Geometry & Topology 21 (2017) 2683–2723

# The nonuniqueness of the tangent cones at infinity of Ricci-flat manifolds

KOTA HATTORI

Colding and Minicozzi established the uniqueness of the tangent cones at infinity of Ricci-flat manifolds with Euclidean volume growth where at least one tangent cone at infinity has a smooth cross section. In this paper, we raise an example of a Ricci-flat manifold implying that the assumption for the volume growth in the above result is essential. More precisely, we construct a complete Ricci-flat manifold of dimension 4 with non-Euclidean volume growth that has infinitely many tangent cones at infinity where one of them has a smooth cross section.

53C23

## 1 Introduction

For a complete Riemannian manifold $(X, g)$ with nonnegative Ricci curvature, it is shown by Gromov's compactness theorem that if one takes a sequence

$$a_1 > a_2 > \cdots > a_i > \cdots > 0$$

such that $\lim_{i\to\infty} a_i = 0$, then there is a subsequence $\{a_{i(j)}\}_j$ such that $(X, a_{i(j)}g, p)$ converges to a pointed metric space $(Y, d, q)$ as $j \to \infty$ in the sense of the pointed Gromov–Hausdorff topology; see Gromov [9; 10]. The limit $(Y, d, q)$ is called the tangent cone at infinity of $(X, g)$. In general, the pointed Gromov–Hausdorff limit might depend on the choice of $\{a_i\}_i$ or its subsequences.

The tangent cone at infinity is said to be unique if the isometry classes of the limits are independent of the choice of $\{a_i\}$ and its subsequences, and Colding and Minicozzi showed the next uniqueness theorem under the given assumptions.

**Theorem 1.1** [6] *Let $(X, g)$ be a Ricci-flat manifold with Euclidean volume growth, and suppose that one of the tangent cones at infinity has a smooth cross section. Then the tangent cone at infinity of $(X, g)$ is unique.*

Among the assumptions in Theorem 1.1, the Ricci-flat condition is essential since there are several examples of complete Riemannian manifolds with nonnegative Ricci

Published: 15 August 2017 DOI: 10.2140/gt.2017.21.2683

curvature and Euclidean volume growth where one of the tangent cones at infinity has smooth cross section, but the tangent cone at infinity is not unique; see Perelman [12] and Colding and Naber [7].

Here, let $\mathcal{T}(X,g)$ be the set of all of the isometry classes of the tangent cones at infinity of $(X,g)$. In this paper, an isometry between pointed metric spaces means a bijective map preserving the metrics and the base points. It is known that $\mathcal{T}(X,g)$ is closed with respect to the pointed Gromov–Hausdorff topology, and has the natural continuous $\mathbb{R}^+$–action defined by the rescaling of metrics. The uniqueness of the tangent cone at infinity means that $\mathcal{T}(X,g)$ consists of only one point.

In this paper, we show that the assumption for the volume growth in Theorem 1.1 is essential. More precisely, we obtain the next main result.

**Theorem 1.2** *There is a complete Ricci-flat manifold $(X,g)$ of dimension 4 such that $\mathcal{T}(X,g)$ is homeomorphic to $S^1$. Moreover, the $\mathbb{R}^+$–action on $\mathcal{T}(X,g)$ fixes $(\mathbb{R}^3, d_0^\infty, 0)$, $(\mathbb{R}^3, h_0, 0)$ and $(\mathbb{R}^3, h_1, 0)$, where $h_0=\sum_{i=1}^3 (d\zeta_i)^2$ is the Euclidean metric, $d_0^\infty$ is the completion of the Riemannian metric*

$$\int_0^\infty \frac{dx}{|\zeta-(x^\alpha,0,0)|}\cdot h_0,$$

*$h_1=(1/|\zeta|)h_0$, and $\mathbb{R}^+$ acts freely on*

$$\mathcal{T}(X,g)\backslash\{(\mathbb{R}^3, d_0^\infty, 0), (\mathbb{R}^3, h_0, 0), (\mathbb{R}^3, h_1, 0)\}.$$

*Here, $\zeta=(\zeta_1,\zeta_2,\zeta_3)$ is the Cartesian coordinate on $\mathbb{R}^3$.*

Here, we mention more about the metric spaces appearing in Theorem 1.2. For $0\le S<T\le\infty$, denote by $d_S^T$ the metric on $\mathbb{R}^3$ induced by the Riemannian metric

$$\int_S^T \frac{dx}{|\zeta-(x^\alpha,0,0)|}\cdot h_0.$$

For $(X,g)$ in Theorem 1.2, we show that $\mathcal{T}(X,g)$ contains $\{(\mathbb{R}^3, d_0^T, 0) : T\in\mathbb{R}^+\}$, $\{(\mathbb{R}^3, d_S^\infty, 0) : S\in\mathbb{R}^+\}$ and $\{(\mathbb{R}^3, h_0+\theta h_1, 0) : \theta\in\mathbb{R}^+\}$. Here, we can check easily that $d_0^T$ and $d_S^\infty$ are homothetic to $d_0^1$ and $d_1^\infty$, respectively. We can show that

$$(\mathbb{R}^3, d_0^T, 0)\xrightarrow[T\to\infty]{\mathrm{GH}}(\mathbb{R}^3, d_0^\infty, 0), \qquad (\mathbb{R}^3, d_0^T, 0)\xrightarrow[T\to 0]{\mathrm{GH}}(\mathbb{R}^3, h_1, 0),$$

$$(\mathbb{R}^3, d_S^\infty, 0)\xrightarrow[S\to\infty]{\mathrm{GH}}(\mathbb{R}^3, h_0, 0), \qquad (\mathbb{R}^3, d_S^\infty, 0)\xrightarrow[S\to 0]{\mathrm{GH}}(\mathbb{R}^3, d_0^\infty, 0),$$

$$(\mathbb{R}^3, h_0+\theta h_1, 0)\xrightarrow[\theta\to\infty]{\mathrm{GH}}(\mathbb{R}^3, h_1, 0), \qquad (\mathbb{R}^3, h_0+\theta h_1, 0)\xrightarrow[\theta\to 0]{\mathrm{GH}}(\mathbb{R}^3, h_0, 0).$$

Both $(\mathbb{R}^3, h_0)$ and $(\mathbb{R}^3, h_1)$ can be regarded as the Riemannian cones with respect to the dilation $\zeta \mapsto \lambda\zeta$ on $\mathbb{R}^3$. Although the dilation also pulls back $d_0^\infty$ to $\lambda^{(\alpha+1)/(2\alpha)} d_0^\infty$, $(\mathbb{R}^3, d_0^\infty)$ does not become the metric cone with respect to this dilation since $\boldsymbol{l} = \{(t,0,0) \in \mathbb{R}^3 : t \geq 0\}$ is not a ray. In fact, any open intervals contained in $\boldsymbol{l}$ have infinite length with respect to $d_0^\infty$.

In general, tangent cones at infinity of complete Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth are metric cones; see Cheeger and Colding [4]. In our case, it is shown in Section 9 that $(\mathbb{R}^3, d_0^\infty, 0)$ never becomes the metric cone of any metric space.

The Ricci-flat manifold $(X, g)$ appearing in Theorem 1.2 is one of the hyper-Kähler manifolds of type $A_\infty$, constructed by Anderson, Kronheimer and LeBrun in [1] applying Gibbons–Hawking ansatz, and by Goto in [8] as hyper-Kähler quotients. Combining Theorems 1.1 and 1.2, we can see that the volume growth of $(X, g)$ should not be Euclidean. In fact, the author [11] has computed the volume growth of the hyper-Kähler manifolds of type $A_\infty$ and showed that they are always greater than cubic growth and less than Euclidean growth. To construct $(X, g)$, we "mix" the hyper-Kähler manifold of type $A_\infty$ whose volume growth is $r^a$ for some $3 < a < 4$, and $\mathbb{R}^4$ equipped with the standard hyper-Kähler structure. Unfortunately, the author could not compute the volume growth of $(X, g)$ in Theorem 1.2 explicitly.

In this paper, we can show that many metric spaces may arise as the Gromov–Hausdorff limit of hyper-Kähler manifolds of type $A_\infty$. Let

$$I \in \mathcal{B}_+(\mathbb{R}^+) := \{J \subset \mathbb{R}^+ : J \text{ is a Borel set of nonzero Lebesgue measure}\},$$

and denote by $d_I$ the metric on $\mathbb{R}^3$ induced by the Riemannian metric

$$\int_I \frac{dx}{|\zeta - (x^\alpha, 0, 0)|} \cdot h_0.$$

Then we have the following result.

**Theorem 1.3** *There is a complete Ricci-flat manifold $(X, g)$ of dimension 4 such that $\mathcal{T}(X, g)$ contains*

$$\{(\mathbb{R}^3, d_I, 0) : I \in \mathcal{B}_+(\mathbb{R}^+)\}/\text{isometry}.$$

Since $d_S^\infty$ and $d_0^T$ are contained in $\mathcal{T}(X, g)$ in the above theorem, their limits $h_0$ and $(1/|\zeta|)h_0$ are also contained in $\mathcal{T}(X, g)$. The author does not know whether any other metric spaces are contained in $\mathcal{T}(X, g)$.

Theorems 1.2 and 1.3 are shown along the following process. The aforementioned hyper-Kähler manifolds are constructed from infinitely countable subsets $\Lambda$ in $\mathbb{R}^3$

such that $\sum_{\lambda\in\Lambda} 1/(1+|\lambda|) < \infty$. We denote such a manifold by $(X, g_\Lambda)$ and fix the base point $p \in X$. From the construction, $(X, g_\Lambda)$ has a natural $S^1$–action preserving $g_\Lambda$ and the hyper-Kähler structure; then we obtain a hyper-Kähler moment map $\mu_\Lambda\colon X \to \mathbb{R}^3$ such that $\mu_\Lambda(p) = 0$, which is a surjective map whose generic fibers are $S^1$. There is a unique distance function $d_\Lambda$ on $\mathbb{R}^3$ such that $\mu_\Lambda$ is a submetry. Here, submetries are the generalizations of Riemannian submersions to the category of metric spaces. For $a > 0$, we can see $ag_\Lambda = g_{a\Lambda}$; hence by taking $a_i > 0$ such that $\lim_{i\to\infty} a_i = 0$, we obtain a sequence of submetries $\mu_{a_i\Lambda}\colon X \to \mathbb{R}^3$. Now, assume that $\{(\mathbb{R}^3, d_{a_i\Lambda}, 0)\}_i$ converges to a metric space $(\mathbb{R}^3, d_\infty, 0)$ for some $d_\infty$ in the pointed Gromov–Hausdorff topology, and the diameters of fibers of $\mu_{a_i\Lambda}$ converge to 0 in some sense. Then we can show that $(\mathbb{R}^3, d_\infty, 0)$ is the Gromov–Hausdorff limit of $\{(X, g_{a_i\Lambda}, p)\}_i$. We raise a concrete example of $\Lambda$ and sequences $\{a_i\}_i$, then obtain several limit spaces. Among them, it is shown in Section 9 that $(\mathbb{R}^3, d_0^\infty)$ is not a polar space in the sense of Cheeger and Colding [5].

This paper is organized as follows. We review the construction of hyper-Kähler manifolds of type $A_\infty$ and the hyper-Kähler moment map $\mu_\Lambda$ in Section 2. Then we review the notion of submetry in Section 3, and the notion of Gromov–Hausdorff topology in Section 4. In Section 5, we construct a submetry $\mu_a$ from $(X, g_{a\Lambda})$ to $(\mathbb{R}^3, d_a)$ by using $\mu_\Lambda$ and dilation, where $d_a$ is the metric induced by the Riemannian metric $\Phi_a(\zeta)h_0$. Here, $\Phi_a$ is a positive valued harmonic function determined by $\Lambda$ and some constants. Then we see that the convergence of $\{(X, g_{a_i\Lambda})\}_i$ can be reduced to the convergence of $\{(\mathbb{R}^3, d_{a_i})\}_i$. In Sections 6 and 7, we raise concrete examples of $\Lambda$ and fix $a > 0$, and then we estimate the difference of $\Phi_a$ and another positive valued harmonic function $\Phi_\infty$, which induces the metric $d_\infty$ on $\mathbb{R}^3$. In Section 8, we observe some examples by applying the results in Sections 6 and 7, and then we show Theorems 1.2 and 1.3. In Section 9, we prove that $(\mathbb{R}^3, d_0^\infty)$ is not a polar space.

**Acknowledgments** The author would like to thank Professor Shouhei Honda who invited the author to this attractive topic, and also for advice on this paper. The author also would like to thank the referee for careful reading and several useful comments. Thanks to these, the author could make the main results much stronger. The author was supported by Grant-in-Aid for Young Scientists (B) Grant Number 16K17598. The author was partially supported by JSPS Core-to-Core Program, "Foundation of a Global Research Cooperative Center in Mathematics focused on Number Theory and Geometry".

## 2 Hyper-Kähler manifolds of type $A_\infty$

Here we review briefly the construction of hyper-Kähler manifolds of type $A_\infty$, along [1].

Let $\Lambda \subset \mathbb{R}^3$ be a countably infinite subset satisfying the convergence condition

$$\sum_{\lambda\in\Lambda} \frac{1}{1+|\lambda|} < \infty,$$

and take a positive valued harmonic function $\Phi_\Lambda$ over $\mathbb{R}^3\backslash\Lambda$ defined by

$$\Phi_\Lambda(\zeta) := \sum_{\lambda\in\Lambda} \frac{1}{|\zeta-\lambda|}.$$

Then $*d\Phi_\Lambda \in \Omega^2(\mathbb{R}^3\backslash\Lambda)$ is a closed 2–form, where $*$ is the Hodge star operator of the Euclidean metric. We have an integrable cohomology class $[1/(4\pi) * d\Phi_\Lambda] \in H^2(\mathbb{R}^3\backslash\Lambda, \mathbb{Z})$, which is equal to the 1st Chern class of a principal $S^1$–bundle $\mu = \mu_\Lambda\colon X^* \to \mathbb{R}^3\backslash\Lambda$. For every $\lambda\in\Lambda$, we can take a sufficiently small open ball $B\subset\mathbb{R}^3$ centered at $\lambda$ which does not contain any other elements in $\Lambda$. Then $\mu\colon \mu^{-1}(B\backslash\{\lambda\}) \to B\backslash\{\lambda\}$ is isomorphic to the Hopf fibration $\mu_0\colon \mathbb{R}^4\backslash\{0\} \to \mathbb{R}^3\backslash\{0\}$ as principal $S^1$–bundles; hence there exists a $C^\infty$ 4–manifold $X$ and an open embedding $X^*\subset X$, and $\mu$ can be extended to an $S^1$–fibration

$$\mu = (\mu_1, \mu_2, \mu_3)\colon X \to \mathbb{R}^3.$$

Moreover, we may write $X\backslash X^* = \{p_\lambda : \lambda\in\Lambda\}$ and $\mu(p_\lambda) = \lambda$. Next we take an $S^1$–connection $\Gamma \in \Omega^1(X^*)$ on $X^* \to \mathbb{R}^3\backslash\Lambda$, whose curvature form is given by $*d\Phi_\Lambda$. Then $\Gamma$ is uniquely determined up to an exact 1–form on $\mathbb{R}^3\backslash\Lambda$. Now, we obtain a Riemannian metric

$$g_\Lambda := (\mu^*\Phi_\Lambda)^{-1}\Gamma^2 + \mu^*\Phi_\Lambda \sum_{i=1}^{3} (d\mu_i)^2$$

on $X^*$, which can be extended to a smooth Riemannian metric $g_\Lambda$ over $X$ by taking $\Gamma$ appropriately.

**Theorem 2.1** [1] *Let $(X, g_\Lambda)$ be as above. Then it is a complete hyper-Kähler (hence Ricci-flat) metric of dimension 4.*

Since $S^1$ acts on $(X, g_\Lambda)$ isometrically, it is easy to check that

$$\mu\colon (X^*, g_\Lambda) \to (\mathbb{R}^3\backslash\Lambda, \Phi_\Lambda\cdot h_0)$$

is a Riemannian submersion, where $h_0$ is the Euclidean metric on $\mathbb{R}^3$.

Next we consider the rescaling of $(X, g_\Lambda)$. For $a > 0$, put $a\Lambda := \{a\lambda \in \mathbb{R}^3 : \lambda \in \Lambda\}$. Then it is easy to see

$$\Phi_{a\Lambda}(\zeta) = \sum_{\lambda\in\Lambda} \frac{1}{|\zeta - a\lambda|} = a^{-1} \sum_{\lambda\in\Lambda} \frac{1}{|a^{-1}\zeta - \lambda|} = a^{-1}\Phi_\Lambda(a^{-1}\zeta),$$

and $\mu_{a\Lambda} = a\mu_\Lambda$; hence $\mu_{a\Lambda}^*\Phi_{a\Lambda} = a^{-1}\mu_\Lambda^*\Phi_\Lambda$ holds. Thus we have

$$\begin{aligned} g_{a\Lambda} &= (\mu_{a\Lambda}^*\Phi_{a\Lambda})^{-1}\Gamma^2 + \mu_{a\Lambda}^*\Phi_{a\Lambda}\sum_{i=1}^{3}(d\mu_{a\Lambda,i})^2 \\ &= a(\mu_\Lambda^*\Phi_\Lambda)^{-1}\Gamma^2 + a\mu_\Lambda^*\Phi_\Lambda\sum_{i=1}^{3}(d\mu_{\Lambda,i})^2 = ag_\Lambda. \end{aligned}$$

# 3 Submetry

Throughout this paper, the distance between $x$ and $y$ in a metric space $(X, d)$ is denoted by $d(x, y)$. If it is clear which metric is used, we often write $|xy| = d(x, y)$

The map $\mu\colon X \to \mathbb{R}^3$ appearing in the previous section is not a Riemannian submersion since $d\mu$ degenerates on $X \backslash X^*$ and $\Phi_\Lambda \cdot h_0$ is not defined on the whole of $\mathbb{R}^3$. However, we can regard $\mu$ as a submetry, which is a notion introduced in [3].

**Definition 3.1** [3] Let $X, Y$ be metric spaces and $\mu\colon X \to Y$ a map which is not necessarily continuous. Then $\mu$ is said to be a *submetry* if $\mu(D(p, r)) = D(\mu(p), r)$ for every $p \in X$ and $r > 0$, where $D(p, r)$ is the closed ball of radius $r$ centered at $p$.

Any proper Riemannian submersions between smooth Riemannian manifolds are known to be submetries. Conversely, a submetry between smooth complete Riemannian manifolds becomes a $C^{1,1}$ Riemannian submersion [2].

Now we go back to the setting in Section 2. Denote by $d_\Lambda$ the metric on $\mathbb{R}^3$ defined as the completion of the Riemannian distance induced from $\Phi_\Lambda \cdot h_0$. Since $\mu\colon (X^*, g_\Lambda) \to (\mathbb{R}^3 \backslash \Lambda, \Phi_\Lambda \cdot h_0)$ is a Riemannian submersion, we have the following proposition.

**Proposition 3.2** *Let $(X, g_\Lambda)$ be a hyper-Kähler manifold of type $A_\infty$. The map $\mu\colon (X, d_{g_\Lambda}) \to (\mathbb{R}^3, d_\Lambda)$ is a submetry, where $d_{g_\Lambda}$ is the Riemannian distance induced from $g_\Lambda$. Moreover, for any $p_0 \in \mu^{-1}(q_0)$, we have*

$$d_\Lambda(q_0, q_1) = \inf_{p_1\in\mu^{-1}(q_1)} d_{g_\Lambda}(p_0, p_1).$$

# 4 The Gromov–Hausdorff convergence

In this section, we discuss the pointed Gromov–Hausdorff convergence of a sequence of pointed metric spaces equipped with submetries. First we review the definition of pointed Gromov–Hausdorff convergence of pointed metric spaces. Denote by $B(p,r) = B_X(p,r)$ the open ball of radius $r$ centered at $p$ in a metric space $X$.

**Definition 4.1** Let $(X,p)$ and $(X',p')$ be pointed metric spaces, and let $r$ and $\varepsilon$ be positive real numbers. Then $f\colon B(p,r) \to X'$ is said to be an $(r,\varepsilon)$*–isometry* from $(X,p)$ to $(X',p')$ if

(1) $f(p) = p'$,

(2) $\big||xy| - |f(x)f(y)|\big| < \varepsilon$ holds for any $x, y \in B(p,r)$, and

(3) $B(f(B(p,r)),\varepsilon)$ contains $B(p', r-\varepsilon)$.

**Definition 4.2** Let $\{(X_i,p_i)\}_i$ be a sequence of pointed metric spaces. Then we say $\{(X_i,p_i)\}_i$ *converges to a metric space* $(X,p)$ *in the pointed Gromov–Hausdorff topology*, or $\{(X_i,p_i)\}_i \xrightarrow{\text{GH}} (X,p)$, if for any $r,\varepsilon > 0$ there exists a positive integer $N_{(r,\varepsilon)}$ such that an $(r,\varepsilon)$–isometry from $(X_i,p_i)$ to $(X,p)$ exists for every $l \geq N_{(r,\varepsilon)}$.

For metric spaces $X$, $Y$, a map $\mu\colon X \to Y$ and $q \in Y$, define $\delta_{q,\mu}(r) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$\delta_{q,\mu}(r) := \sup_{y \in B(q,r)} \operatorname{diam}(\mu^{-1}(y)) = \sup_{\substack{y \in B(q,r) \\ x,x' \in \mu^{-1}(y)}} |xx'|.$$

**Proposition 4.3** *Let $(X,p)$ and $(Y,q)$ be pointed metric spaces equipped with submetries $\mu\colon X \to Y$ satisfying $\mu(p) = q$, and let $(Y_\infty, q_\infty)$ be another pointed metric space. Assume that $\delta_{q,\mu}(r) < \infty$ and we have an $(r,\delta)$–isometry from $(Y,q)$ to $(Y_\infty,q_\infty)$. Then there exists an $(r,\delta+\delta_{q,\mu})$–isometry from $(X,p)$ to $(Y_\infty,q_\infty)$.*

**Proof** There is an $(r,\delta)$–isometry $f$ from $(Y,q)$ to $(Y_\infty,q_\infty)$. It is easy to check that the composition $\hat{f} := f \circ \mu$ is an $(r,\delta+\delta_{q,\mu})$–isometry from $(X,p)$ to $(Y_\infty,q_\infty)$. □

# 5 Tangent cones at infinity

Let $(X,d)$ be a metric space and $\{a_i\}_i$ a decreasing sequence of positive numbers converging to $0$. If $(Y,q)$ is the pointed Gromov–Hausdorff limit of $\{(X,a_i d,p)\}_i$, then it is called an tangent cone at infinity of $X$. It is clear that the limit does not depend on $p \in X$, but may depend on the choice of the sequence $\{a_i\}_i$.

In this paper, we are considering the tangent cones at infinity of $(X,d_{g_\Lambda})$. In Section 2, we have seen that $\sqrt{a}d_{g_\Lambda} = d_{g_{a\Lambda}}$ for $a > 0$; hence $\mu_{a\Lambda}\colon (X,\sqrt{a}d_{g_\Lambda}) \to (\mathbb{R}^3, d_{a\Lambda})$

is a submetry. By taking $N \in \mathbb{R}^+$ and the dilation $I_N\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $I_N(\zeta) := (1/N)\zeta$, we have another submetry

$$\mu_a := I_N^{-1} \circ \mu_{a\Lambda}\colon (X, \sqrt{a} d_{g_\Lambda}) \to (\mathbb{R}^3, d_a := {I_N}^* d_{a\Lambda}).$$

Here, ${I_N}^* d_{a\Lambda}$ is the completion of the Riemannian distance of

$${I_N}^*(\Phi_{a\Lambda} \cdot h_0) = {I_N}^* \Phi_{a\Lambda} \cdot \frac{1}{N^2} h_0 = N \Phi_{Na\Lambda} \cdot \frac{1}{N^2} h_0 = \frac{1}{N} \Phi_{Na\Lambda} \cdot h_0.$$

Thus we obtain $d_a$, which is the completion of the Riemannian metric $\Phi_a \cdot h_0$, where

$$\Phi_a := \frac{1}{N} \Phi_{Na\Lambda}.$$

In other words, $d_a$ is given by

$$d_a(x, y) = \inf_{\gamma \in \mathrm{Path}(x,y)} l_a(\gamma), \tag{1}$$

where $\mathrm{Path}(x, y)$ is the set of smooth paths in $\mathbb{R}^3$ joining $x, y \in \mathbb{R}^3$, and

$$l_a(\gamma) = \int_{t_0}^{t_1} \sqrt{\Phi_a(\gamma(t))}\, |\gamma'(t)|\, dt. \tag{2}$$

By the definition of $g_\Lambda$, one can see that the diameter of the fiber $\mu_\Lambda^{-1}(\zeta)$ is given by $\pi/\sqrt{\Phi_\Lambda(\zeta)}$. Accordingly, the diameter of $\mu_a^{-1}(\zeta)$ is given by $\pi/(N\sqrt{\Phi_a(\zeta)})$.

For a metric $d_\infty$ on $\mathbb{R}^3$ and constants $r, \delta, \delta' > 0$, we introduce the next assumptions.

**(A1)** The identity map $\mathrm{id}_{\mathbb{R}^3}\colon (\mathbb{R}^3, d_a, 0) \to (\mathbb{R}^3, d_\infty, 0)$ is an $(r, \delta)$–isometry.

**(A2)** $$\sup_{\zeta \in B_{d_a}(0,r)} \frac{\pi}{N\sqrt{\Phi_a(\zeta)}} < \delta'.$$

Then we obtain the next proposition by Proposition 4.3.

**Proposition 5.1** *Let $(X, g_\Lambda)$ and $\mu_a$ be as above, $p \in X$ satisfy $\mu_\Lambda(p) = 0$ and $d_\infty$ be a metric on $\mathbb{R}^3$. If* **(A1)** *and* **(A2)** *are satisfied for given constants $r, \delta, \delta' > 0$, then $\mu_a$ is an $(r, \delta+\delta')$–isometry from $(X, ag_\Lambda, p)$ to $(\mathbb{R}^3, d_\infty, 0)$.*

# 6 Construction

Fix $\alpha > 1$, and let

$$\Lambda^\alpha := \{(k^\alpha, 0, 0) : k \in \mathbb{Z}_{\geq 0}\}.$$

Take an increasing sequence of integers $0 < K_0 < K_1 < K_2 < \cdots$.

In this paper, many constants will appear, and they may depend on $\alpha$ or $\{K_n\}$. However, we do not mind the dependence on these parameters. Put

$$\Lambda_{2n} := \{(k^\alpha, 0, 0) \in \Lambda^\alpha : K_{2n} \le k < K_{2n+1}\}, \quad \Lambda := \bigcup_{n=0}^{\infty} \Lambda_{2n}.$$

Since $\Lambda \subset \Lambda^\alpha$, we can see that $\sum_{\lambda\in\Lambda} 1/(1+|\lambda|) < \infty$; accordingly, we obtain a hyper-Kähler manifold $(X, g_\Lambda)$.

From now on, we fix $a > 0$, $n \in \mathbb{N}$ and $P > 0$, then put $N := a^{-1/(1+\alpha)} P^{1/(1+\alpha)}$ and

$$\Phi_a(\zeta) := \frac{1}{N}\Phi_{Na\Lambda}(\zeta) = \sum_{\lambda\in\Lambda} \frac{1}{N|\zeta - PN^{-\alpha}\lambda|}.$$

Let $\boldsymbol{l} := \{(t, 0, 0) \in \mathbb{R}^3 : t \ge 0\}$, and put

$$K(R, D) := \Big\{\zeta \in \mathbb{R}^3 : |\zeta| \le R,\ \inf_{y\in \boldsymbol{l}} |\zeta - y| \ge D\Big\}.$$

Here, $\inf_{y\in\boldsymbol{l}} |\zeta - y|$ is given by

$$\inf_{y\in\boldsymbol{l}} |\zeta - y| = \begin{cases} \sqrt{|\zeta_{\mathbb{C}}|^2} & \text{if } \zeta_{\mathbb{R}} \ge 0, \\ |\zeta| & \text{if } \zeta_1 < 0 \end{cases}$$

for $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$. For $0 \le S < T \le \infty$, define a positive valued function $\Phi_{S,P}^T : \mathbb{R}^3 \backslash \boldsymbol{l} \to \mathbb{R}$ by

$$\Phi_{S,P}^T(\zeta) := \int_S^T \frac{dx}{|\zeta - P(x^\alpha, 0, 0)|}.$$

Throughout this section, we put

$$S_n := \frac{K_{2n}}{N} = a^{\frac{1}{1+\alpha}} P^{\frac{-1}{1+\alpha}} K_{2n}, \quad T_n := \frac{K_{2n+1}}{N} = a^{\frac{1}{1+\alpha}} P^{\frac{-1}{1+\alpha}} K_{2n+1}.$$

**Proposition 6.1** *For any $\zeta \in K(R, D)$, we have*

$$\left|\Phi_a(\zeta) - \sum_{n=0}^{\infty} \Phi_{S_n,P}^{T_n}(\zeta)\right| \le \frac{2}{ND} = \frac{2}{D}\left(\frac{a}{P}\right)^{\frac{1}{1+\alpha}}.$$

**Proof** Since

$$\Lambda_{2n} = \{(k^\alpha, 0, 0) : K_{2n} \le k < K_{2n+1}\},$$

we have

$$\sum_{\lambda\in\Lambda_{2n}} \frac{1}{N|\zeta - PN^{-\alpha}\lambda|} = \sum_{k=K_{2n}}^{K_{2n+1}-1} \frac{1}{N|\zeta - PN^{-\alpha}(k^\alpha, 0, 0)|}.$$

Then we obtain

(3) $$\left|\sum_{n=0}^{\infty}\left(\sum_{\lambda\in\Lambda_{2n}}\frac{1}{N|\zeta-PN^{-\alpha}\lambda|}-\int_{K_{2n}/N}^{K_{2n+1}/N}\frac{dx}{|\zeta-P(x^\alpha,0,0)|}\right)\right|\leq\frac{2}{ND}.$$

The above inequality holds since the function $x\mapsto 1/|\zeta-P(x^\alpha,0,0)|$ has at most one critical point, and for all $\zeta\in K(R,D)$,

$$\sup_{x\in\mathbb{R}}\frac{1}{|\zeta-P(x^\alpha,0,0)|}-\inf_{x\in\mathbb{R}}\frac{1}{|\zeta-P(x^\alpha,0,0)|}\leq\frac{1}{D}.\qquad\square$$

Next we obtain the lower estimate of $\Phi_a$ as follows.

**Proposition 6.2** *We have*

(4) $$\Phi_{S_n,P}^{T_n}(\zeta)\geq\left(\int_{S_n}^{T_n}\frac{dx}{1+Px^\alpha}\right)\min\left\{\frac{1}{|\zeta|},1\right\},$$

(5) $$\Phi_a(\zeta)\geq\left(\sum_{n=0}^{\infty}\int_{S_n}^{T_n}\frac{dx}{1+Px^\alpha}-2(aP^{-1})^{\frac{1}{1+\alpha}}\right)\min\left\{\frac{1}{|\zeta|},1\right\},$$

(6) $$\sum_{n=0}^{\infty}\Phi_{S_n,P}^{T_n}(\zeta)\leq P^{-\frac{1}{\alpha}}\frac{\alpha 2^{\frac{1}{\alpha}}}{\alpha-1}\frac{|\zeta|^{\frac{1}{\alpha}}}{|\zeta_{\mathbb{C}}|},$$

(7) $$\sum_{n=n_0}^{\infty}\Phi_{S_n,P}^{T_n}(\zeta)\leq\frac{2S_{n_0}^{-\alpha+1}}{P(\alpha-1)}\quad \textit{if } S_{n_0}\geq\left(\frac{2|\zeta|}{P}\right)^{\frac{1}{\alpha}},$$

(8) $$\sum_{n=0}^{n_0}\Phi_{S_n,P}^{T_n}(\zeta)\leq\frac{T_{n_0}}{D}\quad \textit{if } \zeta\in K(R,D).$$

**Proof** First of all, one can see

$$\Phi_{S_n,P}^{T_n}(\zeta)\geq\int_{S_n}^{T_n}\frac{dx}{|\zeta|+Px^\alpha}\geq\frac{1}{|\zeta|}\int_{S_n}^{T_n}\frac{dx}{1+Px^\alpha}$$

if $|\zeta|\geq 1$, and

$$\Phi_{S_n,P}^{T_n}(\zeta)\geq\int_{S_n}^{T_n}\frac{dx}{|\zeta|+Px^\alpha}\geq\int_{S_n}^{T_n}\frac{dx}{1+Px^\alpha}$$

if $|\zeta|\leq 1$. Next we have

$$\Phi_a(\zeta)\geq\sum_{n=0}^{\infty}\sum_{k=K_{2n}}^{K_{2n+1}-1}\frac{1}{N(|\zeta|+PN^{-\alpha}k^\alpha)},$$

and a similar argument to the proof of Proposition 6.1 gives

$$\left|\sum_{n=0}^{\infty}\left(\sum_{k=K_{2n}}^{K_{2n+1}-1}\frac{1}{N(|\zeta|+PN^{-\alpha}k^{\alpha})}-\int_{S_n}^{T_n}\frac{dx}{|\zeta|+Px^{\alpha}}\right)\right|\le\frac{2}{N|\zeta|}.$$

Combining these inequalities, one has the second assertion if $|\zeta|\ge 1$. If $|\zeta|\le 1$, then

$$\Phi_a(\zeta)\ge\sum_{n=0}^{\infty}\sum_{k=K_{2n}}^{K_{2n+1}-1}\frac{1}{N(1+PN^{-\alpha}k^{\alpha})},$$

and by a similar argument, we obtain the assertion.

Next we consider (6). If $t\ge(2|\zeta|/P)^{1/\alpha}$, then

$$\int_t^{\infty}\frac{dx}{|\zeta-P(x^{\alpha},0,0)|}\le\int_t^{\infty}\frac{2dx}{Px^{\alpha}}=\frac{2}{P(\alpha-1)}t^{-\alpha+1} \tag{9}$$

holds. Hence one can see

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_{S_n,P}^{T_n}(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})&\le\int_0^{\infty}\frac{dx}{|\zeta-P(x^{\alpha},0,0)|}\\&=\int_0^{\left(\frac{2|\zeta|}{P}\right)^{\frac{1}{\alpha}}}\frac{dx}{|\zeta-P(x^{\alpha},0,0)|}+\int_{\left(\frac{2|\zeta|}{P}\right)^{\frac{1}{\alpha}}}^{\infty}\frac{dx}{|\zeta-P(x^{\alpha},0,0)|}\\&\le\frac{(2|\zeta|/P)^{\frac{1}{\alpha}}}{|\zeta_{\mathbb{C}}|}+\frac{2}{P(\alpha-1)}\left(\frac{2|\zeta|}{P}\right)^{\frac{1}{\alpha}(-\alpha+1)}\\&=P^{-\frac{1}{\alpha}}\left(\frac{(2|\zeta|)^{\frac{1}{\alpha}}}{|\zeta_{\mathbb{C}}|}+\frac{(2|\zeta|)^{\frac{1}{\alpha}}}{\alpha-1}\frac{1}{|\zeta_{\mathbb{C}}|}\right).\end{aligned}$$

The statement (7) follows from (9), and (8) is obvious. □

Put

$$A_{S,P}^{T}:=\int_S^T\frac{dx}{1+Px^{\alpha}}.$$

By Proposition 6.2, we have the following.

**Proposition 6.3** *Let $\Phi_a$ be as above. Then for every $R\ge 1$,*

$$\sup_{|\zeta|\le R}\frac{1}{N\sqrt{\Phi_a(\zeta)}}\le\left(\frac{a}{P}\right)^{\frac{1}{1+\alpha}}\left(\sum_{n=0}^{\infty}A_{S_n,P}^{T_n}-2\left(\frac{a}{P}\right)^{\frac{1}{1+\alpha}}\right)^{-\frac{1}{2}}\sqrt{R}.$$

# 7 Distance

In the previous section, we estimated $\left|\Phi_a - \sum_n \Phi^{T_n}_{S_n,P}\right|$ on $K(R,D)$.

In this section, we introduce more general positive functions $\Phi$ and $\Phi_\infty$, and induced metrics $d$ and $d_\infty$ on $\mathbb{R}^3$, respectively. What we hope to show in this section is that if we fix a very large $R \geq 1$ and assume that $\sup_{K(R,D)}|\Phi - \Phi_\infty| \leq \varepsilon/D$ holds for a very small $\varepsilon$ and every $D \leq 1$, then the identity map of $\mathbb{R}^3$ becomes an $(r,\delta)$–isometry from $(\mathbb{R}^3, d, 0)$ to $(\mathbb{R}^3, d_\infty, 0)$ for a large $r$ and a small $\delta$. Here, we explain the difficulty in showing it.

We hope to show that $\sup_{K(R,D)} |d - d_\infty|$ is small for every $R \geq 1$ and $0 < D \leq 1$. By the estimate of $\sup_{K(R,D)} |\Phi - \Phi_\infty|$, it is easy to see that $\sup_{K(R,D)} |d_{R,D} - d_{\infty,R,D}|$ is small, where $d_{R,D}$ (resp. $d_{\infty,R,D}$) is the Riemannian distance of the Riemannian metric $\Phi h_0|_{K(R,D)}$ (resp. $\Phi_\infty h_0|_{K(R,D)}$). However, $d_{R,D}$ may not equal $d$ in general since the geodesic of $\Phi h_0$ joining two points in $K(R,D)$ might leave $K(R,D)$. To see that $\sup_{K(R,D)} |d_{R,D} - d|$ is sufficiently small, we have to observe that a path joining two points in $K(R,D)$ which leaves $K(R,D)$ can be replaced by a shorter path included in $K(R,D)$.

In this section, we consider positive valued functions $\Phi, \Phi_\infty \in C^\infty(\mathbb{R}^3 \backslash \boldsymbol{l})$ satisfying the following conditions for given constants $R \geq 1$, $m, \varepsilon, C_0, C_1 > 0$ and $\kappa \geq 0$:

**(A3)** $$|\Phi(\zeta) - \Phi_\infty(\zeta)| \leq \frac{\varepsilon}{D^m} \quad \text{and} \quad |\Phi(\zeta) - \Phi_\infty(\zeta)| \leq \frac{C_1}{D}$$

hold for any $D \leq 1$ and $\zeta \in K(R,D)$.

**(A4)** Along the decomposition $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$, put $\zeta = (\zeta_\mathbb{R}, \zeta_\mathbb{C}) \in \mathbb{R} \oplus \mathbb{C}$. Then

$$\Phi(\zeta_\mathbb{R}, e^{i\theta}\zeta_\mathbb{C}) = \Phi(\zeta_\mathbb{R}, \zeta_\mathbb{C}), \qquad \Phi(\zeta_\mathbb{R}, \zeta_\mathbb{C}) \leq \Phi(\zeta_\mathbb{R}, \zeta'_\mathbb{C}),$$
$$\Phi_\infty(\zeta_\mathbb{R}, e^{i\theta}\zeta_\mathbb{C}) = \Phi_\infty(\zeta_\mathbb{R}, \zeta_\mathbb{C}), \qquad \Phi_\infty(\zeta_\mathbb{R}, \zeta_\mathbb{C}) \leq \Phi_\infty(\zeta_\mathbb{R}, \zeta'_\mathbb{C})$$

hold for any $e^{i\theta} \in S^1$, if $|\zeta_\mathbb{C}| \geq |\zeta'_\mathbb{C}|$.

**(A5)** $$\min\{\Phi(\zeta), \Phi_\infty(\zeta)\} \geq \begin{cases} C_0/|\zeta| & \text{if } |\zeta| \geq 1, \\ C_0 & \text{if } |\zeta| \leq 1. \end{cases}$$

**(A6)** For any $u \geq 1$ and $\zeta \in \mathbb{R}^3 \backslash \boldsymbol{l}$ with $|\zeta| \leq u$,

$$\Phi_\infty(\zeta) \leq \frac{C_1 u^\kappa}{|\zeta_\mathbb{C}|}.$$

**Remark 7.1** Let $\Phi = \Phi_a$ and $\Phi_\infty = \Phi^T_{S,P}$ be as in Section 6. Then they satisfy **(A4)**, and also satisfy **(A3)**, **(A5)** and **(A6)** for appropriate constants $\varepsilon$, $C_0$ and $C_1$ given by Propositions 6.1 and 6.2.

From now on, let $\Phi$, $\Phi_\infty$ satisfy **(A3)–(A6)** for constants $R, \varepsilon, C_0, C_1, \kappa$. Denote by $d$, $d_\infty$ the metrics on $\mathbb{R}^3$ induced by $\Phi \cdot h$, $\Phi_\infty \cdot h$, and by $l$, $l_\infty$ the lengths of paths with respect to $d$, $d_\infty$, respectively.

## 7.1 Estimates (1)

Let $\boldsymbol{B}(u) := \{\zeta \in \mathbb{R}^3 : |\zeta| < u\}$ and $\mathrm{Path}(u, x, y)$ be the set of smooth paths in $\overline{\boldsymbol{B}(u)}$ joining $x, y \in \overline{\boldsymbol{B}(u)}$; then put

$$d_u(x, y) = \inf_{\gamma \in \mathrm{Path}(u,x,y)} l(\gamma), \qquad d_{\infty,u}(x, y) = \inf_{\gamma \in \mathrm{Path}(u,x,y)} l_\infty(\gamma)$$

By the definition, $d(x, y) \le d_u(x, y)$ and $d_\infty(x, y) \le d_{\infty,u}(x, y)$ always hold. However, the opposite inequality may not hold since the minimizing geodesic $\gamma$ joining $x, y \in \overline{\boldsymbol{B}(u)}$ may leave $\overline{\boldsymbol{B}(u)}$. The goal of this subsection is to show $d_{\rho(u)}(x, y) \le d(x, y)$ and $d_{\infty,\rho(u)}(x, y) \le d_\infty(x, y)$ for a sufficiently large $\rho(u)$.

**Proposition 7.2** *Suppose $\Phi$ and $\Phi_\infty$ satisfy* **(A3)–(A6)**. *Let $D_u$ and $D_{u,u'}$ be the diameters of $\overline{\boldsymbol{B}(t)}$ with respect to $d$ and $d_{u'}$, respectively, where $0 < u \le u'$. Define $D_{\infty,u}$ and $D_{\infty,u,u'}$ in the same way. Then the inequality*

$$2\sqrt{C_0}(\sqrt{|\zeta|} - 1) \le \min\{d(0, \zeta), d_\infty(0, \zeta)\}$$

*holds for all $\zeta \in \mathbb{R}^3$, and*

$$d(0, \zeta) \le D_u \le D_{u,u} \le C_2 u^{\kappa'}, \quad d_\infty(0, \zeta) \le D_{\infty,u} \le D_{\infty,u,u} \le C_2 u^{\kappa'}$$

*hold for all $\zeta \in \mathbb{R}^3$ and $u \ge 1$ with $|\zeta| \le u \le R$, where $C_2$ is the constant depending only on $C_1$ and $\kappa' = \frac{1}{2}(1 + \kappa)$.*

**Proof** First we show the first inequality. Let $\gamma \colon [a, b] \to \mathbb{R}^3$ be a smooth path such that $\gamma(a) = 0$ and $\gamma(b) = \zeta$. We may suppose $|\zeta| \ge 1$ since it is obviously satisfied when $|\zeta| < 1$. Then there is $s \in [a, b]$ such that $|\gamma(s)| = 1$ and $|\gamma(t)| \ge 1$ for any $t \in [s, b]$. Then by the assumption **(A5)**, one can see

$$l(\gamma) = \int_a^b \sqrt{\Phi(\gamma(t))}\, |\gamma'(t)|\, dt \ge \int_s^b \sqrt{\Phi(\gamma)}\, |\gamma'|\, dt \ge \int_s^b \sqrt{\frac{C_0}{|\gamma|}}\, |\gamma'|\, dt.$$

Since we have $|\gamma'| \ge |\gamma|'$, we obtain, for all $\zeta \in \mathbb{R}^3$ with $|\zeta| \ge 1$,

$$l(\gamma) \ge \int_s^b \sqrt{\frac{C_0}{|\gamma|}}\, |\gamma|'\, dt \ge 2\sqrt{C_0} \int_s^b \frac{d}{dt}\sqrt{|\gamma|}\, dt \ge 2\sqrt{C_0}(\sqrt{|\zeta|} - 1).$$

By the definition, $d(0,\zeta)\le D_u\le D_{u,R_1}\le D_{u,R_0}$ always hold for any $u\le R_0\le R_1$ and $\zeta\in\mathbb{R}^3$ with $|\zeta|\le u$. Next we estimate $D_{u,u}$ from the above. For every $\zeta$, we will prepare the piecewise smooth paths $\gamma_\zeta$ in $\overline{\boldsymbol{B}(u)}$ joining 0 and $\zeta$ as described below. Then we will have an upper bound

$$D_{u,u}\le 2\sup_{\zeta\in\overline{\boldsymbol{B}(u)}} l(\gamma_\zeta).$$

Here we define $\gamma_\zeta$ as follows. We have the isometric $S^1$–action on $\mathbb{R}^3$ with respect to $d$ and $d_\infty$ by **(A4)**. By supposing $\gamma_{e^{i\theta}\zeta}=e^{i\theta}\gamma_\zeta$, it suffices to consider $\gamma_\zeta$ in the case of $\zeta=r(\sin s,-\cos s,0)$, where $r>0$ and $-\pi<s\le\pi$. Let

$$\begin{aligned}\gamma_\zeta|_{[0,1]}(t)&:=(0,-rt,0),\\ \gamma_\zeta|_{[1,2]}(t)&:=r\big(\sin(s(t-1)),-\cos(s(t-1)),0\big).\end{aligned}$$

Since $\zeta\in K(R,|\zeta_{\mathbb{C}}|)$ holds, **(A3)** gives $|\Phi(\zeta)-\Phi_\infty(\zeta)|\le C_1/|\zeta_{\mathbb{C}}|$, and **(A6)** gives $\Phi_\infty(\zeta)\le C_1u^\kappa/|\zeta_{\mathbb{C}}|$. Then we can see

$$\begin{aligned}l(\gamma_\zeta|_{[0,1]})&=\int_0^1\sqrt{\Phi(\gamma_\zeta)}\,|\gamma_\zeta'|\,dt\\ &\le\int_0^1 r\sqrt{|\Phi(\gamma_\zeta)-\Phi_\infty(\gamma_\zeta)|}\,dt+\int_0^1 r\sqrt{|\Phi_\infty(\gamma_\zeta)|}\,dt\\ &\le\int_0^1 r\sqrt{\frac{C_1}{rt}}\,dt+\int_0^1 r\sqrt{\frac{C_1u^\kappa}{rt}}\,dt\\ &\le 2\sqrt{C_1u}+2\sqrt{C_1}\,u^{\frac{\kappa+1}{2}}.\end{aligned}$$

Simultaneously, we also have

$$\begin{aligned}l(\gamma_\zeta|_{[1,2]})&\le\int_1^2|\gamma_\zeta'|\sqrt{|\Phi(\gamma_\zeta)-\Phi_\infty(\gamma_\zeta)|}\,dt+\int_1^2|\gamma_\zeta'|\sqrt{|\Phi_\infty(\gamma_\zeta)|}\,dt\\ &\le\int_0^1 r|s|\sqrt{\frac{C_1}{r|\cos st|}}\,dt+\int_0^1 r|s|\sqrt{\frac{C_1u^\kappa}{r|\cos st|}}\,dt\\ &\le\sqrt{C_1u}+\sqrt{C_1u^{1+\kappa}}\int_0^{|s|}\sqrt{\frac{1}{\cos t}}\,dt.\end{aligned}$$

Here, $\int_0^\pi\sqrt{1/\cos t}\,dt$ is finite. Since $u\ge 1$ and $\kappa\ge 0$, we may suppose that $\max\{\sqrt{u},\sqrt{u^{1+\kappa}}\}=u^{1+\kappa}$. By combining these estimates and putting

$$C_2=\left(2+\int_0^\pi\sqrt{\frac{1}{\cos t}}\,dt\right)\sqrt{C_1},$$

we have the assertion. We also obtain the estimate of $D_{\infty,u,u}$ by the above argument. □

**Proposition 7.3** *Suppose $\Phi, \Phi_\infty$ satisfy* **(A3)–(A6)**, *and for $t > 0$, let*

$$\rho(t) := \max\{t-1, (1+C_3 t^{\kappa'})^2\},$$

*where $C_3 = 3C_2/(2\sqrt{C_0})$ and $C_2$ is the constant in Proposition 7.2. Then $d_{\rho(u)}(x,y) = d(x,y)$ and $d_{\infty,\rho(u)}(x,y) = d_\infty(x,y)$ for any $x, y \in \boldsymbol{B}(u)$ and $1 \le u \le R$.*

**Proof** By the definition, $d(x,y) \le d_{\rho(u)}(x,y)$ always holds. We assume $d(x,y) < d_{\rho(u)}(x,y)$ for some $x, y \in \boldsymbol{B}(u)$. Then there is a smooth $\gamma\colon [a,b] \to \mathbb{R}^3$ joining $x$ and $y$ such that $d(x,y) \le l(\gamma) < d_{\rho(u)}(x,y)$, which implies the existence of $c \in [a,b]$ satisfying $|\gamma(c)| = \rho(u)$. Then one can see

$$\begin{aligned} l(\gamma) \ge l(\gamma|_{[a,c]}) &\ge d(0,\gamma(c)) - d(0,\gamma(a)) \\ &\ge 2\sqrt{C_0}(\sqrt{\rho(u)} - 1) - D_{u,u} \\ &\ge 2\sqrt{C_0}\Big(\sqrt{(1+C_3u^{\kappa'})^2} - 1\Big) - C_2u^{\kappa'} \\ &\ge 2C_2u^{\kappa'} \end{aligned}$$

by Proposition 7.2. On the other hand, we have

$$d_{\rho(u)}(x,y) \le D_{u,\rho(u)} \le D_{u,u} \le C_2u^{\kappa'}$$

by Proposition 7.2. Therefore, we obtain

$$2C_2u^{\kappa'} \le l(\gamma) < d_{\rho(u)}(x,y) \le C_2u^{\kappa'},$$

a contradiction. We can show $d_\infty(x,y) = d_{\infty,\rho(u)}(x,y)$ in the same way. □

## 7.2 Estimates (2)

In this subsection, let $\gamma\colon [a,b] \to \boldsymbol{B}(u)$ be a smooth path joining $x, y \in \mathbb{R}^3 \backslash L(D)$, where

$$L(D) := \{\zeta \in \mathbb{R}^3 : |\zeta_{\mathbb{C}}| < D\}.$$

Now, we are going to show that if $\gamma$ is a minimizing geodesic joining $x$ and $y$, then it never approaches the axis $\{(t,0,0) \in \mathbb{R}^3 : t \in \mathbb{R}\}$. To show this, if the given $\gamma$ invades $L(D)$, then we modify $\gamma$ and construct the new path $c_\gamma$ to not invade $L(D)$.

**Lemma 7.4** *Suppose $\Phi, \Phi_\infty$ satisfy* **(A4)**. *Let $\gamma = (\gamma_{\mathbb{R}}, \gamma_{\mathbb{C}})\colon [a,b] \to \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$ be a smooth path satisfying $|\gamma_{\mathbb{C}}(a)| = |\gamma_{\mathbb{C}}(b)| = D$ and $|\gamma_{\mathbb{C}}(t)| \le D$ for any $t \in [a,b]$. Define $P_\gamma\colon [a,b] \to \mathbb{R}^3$ by*

$$P_\gamma(t) := (\gamma_{\mathbb{R}}(t), \gamma_{\mathbb{C}}(a)).$$

*Then $l(P_\gamma) \le l(\gamma)$ and $l_\infty(P_\gamma) \le l_\infty(\gamma)$ hold.*

**Proof** Since $\Phi(\gamma(t)) \geq \Phi(P_\gamma(t))$ holds by the second inequality of **(A4)**, and

$$|\gamma'|^2 = |\gamma'_{\mathbb{R}}|^2 + |\gamma'_{\mathbb{C}}|^2 \geq |\gamma'_{\mathbb{R}}|^2 = |P'_\gamma|^2$$

holds, we can deduce

$$l(\gamma) = \int_a^b \sqrt{\Phi(\gamma(t))}\,|\gamma'(t)|\,dt \geq \int_a^b \sqrt{\Phi(P_\gamma(t))}\,|P'_\gamma(t)|\,dt \geq l(P_\gamma). \qquad \square$$

Let $\gamma\colon [a,b] \to \mathbb{R}^3$ be a smooth path joining $x, y \in \mathbb{R}^3 \backslash L(D)$, and assume that $|\gamma_{\mathbb{C}}(a')| = |\gamma_{\mathbb{C}}(b')| = D$ and that $\gamma((a',b'))$ is contained in $\overline{L(D)}$ for some $a \leq a' < b' \leq b$. Then define a new path $\Gamma(\gamma,[a',b'])\colon [a,b] \to \mathbb{R}^3$ by connecting

$$\gamma|_{[a,a']}, \quad P_{\gamma|_{[a',b']}}, \quad e^{i\theta}\gamma|_{[b',b]}.$$

Here, by choosing $e^{i\theta}$ appropriately, $\Gamma(\gamma,[a',b'])$ is continuous and piecewise smooth. By Lemma 7.4, the length of $\Gamma(\gamma,[a',b'])$ is not longer than that of $\gamma$ since $S^1$–rotation preserves $d$ and $d_\infty$.

Put $J := \gamma^{-1}(L(D)) \cap (a,b)$. Since $J$ is open in $(a,b)$, it is decomposed into disjoint open intervals

$$J = \bigsqcup_{q \in \mathcal{Q}} (a_q, b_q)$$

for some $a_q, b_q \in [a,b]$ and countable set $\mathcal{Q}$. If $q \in \mathcal{Q}$, then $|\gamma_{\mathbb{C}}(a_q)| = |\gamma_{\mathbb{C}}(b_q)| = D$ holds. Then we have $\gamma_1 := \Gamma(\gamma,[a_q,b_q])$ for a fixed $q \in \mathcal{Q}$; moreover, we obtain $\gamma_2 := \Gamma(\gamma_1,[a_{q'},b_q])$ for another $q' \in \mathcal{Q}$, and repeating this process for all $q \in \mathcal{Q}$ we finally obtain the piecewise smooth path $c\colon [a,b] \to \mathbb{R}^3$ such that $c(a) = \gamma(a)$, $c(b) = e^{i\theta}\gamma(b)$ for some $e^{i\theta_0}$ and

$$l(c) \leq l(\gamma), \quad l_\infty(c) \leq l_\infty(\gamma).$$

Here, we have to modify $c$ so that the terminal points of both paths coincide. Put $\bar{b} := \sup\{t \in [a,b] : |\gamma_{\mathbb{C}}(t)| = D\}$. Then define a path $\hat{\gamma}$ by connecting $c|_{[a,\bar{b}]}$ and $\gamma|_{[\bar{b},b]}$. Here, to connect $c(\bar{b})$ and $\gamma(\bar{b})$, we add the path $c_{\theta_0}\colon [0,\theta_0] \to \partial L(D)$ defined by $c_{\theta_0}(t) = e^{it}\gamma(\bar{b})$. Then by **(A6)**, we obtain

$$l(c_{\theta_0}) \leq \sqrt{C_1(1+u^\kappa)}\sqrt{D} \quad \text{and} \quad l_\infty(c_{\theta_0}) \leq \sqrt{C_1 u^\kappa}\sqrt{D}$$

if $|\gamma(\bar{b})| \leq u \leq R$. Hence we have the following proposition.

**Proposition 7.5** *Let $D \leq 1$ and $1 \leq u \leq R$, and let $x, y, \gamma, \hat{\gamma}$ be as above. If the image of $\gamma$ is contained in $\boldsymbol{B}(u)$, then we have*

$$l(\hat{\gamma}) - l(\gamma) \leq \sqrt{C_1(1+u^\kappa)}\sqrt{D}, \quad l_\infty(\hat{\gamma}) - l_\infty(\gamma) \leq \sqrt{C_1 u^\kappa}\sqrt{D}.$$

**Proposition 7.6** *Let $x, y, \gamma, \hat{\gamma}$ be as above. If the image of $\gamma$ is contained in $\boldsymbol{B}(u)$, then the image of $\hat{\gamma}$ is contained in $\boldsymbol{B}(u+D)\backslash L(D)$.*

**Proof** It is obvious by the construction that the image of $\hat{\gamma}$ is contained in $\mathbb{R}^3\backslash L(D)$. Since $S^1$–action preserves $\boldsymbol{B}(u)$, and

$$|P_\gamma|^2 \le |\gamma|^2 + D^2, \qquad \left|\left(\gamma_{\mathbb{R}}(t), \frac{D\gamma_{\mathbb{C}}(t)}{|\gamma_{\mathbb{C}}(t)|}\right)\right|^2 \le |\gamma|^2 + D^2$$

hold, we have the assertion. □

## 7.3 Estimates (3)

Let

$$\begin{aligned} \mathrm{Path}(u, D, x, y) &:= \{\gamma \in \mathrm{Path}(x, y) : \mathrm{Im}(\gamma) \subset K(u, D)\}, \\ d_{u,D}(x, y) &:= \inf_{\gamma \in \mathrm{Path}(u,D,x,y)} l(\gamma), \\ d_{\infty,u,D}(x, y) &:= \inf_{\gamma \in \mathrm{Path}(u,D,x,y)} l_\infty(\gamma). \end{aligned}$$

for $x, y \in K(u, D)$. By the definition, $d(x, y) \le d_{u,D}(x, y)$ always holds. In this subsection, we consider the opposite estimate.

**Lemma 7.7** *Let $\hat{\zeta} := (\zeta_{\mathbb{R}}, D\zeta_{\mathbb{C}}/|\zeta_{\mathbb{C}}|)$ if $\zeta_{\mathbb{C}} \neq 0$, and $\hat{\zeta} := (\zeta_{\mathbb{R}}, D)$ if $\zeta_{\mathbb{C}} = 0$. Suppose $\Phi, \Phi_\infty$ satisfy* **(A3)–(A6)**, *and $1 \le u \le R$.*

(1) *If $\zeta \in L(D) \cap \boldsymbol{B}(u-1)$ and $0 < D \le 1$, then*

$$d_u(\zeta, \hat{\zeta}) \le 2\sqrt{C_1(1+u^\kappa)D}, \quad d_{\infty,u}(\zeta, \hat{\zeta}) \le 2\sqrt{C_1 u^\kappa D}.$$

(2) *If $\zeta \in L(D) \cap K(u-1, D)$ and $0 < D \le 1$, then*

$$d_{u,D}(\zeta, \hat{\zeta}) \le 2\sqrt{C_1(1+u^\kappa)D}, \quad d_{\infty,u,D}(\zeta, \hat{\zeta}) \le 2\sqrt{C_1 u^\kappa D}.$$

**Proof** Let $\gamma(t) = (\zeta_{\mathbb{R}}, t\hat{\zeta}_{\mathbb{C}})$ for $t \in [|\zeta_{\mathbb{C}}|/D, 1]$. Then $\gamma$ is joining $\zeta$ and $\hat{\zeta}$, and the image of $\gamma$ is contained in $\boldsymbol{B}(u-1+D) \subset \boldsymbol{B}(u)$. Then by **(A3)** and **(A6)**, we have $\Phi(\gamma(t)) \le C_1(1+u^\kappa)/(tD)$. Then we have

$$d_u(\zeta, \hat{\zeta}) \le l(\gamma) \le 2\sqrt{C_1(1+u^\kappa)D}.$$

Moreover, if $\zeta \in K(u-1, D)$, then the image of $\gamma$ is contained in $K(u, D)$; therefore,

$$d_{u,D}(\zeta, \hat{\zeta}) \le l(\gamma) \le 2\sqrt{C_1(1+u^\kappa)D}.$$

The estimates for $d_{\infty,u}(\zeta, \hat{\zeta})$ and $d_{\infty,u,D}(\zeta, \hat{\zeta})$ follow in the same way. □

**Proposition 7.8** *Suppose $\Phi, \Phi_\infty$ satisfy* **(A3)–(A6)** *and let $\rho$ be as in Proposition 7.3. If $\rho(u+1)+1 \le R$, then*

$$|d_{\rho(u+1)+1,D}(x,y) - d(x,y)| \le \xi(u)\sqrt{D},$$
$$|d_{\infty,\rho(u+1)+1,D}(x,y) - d_\infty(x,y)| \le \xi_\infty(u)\sqrt{D}$$

*hold for any $x, y \in K(u, D)$ and $0 < D \le 1$, where*

$$\xi(u) := \sqrt{C_1(1+(\rho(u+1)+1)^\kappa)} + 8\sqrt{C_1(1+(u+1)^\kappa)} + 2,$$
$$\xi_\infty(u) := \sqrt{C_1(\rho(u+1)+1)^\kappa} + 8\sqrt{C_1(u+1)^\kappa} + 2.$$

**Proof** Since $d(x,y) \le d_{\rho(u+1)+1,D}(x,y)$ always holds, it suffices to show that $d_{\rho(u+1)+1,D}(x,y) - d(x,y) \le \xi(u)\sqrt{D}$. Let $x, y \in K(u,D)$ and $0 < D \le 1$. By the assumption $\rho(u+1)+1 \le R$ and the definition of $\rho$, we have that $u+1 \le R$. Define $\hat{x} \in \mathbb{R}^3$ as in Lemma 7.7 if $x \in L(D)$, and $\hat{x} := x$ if $x \notin L(D)$. Define $\hat{y}$ in the same way. Then we can see $\hat{x}, \hat{y} \in \boldsymbol{B}(u+1)\backslash L(D)$ and $d_{u+1,D}(x,\hat{x}) \le 2\sqrt{C_1(1+(u+1)^\kappa)D}$ by Lemma 7.7; consequently, we obtain

$$d_{u+1,D}(x,\hat{x}) + d_{u+1,D}(y,\hat{y}) \le 4\sqrt{C_1(1+(u+1)^\kappa)D}. \tag{10}$$

For any $\gamma \in \mathrm{Path}(\hat{x},\hat{y})$, we construct $F(\gamma) \in \mathrm{Path}(\rho(u+1)+1, D, \hat{x}, \hat{y})$ as follows. By Proposition 7.3, we can see

$$l(\gamma) \ge d(\hat{x},\hat{y}) = d_{\rho(u+1)}(\hat{x},\hat{y}) = \inf_{c \in \mathrm{Path}(\rho(u+1),x,y)} l(c).$$

Accordingly, we can take $c \in \mathrm{Path}(\rho(u+1), x, y)$ such that $l(c) \le l(\gamma) + \sqrt{D}$. Then we can apply the argument in Section 7.2 to $\hat{x}, \hat{y}$ and $c$ so that we obtain a piecewise smooth path $\hat{c}$ whose image is contained in $\boldsymbol{B}(\rho(u+1)+1)\backslash L(D)$, hence in $K(\rho(u+1)+1, D)$. Then we have

$$\liminf_{k\to\infty} l(\hat{c}) - l(c) \le \sqrt{C_1(1+(\rho(u+1)+1)^\kappa)D}$$

by Proposition 7.5. Therefore, there is a sufficiently large $k$, which may depend on $n$ and $D$, such that $l(\hat{c}) - l(c) \le \sqrt{C_1(1+(\rho(u+1)+1)^\kappa)D} + \sqrt{D}$. Put $F(\gamma) = \hat{c}$. Then we can see

$$\begin{aligned} l(F(\gamma)) - l(\gamma) &\le l(F(\gamma)) - l(c) + l(c) - l(\gamma) \\ &\le \sqrt{C_1(1+(\rho(u+1)+1)^\kappa)D} + \sqrt{D} + \sqrt{D} \\ &= \big(\sqrt{C_1(1+(\rho(u+1)+1)^\kappa)} + 2\big)\sqrt{D}. \end{aligned}$$

Thus we obtain $F(\gamma) \in \mathrm{Path}(\rho(u+1)+1, D, \hat{x}, \hat{y})$ for every $\gamma \in \mathrm{Path}(\hat{x},\hat{y})$ such that

$$l(F(\gamma)) - l(\gamma) \le \big(\sqrt{C_1(1+(\rho(u+1)+1)^\kappa)} + 2\big)\sqrt{D}. \tag{11}$$

By taking the infimum of (11) for all $\gamma \in \mathrm{Path}(\hat{x}, \hat{y})$, we obtain

$$d_{\rho(u+1)+1,D}(\hat{x}, \hat{y}) \le d(\hat{x}, \hat{y}) + \left(\sqrt{C_1(1 + (\rho(u+1)+1)^\kappa)} + 2\right)\sqrt{D}. \tag{12}$$

Since $\rho(u+1) \ge u+1$, we have

$$\begin{aligned}
|d_{\rho(u+1)+1,D}(\hat{x}, \hat{y}) - d_{\rho(u+1)+1,D}(x, y)| &\le d_{\rho(u+1)+1,D}(\hat{x}, x) + d_{\rho(u+1)+1,D}(\hat{y}, y)\\
&\le d_{u+1,D}(\hat{x}, x) + d_{u+1,D}(\hat{y}, y)\\
&\le 4\sqrt{C_1(1 + (u+1)^\kappa)D},\\
|d(\hat{x}, \hat{y}) - d(x, y)| &\le d(\hat{x}, x) + d(\hat{y}, y)\\
&\le d_{u+1,D}(\hat{x}, x) + d_{u+1,D}(\hat{y}, y)\\
&\le 4\sqrt{C_1(1 + (u+1)^\kappa)D}
\end{aligned}$$

by (10); hence

$$\begin{aligned}
d_{\rho(u+1)+1,D}(x, y) &\le d_{\rho(u+1)+1,D}(\hat{x}, \hat{y}) + 4\sqrt{C_1(1 + (u+1)^\kappa)D},\\
d(\hat{x}, \hat{y}) &\le d(x, y) + 4\sqrt{C_1(1 + (u+1)^\kappa)D}
\end{aligned}$$

hold. By combining these inequalities with (12), we obtain

$$d_{\rho(u+1)+1,D}(x, y) \le d(x, y) + \xi(u)\sqrt{D}.$$

The second inequality can be shown in the same way. □

## 7.4 From (A3)–(A6) to (A1) and (A2)

**Proposition 7.9** *Suppose that $\Phi, \Phi_\infty$ satisfy (A3)–(A6), and let $\gamma\colon [a, b] \to K(u, D)$ and $1 \le u \le R$. Then*

$$|l(\gamma) - l_\infty(\gamma)| \le \sqrt{\frac{\varepsilon u}{C_0 D^m}}\, l_\infty(\gamma).$$

**Proof** Since $l(\gamma) = \int_a^b \sqrt{\Phi(\gamma(t))}\,|\gamma'(t)|\,dt$, one can see

$$\begin{aligned}
|l(\gamma) - l_\infty(\gamma)| &\le \int_a^b \sqrt{|\Phi(\gamma) - \Phi_\infty(\gamma)|}\,|\gamma'|\,dt\\
&\le \int_a^b \sqrt{\frac{|\Phi(\gamma) - \Phi_\infty(\gamma)|}{\Phi_\infty(\gamma)}}\sqrt{\Phi_\infty(\gamma)}\,|\gamma'|\,dt\\
&\le \int_a^b \sqrt{\frac{\varepsilon \max\{|\gamma|, 1\}}{C_0 D^m}}\sqrt{\Phi_\infty(\gamma(t))}\,|\gamma'(t)|\,dt
\end{aligned}$$

by **(A3)** and **(A5)**. Since we have assumed $|\gamma| \leq u$ and $u \geq 1$, we have

$$|l(\gamma) - l_\infty(\gamma)| \leq \sqrt{\frac{\varepsilon u}{C_0 D^m}}\, l_\infty(\gamma). \qquad \square$$

**Proposition 7.10** *Suppose that $\Phi$ and $\Phi_\infty$ satisfy* **(A3)**, **(A5)** *and* **(A6)**. *Then*

$$|d_{u,D}(x,y) - d_{\infty,u,D}(x,y)| \leq \sqrt{\frac{\varepsilon u}{C_0 D^m}}\, d_{\infty,u,D}(x,y)$$

*holds for all $1 \leq u \leq R$.*

**Proof** Put $\delta = \sqrt{\varepsilon u/(C_0 D^m)}$. Then Proposition 7.9 gives

$$(1-\delta) l_\infty(\gamma) \leq l(\gamma) \leq (1+\delta) l_\infty(\gamma). \tag{13}$$

Then by taking the infimum of (13) for all $\gamma \in \mathrm{Path}(u, D, x, y)$, we can see

$$(1-\delta) d_{\infty,u,D}(x,y) \leq d_{n,u,D}(x,y) \leq (1+\delta) d_{\infty,u,D}(x,y)$$

for all $u \geq 0$. $\square$

**Proposition 7.11** *Suppose that $\Phi, \Phi_\infty$ satisfy* **(A3)–(A6)** *and $u \leq 1$. Let $u^{(2)} := \rho(u+2)+1 \leq R$. Then for all $x, y \in \boldsymbol{B}(u)$, we have*

$$|d(x,y) - d_\infty(x,y)| \\ \leq 26\sqrt{C_1(1+R^\kappa)D} + 4\sqrt{D} + \sqrt{\frac{\varepsilon R}{C_0 D^m}}\big(C_2 R^{\kappa'} + (9\sqrt{C_1 R^\kappa} + 2)\sqrt{D}\big).$$

**Proof** Put $u^{(1)} = \rho(u+1)+1$ and let $x, y \in K(u, D)$. Then $u^{(1)} \leq R$. By combining Propositions 7.8 and 7.10, we have

$$\begin{aligned}|d(x,y) - d_\infty(x,y)| &\leq |d(x,y) - d_{u^{(1)},D}(x,y)| + |d_{u^{(1)},D}(x,y) - d_{\infty,u^{(1)},D}(x,y)| \\ &\qquad + |d_\infty(x,y) - d_{\infty,u^{(1)},D}(x,y)| \\ &\leq \xi(u)\sqrt{D} + \xi_\infty(u)\sqrt{D} + \sqrt{\frac{\varepsilon u^{(1)}}{C_0 D^m}}\, d_{\infty,u^{(1)},D}(x,y) \\ &\leq 2\xi(u)\sqrt{D} + \sqrt{\frac{\varepsilon u^{(1)}}{C_0 D^m}}\,(d_\infty(x,y) + \xi_\infty(u)\sqrt{D}).\end{aligned}$$

By Proposition 7.2, $D_{\infty,u} < C_2 u^{\kappa'}$ holds if $u \geq 1$; consequently, $d_\infty(x,y)$ is not more than $C_2 u^{\kappa'}$. Therefore, for all $x, y \in K(u, D)$, we obtain

$$|d(x,y) - d_\infty(x,y)| \leq 2\xi(u)\sqrt{D} + \sqrt{\frac{\varepsilon u^{(1)}}{C_0 D^m}}(C_2 u^{\kappa'} + \xi_\infty(u)\sqrt{D}).$$

Next we consider the case of $x \in \boldsymbol{B}(u)$, but $x \notin K(u, D)$. In this case, $x \in \boldsymbol{B}(u) \cap L(D)$ holds, hence we can apply Lemma 7.7. Let $\hat{x}$ be as in Lemma 7.7. Then we can see

$$d(x, \hat{x}) \le 2\sqrt{C_1(1 + (u+1)^\kappa)D},$$

and $\hat{x}$ is contained in $K(u+1, D)$. Here we suppose that $y$ is also contained in $\boldsymbol{B}(u) \cap L(D)$ and follow the same procedure. If $y$ is in $K(u, D)$, then suppose $y = \hat{y}$ in the following discussion. Now we have

$$|d(x, y) - d(\hat{x}, \hat{y})| \le d(x, \hat{x}) + d(y, \hat{y}) \le 4\sqrt{C_1(1 + (u+1)^\kappa)D}.$$

Hence we can see

$$\begin{aligned} &|d(x,y) - d_\infty(x,y)| \\ &\le 8\sqrt{C_1(1+(u+1)^\kappa)D} + |d(\hat{x},\hat{y}) - d_\infty(\hat{x},\hat{y})| \\ &\le 8\sqrt{C_1(1+(u+1)^\kappa)D} + 2\xi(u+1)\sqrt{D} + \sqrt{\frac{\varepsilon u^{(2)}}{C_0 D^m}}\big(C_2(u+1)^{\kappa'} + \xi_\infty(u+1)\sqrt{D}\big). \end{aligned}$$

Since $\xi(u)$ is monotonically increasing and $u + 2 \le u^{(2)} \le R$ holds, we have

$$\xi(u+1) \le 9\sqrt{C_1(1 + R^\kappa)} + 2, \quad \xi_\infty(u+1) \le 9\sqrt{C_1 R^\kappa} + 2. \qquad \square$$

**Corollary 7.12** *Suppose that $\Phi$ and $\Phi_\infty$ satisfy* **(A3)–(A6)** *and that $\varepsilon \le 1$, and let $u^{(2)} := \rho(u+2) + 1 \le R$. Then there exists a constant $C$ independent of any other constants such that, for all $x, y \in \boldsymbol{B}(u)$,*

$$|d(x, y) - d_\infty(x, y)| < C(1 + \sqrt{C_1})(1 + C_0^{-\frac{1}{2}})R^{1+\frac{\kappa}{2}}\varepsilon^{\frac{1}{2(1+m)}}.$$

**Proof** In Proposition 7.11, let $D = \varepsilon^{1/(1+m)} \le 1$. As described in the proof of Proposition 7.2, $C_2$ is linearly dependent on $\sqrt{C_1}$. Then assertion follows by using $R \ge 1$, $\varepsilon \le 1$ and unifying constants. $\square$

**Proposition 7.13** *Suppose that $\Phi(\zeta) \ge A/|\zeta|$ holds for some $A > 0$ and all $\zeta$ with $|\zeta| \le 1$, and let $u(r) := \big(1 + \frac{1}{2}A^{-1/2}r\big)^2$. Then $B(0, r) \subset \boldsymbol{B}(u(r))$ holds for all $r > 0$, where $B(0, r)$ is the metric ball with respect to $d$.*

**Proof** Let $\zeta \in B(0, r)$. Then by the same argument as in the proof of the first inequality of Proposition 7.2, we have

$$2\sqrt{A}(\sqrt{|\zeta|} - 1) \le d(0, \zeta) < r,$$

which gives $|\zeta| < \big(1 + \frac{1}{2}A^{-1/2}r\big)^2 = u(r)$. $\square$

**Proposition 7.14** *Suppose that $\Phi$, $\Phi_\infty$ satisfy (A3)–(A6), and suppose $\varepsilon \le 1$. Then the identity map of $\mathbb{R}^3$ is an $(r,\delta)$–isometry from $(\mathbb{R}^3, d, 0)$ to $(\mathbb{R}^3, d_\infty, 0)$, where $r, \delta > 0$ are defined by*

$$\rho(u(r)+2)+1 = R, \quad \delta = C(1+\sqrt{C_1})(1+C_0^{-\frac{1}{2}})R^{1+\frac{\kappa}{2}}\varepsilon^{\frac{1}{2(1+m)}}.$$

**Proof** Let $x, y \in B(0,r)$. Then $x, y \in \boldsymbol{B}(u(r))$; hence

$$|d(x,y) - d_\infty(x,y)| < C(1+\sqrt{C_1})(1+C_0^{-\frac{1}{2}})R^{1+\frac{\kappa}{2}}\varepsilon^{\frac{1}{2(1+m)}} \tag{14}$$

holds. Next we show $B_\infty(0, r-\delta) \subset B(B(0,r),\delta)$. If $x \in B_\infty(0,r-\delta)$, then $x \in \boldsymbol{B}(u(r))$ holds; therefore, (14) gives

$$\begin{aligned} d(0,x) &< d_\infty(0,x) + C(1+\sqrt{C_1})(1+C_0^{-\frac{1}{2}})R^{1+\frac{\kappa}{2}}\varepsilon^{\frac{1}{2(1+m)}} \\ &< r - \delta + C(1+\sqrt{C_1})(1+C_0^{-\frac{1}{2}})R^{1+\frac{\kappa}{2}}\varepsilon^{\frac{1}{2(1+m)}} = r, \end{aligned}$$

which implies $B_\infty(0,r-\delta) \subset B(0,r)$. □

By Propositions 7.13 and 6.3, the following estimate is obtained.

**Proposition 7.15** *Let $\Phi_a$ be as in Section 6 and assume $\sum_{n=0}^\infty A_{S_n,P}^{T_n} - 2a^{1/(1+\alpha)} > 0$. Then $\sup_{\zeta\in B(0,r)} 1/(N\sqrt{\Phi_a(\zeta)})$ is not more than*

$$\frac{(a/P)^{\frac{1}{1+\alpha}}}{\sqrt{\sum_{n=0}^\infty A_{S_n,P}^{T_n} - 2(a/P)^{\frac{1}{1+\alpha}}}}\left(1 + \frac{r}{2\sqrt{\sum_{n=0}^\infty A_{S_n,P}^{T_n} - 2(a/P)^{\frac{1}{1+\alpha}}}}\right).$$

Combining Propositions 7.14 and 7.15, we obtain the following theorem.

**Theorem 7.16** *Let $a_i, P_i, n_i > 0$, $\lim_{i\to\infty} a_i = 0$ and $\lim_{i\to\infty} n_i \to \infty$. Put $S_{i,n_i}$ and $T_{i,n_i}$ as in Section 6. Suppose that there are constants $\varepsilon = \varepsilon_i(R)$, $C_0$, $C_1$, $\kappa$, $m$ for all $R \ge 1$ such that $\Phi = \Phi_{a_i}$ and $\Phi_\infty$ satisfy (A3)–(A6). If*

$$\lim_{i\to\infty} \varepsilon_i(R) = \lim_{i\to\infty} \frac{a_i}{P_i} = 0, \quad \liminf_{i\to\infty} \sum_{l=0}^\infty A_{S_{i,n_i},P_i}^{T_{i,n_i}} > 0$$

*and $C_0, C_1, \kappa, m$ are independent of $i$, $R$, then*

$$\{(X, a_i g_\Lambda, p)\}_i \xrightarrow[i\to\infty]{\text{GH}} (\mathbb{R}^3, d_\infty, 0).$$

**Proof** Fix $r > 0$ and $\delta > 0$ arbitrarily. Put $R(r) = \rho(u(r)+2)+1$, and let $C > 0$ be the constant in Corollary 7.12. By the assumption, there exists $i(r,\delta) > 0$ such that

$$C(1+\sqrt{C_1})(1+C_0^{-\frac{1}{2}})R(r)^{1+\frac{\kappa}{2}}\varepsilon_i(R(r))^{\frac{1}{2(1+m)}} < \tfrac{1}{2}\delta$$

holds for all $i \geq i(r,\delta)$. Then by Proposition 7.14, $\mathrm{id}_{\mathbb{R}^3}$ is an $\left(r, \frac{1}{2}\delta\right)$–isometry from $(\mathbb{R}^3, d_{a_i}, 0)$ to $(\mathbb{R}^3, d_{a_\infty}, 0)$. By Proposition 7.15, we can take $i'(r,\delta) \geq i(r,\delta)$ such that $\sup_{\zeta\in B(0,r)} 1/(N\sqrt{\Phi_a(\zeta)}) < \frac{1}{2}\delta$ for all $i \geq i'(r,\delta)$. Then Proposition 5.1 gives the assertion. □

# 8 Convergence

In this section, we consider the convergence of $\{(X, a_i g_\Lambda)\}_i$, where $\Lambda$ is as defined in Section 6, and $\{a_i\}_i$ is a sequence with $a_i > 0$ and $\lim_{i\to\infty} a_i$, applying Theorem 7.16. To apply them, we have to estimate constants $\varepsilon, C_0, C_1$ in **(A3)–(A6)** uniformly with respect to $i \in \mathbb{N}$, and show that $\varepsilon \to 0$ as $i \to \infty$. In Section 8.1, we consider the uniform estimate for the case of $P = 1$, which is the simplest case. In Sections 8.2 and 8.3, we suppose $P$ is depending on some parameters. Then we apply them to show Theorems 1.2 and 1.3 in Sections 8.4 and 8.5.

Put $S_{a,n} := a^{1/(1+\alpha)}K_{2n}$ and $T_{a,n} := a^{1/(1+\alpha)}K_{2n+1}$. We take a subsequence

$$\{K_{n_0} < K_{n_1} < K_{n_2} < \cdots\} \subset \{K_0 < K_1 < K_2 < \cdots\}.$$

We are now going to consider the convergence (in several cases according to the rate of the convergence of $\{a_i\}_i$) or the divergence of $\{K_n\}_n$.

From now on, we put

$$\Phi_S^T(\zeta) := \Phi_{S,1}^T(\zeta) = \int_S^T \frac{dx}{|\zeta - (x^\alpha, 0, 0)|}, \quad A_S^T := A_{S,1}^T = \int_S^T \frac{dx}{1+x^\alpha}.$$

## 8.1 Convergence (1)

Fix $a > 0$, $n$ and $0 \leq S < T \leq \infty$, and put $P = 1$.

**Proposition 8.1** *Let $R \geq 1$ and $D \leq 1$. There exists a constant $C_\alpha > 0$ depending only on $\alpha$ such that*

$$|\Phi_a(\zeta) - \Phi_S^T(\zeta)| \leq \frac{C_\alpha \varepsilon_n}{D}$$

*holds for any $\zeta \in K(R, D)$, where*

$$\varepsilon_{a,n} = a^{\frac{1}{1+\alpha}} + \frac{K_{2n-1}}{K_{2n}}S_{a,n} + \left(\frac{K_{2n+2}}{K_{2n+1}}T_{a,n}\right)^{-\alpha+1} + |S_{a,n} - S| + |T_{a,n}^{-\alpha+1} - T^{-\alpha+1}|.$$

**Proof** By combining Proposition 6.1, (7) and (8), we have

$$\left|\Phi_a(\zeta)-\Phi_{S_{a,n}}^{T_{a,n}}(\zeta)\right| \le \frac{2a^{\frac{1}{1+\alpha}}}{D}+\frac{T_{a,n-1}}{D}+\frac{2S_{a,n+1}^{-\alpha+1}}{\alpha-1}$$

if $S_{a,n+1}\ge(2|\zeta|)^{1/\alpha}$. Here $|\zeta|\le R$, and

$$S_{a,n+1}=a^{\frac{1}{1+\alpha}}K_{2n+2}=\frac{K_{2n+2}}{K_{2n+1}}T_{a,n},$$
$$T_{a,n-1}=a^{\frac{1}{1+\alpha}}K_{2n-1}=\frac{K_{2n-1}}{K_{2n}}S_{a,n}.$$

On the other hand, we can see

$$\left|\Phi_{S_{a,n}}^{T_{a,n}}(\zeta)-\Phi_S^T(\zeta)\right| \le \frac{|S_{a,n}-S|}{D}+\frac{2|T_{a,n}^{-\alpha+1}-T^{-\alpha+1}|}{\alpha-1}.$$

Thus we obtain the assertion. □

Now, we put $\Phi=\Phi_a$, $\Phi_\infty=\Phi_S^T$, and suppose $a$, $|S_{a,n}-S|$ and $|T_{a,n}^{-\alpha+1}-T^{-\alpha+1}|$ are sufficiently small. Then the constants in **(A3)–(A6)** can be taken uniformly as

$$C_0=\frac{1}{2}A_S^T,\quad C_1=\frac{\alpha 2^{\frac{1}{\alpha}}}{\alpha-1},\quad m=1,\quad \kappa=\frac{1}{\alpha}.$$

Then by Proposition 8.1, if $\lim_{n\to\infty}K_{2n+1}/(2n)=\infty$, then we have $\varepsilon_{a,n}\to 0$ as $a\to 0$, $n\to\infty$, $|S_{a,n}-S|\to 0$ and $|T_{a,n}^{-\alpha+1}-T^{-\alpha+1}|\to 0$. Hence by Theorem 7.16, we have the next results.

**Theorem 8.2** *Let $(X,g_\Lambda)$ be as in Section 6 and suppose $\lim_{n\to\infty}K_{2n+1}/K_{2n}=\infty$. Assume that $\{a_i\}_i\subset\mathbb{R}^+$ and*

$$\{K_{n_0}<K_{n_1}<K_{n_2}<\cdots\}\subset\{K_0<K_1<K_2<\cdots\}$$

*satisfy*

$$\lim_{i\to\infty}a_i^{\frac{1}{1+\alpha}}K_{2n_i}=S\ge 0,\qquad \lim_{i\to\infty}a_i^{\frac{1}{1+\alpha}}K_{2n_i+1}=T\le\infty,\qquad S<T.$$

*Then $\{(X,a_n g_\Lambda,p)\}_n\xrightarrow{\mathrm{GH}}(\mathbb{R}^3,d_S^T,0)$, where $d_S^T$ is the metric induced by $\Phi_S^T\cdot h_0$.*

## 8.2 Convergence (2)

Let $(X,d_X,p)$, $(Y,d_Y,q)$ be pointed metric spaces and suppose $\lim_{n\to\infty}a_n=0$. Assume that $\{(X,a_n d_X,p)\}_n\xrightarrow{\mathrm{GH}}(Y,d_Y,q)$. It is easy to check that we have

$\{(X, sa_n d_X, p)\}_n \xrightarrow{\text{GH}} (Y, sd_Y, q)$ for any $s > 0$. Moreover, if $\{a_{n,m}\}_{n,m\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} a_{n,m} = 0$ for every $m$, and

$$\{(X, a_{n,m} d_X, p)\}_n \xrightarrow{\text{GH}} (Y_m, d_{Y_m}, q_m), \quad \{(Y_m, d_{Y_m}, q_m)\}_m \xrightarrow{\text{GH}} (Y, d_Y, q)$$

hold for every $m$, then by the diagonal argument one can show there exists a subset $\{a_{n,m(n)}\}_n \subset \{a_{n,m}\}_{n,m}$ such that $\lim_{n\to\infty} a_{n,m(n)} = 0$ and

$$\{(X, a_{n,m(n)} d_X, p)\}_n \xrightarrow{\text{GH}} (Y, d_Y, q).$$

Now, let $\mathcal{T}(X, d)$ be the set of isometry classes of tangent cones at infinity of $(X, d)$. From the above argument, one can see that $\mathcal{T}(X, d)$ is closed with respect to the pointed Gromov–Hausdorff topology, and if $(Y, d') \in \mathcal{T}(X, d)$, then its rescaling $(Y, ad')$ is also contained in $\mathcal{T}(X, d)$.

From Section 8.1, $(\mathbb{R}^3, d_S^T, 0)$ may appear as the tangent cone at infinity of some $(X, g_\Lambda)$, where $\Lambda$ is as in Section 6.

Let $\sigma > 0$, $0 \le S < T \le \infty$ and $I_\sigma\colon \zeta \mapsto \sigma^{-1}\zeta$ be the dilation. Then we have

$$I^*_{P^{\frac{1}{1+\alpha}}}(\Phi_S^T h_0) = P^{\frac{-1}{\alpha}} \Phi_{S''}^{T''}(\zeta) h_0 = \Phi_{S',P}^{T'}(\zeta) h_0,$$

where

$$S' = P^{\frac{-1}{1+\alpha}} S, \quad T' = P^{\frac{-1}{1+\alpha}} T, \quad S'' = P^{\frac{1}{\alpha(1+\alpha)}} S, \quad T'' = P^{\frac{1}{\alpha(1+\alpha)}} T. \tag{15}$$

Hence if $(\mathbb{R}^3, d_S^T, 0) \in \mathcal{T}(X, g_\Lambda)$, then $\{(\mathbb{R}^3, d_{\sigma S}^{\sigma T}, 0)\}_{\sigma\in\mathbb{R}^+}$ is also contained in $\mathcal{T}(X, g_\Lambda)$.

Fix a constant $\theta > 0$, put $P^{1/(1+\alpha)} = \theta\sqrt{S^{-\alpha+1} - T^{-\alpha+1}} > 0$, and let $S', T'$ be defined by (15).

**Proposition 8.3** *Let $R \ge 1$. There is a constant $C > 0$ depending only on $\alpha$ such that*

$$\left|\Phi_{S',P}^{T'}(\zeta) - \frac{1}{\theta^2(\alpha-1)}\right| \le \frac{CR}{\theta^3 S^\alpha \sqrt{S^{-\alpha+1} - T^{-\alpha+1}}}$$

*holds for any $\zeta \in K(R, D)$ if $\theta S^\alpha \sqrt{S^{-\alpha+1} - T^{-\alpha+1}} \ge 2R$.*

**Proof** Let $S''$ and $T''$ be defined by (15). Note that

$$\Phi_{S',P}^{T'}(\zeta) = P^{\frac{-1}{\alpha}} \int_{S''}^{T''} \frac{dx}{|\zeta - (x^\alpha, 0, 0)|}.$$

By the assumption, we have $P^{1/(1+\alpha)}S^{\alpha}=\theta S^{\alpha}\sqrt{S^{-\alpha+1}-T^{-\alpha+1}}\geq 2R$; then we see

$$\begin{aligned}\left|\int_{S''}^{T''}\frac{dx}{|\zeta-(x^{\alpha},0,0)|}-\int_{S''}^{T''}\frac{dx}{x^{\alpha}}\right|
&\leq\int_{S''}^{T''}\left|\frac{1}{|\zeta-(x^{\alpha},0,0)|}-\frac{1}{x^{\alpha}}\right|dx\\
&\leq\int_{S''}^{T''}\frac{2x^{\alpha}|\zeta|+|\zeta|^2}{|\zeta-(x^{\alpha},0,0)|x^{\alpha}(|\zeta-(x^{\alpha},0,0)|+x^{\alpha})}\,dx\\
&\leq\int_{S''}^{T''}\frac{8R}{x^{2\alpha}}\,dx+\int_{S''}^{T''}\frac{4R^2}{x^{3\alpha}}\,dx\\
&\leq\frac{8RP^{\frac{-2\alpha+1}{\alpha(1+\alpha)}}}{2\alpha-1}(S^{-2\alpha+1}-T^{-2\alpha+1})+\frac{4R^2P^{\frac{-3\alpha+1}{\alpha(1+\alpha)}}}{3\alpha-1}(S^{-3\alpha+1}-T^{-3\alpha+1}).\end{aligned}$$

Since we have

$$\int_{S''}^{T''}\frac{dx}{x^{\alpha}}=\frac{P^{\frac{-\alpha+1}{\alpha(1+\alpha)}}}{\alpha-1}(S^{-\alpha+1}-T^{-\alpha+1})=\frac{P^{\frac{1}{\alpha}}}{\theta^2(\alpha-1)},$$

we obtain

$$\begin{aligned}&\left|\Phi_{S',P}^{T'}(\zeta)-\frac{1}{\theta^2(\alpha-1)}\right|\\
&\qquad\leq\frac{8RP^{\frac{-3}{1+\alpha}}}{2\alpha-1}(S^{-2\alpha+1}-T^{-2\alpha+1})+\frac{4R^2P^{\frac{-4}{1+\alpha}}}{3\alpha-1}(S^{-3\alpha+1}-T^{-3\alpha+1}).\end{aligned}$$

Using the assumption $2R\leq P^{1/(1+\alpha)}S^{\alpha}$ once more, we have

$$\begin{aligned}&\left|\Phi_{S',P}^{T'}(\zeta)-\frac{1}{\theta^2(\alpha-1)}\right|\\
&\qquad\leq\frac{8RP^{\frac{-3}{1+\alpha}}}{2\alpha-1}(S^{-2\alpha+1}-T^{-2\alpha+1})+\frac{2RP^{\frac{-3}{1+\alpha}}}{3\alpha-1}(S^{-2\alpha+1}-S^{\alpha}T^{-3\alpha+1})\\
&\qquad\leq\theta^{-3}C_{\alpha}RS^{-\frac{1+\alpha}{2}}\frac{1-(S/T)^{3\alpha-1}}{(1-(S/T)^{\alpha-1})^{\frac{3}{2}}}.\end{aligned}$$

Now, put $f(x):=(1-x^{3\alpha-1})/((1-x^{\alpha-1})^{3/2})$ for $0\leq x<1$. Then there exists a constant $C'_{\alpha}>0$ such that $f(x)\leq C'_{\alpha}(1-x^{\alpha-1})^{-1/2}$ holds for all $0\leq x<1$. Consequently, by replacing $C_{\alpha}$ larger if necessary, we can see

$$\left|\Phi_{S',P}^{T'}(\zeta)-\frac{1}{\theta^2(\alpha-1)}\right|\leq\frac{C_{\alpha}R}{\theta^3S^{\alpha}\sqrt{S^{-\alpha+1}-T^{-\alpha+1}}}.\qquad\square$$

**Proposition 8.4** *Suppose* $\theta S^\alpha\sqrt{S^{-\alpha+1}-T^{-\alpha+1}} \geq 2R$ *for* $R \geq 1$. *Then*

$$A_{S',P}^{T'} \geq \frac{1}{2\theta^2(\alpha-1)}, \quad \Phi_{S',P}^{T'}(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}) \leq \frac{2|\zeta|}{\theta^2(\alpha-1)|\zeta_{\mathbb{C}}|}$$

*holds for any* $\zeta = (\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}) \in \mathbb{R}^3 = \mathbb{R}\oplus\mathbb{C}$ *with* $|\zeta| \leq R$.

**Proof** We have $1 \leq S^{-\alpha}x^\alpha$ for all $x \geq S$, then we can see

$$\begin{aligned} A_{S',P}^{T'} &\geq P^{-\frac{1}{1+\alpha}}\int_S^T \frac{dx}{S^{-\alpha}x^\alpha + P^{\frac{1}{1+\alpha}}x^\alpha} \\ &= \frac{1}{P^{\frac{1}{1+\alpha}}(S^{-\alpha}+P^{\frac{1}{1+\alpha}})}\int_S^T \frac{dx}{x^\alpha} \\ &= \frac{1}{P^{\frac{1}{1+\alpha}}S^{-\alpha}(1+S^\alpha P^{\frac{1}{1+\alpha}})}\,\frac{S^{-\alpha+1}-T^{-\alpha+1}}{\alpha-1}. \end{aligned}$$

Since we have

$$S^\alpha P^{\frac{1}{1+\alpha}} = \theta\sqrt{S^{-\alpha+1}-T^{-\alpha+1}} \geq 2R \geq 1,$$

we obtain

$$A_{S',P}^{T'} \geq \frac{S^{-\alpha+1}-T^{-\alpha+1}}{2(\alpha-1)P^{\frac{1}{1+\alpha}}S^{-\alpha}\cdot S^\alpha P^{\frac{1}{1+\alpha}}} = \frac{1}{2\theta^2(\alpha-1)}.$$

Next we consider the upper estimate of $\Phi_{S',P}^{T'}(\zeta)$. Take $\zeta$ such that $|\zeta| \leq R$; then we have $2|\zeta| \leq P^{1/(1+\alpha)}S^\alpha$ by the assumption. Then one can see

$$\begin{aligned} \Phi_{S',P}^{T'}(\zeta) &\leq P^{-\frac{1}{1+\alpha}}\int_S^T \frac{2dx}{P^{\frac{1}{1+\alpha}}x^\alpha} = P^{\frac{-2}{1+\alpha}}\,\frac{2(S^{-\alpha+1}-T^{-\alpha+1})}{\alpha-1} \\ &\leq \frac{2}{\theta^2(\alpha-1)} \leq \frac{2|\zeta|}{\theta^2(\alpha-1)|\zeta_{\mathbb{C}}|}. \end{aligned}$$

□

**Proposition 8.5** *Let* $\Phi = \Phi_{S',P}^{T'}$ *and* $\Phi_\infty \equiv 1/(\theta^2(\alpha-1))$. *Then there exists* $C > 0$ *such that* $\Phi$, $\Phi_\infty$ *satisfy* **(A3)–(A6)** *for* $R \geq 1$, *and*

$$\varepsilon = \frac{CR}{\theta^3 S^\alpha\sqrt{S^{-\alpha+1}-T^{-\alpha+1}}}, \qquad C_0 = \frac{1}{2\theta^2(\alpha-1)},$$

$$C_1 = \frac{1}{\theta^2}\max\left\{\frac{1}{\alpha-1},\frac{C}{2}\right\}, \quad m = 1, \quad \kappa = 1$$

*if* $\theta S^\alpha\sqrt{S^{-\alpha+1}-T^{-\alpha+1}} \geq 2R$.

**Proof** It is obvious that **(A4)** holds. Proposition 8.4 gives **(A5)** for $C_0 = 1/(2\theta^2(\alpha-1))$ if we take $\theta S^\alpha \sqrt{S^{-\alpha+1} - T^{-\alpha+1}} \geq 2R$. **(A6)** holds for $C_1 = 1/(\theta^2(\alpha-1))$ since

$$\frac{1}{\alpha-1} = \frac{1}{\alpha-1}\frac{|\zeta|}{|\zeta|} \leq \frac{1}{\alpha-1}\frac{|\zeta|}{|\zeta_{\mathbb{C}}|}.$$

Combining $\theta S^\alpha \sqrt{S^{-\alpha+1} - T^{-\alpha+1}} \geq 2R$ and Proposition 8.3, we can see that $\varepsilon \leq C/(2\theta^2)$. □

Now, Propositions 7.14 and 8.5 with $\theta = 1$ give the following theorem.

**Theorem 8.6** *Let $\{S_i\}_i$ and $\{T_i\}_i$ be sequences such that $0 \leq S_i < T_i \leq \infty$ and $\lim_{i\to\infty} S_i^\alpha \sqrt{S_i^{-\alpha+1} - T_i^{-\alpha+1}} = \infty$. Then $\{(\mathbb{R}^3, d_{S_i}^{T_i}, 0)\}_i$ converges to $(\mathbb{R}^3, h_0, 0)$ in the pointed Gromov–Hausdorff topology.*

Next put $P^{1/(1+\alpha)} = \theta|T-S|$ for $0 \leq S < T$ and $\theta > 0$, and let $S', T'$ be as in (15). Then we can show the following similarly to Proposition 8.9.

**Proposition 8.7** *Let $D \geq 1$. For all $\zeta \in K(R, D)$, we have*

$$\left|\Phi_{S',P}^{T'}(\zeta) - \frac{1}{\theta|\zeta|}\right| \leq \frac{2}{\theta D}, \qquad \left|\Phi_{S',P}^{T'}(\zeta) - \frac{1}{\theta|\zeta|}\right| \leq \frac{1+\theta T^\alpha(T-S)}{D^3} T^\alpha(T-S).$$

**Proof** Let $S''$ and $T''$ be defined by (15). The first inequality is obviously shown by $\Phi_{S',P}^{T'}(\zeta) \leq 1/(\theta D)$ and $1/|\zeta| \leq 1/D$. The second inequality follows from

$$\begin{aligned}
&\left|\int_{S''}^{T''} \frac{dx}{|\zeta - (x^\alpha, 0, 0)|} - \int_{S''}^{T''} \frac{dx}{|\zeta|}\right| \\
&\quad \leq \int_{S''}^{T''} \frac{2x^\alpha|\zeta| + x^{2\alpha}}{|\zeta - (x^\alpha,0,0)|\,|\zeta|\,(|\zeta-(x^\alpha,0,0)| + |\zeta|)}\,dx \\
&\quad \leq \int_{S''}^{T''} \frac{2x^\alpha}{D^2}\,dx + \int_{S''}^{T''} \frac{x^{2\alpha}}{D^3}\,dx \\
&\quad \leq C_\alpha \frac{P^{\frac{1}{\alpha}}(T^{\alpha+1} - S^{\alpha+1}) + P^{\frac{2\alpha+1}{\alpha(1+\alpha)}}(T^{2\alpha+1} - S^{2\alpha+1})}{D^3} \\
&\quad = C_\alpha \theta^{1+\frac{1}{\alpha}} T^{\alpha+1}(T-S)^{1+\frac{1}{\alpha}} \frac{1-(S/T)^{\alpha+1} + \theta(T-S)T^\alpha(1-(S/T)^{2\alpha+1})}{D^3}.
\end{aligned}$$

By the similar argument to Proposition 8.3, we can replace either $1-(S/T)^{\alpha+1}$ or $1-(S/T)^{2\alpha+1}$ by $1 - S/T$; hence we obtain the assertion. □

**Proposition 8.8** *For any* $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$,

$$A_{S',P}^{T'} \geq \frac{1}{\theta(1+\theta T^\alpha(T-S))}, \quad \Phi_{S',P}^{T'}(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \leq \frac{1}{\theta|\zeta_{\mathbb{C}}|}.$$

**Proof** One can see

$$\begin{aligned} A_{S',P}^{T'} = P^{-\frac{1}{1+\alpha}} \int_S^T \frac{dx}{1+P^{\frac{1}{1+\alpha}} x^\alpha} &\geq P^{-\frac{1}{1+\alpha}} \int_S^T \frac{dx}{1+P^{\frac{1}{1+\alpha}} T^\alpha} \\ &\geq \frac{T-S}{P^{\frac{1}{1+\alpha}}(1+P^{\frac{1}{1+\alpha}} T^\alpha)} \\ &= \frac{1}{\theta(1+\theta T^\alpha(T-S))}. \end{aligned}$$

We can also obtain

$$\Phi_{S',P}^{T'}(\zeta) \leq \frac{T-S}{P^{\frac{1}{1+\alpha}}|\zeta_{\mathbb{C}}|} = \frac{1}{\theta|\zeta_{\mathbb{C}}|}. \qquad \square$$

Combining Propositions 8.7 and 8.8, the next proposition is obtained.

**Proposition 8.9** *Let* $\Phi = \Phi_{S',P}^{T'}$ *and* $\Phi_\infty(\zeta) = 1/(\theta|\zeta|)$. *Then* $\Phi, \Phi_\infty$ *satisfy* **(A3)**–**(A6)** *for* $R \geq 1$, *and*

$$\varepsilon = (1+\theta T^\alpha(T-S))T^\alpha(T-S), \quad C_0 = \frac{1}{\theta(1+\theta T^\alpha(T-S))},$$
$$C_1 = \frac{2}{\theta}, \qquad m = 3, \qquad \kappa = 0$$

*for any* $0 \leq S < T$.

By Propositions 7.14 and 8.9 for $\theta = 1$, we have the next result.

**Theorem 8.10** *Let* $\{S_i\}_i$ *and* $\{T_i\}_i$ *be a sequence such that* $0 \leq S_i < T_i$ *and* $\lim_{i\to\infty} T_i^\alpha |T_i - S_i| = 0$. *Then* $\{(\mathbb{R}^3, d_{S_i}^{T_i}, 0)\}_i$ *converges to* $(\mathbb{R}^3, (1/|\zeta|)h_0, 0)$ *in the pointed Gromov–Hausdorff topology.*

## 8.3 Convergence (3)

Here, we fix $a > 0$ and $n$, and suppose that $T_{a,n} = a^{1/(1+\alpha)} K_{2n+1}$ is sufficiently small and that $S_{a,n+1} = a^{1/(1+\alpha)} K_{2n+2}$ is sufficiently large. Fix $P$ and $\theta$ such that

$$P^{\frac{1}{1+\alpha}} = \theta(T_{a,n} - S_{a,n}) = \sqrt{S_{a,n+1}^{-\alpha+1} - T_{a,n+1}^{-\alpha+1}}.$$

Put $S_l' = P^{-1/(1+\alpha)} S_{a,l}$ and $T_l' = P^{-1/(1+\alpha)} T_{a,l}$.

**Proposition 8.11** *Let $R \geq 1$ and $D \leq 1$, and let $P$ be as above. Assume that $P^{1/(\alpha(1+\alpha))} S_{a,n+2} \geq (2R)^{1/\alpha}$. Then there exists a constant $C_\alpha > 0$ depending only on $\alpha$ such that*

$$\left|\Phi_a(\zeta) - \Phi^{T'_n}_{S'_n,P}(\zeta) - \Phi^{T'_{n+1}}_{S'_{n+1},P}(\zeta)\right| \leq \frac{C_\alpha \varepsilon_{a,n}}{D},$$

*for any $\zeta \in K(R,D)$, where $\varepsilon_{a,n}$ is the constant defined by*

$$\varepsilon_{a,n} = \frac{1+K_{2n-1}}{\theta(K_{2n+1}-K_{2n})} + \frac{K_{2n+4}^{-\alpha+1}}{K_{2n+2}^{-\alpha+1} - K_{2n+3}^{-\alpha+1}}.$$

**Proof** By Proposition 6.1, (7) and (8), we have

$$|\Phi_a - \Phi^{T'_n}_{S'_n,P} - \Phi^{T'_{n+1}}_{S'_{n+1},P}| \leq \frac{2(a/P)^{\frac{1}{1+\alpha}} + P^{\frac{-1}{1+\alpha}} T_{a,n-1}}{D} + \frac{2S_{a,n+2}^{-\alpha+1}}{P^{\frac{2}{1+\alpha}}(\alpha-1)}$$

if $P^{1/(\alpha(1+\alpha))} S_{a,n+2} \geq (2R)^{1/\alpha}$. Since we have

$$\left(\frac{a}{P}\right)^{\frac{1}{1+\alpha}} = \frac{1}{\theta(K_{2n+1}-K_{2n})},$$

$$P^{\frac{-1}{1+\alpha}} T_{a,n-1} = \frac{K_{2n-1}}{\theta(K_{2n+1}-K_{2n})},$$

$$\frac{S_{a,n+2}^{-\alpha+1}}{P^{\frac{2}{1+\alpha}}} = \frac{K_{2n+4}^{-\alpha+1}}{K_{2n+2}^{-\alpha+1} - K_{2n+3}^{-\alpha+1}},$$

we have the assertion. □

Here, the assumption $P^{1/(\alpha(1+\alpha))} S_{a,n+2} \geq (2R)^{1/\alpha}$ can be replaced by

$$\left(\frac{K_{2n+4}}{K_{2n+2}}\right)^\alpha S_{a,n+1}^\alpha \sqrt{S_{a,n+1}^{-\alpha+1} - T_{a,n+1}^{-\alpha+1}} \geq 2R.$$

We can apply Propositions 8.3 and 8.7 to $\Phi^{T'_n}_{S'_n,P}$ and $\Phi^{T'_{n+1}}_{S'_{n+1},P}$. If we put

$$S = S_{a,n+1}, \quad T = T_{a,n+1}, \quad \theta = 1, \quad P^{\frac{1}{1+\alpha}} = \sqrt{S_{a,n+1}^{-\alpha+1} - T_{a,n+1}^{-\alpha+1}}$$

in Proposition 8.3, then we have

$$\left|\Phi^{T'_{n+1}}_{S'_{n+1},P} - \frac{1}{\alpha-1}\right| \leq \frac{CR}{S_{a,n+1}^\alpha \sqrt{S_{a,n+1}^{-\alpha+1} - T_{a,n+1}^{-\alpha+1}}}$$

for any $\zeta \in K(R,D)$ if $S_{a,n+1}^{\alpha}\sqrt{S_{a,n+1}^{-\alpha+1}-T_{a,n+1}^{-\alpha+1}} \geq 2R$. If we put

$$S = S_{a,n}, \quad T = T_{a,n}, \quad P^{\frac{1}{1+\alpha}} = \theta(T_{a,n}-S_{a,n})$$

in Proposition 8.7, then we have

$$\begin{aligned}\left|\Phi_{S_n',P}^{T_n'} - \frac{1}{\theta|\zeta|}\right| &\leq \frac{2}{\theta D},\\ \left|\Phi_{S_n',P}^{T_n'} - \frac{1}{\theta|\zeta|}\right| &\leq \frac{1+\theta T_{a,n}^{\alpha}(T_{a,n}-S_{a,n})}{D^3} T_{a,n}^{\alpha}(T_{a,n}-S_{a,n}).\end{aligned}$$

Now, we put $\Phi = \Phi_a$, $\Phi_\infty = 1/(\alpha-1)+1/(\theta|\zeta|)$. Combining the above arguments and Proposition 8.11, we can describe $\varepsilon, C_1$ in **(A3)** explicitly with $m=3$. Moreover, by Propositions 8.3, 8.7, 8.4 and 8.8, we obtain $C_0, C_1$ in **(A5)** and **(A6)**, and $\kappa = 1$. Fix a constant $A>0$, suppose that

$$A^{-1} \leq \theta \leq A, \quad S_{a,n+1}^{\alpha}\sqrt{S_{a,n+1}^{-\alpha+1}-T_{a,n+1}^{-\alpha+1}} \geq 2R,$$

and take $P$ as above. Then we can take these constants in **(A3)**–**(A6)** being only depending on $\alpha$, $A$ and $R$, if $\varepsilon_{a,n}$, $S_{a,n+1}^{-\alpha}(S_{a,n+1}^{-\alpha+1}-T_{a,n+1}^{-\alpha+1})^{-1/2}$ and $T_{a,n}^{\alpha}(T_{a,n}-S_{a,n})$ are sufficiently small. Therefore, we obtain the following result.

**Theorem 8.12** *Let $(X, g_\Lambda)$ be as in Section 6, take a subsequence*

$$\{K_{n_0} < K_{n_1} < K_{n_2} < \cdots\} \subset \{K_0 < K_1 < K_2 < \cdots\},$$

*and suppose*

$$\lim_{i\to\infty}\left(\frac{K_{2n_i-1}}{K_{2n_i+1}-K_{2n_i}} + \frac{K_{2n_i+4}^{-\alpha+1}}{K_{2n_i+2}^{-\alpha+1}-K_{2n_i+3}^{-\alpha+1}}\right) = 0. \tag{16}$$

*If a sequence $\{a_i\}_i \subset \mathbb{R}^+$ satisfies*

$$\lim_{i\to\infty}\frac{\sqrt{S_{a_i,n_i+1}^{-\alpha+1}-T_{a_i,n_i+1}^{-\alpha+1}}}{T_{a_i,n_i}-S_{a_i,n_i}} = \theta > 0,$$
$$\lim_{i\to\infty} S_{a_i,n_i+1}^{-\alpha}(S_{a_i,n_i+1}^{-\alpha+1}-T_{a_i,n_i+1}^{-\alpha+1})^{\frac{-1}{2}} = \lim_{i\to\infty} T_{a_i,n_i}^{\alpha}(T_{a_i,n_i}-S_{a_i,n_i}) = 0,$$

*then* $\{(X, a_i g_\Lambda, p)\}_n \xrightarrow{\mathrm{GH}} \big(\mathbb{R}^3, \big(1/(\alpha-1)+1/(\theta|\zeta|)\big)h_0, 0\big)$.

Next we estimate $\Phi_a - 1/(\alpha-1)$ in the same situation, as $\theta \to \infty$. We have

$$\left|\Phi_a - \Phi^{T'_{n+1}}_{S'_{n+1},P}\right| \le \frac{2(a/P)^{\frac{1}{1+\alpha}} + P^{\frac{-1}{1+\alpha}} T_{a,n-1} + P^{\frac{-1}{1+\alpha}}(T_{a,n}-S_{a,n})}{D} + \frac{2S^{-\alpha+1}_{a,n+2}}{P^{\frac{2}{1+\alpha}}(\alpha-1)}$$
$$\le \frac{C_\alpha}{D}\left(\frac{1+K_{2n-1}}{\theta(K_{2n+1}-K_{2n})} + \frac{1}{\theta} + \frac{K^{-\alpha+1}_{2n+4}}{K^{-\alpha+1}_{2n+2} - K^{-\alpha+1}_{2n+3}}\right).$$

Applying Propositions 8.3 and 8.4 with $\theta = 1$ and (5), we have

$$\left|\Phi_a - \frac{1}{\alpha-1}\right|$$
$$\le \frac{C_\alpha}{D}\left(\frac{1+K_{2n-1}}{\theta(K_{2n+1}-K_{2n})} + \frac{1}{\theta} + \frac{K^{-\alpha+1}_{2n+4}}{K^{-\alpha+1}_{2n+2}-K^{-\alpha+1}_{2n+3}} + \frac{R}{S^{\alpha}_{a,n+1}\sqrt{S^{-\alpha+1}_{a,n+1}-T^{-\alpha+1}_{a,n+1}}}\right),$$

$$\Phi_a \ge \left(A^{T'_{n+1}}_{S'_{n+1},P} - \frac{2}{\theta(K_{2n+1}-K_{2n})}\right)\min\left\{\frac{1}{|\zeta|},1\right\}$$
$$\ge \left(\frac{1}{2(\alpha-1)} - \frac{2}{\theta(K_{2n+1}-K_{2n})}\right)\min\left\{\frac{1}{|\zeta|},1\right\}$$

if $D \le 1$, $R \ge 1$ and $|\zeta| \le R$. Therefore, we can take $C_0$, $C_1$, $\kappa$ and $m$ in **(A3)–(A6)** depending only on $\alpha$ and $R$ if $\varepsilon \to 0$, where $\Phi = \Phi_a$ and $\Phi_\infty = 1/(\alpha-1)$. Hence we have the following theorem.

**Theorem 8.13** *Let $(X, g_\Lambda)$ be as in Section 6 and suppose $\{K_{n_i}\}_i$ satisfies (16). If a sequence $\{a_i\}_i \subset \mathbb{R}^+$ satisfies*

$$\lim_{i\to\infty} \frac{\sqrt{S^{-\alpha+1}_{a_i,n_i+1} - T^{-\alpha+1}_{a_i,n_i+1}}}{T_{a_i,n_i} - S_{a_i,n_i}} = \infty, \qquad \lim_{i\to\infty} S^{-\alpha}_{a_i,n_i+1}\left(S^{-\alpha+1}_{a_i,n_i+1} - T^{-\alpha+1}_{a_i,n_i+1}\right)^{\frac{-1}{2}} = 0,$$

*then* $\{(X, a_i g_\Lambda, p)\}_n \xrightarrow{\mathrm{GH}} (\mathbb{R}^3, h_0, 0)$.

By the similar argument, we have the following.

**Theorem 8.14** *Let $(X, g_\Lambda)$ be as in Section 6 and suppose $\{K_{n_i}\}_i$ satisfies (16). If a sequence $\{a_i\}_i \subset \mathbb{R}^+$ satisfies*

$$\lim_{i\to\infty} \frac{\sqrt{S^{-\alpha+1}_{a_i,n_i+1} - T^{-\alpha+1}_{a_i,n_i+1}}}{T_{a_i,n_i} - S_{a_i,n_i}} = 0,$$
$$\lim_{i\to\infty} S^{-\alpha}_{a_i,n_i+1}\left(S^{-\alpha+1}_{a_i,n_i+1} - T^{-\alpha+1}_{a_i,n_i+1}\right)^{\frac{-1}{2}} = \lim_{i\to\infty} T^{\alpha}_{a_i,n_i}(T_{a_i,n_i} - S_{a_i,n_i}) = 0,$$

*then* $\{(X, a_i g_\Lambda, p)\}_n \xrightarrow{\mathrm{GH}} (\mathbb{R}^3, (1/|\zeta|)h_0, 0)$.

**Proof** Put

$$P^{\frac{1}{1+\alpha}} := (T_{a,n} - S_{a,n}) = \theta\sqrt{S_{a,n+1}^{-\alpha+1} - T_{a,n+1}^{-\alpha+1}},$$

$$S'_l = P^{\frac{-1}{1+\alpha}} S_{a,l}, \quad T'_l = P^{\frac{-1}{1+\alpha}} T_{a,l}.$$

An argument similar to (7) gives

$$\Phi_{S'_{n+1},P}^{T'_{n+1}}(\zeta) \le \frac{2((S'_{n+1})^{-\alpha+1} - (T'_{n+1})^{-\alpha+1})}{P(\alpha-1)}$$

if $P(S'_{n+1})^\alpha \ge 2R$, which is equivalent to $\theta S_{a,n+1}^\alpha \sqrt{S_{a,n+1}^{-\alpha+1} - T_{a,n+1}^{-\alpha+1}} \ge 2R$. Then an argument similar to Proposition 8.11 gives

$$\left|\Phi_a(\zeta) - \Phi_{S'_n,P}^{T'_n}(\zeta)\right| \le \frac{C_\alpha \varepsilon_{a,n}}{D} + \frac{2}{(\alpha-1)\theta}$$

for any $\zeta \in K(R,D)$, where $\varepsilon_{a,n}$ is the constant defined by

$$\varepsilon_{a,n} = \frac{1+K_{2n-1}}{K_{2n+1}-K_{2n}} + \frac{K_{2n+4}^{-\alpha+1}}{\theta^2(K_{2n+2}^{-\alpha+1} - K_{2n+3}^{-\alpha+1})}.$$

Moreover, Proposition 8.7 with $\theta = 1$ gives

$$\left|\Phi_{S'_n,P}^{T'_n}(\zeta) - \frac{1}{|\zeta|}\right| \le \frac{2}{D},$$

$$\left|\Phi_{S'_n,P}^{T'_n}(\zeta) - \frac{1}{|\zeta|}\right| \le \frac{1+T_{a,n}^\alpha(T_{a,n}-S_{a,n})}{D^3} T_{a,n}^\alpha(T_{a,n}-S_{a,n}).$$

Then we can see $|\Phi_a - 1/|\zeta|| \le \varepsilon/D^3$ for some $\varepsilon > 0$ if $D \le 1$ and $\zeta \in K(R,D)$. Here, $\varepsilon$ goes to $0$ as

$$\varepsilon_{a,n} \to 0, \quad \theta \to \infty, \quad S_{a,n+1}^\alpha \sqrt{S_{a,n+1}^{-\alpha+1} - T_{a,n+1}^{-\alpha+1}} \to \infty \quad \text{and} \quad T_{a,n}^\alpha(T_{a,n}-S_{a,n}) \to 0.$$

Since one can take $C_0, C_1, m, \kappa$ in **(A3)**–**(A6)** depending only on $\alpha$ if $\varepsilon$ is sufficiently small, by Proposition 8.8 with $\theta = 1$ and (5), we obtain the result. □

## 8.4 Example (1)

Let $\Lambda$ be as in Section 6. Moreover, we take an increasing sequence $\{K_n\}_n$ such that

$$\lim_{n\to\infty} \frac{K_n}{K_{n-1}} = \infty.$$

In this situation, we observe which pointed metric spaces can be contained in $\mathcal{T}(X, g_\Lambda)$ and prove Theorem 1.2.

Take $S>0$ and put $a_i := K_{2i}^{-1-\alpha}S^{1+\alpha}$. Then we have $a_i^{1/(1+\alpha)}K_{2i}=S$ and $\lim_i a_i^{1/(1+\alpha)}K_{2i+1}=\infty$. Hence Theorem 8.2 implies $(X,a_i g_\Lambda,p)\xrightarrow{\mathrm{GH}}(\mathbb{R}^3,d_S^\infty,0)$. Similarly, if we take $a_i := K_{2i+1}^{-1-\alpha}T^{1+\alpha}$ for $T>0$, then we obtain $(\mathbb{R}^3,d_0^T,0)$ as the pointed Gromov–Hausdorff limit.

Next we fix $\theta>0$ and put $a_i=\theta^{-1}K_{2i+1}^{-2}K_{2i+2}^{-\alpha+1}$. Then one can check that the assumptions of Theorem 8.12 are satisfied; hence one obtains $\big(\mathbb{R}^3,\ \big(1/(\alpha-1)+1/(\theta|\zeta|)\big)h_0,\ 0\big)$ as the pointed Gromov–Hausdorff limit. By taking the limit $\theta\to 0$ or $\theta\to\infty$, we obtain $(\mathbb{R}^3,h_0,0)$ or $(\mathbb{R}^3,(1/|\zeta|)h_0,0)$ as the pointed Gromov–Hausdorff limit. In fact, we obtain the next result.

**Theorem 8.15** *Let* $\Lambda,\{K_n\}_n$ *satisfy* $\lim_{n\to\infty}K_n/K_{n-1}=\infty$. *Then* $\mathcal{T}(X,g_\Lambda)$ *is equal to the closure of*

$$\{(\mathbb{R}^3,sd_1^\infty,0):s>0\}\cup\{(\mathbb{R}^3,sd_0^1,0):s>0\}\cup\left\{\left(\mathbb{R}^3,s\left(1+\frac{1}{|\zeta|}\right)h_0,0\right):s>0\right\}$$

*with respect to the Gromov–Hausdorff topology. Moreover, we have*

$$\begin{aligned}\lim_{s\to\infty}(\mathbb{R}^3,sd_1^\infty,0)&=\lim_{s\to0}\left(\mathbb{R}^3,s\left(1+\frac{1}{|\zeta|}\right)h_0,0\right)=(\mathbb{R}^3,h_0,0),\\ \lim_{s\to0}(\mathbb{R}^3,sd_0^1,0)&=\lim_{s\to\infty}\left(\mathbb{R}^3,s\left(1+\frac{1}{|\zeta|}\right),0\right)=\left(\mathbb{R}^3,\frac{1}{|\zeta|}h_0,0\right),\\ \lim_{s\to0}(\mathbb{R}^3,sd_1^\infty,0)&=\lim_{s\to\infty}(\mathbb{R}^3,sd_0^1,0)=(\mathbb{R}^3,d_0^\infty,0).\end{aligned}$$

**Proof** We have already shown that the pointed metric spaces in the above list are contained in $\mathcal{T}(X,g_\Lambda)$. Accordingly, what we have to show is that any other pointed metric spaces may not arise as the tangent cone at infinity of $(X,g_\Lambda)$.

Suppose that a sequence $\{a_i\}_i\subset\mathbb{R}^+$ is given such that $(X,a_i g_\Lambda,p)\xrightarrow{\mathrm{GH}}(Y,d,q)$ as $i\to\infty$. It suffices to show that $(Y,d,q)$ is one of the metric spaces in the list.

First of all, we may assume that for any large $M>0$, there exists $i(M)$ such that

$$\left\{a_i^{\frac{1}{1+\alpha}}K_n\in\mathbb{R}^+ : n\in\mathbb{N}\right\}\cap[M^{-1},M]$$

is empty for any $i\ge i(M)$. If not, there is an $M>0$ and a map $i\mapsto n_i$ such that $M^{-1}\le a_i^{1/(1+\alpha)}K_{n_i}\le M$ holds for infinitely many $i$. Then by taking a subsequence $\{a_{i_j}\}\subset\{a_i\}_i$, we may suppose that $M^{-1}\le a_{i_j}^{1/(1+\alpha)}K_{2n_{i_j}}\le M$ holds for any $j$ or that $M^{-1}\le a_{i_j}^{1/(1+\alpha)}K_{2n_{i_j}+1}\le M$ holds for any $j$. If the former case holds, then by replacing with a subsequence, we may suppose

$$\lim_{i\to\infty} a_i^{\frac{1}{1+\alpha}} K_{2n_i} = S \in [M^{-1}, M],$$
$$\lim_{i\to\infty} a_i^{\frac{1}{1+\alpha}} K_{2n_i+1} = S \lim_{i\to\infty} \frac{K_{2n_i+1}}{K_{2n_i}} = \infty,$$

and we can apply Theorem 8.2; hence we obtain $(Y, d, q) = (\mathbb{R}^3, d_S^{\infty}, 0)$. If the latter case holds, then we have $(Y, d, q) = (\mathbb{R}^3, d_0^T, 0)$ for some $T > 0$.

We may suppose that there exists $l_i \in \mathbb{N}$ for each $i$ such that $\lim_{i\to\infty} a_i^{1/(1+\alpha)} K_{l_i} = 0$ and $\lim_{i\to\infty} a_i^{1/(1+\alpha)} K_{l_i+1} = \infty$ hold. If $\{i \in \mathbb{N} : l_i \text{ is even}\}$ is an infinite set, then we can apply Theorem 8.2 again and obtain $(Y, d, q) = (\mathbb{R}^3, d_0^{\infty}, 0)$. Therefore, replacing with a subsequence, we may suppose

$$\lim_{i\to\infty} a_i^{\frac{1}{1+\alpha}} K_{2n_i+1} = 0, \qquad \lim_{i\to\infty} a_i^{\frac{1}{1+\alpha}} K_{2n_i+2} = \infty.$$

Now, we have

$$\sqrt{S_{a,n+1}^{-\alpha+1} - T_{a,n+1}^{-\alpha+1}} \geq \tfrac{1}{2} S_{a,n+1}^{\frac{-\alpha+1}{2}}, \qquad T_{a,n} - S_{a,n} \geq \tfrac{1}{2} T_{a,n}$$

holds for sufficiently large $n$. Hence if

$$0 < \liminf_{i\to\infty} \frac{S_{a_i,n_i+1}^{\frac{1-\alpha}{2}}}{T_{a_i,n_i}} \leq \limsup_{i\to\infty} \frac{S_{a_i,n_i+1}^{\frac{1-\alpha}{2}}}{T_{a_i,n_i}} < \infty,$$

then Theorem 8.12 can be applied to this situation by taking a subsequence. Then we obtain $(Y, d, q) = (\mathbb{R}^3, (1+\theta/|\zeta|)h_0, 0)$ for some $\theta > 0$. Hence the remaining cases are

$$\lim_{i\to\infty} \frac{S_{a_i,n_i+1}^{\frac{1-\alpha}{2}}}{T_{a_i,n_i}} = 0 \quad \text{or} \quad \lim_{i\to\infty} \frac{S_{a_i,n_i+1}^{\frac{1-\alpha}{2}}}{T_{a_i,n_i}} = \infty.$$

In both of the cases, we can apply Theorems 8.13 or 8.14, and then obtain $(Y, d, q) = (\mathbb{R}^3, h_0, 0)$ or $(\mathbb{R}^3, (1/|\zeta|)h_0, 0)$. □

One can also see that there are no nontrivial isometries between two pointed metric spaces appearing in the list of Theorem 8.15. Here, an isometry of pointed metric spaces means a bijective morphism preserving the metrics and the base points.

Obviously, there is no isometry between $(\mathbb{R}^3, h_0, 0)$ and $(\mathbb{R}^3, (1/|\zeta|)h_0, 0)$. In the next section, we will show that $(\mathbb{R}^3, d_0^{\infty}, 0)$ is isometric to neither $(\mathbb{R}^3, h_0, 0)$ nor $(\mathbb{R}^3, (1/|\zeta|)h_0, 0)$.

| metric | tangent cone at 0 | tangent cone at $\infty$ |
|---|---|---|
| $d_S^T$ $(S < T)$ | $h_0$ | $\frac{1}{\lvert\zeta\rvert}h_0$ |
| $d_S^\infty$ | $h_0$ | $d_0^\infty$ |
| $d_0^T$ | $d_0^\infty$ | $\frac{1}{\lvert\zeta\rvert}h_0$ |
| $d_0^\infty$ | $d_0^\infty$ | $d_0^\infty$ |
| $h_0$ | $h_0$ | $h_0$ |
| $\frac{1}{\lvert\zeta\rvert}h_0$ | $\frac{1}{\lvert\zeta\rvert}h_0$ | $\frac{1}{\lvert\zeta\rvert}h_0$ |
| $(1+\frac{\theta}{\lvert\zeta\rvert})h_0$ | $\frac{1}{\lvert\zeta\rvert}h_0$ | $h_0$ |

Table 1: Tangent cones $(0 < S, T, \theta < \infty)$

Then Table 1 implies that nontrivial isometries may exist between

$$
\begin{aligned}
&(\mathbb{R}^3, d_S^\infty, 0) \quad \text{and} \quad (\mathbb{R}^3, d_{S'}^\infty, 0) && \text{for} \quad S \neq S',\\
&(\mathbb{R}^3, d_0^T, 0) \quad \text{and} \quad (\mathbb{R}^3, d_0^{T'}, 0) && \text{for} \quad T \neq T',\\
&\Big(\mathbb{R}^3, \Big(1+\frac{\theta}{|\zeta|}\Big)h_0, 0\Big) \quad \text{and} \quad \Big(\mathbb{R}^3, \Big(1+\frac{\theta'}{|\zeta|}\Big)h_0, 0\Big) && \text{for} \quad \theta \neq \theta'.
\end{aligned}
$$

Suppose $(\mathbb{R}^3, d_S^\infty, 0)$ is isometric to $(\mathbb{R}^3, d_{S'}^\infty, 0)$ for some $S \neq S'$. Then the topological space

$$\{(\mathbb{R}^3, d_S^\infty, 0) : S \in \mathbb{R}^+\}$$

with respect to pointed Gromov–Hausdorff topology is homeomorphic to $S^1$ or 1–point; hence it is compact. Then its closure is itself; therefore $(\mathbb{R}^3, h_0, 0)$ is isometric to some $(\mathbb{R}^3, d_S^\infty, 0)$, which is a contradiction by Table 1. Similarly, we can show that there are no isometries between $(\mathbb{R}^3, d_0^T, 0)$ and $(\mathbb{R}^3, d_0^{T'}, 0)$, or between $(\mathbb{R}^3, (1+\theta/|\zeta|)h_0, 0)$ and $(\mathbb{R}^3, (1+\theta'/|\zeta|)h_0, 0)$.

## 8.5 Example (2)

Next we suppose that $\{K_n\}_n$ satisfies

$$\lim_{n\to\infty} \frac{K_{2n}}{K_{2n-1}} = \infty, \qquad \frac{K_{2n+1}}{K_{2n}} = \beta > 1.$$

Take $S > 0$ and put $a_n := K_{2n}^{-1-\alpha} S^{1+\alpha}$. Then we have $a_n^{1/(1+\alpha)} K_{2n} = S$ and $a_n^{1/(1+\alpha)} K_{2n+1} = \beta S$. Hence Theorem 8.2 implies that $(X, a_n g_\Lambda, p) \xrightarrow{\text{GH}} (\mathbb{R}^3, d_S^{\beta S})$. By arguing similarly to the proof of Theorem 8.15, we obtain the following.

**Theorem 8.16** *Let* $\Lambda, \{K_n\}_n$ *satisfy*

$$\lim_{n\to\infty} \frac{K_{2n}}{K_{2n-1}} = \infty, \qquad \lim_{n\to\infty} \frac{K_{2n+1}}{K_{2n}} = \beta > 1.$$

*Then* $\mathcal{T}(X, g_\Lambda)$ *is equal to the closure of*

$$\{(\mathbb{R}^3, sd_1^\beta, 0) : s > 0\} \cup \left\{\left(\mathbb{R}^3, s\left(1 + \frac{1}{|\zeta|}\right)h_0, 0\right) : s > 0\right\}$$

*with respect to the Gromov–Hausdorff topology. Moreover, we have*

$$\lim_{s\to\infty} (\mathbb{R}^3, sd_1^\beta, 0) = \lim_{s\to 0} \left(\mathbb{R}^3, s\left(1 + \frac{1}{|\zeta|}\right)h_0, 0\right) = (\mathbb{R}^3, h_0, 0),$$
$$\lim_{s\to 0} (\mathbb{R}^3, sd_1^\beta, 0) = \lim_{s\to\infty} \left(\mathbb{R}^3, s\left(1 + \frac{1}{|\zeta|}\right), 0\right) = \left(\mathbb{R}^3, \frac{1}{|\zeta|}h_0, 0\right).$$

By a similar argument to Section 8.4, we can see that $(\mathbb{R}^3, d_S^{\beta S}, 0)$ is isometric to neither $(\mathbb{R}^3, h_0, 0)$, $(\mathbb{R}^3, (1/|\zeta|)h_0, 0)$ nor $(\mathbb{R}^3, d_{S'}^{\beta S'}, 0)$ for $S' \neq S$.

## 8.6 Example (3)

For $I \subset \mathbb{R}^+$, denote by $d_I$ the metric on $\mathbb{R}^3$ induced by

$$\int_{x\in I} \frac{dx}{|\zeta - (x^\alpha, 0, 0)|} \cdot h_0.$$

Denote by $\mathcal{B}_+(\mathbb{R}^+)$ the set consisting of all Borel subsets of $\mathbb{R}^+$ of nonzero Lebesgue measure. In this subsection, we show the next theorem.

**Theorem 8.17** *There is a sequence* $\{K_n\}_n$ *such that* $\mathcal{T}(X, g_\Lambda)$ *contains*

$$\{(\mathbb{R}^3, d_I, 0) : I \in \mathcal{B}_+(\mathbb{R}^+)\}/\text{isometry}.$$

**Proof** Put

$$\mathcal{O}_0 := \{I \subset \mathbb{R}^+ : I \text{ is nonempty and open}\},$$
$$\mathcal{O}_1 := \left\{\bigcup_{i=1}^k (S_l, T_l) \subset \mathbb{R}^+ : \begin{array}{l} S_l, T_l \in \mathbb{Q},\ 1 \le k < \infty, \\ 0 < S_l < T_l < S_{l+1} < \infty \end{array}\right\}.$$

Then one can see $\mathcal{O}_1 \subset \mathcal{O}_0 \subset \mathcal{B}_+(\mathbb{R}^+)$. Since $\mathcal{O}_1$ is countable, we can label the open sets in $\mathcal{O}_1$ as follows:

$$\mathcal{O}_1 = \{I_1, I_2, I_3, \dots\}, \quad I_m = \bigcup_{l=1}^{k_m} (S_{m,l}, T_{m,l}).$$

Now we fix a bijection $F\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ and write $F(q)=(i(q),m(q))$. Define $L_q>0$ inductively by

$$L_{q+1} := 2^{i(q)+i(q+1)} L_q \cdot \frac{T_{m(q),k_{m(q)}}}{S_{m(q),1}}, \quad L_0 := 1.$$

Then we can define $0<K_0<K_1<\cdots$ such that

$$\{K_0<K_1<\cdots\} = \left\{ L_q \frac{S_{m(q),l}}{S_{m(q),1}}, L_q \frac{T_{m(q),l}}{S_{m(q),1}} : 1 \le l \le k_{m(q)},\ q=0,1,\ldots \right\}.$$

First we show that $(\mathbb{R}^3, d_{I_m}, 0) \in \mathcal{T}(X, g_\Lambda)$ for every $I_m \in \mathcal{O}_1$. Fix $m$. For any $i\in\mathbb{N}$, we can take a unique $q$ such that $i(q)=i$ and $m(q)=m$. Put $a_i^{1/(1+\alpha)} := L_q^{-1} S_{m,1}$; then we have

$$a_i^{\frac{1}{1+\alpha}} L_q \frac{S_{m,l}}{S_{m,1}} = S_{m,l}, \quad a_i^{\frac{1}{1+\alpha}} L_q \frac{T_{m,l}}{S_{m,1}} = T_{m,l}.$$

Note that $L_{q+1} \ge 2^{i(q)+i(q+1)} L_q$ implies $L_q\to\infty$ as $i\to\infty$, hence $a_i\to 0$ as $i\to\infty$. Here, we put $\Phi=\Phi_{a_i}$ and $\Phi_\infty = \sum_{l=1}^{k_m} \Phi_{S_{m,l}}^{T_{m,l}}$. By applying Proposition 6.1 and (4)–(8) with $P=1$, the constants appearing in **(A3)**–**(A6)** are given by

$$\varepsilon = 2a_i^{\frac{1}{1+\alpha}} + 2^{-i} S_{m,1} + \frac{2^{1-(\alpha-1)i} T_{m,k_m}^{-\alpha+1}}{\alpha-1}, \quad C_0 = \frac{1}{2}\sum_{l=1}^{k_m} A_{S_{m,l}}^{T_{m,l}},$$

$$C_1 = \frac{\alpha 2^{\frac{1}{\alpha}}}{\alpha-1}, \qquad m=1, \qquad \kappa = \frac{1}{\alpha}$$

if we suppose $\varepsilon$ is sufficiently small. One can see $\varepsilon\to 0$ as $i\to\infty$, so we obtain $\{(X, a_i g_\Lambda, p)\}_i \xrightarrow{\mathrm{GH}} (\mathbb{R}^3, d_{I_m}, 0)$.

Next we show that $(\mathbb{R}^3, d_I, 0) \in \mathcal{T}(X, g_\Lambda)$ for any $I\in\mathcal{O}_0$. To show it, we apply Vitali's covering theorem. Fix $I\in\mathcal{O}_0$ and put $\mathcal{I} := \{(a,b)\in\mathcal{O}_0 : [a,b]\subset I\}$. Then $\mathcal{I}$ is a Vitali cover of $I$; hence there exists $\{J_n\}_{n\in\mathbb{N}} \subset \mathcal{I}$ such that

$$J_n \ne J_{n'} \text{ if } n\ne n', \quad m\Big(I \setminus \bigsqcup_{n\in\mathbb{N}} J_n\Big) = 0,$$

where $m$ is the Lebesgue measure. Put $\hat{J}_n := \bigsqcup_{k=1}^n J_k$. Since $\hat{J}_n \in \mathcal{O}_1$ holds, we have $(\mathbb{R}^3, d_{\hat{J}_n}, 0) \in \mathcal{T}(X, g_\Lambda)$. If we put

$$\Phi_J(\zeta) := \int_{x\in J} \frac{dx}{|\zeta - (x^\alpha, 0, 0)|},$$

then we can see

$$|\Phi_{\hat{J}_n}(\zeta) - \Phi_I(\zeta)| \le \frac{m(I\setminus \hat{J}_n)}{D} \to 0 \quad \text{as } n\to\infty,$$

and we can take the constants in **(A3)–(A6)** independent of $n$ by using Proposition 6.2. Therefore, we obtain $\{(\mathbb{R}^3, d_{\hat{J}_n}, 0)\}_n \xrightarrow{\mathrm{GH}} (\mathbb{R}^3, d_I, 0)$.

Finally, let $I \in \mathcal{B}_+(\mathbb{R}^+)$. Since the Lebesgue measure is the Radon measure, there exists $U_n \subset \mathcal{O}_1$ for any $n$ such that $I \subset U$ and $m(U) \le m(I) + 1/n$. Then we have $|\Phi_I(\zeta) - \Phi_{U_n}(\zeta)| \le 1/(nD)$, and thus $\{(\mathbb{R}^3, d_{U_n}, 0)\}_n \xrightarrow{\mathrm{GH}} (\mathbb{R}^3, d_I, 0)$ by a similar argument. Here, the positivity of $m(I)$ is necessary since, by (4), $C_0$ in **(A5)** is given by

$$\int_I \frac{dx}{1+x^\alpha} > 0. \qquad \square$$

By Theorem 8.17, we can see that $(\mathbb{R}^3, h_0, 0)$ and $(\mathbb{R}^3, (1/|\zeta|)h_0, 0)$ are also contained in $\mathcal{B}_+(\mathbb{R}^+)$. The author does not know whether any other metric spaces may appear as the tangent cone at infinity of $(X, g_\Lambda)$ or not.

# 9 On the geometry of the limit spaces

In this section, we study the geometry of $(\mathbb{R}^3, d_0^\infty)$ and conclude that there is no isometry between $(\mathbb{R}^3, d_0^\infty)$ and $(\mathbb{R}^3, h_0)$, nor between $(\mathbb{R}^3, d_0^\infty)$ and $(\mathbb{R}^3, (1/|\zeta|)h_0)$.

**Proposition 9.1** *$(\mathbb{R}^3, (1/|\zeta|)h_0)$ is the Riemannian cone $S^2 \times \mathbb{R}^+$, where the Riemannian metric on $S^2$ is the homogeneous one whose area is equal to $\pi$.*

**Proof** Put $\zeta = (\zeta_1, \zeta_2, \zeta_3) \neq 0$ and $r = \sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}$, and let $g_{S^2}$ be the standard Riemannian metric on $S^2$ with constant curvature and volume $4\pi$. Then by putting $R := 2\sqrt{r}$, we have

$$\frac{1}{|\zeta|} h_0 = \frac{1}{r}\big((dr)^2 + r^2 g_{S^2}\big) = (dR)^2 + R^2 \cdot \frac{g_{S^2}}{4}. \qquad \square$$

Next we review the notion of polar spaces, introduced by Cheeger and Colding in [5], and then we show that the metric space $(\mathbb{R}^3, d_0^\infty)$ is never a polar space.

Let $Y$ be a metric space, and suppose that there is a tangent cone $Y_y$ at $y \in Y$. Then we can consider tangent cones at any points in $Y_y$. The tangent cones obtained by repeating this process are called *iterated tangent cones* of $Y$. A point $x$ in a length-space $X$ is called a *pole* if there is a ray $\gamma\colon [0,\infty) \to X$ and $t \ge 0$ for any $\underline{x} \neq x$ such that $\gamma(0) = x$ and $\gamma(t) = \underline{x}$. Here, the ray $\gamma\colon [0,\infty) \to X$ is a continuous curve such that the length of $\gamma|_{[t_0,t_1]}$ is equal to $|\gamma(t_0)\gamma(t_1)|$.

**Definition 9.2** [5] The metric space $Y$ is called a *polar space* if all of the base points of the iterated tangent cones of $Y$ are poles.

For example, let $C(X)$ be a metric cone of a metric space $X$. Then every $\gamma$ defined by $\gamma(t) := (x,t) \in X \times \mathbb{R}^+ = C(X)$ is a ray; hence the base points of any metric cones are poles. Now, since $(\mathbb{R}^3, (1/|\zeta|)h_0)$ is a Riemannian cone of a smooth compact Riemannian manifold, then all of the iterated tangent cones are $(\mathbb{R}^3, (1/|\zeta|)h_0)$ itself or $(\mathbb{R}^3, h_0)$. Consequently, we can conclude that $(\mathbb{R}^3, (1/|\zeta|)h_0)$ is polar. Obviously, $(\mathbb{R}^3, h_0)$ is also polar. We can also see in the similar way that $(\mathbb{R}^3, (1+\theta/|\zeta|)h_0)$ is polar. On the other hand, we can show the next proposition.

**Proposition 9.3** *The origin $0 \in \mathbb{R}^3$ is not a pole of the metric space $(\mathbb{R}^3, d_0^\infty)$. In particular, $(\mathbb{R}^3, d_0^\infty)$ is neither a polar space nor a metric cone of any metric spaces.*

**Proof** First of all we show that $0 \in \mathbb{R}^3$ is not a pole with respect to $d_0^\infty$. Put $p := (1,0,0) \in \mathbb{R}^3$, and suppose that there is a ray $\gamma\colon [0,\infty) \to \mathbb{R}^3$ such that $\gamma(0) = 0$ and $\gamma(t_0) = p$ for some $t_0 > 0$. Then we have

$$d_0^\infty(\gamma(s_0), \gamma(s_1)) = \int_{s_0}^{s_1} \sqrt{\Phi_0^\infty(\gamma(t))}\, |\gamma'(t)|\, dt$$

for any $0 \le s_0 < s_1$. For $\delta > 0$, let

$$A_\delta := \{t \in \mathbb{R} : |\gamma_{\mathbb{C}}(t)| \ge \delta\}.$$

Then there is a sufficiently small $\delta$ such that $A_\delta \cap (0,t_0) \ne \varnothing$ and $A_\delta \cap (t_0,\infty) \ne \varnothing$. This is because the length of $\gamma|_I$ becomes infinity for any small interval $I \subset \mathbb{R}$ if not. Since $A_\delta$ is closed and does not contain $t_0$, we can take a connected component $(a_0,a_1)$ of $\mathbb{R} \backslash A_\delta$ containing $t_0$. Then we can see that $|\gamma_{\mathbb{C}}(a_0)| = |\gamma_{\mathbb{C}}(a_1)| = \delta$ and $|\gamma_{\mathbb{C}}(t)| < \delta$ for any $t \in (a_0,a_1)$. Now define $\tilde{\gamma}\colon [0,a_1] \to X$ by

$$\tilde{\gamma}(t) := \begin{cases} (\gamma_{\mathbb{R}}(t), e^{i\theta}\gamma_{\mathbb{C}}(t)), & 0 \le t \le a_0, \\ e^{i\theta} P_{\gamma|_{[a_0,a_1]}}(t), & a_0 \le t \le a_1, \end{cases}$$

where $\theta$ is defined by $e^{i\theta}\gamma_{\mathbb{C}}(a_0) = \gamma_{\mathbb{C}}(a_1)$. Recall that $P_{\gamma|_{[a_0,a_1]}}$ is already defined in Lemma 7.4. Then by applying Lemma 7.4, we can see that the length of $\tilde{\gamma}$ is strictly less than the length of $\gamma|_{[0,a_1]}$; therefore, $\gamma$ is not a ray, which is a contradiction. Hence $0 \in \mathbb{R}^3$ is not a pole.

Now we can check that the $\mathbb{R}^+$–action on $\mathbb{R}^3$ defined by scalar multiplication is homothetic with respect to $d_0^\infty$; thus the tangent cone of $(\mathbb{R}^3, d_0^\infty)$ at $0$ is itself. Consequently, $(\mathbb{R}^3, d_0^\infty)$ is not a polar space.

Suppose that $(\mathbb{R}^3, d_0^\infty)$ is the metric cone of some metric space $X$; then the origin $0$ is nothing but the base point of the metric cone. Since the base point of the metric cone is always a pole, we have a contradiction. □

**Corollary 9.4** *There is no isometry between* $(\mathbb{R}^3, d_0^\infty)$ *and* $(\mathbb{R}^3, h_0)$, *nor between* $(\mathbb{R}^3, d_0^\infty)$ *and* $(\mathbb{R}^3, (1/|\zeta|)h_0)$.

# References


[1] **M T Anderson**, **P B Kronheimer**, **C LeBrun**, *Complete Ricci-flat Kähler manifolds of infinite topological type*, Comm. Math. Phys. 125 (1989) 637–642 MR

[2] **V N Berestovskii**, **L Guijarro**, *A metric characterization of Riemannian submersions*, Ann. Global Anal. Geom. 18 (2000) 577–588 MR

[3] **V N Berestovskiĭ**, *"Submetries" of three-dimensional forms of nonnegative curvature*, Sibirsk. Mat. Zh. 28 (1987) 44–56, 224 MR In Russian; translated in Siberian Math. J. 28 (1987) 552–562

[4] **J Cheeger**, **T H Colding**, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. 144 (1996) 189–237 MR

[5] **J Cheeger**, **T H Colding**, *On the structure of spaces with Ricci curvature bounded below, I*, J. Differential Geom. 46 (1997) 406–480 MR

[6] **T H Colding**, **W P Minicozzi, II**, *On uniqueness of tangent cones for Einstein manifolds*, Invent. Math. 196 (2014) 515–588 MR

[7] **T H Colding**, **A Naber**, *Characterization of tangent cones of noncollapsed limits with lower Ricci bounds and applications*, Geom. Funct. Anal. 23 (2013) 134–148 MR

[8] **R Goto**, *On hyper-Kähler manifolds of type* $A_\infty$, Geom. Funct. Anal. 4 (1994) 424–454 MR

[9] **M Gromov**, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques 1, CEDIC, Paris (1981) MR

[10] **M Gromov**, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser, Boston (2007) MR

[11] **K Hattori**, *The volume growth of hyper-Kähler manifolds of type* $A_\infty$, J. Geom. Anal. 21 (2011) 920–949 MR

[12] **G Perelman**, *A complete Riemannian manifold of positive Ricci curvature with Euclidean volume growth and nonunique asymptotic cone*, from "Comparison geometry" (K Grove, P Petersen, editors), Math. Sci. Res. Inst. Publ. 30, Cambridge Univ. Press (1997) 165–166 MR


*Department of Mathematics, Keio University*
*3-14-1 Hiyoshi, Kohoku, Yokohama 223-8522, Japan*

hattori@math.keio.ac.jp

Proposed: Tobias H. Colding Received: 10 May 2015
Seconded: Dmitri Burago, Bruce Kleiner Revised: 12 October 2016