pic}^0(A)$ with the subgroup of unitary characters, this means that

$$\{L \in \operatorname{Pic}^0(A) \mid H^i(A, \operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}') \otimes_{\mathscr{O}_A} L) \neq 0\}$$

is contained in the intersection of (15.2) and the subgroup of unitary characters. When $i \geq 1$, this intersection is a finite union of translates of subtori of codimension $\geq i+1$; it follows that

$$\operatorname{codim}_{\operatorname{Pic}^0(A)}\{L \in \operatorname{Pic}^0(A) \mid H^i(A, \operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}') \otimes_{\mathscr{O}_A} L) \neq 0\} \geq i+1.$$

Since the cotangent bundle of A is trivial, a simple induction on k as in the proof of [26, Lemma 1] gives

$$\operatorname{codim}_{\operatorname{Pic}^0(A)}\{L \in \operatorname{Pic}^0(A) \mid H^i(A, \operatorname{gr}_k^F \mathcal{M}' \otimes_{\mathscr{O}_A} L) \neq 0\} \geq i+1,$$

and so each $\operatorname{gr}_k^F \mathcal{M}'$ is indeed M–regular. □

Note In fact, the result still holds without the assumption that $\operatorname{Supp} M$ is projective; this is an easy consequence of the decomposition in (7.2).

16 Chen–Jiang decomposition and generic vanishing

Using the decomposition in Theorem 7.1 and the result of the previous section, we can now prove the most general version of the generic vanishing theorem, namely Theorem D in the introduction.

Proof of Theorem D We apply Theorem 7.1 to the complexification $(M \oplus M, J_M)$ in $\operatorname{HM}_{\mathbb{C}}(T, w)$. Passing to the associated graded in (7.2), we obtain a decomposition of the desired type with $\mathscr{F}_j = \operatorname{gr}_k^F \mathcal{N}_j'$ and $L_j = \mathbb{C}_{\rho_j} \otimes_{\mathbb{C}} \mathscr{O}_T$, where

$$\mathcal{N}_j = \mathcal{N}_j' \oplus \mathcal{N}_j'' = \ker(J_j - i \cdot \mathrm{id}) \oplus \ker(J_j + i \cdot \mathrm{id})$$

is as usual the decomposition into eigenspaces of $J_j \in \operatorname{End}(\mathcal{N}_j)$. Since $\operatorname{Supp} N_j$ is projective and $\chi(T_j, N_j, J_j) > 0$, we conclude from Lemma 15.1 that each coherent $\mathscr{O}_{T_j}$–module $\mathscr{F}_j$ is M–regular. □

Corollary 16.1 *If $M = (\mathcal{M}, F_\bullet\mathcal{M}, M_{\mathbb{R}}) \in \operatorname{HM}_{\mathbb{R}}(T, w)$, then for every $k \in \mathbb{Z}$ the coherent $\mathscr{O}_T$–module $\operatorname{gr}_k^F \mathcal{M}$ is a GV-sheaf.*

Proof This follows immediately from Theorem D and the fact that, if $p\colon T \to T_0$ is a surjective homomorphism of complex tori and $\mathscr{G}$ is a GV-sheaf on T_0, then $\mathscr{F} = f^*\mathscr{G}$ is a GV-sheaf on T. For this last statement and more refined facts (for instance when $\mathscr{G}$ is M–regular), see eg [9, Section 2], especially Proposition 2.6. The arguments in [9] are for abelian varieties, but given the remarks in Section 13, they work equally well on compact complex tori. □

By specializing to the direct image of the canonical Hodge module $\mathbb{R}_X[\dim X]$ along a morphism $f\colon X \to T$, we are finally able to conclude that each $R^j f_* \omega_X$ is a GV-sheaf. In fact, we have the more refined Theorem A; it was first proved for smooth projective varieties of maximal Albanese dimension by Chen and Jiang [9, Theorem 1.2], which was a source of inspiration for us.

Proof of Theorem A Denote by $\mathbb{R}_X[\dim X] \in \mathrm{HM}_{\mathbb{R}}(X, \dim X)$ the polarizable real Hodge module corresponding to the constant real variation of Hodge structure of rank one and weight zero on X. According to [29, Theorem 3.1], each $\mathcal{H}^j f_* \mathbb{R}_X[\dim X]$ is a polarizable real Hodge module of weight $\dim X + j$ on T; it also admits an integral structure [35, Section 1.2.2]. In the decomposition by strict support, let M be the summand with strict support $f(X)$; note that M still admits an integral structure by Lemma 5.2. Now $R^j f_* \omega_X$ is the first nontrivial piece of the Hodge filtration on the underlying regular holonomic $\mathscr{D}$–module [31], and so the result follows directly from Theorem D and Corollary 16.1. For the ampleness, see Corollary 20.1. □

Note Except for the assertion about finite order, Theorem A still holds for arbitrary coherent $\mathscr{O}_T$–modules of the form

$$R^j f_*(\omega_X \otimes L)$$

with $L \in \mathrm{Pic}^0(X)$. The point is that every such L is the holomorphic line bundle associated with a unitary character $\rho \in \mathrm{Char}(X)$; we can therefore apply the same argument as above to the polarizable complex Hodge module $\mathbb{C}_\rho[\dim X]$.

If the given morphism is generically finite over its image, we can say more:

Corollary 16.2 *If $f\colon X \to T$ is generically finite over its image, then $S^0(T, f_*\omega_X)$ is preserved by the involution $L \mapsto L^{-1}$ of $\mathrm{Pic}^0(T)$.*

Proof As before, we define $M = \mathcal{H}^0 f_* \mathbb{R}_X[\dim X] \in \mathrm{HM}_{\mathbb{R}}(T, \dim X)$. Recall from Corollary 7.3 that we have a decomposition

$$(M \oplus M, J_M) \simeq \bigoplus_{j=1}^{n} (q_j^{-1}(N_j, J_j) \otimes_{\mathbb{C}} \mathbb{C}_{\rho_j}).$$

Since f is generically finite over its image, there is a dense Zariski-open subset of $f(X)$ where M is a variation of Hodge structure of type $(0,0)$; the above decomposition shows that the same is true for N_j on $(q_j \circ f)(X)$. If we pass to the underlying regular holonomic $\mathscr{D}$–modules and remember Lemma 6.1, we see that

$$\mathcal{M} \oplus \mathcal{M} \simeq \bigoplus_{j=1}^{n} (q_j^* \mathcal{N}_j' \otimes_{\mathscr{O}_T} (L_j, \nabla_j)) \oplus \bigoplus_{j=1}^{n} (q_j^* \mathcal{N}_j'' \otimes_{\mathscr{O}_T} (L_j, \nabla_j)^{-1}),$$

where (L_j, ∇_j) is the flat bundle corresponding to the character ρ_j. By looking at the first nontrivial step in the Hodge filtration on $\mathcal{M}$, we then get

$$f_*\omega_X \oplus f_*\omega_X \simeq \bigoplus_{j=1}^{n}(q_j^*\mathscr{F}_j' \otimes_{\mathscr{O}_T} L_j) \oplus \bigoplus_{j=1}^{n}(q_j^*\mathscr{F}_j'' \otimes_{\mathscr{O}_T} L_j^{-1}),$$

where $\mathscr{F}_j' = F_{p(M)}\mathcal{N}_j'$ and $\mathscr{F}_j'' = F_{p(M)}\mathcal{N}_j''$, and $p(M)$ is the smallest integer with the property that $F_p\mathcal{M} \neq 0$. Both sheaves are torsion-free on $(q_j \circ f)(X)$, and can therefore be nonzero only when $\operatorname{Supp} N_j = (q_j \circ f)(X)$; after reindexing, we may assume that this holds exactly in the range $1 \leq j \leq m$.

Now we reach the crucial point of the argument: the fact that N_j is generically a polarizable real variation of Hodge structure of type $(0,0)$ implies that $\mathscr{F}_j'$ and $\mathscr{F}_j''$ have the same rank at the generic point of $(q_j \circ f)(X)$. Indeed, on a dense Zariski-open subset of $(q_j \circ f)(X)$, we have $\mathscr{F}_j' = \mathcal{N}_j'$ and $\mathscr{F}_j'' = \mathcal{N}_j''$, and complex conjugation with respect to the real structure on N_j interchanges the two factors.

Since $\mathscr{F}_j'$ and $\mathscr{F}_j''$ are M–regular by Lemma 15.1, we have (for $1 \leq j \leq m$)

$$S^0(T, q_j^*\mathscr{F}_j' \otimes_{\mathscr{O}_T} L_j) = L_j^{-1} \otimes S^0(T_j, \mathscr{F}_j') = L_j^{-1} \otimes \operatorname{Pic}^0(T_j),$$

and similarly for $q_j^*\mathscr{F}_j'' \otimes_{\mathscr{O}_T} L_j^{-1}$; to simplify the notation, we identify $\operatorname{Pic}^0(T_j)$ with its image in $\operatorname{Pic}^0(T)$. The decomposition from above now gives

$$S^0(T, f_*\omega_X) = \bigcup_{j=1}^{m}(L_j^{-1} \otimes \operatorname{Pic}^0(T_j)) \cup \bigcup_{j=1}^{m}(L_j \otimes \operatorname{Pic}^0(T_j)),$$

and the right-hand side is clearly preserved by the involution $L \mapsto L^{-1}$. □

17 Points of finite order on cohomology support loci

Let $f\colon X \to T$ be a holomorphic mapping from a compact Kähler manifold to a compact complex torus. Our goal in this section is to prove that the cohomology support loci of the coherent $\mathscr{O}_T$–modules $R^j f_*\omega_X$ are finite unions of translates of subtori by points of finite order. We consider the refined cohomology support loci

$$S^i_m(T, R^j f_*\omega_X) = \{L \in \operatorname{Pic}^0(T) \mid \dim H^i(T, R^j f_*\omega_X \otimes L) \geq m\} \subseteq \operatorname{Pic}^0(T).$$

The following result is well-known in the projective case:

Corollary 17.1 *Every irreducible component of $S^i_m(T, R^j f_*\omega_X)$ is a translate of a subtorus of $\operatorname{Pic}^0(T)$ by a point of finite order.*

Proof As in the proof of Theorem A (in Section 16), we let $M \in \mathrm{HM}_{\mathbb{R}}(T, \dim X + j)$ be the summand with strict support $f(X)$ in the decomposition by strict support of $\mathcal{H}^j f_* \mathbb{R}_X[\dim X]$; then M admits an integral structure, and

$$R^j f_* \omega_X \simeq F_{p(M)} \mathcal{M},$$

where $p(M)$ again means the smallest integer such that $F_p \mathcal{M} \neq 0$. Since M still admits an integral structure by Lemma 5.2, the result in Corollary 12.1 shows that the sets

$$S^i_m(T, M) = \{\rho \in \mathrm{Char}(T) \mid \dim H^i(T, M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}_\rho) \geq m\}$$

are finite unions of translates of linear subvarieties by points of finite order. As in the proof of Lemma 15.1, the strictness of the complex computing the hypercohomology of $(M \oplus M, J_M) \otimes_{\mathbb{C}} \mathbb{C}_\rho$ implies that

$$\dim H^i(T, M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}_\rho) = \sum_{p \in \mathbb{Z}} \dim H^i(T, \mathrm{gr}^F_p \mathrm{DR}(\mathcal{M}) \otimes_{\mathscr{O}_T} L_\rho)$$

for every unitary character $\rho \in \mathrm{Char}(T)$; here $L_\rho = \mathbb{C}_\rho \otimes_{\mathbb{C}} \mathscr{O}_T$. Note that $\mathrm{gr}^F_p \mathrm{DR}(\mathcal{M})$ is acyclic for $p \gg 0$, and so the sum on the right-hand side is actually finite. Intersecting $S^i_m(T, M)$ with the subgroup of unitary characters, we see that each set

$$\left\{ L \in \mathrm{Pic}^0(T) \;\middle|\; \sum_{p \in \mathbb{Z}} \dim H^i(T, \mathrm{gr}^F_p \mathrm{DR}(\mathcal{M}) \otimes_{\mathscr{O}_T} L) \geq m \right\}$$

is a finite union of translates of subtori by points of finite order. By a standard argument [1, page 312], it follows that the same is true for each of the summands; in other words, for each $p \in \mathbb{Z}$, the set

$$S^i_m(T, \mathrm{gr}^F_p \mathrm{DR}(\mathcal{M})) \subseteq \mathrm{Pic}^0(T)$$

is itself a finite union of translates of subtori by points of finite order. Since

$$\mathrm{gr}^F_{p(M)} \mathrm{DR}(\mathcal{M}) = \omega_T \otimes F_{p(M)} \mathcal{M} \simeq R^j f_* \omega_X,$$

we now obtain the assertion by specializing to $p = p(M)$. □

Note Alternatively, one can deduce Corollary 17.1 from Wang's theorem [41] about cohomology jump loci on compact Kähler manifolds, as follows. Wang shows that the sets $S^{p,q}_m(X) = \{L \in \mathrm{Pic}^0(X) \mid \dim H^q(X, \Omega^p_X \otimes L) \geq m\}$ are finite unions of translates of subtori by points of finite order; in particular, this is true for $\omega_X = \Omega^{\dim X}_X$. Takegoshi's results about higher direct images of ω_X in Theorem 14.1 imply the E_2–degeneration of the spectral sequence

$$E_2^{i,j} = H^i(T, R^j f_* \omega_X \otimes L) \Rightarrow H^{i+j}(X, \omega_X \otimes f^* L)$$

for every $L \in \mathrm{Pic}^0(T)$, which means that

$$\dim H^q(X, \omega_X \otimes f^*L) = \sum_{k+j=q} \dim H^k(T, R^j f_*\omega_X \otimes L).$$

The assertion now follows from Wang's theorem by the same argument as above.

E Applications

18 Bimeromorphic characterization of tori

Our main application of generic vanishing for higher direct images of dualizing sheaves is an extension of the Chen–Hacon birational characterization of abelian varieties [6] to the Kähler case.

Theorem 18.1 *Let X be a compact Kähler manifold with $P_1(X) = P_2(X) = 1$ and $h^{1,0}(X) = \dim X$. Then X is bimeromorphic to a compact complex torus.*

Throughout this section, we take X to be a compact Kähler manifold, and denote by $f\colon X \to T$ its Albanese mapping; by assumption, we have

$$\dim T = h^{1,0}(X) = \dim X.$$

We use the following standard notation, analogous to that in Section 13:

$$S^i(X, \omega_X) = \{L \in \mathrm{Pic}^0(X) \mid H^i(X, \omega_X \otimes L) \neq 0\}$$

To simplify things, we shall identify $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(T)$ in what follows. We begin by recalling a few well-known results.

Lemma 18.2 *If $P_1(X) = P_2(X) = 1$, there cannot be any positive-dimensional analytic subvariety $Z \subseteq \mathrm{Pic}^0(X)$ such that both Z and Z^{-1} are contained in $S^0(X, \omega_X)$. In particular, the origin must be an isolated point in $S^0(X, \omega_X)$.*

Proof This result is due to Ein and Lazarsfeld [12, Proposition 2.1]; they state it only in the projective case, but their proof actually works without any changes on arbitrary compact Kähler manifolds. □

Lemma 18.3 *Assume that $S^0(X, \omega_X)$ contains isolated points. Then the Albanese map of X is surjective.*

Proof By Theorem A (for $j = 0$), $f_*\omega_X$ is a GV-sheaf. Proposition 13.6 shows that any isolated point in $S^0(T, f_*\omega_X) = S^0(X, \omega_X)$ also belongs to $S^{\dim T}(T, f_*\omega_X)$; but this is only possible if the support of $f_*\omega_X$ has dimension at least $\dim T$. □

To prove Theorem 18.1, we follow the general strategy introduced in [20, Section 4], which in turn is inspired by [12; 7]. The crucial new ingredient is of course Theorem A, which had only been known in the projective case. Even in the projective case however, the argument below is substantially cleaner than the existing proofs; this is due to Corollary 16.2.

Proof of Theorem 18.1 The Albanese map $f\colon X \to T$ is surjective by Lemmas 18.2 and 18.3; since $h^{1,0}(X) = \dim X$, this means that f is generically finite. To conclude the proof, we just have to argue that f has degree one; more precisely, we shall use Theorem A to show that $f_*\omega_X \simeq \mathscr{O}_T$.

As a first step in this direction, let us prove that $\dim S^0(T, f_*\omega_X) = 0$. If

$$S^0(T, f_*\omega_X) = S^0(X, \omega_X)$$

had an irreducible component Z of positive dimension, Corollary 16.2 would imply that Z^{-1} is contained in $S^0(X, \omega_X)$ as well. As this would contradict Lemma 18.2, we conclude that $S^0(T, f_*\omega_X)$ is zero-dimensional.

Now $f_*\omega_X$ is a GV-sheaf by Theorem A, and so Proposition 13.6 shows that

$$S^0(T, f_*\omega_X) = S^{\dim T}(T, f_*\omega_X).$$

Since f is generically finite, Theorem 14.1 implies that $R^j f_*\omega_X = 0$ for $j > 0$, which gives

$$S^{\dim T}(T, f_*\omega_X) = S^{\dim T}(X, \omega_X) = S^{\dim X}(X, \omega_X) = \{\mathscr{O}_T\}.$$

Putting everything together, we see that $S^0(T, f_*\omega_X) = \{\mathscr{O}_T\}$.

We can now use the Chen–Jiang decomposition for $f_*\omega_X$ to get more information. The decomposition in Theorem A (for $j = 0$) implies that

$$\{\mathscr{O}_T\} = S^0(T, f_*\omega_X) = \bigcup_{k=1}^{n} L_k^{-1} \otimes \operatorname{Pic}^0(T_k),$$

where we identify $\operatorname{Pic}^0(T_k)$ with its image in $\operatorname{Pic}^0(T)$. This equality forces $f_*\omega_X$ to be a trivial bundle of rank n; but then

$$n = \dim H^{\dim T}(T, f_*\omega_X) = \dim H^{\dim X}(X, \omega_X) = 1,$$

and so $f_*\omega_X \simeq \mathscr{O}_T$. The conclusion is that f is generically finite of degree one, and hence birational, as asserted by the theorem. □

19 Connectedness of the fibers of the Albanese map

As another application, one obtains the following analogue of an effective version of Kawamata's theorem on the connectedness of the fibers of the Albanese map, proved by Jiang [16, Theorem 3.1] in the projective setting. Note that the statement is more general than Theorem 18.1, but uses it in its proof.

Theorem 19.1 *Let X be a compact Kähler manifold with $P_1(X) = P_2(X) = 1$. Then the Albanese map of X is surjective, with connected fibers.*

Proof The proof goes entirely along the lines of [16]. We only indicate the necessary modifications in the Kähler case. We have already seen that the Albanese map $f\colon X \to T$ is surjective. Consider its Stein factorization:

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f"] & \\ Y \arrow[r, "h"] & T \end{tikzcd}$$

Up to passing to a resolution of singularities and allowing h to be generically finite, we can assume that Y is a compact complex manifold. Moreover, by [40, Théorème 3], after performing a further bimeromorphic modification, we can assume that Y is in fact compact Kähler. This does not change the hypothesis $P_1(X) = P_2(X) = 1$.

The goal is to show that Y is bimeromorphic to a torus, which is enough to conclude. If one could prove that $P_1(Y) = P_2(Y) = 1$, then Theorem 18.1 would do the job. In fact, one can show precisely as in [16, Theorem 3.1] that $H^0(X, \omega_{X/Y}) \neq 0$, and consequently that

$$P_m(Y) \leq P_m(X) \quad \text{for all } m \geq 1.$$

The proof of this statement needs the degeneration of the Leray spectral sequence for $g_*\omega_X$, which follows from Theorem 14.1, and the fact that $f_*\omega_X$ is a GV-sheaf, which follows from Theorem A. Besides this, the proof is purely Hodge-theoretic, and hence works equally well in the Kähler case. □

20 Semipositivity of higher direct images

In the projective case, GV-sheaves automatically come with positivity properties; more precisely, on abelian varieties it was proved in [10, Corollary 3.2] that M–regular sheaves are ample, and in [24, Theorem 4.1] that GV-sheaves are nef. Due to Theorem D a stronger result in fact holds true for arbitrary graded quotients of Hodge modules on compact complex tori.

Recall that to a coherent sheaf $\mathscr{F}$ on a compact complex manifold one can associate the analytic space $\mathbb{P}(\mathscr{F}) = \mathbb{P}(\mathrm{Sym}^\bullet \mathscr{F})$, with a natural mapping to X and a line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{F})}(1)$. If X is projective, the sheaf $\mathscr{F}$ is called *ample* if the line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{F})}(1)$ is ample on $\mathbb{P}(\mathscr{F})$.

Corollary 20.1 *Let $M = (\mathcal{M}, F_\bullet\mathcal{M}, M_\mathbb{R})$ be a polarizable real Hodge module on a compact complex torus T. Then, for each $k \in \mathbb{Z}$, the coherent $\mathscr{O}_T$–module $\operatorname{gr}_k^F \mathcal{M}$ admits a decomposition*

$$\operatorname{gr}_k^F \mathcal{M} \simeq \bigoplus_{j=1}^{n} (q_j^* \mathscr{F}_j \otimes_{\mathscr{O}_T} L_j),$$

where $q_j \colon T \to T_j$ is a quotient torus, $\mathscr{F}_j$ is an ample coherent $\mathscr{O}_{T_j}$–module whose support $\operatorname{Supp} \mathscr{F}_j$ is projective, and $L_j \in \operatorname{Pic}^0(T)$.

Proof By Theorem D we have a decomposition as in the statement, where each $\mathscr{F}_j$ is an M–regular sheaf on the abelian variety generated by its support. But then [10, Corollary 3.2] implies that each $\mathscr{F}_j$ is ample. □

The ampleness part in Theorem A is then a consequence of the proof in Section 16 and the statement above. It implies that higher direct images of canonical bundles have a strong semipositivity property (corresponding to semiampleness in the projective setting). Even the following very special consequence seems to go beyond what can be said for arbitrary holomorphic mappings of compact Kähler manifolds (see eg [19]).

Corollary 20.2 *Let $f \colon X \to T$ be a surjective holomorphic mapping from a compact Kähler manifold to a complex torus. If f is a submersion outside of a simple normal crossings divisor on T, then each $R^i f_* \omega_X$ is locally free and admits a smooth hermitian metric with semipositive curvature (in the sense of Griffiths).*

Proof Note that if f is surjective, then Theorem 14.1 implies that $R^i f_* \omega_X$ are all torsion-free. If one assumes in addition that f is a submersion outside of a simple normal crossings divisor on T, then they are locally free; see [38, Theorem V]. Because of the decomposition in Theorem A, it is therefore enough to show that an M–regular locally free sheaf on an abelian variety always admits a smooth hermitian metric with semipositive curvature. But this is an immediate consequence of the fact that M–regular sheaves are continuously globally generated [22, Proposition 2.19]. □

The existence of a metric with semipositive curvature on a vector bundle E implies that the line bundle $\mathscr{O}_{\mathbb{P}(E)}(1)$ is nef, but is in general known to be a strictly stronger condition. Corollary 20.2 suggests the following question:

Problem Let T be a compact complex torus. Suppose that a locally free sheaf $\mathscr{E}$ on T admits a smooth hermitian metric with semipositive curvature (in the sense of Griffiths or Nakano). Does this imply the existence of a decomposition

$$\mathscr{E} \simeq \bigoplus_{k=1}^{n} (q_k^* \mathscr{E}_k \otimes L_k)$$

as in Theorem A, in which each locally free sheaf $\mathscr{E}_k$ has a smooth hermitian metric with strictly positive curvature?

21 Leray filtration

Let $f\colon X \to T$ be a holomorphic mapping from a compact Kähler manifold X to a compact complex torus T. We use Theorem A to describe the Leray filtration on the cohomology of ω_X, induced by the Leray spectral sequence associated to f. Recall that, for each k, the Leray filtration on $H^k(X, \omega_X)$ is a decreasing filtration $L^\bullet H^k(X, \omega_X)$ with the property that

$$\operatorname{gr}_L^i H^k(X, \omega_X) = H^i(T, R^{k-i} f_* \omega_X).$$

On the other hand, one can define a natural decreasing filtration $F^\bullet H^k(X, \omega_X)$ induced by the action of $H^1(T, \mathscr{O}_T)$, namely

$$F^i H^k(X, \omega_X) = \operatorname{Im}\bigl(\textstyle\bigwedge^i H^1(T, \mathscr{O}_T) \otimes H^{k-i}(X, \omega_X) \to H^k(X, \omega_X)\bigr).$$

It is obvious that the image of the cup product mapping

$$H^1(T, \mathscr{O}_T) \otimes L^i H^k(X, \omega_X) \to H^{k+1}(X, \omega_X) \tag{21.1}$$

is contained in the subspace $L^{i+1} H^{k+1}(X, \omega_X)$. This implies that

$$F^i H^k(X, \omega_X) \subseteq L^i H^k(X, \omega_X) \quad \text{for all } i \in \mathbb{Z}.$$

This inclusion is actually an equality, as shown by the following result:

Theorem 21.2 *The image of the mapping in* (21.1) *is equal to* $L^{i+1} H^{k+1}(X, \omega_X)$. *Consequently, the two filtrations* $L^\bullet H^k(X, \omega_X)$ *and* $F^\bullet H^k(X, \omega_X)$ *coincide.*

Proof By [18, Theorem A], the graded module

$$Q_X^j = \bigoplus_{i=0}^{\dim T} H^i(T, R^j f_* \omega_X)$$

over the exterior algebra on $H^1(T, \mathscr{O}_T)$ is 0–regular, hence generated in degree 0. (Since each $R^j f_* \omega_X$ is a GV-sheaf by Theorem A, the proof in [18] carries over to the case where X is a compact Kähler manifold.) This means that the cup product mappings

$$\textstyle\bigwedge^i H^1(T, \mathscr{O}_T) \otimes H^0(T, R^j f_* \omega_X) \to H^i(T, R^j f_* \omega_X)$$

are surjective for all i and j, which in turn implies that the mappings

$$H^1(T, \mathscr{O}_T) \otimes \operatorname{gr}_L^i H^k(X, \omega_X) \to \operatorname{gr}_L^{i+1} H^{k+1}(X, \omega_X)$$

are surjective for all i and k. This implies the assertion by ascending induction. □

If we represent cohomology classes by smooth forms, Hodge conjugation and Serre duality provide for each $k \geq 0$ a hermitian pairing

$$H^0(X, \Omega_X^{n-k}) \times H^k(X, \omega_X) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \overline{\beta},$$

where $n = \dim X$. The Leray filtration on $H^k(X, \omega_X)$ therefore induces a filtration on $H^0(X, \Omega_X^{n-k})$; concretely, with a numerical convention which again gives us a decreasing filtration with support in the range $0, \ldots, k$, we have

$$L^i H^0(X, \Omega_X^{n-k}) = \{\alpha \in H^0(X, \Omega_X^{n-k}) \mid \alpha \perp L^{k+1-i} H^k(X, \omega_X)\}.$$

Using the description of the Leray filtration in Theorem 21.2, and the elementary fact that

$$\int_X \alpha \wedge \overline{\theta \wedge \beta} = \int_X \alpha \wedge \overline{\theta} \wedge \overline{\beta}$$

for all $\theta \in H^1(X, \mathscr{O}_X)$, we can easily deduce that $L^i H^0(X, \Omega_X^{n-k})$ consists of those holomorphic $(n-k)$–forms whose wedge product with

$$\bigwedge^{k+1-i} H^0(X, \Omega_X^1)$$

vanishes. In other words, for all j we have:

Corollary 21.3 *The induced Leray filtration on $H^0(X, \Omega_X^j)$ is given by*

$$L^i H^0(X, \Omega_X^j) = \left\{\alpha \in H^0(X, \Omega_X^j) \,\middle|\, \alpha \wedge \bigwedge^{n+1-i-j} H^0(X, \Omega_X^1) = 0\right\}.$$

Remark It is precisely the fact that we do not know how to obtain this basic description of the Leray filtration using standard Hodge theory that prevents us from giving a proof of Theorem A in the spirit of [13], and forces us to appeal to the theory of Hodge modules for the main results.

Acknowledgements We thank Christopher Hacon, who first asked us about the behavior of higher direct images in the Kähler setting some time ago, and with whom we have had numerous fruitful discussions about generic vanishing over the years. We also thank Claude Sabbah for advice about the definition of polarizable complex Hodge modules, and Jungkai Chen, János Kollár, Thomas Peternell and Valentino Tosatti for useful discussions.

During the preparation of this paper Schnell has been partially supported by the NSF grant DMS-1404947, Popa by the NSF grant DMS-1405516 and Pareschi by the MIUR PRIN project *Geometry of algebraic varieties*.

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Proposed: Richard Thomas — Received: 3 March 2016
Seconded: Jim Bryan, Dan Abramovich — Accepted: 8 October 2016