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Smooth Kuranishi atlases with isotropy

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Kuranishi structures were introduced in the 1990s by Fukaya and Ono for the purpose of assigning a virtual cycle to moduli spaces of pseudoholomorphic curves that cannot be regularized by geometric methods. Their core idea was to build such a cycle by patching local finite-dimensional reductions, given by smooth sections that are equivariant under a finite isotropy group.

Building on our notions of topological Kuranishi atlases and perturbation constructions in the case of trivial isotropy, we develop a theory of Kuranishi atlases and cobordisms that transparently resolves the challenges posed by nontrivial isotropy. We assign to a cobordism class of weak Kuranishi atlases both a virtual moduli cycle (a cobordism class of weighted branched manifolds) and a virtual fundamental class (a Čech homology class).

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1 Introduction

1.1 Overview

This is the third in a series [13; 14] of papers that construct a fundamental class for compact spaces X that are modeled locally by the zero sets of smooth sections $s_i\colon U_i \to E_i$ in finite rank bundles over finite-dimensional manifolds. While these obstruction bundles have fixed index, they may have varying rank, and thus an ambient space $\bigcup U_i/\!\sim$ naively constructed from the ambient manifolds of the local zero sets $s_i^{-1}(0)$ modulo transition data is lacking all topological controls (Hausdorffness, local compactness, in fact existence) that are needed for a perturbative construction $[X] := \bigcup (s_i + \nu_i)^{-1}(0)/\!\sim$ of the fundamental class. Moreover, most interesting cases involve nontrivial isotropy groups that are captured in the local charts as finite symmetry groups Γ_i of the sections s_i, so that X is locally modeled by the quotients $s_i^{-1}(0)/\,\Gamma_i$.

Pioneered by Fukaya et al [6; 3], this problem has been considered by symplectic topologists since the 1990s as a tool for "counting curves", ie assigning homological information to moduli spaces of pseudoholomorphic curves, such as the Gromov–Witten

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moduli spaces (in which isotropy arises from components that are multiply covered). In the case of trivial isotropy, a comprehensive solution was developed in [13; 14] by introducing notions of Kuranishi atlases, which on the one hand can in practice be constructed from moduli spaces, and on the other hand have sufficient compatibility between the local models for the construction of a virtual fundamental class. This paper extends these techniques to the case of nontrivial isotropy, proving the following result.

Theorem A *Let $\mathcal{K}$ be an oriented, d–dimensional, additive, smooth weak Kuranishi atlas on a compact metrizable space X. Then $\mathcal{K}$ determines*

- *a **virtual moduli cycle (VMC)** as cobordism class of weighted branched manifolds,*
- *a **virtual fundamental class (VFC)** $[X]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}_d(X;\mathbb{Q})$ in Čech homology,*

both of which depend only on the cobordism class of $\mathcal{K}$.

A more precise statement that also applies when $\mathcal{K}$ is a cobordism from an atlas $\mathcal{K}^0$ on X^0 to an atlas $\mathcal{K}^1$ on X^1 is given in Theorem 3.3.5. Notice further that the VMC contains more information than the VFC since cobordism classes of weighted branched manifolds contain more information than just their fundamental class; for example, Pontryagin numbers are invariants of weighted branched cobordism by [11, Remark 4.7].

The guiding idea of a Kuranishi atlas $\mathcal{K}$ is to start with a family of basic charts $(\boldsymbol{K}_i)_{i=1,\dots,N}$, where each basic chart

$$\boldsymbol{K}_i = (U_i, E_i, \Gamma_i, s_i, \psi_i)$$

is a tuple consisting of a domain U_i, an obstruction space E_i, a group Γ_i, a section $s_i\colon U_i \to E_i$, and a footprint map $\psi_i\colon s_i^{-1}(0) \to X$ inducing a homeomorphism from $s_i^{-1}(0)/\Gamma_i$ onto the "footprint", an open subset $F_i \subset X$ such that $(F_i)_{i=1,\dots,N}$ covers X. The compatibility of these charts then involves transition charts $\boldsymbol{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I)$ of the same type as the basic charts, but with $I \subset \{1,\dots,N\}$ such that $F_I := \bigcap_{i\in I} F_i \neq \varnothing$. Finally, the basic and transition charts are related by coordinate changes from $\boldsymbol{K}_I$ to $\boldsymbol{K}_J$ whenever $I \subset J$. This gives rise to an "étale-like" category $\boldsymbol{B}_{\mathcal{K}}$ whose space of objects is $\bigsqcup_I U_I$, and whose morphisms are determined by the local group actions and the coordinate changes. The category $\boldsymbol{B}_{\mathcal{K}}$ is not a groupoid since some morphisms (those relating the different charts) are not invertible. On the other hand, its spaces of objects and morphisms are very closely controlled, which enables us to carry out various geometric constructions, in particular the construction of perturbations, very explicitly. The realization $|\mathcal{K}|$ of $\boldsymbol{B}_{\mathcal{K}}$ (the space of objects modulo the equivalence relation generated by the morphisms) is much larger than X, though it does contain a homeomorphic image of X formed from the zero sets of the local sections s_I. As in [14], the class $[X]^{\mathrm{vir}}_{\mathcal{K}}$ is constructed from the zero sets of suitable perturbations $\mathfrak{s}_{\mathcal{K}}+\nu$ of the basic section $\mathfrak{s}_{\mathcal{K}}=(s_I)$ of $\mathcal{K}$.

Even if X is an orbifold so that no obstruction spaces are needed, our formulations are new.[1] Rather than being given by inclusions $U_I\supset U_{IJ}\hookrightarrow U_J$ as in the case with trivial isotropy, our notion of coordinate changes in the presence of isotropy involves equivariant covering maps $\widetilde{\rho}_{IJ}\colon(\widetilde{U}_{IJ},\Gamma_J)\to(U_{IJ},\Gamma_I)\subset(U_I,\Gamma_I)$, where $\widetilde{U}_{IJ}$ is a suitable subset of the domain U_J and $\widetilde{U}_{IJ}\to U_{IJ}$ is a principal Γ_J/Γ_I-bundle. As the following result from [11, Proposition 3.3] shows, every orbifold has a structure of this kind.

Proposition *Every compact orbifold Y has an orbifold atlas $\mathcal{K}$ with trivial obstruction spaces whose associated groupoid $\boldsymbol{G}_{\mathcal{K}}$ is an orbifold structure on Y. Moreover, there is a bijective correspondence between commensurability classes of such Kuranishi atlases and Morita equivalence classes of ep groupoids.*

To apply the above theory to moduli spaces X that arise in geometric examples, one needs to develop methods for constructing Kuranishi atlases on such X. Some parts of this construction were detailed in the 2012 preprint [12], and now appear in [14]. They will be extended in [10] to include multiply covered curves (and hence nontrivial isotropy) as well as nodal curves. Both McDuff [10] and Pardon [16] outline the needed construction for moduli spaces of closed stable maps, though neither approach is sufficient to give the smooth charts whose existence is assumed in the current paper. In [10] we will combine the same setup with an implicit function theorem from polyfold theory (see Hofer, Wysocki and Zehnder [7]) to obtain compatible choices of smooth structures near nodal curves. An alternative approach is to extend the VMC/VFC construction to less smooth sections. In fact, Castellano [2] proves a gluing theorem for Gromov–Witten moduli spaces that allows the construction of stratified smooth Kuranishi atlases with $\mathcal{C}^1$–differentiability across strata, to which our construction applies with minor modifications. He moreover shows that the resulting genus zero Gromov–Witten invariants satisfy the standard axioms.

1.2 Outline of the construction

This paper contains all relevant definitions and a fair amount of review so that it can be read independently of the previous papers in this series. This outline will also be

[1] Our construction was outlined in [9]. In [16], Pardon independently takes a similar approach to handling the isotropy groups.

rather brief since the earlier papers give extensive explanation and justification for our approach:

- The first part of [14] is a general discussion of different approaches to regularizing moduli spaces — eg as VMC/VFC — and explains important analytic background.

- The paper [13] starts with an overview of the topological challenges that need to be addressed in constructing a VMC/VFC, and then proves the basic topological results needed to show that a filtered weak Kuranishi atlas determines a tame Kuranishi atlas $\mathcal{K}$, well-defined up to cobordism, whose realization $|\mathcal{K}|$ is Hausdorff, contains a homeomorphic copy of the moduli space X, and can be equipped with a metric that is compatible with local charts (but generally induces a different topology on $|\mathcal{K}|$).

- The second part of [14] carries out the full construction of the VMC as the zero set of a suitable perturbation of the canonical section $\mathfrak{s}_{\mathcal{K}}$ in the case of trivial isotropy.

We now discuss the main steps in the construction below in more detail, highlighting the new features needed to deal with nontrivial isotropy.

- In order to simplify the abstract discussion, we decided to give a rather narrow definition of a Kuranishi atlas $\mathcal{K}$. Thus the domains of both the basic and transition charts are group quotients (U_I, Γ_I), and the coordinate changes are determined by rather special equivariant covering maps $(\widetilde{U}_{IJ}, \Gamma_J) \to (U_{IJ}, \Gamma_I)$. The basic theory is set up in Section 2.1; see in particular Definition 2.1.4 and Lemma 2.1.5. If there were a need, one could no doubt replace these group quotients by more general étale groupoids and use more general covering maps and obstruction bundles, at the expense of revisiting the construction of perturbations.

- Smooth atlases and coordinate changes are defined in Sections 2.2 and 2.3. Though in general the definitions are similar to those in the case with trivial isotropy, there is an important difference in the notion of coordinate change: when $I \subset J$ this is now given by a covering map from an appropriate submanifold $\widetilde{U}_{IJ}$ of the domain of the higher dimensional domain U_J onto an open subset U_{IJ} of the lower dimensional domain U_I. If the isotropy groups are trivial, this map is a diffeomorphism with inverse equal to the coordinate changes $\phi_{IJ}\colon U_{IJ} \to U_J$ considered in [13; 14]. Another small difference is that we build in the notion of additivity since at least some version of this is needed for the taming construction discussed below. (In some situations, for example when considering products, this formulation is too rigid; for appropriate generalizations see [10].)

- An important feature of our definitions is that the quotients $\underline{U}_I := U_I/\Gamma_I$ fit together to form an *intermediate atlas*, which Lemma 2.3.4 shows to be a filtered topological atlas in the sense of [13]. In particular it has an associated category $\boldsymbol{B}_{\underline{\mathcal{K}}}$ with space of objects the orbifold $\mathrm{Obj}_{\boldsymbol{B}_{\underline{\mathcal{K}}}} := \bigsqcup_I \underline{U}_I$, and identical realization $|\underline{\mathcal{K}}| = |\mathcal{K}|$.

- One difficulty in constructing a VFC for a given moduli space X is that in practice one cannot usually construct an atlas on X. Instead one constructs a weak atlas, which is like an atlas except that one has less control of the domains of the charts and coordinate changes; cf the various *cocycle conditions* discussed in Definition 2.2.12 and Lemma 2.2.13. But a weak atlas does not even define a category, let alone one whose realization $|\boldsymbol{B}_{\mathcal{K}}| =: |\mathcal{K}|$ has good topological properties. For example, we would like $|\mathcal{K}|$ to be Hausdorff and (in order to make local constructions possible) for the projection $\pi_{\mathcal{K}}\colon U_I \to |\mathcal{K}|$ to be a homeomorphism to its image. Theorem 2.5.3 summarizes the main topological facts about $\mathcal{K}$ that are needed for subsequent constructions. We achieve these via *shrinking* and *taming*. Our definitions were designed so that all the topological constructions of [13], such as the taming, cobordism and reduction constructions, apply to the intermediate atlas $\underline{\mathcal{K}}$ and then lift to $\mathcal{K}$ because the quotient maps $U_I \to \underline{U}_I$ are proper. However, we do need to take some care with the proof of the linearity properties of the projection $\mathrm{pr}\colon |\boldsymbol{E}_{\mathcal{K}}| \to |\boldsymbol{B}_{\mathcal{K}}|$.

- Another important part of Theorem 2.5.3 is the claim that any two tame shrinkings of a weak atlas $\mathcal{K}$ are *concordant*, ie cobordant over $[0,1]\times X$, which is required to show independence of the VMC/VFC from the choice of shrinking. In Section 2.4 we give the precise definition of a cobordism atlas. This is an immediate generalization of the notion of cobordism in [13; 14], and the relevant proofs generalize easily.

- Given a weak atlas, the taming procedure gives us two categories $\boldsymbol{B}_{\mathcal{K}}$ and $\boldsymbol{E}_{\mathcal{K}}$ with a projection functor $\mathrm{pr}\colon \boldsymbol{E}_{\mathcal{K}} \to \boldsymbol{B}_{\mathcal{K}}$ and section functor $\mathfrak{s}_{\mathcal{K}}\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}}$. However, even when the isotropy is trivial, the category has too many morphisms (ie the chart domains overlap too much) for us to be able to construct a perturbation $\nu\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}}$ that is transverse to 0 (written $\mathfrak{s}_{\mathcal{K}}+\nu \pitchfork 0$). We therefore pass to a full subcategory $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}$ of $\boldsymbol{B}_{\mathcal{K}}$ with objects $\mathcal{V} := \bigsqcup V_I$ that does support suitable perturbations $\nu\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}$. This subcategory $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}$ is called a *reduction* of $\mathcal{K}$; cf Definition 3.2.1. Constructing it is akin to passing from the covering of a triangulated space by the stars of its vertices to the covering by the stars of its first barycentric subdivision. Again this construction can be done at the level of the intermediate category, so that the methods of [13] immediately give us the required reductions.

- In the presence of nontrivial isotropy, we may still not be able to construct a transverse perturbation $\nu\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}$ as a functor, since local perturbations ν_I are required to be Γ_I–equivariant. In general, this can be resolved by using multivalued perturbations. Our setup allows for a simplified approach: we define perturbations $\nu = (\nu_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ to be families of maps that are compatible with the covering maps ρ_{IJ}

but need not be Γ_I–equivariant. We show in Section 3.2 that this construction inherits enough equivariance to yield an étale category that represents the zero set of the perturbed section $\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}} + \nu$, assuming that this is transverse to 0. The remaining morphisms are then added back in at the expense of weighting functions, which give the perturbed zero set the structure of a *weighted branched manifold*. More precisely, we construct the perturbed zero set in Theorem 3.2.8 as the Hausdorff realization $|\boldsymbol{Z}^\nu|_{\mathcal{H}}$ of an étale (but nonproper) category $\boldsymbol{Z}^\nu$ whose space of objects has one component $Z_I = (s_I|_{V_I} + \nu)^{-1}(0)$ for each $I \in \mathcal{I}_{\mathcal{K}}$, and whose branching locus and weighting function are explicitly determined by the reduction $\mathcal{V}$ and the isotropy groups.

• For the convenience of the reader we prove the needed results about weighted branched manifolds and cobordisms in the appendix. Moreover, the short paper [11] explains the construction of $\boldsymbol{Z}^\nu$ in the orbifold case. This is much simpler, since the obstruction spaces, and hence also the sections $\mathfrak{s}_{\mathcal{K}}$, ν are zero.

• Moreover, we must ensure that the perturbed zero sets are compact and unique up to cobordism. As we show in Proposition 3.3.3 the rather intricate construction in [14] carries through in the current situation without essential change.

• In Section 3.1 we extend the notion of orientation to atlases with nontrivial isotropy. As in [14], we define the orientation line bundle of $\mathcal{K}$ in two equivalent ways, showing in Proposition 3.1.13 that the bundle $\det \mathfrak{s}_{\mathcal{K}}$ (with local bundles $(\det s_I)_{I \in \mathcal{I}_{\mathcal{K}}}$) is isomorphic to $\Lambda_{\mathcal{K}}$ (with local bundles $(\Lambda^{\max} U_I \otimes (\Lambda^{\max} E_I)^*)_{I \in \mathcal{I}_{\mathcal{K}}}$). Most of the needed proofs can again be quoted directly from [14]. Lemma 3.1.14 explains how these bundles are used to orient local zero sets of sections.

• The final step is to build the homology class $[X]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}_d(X;\mathbb{Q})$ from the zero set $(\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}} + \nu)^{-1}(0)$. Many of the details here are again the same as in [14]. In particular, we build a geometric representative $|\boldsymbol{Z}^\nu|_{\mathcal{H}}$ for this class that maps to the precompact "neighborhood"[2] $|\mathcal{V}| = \bigcup_I \pi_{\mathcal{K}}(V_I) \subset |\mathcal{K}|$ of $\iota_{\mathcal{K}}(X) = |\mathfrak{s}_{\mathcal{K}}^{-1}(0)|$, and then define $[X]^{\mathrm{vir}}_{\mathcal{K}}$ by taking an appropriate inverse limit in rational Čech homology.

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[2] In fact, $\iota_{\mathcal{K}}(X)$ does *not* have a compact neighborhood in $|\mathcal{K}|$; we should think of $|\mathcal{V}|$ as the closest we can come to such a neighborhood.

2 Smooth Kuranishi atlases with isotropy

In this section we extend the notions of smooth Kuranishi charts and transition data introduced in [14] to nontrivial isotropy and then discuss cobordisms and taming. The main result is Theorem 2.5.3.

Throughout this section we fix X to be a compact metrizable space. The main change from [14] is that the domains of the charts are no longer smooth manifolds, but rather group quotients. We begin by setting up notation for the latter. As in [14, Remark 5.1.2] we assume all manifolds are smooth and second countable.

2.1 Group quotients

Definition 2.1.1 A *group quotient* is a pair (U, Γ) consisting of a smooth manifold U and a finite group Γ together with a smooth action $\Gamma \times U \to U$. We will denote the quotient space by

$$\underline{U} := U/\Gamma,$$

giving it the quotient topology, and write $\pi\colon U \to \underline{U}$ for the associated projection. Moreover, we denote the *stabilizer* of each $x \in U$ by

$$\Gamma^x := \{\gamma \in \Gamma \mid \gamma x = x\} \subset \Gamma.$$

We could consider a group quotient as a topological category with space of objects U and morphisms $U \times \Gamma$, but in the interest of simplicity will often avoid doing this.

Both the basic and transition charts of Kuranishi atlases will be group quotients, related by coordinate changes that are composites of the following kinds of maps.

Definition 2.1.2 Let (U, Γ), (U', Γ') be group quotients. A *group embedding*

$$(\phi, \phi^\Gamma)\colon (U, \Gamma) \to (U', \Gamma')$$

is a smooth embedding $\phi\colon U \to U'$ that is equivariant with respect to an injective group homomorphism $\phi^\Gamma\colon \Gamma \to \Gamma'$ and induces an injection $\underline{\phi}\colon \underline{U} \to \underline{U}'$ on the quotient spaces. We call a group embedding *equidimensional* if $\dim U = \dim U'$.

In a Kuranishi atlas we often consider embeddings $(\phi, \phi^\Gamma)\colon (U, \Gamma) \to (U', \Gamma)$ where $\dim U < \dim U'$ and $\phi^\Gamma\colon \Gamma \to \Gamma' := \Gamma$ is the identity map. On the other hand, group quotients of the same dimension are usually related either by restriction or by coverings as follows.

Definition 2.1.3 Let (U, Γ) be a group quotient and $\underline{V} \subset \underline{U}$ an open subset. Then the *restriction of* (U, Γ) *to* $\underline{V}$ is the group quotient $(\pi^{-1}(\underline{V}), \Gamma)$.

Note that the inclusion $\pi^{-1}(\underline{V}) \to U$ induces an equidimensional group embedding $(\pi^{-1}(\underline{V}), \Gamma) \to (U, \Gamma)$ that covers the inclusion $\underline{V} \to \underline{U}$. The third kind of map that occurs in a coordinate change is a group covering. This notion is less routine; notice in particular the requirement in (ii) that $\ker \rho^\Gamma$ act freely. Further, the two domains $\widetilde{U}, U$ will necessarily have the same dimension since they are related by a regular covering ρ.

Definition 2.1.4 Let (U, Γ) be a group quotient. A *group covering* of (U, Γ) is a tuple $(\widetilde{U}, \widetilde{\Gamma}, \rho, \rho^\Gamma)$ consisting of

(i) a surjective group homomorphism $\rho^\Gamma \colon \widetilde{\Gamma} \to \Gamma$,

(ii) a group quotient $(\widetilde{U}, \widetilde{\Gamma})$, where $\ker \rho^\Gamma$ acts freely,

(iii) a regular covering $\rho \colon \widetilde{U} \to U$ that is the quotient map $\widetilde{U} \to \widetilde{U} / \ker \rho^\Gamma$ composed with a diffeomorphism $\widetilde{U} / \ker \rho^\Gamma \cong U$ that is equivariant with respect to the induced $\Gamma = \operatorname{im}(\rho^\Gamma)$ action on both spaces.

Thus $\rho \colon \widetilde{U} \to U$ is equivariant with respect to $\rho^\Gamma \colon \widetilde{\Gamma} \to \Gamma$ and ρ^Γ acts transitively on the fibers of ρ. We denote by $\underline{\rho} \colon \underline{\widetilde{U}} \to \underline{U}$ the induced map on quotients.

Next, we establish some basic properties of group quotients, in particular the fact that coverings induce homeomorphisms between the quotients. Here and subsequently we denote a precompact inclusion by $V \sqsubset U$.

Lemma 2.1.5 *Let (U, Γ) be a group quotient.*

(i) *The projection $\pi \colon U \to \underline{U}$ is open, closed and proper. In particular, any precompact set $P \sqsubset \underline{U}$ has precompact preimage $\pi^{-1}(P) \sqsubset U$, Moreover, $\underline{U}$ is a separable, locally compact metric space.*

(ii) *Every point $x \in U$ has a neighborhood U_x that is invariant under Γ^x and is such that inclusion $U_x \hookrightarrow U$ induces a homeomorphism from U_x / Γ^x to $\pi(U_x)$. In particular, the inclusion $(U_x, \Gamma^x) \to (\pi^{-1}(\pi(U_x)), \Gamma)$ is a group embedding.*

(iii) *If $(\widetilde{U}, \widetilde{\Gamma}, \rho, \rho^\Gamma)$ is a group covering of (U, Γ), then $\underline{\rho} \colon \underline{\widetilde{U}} \to \underline{U}$ is a homeomorphism and ρ^Γ induces isomorphisms between the stabilizers $\widetilde{\Gamma}^y \to \Gamma^{\rho(y)}$ for all $y \in \widetilde{U}$.*

Proof Let $W \subset U$ be open. Then $\pi^{-1}(\pi(W)) = \bigcup_{\gamma \in \Gamma} \gamma W$ is open since each γW is the preimage, under the continuous action of γ^{-1}, of the open set W. Hence, by definition of the quotient topology, $\pi(W)$ is open. This shows that π is open. The same argument applied to the complement of a closed set shows that π is closed.

To see that π is proper, consider a compact set $\underline{V} \subset \underline{U}$. Given any open cover $(U_\alpha)_{\alpha\in A}$ of $\pi^{-1}(\underline{V})$, choose for each $x \in \pi^{-1}(\underline{V})$ an element $\alpha_x \in A$ such that $x \in U_{\alpha_x}$. Then for each $\underline{x} \in \underline{V}$ define

$$\underline{W}_{\underline{x}} := \bigcap_{x\in\pi^{-1}(\underline{x})} \pi(U_{\alpha_x}) \subset \underline{U}.$$

These are open sets since $\pi^{-1}(\underline{x})$ is finite and the map π is open, and they cover the compact set $\underline{V}$. So we may choose a finite subcover $(\underline{W}_{\underline{x_i}})_{i=1,\dots,n}$ of $\underline{V}$. Then $(U_{\alpha_x})_{x\in\pi^{-1}\{\underline{x_1},\dots,\underline{x_n}\}}$ is a finite subcover of $\pi^{-1}(\underline{V})$. This shows that preimages of compact sets are compact, ie π is proper.

To see that preimages of precompact sets $P \sqsubset \underline{U}$ are precompact, it suffices to note that the continuity of π gives $\overline{\pi^{-1}(P)} \subset \pi^{-1}(\overline{P})$, so that $\overline{\pi^{-1}(P)}$ is compact because it is a closed subset of $\pi^{-1}(\overline{P})$, which is compact as preimage of the compact set $\overline{P} \subset \underline{U}$.

To finish the proof of (i) we must show that $\underline{U}$ is a separable, locally compact metric space. But $\underline{U}$ inherits these properties from U by [15, Exercise 31.7] which applies to closed continuous surjective maps $\pi\colon X \to Y$ such that $\pi^{-1}(y)$ is compact for all $y \in Y$.

To prove (ii), first choose any open neighborhood $V_x \subset U$ of x that is disjoint from its images under the elements of $\Gamma \setminus \Gamma^x$, and then set

$$U_x := \bigcap_{\gamma\in\Gamma^x} \gamma V_x.$$

Then U_x is open since Γ^x is finite and each γV_x is open. Moreover, U_x is invariant under Γ^x, and has the property that its intersection with each Γ–orbit is either empty or is a Γ^x–orbit. Thus the restriction of π to U_x is simply the quotient by the Γ^x action, so that $U_x/\Gamma^x \to \pi(U_x)$ is the identity.

To prove the first claim in (iii), note that Γ acts on the partial quotient $\widetilde{U}/\ker\rho^\Gamma$ via its identification with $\operatorname{im}\rho^\Gamma = \widetilde{\Gamma}/\ker\rho^\Gamma$ to induce a homeomorphism $\widetilde{U}/\widetilde{\Gamma} \cong (\widetilde{U}/\ker\rho^\Gamma)/\Gamma$. Now $\underline{\rho}$ is this identification composed with the homeomorphism $(\widetilde{U}/\ker\rho^\Gamma)/\Gamma \to U/\Gamma$ induced by the Γ–equivariant diffeomorphism $\widetilde{U}/\ker\rho^\Gamma \cong U$.

As for the statement about stabilizers, notice that we have $\widetilde{\Gamma}^y \cap (\ker\rho^\Gamma) = \mathrm{id}$, because $\ker\rho^\Gamma$ acts freely. Thus $\rho^\Gamma|_{\widetilde{\Gamma}^y}$ is injective. It takes values in Γ^x for $x := \rho(y)$ by the equivariance of ρ with respect to ρ^Γ. To see that $\rho^\Gamma|_{\widetilde{\Gamma}^y}\colon \widetilde{\Gamma}^y \to \Gamma^x$ is surjective, fix an element $\delta \in \Gamma^x$. By surjectivity of $\rho^\Gamma\colon \widetilde{\Gamma} \to \Gamma$ we can choose a lift $\widetilde{\delta} \in (\rho^\Gamma)^{-1}(\delta)$. Since $\rho(\widetilde{\delta}y) = \rho^\Gamma(\widetilde{\delta})\rho(y) = \delta x = \rho(y)$ and the fibers of ρ are $\ker\rho^\Gamma$ orbits, there is a unique $\gamma \in \ker \rho^\Gamma$ such that $\gamma\widetilde{\delta}y = y$, and hence $\gamma\widetilde{\delta} \in \widetilde{\Gamma}^y$. Since $\rho^\Gamma(\gamma\widetilde{\delta}) = \rho^\Gamma(\widetilde{\delta}) = \delta$, this shows that the induced map on stabilizers $\widetilde{\Gamma}^y \to \Gamma^x$ is surjective and hence an isomorphism. □

Remark 2.1.6 In order to make our presentation more accessible we have chosen to require that the domains of our Kuranishi charts are explicit group quotients (U, Γ). Instead we could have worked with étale proper groupoids $\mathcal{G}$ with the additional property that the realization map $\mathrm{Obj}_{\mathcal{G}} \to \mathrm{Obj}_{\mathcal{G}}/{\sim}$, that identifies two objects if and only if there is a morphism between them, is proper. This extra properness assumption is proved for group quotients in Lemma 2.1.5(i). We will see below that this properness allows us to deduce results about a Kuranishi atlas $\mathcal{K}$ from results of [13] applied to the intermediate atlas $\underline{\mathcal{K}}$ in which the charts have domains $\underline{U} = U/\Gamma$. ◇

2.2 Kuranishi charts and coordinate changes

We begin by generalizing the notion of smooth Kuranishi chart (with trivial isotropy) from [14] to the case of nontrivial finite isotropy.

Remark 2.2.1 To simplify language, we will not add the specifications "smooth", "nontrivial isotropy" or "additive" to Kuranishi charts, coordinate changes, and atlases in this paper. Hence a Kuranishi atlas in this paper is a generalization (allowing nontrivial isotropy) of the notion of smooth additive Kuranishi atlas in [14]. We will see that it induces a filtered topological Kuranishi atlas in the sense of [13], given by the "intermediate charts and coordinate changes" introduced in Definition 2.2.3 and Remark 2.2.11 below. So in this paper we will take "intermediate" to include the specification "topological". ◇

Definition 2.2.2 A Kuranishi chart for X is a tuple $\boldsymbol{K} = (U, E, \Gamma, s, \psi)$ consisting of

- the *domain* U, which is a smooth finite-dimensional manifold;
- a finite-dimensional vector space E called the *obstruction space*;
- a finite *isotropy group* Γ with a smooth action on U and a linear action on E;
- a smooth Γ–equivariant function $s\colon U \to E$, called the *section*;
- a continuous map $\psi\colon s^{-1}(0) \to X$ that induces a homeomorphism

$$\underline{\psi}\colon \underline{s^{-1}(0)} := s^{-1}(0)/\Gamma \;\to\; F$$

  with open image $F \subset X$, called the *footprint* of the chart.

The *dimension* of $\boldsymbol{K}$ is $\dim(\boldsymbol{K}) := \dim U - \dim E$.

In order to extend topological constructions from [13] to the case of nontrivial isotropy, we will also consider the following notion of intermediate Kuranishi charts which have trivial isotropy but less smooth structure.

Definition 2.2.3 We associate to each Kuranishi chart $\boldsymbol{K} = (U, E, \Gamma, s, \psi)$ the *intermediate chart* $\underline{\boldsymbol{K}} := (\underline{U}, \mathbb{E}, \mathfrak{s}, \underline{\psi})$ consisting of

- the *intermediate domain* $\underline{U} := U/\Gamma$;
- the *intermediate obstruction "bundle"*, whose total space $\mathbb{E} := \underline{U \times E}$ is the quotient by the diagonal action of Γ, with the projection map $\mathrm{pr}\colon \mathbb{E} \to \underline{U}$, $\Gamma(u,e) \mapsto \Gamma u$, and zero section $0\colon \underline{U} \to \mathbb{E}$, $\Gamma u \mapsto \Gamma(u,0)$;
- the *intermediate section* $\underline{\mathfrak{s}}\colon \underline{U} \to \mathbb{E}$ induced by $\mathfrak{s} = \mathrm{id}_U \times s\colon U \to U \times E$;
- the *intermediate footprint map* $\underline{\psi}\colon \underline{\mathfrak{s}}^{-1}(\mathrm{im}\, 0) \to X$ induced by $\psi\colon s^{-1}(0) \to X$.

We write $\pi\colon U \to \underline{U}$ for the projection from the Kuranishi domain. Moreover if a chart $\boldsymbol{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I)$ has the label I, then $\underline{\boldsymbol{K}}_I = (\underline{U}_I, \mathbb{E}_I, \underline{\mathfrak{s}}_I, \underline{\psi}_I)$ and $\pi_I\colon U_I \to \underline{U}_I$ denote the corresponding intermediate chart and projection.

The intermediate charts and coordinate changes of a Kuranishi atlas (with isotropy) will form a topological Kuranishi atlas (without isotropy). For the charts, the following is a direct consequence of Lemma 2.1.5.

Lemma 2.2.4 *The intermediate chart* $\underline{\boldsymbol{K}}$ *is a topological chart in the sense of* [13, Definition 2.1.3]. *In other words,*

- *the intermediate domain* $\underline{U}$ *is a separable, locally compact metric space;*
- *the intermediate obstruction "bundle"* $\mathrm{pr}\colon \mathbb{E} \to \underline{U}$ *is a continuous map between separable, locally compact metric spaces, so that the zero section* $0\colon \underline{U} \to \mathbb{E}$ *is a continuous map with* $\mathrm{pr} \circ 0 = \mathrm{id}_{\underline{U}}$*;*
- *the intermediate section* $\underline{\mathfrak{s}}\colon \underline{U} \to \mathbb{E}$ *is a continuous map with* $\mathrm{pr} \circ \underline{\mathfrak{s}} = \mathrm{id}_{\underline{U}}$*;*
- *the intermediate footprint map* $\underline{\psi}\colon \underline{\mathfrak{s}}^{-1}(0) \to X$ *is a homeomorphism onto the footprint* $\underline{\psi}(\mathfrak{s}^{-1}(0)) = F$*, which is an open subset of* X*.*

Remark 2.2.5 (i) The intermediate bundle $\mathrm{pr}\colon \mathbb{E} \to \underline{U}$ is an orbibundle and hence has more structure than a general topological chart. In particular, it has a natural zero section $0\colon \underline{U} \to \mathbb{E}$. Hence, when working with labeled charts $\underline{\boldsymbol{K}}_I$, we will usually simply denote the projection and zero section by pr and 0 rather than $\mathrm{pr}_I, 0_I$.

(ii) We will find that many results from [13], in particular the taming constructions, carry over to nontrivial isotropy via the intermediate charts, since precompact subsets of $\underline{U}$ lift to precompact subsets of U by Lemma 2.1.5(i). An important exception is the construction of perturbations which must be done on the smooth spaces U. $\diamond$

Next, as in [13; 14], compatibility of Kuranishi charts will require restrictions and embeddings to common transition charts.

Definition 2.2.6 Let $\boldsymbol{K} = (U, E, \Gamma, s, \psi)$ be a Kuranishi chart and $F' \subset F$ an open subset of its footprint. A *restriction of* $\boldsymbol{K}$ *to* F' is a Kuranishi chart of the form

$$\boldsymbol{K}' = \boldsymbol{K}|_{\underline{U}'} := (U', E, \Gamma, s' = s|_{U'}, \psi' = \psi|_{s'^{-1}(0)}) \quad \text{with } U' := \pi^{-1}(\underline{U}')$$

given by a choice of open subset $\underline{U}' \subset \underline{U}$ such that $\underline{U}' \cap \underline{\psi}^{-1}(F) = \underline{\psi}^{-1}(F')$. We call $\underline{U}'$ the *domain* of the restriction.

Note that the restriction $\boldsymbol{K}'$ in the above definition has footprint $\psi'(s'^{-1}(0)) = F'$, and its domain group quotient (U', Γ) is the restriction of (U, Γ) to $\underline{U}'$ in the sense of Definition 2.1.3. Moreover, because the restriction of a chart is determined by a subset of the intermediate domain $\underline{U}$, we can in the following use the existence result in [13] for restrictions of topological charts to obtain restrictions of charts with isotropy. Here we use the notation $\sqsubset$ to denote a precompact inclusion and we write $\mathrm{cl}_V(V')$ for the closure of a subset $V' \subset V$ in the relative topology of V.

Lemma 2.2.7 *Let* $\boldsymbol{K}$ *be a Kuranishi chart. Then for any open subset* $F' \subset F$ *there is a restriction* $\boldsymbol{K}'$ *to* F' *with domain* $\underline{U}'$ *such that* $U' := \pi^{-1}(\underline{U}')$ *satisfies* $\mathrm{cl}_U(U') \cap s^{-1}(0) = \psi^{-1}(\mathrm{cl}_X(F'))$*. Moreover, if* $F' \sqsubset F$ *is precompact, then* $\underline{U}' \sqsubset \underline{U}$ *can be chosen precompact so that* $U' \sqsubset U$*.*

Proof By [13, Lemma 2.1.6] applied to the intermediate chart $\underline{\boldsymbol{K}}$, there is a subset $\underline{U}' \subset \underline{U}$ that defines a restriction of this topological chart, and in particular satisfies $\underline{U}' \cap \underline{\mathfrak{s}}^{-1}(0) = \underline{\psi}^{-1}(F')$, with the additional property $\mathrm{cl}_{\underline{U}}(\underline{U}') \cap \underline{\mathfrak{s}}^{-1}(0) = \underline{\psi}^{-1}(\mathrm{cl}_X(F'))$. Further, we may assume that $\underline{U}'$ is precompact in $\underline{U}$ if $F' \sqsubset F$. Then $U' = \pi^{-1}(\underline{U}')$ is the required domain. It inherits precompactness by Lemma 2.1.5(i). Further, the same lemma shows that $\pi^{-1}(\mathrm{cl}_{\underline{U}}(\underline{U}')) = \mathrm{cl}_U(U')$. Hence applying π^{-1} to the identity $\mathrm{cl}_{\underline{U}}(\underline{U}') \cap \underline{\mathfrak{s}}^{-1}(0) = \underline{\psi}^{-1}(\mathrm{cl}_X(F'))$ implies that $\mathrm{cl}_U(U') \cap s^{-1}(0) = \psi^{-1}(\mathrm{cl}_X(F'))$. □

Most definitions in [14] extend, as the previous ones, with only minor changes to the case of nontrivial isotropy. However, the notion of smooth coordinate change [14, Definition 5.2.2] needs to be generalized significantly to include a covering map. For simplicity we will formulate the definition in the situation that is relevant to additive Kuranishi atlases.[3] That is, we suppose that a finite set of basic Kuranishi charts $(\boldsymbol{K}_i)_{i \in \{1,\dots,N\}}$ is given so that for each $I \subset \{1, \dots, N\}$ with $F_I := \bigcap_{i \in I} F_i \neq \emptyset$ we have another Kuranishi chart $\boldsymbol{K}_I$ with

[3] While additivity was introduced as separate property in [12], it is both so crucial and natural that below in Section 2.3 we will define the notion of Kuranishi atlas to be automatically additive.

- isotropy group $\Gamma_I := \prod_{i\in I}\Gamma_i$,
- obstruction space $E_I := \prod_{i\in I} E_i$ on which Γ_I acts with the product action,
- footprint $F_I := \bigcap_{i\in I} F_i$.

Then for $I\subset J$ we have the natural splitting $\Gamma_J \cong \Gamma_I\times\Gamma_{J\setminus I}$ with induced inclusion $\Gamma_I \hookrightarrow \Gamma_I\times\{\mathrm{id}\}\subset\Gamma_J$ and projection $\rho^\Gamma_{IJ}\colon \Gamma_J\to\Gamma_I$ with kernel $\Gamma_{J\setminus I}$. (Here we include the case $I=J$, interpreting $\Gamma_\varnothing := \{\mathrm{id}\}$.) Moreover, we have the natural inclusion $\widehat\phi_{IJ}\colon E_I\to E_J$, which is equivariant with respect to the inclusion $\Gamma_I\hookrightarrow\Gamma_J$ and such that the complement of this inclusion $\Gamma_{J\setminus I}$ acts trivially on the image $\widehat\phi_{IJ}(E_I)\subset E_J$.

Definition 2.2.8 Given $I\subset J\subset\{1,\dots,N\}$ let $\boldsymbol{K}_I$ and $\boldsymbol{K}_J$ be Kuranishi charts as above with $F_I\supset F_J$. A *smooth coordinate change* $\widehat\Phi_{IJ}$ from $\boldsymbol{K}_I$ to $\boldsymbol{K}_J$ consists of

- a choice of domain $\underline{U}_{IJ}\subset\underline{U}_I$ such that $\boldsymbol{K}_I|_{\underline{U}_{IJ}}$ is a restriction of $\boldsymbol{K}_I$ to F_J,
- the splitting $\Gamma_J\cong\Gamma_I\times\Gamma_{J\setminus I}$ as above, and the induced inclusion $\Gamma_I\hookrightarrow\Gamma_J$ and projection $\rho^\Gamma_{IJ}\colon\Gamma_J\to\Gamma_I$,
- a Γ_J–invariant submanifold $\widetilde U_{IJ}\subset U_J$ on which $\Gamma_{J\setminus I}$ acts freely, and the induced Γ_J–equivariant inclusion $\widetilde\phi_{IJ}\colon\widetilde U_{IJ}\hookrightarrow U_J$,
- a group covering $(\widetilde U_{IJ},\Gamma_J,\rho_{IJ},\rho^\Gamma_{IJ})$ of the group quotient (U_{IJ},Γ_I), where $U_{IJ} := \pi_I^{-1}(\underline{U}_{IJ})\subset U_I$,
- the linear equivariant injection $\widehat\phi_{IJ}\colon E_I\to E_J$ as above,

such that the inclusions $\widetilde\phi_{IJ},\widehat\phi_{IJ}$ and covering ρ_{IJ} intertwine the sections and footprint maps,

$$\begin{aligned} s_J\circ\widetilde\phi_{IJ} &= \widehat\phi_{IJ}\circ s_I\circ\rho_{IJ} && \text{on } \widetilde U_{IJ},\\ \psi_J\circ\widetilde\phi_{IJ} &= \psi_I\circ\rho_{IJ} && \text{on } s_J^{-1}(0)\cap\widetilde U_{IJ} = \rho_{IJ}^{-1}(s_I^{-1}(0)).\end{aligned}\tag{2.2.1}$$

Moreover, we denote $s_{IJ} := s_I\circ\rho_{IJ}\colon\widetilde U_{IJ}\to E_I$ and require the *index condition*:

(i) The embedding $\widetilde\phi_{IJ}\colon\widetilde U_{IJ}\hookrightarrow U_J$ identifies the kernels:

$$\mathrm{d}_u\widetilde\phi_{IJ}(\ker \mathrm{d}_u s_{IJ}) = \ker \mathrm{d}_{\widetilde\phi_{IJ}(u)} s_J \quad \forall u\in\widetilde U_{IJ}.$$

(ii) The linear embedding $\widehat\phi_{IJ}\colon E_I\to E_J$ identifies the cokernels:

$$E_I = \mathrm{im}(\mathrm{d}_u s_{IJ})\oplus C_{u,I} \implies E_J = \mathrm{im}(\mathrm{d}_{\widetilde\phi_{IJ}(u)} s_J)\oplus\widehat\phi_{IJ}(C_{u,I}) \quad \forall u\in\widetilde U_{IJ}.$$

The subset $\underline{U}_{IJ}\subset\underline{U}_I$ is called the *domain* of the coordinate change, while $\widetilde U_{IJ}\subset U_J$ is its *lifted domain*.

Recall that we have $\dim \widetilde{U}_{IJ} = \dim U_I$ since $\rho_{IJ}\colon \widetilde{U}_{IJ} \to U_{IJ}$ is a regular covering. Moreover, ρ_{IJ} identifies the kernels and images of $\mathrm{d}s_{IJ}$ and $\mathrm{d}s_I$, in other words

$$\text{(2.2.2)} \qquad \mathrm{d}_u\rho_{IJ}(\ker \mathrm{d}_u s_{IJ}) = \ker \mathrm{d}_{\rho_{IJ}(u)} s_I, \quad \operatorname{im}(\mathrm{d}_u s_{IJ}) = \operatorname{im}(\mathrm{d}_{\rho_{IJ}(u)} s_I) \subset E_I.$$

Hence the index condition is equivalent to kernels and cokernels of $\mathrm{d}_{\rho_{IJ}(u)} s_I$ and $\mathrm{d}_u s_J$ being identified by the coordinate change. As in [14, Lemma 5.2.5] it is also equivalent to the *tangent bundle condition*

$$\text{(2.2.3)} \qquad \mathrm{d}_{\widetilde{\phi}_{IJ}(u)} s_J\colon \mathrm{T}_{\widetilde{\phi}_{IJ}(u)} U_J \big/ \mathrm{d}_u\widetilde{\phi}_{IJ}(\mathrm{T}_u\widetilde{U}_{IJ}) \xrightarrow{\cong} E_J \big/ \widehat{\phi}_{IJ}(E_I) \quad \forall u \in \widetilde{U}_{IJ}.$$

This also shows that any two charts that are related by a coordinate change have the same dimension. To keep our language similar to that in [14], we denote a coordinate change by $\widehat{\Phi}_{IJ} = (\widetilde{\phi}_{IJ}, \widehat{\phi}_{IJ}, \rho_{IJ})\colon \boldsymbol{K}_I|_{\underline{U}_{IJ}} \to \boldsymbol{K}_J$. However, since the linear map $\widehat{\phi}_{IJ}$ is fixed by our conventions, the coordinate change $\widehat{\Phi}_{IJ}$ is in fact determined by a group covering $(\widetilde{U}_{IJ}, \Gamma_J, \rho_{IJ}, \rho^{\Gamma}_{IJ})$ of $(\pi_I^{-1}(\underline{U}_{IJ}), \Gamma_I)$, where $\underline{U}_{IJ} \subset \underline{U}_I$ is a choice of domain for which $\underline{U}_{IJ} \cap \underline{\psi}_I^{-1}(F_I) = \underline{\psi}_I^{-1}(F_J)$.

Remark 2.2.9 (i) In the case of trivial isotropy and with trivial covering $\rho_{IJ} =: \phi_{IJ}^{-1}$, this definition is the notion of coordinate change in [14] with $\widetilde{U}_{IJ} = \phi_{IJ}(U_{IJ})$. Because $U_{IJ} \subset U_I$ is open, the index condition together with the condition that $\widetilde{U}_{IJ}$ is a submanifold of U_J implies that $\widetilde{U}_{IJ}$ is an open subset of $s_J^{-1}(E_I)$.

(ii) The following diagram of group embeddings and group coverings is associated to each coordinate change:

$$\text{(2.2.4)} \qquad \begin{tikzcd} & (\widetilde{U}_{IJ}, \Gamma_J) \arrow[r, hook, "(\widetilde{\phi}_{IJ}, \mathrm{id})"] \arrow[d, "(\rho_{IJ}, \rho^{\Gamma}_{IJ})"] & (U_J, \Gamma_J) \\ (U_I, \Gamma_I) & (U_{IJ}, \Gamma_I) \arrow[l, hook'] & \end{tikzcd}$$

(iii) Since $\underline{\rho}_{IJ}\colon \underline{\widetilde{U}}_{IJ} \to \underline{U}_{IJ}$ is a homeomorphism by Lemma 2.1.5(iii), each coordinate change $(\phi_{IJ}, \widehat{\phi}_{IJ}, \rho_{IJ})\colon \boldsymbol{K}_I|_{\underline{U}_{IJ}} \to \boldsymbol{K}_J$ induces an injective map

$$\underline{\phi}_{IJ} := \underline{\widetilde{\phi}}_{IJ} \circ \underline{\rho}_{IJ}^{-1}\colon \underline{U}_{IJ} \to \underline{U}_J$$

on the domain of the intermediate chart. In fact there is an induced coordinate change $\underline{\widehat{\Phi}}_{IJ}\colon \underline{\boldsymbol{K}}_I|_{\underline{U}_{IJ}} \to \underline{\boldsymbol{K}}_J$ between the intermediate charts, given by the bundle map $\underline{\widehat{\Phi}}_{IJ}\colon \underline{U_{IJ} \times E_I} \to \underline{U_J \times E_J}$ which is induced by the multivalued map $(\widetilde{\phi}_{IJ} \circ \rho_{IJ}^{-1}) \times \widehat{\phi}_{IJ}$ and thus covers $\underline{\widetilde{\phi}}_{IJ} \circ \underline{\rho}_{IJ}^{-1} =: \underline{\phi}_{IJ}$. This is a topological coordinate change in the sense of [13, Definition 2.2.1]. This means in particular that the map

$$\underline{\widehat{\Phi}}_{IJ}\colon \underline{U_{IJ} \times E_I} =: \underline{\mathbb{E}}_I|_{\underline{U}_{IJ}} := \mathrm{pr}_I^{-1}(\underline{U}_{IJ}) \to \underline{\mathbb{E}}_J$$

is a topological embedding (ie homeomorphism to its image) that satisfies the following:

- It is a bundle map, ie we have $\mathrm{pr}_J \circ \widehat{\underline{\Phi}}_{IJ} = \underline{\phi}_{IJ} \circ \mathrm{pr}_I|_{\mathrm{pr}_I^{-1}(\underline{U}_{IJ})}$ for a topological embedding $\underline{\phi}_{IJ} \colon \underline{U}_{IJ} \to \underline{U}_J$, and it is linear in the sense that $0_J \circ \underline{\phi}_{IJ} = \widehat{\underline{\Phi}}_{IJ} \circ 0_I|_{U_{IJ}}$, where 0_I denotes the zero section $0_I \colon \underline{U}_I \to \underline{\mathbb{E}}_I$ in the chart $\underline{\boldsymbol{K}}_I$.
- It intertwines the sections and footprints maps, ie

$$\underline{\mathfrak{s}}_J \circ \underline{\phi}_{IJ} = \widehat{\underline{\Phi}}_{IJ} \circ \underline{\mathfrak{s}}_I|_{U_{IJ}}, \quad \underline{\phi}_{IJ}|_{\underline{\psi}_I^{-1}(F_I \cap F_J)} = \underline{\psi}_J^{-1} \circ \underline{\psi}_I.$$

However, $\widehat{\underline{\Phi}}_{IJ}$ has more smooth structure than a general topological coordinate change since $\underline{\phi}_{IJ} \colon \underline{U}_{IJ} \to \underline{U}_J$ preserves the orbifold structure and $\widehat{\underline{\Phi}}_{IJ}$ is a map of orbibundles.

(iv) Conversely, suppose we are given a topological coordinate change $\widehat{\underline{\Phi}}_{IJ} \colon \underline{\boldsymbol{K}}_I \to \underline{\boldsymbol{K}}_J$ with domain $\underline{U}_{IJ}$. Then any coordinate change from $\boldsymbol{K}_I$ to $\boldsymbol{K}_J$ that induces $\widehat{\underline{\Phi}}_{IJ}$ is determined by the Γ_J–invariant set $\widetilde{U}_{IJ} := \pi_J^{-1}(\underline{\phi}_{IJ}(\underline{U}_{IJ}))$ and a choice of Γ_I–equivariant homeomorphism between $\widetilde{U}_{IJ}/\Gamma_{J\setminus I}$ and $U_{IJ} := \pi_I^{-1}(\underline{U}_{IJ})$. If we can choose this homeomorphism to be smooth, then we obtain a smooth coordinate change $\boldsymbol{K}_I \to \boldsymbol{K}_J$ with domain $\underline{U}_{IJ}$ provided that the index condition is satisfied, which is a condition on the relation between the set $\widetilde{U}_{IJ}$ and the section s_J. When constructing coordinate changes in the Gromov–Witten setting in [10], we will see that there is a natural choice of this diffeomorphism since the covering maps ρ_{IJ} are given by forgetting certain added marked points. Further, the index condition is automatically satisfied in this setting.

(v) Because $\widetilde{U}_{IJ}$ is defined to be a subset of U_J, it is sometimes convenient to think of an element $\widetilde{x} \in \widetilde{U}_{IJ}$ as an element in U_J, omitting the notation for the inclusion map $\widetilde{\phi}_{IJ} \colon \widetilde{U}_{IJ} \to U_J$. ◇

The next step is to consider restrictions and composites of coordinate changes. Restrictions exist analogously to [14, Lemma 5.2.6]: for $I \subset J$, given a coordinate change $\widehat{\Phi}_{IJ} \colon \boldsymbol{K}_I|_{\underline{U}_{IJ}} \to \boldsymbol{K}_J$ and restrictions $\boldsymbol{K}'_I := \boldsymbol{K}_I|_{\underline{U}'_I}$ and $\boldsymbol{K}'_J := \boldsymbol{K}_J|_{\underline{U}'_J}$ whose footprints $F'_I \cap F'_J$ have nonempty intersection, there is an induced *restricted coordinate change* $\widehat{\Phi}_{IJ}|_{\underline{U}'_{IJ}} \colon \boldsymbol{K}'_I|_{\underline{U}'_{IJ}} \to \boldsymbol{K}'_J$ for any open subset $\underline{U}'_{IJ} \subset \underline{U}_{IJ}$ satisfying the conditions

$$\underline{U}'_{IJ} \subset \underline{U}'_I \cap \underline{\phi}_{IJ}^{-1}(\underline{U}'_J), \quad \underline{U}'_{IJ} \cap \underline{\mathfrak{s}}_I^{-1}(0) = \underline{\psi}_I^{-1}(F'_I \cap F'_J). \tag{2.2.5}$$

However, coordinate changes now do not directly compose due to the coverings involved. The induced coordinate changes on the intermediate charts still compose directly, but the analog of [14, Lemma 5.2.7] is the following.

Lemma 2.2.10 *Let $I \subset J \subset K$ (so that automatically $F_I \supset F_J \supset F_K$) and suppose that $\widehat{\Phi}_{IJ}\colon \boldsymbol{K}_I \to \boldsymbol{K}_J$ and $\widehat{\Phi}_{JK}\colon \boldsymbol{K}_J \to \boldsymbol{K}_K$ are coordinate changes with domains $\underline{U}_{IJ}$ and $\underline{U}_{JK}$ respectively. Then:*

(i) *The domain $\underline{U}_{IJK} := \underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK}) \subset \underline{U}_I$ defines a restriction $\boldsymbol{K}_I|_{\underline{U}_{IJK}}$ of $\boldsymbol{K}_I$ to F_K.*

(ii) *The composite $\rho_{IJK} := \rho_{IJ} \circ \rho_{JK}\colon \widetilde{U}_{IJK} \to U_{IJK} := \pi_I^{-1}(\underline{U}_{IJK})$ is defined on $\widetilde{U}_{IJK} := \pi_K^{-1}((\underline{\phi}_{JK} \circ \underline{\phi}_{IJ})(\underline{U}_{IJK}))$ via the natural identification of $\rho_{JK}(\widetilde{U}_{IJK}) \subset U_J$ with a subset of $\widetilde{U}_{IJ}$. Together with the natural projection $\rho^\Gamma_{IK}\colon \Gamma_K \to \Gamma_I$ with kernel $\Gamma_{K\setminus I}$, which factors $\rho^\Gamma_{IK} = \rho^\Gamma_{IJ} \circ \rho^\Gamma_{JK}$, this forms a group covering $(\widetilde{U}_{IJK}, \Gamma_K, \rho_{IJK}, \rho^\Gamma_{IK})$ of (U_{IJK}, Γ_I).*

(iii) *The inclusion $\widetilde{\phi}_{IJK}\colon \widetilde{U}_{IJK} \hookrightarrow U_K$, taken together with the natural inclusion $\widehat{\phi}_{IK}\colon E_I \to E_K$ (which factors $\widehat{\phi}_{IK} = \widehat{\phi}_{JK} \circ \widehat{\phi}_{IJ}$) and ρ_{IJK}, satisfies (2.2.1) and the index condition with respect to the indices I, K.*

Hence this defines a ***composite coordinate change***

$$\widehat{\Phi}_{JK} \circ \widehat{\Phi}_{IJ} := \widehat{\Phi}_{IJK} = (\widetilde{\phi}_{IJK}, \widehat{\phi}_{IK}, \rho_{IJK})$$

from $\boldsymbol{K}_I$ to $\boldsymbol{K}_K$ with domain $\underline{U}_{IJK}$.

Proof The corresponding statement for the induced coordinate changes for the intermediate charts is proved in [13, Lemma 2.2.5]. Thus claim (i) follows from part (i) of [13, Lemma 2.2.5].

To see that ρ_{IJK} in (ii) is well defined, we need to verify that $\rho_{JK}(\widetilde{U}_{IJK}) \subset \widetilde{U}_{IJ}$, or (due to equivariance) equivalently $\underline{\rho}_{JK}(\underline{\widetilde{U}}_{IJK}) \subset \underline{\widetilde{U}}_{IJ}$. For that purpose we drop the natural identifications $\underline{\widetilde{\phi}}_{IJ}\colon \underline{\widetilde{U}}_{IJ} \to \underline{U}_J$ from the notation so that the intermediate coordinate changes are $\underline{\phi}_{IJ} = \underline{\rho}_{IJ}^{-1}\colon \underline{U}_{IJ} \to \underline{\widetilde{U}}_{IJ} \subset \underline{U}_J$ and the inclusion follows from

$$\begin{aligned}\underline{\rho}_{JK}(\underline{\widetilde{U}}_{IJK}) &= \underline{\rho}_{JK}((\underline{\phi}_{JK} \circ \underline{\phi}_{IJ})(\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK}))) \\ &= (\underline{\rho}_{JK} \circ \underline{\phi}_{JK})(\underline{\widetilde{U}}_{IJ} \cap \underline{U}_{JK}) \\ &= \underline{\widetilde{U}}_{IJ} \cap \underline{U}_{JK}.\end{aligned}$$

Next, observe that composites of group covering maps are also group covering maps. In particular, since $\Gamma_{K\setminus J}$ acts freely on $\widetilde{U}_{IJK} \subset \widetilde{U}_{JK}$ and $\Gamma_{J\setminus I}$ acts freely on the quotient $\widetilde{U}_{IJK}/\Gamma_{K\setminus J}$ (because it is identified Γ_J–equivariantly with a subset of $\widetilde{U}_{IJ}$), the group $\Gamma_{K\setminus I} \cong \Gamma_{K\setminus J} \times \Gamma_{J\setminus I}$ acts freely on $\widetilde{U}_{IJK}$.

To prove (iii), first observe that (2.2.1) holds for the index pair IK because it holds for IJ and JK:

$$\begin{aligned} s_K \circ \widetilde{\phi}_{IJK} &= \widehat{\phi}_{JK} \circ s_J \circ \rho_{JK}|_{\widetilde{U}_{IJK}} \\ &= \widehat{\phi}_{JK} \circ (\widehat{\phi}_{IJ} \circ s_I \circ \rho_{IJ}) \circ \rho_{JK}|_{\widetilde{U}_{IJK}} \\ &= \widehat{\phi}_{IK} \circ s_I \circ \rho_{IJK} && \text{on } \widetilde{U}_{IJK}, \\ \psi_K \circ \widetilde{\phi}_{IJK} &= \psi_J \circ \rho_{JK} \\ &= \psi_I \circ \rho_{IJ} \circ \rho_{JK} \\ &= \psi_I \circ \rho_{IJK} && \text{on } s_K^{-1}(0) \cap \widetilde{U}_{IJK}. \end{aligned}$$

Finally, it is easiest to check the index condition in the form given in (2.2.3), ie we need to establish isomorphisms for all $u \in \widetilde{U}_{IJK}$,

$$\mathrm{d}_{\widetilde{\phi}_{IJK}(u)} s_K\colon \mathrm{T}_{\widetilde{\phi}_{IJK}(u)} U_K / \mathrm{d}_u \widetilde{\phi}_{IJK}(\mathrm{T}_u \widetilde{U}_{IJK}) \xrightarrow{\cong} E_K / \widehat{\phi}_{IK}(E_I). \tag{2.2.6}$$

Here and below we will suppress the natural embedding $\widetilde{\phi}_{IJK}\colon \widetilde{U}_{IJK} \to U_K$ from the notation, hence identifying eg $u \in \widetilde{U}_{IJK}$ with $\widetilde{\phi}_{IJK}(u) \in U_K$. With that, the quotient on the left is naturally identified with the normal fiber $\mathrm{T}_u U_K / \mathrm{T}_u \widetilde{U}_{IJK}$ to the submanifold $\widetilde{U}_{IJK}$ of U_K. Next, $\widetilde{U}_{IJK}$ is by construction a submanifold of $\widetilde{U}_{JK}$, which in turn is a submanifold of U_K, hence this normal fiber is isomorphic to the direct sum of the normal fiber of $\widetilde{U}_{IJK}$ in $\widetilde{U}_{JK}$ together with that of $\widetilde{U}_{JK}$ in U_K,

$$\mathrm{T}_u U_K / \mathrm{T}_u \widetilde{U}_{IJK} \cong \mathrm{T}_u U_K / \mathrm{T}_u \widetilde{U}_{JK} \oplus \mathrm{T}_u \widetilde{U}_{JK} / \mathrm{T}_u \widetilde{U}_{IJK}.$$

By the index condition for $\widehat{\Phi}_{JK}$, the map $\mathrm{d}_u s_K$ restricted to the first summand induces an isomorphism $\mathrm{T}_u U_K / \mathrm{T}_u \widetilde{U}_{JK} \xrightarrow{\cong} E_K / \widehat{\Phi}_{JK}(E_J)$. Considering the second summand, recall that on $\widetilde{U}_{JK}$ we have $s_K = s_J \circ \rho_{JK}$, where $\rho_{JK}\colon \widetilde{U}_{JK} \to U_{JK}$ is a local diffeomorphism onto an open subset of U_J. It maps $\widetilde{U}_{IJK}$ to $\rho_{JK}(\widetilde{U}_{IJK}) = \widetilde{U}_{IJ} \cap U_{JK}$ so that, with $v := \rho_{JK}(u)$, the map $\mathrm{d}_u \rho_{JK}$ induces an isomorphism $\mathrm{T}_u \widetilde{U}_{JK} / \mathrm{T}_u \widetilde{U}_{IJK} \xrightarrow{\cong} \mathrm{T}_v U_J / \mathrm{T}_v \widetilde{U}_{IJ}$. Thus the restriction of $\mathrm{d}_u s_K$ to the second summand induces the isomorphism

$$\mathrm{d}_v s_J \circ \mathrm{d}_u \rho_{JK}\colon \mathrm{T}_u \widetilde{U}_{JK} / \mathrm{T}_u \widetilde{U}_{IJK} \xrightarrow{\cong} \mathrm{T}_v U_J / \mathrm{T}_v \widetilde{U}_{IJ} \xrightarrow{\cong} E_J / \widehat{\Phi}_{IJ}(E_I),$$

where the second isomorphism results from the index condition for $\widehat{\Phi}_{IJ}$. Putting this all together, $\mathrm{d}_u s_K$ induces an isomorphism from $\mathrm{T}_u U_K / \mathrm{T}_u \widetilde{U}_{IJK}$ to

$$E_K / \widehat{\Phi}_{JK}(E_J) \oplus E_J / \widehat{\Phi}_{IJ}(E_I) \cong E_K / \widehat{\Phi}_{IK}(E_I),$$

where in the last step we used the fact that $\widehat{\Phi}_{JK}\colon E_J \to E_K$ is the natural inclusion. This establishes the isomorphism (2.2.6) and thus completes the proof. □

Remark 2.2.11 The composition $\widehat{\Phi}_{IJK}\colon \boldsymbol{K}_I \to \boldsymbol{K}_K$ induces a coordinate change $\underline{\widehat{\Phi}}_{IJK}\colon \underline{\boldsymbol{K}}_I \to \underline{\boldsymbol{K}}_K$ on the intermediate charts. This agrees with the composition of the intermediate coordinate changes $\underline{\widehat{\Phi}}_{IJ}, \underline{\widehat{\Phi}}_{JK}$ as defined for topological charts in [13, Lemma 2.2.5]. $\diamond$

Next, the cocycle conditions from [13, Definition 2.3.2] have direct generalizations.

Definition 2.2.12 Let $\boldsymbol{K}_\alpha$ for $\alpha = I, J, K$ be Kuranishi charts with $I \subset J \subset K$, and let $\widehat{\Phi}_{\alpha\beta}\colon \boldsymbol{K}_\alpha|_{\underline{U}_{\alpha\beta}} \to \boldsymbol{K}_\beta$ for $(\alpha,\beta) \in \{(I,J),(J,K),(I,K)\}$ be coordinate changes. We say that this triple $\widehat{\Phi}_{IJ}, \widehat{\Phi}_{JK}, \widehat{\Phi}_{IK}$ satisfies the

- *weak cocycle condition* if $\widehat{\Phi}_{JK} \circ \widehat{\Phi}_{IJ} \approx \widehat{\Phi}_{IK}$ are equal on the overlap, in the sense that

$$\rho_{IK} = \rho_{IJ} \circ \rho_{JK} \quad \text{on } \widetilde{U}_{IK} \cap \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}); \tag{2.2.7}$$

- *cocycle condition* if $\widehat{\Phi}_{JK} \circ \widehat{\Phi}_{IJ} \subset \widehat{\Phi}_{IK}$, ie $\widehat{\Phi}_{IK}$ extends the composed coordinate change in the sense that (2.2.7) holds and

$$\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK}) \subset \underline{U}_{IK}; \tag{2.2.8}$$

- *strong cocycle condition* if $\widehat{\Phi}_{JK} \circ \widehat{\Phi}_{IJ} = \widehat{\Phi}_{IK}$ are equal as coordinate changes, that is if (2.2.7) holds and

$$\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK}) = \underline{U}_{IK}. \tag{2.2.9}$$

We stated these last two conditions on the level of the intermediate category because, as we now show, they imply corresponding identities on the level of the Kuranishi atlas.

Lemma 2.2.13 (i) *Condition* (2.2.7) *implies*

$$\underline{\phi}_{IK} = \underline{\phi}_{JK} \circ \underline{\phi}_{IJ} \quad \text{on } \underline{U}_{IK} \cap (\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK}));$$

(ii) *The cocycle condition* (2.2.8) *implies that*

$$\rho_{IK} = \rho_{IJ} \circ \rho_{JK} \quad \text{on } \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}) \subset \widetilde{U}_{IK}.$$

(iii) *The strong cocycle condition* (2.2.9) *implies that*

$$\rho_{IK} = \rho_{IJ} \circ \rho_{JK} \quad \text{on } \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}) = \widetilde{U}_{IK}.$$

Proof By definition, $\underline{\rho}_{\alpha\beta} \circ \pi_\beta = \pi_\alpha \circ \rho_{\alpha\beta}$ when $\alpha \subset \beta$, so condition (2.2.7) implies

$$\underline{\rho}_{IK} = \underline{\rho}_{IJ} \circ \underline{\rho}_{JK} \quad \text{on } \pi_K(\widetilde{U}_{IK} \cap \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK})).$$

The identity $\underline{\phi}_{\alpha\beta} = \underline{\rho}_{\alpha\beta}^{-1}$ from Remark 2.2.9(iii) then implies $\underline{\phi}_{IK} = \underline{\phi}_{JK} \circ \underline{\phi}_{IJ}$ on

$$\begin{aligned}
\underline{\rho}_{IK}(\pi_K(\widetilde{U}_{IK} \cap \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}))) &= \pi_I(\rho_{IK}(\widetilde{U}_{IK} \cap \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}))) \\
&= \pi_I(\rho_{IK}(\widetilde{U}_{IK}) \cap \rho_{IJ} \circ \rho_{JK}(\rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}))) \\
&= \pi_I(U_{IK} \cap \rho_{IJ}(\widetilde{U}_{IJ} \cap U_{JK})) \\
&= \underline{U}_{IK} \cap (\underline{U}_{IJ} \cap \underline{\rho}_{IJ}(\underline{U}_{JK})) \\
&= \underline{U}_{IK} \cap (\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK})),
\end{aligned}$$

where the second equality uses $\rho_{IK} = \rho_{IJ} \circ \rho_{JK}$ on $\widetilde{U}_{IK} \cap \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK})$, and the last uses $\underline{\rho}_{IJ} = \underline{\phi}_{IJ}^{-1}$. This proves (i).

Using in addition the identities $U_{\alpha\beta} = \pi_\alpha^{-1}(\underline{U}_{\alpha\beta})$ and $\widetilde{U}_{\alpha\beta} = \pi_\beta^{-1}(\underline{\phi}_{\alpha\beta}(\underline{U}_{\alpha\beta}))$, the cocycle condition (2.2.8) implies the inclusion claimed in (ii),

$$\begin{aligned}
\rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK}) &= (\pi_J \circ \rho_{JK})^{-1}(\underline{\phi}_{IJ}(\underline{U}_{IJ}) \cap \underline{U}_{JK}) \\
&= (\underline{\phi}_{IJ} \circ \underline{\rho}_{IK} \circ \pi_K)^{-1}(\underline{\phi}_{IJ}(\underline{U}_{IJ}) \cap \underline{U}_{JK}) \\
&= (\underline{\rho}_{IK} \circ \pi_K)^{-1}(\underline{U}_{IJ} \cap \underline{\phi}_{IJ}^{-1}(\underline{U}_{JK})) \subset \pi_K^{-1}(\underline{\rho}_{IK}^{-1}(\underline{U}_{IK})) \\
&= \widetilde{U}_{IK}.
\end{aligned}$$

The proof of (iii) is the same, with the strong cocycle condition implying equality in the second to last step. □

2.3 Kuranishi atlases

With the notions of Kuranishi charts and coordinate changes with nontrivial isotropy in place, we can now directly extend the notion of smooth Kuranishi atlas from [14, Definition 6.1.3]. For comparison with the notions of smooth and topological Kuranishi atlas from [13; 14], see Remark 2.2.1.

Definition 2.3.1 A *(weak) Kuranishi atlas of dimension d* on a compact metrizable space X is a tuple

$$\mathcal{K} = (\boldsymbol{K}_I, \widehat{\Phi}_{IJ})_{I,J \in \mathcal{I}_\mathcal{K}, I \subsetneq J}$$

consisting of a covering family of basic charts $(\boldsymbol{K}_i)_{i=1,\dots,N}$ of dimension d and transition data $(\boldsymbol{K}_J)_{|J| \ge 2}$, $(\widehat{\Phi}_{IJ})_{I \subsetneq J}$ for $(\boldsymbol{K}_i)_{i=1,\dots,N}$, where:

- A *covering family of basic charts* for X is a finite collection $(\boldsymbol{K}_i)_{i=1,\dots,N}$ of Kuranishi charts for X whose footprints cover $X = \bigcup_{i=1}^N F_i$.
- *Transition data* for a covering family $(\boldsymbol{K}_i)_{i=1,\dots,N}$ is a collection of Kuranishi charts $(\boldsymbol{K}_J)_{J \in \mathcal{I}_\mathcal{K}, |J| \ge 2}$ and coordinate changes $(\widehat{\Phi}_{IJ})_{I,J \in \mathcal{I}_\mathcal{K}, I \subsetneq J}$ as follows:

(i) $\mathcal{I}_{\mathcal{K}}$ denotes the set of subsets $I \subset \{1, \dots, N\}$ for which the intersection of footprints is nonempty,

$$F_I := \bigcap_{i \in I} F_i \neq \varnothing.$$

(ii) For each $J \in \mathcal{I}_{\mathcal{K}}$ with $|J| \geq 2$, $\boldsymbol{K}_J$ is a Kuranishi chart for X with footprint $F_J = \bigcap_{i \in J} F_i$, group $\Gamma_J = \prod_{j \in J} \Gamma_j$, and obstruction space $E_J = \prod_{j \in J} E_j$. Further, for one element sets $J = \{i\}$ we denote $\boldsymbol{K}_{\{i\}} := \boldsymbol{K}_i$.

(iii) $\widehat{\Phi}_{IJ} = (\rho_{IJ}, \rho^{\Gamma}_{IJ}, \widehat{\phi}_{IJ})$ is a coordinate change $\boldsymbol{K}_I \to \boldsymbol{K}_J$ for every $I, J \in \mathcal{I}_{\mathcal{K}}$ with $I \subsetneq J$, where $\rho^{\Gamma}_{IJ} \colon \Gamma_J \to \Gamma_I$ is the natural projection $\prod_{j \in J} \Gamma_j \to \prod_{i \in I} \Gamma_i$ and $\widehat{\phi}_{IJ} \colon E_I \to E_J$ is the natural inclusion $\prod_{i \in I} E_j \to \prod_{j \in J} E_j$.

For a weak atlas we require that the weak cocycle condition in Definition 2.2.12 hold for every triple $I, J, K \in \mathcal{I}_K$ with $I \subsetneq J \subsetneq K$, while for an atlas the cocycle condition must hold for all such triples.

Remark 2.3.2 Note that we have built *additivity* in the sense of [14, Definition 6.1.5] into the above definitions. Namely, the natural embeddings $\widehat{\phi}_{iI} \colon E_i \to E_I = \prod_{\ell \in I} E_\ell$ for each $I \in \mathcal{I}_{\mathcal{K}}$ induce the identity isomorphism

$$\prod_{i \in I} \widehat{\phi}_{iI} \colon \prod_{i \in I} E_i \xrightarrow{\cong} E_I = \prod_{\ell \in I} E_\ell, \tag{2.3.1}$$

and for $I \subset J$ the linear map $\widehat{\phi}_{IJ} \colon E_I \to E_J$ is the induced inclusion $\prod_{i \in I} E_i \to \prod_{i \in J} E_i$. Further, each group Γ_I is the product $\prod_{i \in I} \Gamma_i$ and we use the natural projections $\rho^{\Gamma}_{IJ} \colon \Gamma_J \to \Gamma_I$ in the group covering maps of the coordinate changes. Hence, when $I \subset J \subset K$ the projections $\rho^{\Gamma}_{\bullet\bullet}$ and linear inclusions $\widehat{\phi}_{\bullet\bullet}$ are automatically compatible:

$$\rho^{\Gamma}_{IK} = \rho^{\Gamma}_{IJ} \circ \rho^{\Gamma}_{JK}, \quad \widehat{\phi}_{IK} = \widehat{\phi}_{JK} \circ \widehat{\phi}_{IJ}.$$

So when $I \subset J$ we will almost always write $E_I \subset E_J$ for the subspace $\widehat{\phi}_{IJ}(E_I) \subset E_J$, and similarly we have a natural identification of Γ_J with $\Gamma_I \times \Gamma_{J \setminus I}$. $\diamond$

Remark 2.3.3 Although it seems that many interdependent choices are needed in order to construct a Kuranishi atlas, this is somewhat deceptive. For example, in the Gromov–Witten case considered in [10] (see also [10]), the geometric choices involved in the construction of a family of basic charts $(\boldsymbol{K}_i)_{i=1,\dots,N}$ essentially induce the transition data as well. Namely, each basic chart $\boldsymbol{K}_i$ is constructed by adding a certain tuple $\vec{w}_i$ of marked points to the domains of the stable maps (f, z), given by the preimages of a fixed hypersurface of M in a fixed set of disjoint disks. The group Γ_i acts by permuting these disks, which has a rather nontrivial effect when viewing the chart in a local slice, in which the first three marked points are fixed. However, the transition charts $\boldsymbol{K}_J$ are constructed very similarly: Elements of the domain U_J consist of stable maps $(f, \boldsymbol{z})$ together with $|J|$ sets of added tuples of marked points $(\vec{w}_j)_{j\in J}$, each lying in an appropriate set of disks and mapping to certain hypersurfaces. Each factor Γ_j of the group Γ_J acts by permuting the components of the j^{th} set of disks, leaving the others alone. Moreover, the covering map $\widetilde{U}_{IJ} \to U_I$ simply forgets the extra tuples $(\vec{w}_j)_{j\in J\setminus I}$. Thus it is immediate from the construction that the group $\Gamma_{J\setminus I}$ acts freely on the subset $\widetilde{U}_{IJ}$ of U_J, and that the covering map is equivariant in the appropriate sense. Further, when $I \subset J \subset K$ the compatibility condition $\rho_{IK} = \rho_{IJ} \circ \rho_{JK}$ holds whenever both sides are defined.

Furthermore, the stabilization process explained in [10] (see also [13, Remark 6.1.6]) allows us to directly work with products of obstruction spaces $E_I := \prod_{i\in I} E_i$; there is no need for a transversality requirement such as Sum Condition II$'$ in [13, Section 4.3]. In fact, already each E_i is a product of the form $E_i = \prod_{\gamma\in\Gamma_i}(E_{0i})_\gamma$, on which Γ_i acts by permutation of the $|\Gamma_i|$ copies of a vector space E_{0i}. Therefore, just as in the case with no isotropy, once given the geometric choices that determine the basic charts, we naturally obtain an additive weak Kuranishi atlas in which the only new choices are those of the domains $\underline{U}_I = \underline{U}_{II}$ and $\underline{U}_{IJ}$ of the transition charts and coordinate changes which are required to intersect the zero set $\underline{\mathfrak{s}}_I^{-1}(0)$ in $\underline{\psi}_I^{-1}(F_J)$. Note that there is no simple hierarchy by which one could organize these choices to automatically fulfill the cocycle condition. Hence concrete constructions will usually only satisfy a weak cocycle condition. However, we show below that any weak (automatically additive) atlas can be "tamed" so that it satisfies the strong cocycle condition, and hence in particular gives a Kuranishi atlas. $\diamond$

Given a (weak) atlas $\mathcal{K} = (\boldsymbol{K}_I, \widehat{\Phi}_{IJ})_{I,J\in\mathcal{I}_\mathcal{K},\, I\subsetneq J}$, we define the associated *intermediate atlas* $\underline{\mathcal{K}} := (\underline{\boldsymbol{K}}_I, \underline{\widehat{\Phi}}_{IJ})_{I,J\in\mathcal{I}_\mathcal{K},\, I\subsetneq J}$ to consist of the intermediate charts and coordinate changes. The next lemma shows that the intermediate atlas is a (weak) topological atlas in the sense of [13, Definition 3.1.1], and that it is *filtered* in the sense that there are closed sets $\underline{\mathbb{E}}_{IJ} \subset \underline{\mathbb{E}}_J := \underline{U_J \times E_J}$ for each $I \subset J$ that satisfy the following conditions (cf [13, Definition 3.1.3]):

(i) $\underline{\mathbb{E}}_{JJ} = \underline{\mathbb{E}}_J$ and $\underline{\mathbb{E}}_{\varnothing J} = \operatorname{im} 0_J$ for all $J \in \mathcal{I}_\mathcal{K}$;

(ii) $\underline{\widehat{\Phi}}_{JK}(\mathrm{pr}_J^{-1}(\underline{U}_{JK}) \cap \underline{\mathbb{E}}_{IJ}) = \underline{\mathbb{E}}_{IK} \cap \mathrm{pr}_K^{-1}(\operatorname{im} \underline{\phi}_{JK})$ for all $I, J, K \in \mathcal{I}_\mathcal{K}$ with $I \subset J \subsetneq K$;

(iii) $\underline{\mathbb{E}}_{IJ} \cap \underline{\mathbb{E}}_{HJ} = \underline{\mathbb{E}}_{(I\cap H)J}$ for all $I, H, J \in \mathcal{I}_\mathcal{K}$ with $I, H \subset J$;

(iv) $\operatorname{im} \underline{\phi}_{IJ}$ is an open subset of $\mathfrak{s}_J^{-1}(\underline{\mathbb{E}}_{IJ})$ for all $I, J \in \mathcal{I}_\mathcal{K}$ with $I \subsetneq J$.

Lemma 2.3.4 *Let $\mathcal{K}$ be a weak Kuranishi atlas. Then the intermediate atlas $\underline{\mathcal{K}}$ is a filtered weak topological Kuranishi atlas, with filtration $\underline{\mathbb{E}}_{IJ} := \underline{U_J \times \widehat{\phi}_{IJ}(E_I)}$, using the conventions $E_\varnothing := \{0\}$ and $\widehat{\phi}_{JJ} := \mathrm{id}_{E_J}$.*

Proof Lemma 2.2.4 and Remark 2.2.9(iii) assert that $\underline{\mathcal{K}}$ consists of topological Kuranishi charts and coordinate changes. The intermediate basic charts cover X since they have the same footprints as the basic charts of $\mathcal{K}$, and this also implies that the intermediate transition charts have the prescribed footprints. Moreover, the weak cocycle condition for $\mathcal{K}$ transfers to $\underline{\mathcal{K}}$ by Lemma 2.2.13(i), and the same holds for the cocycle condition since its definition (2.2.8) is in terms of the intermediate domains.

Next, to see that $\underline{\mathbb{E}}_{IJ}$ defines a filtration on $\underline{\mathcal{K}}$, we need a mild generalization of [14, Lemma 6.3.1]. First note that $\underline{U_J{\times}\widehat{\phi}_{IJ}(E_I)} \subset \underline{U_J{\times}E_J}$ is closed since $U_J{\times}\widehat{\phi}_{IJ}(E_I) \subset U_J{\times}E_J$ is closed and the projection $U_J \times E_J \to \underline{U_J \times E_J}$ is a closed map by Lemma 2.1.5(i). The filtration property (i) above holds by definition, and property (iii) holds because additivity implies

$$\widehat{\phi}_{IJ}(E_I) \,\cap\, \widehat{\phi}_{HJ}(E_H) \;=\; \widehat{\phi}_{(I\cap H)J}(E_{I\cap H}).$$

Moreover, because $\underline{\widehat{\Phi}}_{JK} = \underline{\phi_{JK} \times \widehat{\phi}_{JK}}$, property (ii) follows by quotienting the next identity by the group Γ_K,

$$\begin{aligned}\widehat{\Phi}_{JK}(U_{JK} \times \widehat{\phi}_{IJ}(E_I)) &= \operatorname{im}\phi_{JK} \times \widehat{\phi}_{JK}(\widehat{\phi}_{IJ}(E_I))\\ &= \operatorname{im}\phi_{JK} \times \widehat{\phi}_{IK}(E_I)\\ &= (U_K \times \widehat{\phi}_{IK}(E_I)) \cap (\operatorname{im}\phi_{JK} \times E_K).\end{aligned}$$

Finally, to check property (iv) we first apply [14, Lemma 5.2.5] to the embedding $\widetilde{\phi}_{IJ}\colon \widetilde{U}_{IJ} \to U_J$, which satisfies the index condition, ie identifies kernel and cokernel of $\mathrm{d}s_J$ and $\mathrm{d}s_I$ (the latter being pulled back with the covering ρ_{IJ}). It implies that $\operatorname{im}\widetilde{\phi}_{IJ}$ is an open subset of $s_J^{-1}(E_I)$. This openness is preserved in the Γ_J quotient, since Lemma 2.1.5 applies to the projection

$$s_J^{-1}(E_I) \to s_J^{-1}(E_I)/\Gamma_J = \mathfrak{s}_J^{-1}(\underline{U_J \times E_I}) = \mathfrak{s}_J^{-1}(\underline{\mathbb{E}}_{IJ}),$$

which maps $\operatorname{im}\phi_{IJ}$ to $\operatorname{im}\underline{\phi}_{IJ}$. □

If $\mathcal{K}$ is a Kuranishi atlas, then the topological atlas $\underline{\mathcal{K}}$ also satisfies the cocycle conditions, and hence by [13, Lemma 2.3.7] there is an *intermediate domain category* $\boldsymbol{B}_{\underline{\mathcal{K}}}$ with objects $\mathrm{Obj}_{\boldsymbol{B}_{\underline{\mathcal{K}}}} := \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} \underline{U}_I$ equal to the disjoint union of the intermediate domains, and morphisms

$$\mathrm{Mor}_{\boldsymbol{B}_{\underline{\mathcal{K}}}} := \bigsqcup_{I\subset J} \underline{U}_{IJ}$$

given by the intermediate coordinate changes $\underline{\phi}_{IJ}\colon \underline{U}_{IJ} \to \underline{U}_J$, where the identity maps $\underline{\phi}_{II}$ on $\underline{U}_{II} = \underline{U}_I$ are included. Thus the source and target maps are

$$s \times t\colon\ \underline{U}_{IJ} \to \underline{U}_I \times \underline{U}_J \subset \mathrm{Obj}_{\boldsymbol{B}_{\underline{\mathcal{K}}}} \times \mathrm{Obj}_{\boldsymbol{B}_{\underline{\mathcal{K}}}}, \quad (I,x) \mapsto ((I,x),(J,\underline{\phi}_{IJ}(x))).$$

The following gives the analogous categorical interpretation for the Kuranishi atlas itself.

Definition 2.3.5 Given a Kuranishi atlas $\mathcal{K}$ we define its *domain category* $\boldsymbol{B}_{\mathcal{K}}$ to consist of the space of objects

$$\mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}} := \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} U_I = \{(I,x) \mid I \in \mathcal{I}_{\mathcal{K}}, x \in U_I\}$$

and the space of morphisms

$$\mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}} := \bigsqcup_{I,J \in \mathcal{I}_{\mathcal{K}},\, I \subset J} \widetilde{U}_{IJ} \times \Gamma_I = \{(I,J,y,\gamma) \mid I \subset J, y \in \widetilde{U}_{IJ}, \gamma \in \Gamma_I\}.$$

Here we denote $\widetilde{U}_{II} := U_I$ for $I = J$, and for $I \subsetneq J$ use the lifted domain $\widetilde{U}_{IJ} \subset U_J$ of the restriction $\boldsymbol{K}_I|_{\underline{U}_{IJ}}$ to F_J that is part of the coordinate change $\widehat{\Phi}_{IJ}\colon \boldsymbol{K}_I|_{\underline{U}_{IJ}} \to \boldsymbol{K}_J$. Source and target of these morphisms are given by

$$(I,J,y,\gamma) \in \mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}}\big((I,\gamma^{-1}\rho_{IJ}(y)),(J,\widetilde{\phi}_{IJ}(y))\big), \tag{2.3.2}$$

where we denote $\widetilde{\phi}_{II} = \mathrm{id}$. Composition[4] is defined by

$$(I,J,y,\gamma) \circ (J,K,z,\delta) := (I,K,z{=}\widetilde{\phi}_{IK}^{-1}(\widetilde{\phi}_{JK}(z)), \rho^{\Gamma}_{IJ}(\delta)\gamma)$$

whenever $\delta^{-1}\rho_{JK}(z) = \widetilde{\phi}_{IJ}(y)$.

The *obstruction category* $\boldsymbol{E}_{\mathcal{K}}$ is defined in complete analogy to $\boldsymbol{B}_{\mathcal{K}}$ to consist of the spaces of objects $\mathrm{Obj}_{\boldsymbol{E}_{\mathcal{K}}} := \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} U_I \times E_I$ and morphisms

$$\mathrm{Mor}_{\boldsymbol{E}_{\mathcal{K}}} := \bigsqcup_{I \subset J,\, I,J \in \mathcal{I}_{\mathcal{K}}} \widetilde{U}_{IJ} \times E_I \times \Gamma_I,$$

with source and target maps

$$(I,J,y,e,\gamma) \mapsto (I,\gamma^{-1}\rho_{IJ}(y),\gamma^{-1}e), \quad (I,J,y,e,\gamma) \mapsto (J,\widetilde{\phi}_{IJ}(y),\widehat{\phi}_{IJ}(e)),$$

[4] Note that we write compositions in the categorical ordering here. Recall that $\widetilde{\phi}_{JK}\colon \widetilde{U}_{JK} \to U_K$ is the canonical inclusion of the subset $\widetilde{U}_{JK} \subset U_K$. We then identify $z = \widetilde{\phi}_{IK}^{-1}(\widetilde{\phi}_{JK}(z))$, since composability of the morphisms implies $z \in \rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK})$ and the cocycle condition ensures that $\rho_{JK}^{-1}(\widetilde{U}_{IJ} \cap U_{JK})$ is contained in $\widetilde{U}_{IK}$, where both are considered as subsets of U_K.

and composition defined by

$$(I,J,y,e,\gamma)\circ(J,K,z,f,\delta) := \big(I,K,\widetilde{\phi}_{IK}^{-1}(\widetilde{\phi}_{JK}(z)), f, \rho^{\Gamma}_{IJ}(\delta)\gamma\big)$$

for any $I\subset J\subset K$ and $(y,e,\gamma)\in \widetilde{U}_{IJ}\times E_I\times\Gamma_I$, $(z,f,\delta)\in\widetilde{U}_{JK}\times E_J\times\Gamma_J$ such that $\rho^{\Gamma}_{IJ}(\delta^{-1})\rho_{JK}(z)=\widetilde{\phi}_{IJ}(y)$ and $\delta^{-1}f=e$.

Lemma 2.3.6 *If $\mathcal{K}$ is a Kuranishi atlas, then the categories $\boldsymbol{B}_{\mathcal{K}}$, $\boldsymbol{E}_{\mathcal{K}}$ are well defined.*

Proof We must check that the composition of morphisms in $\boldsymbol{B}_{\mathcal{K}}$ is well defined, has identities, and is associative; the proof for $\boldsymbol{E}_{\mathcal{K}}$ is analogous. We begin by checking that $z=\widetilde{\phi}_{IK}^{-1}(\widetilde{\phi}_{JK}(z))$ lies in the lifted domain $\widetilde{U}_{IK}$ of $\widehat{\Phi}_{IK}$. For that purpose we drop the natural inclusions $\widetilde{\phi}_{**}$ from the notation and note that the composition $(I,J,y,\gamma)\circ(J,K,z,\delta)$ is defined only when the target of (I,J,y,γ) equals the source of (J,K,z,δ); ie when $y=\delta^{-1}\rho_{JK}(z)$. So the cocycle condition in Lemma 2.2.13(ii) implies that $z\in\rho_{JK}^{-1}(\delta y)$ is contained in $\rho_{JK}^{-1}(\widetilde{U}_{IJ}\cap U_{JK})\subset\widetilde{U}_{IK}$, as claimed. This means that $(I,K,z,\rho^{\Gamma}_{IJ}(\delta)\gamma)$ is a well-defined morphism of $\boldsymbol{B}_{\mathcal{K}}$. Its source is

$$(\rho^{\Gamma}_{IJ}(\delta)\gamma)^{-1}\rho_{IK}(z)=\gamma^{-1}\rho^{\Gamma}_{IJ}(\delta)^{-1}\rho_{IJ}(\delta y)=\gamma^{-1}\rho_{IJ}(y),$$

which coincides with the source of (I,J,y,γ) as required. Finally, the target of the composed morphism, $z=\widetilde{\phi}_{IK}(\widetilde{\phi}_{IK}^{-1}(\widetilde{\phi}_{JK}(z))$ coincides with the target $\widetilde{\phi}_{JK}(z)$ of (J,K,z,δ). This shows that composition is well defined. The identity morphisms are given by (I,I,x,id) for all $x\in U_{II}:=U_I$. To check associativity we consider $I\subset J\subset K\subset L$ and suppose that the three morphisms (I,J,y,γ), (J,K,z,δ), (K,L,w,σ) are composable. Then we have

$$\begin{aligned}(I,J,y,\gamma)\circ\big((J,K,z,\delta)\circ(K,L,w,\sigma)\big)&=(I,J,y,\gamma)\circ(J,L,w,\rho^{\Gamma}_{JK}(\sigma)\delta)\\&=(I,L,w,\rho^{\Gamma}_{IJ}(\rho^{\Gamma}_{JK}(\sigma)\delta)\gamma),\end{aligned}$$

and associativity follows from comparing this expression with

$$\begin{aligned}\big((I,J,y,\gamma)\circ(J,K,z,\delta)\big)\circ(K,L,w,\sigma)&=(I,K,z,\rho^{\Gamma}_{IJ}(\delta)\gamma)\circ(K,L,w,\sigma)\\&=(I,L,w,\rho^{\Gamma}_{IK}(\sigma)\rho^{\Gamma}_{IJ}(\delta)\gamma).\end{aligned}$$

This completes the proof. □

For the rest of this subsection we will make the standing assumption that $\mathcal{K}$ is a Kuranishi atlas, ie satisfies the cocycle condition (not just the weak cocycle condition). Given the categorical interpretation of domains and obstruction spaces of Kuranishi charts, we can now express the bundles, sections and footprint maps as functors:

- The obstruction category $\boldsymbol{E}_{\mathcal{K}}$ is a bundle over $\boldsymbol{B}_{\mathcal{K}}$ in the sense that there is a functor $\mathrm{pr}_{\mathcal{K}}\colon \boldsymbol{E}_{\mathcal{K}} \to \boldsymbol{B}_{\mathcal{K}}$ that is given on objects and morphisms by projection $(I,x,e) \mapsto (I,x)$ and $(I,J,y,e,\gamma) \mapsto (I,J,y,\gamma)$.
- The sections s_I induce a smooth section of this bundle, ie a functor $\mathfrak{s}_{\mathcal{K}}\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}}$ which acts smoothly on the spaces of objects and morphisms, and whose composite with the projection $\mathrm{pr}_{\mathcal{K}}\colon \boldsymbol{E}_{\mathcal{K}} \to \boldsymbol{B}_{\mathcal{K}}$ is the identity. More precisely, $\mathfrak{s}_{\mathcal{K}}$ is given by $(I,x) \mapsto (I,x,s_I(x))$ on objects and by $(I,J,y,\gamma) \mapsto (I,J,y,s_I(y),\gamma)$ on morphisms.
- The zero sections also fit together to give a functor $0_{\mathcal{K}}\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}}$ given by $(I,x) \mapsto (I,x,0)$ on objects and by $(I,J,y,\gamma) \mapsto (I,J,y,0,\gamma)$ on morphisms.
- The footprint maps ψ_I induce a surjective functor
$$\psi_{\mathcal{K}}\colon \mathfrak{s}_{\mathcal{K}}^{-1}(0) := \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} s_I^{-1}(0) \to \boldsymbol{X}$$
to the category $\boldsymbol{X}$ with object space X and trivial morphism spaces. It is given by $(I,x) \mapsto \psi_I(x)$ on objects and by $(I,J,y,\gamma) \mapsto \mathrm{id}_{\psi_J(\tilde{\phi}_{IJ}(y))} = \mathrm{id}_{\psi_I(\gamma^{-1}\rho_{IJ}(y))}$ on morphisms.

As in [13] we denote by $|\mathcal{K}|$ (resp. $|\underline{\mathcal{K}}|$) the *realization* of the category $\boldsymbol{B}_{\mathcal{K}}$ (resp. $\boldsymbol{B}_{\underline{\mathcal{K}}}$). This is the topological space obtained as the quotient of the object space by the equivalence relation generated by the morphisms. The next lemma fits the quotient maps $\pi_{\mathcal{K}}\colon \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}} \to |\mathcal{K}|$, $(I,x) \mapsto [I,x]$ and $\pi_{\underline{\mathcal{K}}}\colon \mathrm{Obj}_{\boldsymbol{B}_{\underline{\mathcal{K}}}} \to |\underline{\mathcal{K}}|$, $(I,\underline{x}) \mapsto [I,\underline{x}]$ into a commutative diagram that will allow us to identify the realizations $|\mathcal{K}| \cong |\underline{\mathcal{K}}|$ as topological spaces.

Lemma 2.3.7 *If $\mathcal{K}$ is a Kuranishi atlas, then there is a functor $\rho_{\mathcal{K}}\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{B}_{\underline{\mathcal{K}}}$ that is given on objects by the quotient maps $U_I \to \underline{U}_I$, $x \mapsto \underline{x}$, and on morphisms by the group coverings ρ_{IJ} together with a quotient,*
$$\widetilde{U}_{IJ} \times \Gamma_I \to \underline{U}_{IJ}, \quad (I,J,y,\gamma) \mapsto (I,J,\underline{\rho_{IJ}(y)}).$$
It induces a homeomorphism $|\rho_{\mathcal{K}}|\colon |\mathcal{K}| \to |\underline{\mathcal{K}}|$ between the realizations that fits into a commutative diagram:
$$\begin{tikzcd} \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}} \arrow[r, "\rho_{\mathcal{K}}"] \arrow[d, "\pi_{\mathcal{K}}"] & \mathrm{Obj}_{\boldsymbol{B}_{\underline{\mathcal{K}}}} \arrow[d, "\pi_{\underline{\mathcal{K}}}"] \\ |\mathcal{K}| \arrow[r, "|\rho_{\mathcal{K}}|"] & |\underline{\mathcal{K}}| \end{tikzcd}$$

Proof To see that $\rho_{\mathcal{K}}$ is a functor, recall that $(y,\gamma) \in \widetilde{U}_{IJ} \times \Gamma_I$ represents a morphism from $\gamma^{-1}\rho_{IJ}(y)$ to $y \in U_J$. On the other hand, $\underline{\rho_{IJ}(y)} = \underline{\rho}_{IJ}(\underline{y}) \in \underline{U}_{IJ}$ represents a morphism from $\underline{\rho_{IJ}(y)} = \underline{\gamma^{-1}\rho_{IJ}(y)}$ to $\underline{\phi}_{IJ}(\underline{\rho_{IJ}(y)}) = \underline{y}$, which shows compatibility of $\rho_{\mathcal{K}}$ with source and target maps. Compatibility with composition as in (2.3.2) follows from $\underline{\rho}_{IK}(\underline{z}) = \underline{\rho}_{IJ}(\underline{y})$ when $\underline{y} = \underline{\rho}_{JK}(\underline{z})$.

Next, any functor such as $\rho_{\mathcal{K}}$ induces a map $|\rho_{\mathcal{K}}|$ between the realizations that is defined exactly by the above commutative diagram. The map $|\rho_{\mathcal{K}}|$ is surjective because the functor $\rho_{\mathcal{K}}$ is surjective on the level of objects. It is injective because $\rho_{\mathcal{K}}$ is surjective on the level of morphisms.

To check that $|\rho_{\mathcal{K}}|$ is open and continuous note that $|\rho_{\mathcal{K}}|(U) = V$ is equivalent to $\rho_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}^{-1}(U)) = \pi_{\underline{\mathcal{K}}}^{-1}(V)$. Since $\rho_{\mathcal{K}}$ is continuous and open by Lemma 2.1.5(i), and $|\mathcal{K}|$, $|\underline{\mathcal{K}}|$ are equipped with the quotient topologies, the openness of $U \subset |\mathcal{K}|$, $\pi_{\mathcal{K}}^{-1}(U)$, $\pi_{\underline{\mathcal{K}}}^{-1}(V)$ and $V \subset |\underline{\mathcal{K}}|$ are all equivalent. This proves that $|\rho_{\mathcal{K}}|$ is a homeomorphism. □

Remark 2.3.8 (i) If $\mathcal{K}$ is a Kuranishi atlas with trivial isotropy groups $\Gamma_I = \{\mathrm{id}\}$, then the intermediate atlas $\underline{\mathcal{K}}$ has the exact same object space and naturally diffeomorphic morphism spaces, only the direction of the maps in the coordinate changes are reversed from $\rho_{IJ}\colon \widetilde{U}_{IJ} \to U_{IJ} \subset U_I$ to $\underline{\phi}_{IJ} = \rho_{IJ}^{-1}\colon U_{IJ} \to \widetilde{U}_{IJ} \subset U_J$. In this special case, $\underline{\mathcal{K}}$ is a Kuranishi atlas in the sense of [14], and Lemma 2.3.7 identifies the atlases and their realizations.

(ii) In general, the spaces of objects and morphisms of the intermediate category are orbifolds, and there is at most one morphism between any pair of objects. However, just as in the case of trivial isotropy, we do not attempt to make this category into a groupoid by formally inverting the morphisms and then adding all resulting composites, since doing so would in general give components of the morphism space without orbifold structure; cf [14, Remark 6.1.8]. This objection does not apply if all the obstruction spaces are trivial. It is shown in [10; 11] that every such atlas can be completed to a groupoid without changing its realization. ◇

In complete analogy to Lemma 2.3.7, the obstruction categories $\boldsymbol{E}_{\mathcal{K}}$ and $\boldsymbol{E}_{\underline{\mathcal{K}}}$ of the Kuranishi atlas $\mathcal{K}$ and the intermediate atlas $\underline{\mathcal{K}}$ also fit into a commutative diagram that identifies their realizations $|\boldsymbol{E}_{\mathcal{K}}| \cong |\boldsymbol{E}_{\underline{\mathcal{K}}}|$. Moreover, these two diagrams also intertwine the section functors $\mathfrak{s}_{\mathcal{K}}$, $\mathfrak{s}_{\underline{\mathcal{K}}}$ and their realizations:

$$
\begin{tikzcd}
 & \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}} \arrow[l, "\rho_{\mathcal{K}}"'] \arrow[r, "\mathfrak{s}_{\mathcal{K}}"] \arrow[d, "\pi_{\mathcal{K}}"] & \mathrm{Obj}_{\boldsymbol{E}_{\mathcal{K}}} \arrow[r] \arrow[d, "\pi_{\boldsymbol{E}_{\mathcal{K}}}"] & \mathrm{Obj}_{\boldsymbol{E}_{\underline{\mathcal{K}}}} \arrow[d, "\pi_{\boldsymbol{E}_{\underline{\mathcal{K}}}}"] & \mathrm{Obj}_{\boldsymbol{B}_{\underline{\mathcal{K}}}} \arrow[l, "\mathfrak{s}_{\underline{\mathcal{K}}}"'] \arrow[d, "\pi_{\underline{\mathcal{K}}}"] & \arrow[l, "\rho_{\mathcal{K}}"'] \\
 & |\mathcal{K}| \arrow[l, "|\rho_{\mathcal{K}}|"'] \arrow[r, "|\mathfrak{s}_{\mathcal{K}}|"] & |\boldsymbol{E}_{\mathcal{K}}| \arrow[r] & |\boldsymbol{E}_{\underline{\mathcal{K}}}| & |\underline{\mathcal{K}}| \arrow[l, "|\mathfrak{s}_{\underline{\mathcal{K}}}|"'] & \arrow[l, "|\rho_{\mathcal{K}}|"']
\end{tikzcd}
\tag{2.3.3}
$$

There are analogous diagrams for the projection functors $\mathrm{pr}_{\mathcal{K}}$, $\mathrm{pr}_{\underline{\mathcal{K}}}$ and zero sections $0_{\mathcal{K}}$ and $0_{\underline{\mathcal{K}}}$, which identify the induced maps between the realizations as stated below.

Lemma 2.3.9 *Let $\mathcal{K}$ be a Kuranishi atlas.*

(i) *The functors $\mathrm{pr}_{\mathcal{K}}\colon \boldsymbol{E}_{\mathcal{K}} \to \boldsymbol{B}_{\mathcal{K}}$, $\mathrm{pr}_{\underline{\mathcal{K}}}\colon \underline{\boldsymbol{E}}_{\mathcal{K}} \to \underline{\boldsymbol{B}}_{\mathcal{K}}$ induce the same continuous map*

$$|\mathrm{pr}_{\mathcal{K}}|\colon |\boldsymbol{E}_{\mathcal{K}}| \to |\mathcal{K}|,$$

*which we call the **obstruction bundle** of $\mathcal{K}$, although its fibers generally do not have the structure of a vector space.*

(ii) *The zero sections $0_{\mathcal{K}}\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}}$, $0_{\underline{\mathcal{K}}}\colon \boldsymbol{B}_{\underline{\mathcal{K}}} \to \boldsymbol{E}_{\underline{\mathcal{K}}}$ as well as the section functors $\mathfrak{s}_{\mathcal{K}}\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}}$, $\mathfrak{s}_{\underline{\mathcal{K}}}\colon \boldsymbol{B}_{\underline{\mathcal{K}}} \to \boldsymbol{E}_{\underline{\mathcal{K}}}$ induce the same continuous maps*

$$|0_{\mathcal{K}}| \cong |0_{\underline{\mathcal{K}}}|\colon |\mathcal{K}| \to |\boldsymbol{E}_{\mathcal{K}}|, \quad |\mathfrak{s}_{\mathcal{K}}| \cong |\mathfrak{s}_{\underline{\mathcal{K}}}|\colon |\mathcal{K}| \to |\boldsymbol{E}_{\mathcal{K}}|,$$

which are sections in the sense that $|\mathrm{pr}_{\mathcal{K}}| \circ |0_{\mathcal{K}}| = \mathrm{id}_{|\mathcal{K}|} = |\mathrm{pr}_{\mathcal{K}}| \circ |\mathfrak{s}_{\mathcal{K}}|$.

(iii) *There is a natural homeomorphism from the realization of the subcategory $\mathfrak{s}_{\mathcal{K}}^{-1}(0)$ to the zero set of $|\mathfrak{s}_{\mathcal{K}}|$, with the relative topology induced from $|\mathcal{K}|$,*

$$|\mathfrak{s}_{\mathcal{K}}^{-1}(0)| = \mathfrak{s}_{\mathcal{K}}^{-1}(0)/\sim \xrightarrow{\cong} |\mathfrak{s}_{\mathcal{K}}|^{-1}(|0_{\mathcal{K}}|) := \bigl\{[I,x] \bigm| |\mathfrak{s}_{\mathcal{K}}|([I,x]) = |0_{\mathcal{K}}|([I,x])\bigr\} \subset |\mathcal{K}|.$$

Proof The induced maps on the realizations are identified by commutative diagrams such as (2.3.3). The continuity and other identities are proven exactly as in [14, Lemma 6.1.10] for the case of trivial isotropy. □

Next, we extend the notion of metrizability to Kuranishi atlases with nontrivial isotropy. In the case of trivial isotropy, recall from [14, Definition 6.1.14] that an admissible metric is a bounded metric d on the set $|\mathcal{K}|$ such that for each $I \in \mathcal{I}_{\mathcal{K}}$ the pullback metric $d_I := (\pi_{\mathcal{K}}|_{U_I})^* d$ on U_I induces the given topology on the manifold U_I. However, in the presence of isotropy, it makes no sense to try to pull this metric back to U_I since the pullback of a metric by a noninjective map is no longer a metric. Instead, we use the fact that the realizations $|\mathcal{K}| \cong |\underline{\mathcal{K}}|$ of the Kuranishi atlas and its intermediate atlas are canonically identified, which allows us to work with admissible metrics on $|\underline{\mathcal{K}}|$, which is the realization of a topological Kuranishi atlas $\underline{\mathcal{K}}$ with trivial isotropy and given metrizable topologies on the domains $\underline{U}_I = U_I/\Gamma_I$.

Definition 2.3.10 Let $\mathcal{K}$ be a Kuranishi atlas. Then an *admissible metric* on $|\mathcal{K}| \cong |\underline{\mathcal{K}}|$ is a bounded metric on this set (not necessarily compatible with the topology of the realization) such that for each $I \in \mathcal{I}_{\mathcal{K}}$ the pullback metric $\underline{d}_I := (\pi_{\underline{\mathcal{K}}}|_{\underline{U}_I})^* d$ on $\underline{U}_I$ induces the given quotient topology on $\underline{U}_I = U_I/\Gamma_I$.

A *metric Kuranishi atlas* is a pair $(\mathcal{K}, d)$ consisting of a Kuranishi atlas $\mathcal{K}$ together with a choice of admissible metric d on $|\mathcal{K}|$.

We finish this subsection with two comparisons of our notion of Kuranishi atlas: on the one hand with orbifolds, and on the other hand with Kuranishi structures.

Example 2.3.11 If the obstruction spaces are trivial, ie $E_I = \{0\}$ for all I, then the two categories $\boldsymbol{B}_{\mathcal{K}}$, $\boldsymbol{E}_{\mathcal{K}}$ are equal, and their realization is an orbifold. A first nontrivial example is a "football" $X = S^2$ with two basic Kuranishi charts

$$(U_1, \Gamma_1 = \mathbb{Z}_2, \psi_1), \quad (U_2, \Gamma_2 = \mathbb{Z}_3, \psi_2),$$

covering neighborhoods $\underline{\psi}_i(\underline{U}_i) \subset S^2$ of the northern (resp. southern) hemisphere with isotropy of order 2 (resp. 3) at the north (resp. south) pole. We may moreover assume that the overlap $\underline{\psi}_1(\underline{U}_1) \cap \underline{\psi}_2(\underline{U}_2) = \underline{A}$ is an annulus around the equator. The restrictions of the basic charts to $\underline{A} \subset X$ are $(A_1, \mathbb{Z}_2)$ and $(A_2, \mathbb{Z}_3)$, where both $A_i = \psi_i^{-1}(\underline{A})$ are annuli, but the freely acting isotropy groups are different. There is no functor between these restrictions because the coverings $A_1 \to \underline{A}$ and $A_2 \to \underline{A}$ are incompatible. However, they both have functors (ie coordinate changes) to a common free covering, namely the pullback defined by the diagram

$$\begin{tikzcd} U_{12} \arrow[r] \arrow[d] & A_1 \arrow[d, "\pi_1"] \\ A_2 \arrow[r, "\pi_2"] & \underline{A} \subset X \end{tikzcd}$$

ie $U_{12} := \{(x, y) \in A_1 \times A_2 \mid \pi_1(x) = \pi_2(y)\}$ with group $\Gamma_{12} := \Gamma_1 \times \Gamma_2 = \mathbb{Z}_2 \times \mathbb{Z}_3$. The corresponding footprint map $\psi_{12} \colon U_{12} \to \underline{A}$ is the 6–fold covering of the annulus, and the coordinate changes from $(U_i, \Gamma_i, \psi_i)|_{\underline{A}}$ to $(U_{12}, \Gamma_{12}, \psi_{12})$ are the coverings $\widetilde{U}_{i,12} := U_{12} \to A_i =: U_{i,12}$ in the diagram. Therefore the category $\boldsymbol{B}_{\mathcal{K}}$ in this example has index set $\mathcal{I}_{\mathcal{K}} = \{1, 2, 12\}$, objects the disjoint union $\bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} U_I$, and morphisms

$$\Bigl(\bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} U_I \times \Gamma_I\Bigr) \cup \Bigl(\bigsqcup_{i=1,2} U_{12} \times \Gamma_i\Bigr),$$

where for $i = 1, 2$ the elements in $U_{12} \times \Gamma_i$ represent the morphisms from U_i to U_{12}.

This simple construction does not work for arbitrary orbifolds since the (set-theoretic) pullback U_{12} considered above will not be a smooth manifold if any point in $\psi_1(U_1) \cap \psi_2(U_2)$ has nontrivial stabilizer. However, we show in [11, Proposition 3.3] that the construction can be generalized to show that every orbifold has a Kuranishi atlas with trivial obstruction spaces. ◇

Remark 2.3.12 (relation to Kuranishi structures) A Kuranishi structure in the sense of [3, Appendix A] and [4] consists of a Kuranishi chart $\boldsymbol{K}_p$ at every point $p \in X$ and coordinate changes $\boldsymbol{K}_q|_{U_{qp}} \to \boldsymbol{K}_p$ whenever $q \in F_p$, that satisfy a suitable weak cocycle condition. Much as in the case of Kuranishi atlases with trivial isotropy (see [14, Remark 6.1.16]), a weak Kuranishi atlas in the sense of Definition 2.3.1 induces a Kuranishi structure. Indeed, given a covering family of basic charts $(\boldsymbol{K}_i)_{i=1,\dots,N}$ with footprints F_i, we may choose a family of compact subsets $C_i \subset F_i$ that also cover X. Then we use the transition data $(\boldsymbol{K}_I, \widehat{\Phi}_{IJ})$ and weak cocycle conditions to obtain a Kuranishi structure as follows:

- For any $p \in X$, we define $\boldsymbol{K}_p := \boldsymbol{K}_{I_p}|_{U_p}$ to be a restriction of $\boldsymbol{K}_{I_p}$, where $I_p := \{i \mid p \in C_i\}$ and $U_p \subset U_{I_p}$ is an open subset such that the footprint

$$F_p := \psi_{I_p}(s_{I_p}^{-1}(0) \cap U_p)$$

is a neighborhood of p and contained in $\bigcap_{i\in I_p} F_i \setminus \bigcup_{i\notin I_p} C_i$. Here we use a more general notion of restriction than Definition 2.2.6, in that we allow for a domain U_p that is invariant only under a subgroup $\Gamma_p \subset \Gamma_{I_p}$ such that the induced map $U_p/\Gamma_p \to U_{I_p}/\Gamma_{I_p}$ is a homeomorphism to its image. More precisely, to satisfy the minimality requirements of [3, Appendix A1.1], we choose a lift $x_p \in \pi^{-1}(p) \cap U_{I_p}$, set $\Gamma_p := \Gamma_{I_p}^{x_p}$ to be its stabilizer in Γ_{I_p}, and take the domain $U_p \subset U_{I_p}$ to be a $\Gamma_{I_p}^{x_p}$–invariant neighborhood of x_p, which exists with the required topological properties by Lemma 2.1.5(ii).

- For $q \in F_p$ we have $I_q \subset I_p$, since by construction $F_p \cap C_i = \varnothing$ for $i \notin I_p$. So we obtain a coordinate change[5] $\widehat{\Phi}_{qp}\colon \boldsymbol{K}_q \to \boldsymbol{K}_p$ from a suitable restriction of $\widehat{\Phi}_{I_q I_p}$ to a $\Gamma_q^{x_q}$–invariant neighborhood $U_{qp} \subset U_q$ of x_q. More precisely, we choose $U_{qp} \subset U_q$ small enough so that the projection $\rho_{I_q I_p}\colon U_p \cap \widetilde{U}_{I_q I_p} \to U_{I_q I_p}$ has a continuous section over U_{qp}. Writing $\widetilde{U}_{qp}$ for its image we thus obtain an embedding $\phi_{qp} := \rho_{I_q I_p}^{-1}\colon U_{qp} \to \widetilde{U}_{qp} \subset U_p \cap \widetilde{U}_{I_q I_p}$. Since the projection $\rho_{I_q I_p}$ induces an isomorphism on stabilizer subgroups by Lemma 2.1.5(iii), this is equivariant with respect to a suitable injective homomorphism $h_{qp}\colon \Gamma_q \to \Gamma_p$ and induces an injection

$$\underline{\phi}_{qp}\colon \underline{U}_{qp} := U_{qp}/\Gamma_q \to \underline{U}_p := U_p/\Gamma_p.$$

By construction of $\underline{U}_q \to \underline{U}_{I_q}$ above, the map $\underline{U}_{qp} = U_{qp}/\Gamma_q \to \underline{U}_{I_q} = U_{I_q}/\Gamma_{I_q}$ is a homeomorphism to its image, and similarly for p. Thus we can identify $\underline{\phi}_{qp}$ with a suitable restriction of the map $\underline{\phi}_{I_q I_p}$ underlying the coordinate change $\widehat{\Phi}_{I_q I_p}$ in the given Kuranishi atlas. The coordinate change $\widehat{\Phi}_{qp} = (U_{qp}, \underline{\phi}_{qp})$ is then given by the domain U_{qp} and the restriction of $\underline{\phi}_{I_q I_p}$ to $\underline{U}_{qp} \subset \underline{U}_q$.

Further, the weak cocycle condition for $\mathcal{K}$ implies the compatibility condition required by [3], namely for all triples $p, q, r \in X$ with $q \in F_p$ and

$$r \in \psi_q(U_{qp} \cap s_q^{-1}(0)) \subset F_q \cap F_p,$$

[5] While [3] denotes this coordinate change by ϕ_{pq}, we will write $\widehat{\Phi}_{qp}$ for consistency with our notation $\Phi_{IJ}\colon \boldsymbol{K}_I \to \boldsymbol{K}_J$.

the equality $\underline{\phi}_{qp} \circ \underline{\phi}_{rq} = \underline{\phi}_{rp}$ holds on the common domain $\underline{\phi}_{rq}^{-1}(\underline{U}_{qp}) \cap \underline{U}_{rp}$ of the maps in this equation.

- This atlas satisfies the effectivity condition required by [3] only if the action of Γ_p on U_p is locally effective in the sense that $s_p^{-1}(0)$ has a Γ_p–invariant open neighborhood that is disjoint from the interior of the fixed point set $\mathrm{Fix}(\gamma) \subset U_p$ for each $\gamma \in \Gamma_p \setminus \{\mathrm{id}\}$.

With this construction, we lose the distinction between basic charts and transition charts, and also in general can no longer recover the original transition charts with their group actions from the Kuranishi structure. Indeed, [4] works with a "good coordinate system" (an analog of our notion of reduction in Definition 3.2.1) that is defined on the orbifold level, ie on the level of the intermediate category. Notice also that the construction of a Kuranishi structure given for example in [4] essentially follows the above outline, and in particular starts with a finite covering family of basic charts and uses transition charts much like ours, though they are more localized and are not required to cover the full footprint F_I. However, the properties of these charts are never explicitly formulated. Indeed our work started by trying to understand precisely this point in their construction. Though it is not clear how relevant the extra information contained in a Kuranishi atlas is to the question of how to define Gromov–Witten invariants for closed curves, it might prove useful in other situations, for example in the case of orbifold Gromov–Witten invariants, or in the recent work of Fukaya et al [5], where the authors consider a process that rebuilds a Kuranishi structure from a coordinate system. Further, our categorical formulation makes it very easy to give an explicit description and construction for sections as in Definition 3.2.4. $\diamond$

2.4 Kuranishi cobordisms and concordance

This section extends the notions of cobordism and concordance developed in [13, Section 4] and [14, Section 6.2] to the case of smooth Kuranishi atlases with nontrivial isotropy. It is a straightforward generalization that can be skipped until precise concordance notions are needed in the proof of Theorem 2.5.3. We begin by summarizing the topological cobordism notions from [13, Section 4.1].

A *collared cobordism* $(Y, \iota_Y^0, \iota_Y^1)$ is a separable, locally compact, metrizable space Y together with disjoint (possibly empty) closed subsets $\partial^0 Y$, $\partial^1 Y \subset Y$ and equipped with *collared neighborhoods*

$$\iota_Y^0\colon [0,\varepsilon) \times \partial Y^0 \to Y, \quad \iota_Y^1\colon (1-\varepsilon, 1] \times \partial Y^1 \to Y,$$

for some $\varepsilon > 0$. The latter are homeomorphisms onto disjoint open neighborhoods of $\partial^\alpha Y \subset Y$, extending the inclusions $\iota_Y^\alpha(\alpha, \cdot)\colon \partial^\alpha Y \hookrightarrow Y$ for $\alpha = 0, 1$. We call $\partial^0 Y$ and $\partial^1 Y$ the *boundary components* of $(Y, \iota_Y^0, \iota_Y^1)$. The main example is the *trivial* *cobordism* $Y = [0,1] \times X$ with the natural inclusions $\iota^\alpha_Y \colon A^\alpha_\varepsilon \times X \to [0,1] \times X$, where we denote

$$A^0_\varepsilon := [0,\varepsilon) \quad \text{and} \quad A^1_\varepsilon := (1-\varepsilon, 1], \quad \text{for } 0 < \varepsilon < \tfrac{1}{2}.$$

Next, a subset $F \subset Y$ is *collared* if there is $0 < \delta \le \varepsilon$ such that for $\alpha = 0, 1$ we have

$$F \cap \operatorname{im}(\iota^\alpha_Y) \neq \varnothing \iff F \cap \iota^\alpha_Y(A^\alpha_\delta \times \partial^\alpha Y) = \iota^\alpha_Y(A^\alpha_\delta \times \partial^\alpha F), \tag{2.4.1}$$

where the intersections with the boundary components $\partial^\alpha F := F \cap \partial^\alpha Y$ may be empty.

In the notion of Kuranishi cobordism, we will require all charts and coordinate changes to be of product form in sufficiently small collars, as follows.

Definition 2.4.1 Let $(Y, \iota^0_Y, \iota^1_Y)$ be a compact collared cobordism.

- Given a Kuranishi chart $\boldsymbol{K}^\alpha = (U^\alpha, E^\alpha, \Gamma^\alpha, s^\alpha, \psi^\alpha)$ for $\partial^\alpha Y$ and an open subset $A \subset [0,1]$, the *product chart* for $[0,1] \times \partial^\alpha Y$ with footprint $A \times F^\alpha$ is

$$A \times \boldsymbol{K}^\alpha := (A \times U^\alpha, E^\alpha, \ \Gamma^\alpha, s^\alpha \circ \mathrm{pr}_{U^\alpha}, \mathrm{id}_A \times \psi^\alpha),$$

where Γ^α acts trivially on the first factor of $A \times U^\alpha$ and $\mathrm{pr}_{U^\alpha} \colon A \times U^\alpha \to U^\alpha$ is the evident projection.

- Given a coordinate change $\widehat{\Phi}^\alpha_{IJ} = (\widetilde{\phi}^\alpha_{IJ}, \widehat{\phi}^\alpha_{IJ}, \rho^\alpha_{IJ}) \colon \boldsymbol{K}^\alpha_I \to \boldsymbol{K}^\alpha_J$ between Kuranishi charts for $\partial^\alpha Y$ with lifted domain $\widetilde{U}^\alpha_{IJ}$, and open subsets $A_I, A_J \subset [0,1]$, the *product coordinate change* $(A_I \cap A_J) \times \boldsymbol{K}^\alpha_I \to A_J \times \boldsymbol{K}^\alpha_J$ is

$$\mathrm{id}_{A_I \cap A_J} \times \widehat{\Phi}^\alpha_{IJ} : (\mathrm{id}_{A_I \cap A_J} \times \widetilde{\phi}^\alpha_{IJ}, \widehat{\phi}_{IJ} := \widehat{\phi}^\alpha_{IJ}, \ \mathrm{id}_{A_I \cap A_J} \times \rho^\alpha_{IJ})$$

with the lifted domain $(A_I \cap A_J) \times \widetilde{U}^\alpha_{IJ}$.

- A *Kuranishi chart with collared boundary* for $(Y, \iota^0_Y, \iota^1_Y)$ is given by a tuple $\boldsymbol{K} = (U, E, \Gamma, s, \psi)$ as in Definition 2.2.2, with the following collar form requirements:

  (i) The footprint $F \subset Y$ is collared with at least one nonempty boundary $\partial^\alpha F$.

  (ii) The domain is a collared cobordism $(U, \iota^0_U, \iota^1_U)$ whose boundary components $\partial^\alpha U$ are nonempty if and only if $\partial^\alpha F \neq \varnothing$. It is smooth in the sense that U is a manifold with boundary $\partial U = \partial^0 U \sqcup \partial^1 U$ and the ι^α_U are tubular neighborhood diffeomorphisms.

  (iii) If $\partial^\alpha F \neq \varnothing$ then there is a *restriction of* $\boldsymbol{K}$ *to the boundary* $\partial^\alpha Y$; that is, a Kuranishi chart $\partial^\alpha \boldsymbol{K} = (\partial^\alpha U^\alpha, E, \Gamma, s^\alpha, \psi^\alpha)$ for $\partial^\alpha Y$, with the isotropy group Γ and obstruction space E of $\boldsymbol{K}$ and footprint $\partial^\alpha F$, and an embedding of the product chart $A^\alpha_\varepsilon \times \partial^\alpha \boldsymbol{K}$ into $\boldsymbol{K}$ for some $\varepsilon > 0$, in the sense that the boundary embedding ι^α_U is Γ–equivariant and the following diagrams commute:

$$\begin{tikzcd} A^\alpha_\varepsilon \times \partial^\alpha U \arrow[r, "\iota^\alpha_U"] \arrow[d, "s^\alpha \circ \mathrm{pr}_{\partial^\alpha U}"'] & U \arrow[d, "s"] \\ E \arrow[r, "\mathrm{id}_E"] & E \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{id}_{A^\alpha_\varepsilon} \times s^\alpha)^{-1}(0) \arrow[r, "\iota^\alpha_U"] \arrow[d, "\mathrm{id}_{A^\alpha_\varepsilon} \times \psi^\alpha"'] & s^{-1}(0) \arrow[d, "\psi"] \\ A^\alpha_\varepsilon \times \partial^\alpha Y \arrow[r, "\iota^\alpha_Y"] & Y \end{tikzcd}$$

- Let $\boldsymbol{K}_I, \boldsymbol{K}_J$ be Kuranishi charts for $(Y, \iota^0_Y, \iota^1_Y)$ such that only $\boldsymbol{K}_I$ or both $\boldsymbol{K}_I, \boldsymbol{K}_J$ have collared boundary. A *coordinate change with collared boundary* $\widehat{\Phi}_{IJ}\colon \boldsymbol{K}_I \to \boldsymbol{K}_J$ with domain $\underline{U}_{IJ}$ satisfies the conditions in Definition 2.2.8, with the following collar form requirements:

(i) The lifted domain $\widetilde{U}_{IJ} \subset U_J$, as well as $U_{IJ} \subset U_I$, are collared subsets.

(ii) If $F_J \cap \partial^\alpha Y \neq \varnothing$ then $F_I \cap \partial^\alpha Y \neq \varnothing$ and there is a *restriction of* $\widehat{\Phi}_{IJ}$ *to the boundary* $\partial^\alpha Y$; that is, a coordinate change $\partial^\alpha \widehat{\Phi}_{IJ}\colon \partial^\alpha \boldsymbol{K}_I \to \partial^\alpha \boldsymbol{K}_J$ such that the restriction of $\widehat{\Phi}_{IJ}$ to

$$\underline{U_{IJ} \cap \iota^\alpha_{U_I}(A^\alpha_\varepsilon \times \partial^\alpha U_I)}$$

pulls back via the collar inclusions $\iota^\alpha_{U_I}$, $\iota^\alpha_{U_J}$ to the product coordinate change $\mathrm{id}_{A^\alpha_\varepsilon} \times \partial^\alpha \widehat{\Phi}_{IJ}$ for some $\varepsilon > 0$. In particular we have

$$(\iota^\alpha_{U_J})^{-1}(\widetilde{U}_{IJ}) \cap (A^\alpha_\varepsilon \times \partial^\alpha U_J) = A^\alpha_\varepsilon \times \partial^\alpha \widetilde{U}_{IJ},$$
$$(\iota^\alpha_{U_I})^{-1}(U_{IJ}) \cap (A^\alpha_\varepsilon \times \partial^\alpha U_I) = A^\alpha_\varepsilon \times \partial^\alpha U_{IJ}.$$

(iii) If $F_J \cap \partial^\alpha Y = \varnothing$ but $F_I \cap \partial^\alpha Y \neq \varnothing$, then $U_{IJ} \subset U_I$ is collared with $\partial^\alpha U_{IJ} = \varnothing$. As a consequence we have $U_{IJ} \cap \iota^\alpha_{U_I}(A^\alpha_\varepsilon \times \partial^\alpha \widetilde{U}_I) = \varnothing$ for some $\varepsilon > 0$.

Definition 2.4.2 A *(weak) Kuranishi cobordism* on a compact collared cobordism $(Y, \iota^0_Y, \iota^1_Y)$ is a tuple $\mathcal{K} = (\boldsymbol{K}_I, \widehat{\Phi}_{IJ})_{I,J \in \mathcal{I}_{\mathcal{K}}}$ of basic charts and transition data as in Definition 2.3.1, with the following collar form requirements:

- The charts of $\mathcal{K}$ are either Kuranishi charts with collared boundary or standard Kuranishi charts whose footprints are precompactly contained in $Y \setminus (\partial^0 Y \cup \partial^1 Y)$.
- The coordinate changes $\widehat{\Phi}_{IJ}\colon \boldsymbol{K}_I \to \boldsymbol{K}_J$ are either standard coordinate changes on $Y \setminus (\partial^0 Y \cup \partial^1 Y)$ between pairs of standard charts, or coordinate changes with collared boundary between pairs of charts, of which at least the first has collared boundary.

We say that $\mathcal{K}$ has *uniform collar width* $\delta > 0$ if all domains and coordinate changes have the required collar form over intervals A^α_ε of length $\varepsilon > \delta$.

Remark 2.4.3 Let $\mathcal{K}$ be a (weak) Kuranishi cobordism on $(Y, \iota^0_Y, \iota^1_Y)$.

(i) $\mathcal{K}$ induces by restriction (weak) Kuranishi atlases $\partial^\alpha \mathcal{K}$ on the boundary components $\partial^\alpha Y$ for $\alpha = 0, 1$ with

- basic charts $\partial^\alpha \boldsymbol{K}_i$ given by restriction of basic charts of $\mathcal{K}$ with $F_i \cap \partial^\alpha Y \neq \varnothing$;
- index set $\mathcal{I}_{\partial^\alpha \mathcal{K}} = \{ I \in \mathcal{I}_{\mathcal{K}} \mid F_I \cap \partial^\alpha Y \neq \varnothing \}$;
- transition charts $\partial^\alpha \boldsymbol{K}_I$ given by restriction of transition charts of $\mathcal{K}$;
- coordinate changes $\partial^\alpha \widehat{\Phi}_{IJ}$ given by restriction of coordinate changes of $\mathcal{K}$.

(ii) The charts and coordinate changes of $\mathcal{K}$ induce intermediate charts and coordinate changes as in Definition 2.2.3 and Remark 2.2.9(iii). These fit together to form a filtered (weak) topological cobordism $\underline{\mathcal{K}}$ in the sense of [13, Definitions 4.1.12] by a direct generalization of Lemma 2.3.4. Its boundary restrictions are the intermediate Kuranishi atlases $\partial^\alpha \underline{\mathcal{K}} = \underline{\partial^\alpha \mathcal{K}}$ induced by the boundary restrictions $\partial^\alpha \mathcal{K}$.

(iii) As in [13, Remark 4.1.11] we can think of the virtual neighborhood $|\mathcal{K}|$ as a collared cobordism with boundary components $\partial^0 |\mathcal{K}| \cong |\partial^0 \mathcal{K}|$ and $\partial^1 |\mathcal{K}| \cong |\partial^1 \mathcal{K}|$, with the exception that $|\mathcal{K}|$ is usually not locally compact or metrizable. More precisely, if $\mathcal{K}$ has collar width $\varepsilon > 0$, then the inclusions $\iota^\alpha_{U_I}\colon A^\alpha_\varepsilon \times U^\alpha_I \hookrightarrow U_I$ induce topological embeddings

$$\iota^0_{|\mathcal{K}|}\colon [0, \varepsilon) \times |\partial^0 \mathcal{K}| \hookrightarrow |\mathcal{K}|, \quad \iota^1_{|\mathcal{K}|}\colon (1-\varepsilon, 1] \times |\partial^1 \mathcal{K}| \hookrightarrow |\mathcal{K}|$$

to open neighborhoods of the closed subsets

$$\partial^\alpha |\mathcal{K}| := \bigsqcup_{I \in \mathcal{I}_{\partial^\alpha \mathcal{K}}} \iota^\alpha_{U_I}(\{\alpha\} \times U^\alpha_I)/\sim \; \subset |\mathcal{K}|. \qquad \diamond$$

With this language in hand, one obtains cobordism relations between (weak) Kuranishi atlases in complete analogy with [13, Definition 4.1.8] and [14, Definition 6.2.10]. For the uniqueness results in this paper, the more important notion is the following. Here we use the notion of tameness, a refinement of the strong cocycle condition that is formalized in Definition 2.5.1 below.

Definition 2.4.4 Two (weak/tame) Kuranishi atlases $\mathcal{K}^0$, $\mathcal{K}^1$ on the same compact metrizable space X are said to be *(weakly/tamely) concordant* if there exists a (weak/tame) Kuranishi cobordism $\mathcal{K}$ on the trivial cobordism $Y = [0,1] \times X$ whose boundary restrictions are $\partial^0 \mathcal{K} = \mathcal{K}^0$ and $\partial^1 \mathcal{K} = \mathcal{K}^1$. More precisely, there are injections $\iota^\alpha\colon \mathcal{I}_{\mathcal{K}^\alpha} \hookrightarrow \mathcal{I}_{\mathcal{K}}$ for $\alpha = 0, 1$ such that $\operatorname{im} \iota^\alpha = \mathcal{I}_{\partial^\alpha \mathcal{K}}$ and we have

$$\boldsymbol{K}^\alpha_I = \partial^\alpha \boldsymbol{K}_{\iota^\alpha(I)}, \quad \widehat{\Phi}^\alpha_{IJ} = \partial^\alpha \widehat{\Phi}_{\iota^\alpha(I) \iota^\alpha(J)} \quad \forall I, J \in \mathcal{I}_{\mathcal{K}^\alpha}.$$

Moreover, two metric Kuranishi atlases $(\mathcal{K}^0, d_0)$, $(\mathcal{K}^1, d_1)$ are called *metric concordant* if they are concordant as above with $\mathcal{K}$ a Kuranishi cobordism whose realization $|\mathcal{K}| \cong |\underline{\mathcal{K}}|$ supports an admissible, ε–collared metric d in the sense of [13, Definition 4.2.1] for the intermediate cobordism atlas $\underline{\mathcal{K}}$ such that $d|_{\partial^\alpha|\mathcal{K}|} = d_\alpha$ for $\alpha = 0, 1$.

2.5 Tameness and shrinkings

As in the case of trivial isotropy, we must adjust the Kuranishi atlas in order for its realization $|\mathcal{K}|$ to have good topological properties; for example, so that it is Hausdorff and has "enough" compact subsets. We essentially already dealt with these problems in [13] by

- introducing notions of tameness and preshrunk shrinking for topological Kuranishi atlases, which ensure the desired topological properties of the realization;
- constructing tame shrinkings of filtered weak topological Kuranishi atlases;
- proving that tame shrinkings are unique up to tame concordance.

In order to apply these results to smooth Kuranishi atlases with nontrivial isotropy, recall first that we built additivity into the notion of Kuranishi atlas, and showed in Lemma 2.3.4 that the resulting intermediate atlases are naturally filtered by

$$(\mathbb{E}_{IJ} := \underline{U_J \times \widehat{\phi}_{IJ}(E_I)})_{I \subset J}.$$

The same holds for Kuranishi cobordisms by Remark 2.4.3(ii). We can thus extend the notions of tameness to the case of nontrivial isotropy by working at the level of the intermediate category.

Definition 2.5.1 A weak Kuranishi atlas or cobordism is *tame* if its intermediate atlas is tame in the sense of [13, Definition 3.1.10]; that is, for all $I, J, K \in \mathcal{I}_{\mathcal{K}}$ we have

$$\underline{U}_{IJ} \cap \underline{U}_{IK} = \underline{U}_{I(J \cup K)} \qquad \forall I \subset J, K, \tag{2.5.1}$$

$$\underline{\phi}_{IJ}(\underline{U}_{IK}) = \underline{U}_{JK} \cap \underline{\mathfrak{s}}_J^{-1}(\underline{\mathbb{E}}_{IK}) \qquad \forall I \subset J \subset K. \tag{2.5.2}$$

Here we allow equalities between I, J and K using the notation $\underline{U}_{II} := \underline{U_I}$ and $\underline{\phi}_{II} := \mathrm{Id}_{\underline{U}_I}$.

Similarly, a shrinking of a Kuranishi atlas or cobordism will arise exactly from a shrinking $(\underline{U}'_I \sqsubset \underline{U}_I)_{I \in \mathcal{I}_{\mathcal{K}}}$ of the intermediate atlas in the sense of [13, Definition 3.3.2]. Recall that shrinkings of cobordisms are necessarily given by collared subsets $\underline{U}'_I \sqsubset \underline{U}_I$.

Definition 2.5.2 Let $\mathcal{K} = (\boldsymbol{K}_I, \widehat{\Phi}_{IJ})_{I,J \in \mathcal{I}_{\mathcal{K}}, I \subsetneq J}$ be a weak Kuranishi atlas or cobordism. Then a weak Kuranishi atlas or cobordism $\mathcal{K}' = (\boldsymbol{K}'_I, \widehat{\Phi}'_{IJ})_{I,J \in \mathcal{I}_{\mathcal{K}'}, I \subsetneq J}$ is a *shrinking* of $\mathcal{K}$ if:

(i) The footprint cover $(F_i')_{i=1,\dots,N}$ is a shrinking of the cover $(F_i)_{i=1,\dots,N}$; that is, $F_i' \sqsubset F_i$ are precompact open subsets such that $X = \bigcup_{i=1,\dots,N} F_i'$ and $F_I' := \bigcap_{i\in I} F_i'$ is nonempty whenever F_I is, so that the index sets $\mathcal{I}_{\mathcal{K}'} = \mathcal{I}_{\mathcal{K}}$ agree.

(ii) For each $I \in \mathcal{I}_{\mathcal{K}}$ the chart $\boldsymbol{K}_I'$ is the restriction of $\boldsymbol{K}_I$ to a precompact domain $\underline{U}'_I \sqsubset \underline{U}_I$ as in Definition 2.2.6.

(iii) For each $I, J \in \mathcal{I}_{\mathcal{K}}$ with $I \subsetneq J$ the coordinate change $\widehat{\Phi}'_{IJ}$ is the restriction of $\widehat{\Phi}_{IJ}$ to the open subset $\underline{U}'_{IJ} := \underline{\phi}_{IJ}^{-1}(\underline{U}'_J) \cap \underline{U}'_I$ as in Equation (2.2.5).

A *tame shrinking* of $\mathcal{K}$ is a shrinking that is tame in the sense of Definition 2.5.1. Finally, a *preshrunk tame shrinking* of $\mathcal{K}$ is a tame shrinking $\mathcal{K}''$ that is obtained as a shrinking of a tame shrinking $\mathcal{K}'$ of $\mathcal{K}$.

With this language in place, we can directly generalize [14, Theorem 6.3.9]. Recall here that by [13, Example 2.4.5] the quotient topology on $|\mathcal{K}|$ is never metrizable except in the most trivial cases. In fact, for any point $x \in \overline{U_{IJ}} \setminus U_{IJ}$ where $\dim U_I < \dim U_J$, the projection $\pi_{\mathcal{K}}(x)$ does not have a countable neighborhood basis in $|\mathcal{K}|$ with respect to the quotient topology. So an admissible metric will almost always induce a different topology on $|\mathcal{K}|$, which we will make no use of in the following statement.

Theorem 2.5.3 (i) *Any weak Kuranishi atlas or cobordism $\mathcal{K}$ has a preshrunk tame shrinking $\mathcal{K}'$.*

(ii) *For any tame Kuranishi atlas or cobordism $\mathcal{K}'$, the realizations $|\mathcal{K}'|$ and $|\boldsymbol{E}_{\mathcal{K}'}|$ are Hausdorff in the quotient topology, and for each $I \in \mathcal{I}_{\mathcal{K}'}$ the projection maps $\pi_{\underline{\mathcal{K}}'}\colon \underline{U}'_I \to |\mathcal{K}'|$ and $\pi_{\boldsymbol{E}_{\underline{\mathcal{K}}'}}\colon \underline{U_I' \times E_I} \to |\boldsymbol{E}_{\mathcal{K}'}|$ are homeomorphisms onto their images.*

(iii) *For any preshrunk tame shrinking $\mathcal{K}'$ as in (i), there exists an admissible metric on the set $|\mathcal{K}'|$. If $\mathcal{K}$ is a cobordism, then the metric can also be taken to be collared.*

(iv) *Any two metric preshrunk tame shrinkings of a weak Kuranishi atlas are metric tame concordant.*

Proof Since tameness, shrinking and admissible metrics are all defined on the level of intermediate atlases, and we are only concerned with homeomorphism properties of the intermediate projections, in the case of Kuranishi atlases we can simply quote [13, Proposition 3.3.5] for (i), [13, Proposition 3.1.13] for (ii), and [13, Proposition 3.3.8] for (iii). Moreover, [13, Proposition 4.2.3] proves (iv), as well as (i) and (iii) for Kuranishi cobordisms, and (ii) for cobordisms is established in [13, Lemma 4.1.15]. □

3 From Kuranishi atlases to the virtual fundamental class

In this section, Section 3.1 discusses orientations, Section 3.2 establishes the notions of reductions and perturbations. The main result here is Theorem 3.2.8, which shows that the zero set of a suitable perturbation $\mathfrak{s}_{\mathcal{K}} + \nu$ of the canonical section $\mathfrak{s}_{\mathcal{K}}$ has the structure of a compact weighted branched manifold. The construction of such perturbations is deferred to Proposition 3.3.3, and is followed by the construction of the VMC and VFC in Theorem 3.3.5.

3.1 Orientations

This section extends the theory of orientations of weak Kuranishi atlases from [14, Section 8.1] to the case with nontrivial isotropy. Since we use the method of determinant bundles, we first need to generalize the notions of vector bundles and isomorphisms.

Definition 3.1.1 A *vector bundle* $\Lambda = (\Lambda_I, \widetilde{\phi}^{\Lambda}_{IJ})_{I,J\in\mathcal{I}_{\mathcal{K}}}$ *over a weak Kuranishi atlas* $\mathcal{K}$ consists of local bundles and compatible transition maps as follows:

- For each $I \in \mathcal{I}_{\mathcal{K}}$, a vector bundle $\Lambda_I \to U_I$ with an action of Γ_I on Λ_I that covers the given action on U_I.
- For each $I \subsetneq J$, a Γ_J–equivariant map $\widetilde{\phi}^{\Lambda}_{IJ}\colon \rho^*_{IJ}(\Lambda_I|_{U_{IJ}}) \to \Lambda_J$ that is a linear isomorphism on each fiber and covers the embedding $\widetilde{\phi}_{IJ}\colon \widetilde{U}_{IJ} \to U_J$. Here $\Gamma_J \cong \Gamma_I \times \Gamma_{J\setminus I}$ acts on $\rho^*_{IJ}(\Lambda_I|_{U_{IJ}}) \to \widetilde{U}_{IJ}$ by the pullback action of Γ_I together with the natural identification of the fibers of $\rho^*_{IJ}(\Lambda_I|_{U_{IJ}})$ along $\Gamma_{J\setminus I}$–orbits in $\widetilde{U}_{IJ}$.
- For each $I \subsetneq J \subsetneq K$, we have the weak cocycle condition
$$\widetilde{\phi}^{\Lambda}_{IK} = \widetilde{\phi}^{\Lambda}_{JK} \circ \rho^*_{JK}(\widetilde{\phi}^{\Lambda}_{IJ}) \quad \text{on } \rho^{-1}_{JK}(\widetilde{\phi}_{IJ}(\widetilde{U}_{IJ})) \cap \widetilde{U}_{IK}.$$

A *section* of a vector bundle Λ over $\mathcal{K}$ is a collection of smooth Γ_I–equivariant sections $\sigma = (\sigma_I\colon U_I \to \Lambda_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ that are compatible with the pullbacks ρ^*_{IJ} and bundle maps $\widetilde{\phi}^{\Lambda}_{IJ}$ in the sense that there are commutative diagrams for each $I \subsetneq J$:

$$\begin{tikzcd}
\Lambda_I|_{U_{IJ}} & \rho^*_{IJ}(\Lambda_I|_{U_{IJ}}) \arrow[l, "\rho_{IJ}"'] \arrow[r, "\widetilde{\phi}^{\Lambda}_{IJ}"] & \Lambda_J \\
U_{IJ} \arrow[u, "\sigma_I"] & \widetilde{U}_{IJ} \arrow[l, "\rho_{IJ}"'] \arrow[u, "\rho^*_{IJ}(\sigma_I)"'] \arrow[r, "\widetilde{\phi}_{IJ}"] & U_J. \arrow[u, "\sigma_J"']
\end{tikzcd}$$

Definition 3.1.2 If $\Lambda = (\Lambda_I, \widetilde{\phi}^{\Lambda}_{IJ})_{I,J\in\mathcal{I}_{\mathcal{K}}}$ is a bundle over $\mathcal{K}$ and $A \subset [0,1]$ an interval, then the *product bundle* $A \times \Lambda$ over $A \times \mathcal{K}$ is the tuple $(A \times \Lambda_I, \mathrm{id}_A \times \widetilde{\phi}^{\Lambda}_{IJ})_{I,J\in\mathcal{I}_{\mathcal{K}}}$. Here and in the following we denote by $A \times \Lambda_I \to A \times U_I$ the pullback bundle of $\Lambda_I \to U_I$ under the projection $\mathrm{pr}_{U_I}\colon A \times U_I \to U_I$.

Definition 3.1.3 A *vector bundle over a weak Kuranishi cobordism* $\mathcal{K}$ is a collection $\Lambda = (\Lambda_I, \widetilde{\phi}^{\Lambda}_{IJ})_{I,J\in\mathcal{I}_{\mathcal{K}}}$ of vector bundles and bundle maps as in Definition 3.1.1, together with a choice of isomorphism from its collar restriction to a product bundle. More precisely, this requires for $\alpha = 0, 1$ the choice of a *restricted vector bundle*

$$\Lambda|_{\partial^\alpha\mathcal{K}} = (\Lambda^\alpha_I \to \partial^\alpha U_I, \widetilde{\phi}^{\Lambda,\alpha}_{IJ})_{I,J\in\mathcal{I}_{\partial^\alpha\mathcal{K}}}$$

over $\partial^\alpha\mathcal{K}$, and, for some $\varepsilon > 0$ less than the collar width of $\mathcal{K}$, a choice of lifts of the embeddings ι^α_I for $I \in \mathcal{I}_{\partial^\alpha\mathcal{K}}$ to Γ_I–equivariant bundle isomorphisms

$$\widetilde{\iota}^{\Lambda,\alpha}_I \colon A^\alpha_\varepsilon \times \Lambda^\alpha_I \to \Lambda_I|_{\operatorname{im}\iota^\alpha_I}$$

such that, with $A := A^\alpha_\varepsilon$ and $\rho^*\widetilde{\iota}^{\Lambda,\alpha}_I := \rho^*_{IJ} \circ \widetilde{\iota}^{\Lambda,\alpha}_I \circ (\mathrm{id}_A \times (\rho^\alpha_{IJ})_*)$, the following diagrams commute:

$$\begin{tikzcd} A \times \Lambda^\alpha_I \arrow[r, "\widetilde{\iota}^{\Lambda,\alpha}_I"] \arrow[d] & \Lambda_I|_{\operatorname{im}\iota^\alpha_I} \arrow[d] \\ A \times \partial^\alpha U_I \arrow[r, "\iota^\alpha_I"] & \operatorname{im}\iota^\alpha_I \subset U_I \end{tikzcd} \qquad \begin{tikzcd} A \times (\rho^\alpha_{IJ})^*(\Lambda^\alpha_I|_{\partial^\alpha U_{IJ}}) \arrow[r, "\rho^*\widetilde{\iota}^{\Lambda,\alpha}_I"] \arrow[d, "\mathrm{id}_A \times \widetilde{\phi}^{\Lambda,\alpha}_{IJ}"'] & \rho^*_{IJ}(\Lambda_I|_{\iota^\alpha_I(A\times\partial^\alpha U_{IJ})}) \arrow[d, "\widetilde{\phi}^{\Lambda}_{IJ}"] \\ A \times \Lambda^\alpha_J \arrow[r, "\widetilde{\iota}^{\Lambda,\alpha}_J"] & \Lambda_J|_{\operatorname{im}\iota^\alpha_J} \end{tikzcd}$$

A *section* of a vector bundle Λ over a Kuranishi cobordism as above is a compatible collection $(\sigma_I \colon U_I \to \Lambda_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ of equivariant sections as in Definition 3.1.1 that in addition are of product form in the collar. That is, we require that for each $\alpha = 0, 1$ there is a *restricted section* $\sigma|_{\partial^\alpha\mathcal{K}} = (\sigma^\alpha_I \colon \partial_\alpha U_I \to \Lambda^\alpha_I)_{I\in\mathcal{I}_{\partial^\alpha\mathcal{K}}}$ of $\Lambda|_{\partial^\alpha\mathcal{K}}$ such that for $\varepsilon > 0$ sufficiently small, $(\widetilde{\iota}^{\Lambda,\alpha}_I)^*\sigma_I = \mathrm{id}_{A^\alpha_\varepsilon} \times \sigma^\alpha_I$.

In the above definition we implicitly work with an isomorphism $(\widetilde{\iota}^{\Lambda,\alpha}_I)_{I\in\mathcal{I}_{\partial^\alpha\mathcal{K}}}$ that satisfies all but the product structure requirements of the following notion of isomorphism on Kuranishi cobordisms.

Definition 3.1.4 An *isomorphism* $\Psi \colon \Lambda \to \Lambda'$ between vector bundles over $\mathcal{K}$ is a collection $(\Psi_I \colon \Lambda_I \to \Lambda'_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ of Γ_I–equivariant bundle isomorphisms covering the identity on U_I that intertwine the transition maps, ie $\widetilde{\phi}^{\Lambda'}_{IJ} \circ \rho^*_{IJ}(\Psi_I) = \Psi_J \circ \widetilde{\phi}^{\Lambda}_{IJ}|_{\widetilde{U}_{IJ}}$ for all $I \subsetneq J$.

If $\mathcal{K}$ is a Kuranishi cobordism then we additionally require Ψ to be of product form in the collar. That is, we require that for each $\alpha = 0, 1$ there is a restricted isomorphism $\Psi|_{\partial^\alpha\mathcal{K}} = (\Psi^\alpha_I \colon \Lambda^\alpha_I \to {\Lambda'}^\alpha_I)_{I\in\mathcal{I}_{\partial^\alpha\mathcal{K}}}$ from $\Lambda|_{\partial^\alpha\mathcal{K}}$ to $\Lambda'|_{\partial^\alpha\mathcal{K}}$ such that for $\varepsilon > 0$ sufficiently small we have $(\widetilde{\iota}'_I)^{\Lambda,\alpha} \circ (\mathrm{id}_A \times \Psi^\alpha_I) = \Psi_I \circ \widetilde{\iota}^{\Lambda,\alpha}_I$ on $A^\alpha_\varepsilon \times \partial^\alpha U_I$.

Note that although the compatibility conditions are the same, the canonical section $\mathfrak{s}_{\mathcal{K}} = (s_I \colon U_I \to E_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ of a Kuranishi atlas does not form a section of a vector bundle since the obstruction spaces E_I are in general not of the same dimension, hence no bundle isomorphisms $\widetilde{\phi}^{\Lambda}_{IJ}$ as above exist. Nevertheless, we will see that there is a natural bundle associated with the section $\mathfrak{s}_{\mathcal{K}}$, namely its determinant line bundle, and that this line bundle is isomorphic to a bundle constructed by combining the determinant lines of the obstruction spaces E_I and the domains U_I.

Remark 3.1.5 If Λ is a bundle over a Kuranishi atlas $\mathcal{K}$ (rather than a weak atlas), then it is straightforward to verify that the union $\bigsqcup_I \Lambda_I$ of the local bundles form the objects of a category with projection to the Kuranishi category $\boldsymbol{B}_{\mathcal{K}}$. We did not formulate the above definitions in this language since orientations in applications to moduli spaces (eg Gromov–Witten as in [10]) will usually be constructed on a weak Kuranishi atlas, which does not form a category. $\Diamond$

Here and in the following we will exclusively work with finite-dimensional vector spaces. First recall that the determinant line of a vector space V is its maximal exterior power $\Lambda^{\max} V := \bigwedge^{\dim V} V$, with $\bigwedge^0 \{0\} := \mathbb{R}$. More generally, the *determinant line of a linear map* $D\colon V \to W$ is defined to be

$$\det(D) := \Lambda^{\max} \ker D \otimes (\Lambda^{\max}(W/\operatorname{im} D))^*. \tag{3.1.1}$$

In order to construct isomorphisms between determinant lines, we will need to fix various conventions, in particular pertaining to the ordering of factors in their domains and targets. We begin by noting that every isomorphism $F\colon Y \to Z$ between finite-dimensional vector spaces induces an isomorphism

$$\Lambda_F\colon \Lambda^{\max} Y \xrightarrow{\cong} \Lambda^{\max} Z, \quad y_1 \wedge \dots \wedge y_k \mapsto F(y_1) \wedge \dots \wedge F(y_k). \tag{3.1.2}$$

For example, the fact that $\gamma \circ s_I := s_I \circ \gamma\colon U_I \to E_I$ for all $\gamma \in \Gamma_I$, implies that

$$\gamma_\Lambda := \Lambda_{\mathrm{d}_x \gamma} \otimes (\Lambda_{[\gamma]^{-1}})^*\colon \det(\mathrm{d}_x s_I) \to \det(\mathrm{d}_{\gamma x} s_I) \tag{3.1.3}$$

is an isomorphism, where $[\gamma]\colon E_I/\operatorname{im} \mathrm{d}_x s_I \to E_I/\operatorname{im} \mathrm{d}_{\gamma x} s_I$ is the induced map. Further, if $I \subsetneq J$ and $\widetilde{x} \in \widetilde{U}_{IJ}$ is such that $\rho_{IJ}(\widetilde{x}) = x$, then because

$$s_I \circ \rho_{IJ} =: s_{IJ}\colon \widetilde{U}_{IJ} \to E_I,$$

the derivative $\mathrm{d}_{\widetilde{x}} \rho_{IJ}\colon \ker \mathrm{d}s_{IJ} \to \ker \mathrm{d}s_I$ induces an isomorphism

$$\Lambda_{\mathrm{d}_{\widetilde{x}} \rho_{IJ}} \otimes \Lambda_{\mathrm{Id}}\colon \det(\mathrm{d}_{\widetilde{x}} s_{IJ}) \to \det(\mathrm{d}_x s_I)$$

and composition with pullback by ρ_{IJ} defines an isomorphism

$$P_{IJ}(\widetilde{x})\colon \det(\mathrm{d}_{\widetilde{x}} s_{IJ}) \to \rho^*_{IJ}(\det \mathrm{d}_x s_I). \tag{3.1.4}$$

Further, it follows from the index condition in Definition 2.2.8 that with $y := \widetilde{\phi}_{IJ}(\widetilde{x})$, the map

$$\widetilde{\Lambda}_{IJ}(\widetilde{x}) := \Lambda_{\mathrm{d}_{\widetilde{x}}\widetilde{\phi}_{IJ}} \otimes (\Lambda_{[\widehat{\phi}_{IJ}]^{-1}})^* \colon \det(\mathrm{d}_{\widetilde{x}} s_{IJ}) \to \det(\mathrm{d}_y s_J) \tag{3.1.5}$$

is an isomorphism, induced by the isomorphisms $\mathrm{d}\widetilde{\phi}_{IJ} \colon \ker \mathrm{d}s_{IJ} \to \ker \mathrm{d}s_J$ and $[\widehat{\phi}_{IJ}] \colon E_I / \operatorname{im} \mathrm{d}s_I \to E_J / \operatorname{im} \mathrm{d}s_J$. We can therefore define the determinant bundle $\det(\mathfrak{s}_{\mathcal{K}})$ of a Kuranishi atlas. A second, isomorphic, determinant line bundle $\det(\mathcal{K})$ with fibers $\Lambda^{\max} \mathrm{T}_x U_I \otimes (\Lambda^{\max} E_I)^*$ will be constructed in Proposition 3.1.13.

Definition 3.1.6 The *determinant line bundle* of a weak Kuranishi atlas (or cobordism) $\mathcal{K}$ is the vector bundle $\det(\mathfrak{s}_{\mathcal{K}})$ given by the line bundles

$$\det(\mathrm{d}s_I) := \bigcup_{x \in U_I} \det(\mathrm{d}_x s_I) \to U_I \qquad \text{for all } I \in \mathcal{I}_{\mathcal{K}},$$

with Γ_I actions given by the isomorphisms γ_Λ of (3.1.3), and the isomorphisms $\widetilde{\phi}^{\Lambda}_{IJ}(\widetilde{x}) := \Lambda_{IJ}(\widetilde{x}) \circ P_{IJ}(\widetilde{x})^{-1}$ in (3.1.4) and (3.1.5) for $I \subsetneq J$ and $\widetilde{x} \in U_{IJ}$.

To show that $\det(\mathfrak{s}_{\mathcal{K}})$ is well defined, in particular that $\widetilde{x} \mapsto \Lambda_{IJ}(\widetilde{x})$ is smooth, we introduce some further natural[6] isomorphisms and fix various ordering conventions.

- For any subspace $V' \subset V$ the *splitting isomorphism*

$$\Lambda^{\max} V \cong \Lambda^{\max} V' \otimes \Lambda^{\max}(V/V') \tag{3.1.6}$$

is given by completing a basis $v_1, \dots, v_k$ of V' to a basis $v_1, \dots, v_n$ of V and mapping $v_1 \wedge \cdots \wedge v_n \mapsto (v_1 \wedge \cdots \wedge v_k) \otimes ([v_{k+1}] \wedge \cdots \wedge [v_n])$.

- For each isomorphism $F \colon Y \xrightarrow{\cong} Z$ the *contraction isomorphism*

$$\mathfrak{c}_F \colon \Lambda^{\max} Y \otimes (\Lambda^{\max} Z)^* \xrightarrow{\cong} \mathbb{R}, \tag{3.1.7}$$

is given by the map $(y_1 \wedge \cdots \wedge y_k) \otimes \eta \mapsto \eta(F(y_1) \wedge \cdots \wedge F(y_k))$.

- For any space V we use the *duality isomorphism*

$$\Lambda^{\max} V^* \xrightarrow{\cong} (\Lambda^{\max} V)^*, \quad v_1^* \wedge \cdots \wedge v_n^* \mapsto (v_1 \wedge \cdots \wedge v_n)^*, \tag{3.1.8}$$

which corresponds to the natural pairing

$$\Lambda^{\max} V \otimes \Lambda^{\max} V^* \xrightarrow{\cong} \mathbb{R}, \quad (v_1 \wedge \cdots \wedge v_n) \otimes (\eta_1 \wedge \cdots \wedge \eta_n) \mapsto \prod_{i=1}^{n} \eta_i(v_i)$$

[6] Here a "natural" isomorphism is one that is functorial, ie it commutes with the action on both sides induced by a vector space isomorphism.

via the general identification

$$\text{(3.1.9)} \qquad \mathrm{Hom}(A\otimes B,\mathbb{R}) \xrightarrow{\cong} \mathrm{Hom}(B,A^*), \quad H\mapsto (b\mapsto H(\cdot\otimes b)),$$

which in the case of line bundles A, B maps $\eta\neq 0$ to a nonzero homomorphism, ie an isomorphism. Next, we combine the above isomorphisms to obtain a more elaborate contraction isomorphism.

Lemma 3.1.7 [14, Lemma 8.1.7] *Every linear map $F\colon V\to W$ together with an isomorphism $\phi\colon K\to \ker F$ induces an isomorphism*

$$\text{(3.1.10)} \qquad \mathfrak{C}_F^{\phi}\colon \Lambda^{\max}V\otimes(\Lambda^{\max}W)^* \xrightarrow{\cong} \Lambda^{\max}K\otimes(\Lambda^{\max}(W/F(V)))^*$$

given by

$$(v_1\wedge\cdots\wedge v_n)\otimes(w_1\wedge\cdots\wedge w_m)^*\mapsto(\phi^{-1}(v_1)\wedge\cdots\wedge\phi^{-1}(v_k))\otimes([w_1]\wedge\cdots\wedge[w_{m-n+k}])^*,$$

where $v_1,\dots,v_n$ is a basis for V with $\mathrm{span}(v_1,\dots,v_k)=\ker F$, and $w_1,\dots,w_m$ is a basis for W whose last $n-k$ vectors are $w_{m-n+i}=F(v_i)$ for $i=k+1,\dots,n$.

In particular, for every linear map $D\colon V\to W$ we may pick ϕ as the inclusion $K=\ker D\hookrightarrow V$ to obtain an isomorphism

$$\mathfrak{C}_D\colon \Lambda^{\max}V\otimes(\Lambda^{\max}W)^* \xrightarrow{\cong} \det(D).$$

Remark 3.1.8 If F is equivariant with respect to actions of the group Γ on V and W, and we equip K with the induced Γ action so that ϕ is also equivariant, then the above isomorphism $\mathfrak{C}_F^{\phi}$ is equivariant with respect to the action of Γ on $\Lambda^{\max}V\otimes(\Lambda^{\max}W)^*$ given by the maps $\Lambda_\gamma\otimes(\Lambda_{\gamma^{-1}})^*$ on $\Lambda^{\max}V\otimes(\Lambda^{\max}W)^*$ and by the corresponding maps $\Lambda_\gamma\otimes(\Lambda_{[\gamma]^{-1}})^*$ on $\Lambda^{\max}K\otimes(\Lambda^{\max}(W/F(V)))^*$, with $\Lambda_{[\gamma]}$ as in (3.1.3). ◇

With this notation in hand, we can now prove one of the main results of this section.

Proposition 3.1.9 *For any weak Kuranishi atlas, $\det(\mathfrak{s}_{\mathcal{K}})$ is a well-defined line bundle over $\mathcal{K}$. Further, if $\mathcal{K}$ is a weak Kuranishi cobordism, then $\det(\mathfrak{s}_{\mathcal{K}})$ can be given product form on the collar of $\mathcal{K}$ with restrictions $\det(\mathfrak{s}_{\mathcal{K}})|_{\partial^\alpha\mathcal{K}}=\det(\mathfrak{s}_{\partial^\alpha\mathcal{K}})$ for $\alpha=0,1$. The required bundle isomorphisms from the product $A_\varepsilon^\alpha\times\det(\mathfrak{s}_{\partial^\alpha\mathcal{K}})$ to the collar restriction $(\iota_\varepsilon^\alpha)^*\det(\mathfrak{s}_{\mathcal{K}})$ are given in (3.1.12).*

Proof We use the same local trivializations of $\det(\mathrm{d}s_I)$ as in the proof of the analogous result [14, Proposition 8.1.8] for trivial isotropy, and must check that these are compatible with the isotropy group actions and coordinate changes. We will begin by defining these trivializations, referring to [14] for many details of proofs.

Let $x_0 \in U_I$, and denote its stabilizer subgroup in Γ_I by $\Gamma_I^{x_0}$. Take a subspace of E_I that covers the cokernel of $\mathrm{d}_{x_0} s_I$, sweep it out to obtain a $\Gamma_I^{x_0}$–invariant subspace $E' \subset E_I$, then choose an isomorphism $\mathbb{R}^N \cong E'$ and equip $\mathbb{R}^N$ with the pullback action of $\Gamma_I^{x_0}$ denoted $(\gamma, v) \mapsto \gamma \cdot_{x_0} v$. The resulting equivariant map $R_I \colon (\mathbb{R}^N, \Gamma_I^{x_0}) \to (E_I, \Gamma_I^{x_0})$ covers the cokernel of $\mathrm{d}_x s_I$ for all x in some neighborhood O of x_0.

Thus $\mathrm{d}_x s_I \oplus R_I$ is surjective for $x \in O$, and as in [14, Equation 8.1.9] we may define a trivialization of $\det(\mathrm{d}s_I)|_O$ by

$$(3.1.11)\quad \begin{aligned} &\widehat{T}_{I,x} \colon \Lambda^{\max} \ker(\mathrm{d}_x s_I \oplus R_I) \xrightarrow{\cong} \det(\mathrm{d}_x s_I), \\ &\overline{v}_1 \wedge \cdots \wedge \overline{v}_n \mapsto (v_1 \wedge \cdots \wedge v_k) \otimes ([R_I(e_1)] \wedge \cdots \wedge [R_I(e_{N-n+k})])^*, \end{aligned}$$

where $\overline{v}_i = (v_i, r_i)$ is a basis of $\ker(\mathrm{d}_x s_I \oplus R_I) \subset \mathrm{T}_x U_I \times \mathbb{R}^N$ such that $v_1, \dots, v_k$ span $\ker \mathrm{d}_x s_I$ (and hence $r_1 = \cdots = r_k = 0$), and $e_1, \dots, e_N$ is a positively ordered normalized basis of $\mathbb{R}^N$ (that is, $e_1 \wedge \cdots \wedge e_N = 1 \in \mathbb{R} \cong \Lambda^{\max}\mathbb{R}^N$) such that $R_I(e_{N-n+i}) = \mathrm{d}_x s_I(v_i)$ for $i = k+1, \dots, n$. In particular, the last $n-k$ vectors span $\operatorname{im} \mathrm{d}_x s_I \cap \operatorname{im} R_I \subset E_I$, and thus the first $N-n+k$ vectors $[R_I(e_1)], \dots, [R_I(e_{N-n+k})]$ span the cokernel $E_I / \operatorname{im} \mathrm{d}_x s_I \cong \operatorname{im} R_I / \operatorname{im} \mathrm{d}_x s_I \cap \operatorname{im} R_I$. In [14, Proposition 8.1.8] we prove that these trivializations do not depend on the choice of injection $R_I \colon \mathbb{R}^N \to E_I$. In other words, if $R'_I \colon \mathbb{R}^{N'} \to E_I$ is another Γ_I–equivariant injection that also maps onto the cokernel of $\mathrm{d}_{x_0} s_I$, then there is a bundle isomorphism

$$\Psi \colon \Lambda^{\max} \ker(\mathrm{d}s_I \oplus R_I)|_O \to \Lambda^{\max} \ker(\mathrm{d}s_I \oplus R'_I)|_O$$

which is necessarily Γ_I–equivariant and such that $\widehat{T}_I = \widehat{T}'_I \circ \Psi$. Thus $\det(\mathrm{d}s_I)$ is a smooth line bundle over U_I for each $I \in \mathcal{I}_{\mathcal{K}}$.

It remains to check that the action $\gamma \in \Gamma_I$ on

$$\det(\mathrm{d}s_I) = \Lambda^{\max}(\ker \mathrm{d}s_I) \otimes \Lambda^{\max}(E_I / \operatorname{im} \mathrm{d}s_I)^*$$

is smooth. We prove this by choosing suitable trivializations near x_0 and γx_0 and then lifting the action of γ to a smooth action on the domains $\ker(\mathrm{d}s_I \oplus R_I)$ of the trivializations. To this end, first consider the trivialization $T_{I,x}$ defined near $x_0 \in U_I$ by a $\Gamma_I^{x_0}$–equivariant injection $R_I \colon (\mathbb{R}^N, \Gamma_I^{x_0}) \to (E_I, \Gamma_I^{x_0})$, and for $\gamma \in \Gamma_I$ the associated trivialization $T'_{I,\gamma x}$ defined near $\gamma x_0 \in U_I$ by

$$R'_I := \gamma \circ R_I \colon (\mathbb{R}^N, \Gamma_I^{\gamma x_0}) \to (E_I, \Gamma_I^{\gamma x_0}),$$

where $(\mathbb{R}^N, \Gamma_I^{x_0})$ denotes $\mathbb{R}^N$ with the $\Gamma_I^{x_0}$–action $\delta \colon v \mapsto \delta \cdot_{x_0} v$, and $(\mathbb{R}^N, \Gamma_I^{\gamma x_0})$ denotes $\mathbb{R}^N$ with the $\Gamma_I^{\gamma x_0}$–action

$$\delta' \colon v \mapsto \delta' \cdot_{\gamma x_0} v := \gamma^{-1} \delta' \gamma \cdot_{x_0} v,$$

which is well defined since conjugation by γ defines an isomorphism $c_\gamma\colon \Gamma_I^{\gamma x_0} \to \Gamma_I^{x_0}$, $\delta' \mapsto \gamma^{-1}\delta'\gamma$. Then $R_I' = \gamma \circ R_I$ is $\Gamma_I^{\gamma x_0}$–equivariant because when $\delta' \in \Gamma_I^{\gamma x_0}$,

$$\begin{aligned} R_I'(\delta' \cdot_{\gamma x_0} v) = R_I'(\gamma^{-1}\delta'\gamma \cdot_{x_0} v) &= \gamma R_I(\gamma^{-1}\delta'\gamma \cdot_{x_0} v) \\ &= \gamma(\gamma^{-1}\delta'\gamma) R_I(v) \\ &= \gamma\gamma^{-1}\delta'\gamma R_I(v) = \delta'\gamma R_I(v) = \delta' \circ R_I'(v), \end{aligned}$$

where the fourth equality holds because the full group Γ_I acts on E_I. Thus the diagram

$$\begin{tikzcd} (\mathbb{R}^N, \Gamma_I^{x_0}) \arrow[r, "{(R_I,\mathrm{id})}"] \arrow[d, "{(\mathrm{id},c_\gamma^{-1})}"] & (E_I, \Gamma_I^{x_0}) \arrow[d, "{(\gamma,c_\gamma^{-1})}"] \\ (\mathbb{R}^N, \Gamma_I^{\gamma x_0}) \arrow[r, "{(R_I',\mathrm{id})}"] & (E_I, \Gamma_I^{\gamma x_0}) \end{tikzcd}$$

commutes; in other words, the action of the element $\gamma \in \Gamma_I$ on E_I lifts to the identity map of $\mathbb{R}^N$. Hence the definition (3.1.11) of the maps $\widehat{T}_{I,x}$ implies that the following diagram commutes:

$$\begin{tikzcd} \Lambda^{\max}(\ker(\mathrm{d}_x s_I \oplus R_I)) \arrow[r, "\widehat{T}_{I,x}"] \arrow[d, "\Lambda_{\mathrm{d}_x\gamma \oplus \mathrm{id}_{\mathbb{R}^N}}"] & \det \mathrm{d}_x s_I \arrow[d, "\Lambda_{\mathrm{d}_x\gamma} \otimes (\Lambda_{[\gamma^{-1}]})^*"] \\ \Lambda^{\max}(\ker(\mathrm{d}_{\gamma x} s_I \oplus R_I')) \arrow[r, "\widehat{T}_{I,\gamma x}"] & \det \mathrm{d}_{\gamma x} s_I \end{tikzcd}$$

Since the map $\Lambda_{\mathrm{d}_x\gamma \oplus \mathrm{id}_{\mathbb{R}^N}}$ is smooth, so is $\gamma_\Lambda := \Lambda_{\mathrm{d}_x\gamma} \otimes (\Lambda_{[\gamma^{-1}]})^*$. Thus $\det(\mathrm{d}s_I)$ is a Γ_I–equivariant smooth line bundle over U_I for each $I \in \mathcal{I}_\mathcal{K}$.

Next note that because $\Gamma_{J\setminus I}$ acts freely on $\widetilde{U}_{IJ}$, the stabilizer subgroup $\Gamma_J^{\widetilde{x}_0}$ of a point $\widetilde{x}_0 \in \rho_{IJ}^{-1}(x_0)$ is taken isomorphically to $\Gamma_I^{x_0}$ by the projection $\rho_{IJ}^\Gamma\colon \Gamma_J \to \Gamma_I$. For simplicity we will identify these groups. Since $s_{IJ}\colon \widetilde{U}_{IJ} \to E_I$ is the composite $s_I \circ \rho_{IJ}$, we may therefore trivialize the bundle $\det(\mathrm{d}_{\widetilde{x}} s_{IJ})$ near $\widetilde{x}_0 \in \rho_{IJ}^{-1}(x_0)$ by using the same injection $R_I\colon \mathbb{R}^N \to E_I$, now considered as a $\Gamma_J^{\widetilde{x}_0}$–equivariant map. Since the diagram

$$\begin{tikzcd} \Lambda^{\max}(\ker(\mathrm{d}_{\widetilde{x}} s_{IJ} \oplus R_I)) \arrow[r, "\widehat{T}_{I,\widetilde{x}}"] \arrow[d, "\Lambda_{\rho_{IJ} \oplus \mathrm{id}_{\mathbb{R}^N}}"] & \det \mathrm{d}_{\widetilde{x}} s_{IJ} \arrow[d, "\Lambda_{\rho_{IJ}} \otimes (\Lambda_{\mathrm{id}})^*"] \\ \Lambda^{\max}(\ker(\mathrm{d}_x s_I \oplus R_I)) \arrow[r, "\widehat{T}_{I,x}"] & \det \mathrm{d}_x s_I \end{tikzcd}$$

commutes, the isomorphism $P_{IJ}(\widetilde{x})\colon \det \mathrm{d}_{\widetilde{x}} s_{IJ} \to \det \mathrm{d}_x s_I$ of (3.1.4) is smooth. Moreover the equivariance of the covering map $\rho_{IJ}\colon (\widetilde{U}_{IJ}, \Gamma_J) \to (U_{IJ}, \Gamma_I)$ and the identity

$s_I \circ \rho_{IJ} = s_{IJ} \colon \widetilde{U}_{IJ} \to E_I$ imply that it is equivariant. Therefore, to complete the proof that the transition maps $\widetilde{\phi}^{\Lambda}_{IJ}$ are smooth, we must check that the map

$$\widetilde{\Lambda}_{IJ}(\widetilde{x}) := \Lambda_{\mathrm{d}_{\widetilde{x}}\widetilde{\phi}_{IJ}} \otimes (\Lambda_{[\widehat{\phi}_{IJ}]^{-1}})^* \colon \det(\mathrm{d}_{\widetilde{x}} s_{IJ}) \to \det(\mathrm{d}_y s_J)$$

in (3.1.5) is equivariant and smooth. Its equivariance follows from the equivariance of its constituent maps $\widetilde{\phi}_{IJ}$ and $\widehat{\phi}_{IJ}$. To see that it is smooth, it suffices to show that the composite $\Lambda_{IJ}(x) := \widetilde{\Lambda}_{IJ}(\rho_{IJ}^{-1}(x))$ is smooth in some neighborhood O of $x_0 \in U_{IJ}$, where $\rho_{IJ}^{-1} \colon O \to \widetilde{U}_{IJ}$ is a local inverse for the covering map ρ_{IJ}. But if we define $\phi_{IJ}(x) := \widetilde{\phi}_{IJ}(\rho_{IJ}^{-1}(x)) \colon O \to U_J$, then $\Lambda_{IJ}(x) = \Lambda_{\mathrm{d}_x \phi_{IJ}} \otimes (\Lambda_{[\widehat{\phi}_{IJ}]^{-1}})^*$ is identical to the map of the same name in [14, Equation 8.1.11], so that smoothness follows by the Claim proved as part of [14, Proposition 5.1.8]. This completes the proof that $\det(s_{\mathcal{K}})$ is a vector bundle over $\mathcal{K}$.

In the case of a weak Kuranishi cobordism $\mathcal{K}$, Proposition 8.1.8 in [14] also constructs smooth bundle isomorphisms from the collar restrictions to the product bundles $A^\alpha_\varepsilon \times \det(\mathfrak{s}_{\partial^\alpha \mathcal{K}})$ of the form

$$\widetilde{\iota}^{\Lambda,\alpha}_I(t,x) := (\Lambda_{\mathrm{d}_{(t,x)}\iota^\alpha_I} \circ \wedge_1) \otimes (\Lambda_{\mathrm{id}_{E_I}})^* \colon A^\alpha_\varepsilon \times \det(\mathrm{d}_x s^\alpha_I) \to \det(\mathrm{d}_{\iota^\alpha_I(x,t)} s_I), \tag{3.1.12}$$

where $\wedge_1 \colon \Lambda^{\max} \ker \mathrm{d}_x s^\alpha_I \to \Lambda^{\max}(\mathbb{R} \times \ker \mathrm{d}_x s^\alpha_I)$ is given by $\eta \mapsto 1 \wedge \eta$. These are equivariant because they are induced by the equivariant map ι^α_I, and are compatible with the coordinate changes because the collar embeddings ι^α_I are. $\square$

We next use the determinant bundle $\det(s_{\mathcal{K}})$ to define the notion of an orientation of a Kuranishi atlas.

Definition 3.1.10 A weak Kuranishi atlas or Kuranishi cobordism $\mathcal{K}$ is *orientable* if there exists a nonvanishing section σ of the bundle $\det(\mathfrak{s}_{\mathcal{K}})$, ie with $\sigma_I^{-1}(0) = \varnothing$ for all $I \in \mathcal{I}_{\mathcal{K}}$. An *orientation* of $\mathcal{K}$ is a choice of nonvanishing section σ of $\det(\mathfrak{s}_{\mathcal{K}})$. An *oriented Kuranishi atlas or cobordism* is a pair $(\mathcal{K}, \sigma)$ consisting of a Kuranishi atlas or cobordism and an orientation σ of $\mathcal{K}$.

For an oriented Kuranishi cobordism $(\mathcal{K}, \sigma)$ the *induced orientation of the boundary* $\partial^\alpha \mathcal{K}$ for $\alpha = 0, 1$ is the orientation of $\partial^\alpha \mathcal{K}$

$$\partial^\alpha \sigma := \big(((\widetilde{\iota}^{\Lambda,\alpha}_I)^{-1} \circ \sigma_I \circ \iota^\alpha_I)\big|_{\{\alpha\} \times \partial^\alpha U_I} \big)_{I \in \mathcal{I}_{\partial^\alpha \mathcal{K}}}$$

given by the isomorphism $(\widetilde{\iota}^{\Lambda,\alpha}_I)_{I \in \mathcal{I}_{\partial^\alpha \mathcal{K}}}$ in (3.1.12) between a collar neighborhood of the boundary in $\mathcal{K}$ and the product Kuranishi atlas $A^\alpha_\varepsilon \times \partial^\alpha \mathcal{K}$, followed by restriction to the boundary $\partial^\alpha \mathcal{K} = \partial^\alpha (A^\alpha_\varepsilon \times \partial^\alpha \mathcal{K})$, where we identify $\{\alpha\} \times \partial^\alpha U_I \cong \partial^\alpha U_I$.

With that, we say that two oriented weak Kuranishi atlases $(\mathcal{K}^0, \sigma^0)$ and $(\mathcal{K}^1, \sigma^1)$ are *oriented cobordant* if there exists a weak Kuranishi cobordism $\mathcal{K}$ from $\mathcal{K}^0$ to $\mathcal{K}^1$ and a section σ of $\det(\mathfrak{s}_{\mathcal{K}})$ such that $\partial^\alpha \sigma = \sigma^\alpha$ for $\alpha = 0, 1$.

Remark 3.1.11 Here we have defined the induced orientation on the boundary $\partial^\alpha \mathcal{K}$ of a cobordism so that it is completed to an orientation of the collar by adding the positive unit vector 1 along $A^\alpha_\varepsilon \subset \mathbb{R}$ rather than the more usual outward normal vector. In particular, by [14, Equation (8.1.12)], $\eta_1, \dots, \eta_n$ is a positively ordered basis for $\mathrm{T}_x U^\alpha_I$ exactly if $1, \eta_1, \dots, \eta_n$ is a positively ordered basis for $\mathrm{T}_x(A^\alpha_\varepsilon \times U^\alpha_I)$. ◇

Lemma 3.1.12 *Let $(\mathcal{K}, \sigma)$ be an oriented weak Kuranishi atlas or cobordism.*

(i) *The orientation σ induces a canonical orientation $\sigma|_{\mathcal{K}'} := (\sigma_I|_{U'_I})_{I \in \mathcal{I}_{\mathcal{K}'}}$ on each shrinking $\mathcal{K}'$ of $\mathcal{K}$ with domains $(U'_I \subset U_I)_{I \in \mathcal{I}_{\mathcal{K}'}}$.*

(ii) *In the case of a Kuranishi cobordism $\mathcal{K}$, the restrictions to boundary and shrinking commute; that is, $(\sigma|_{\mathcal{K}'})|_{\partial^\alpha \mathcal{K}'} = (\sigma|_{\partial^\alpha \mathcal{K}})|_{\partial^\alpha \mathcal{K}'}$.*

(iii) *In the case of a weak Kuranishi atlas $\mathcal{K}$, the orientation σ on $\mathcal{K}$ induces an orientation $\sigma^{[0,1]}$ on $[0,1] \times \mathcal{K}$, which induces the given orientation $\partial^\alpha \sigma^{[0,1]} = \sigma$ of the boundaries $\partial^\alpha([0,1] \times \mathcal{K}) = \mathcal{K}$ for $\alpha = 0, 1$.*

Proof See the proof of [14, Lemma 8.1.11]. □

As in [14], in order to orient the zero sets of a perturbed section $\mathfrak{s}_{\mathcal{K}} + \nu$ we will work with a "more universal" determinant bundle $\det(\mathcal{K})$ over $\mathcal{K}$ that is constructed from the determinant bundles of the zero sections in each chart. Since the zero section $0_{\mathcal{K}}$ does not satisfy the index condition, we need to construct different transition maps for $\det(\mathcal{K})$, which will now depend on the section $\mathfrak{s}_{\mathcal{K}}$. For this purpose, we again use contraction isomorphisms from Lemma 3.1.7.

On the one hand, this provides families of isomorphisms

$$\mathfrak{C}_{\mathrm{d}_x s_I} \colon \Lambda^{\max} \mathrm{T}_x U_I \otimes (\Lambda^{\max} E_I)^* \xrightarrow{\cong} \det(\mathrm{d}_x s_I) \quad \text{for } x \in U_I, \tag{3.1.13}$$

which, by Remark 3.1.8, are equivariant with the respect to the action of $\gamma \in \Gamma_I$ on $\Lambda^{\max} \mathrm{T} U_I \otimes (\Lambda^{\max} E_I)^*$ given by

$$\begin{aligned} \hat{\gamma}_\Lambda := \Lambda_{\mathrm{d}_x \gamma} \otimes (\Lambda_{\gamma^{-1}})^* \colon \Lambda^{\max} \mathrm{T}_x U_I \otimes (\Lambda^{\max} E_I)^* \\ \to \Lambda^{\max} \mathrm{T}_{\gamma x} U_I \otimes (\Lambda^{\max} E_I)^* \end{aligned} \tag{3.1.14}$$

and the corresponding action on $\det(\mathrm{d}_x s_I)$ in Equation (3.1.3).

On the other hand, recall that the tangent bundle condition (2.2.3) implies that $\mathrm{d}s_J$ restricts to an isomorphism

$$\mathrm{T}_y U_J/\mathrm{d}_{\tilde{x}}\tilde{\phi}_{IJ}(\mathrm{T}_{\tilde{x}}\tilde{U}_{IJ}) \xrightarrow{\cong} E_J/\hat{\phi}_{IJ}(E_I)$$

for $y=\tilde{\phi}_{IJ}(\tilde{x})$.[7] Therefore, if we choose a Γ_J–equivariant smooth normal bundle

$$N_{IJ} = \bigcup_{y\in \mathrm{im}\,\tilde{\phi}_{IJ}} N_{IJ,y} \subset \mathrm{T}_y U_J$$

to the submanifold $\mathrm{im}\,\tilde{\phi}_{IJ}\subset U_J$, then the subspaces $\mathrm{d}_y s_J(N_{IJ,y})$ form a smooth family of subspaces of E_J that are complements to $\hat{\phi}_{IJ}(E_I)$. Hence, if we write $\mathrm{pr}_{N_{IJ}}(y)\colon E_J\to \mathrm{d}_y s_J(N_{IJ,y})\subset E_J$ for the smooth family of projections with kernel $\hat{\phi}_{IJ}(E_I)$, we obtain a smooth family of linear maps

$$F_{\tilde{x}} := \mathrm{pr}_{N_{IJ}}(y)\circ \mathrm{d}_y s_J\colon \mathrm{T}_y U_J \longrightarrow E_J \quad \text{for } y=\tilde{\phi}_{IJ}(\tilde{x}),$$

with images $\mathrm{im}\,F_{\tilde{x}} = \mathrm{d}_y s_J(N_{IJ,y})$, and also isomorphisms to their kernel

$$\phi_{\tilde{x}} := \mathrm{d}_{\tilde{x}}\tilde{\phi}_{IJ}\colon \mathrm{T}_{\tilde{x}}\tilde{U}_{IJ} \xrightarrow{\cong} \ker F_{\tilde{x}} = \mathrm{T}_y(\mathrm{im}\,\tilde{\phi}_{IJ})\subset \mathrm{T}_y U_J.$$

By Lemma 3.1.7 these induce isomorphisms

$$\mathfrak{C}^{\phi_{\tilde{x}}}_{F_{\tilde{x}}}\colon \Lambda^{\max}\mathrm{T}_{\tilde{\phi}_{IJ}(\tilde{x})}U_J\otimes(\Lambda^{\max}E_J)^* \xrightarrow{\cong} \Lambda^{\max}\mathrm{T}_{\tilde{x}}\tilde{U}_{IJ}\otimes(\Lambda^{\max}(E_J/\,\mathrm{im}\,F_{\tilde{x}}))^*.$$

We may combine this with the isomorphism $\Lambda^{\max}\mathrm{T}_{\tilde{x}}\tilde{U}_{IJ}\to \rho_{IJ}^*(\Lambda^{\max}\mathrm{T}_x U_I)$ induced by $\mathrm{d}_{\tilde{x}}\rho_{IJ}$, where $x:=\rho_{IJ}(\tilde{x})$, and the dual of the isomorphism

$$\Lambda^{\max}(E_J/\mathrm{d}_y s_J(N_{IJ,y}))\cong \Lambda^{\max}E_I$$

induced via (3.1.2) by $\mathrm{pr}^{\perp}_{N_{IJ}}(y)\circ\hat{\phi}_{IJ}\colon E_I\to E_J/\mathrm{d}_y s_J(N_{IJ,y})$, to obtain for each $\tilde{x}\in\tilde{U}_{IJ}$ an isomorphism

$$\tilde{\mathfrak{C}}_{IJ}(\tilde{x})\colon \Lambda^{\max}\mathrm{T}_y U_J\otimes(\Lambda^{\max}E_J)^* \xrightarrow{\cong} \rho_{IJ}^*(\Lambda^{\max}\mathrm{T}_x U_I)\otimes(\Lambda^{\max}E_I)^* \tag{3.1.15}$$

with

$$y:=\tilde{\phi}_{IJ}(\tilde{x}),\qquad x:=\rho_{IJ}(\tilde{x}),$$

given by the composite of $\mathfrak{C}^{\phi_{\tilde{x}}}_{F_{\tilde{x}}}$ with the map

$$\begin{aligned}(\Lambda_{\mathrm{d}_{\tilde{x}}\rho_{IJ}})\otimes\big(\Lambda_{(\mathrm{pr}^{\perp}_{N_{IJ}}(y)\circ\hat{\phi}_{IJ})^{-1}}\big)^*\colon \Lambda^{\max}\mathrm{T}_{\tilde{x}}\tilde{U}_{IJ}\otimes(\Lambda^{\max}(E_J/\,\mathrm{im}\,F_{\tilde{x}}))^*&\\ \to \rho_{IJ}^*(\Lambda^{\max}\mathrm{T}_x U_I)\otimes(\Lambda^{\max}E_I)^*.&\end{aligned}$$

[7] Here and subsequently, we will distinguish between the manifold $\tilde{U}_{IJ}$ and its image $\mathrm{im}\,\tilde{\phi}_{IJ}$ in U_J, denoting points of $\tilde{U}_{IJ}$ by $\tilde{x}$, with $y=\tilde{\phi}_{IJ}(x)\in U_J$ and $x=\rho_{IJ}(\tilde{x})\in U_{IJ}$.

Proposition 3.1.13 (i) *Let $\mathcal{K}$ be a weak Kuranishi atlas. Then there is a well-defined line bundle* $\det(\mathcal{K})$ *over* $\mathcal{K}$ *given by the line bundles*

$$\Lambda_I^{\mathcal{K}} := \Lambda^{\max}\mathrm{T}U_I \otimes (\Lambda^{\max}E_I)^* \to U_I \quad \text{for } I \in \mathcal{I}_{\mathcal{K}},$$

with group actions as in (3.1.14) *and the transition maps*

$$\widetilde{\mathfrak{C}}_{IJ}^{-1}\colon \rho_{IJ}^*(\Lambda_I^{\mathcal{K}}|_{U_{IJ}}) \to \Lambda_J^{\mathcal{K}}|_{\mathrm{im}\,\widetilde{\phi}_{IJ}}$$

from (3.1.15) *for $I \subsetneq J$. In particular, the latter isomorphisms are independent of the choice of normal bundle N_{IJ}.*
Furthermore, the contractions $\mathfrak{C}_{\mathrm{d}s_I}\colon \Lambda_I^{\mathcal{K}} \to \det(\mathrm{d}s_I)$ from (3.1.13) *define an isomorphism $\Psi^{s_{\mathcal{K}}} := (\mathfrak{C}_{\mathrm{d}s_I})_{I\in\mathcal{I}_{\mathcal{K}}}$ from $\det(\mathcal{K})$ to $\det(\mathfrak{s}_{\mathcal{K}})$.*

(ii) *If $\mathcal{K}$ is a weak Kuranishi cobordism, then the determinant bundle $\det(\mathcal{K})$ defined as in (i) can be given a product structure on the collar so that its boundary restrictions are $\det(\mathcal{K})|_{\partial^\alpha\mathcal{K}} = \det(\partial^\alpha\mathcal{K})$ for $\alpha = 0, 1$.*
Further, the isomorphism $\Psi^{s_{\mathcal{K}}}\colon \det(\mathcal{K}) \to \det(s_{\mathcal{K}})$ defined as in (i) has product structure on the collar with restrictions $\Psi^{s_{\mathcal{K}}}|_{\partial^\alpha\mathcal{K}} = \Psi^{s_{\partial^\alpha\mathcal{K}}}$ for $\alpha = 0, 1$.

Proof To begin, note that each $\Lambda_I^{\mathcal{K}} = \Lambda^{\max}\mathrm{T}U_I \otimes (\Lambda^{\max}E_I)^*$ is a smooth line bundle over U_I, since it inherits local trivializations from the tangent bundle $\mathrm{T}U_I \to U_I$. Moreover the action of Γ_I on $U_I \times E_I$ induces a smooth action on $\Lambda_I^{\mathcal{K}}$ given by (3.1.14) that covers its action on U_I. Thus $\Lambda_I^{\mathcal{K}} \to U_I$ is a smooth Γ_I–equivariant bundle. We showed in [14, Proposition 8.1.12] that the isomorphisms $\mathfrak{C}_{\mathrm{d}_x s_I}$ from (3.1.13) are smooth in this trivialization, where $\det(\mathrm{d}s_I)$ is trivialized via the maps $\widehat{T}_{I,x}$ as in Proposition 3.1.9. Since $\mathfrak{C}_{\mathrm{d}s_I}$ is equivariant, we can define preliminary transition maps

$$\widetilde{\phi}_{IJ}^{\Lambda} := \mathfrak{C}_{\mathrm{d}s_J}^{-1} \circ \widetilde{\Lambda}_{IJ} \circ \rho_{IJ}^*(\mathfrak{C}_{\mathrm{d}s_I})\colon \rho_{IJ}^*(\Lambda_I^{\mathcal{K}}|_{U_{IJ}}) \to \Lambda_J^{\mathcal{K}} \quad \text{for } I \subsetneq J \in \mathcal{I}_{\mathcal{K}} \tag{3.1.16}$$

by the transition maps (3.1.5) of $\det(\mathfrak{s}_{\mathcal{K}})$, the isomorphisms (3.1.13) and the pullback by ρ_{IJ}. These define a line bundle

$$\Lambda^{\mathcal{K}} := (\Lambda_I^{\mathcal{K}}, \widetilde{\phi}_{IJ}^{\Lambda})_{I,J\in\mathcal{I}_{\mathcal{K}}}$$

since the weak cocycle condition follows directly from that for the $\widetilde{\Lambda}_{IJ}$. Moreover, this automatically makes the family of bundle isomorphisms $\Psi^{\mathcal{K}} := (\widetilde{\mathfrak{C}}_{\mathrm{d}s_I})_{I\in\mathcal{I}_{\mathcal{K}}}$ an isomorphism from $\Lambda^{\mathcal{K}}$ to $\det(\mathfrak{s}_{\mathcal{K}})$. It remains to see that $\Lambda^{\mathcal{K}} = \det(\mathcal{K})$ and $\Psi^{\mathcal{K}} = \Psi^{\mathfrak{s}_{\mathcal{K}}}$, ie we claim equality of transition maps $\widetilde{\phi}_{IJ}^{\Lambda} = \widetilde{\mathfrak{C}}_{IJ}^{-1}$. This also shows that $\widetilde{\mathfrak{C}}_{IJ}^{-1}$ and thus $\det(\mathcal{K})$ is independent of the choice of normal bundle N_{IJ} in (3.1.15).

So to finish the proof of (i), it suffices to establish the following commuting diagram at a fixed $\widetilde{x} \in U_{IJ}$ with $x = \rho_{IJ}(\widetilde{x})$, $y = \widetilde{\phi}_{IJ}(\widetilde{x})$:

$$\begin{tikzcd}
\Lambda^{\max}\mathrm{T}_x U_I \otimes (\Lambda^{\max} E_I)^* \arrow[r, "\mathfrak{C}_{\mathrm{d}_x s_I}"] & \det(\mathrm{d}_x s_I) \\
\rho^*_{IJ}(\Lambda^{\max}\mathrm{T}_x U_I) \otimes (\Lambda^{\max} E_I)^* \arrow[u, "\rho_{IJ}"] \arrow[r, "\rho^*_{IJ}(\mathfrak{C}_{\mathrm{d}_x s_I})"] & \rho^*_{IJ}(\det(\mathrm{d}_x s_I)) \arrow[u, "\rho_{IJ}"'] \arrow[d, "\widetilde{\Lambda}_{IJ}(\widetilde{x})"] \\
\Lambda^{\max}\mathrm{T}_y U_J \otimes (\Lambda^{\max} E_J)^* \arrow[u, "\widetilde{\mathfrak{C}}_{IJ}(\widetilde{x})"] \arrow[r, "\mathfrak{C}_{\mathrm{d}_y s_J}"] & \det(\mathrm{d}_y s_J)
\end{tikzcd} \tag{3.1.17}$$

However, the composition $y \mapsto \rho_{IJ} \circ \widetilde{\mathfrak{C}}_{IJ}(\widetilde{\phi}_{IJ}^{-1}(\widetilde{x}))$ of the left-hand vertical maps is precisely the map denoted by $y \mapsto \mathfrak{C}_{IJ}(x)$ in [14, Equation (8.1.15)], while, as in the proof of Proposition 3.1.9 above, the right-hand vertical maps combine to $\Lambda_{IJ}(x) = \widetilde{\Lambda}(\rho_{IJ}^{-1}(x))$: $\det(\mathrm{d}_x s_I) \to \det(\mathrm{d}_y s_J)$, where ρ_{IJ}^{-1} is a local inverse to ρ_{IJ}. Therefore the desired result follows from the commutativity of the diagram

$$\begin{tikzcd}
\Lambda^{\max}\mathrm{T}_x U_I \otimes (\Lambda^{\max} E_I)^* \arrow[r, "\mathfrak{C}_{\mathrm{d}_x s_I}"] & \det(\mathrm{d}_x s_I) \arrow[d, "\Lambda_{IJ}(x)"] \\
\Lambda^{\max}\mathrm{T}_y U_J \otimes (\Lambda^{\max} E_J)^* \arrow[u, "\mathfrak{C}_{IJ}(x)"] \arrow[r, "\mathfrak{C}_{\mathrm{d}_y s_J}"] & \det(\mathrm{d}_y s_J)
\end{tikzcd}$$

which is established in [14, Proposition 8.1.12].

For part (ii) the same arguments apply to define a bundle $\det(\mathcal{K})$ and isomorphism $\Psi^{\mathfrak{s}_\mathcal{K}}$. The required product structure on a collar follows as in [14]. □

We end this section by explaining how orientations of a Kuranishi atlas induce compatible orientations on local zero sets of transverse sections.

Lemma 3.1.14 *Let $(\mathcal{K}, \sigma)$ be a d–dimensional oriented, tame Kuranishi atlas or cobordism, and for some $I \in \mathcal{I}_\mathcal{K}$ let $f\colon W \to E_I$ be a smooth section over an open subset $W \subset U_I$ that is transverse to 0.*

(i) *The zero set $Z_f := f^{-1}(0) \subset U_I$ inherits the structure of a smooth oriented d–dimensional submanifold.*

(ii) *The action of any $\gamma \in \Gamma_I$ on U_I induces an orientation-preserving diffeomorphism $Z_f \to Z_{\gamma * f}$ to the zero set of $\gamma * f\colon \gamma(W) \to E_I$, $x \mapsto \gamma f(\gamma^{-1}(x))$.*

(iii) *Suppose further that $f(W) \subset \widehat{\phi}_{HI}(E_H)$, $\widetilde{W}_{HI} := W \cap \widetilde{U}_{HI} \neq \varnothing$ and $\rho_{HI}|_{\widetilde{W}_{HI}}$ is injective for some $H \subset I$. Then ρ_{HI} induces an orientation-preserving diffeomorphism $Z_f \to Z_{\rho_{HI} * f}$ to the zero set of $\rho_{HI} * f\colon \rho_{HI}(W) \to E_H$, $x \mapsto \widehat{\phi}_{HI}^{-1}(f(\rho_{HI}^{-1}(x)))$.*

(iv) *If $\mathcal{K}$ is a cobordism, suppose in addition that $\boldsymbol{K}_I$ is a chart that intersects the boundary $\partial^\alpha\mathcal{K}$, with $W = \iota_I^\alpha(A_\varepsilon^\alpha \times W^\alpha)$ for some $W^\alpha \subset \partial^\alpha W$, and $f(\iota_I^\alpha(t,x)) = f^\alpha(x)$ for some transverse section $f^\alpha\colon W^\alpha \to E_I$. Then the atlas $(\partial^\alpha\mathcal{K}, \partial^\alpha\sigma)$ induces an oriented smooth structure on $Z_{f^\alpha} \subset W^\alpha$ by* (i)*, $Z_f \subset U_I$ is a submanifold with boundary and $j_I^\alpha := \iota_I(\alpha,\cdot)$ is a diffeomorphism $Z_{f^\alpha} \to \partial Z_f$ that preserves (resp. reverses) orientations when $\alpha = 1$ (resp. $\alpha = 0$).*

Proof Except for (ii) these local claims follow directly from the corresponding parts of the proof of [14, Proposition 8.1.13]. For (iii) note that the injectivity assumption allows us to write $\rho_{HI}{*}f = \phi_{HI}^* f$ for an embedding $\phi_{HI}\colon \rho_{HI}(\widetilde{W}_{HI}) \to W$. Before we can prove (ii), recall that the orientation on Z_f is induced from the orientation of the Kuranishi atlas/cobordism $\sigma_I\colon U_I \to \det(\mathrm{d}s_I)$ via the isomorphisms for $z \in Z_f$,

$$\Lambda^{\max}\mathrm{T}_z Z_f = \Lambda^{\max}\ker \mathrm{d}_z f \cong \Lambda^{\max}\ker \mathrm{d}_z f \otimes \mathbb{R} = \det(\mathrm{d}_z f),$$

$$\mathfrak{C}_{\mathrm{d}_z f}\colon \det(\mathrm{d}_z f) \to \Lambda^{\max}\mathrm{T}_z U_I \otimes (\Lambda^{\max} E_I)^*,$$

$$\mathfrak{C}_{\mathrm{d}_z s_I}\colon \det(\mathrm{d}_z s_I) \to \Lambda^{\max}\mathrm{T}_z U_I \otimes (\Lambda^{\max} E_I)^*.$$

Now to prove that $\gamma \in \Gamma_I$ acts by an orientation-preserving diffeomorphism, note that a smooth group action always acts by diffeomorphisms. Restriction to Z_f of the action by $\gamma \in \Gamma_I$ thus yields a diffeomorphism to its image, which is easily seen to be the zero set of $\gamma{*}f$. To show that this diffeomorphism is compatible with the induced orientations at $z \in Z_f$ and $\gamma z \in Z_{\gamma * f}$, we begin by noting that the action of γ is $\Lambda_{\mathrm{d}_z\gamma}\colon \Lambda^{\max}\mathrm{T}_z Z_f \to \Lambda^{\max}\mathrm{T}_{\gamma z} Z_{\gamma * f}$. On the other hand, the orientations $\sigma_I(z)$ and $\sigma_I(\gamma z)$ are by assumption intertwined by the isomorphism $\Lambda_{\mathrm{d}_z\gamma} \otimes (\Lambda_{[\gamma^{-1}]})^*\colon \det \mathrm{d}_z s_I \to \det \mathrm{d}_{\gamma z} s_I$, and by Proposition 3.1.13(i) this implies that their pullbacks to $\Lambda^{\max}\mathrm{T}_x U_I \otimes (\Lambda^{\max} E_I)^*$ for $x = z, \gamma z$ are intertwined by $\Lambda_{\mathrm{d}_z\gamma} \otimes (\Lambda_{\gamma^{-1}})^*$. Thus it remains to prove that the following diagram commutes:

$$\begin{tikzcd}
\Lambda^{\max}\mathrm{T}_z U_I \otimes (\Lambda^{\max} E_I)^* \arrow[r, "\mathfrak{C}_{\mathrm{d}_z f}"] \arrow[d, "\Lambda_{\mathrm{d}_z\gamma}\otimes(\Lambda_{\gamma^{-1}})^*"'] & \Lambda^{\max}\ker(\mathrm{d}_z f) \otimes \mathbb{R} \cong \Lambda^{\max}\mathrm{T}_z Z_f \arrow[d, "\Lambda_{\mathrm{d}_z\gamma}\otimes \mathrm{id}_{\mathbb{R}}"] \\
\Lambda^{\max}\mathrm{T}_{\gamma z} U_I \otimes (\Lambda^{\max} E_I)^* \arrow[r, "\mathfrak{C}_{\mathrm{d}_{\gamma z}(\gamma * f)}"] & \Lambda^{\max}\ker(\mathrm{d}_{\gamma z}(\gamma * f)) \otimes \mathbb{R} \cong \Lambda^{\max}\mathrm{T}_{\gamma z} Z_{\gamma * f}
\end{tikzcd}$$

By Lemma 3.1.7, $\mathfrak{C}_{\mathrm{d}_z f}$ is given by $(v_1 \wedge \cdots \wedge v_n) \otimes (w_1 \wedge \cdots \wedge w_m)^* \mapsto v_1 \wedge \cdots \wedge v_k$, where $v_1, \dots, v_n$ is any basis for $\mathrm{T}_z U_I$ whose first k elements span $\ker \mathrm{d}_z f$, and $w_1, \dots, w_m$ is a basis for E_I, and similarly for $\mathfrak{C}_{\mathrm{d}_{\gamma z}(\gamma * f)}$. Therefore, if we denote $v_i' := \mathrm{d}_z\gamma(v_i)$ and $w_j' := \gamma w_j$, we find that $(\Lambda_{\gamma^{-1}})^*(w_1 \wedge \cdots \wedge w_m)^* = (w_1' \wedge \cdots \wedge w_m')^*$

and thus the diagram commutes as required:

$$\begin{aligned}\mathfrak{C}_{\mathrm{d}_{\gamma z}(\gamma * f)}\big(\Lambda_{\mathrm{d}_z\gamma} \otimes (\Lambda_{\gamma^{-1}})^*((v_1\wedge\cdots\wedge v_n)\otimes(w_1\wedge\cdots\wedge w_m)^*)\big)&\\ &= \mathfrak{C}_{\mathrm{d}_{\gamma z}(\gamma * f)}((v_1'\wedge\cdots\wedge v_n')\otimes(w_1'\wedge\cdots\wedge w_m')^*)\\ &= v_1'\wedge\cdots\wedge v_k'\\ &= \Lambda_{\mathrm{d}_z\gamma}(v_1\wedge\cdots\wedge v_k)\\ &= \Lambda_{\mathrm{d}_z\gamma}\big(\mathfrak{C}_{\mathrm{d}_z f}((v_1\wedge\cdots\wedge v_n)\otimes(w_1\wedge\cdots\wedge w_m)^*)\big). \qquad \square\end{aligned}$$

3.2 Perturbed zero sets

With Theorem 2.5.3 providing existence and uniqueness of tame shrinkings, the second part of the proof of Theorem A is the construction of the VMC/VFC from the zero sets of suitable perturbations $\mathfrak{s}_{\mathcal{K}} + \nu$ of the canonical section $\mathfrak{s}_{\mathcal{K}}$ of a tame Kuranishi atlas or cobordism. In this section, we describe a suitable class of perturbations ν, and prove that the corresponding perturbed zero sets are compact weighted branched manifolds, a notion from [8] that we review in the appendix. The existence and uniqueness of such perturbations will be established in Section 3.3, as part of the perturbative construction of VMC and VFC. The main work is done by the setup in this section, which will put us into a situation in which the construction of perturbations and the resulting VMC/VFC can essentially be copied from [14]. Since the construction of perturbations requires tameness and the notion of weighted branched manifolds requires an orientation in [8], we will — unless otherwise stated — work with an oriented tame Kuranishi atlas or cobordism $\mathcal{K}$.

As in the case of trivial isotropy, one cannot in general find transverse perturbations $s_I + \nu_I \pitchfork 0$ that are also compatible with the coordinate changes $\widehat{\Phi}_{IJ}$. Instead, we will construct perturbations over the following notion of a reduced atlas that still covers X but generally does not form a Kuranishi atlas.

Definition 3.2.1 [13, Definition 5.1.2] A *(cobordism) reduction* of a tame Kuranishi atlas or cobordism $\mathcal{K}$ is an open subset $\mathcal{V} = \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} V_I \subset \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}}$, ie a tuple of (possibly empty) open subsets $V_I \subset U_I$ satisfying the following conditions:

- (i) $V_I = \pi_I^{-1}(\underline{V}_I)$ for each $I \in \mathcal{I}_{\mathcal{K}}$, ie V_I is pulled back from the intermediate category and so is Γ_I–invariant.
- (ii) $V_I \sqsubset U_I$ for all $I \in \mathcal{I}_{\mathcal{K}}$, and if $V_I \neq \varnothing$ then $V_I \cap s_I^{-1}(0) \neq \varnothing$.
- (iii) If $\pi_{\mathcal{K}}(\overline{V_I}) \cap \pi_{\mathcal{K}}(\overline{V_J}) \neq \varnothing$ then $I \subset J$ or $J \subset I$.
- (iv) The zero set $\iota_{\mathcal{K}}(X) = |s_{\mathcal{K}}|^{-1}(0)$ is contained in $\pi_{\mathcal{K}}(\mathcal{V}) = \bigcup_{I\in\mathcal{I}_{\mathcal{K}}} \pi_{\mathcal{K}}(V_I)$.

If $\mathcal{K}$ is a cobordism, we require in addition that $\mathcal{V}$ is collared in the following sense:

(v) For each $\alpha \in \{0,1\}$ and $I \in \mathcal{I}_{\partial^\alpha \mathcal{K}} \subset \mathcal{I}_{\mathcal{K}}$, there exists an $\varepsilon > 0$ and a subset $\partial^\alpha V_I \subset \partial^\alpha U_I$ such that $\partial^\alpha V_I \neq \varnothing$ if and only if $V_I \cap \psi_I^{-1}(\partial^\alpha F_I) \neq \varnothing$, and

$$(\iota_I^\alpha)^{-1}(V_I) \cap (A_\varepsilon^\alpha \times \partial^\alpha U_I) = A_\varepsilon^\alpha \times \partial^\alpha V_I.$$

We call $\partial^\alpha \mathcal{V} := \bigsqcup_{I \in \mathcal{I}_{\partial^0 \mathcal{K}}} \partial^\alpha V_I \subset \mathrm{Obj}_{\boldsymbol{B}_{\partial^\alpha \mathcal{K}}}$ the *boundary restriction* of $\mathcal{V}$ to $\partial^\alpha \mathcal{K}$.

Remark 3.2.2 (i) The notion of (cobordism) reduction is equivalent to saying that $\mathcal{V} := \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} V_I \subset \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}}$ is the lift $V_I := \pi_I^{-1}(\underline{V}_I)$ of a (cobordism) reduction $\underline{\mathcal{V}} = \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} \underline{V}_I \subset \mathrm{Obj}_{\boldsymbol{B}_{\underline{\mathcal{K}}}}$ of the intermediate Kuranishi atlas/cobordism. Thus existence and uniqueness of reductions is proven in [13, Theorem 5.1.6].

(ii) The restrictions $\partial^\alpha \mathcal{V}$ of a cobordism reduction $\mathcal{V}$ of a Kuranishi cobordism $\mathcal{K}$ are reductions of the restricted Kuranishi atlases $\partial^\alpha \mathcal{K}$ for $\alpha = 0, 1$. In particular, condition (ii) holds because part (v) of Definition 3.2.1 implies that if $\partial^\alpha V_I \neq \varnothing$ then $\partial^\alpha V_I \cap \psi_I^{-1}(\partial^\alpha F_I) \neq \varnothing$. Note that condition (v) also implies that $V_I \subset U_I$ is a collared subset in the sense of (2.4.1). ◇

Given a reduction $\mathcal{V}$, we define the *reduced domain category* $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}$ and the *reduced obstruction category* $\boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}$ to be the full subcategories of $\boldsymbol{B}_{\mathcal{K}}$ and $\boldsymbol{E}_{\mathcal{K}}$ with objects $\bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} V_I$ and $\bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} V_I \times E_I$ respectively, and denote by $\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}} \colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}$ the section given by restriction of $\mathfrak{s}_{\mathcal{K}}$. Now one might hope to find transverse perturbation functors $\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}} + \nu \colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}$ by iteratively constructing $\nu_I \colon V_I \to E_I$ as in [14], where compatibility with the morphisms can be ensured by working along the partial order $\subsetneq$ on $\mathcal{I}_{\mathcal{K}}$, using the separation property (iii) of a reduction. However, we also have to ensure compatibility with the morphisms given by the action of nontrivial isotropy groups Γ_I. Depending on their action, we might not even be able to even find a Γ_I–equivariant perturbation ν_I in a single chart such that $s_I + \nu_I \pitchfork 0$. In general, this can be resolved by using multivalued perturbations such as in the perturbative construction of the Euler class of an orbibundle, explained for example in [6] as motivation for perturbations in Kuranishi structures. We could also formulate our perturbation scheme in these terms, but due to the particularly simple setup — notably additivity $\Gamma_I = \prod_{i \in I} \Gamma_i$ of the isotropy groups — we can construct the "multivalued perturbations" as single-valued section functors $\nu \colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ over a *pruned domain category* $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$, which is obtained in Lemma 3.2.3 from the reduced domain category $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}$ by forgetting sufficiently many morphisms to obtain trivial isotropy. It is to this category that the iterative perturbation scheme of [14] will be applied in Section 3.3 to obtain a suitable class of transverse perturbations ν. Once a zero set is cut out transversely from $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$, we will then show in Theorem 3.2.8 that adding some of the isotropy morphisms back in — at the expense of adding weights to corresponding branches of the solution set — yields the structure of a weighted branched manifold on the Hausdorff quotient of the perturbed solution set $\big|(\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}+\nu)^{-1}(0)\big| \subset \big|\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}\big|$. This perturbed solution set is not a subset of the virtual neighborhood $|\mathcal{K}|$, but its Hausdorff quotient supports a fundamental class by Proposition A.7, and the inclusion $\iota^{\nu}\colon (\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}+\nu)^{-1}(0) \to \mathrm{Obj}_{\mathcal{B}_{\mathcal{K}}}$ induces a continuous map $|\iota^{\nu}|_{\mathcal{H}}\colon \big|(\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}+\nu)^{-1}(0)\big|_{\mathcal{H}} \to |\mathcal{K}|$ that will represent the virtual fundamental cycle of $\mathcal{K}$.

We will describe the pruned categories in terms of the sets

$$\widetilde{V}_{IJ} := V_J \cap \rho_{IJ}^{-1}(V_I) = V_J \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_I)) \subset \widetilde{U}_{IJ}.$$

Note that $\widetilde{V}_{IJ}$ is invariant under the action of Γ_J, and is an open subset of the closed submanifold $\widetilde{U}_{IJ} = s_J^{-1}(E_I)$ of V_J, where the last equality holds by the tameness condition (2.5.2). Further if $F \subset I \subset J$,

$$V_J \cap \rho_{IJ}^{-1}(\widetilde{V}_{FI}) = \widetilde{V}_{IJ} \cap \widetilde{V}_{FJ} = V_J \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_I) \cap \pi_{\mathcal{K}}(V_F)) \subset \widetilde{U}_{FJ}. \tag{3.2.1}$$

In fact, the second equality above holds for any pair of subsets $F, I \subset J$. However, because $\mathcal{V}$ is a reduction, the intersection is empty unless F and I are nested, ie either $F \subset I$ or $I \subset F$. Finally, the group $\Gamma_{I\setminus F}$ acts freely on $\widetilde{U}_{FI}$ (by Definition 2.2.8 for a coordinate change), and hence also on $\widetilde{V}_{FI}$. If $I = F$ we define $\Gamma_{I\setminus F} := \Gamma_{\varnothing} := \{\mathrm{id}\}$.

Lemma 3.2.3 *Let $\mathcal{V}$ be a (cobordism) reduction of a tame Kuranishi atlas or cobordism $\mathcal{K}$. Then there are well-defined categories — the* ***pruned domain category*** $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ *and the* ***pruned obstruction category*** $\boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ *— obtained from $\boldsymbol{B}_{\mathcal{K}}$ and $\boldsymbol{E}_{\mathcal{K}}$ as follows:*

- *Object spaces are given by restriction to the reduction $\mathcal{V} = \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} V_I \subset \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}}$:*

$$\mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}} := \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} V_I \subset \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}}, \quad \mathrm{Obj}_{\boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}} := \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} V_I \times E_I \subset \mathrm{Obj}_{\boldsymbol{E}_{\mathcal{K}}}.$$

- *Morphism spaces are open subsets of $\mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}}$ and $\mathrm{Mor}_{\boldsymbol{E}_{\mathcal{K}}}$ respectively, with components*

$$\mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}} := \bigsqcup_{I,J\in\mathcal{I}_{\mathcal{K}}} \mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}}(V_I, V_J), \quad \mathrm{Mor}_{\boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}} := \bigsqcup_{I,J\in\mathcal{I}_{\mathcal{K}}} \mathrm{Mor}_{\boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}}(V_I, V_J),$$

given by $\mathrm{Mor}_{\dots}(V_I, V_J) = \varnothing$ unless $I \subset J$, in which case the morphisms are given in terms of the open subsets $\widetilde{V}_{IJ} := V_J \cap \rho_{IJ}^{-1}(V_I) \subset \widetilde{U}_{IJ}$ as

$$\mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}}(V_I, V_J) := \widetilde{V}_{IJ} \times \{\mathrm{id}\} \subset \widetilde{U}_{IJ} \times \Gamma_I = \mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}}(U_I, U_J),$$
$$\mathrm{Mor}_{\boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}}(V_I, V_J) := \widetilde{V}_{IJ} \times E_I \times \{\mathrm{id}\} \subset \widetilde{U}_{IJ} \times E_I \times \Gamma_I = \mathrm{Mor}_{\boldsymbol{E}_{\mathcal{K}}}(U_I, U_J).$$

- *All structure maps (source, target, identity, and composition) are given by restriction of the respective structure maps of $\boldsymbol{B}_{\mathcal{K}}$ and $\boldsymbol{E}_{\mathcal{K}}$ in Definition 2.3.5.*

These pruned categories are nonsingular in the sense that there is at most one morphism between any two objects. Moreover, the projection and section functors $\mathrm{pr}_{\mathcal{K}}\colon \boldsymbol{E}_{\mathcal{K}} \to \boldsymbol{B}_{\mathcal{K}}$ *and* $\mathfrak{s}_{\mathcal{K}}\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}}$ *restrict to well-defined functors* $\mathrm{pr}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ *and* $\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ *with* $\mathrm{pr}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} \circ \mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} = \mathrm{id}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}}$.

Proof Recall that $(I, J, y, \mathrm{id}) \in \mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}}$ has source $(I, \rho_{IJ}(y))$ and target (J, y) (where, as in Lemma 2.3.6, we suppress mention of the inclusion $\tilde{\phi}_{IJ}$). Now morphisms are closed under composition because the strong cocycle condition guarantees that $\rho_{IJ} \circ \rho_{JK} = \rho_{IK}$, with identical domains whenever $I \subset J \subset K$. Moreover, the category is nonsingular because source and target determine the morphism uniquely. Similar arguments apply to $\boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$. Finally, the projection and section functors of $\mathcal{K}$ act trivially on the isotropy groups Γ_I, and thus restrict to well-defined functors when we drop these. □

The following combines Definitions 7.2.1, 7.2.5, 7.2.6 and 7.2.9 from [14].

Definition 3.2.4 A *(cobordism) perturbation* of $\mathcal{K}$ is a smooth functor

$$\nu\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$$

between the pruned domain and obstruction categories of some (cobordism) reduction $\mathcal{V}$ of $\mathcal{K}$, such that $\mathrm{pr}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} \circ \nu = \mathrm{id}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}}$.

That is, $\nu = (\nu_I)_{I \in \mathcal{I}_{\mathcal{K}}}$ is given by a family of smooth maps $\nu_I\colon V_I \to E_I$ that are compatible with coordinate changes in the sense that for all $I \subsetneq J$ we have

$$\nu_J|_{\tilde{V}_{IJ}} = \hat{\phi}_{IJ} \circ \nu_I \circ \rho_{IJ}|_{\tilde{V}_{IJ}} \quad \text{on} \quad \tilde{V}_{IJ} = V_J \cap \rho_{IJ}^{-1}(V_I). \tag{3.2.2}$$

If $\mathcal{K}$ is a Kuranishi cobordism we require in addition that ν has product form in a collar neighborhood of the boundary. That is, for $\alpha = 0, 1$ and $I \in \mathcal{I}_{\mathcal{K}^\alpha} \subset \mathcal{I}_{\mathcal{K}}$ there is an $\varepsilon > 0$ and a map $\nu_I^\alpha\colon \partial^\alpha V_I \to E_I$ such that

$$\nu_I(\iota_I^\alpha(t, x)) = \nu_I^\alpha(x) \quad \forall x \in \partial^\alpha V_I,\ t \in A_\varepsilon^\alpha.$$

We say that a (cobordism) perturbation ν is

- *admissible* if we have $\mathrm{d}_y \nu_J(\mathrm{T}_y V_J) \subset \mathrm{im}\, \hat{\phi}_{IJ}$ for all $I \subsetneq J$ and $y \in \tilde{V}_{IJ}$;
- *transverse* if $s_I|_{V_I} + \nu_I\colon V_I \to E_I$ is transverse to 0 for each $I \in \mathcal{I}_{\mathcal{K}}$;
- *precompact* if there is a precompact open subset $\mathcal{C} \sqsubset \mathcal{V}$ which itself is a (cobordism) reduction, such that

$$\pi_{\mathcal{K}}\Bigl(\bigcup_{I \in \mathcal{I}_{\mathcal{K}}} (s_I|_{V_I} + \nu_I)^{-1}(0)\Bigr) \subset \pi_{\mathcal{K}}(\mathcal{C}). \tag{3.2.3}$$

Remark 3.2.5 Although $\pi_{\mathcal{K}}\colon \mathrm{Obj}_{\mathcal{B}_{\mathcal{K}}} \to |\mathcal{K}|$ is not induced by a functor on $\mathcal{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$, we will work with $\pi_{\mathcal{K}}\colon \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I \to |\mathcal{K}|$ as continuous map, in particular for the notion of precompactness. As in the case of trivial isotropy, we do not have a nicely controlled cover of sets $U_J \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(\mathcal{C}))$ for $\mathcal{C} \subset \bigsqcup U_I$. However, because $\mathcal{C} = \bigsqcup C_I \subset \mathcal{V} = \bigsqcup V_I \subset \bigsqcup U_I$ are lifts of reductions of $|\underline{\mathcal{K}}|$ as in Remark 3.2.2, the morphisms between V_J and $\mathcal{C}$ are better understood, yielding

$$V_J \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(\mathcal{C})) = V_J \cap \Bigl(\bigcup_{H\supset J} \rho_{JH}(C_H) \cup \bigcup_{H\subsetneq J} \rho_{HJ}^{-1}(C_H) \Bigr). \tag{3.2.4}$$

Indeed, by the reduction property, $\pi_{\mathcal{K}}(V_J)$ only intersects $\pi_{\mathcal{K}}(C_H)$ for $H \supset J$ or $H \subset J$. The morphisms between V_J and C_H are then given by ρ_{JH} and Γ_J in the first case, or ρ_{HJ} and Γ_H in the second, and the isotropy groups are absorbed by the equivariance $\Gamma_J \rho_{JH}(C_H) = \rho_{JH}(\Gamma_H C_H)$ and fact that $\Gamma_H C_H = C_H = \pi_H^{-1}(\underline{C}_H)$. As a result, we can write (3.2.3) in terms of the covering maps $(\rho_{IJ})_{I,J\in\mathcal{I}_{\mathcal{K}}}$, without explicit reference to the isotropy groups Γ_I, as

$$(s_J|_{V_J} + \nu_J)^{-1}(0) \subset \bigcup_{H\supset J} \rho_{JH}(C_H) \cup \bigcup_{H\subsetneq J} \rho_{HJ}^{-1}(C_H) \quad \forall J \in \mathcal{I}_{\mathcal{K}}. \tag{3.2.5}$$ ◇

Definition 3.2.6 Given a (cobordism) perturbation ν, the *perturbed zero set* $|\boldsymbol{Z}^\nu|$ is defined to be the realization of the full subcategory $\boldsymbol{Z}^\nu$ of $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ with object space

$$\bigl(\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} + \nu\bigr)^{-1}(0) := \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} (s_I|_{V_I} + \nu_I)^{-1}(0) \subset \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}}$$

given by the local zero sets $Z_I := (s_I|_{V_I} + \nu_I)^{-1}(0)$. That is, we equip

$$|\boldsymbol{Z}^\nu| := \bigl|(\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} + \nu)^{-1}(0)\bigr| = \Bigl(\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} Z_I\Bigr)\Big/{\sim}$$

with the quotient topology generated by the morphisms of $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$. Moreover, we denote by $\iota^\nu\colon \boldsymbol{Z}^\nu \to \boldsymbol{B}_{\mathcal{K}}$ the functor induced by the inclusion $(\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} + \nu)^{-1}(0) \to \mathrm{Obj}_{\boldsymbol{B}_{\mathcal{K}}}$ and corresponding inclusion of morphism spaces (to a generally not full subcategory), with resulting continuous map

$$|\iota^\nu|\colon |\boldsymbol{Z}^\nu| \to |\mathcal{K}|. \tag{3.2.6}$$

Remark 3.2.7 If $\nu\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ is a cobordism perturbation of a tame Kuranishi cobordism $\mathcal{K}$, then each *restriction* $\nu|_{\partial^\alpha\mathcal{V}} := (\nu_I^\alpha)_{I\in\mathcal{I}_{\mathcal{K}^\alpha}}$ for $\alpha = 0, 1$ forms a perturbation of the Kuranishi atlas $\partial^\alpha\mathcal{K}$ with respect to the boundary restriction $\partial^\alpha\mathcal{V}$ of the reduction.

If in addition ν is admissible/transverse/precompact, then so are the restrictions $\nu|_{\partial^\alpha\mathcal{V}}$. Moreover, in the case of transversality each perturbed section $s_I|_{V_I}+\nu_I\colon V_I\to E_I$ for $I\in\mathcal{I}_{\partial^0\mathcal{K}}\cup\mathcal{I}_{\partial^1\mathcal{K}}\subset\mathcal{I}_{\mathcal{K}}$ is transverse to 0 as a map on a domain with boundary; ie the kernel of its differential is transverse to the boundary

$$\partial V_I=\bigsqcup_{\alpha=0,1}\iota_I^\alpha(\{\alpha\}\times\partial^\alpha V_I). \qquad \diamond$$

Theorem 3.2.8 *Let $(\mathcal{K},\sigma)$ be an oriented tame Kuranishi atlas/cobordism of dimension d and let ν be an admissible, transverse, precompact (cobordism) perturbation of $\mathcal{K}$ with respect to nested (cobordism) reductions $\mathcal{C}\sqsubset\mathcal{V}\sqsubset\mathrm{Obj}_{\mathcal{B}_\mathcal{K}}$. Then $\boldsymbol{Z}^\nu$ can be completed to a compact, d–dimensional wnb (cobordism) groupoid $\widehat{\boldsymbol{Z}}^\nu$, in the sense of Definition A.4, with the same realization $|\widehat{\boldsymbol{Z}}^\nu|=|\boldsymbol{Z}^\nu|$. In addition,*

$$\Lambda^\nu(p):=|\Gamma_I|^{-1}\#\{z\in Z_I\mid\pi_H(|z|)=p\},\quad \text{for } p\in|Z_I|_{\mathcal{H}},$$

defines a weighting function $\Lambda^\nu\colon|\boldsymbol{Z}^\nu|_{\mathcal{H}}\to\mathbb{Q}^+$ on the Hausdorff quotient of the perturbed zero set $|\boldsymbol{Z}^\nu|_{\mathcal{H}}$. Together, these give $(|\widehat{\boldsymbol{Z}}^\nu|_{\mathcal{H}},\Lambda^\nu)$ the structure of a compact, d–dimensional weighted branched manifold/cobordism, in the sense of Definition A.5. It defines a cycle in $|\mathcal{C}|$ in the sense that the map $|\iota^\nu|_{\mathcal{H}}\colon|\widehat{\boldsymbol{Z}}^\nu|_{\mathcal{H}}\to|\mathcal{K}|$ induced by (3.2.6) has image in $|\mathcal{C}|$.

Moreover, if $\mathcal{K}$ is a Kuranishi cobordism and the boundary restrictions of ν are denoted $\nu^\alpha:=\nu|_{\partial^\alpha\mathcal{V}}$, then $(\widehat{\boldsymbol{Z}}^\nu,\Lambda^\nu)$ has oriented boundaries $(\widehat{\boldsymbol{Z}}^{\nu^0},\Lambda^{\nu^0})$ and $(\widehat{\boldsymbol{Z}}^{\nu^1},\Lambda^{\nu^1})$, and the cycle $|\iota^\nu|_{\mathcal{H}}\colon|\widehat{\boldsymbol{Z}}^\nu|_{\mathcal{H}}\to|\mathcal{C}|$ restricts on the boundaries to $|\iota^{\nu^\alpha}|_{\mathcal{H}}\colon|\widehat{\boldsymbol{Z}}^{\nu^\alpha}|_{\mathcal{H}}\to|\partial^\alpha\mathcal{C}|$.

We begin the proof of Theorem 3.2.8 by explaining the structure of the groupoid completion $\widehat{\boldsymbol{Z}}^\nu$. Note that the compatibility condition (3.2.2) implies partial equivariance of the perturbation: $\nu_J(\alpha y)=\nu_J(y)$ for $y\in\widetilde{V}_{IJ},\alpha\in\Gamma_{J\setminus I}$. This fact is reflected in the structure of the morphisms in the groupoid $\widehat{\boldsymbol{Z}}^\nu$, which contain this action of $\Gamma_{J\setminus I}$ on $\widetilde{V}_{IJ}\cap Z_J$ as part of the morphism space $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_J,Z_J)$.

Lemma 3.2.9 *Let ν be any (cobordism) perturbation of a tame d–dimensional Kuranishi atlas/cobordism $\mathcal{K}$.*

(i) *There is a unique nonsingular groupoid $\widehat{\boldsymbol{Z}}^\nu$ with the same objects and realization as $\boldsymbol{Z}^\nu$. Its morphism space for $I\subset J$ is given by*

$$\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_I,Z_J):=\bigcup_{\varnothing\ne F\subset I}(Z_J\cap\widetilde{V}_{IJ}\cap\widetilde{V}_{FJ})\times\Gamma_{I\setminus F}\subset\widetilde{U}_{IJ}\times\Gamma_I=\mathrm{Mor}_{\boldsymbol{B}_\mathcal{K}}(U_I,U_J).$$

(ii) *If ν is admissible and transverse, then the subsets $Z_J\cap\widetilde{V}_{IJ}\subset Z_J$ are open for all $I\subset J$ and the groupoid $\widehat{\boldsymbol{Z}}^\nu$ is étale and has dimension d. Further, $\widehat{\boldsymbol{Z}}^\nu$ is oriented if in addition $\mathcal{K}$ is oriented.*

(iii) *If $\mathcal{K}$ is an oriented tame Kuranishi cobordism and ν is admissible and transverse, then $\widehat{\boldsymbol{Z}}^\nu$ satisfies all conditions given in the appendix for being an étale, oriented, cobordism groupoid, except possibly that of compactness.*

Proof First note that there is at most one nonsingular groupoid with the same objects and realization as $\boldsymbol{Z}^\nu$, since any such groupoid has a unique morphism $(I,x)\mapsto(J,y)$ whenever $(I,x)\sim(J,y)$, where $\sim$ is the equivalence relation on $\mathrm{Obj}_{\boldsymbol{Z}^\nu}$ generated by $\mathrm{Mor}_{\boldsymbol{Z}^\nu}$. To prove existence of such a groupoid, we show below that when $I\subset J$,

(a) each element in $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_I,Z_J)$ is uniquely determined by its source and target;

(b) if there is a morphism $(I,J,y,\alpha)\in\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_I,Z_J)$ with source (I,x) and target (J,y), then $(I,x)\sim(J,y)$;

(c) the set of morphisms $\bigcup_{I\subset J}\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_I,Z_J)$ together with their inverses (which are uniquely defined by (a)) is closed under composition.

Parts (a) and (c) show that there is a nonsingular groupoid $\widehat{\boldsymbol{Z}}^\nu$ with the given morphisms. Moreover, since the equivalence relation $\sim$ is generated by the morphisms $(I,J,y,\mathrm{id})\in\mathrm{Mor}_{\boldsymbol{Z}^\nu}(Z_I,Z_J)\subset\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_I,Z_J)$, (c) shows that if $(I,x)\sim(J,y)$, where $I\subset J$, then $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}((I,x),(J,y))\neq\varnothing$. Together with (b) this implies that $\widehat{\boldsymbol{Z}}^\nu$ has realization $|\boldsymbol{Z}^\nu|$.

To prove (a) we must check that given $x\in U_I$, $y\in Z_J\cap\widetilde{V}_{IJ}$, where $I\subset J$, there is at most one element $\alpha\in\Gamma_I$ such that

- $x=\alpha^{-1}\rho_{IJ}(y)$;
- there is $F\subset I$ such that $\alpha\in\Gamma_{I\setminus F}$ and $y\in\widetilde{V}_{FJ}\cap\widetilde{V}_{IJ}$.

But if α_1 and α_2 are two such elements, corresponding to F_1 and F_2, then $\alpha_1^{-1}\alpha_2$ fixes the point $\rho_{IJ}(y)$. On the other hand, because the set of F such that $y\in\widetilde{V}_{FJ}$ is nested, we can suppose that $F_1\subset F_2$. Then $\rho_{IJ}(y)\in\widetilde{V}_{F_1I}$ and $\alpha_1^{-1}\alpha_2\in\Gamma_{I\setminus F_1}$. Since $\Gamma_{I\setminus F_1}$ acts freely on $\widetilde{V}_{F_1I}$, this implies that $\alpha_1=\alpha_2$ as required.

To prove (b), observe that if $I\subset J$ and $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}((I,x),(J,y))\neq\varnothing$ then there is $F\subset I$ and $\alpha\in\Gamma_{I\setminus F}$ such that $x=\alpha^{-1}\rho_{IJ}(y)$, which implies that

$$\rho_{FI}(x)=\rho_{FI}(\alpha^{-1}\rho_{IJ}(y))=\rho_{FI}(\rho_{IJ}(y))=\rho_{FJ}(y).$$

Hence, the composite $(F,I,x,\mathrm{id})\circ(I,J,y,\alpha)$ is well defined and equal to (F,J,y,id). Therefore $(F,\rho_{FI}(x))\sim(I,x)$ and $(F,\rho_{FI}(x))=(F,\rho_{FJ}(y))\sim(J,y)$, which gives $(I,x)\sim(J,y)$ since $\sim$ is an equivalence relation.

Finally, to prove (c), it is convenient to consider two special kinds of morphisms: morphisms denoted μ^A with $I=J$, and morphisms denoted μ^B with $I\subsetneq J$ and $\alpha=\mathrm{id}$

that therefore belong to $\mathrm{Mor}_{\boldsymbol{Z}^\nu}$. We first observe that every morphism (I, J, y, α) in $\mathrm{Mor}_{\hat{\boldsymbol{Z}}^\nu}(Z_I, Z_J)$ can be written in two ways as a composite of morphisms of types (A) and (B). More precisely, the identity $\mu_1^A \circ \mu_1^B = \mu_2^B \circ \mu_2^A$ holds, where μ_i^A, μ_j^B are the morphisms in the following diagram:

$$\begin{tikzcd} (I, \alpha^{-1}\rho_{IJ}(y)) \arrow[r, "\mu_2^B"] \arrow[d, "\mu_1^A"] & (J, \alpha^{-1} y) \arrow[d, "\mu_2^A"] \\ (I, \rho_{IJ}(y)) \arrow[r, "\mu_1^B"] & (J, y) \end{tikzcd} \tag{3.2.7}$$

Therefore these morphisms μ^A, μ^B and their inverses generate $\mathrm{Mor}_{\hat{\boldsymbol{Z}}^\nu}$. The commutativity of the above diagram also shows that we can interchange their order: ie every morphism of the form $\mu_1^A \circ \mu_1^B$ can also be written as $\mu_2^B \circ \mu_2^A$, which we abbreviate below as the identity $\mu_1^A \circ \mu_1^B = \mu_2^B \circ \mu_2^A$.

Next let us consider the other composites. Morphisms of type (A) with fixed $F \subset I$ are closed under composition since they are given by the action of $\Gamma_{I \setminus F}$. Moreover, two morphisms of this type corresponding to different subgroups F_1, F_2 can be composed only if the sets $\pi_{\mathcal{K}}(V_I), \pi_{\mathcal{K}}(V_{F_1}), \pi_{\mathcal{K}}(V_{F_2})$ intersect. Hence the sets F_1, F_2 are nested, either $F_1 \subset F_2$ or $F_2 \subset F_1$, and in either case the composite is another morphism of this type. The situation for morphisms of type (B) is more complicated (which is precisely why we needed to add the morphisms of type (A) to obtain a groupoid). We have:

- $\mu_1^B \circ \mu_2^B = \mu_3^B$: ie if $I \subset J \subset K$ and $y = \rho_{JK}(z)$ then the following identity holds (this statement includes the claim that the left-hand composite is well defined):

$$(I, J, y, \mathrm{id}) \circ (J, K, z, \mathrm{id}) = (I, K, z, \mathrm{id}) \in \mathrm{Mor}_{\hat{\boldsymbol{Z}}^\nu}((I, \rho_{IK}(z)),\ (K, z)).$$

- $(\mu_1^B)^{-1} \circ \mu_2^B = \mu^A \circ \mu_3^B$ or $= \mu^A \circ (\mu_3^B)^{-1}$:
  - If $I \subset J \subset K$ and $\rho_{IJ}(y') = \rho_{IK}(y) = \rho_{IJ} \circ \rho_{JK}(y)$, then $\rho_{JK}(y)$ and y' lie in the same $\Gamma_{J \setminus I}$–orbit so that $y' = \alpha^{-1}\rho_{JK}(y)$ for some $\alpha \in \Gamma_{J \setminus I}$, and

$$\begin{aligned}(I, J, y', \mathrm{id})^{-1} \circ (I, K, y, \mathrm{id}) &= (J, J, \rho_{JK}(y), \alpha) \circ (J, K, y, \mathrm{id}) \\ &\in \mathrm{Mor}_{\hat{\boldsymbol{Z}}^\nu}((J, \alpha^{-1}\rho_{JK}(y)), (K, y)).\end{aligned}$$

  - If $I \subset K \subset J$ and there are $y' \in \widetilde{V}_{IJ} \cap Z_J,\ y \in \widetilde{V}_{IK} \cap Z_K$ with

$$\rho_{IJ}(y') = \rho_{IK}(\rho_{KJ}(y')) = \rho_{IK}(y) \in Z_I,$$

  then there is $\beta \in \Gamma_{K \setminus I}$ such that $y = \beta\, \rho_{KJ}(y') = \rho_{KJ}(\beta y') \in Z_K$, and

$$\begin{aligned}(I, J, y', \mathrm{id})^{-1} \circ (I, K, y, \mathrm{id}) &= (J, J, \beta\, y', \beta) \circ (K, J, \beta\, y', \mathrm{id})^{-1} \\ &\in \mathrm{Mor}_{\hat{\boldsymbol{Z}}^\nu}((J, y'), (K, y)).\end{aligned}$$

- One can check similarly that if $\mu_1^B = (I, J, y, \mathrm{id})$ and $\mu_2^B = (K, J, y, \mathrm{id})$ then
$$\mu_1^B \circ (\mu_2^B)^{-1} = \begin{cases} \mu^A \circ \mu_3^B & \text{if } I \subset K, \\ \mu^A \circ (\mu_3^B)^{-1} & \text{if } K \subset I. \end{cases}$$

Combining these identities with $\mu_1^A \circ \mu_1^B = \mu_2^B \circ \mu_2^A$ and its inverse, we see that if $I \subset J$, every composite morphism $Z_I \to Z_J \to Z_K$ can be written in the form $\mu^B \circ \mu^A$ if $I \subset K$, and in the form $\mu_1^A \circ (\mu_1^B)^{-1} = (\mu_2^B)^{-1} \circ \mu_2^A$ if $K \subset I$. This completes the proof of (c) and hence of part (i) of the lemma.

The claims in (ii) are proved by applying Lemma 3.1.14 with $f\colon W \to E_I$ given by $s_I + \nu_I\colon V_I \to E_I$. Since $s_I + \nu_I \pitchfork 0$, Lemma 3.1.14(i) shows that Z_I is a manifold, while the admissibility of ν implies that the hypothesis of Lemma 3.1.14(iii) holds on $\widetilde{V}_{HI}$ so that the subset $Z_I \cap \widetilde{V}_{HI}$ of Z_I is open and ρ_{HI} induces a local diffeomorphism from $Z_I \cap \widetilde{V}_{HI}$ to $Z_H \cap \rho_{HI}(\widetilde{V}_{HI})$. Further, by the compatibility condition (3.2.2) we can identify with the zero set of $\rho_{HI} * (s_I + \nu_I) = s_H + \rho_{HI} * (s_I)$. Since the maps ρ_{IJ} together with their inverses generate the structure maps in $\widehat{\boldsymbol{Z}}^\nu$, this shows that this groupoid is étale. Moreover, if $\mathcal{K}$ is oriented, then Lemma 3.1.14(ii)–(iii) also implies that the structure maps in $\widehat{\boldsymbol{Z}}^\nu$ are orientation-preserving.

Finally, (iii) holds by Lemma 3.1.14(iv). □

In order to show that $\widehat{\boldsymbol{Z}}^\nu$ represents a weighted branched manifold, we must understand its maximal Hausdorff quotient $|\widehat{\boldsymbol{Z}}^\nu|_{\mathcal{H}}$ as defined in Lemma A.2. The morphisms in a nonsingular groupoid $\boldsymbol{G}$ correspond bijectively to the equivalence relation $\sim_{\boldsymbol{G}}$ on $\mathrm{Obj}_{\boldsymbol{G}}$ where $x \sim_{\boldsymbol{G}} y$ if and only if $\mathrm{Mor}_{\boldsymbol{G}}(x, y) \neq \varnothing$. A necessary condition for the quotient $|\boldsymbol{G}| := \mathrm{Obj}_{\boldsymbol{G}} / {\sim_{\boldsymbol{G}}}$ to be Hausdorff is that this equivalence relation be given by a closed subset of $\mathrm{Obj}_{\boldsymbol{G}} \times \mathrm{Obj}_{\boldsymbol{G}}$; in other words, we need the map $s \times t\colon \mathrm{Mor}_{\boldsymbol{G}} \to \mathrm{Obj}_{\boldsymbol{G}} \times \mathrm{Obj}_{\boldsymbol{G}}$ that takes a morphism to its source and target to have closed image. The following lemma shows that in the special case of the groupoid $\widehat{\boldsymbol{Z}}^\nu$ this necessary condition is also sufficient.

Lemma 3.2.10 *Let ν be an admissible, transverse, (cobordism) perturbation of a tame Kuranishi atlas/cobordism $\mathcal{K}$. Then:*

(i) *Let $\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}$ be the groupoid obtained from $\widehat{\boldsymbol{Z}}^\nu$ by closing the relation $\sim$ on $\mathrm{Obj}_{\widehat{\boldsymbol{Z}}^\nu}$. Then we have that $\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}$ is nonsingular and $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is Hausdorff. Further, we can identify $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ with the maximal Hausdorff quotient $|\widehat{\boldsymbol{Z}}^\nu|_{\mathcal{H}}$ in such a way that the canonical quotient map $|\widehat{\boldsymbol{Z}}^\nu| \to |\widehat{\boldsymbol{Z}}^\nu|_{\mathcal{H}} = |\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is induced by the functor $\iota_H\colon \widehat{\boldsymbol{Z}}^\nu \to \widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}$.*

(ii) *For each* $I \in \mathcal{I}_{\mathcal{K}}$, *the projection*

$$\pi_{\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}} \colon \mathrm{Obj}_{\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}} \to |\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}|$$

takes Z_I *onto a subset of* $|\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}|$ *that is open with respect to the quotient topology. This topology on* $|\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}|$ *is metrizable.*

(iii) *If* $x \in Z_I$ *and* $p = \pi_{\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}}(x) \in |\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}|$, *then* $\{x' \in Z_I \mid \pi_{\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}}(I, x') = p\}$ *is the* $(\Gamma_{I \setminus F_x})$*–orbit of* x, *so*

$$\#\{x \in Z_I \mid \pi_{\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}}(I, x) = p\} = |\Gamma_{I \setminus F_x}|,$$

where $F_x = \min\{F : Z_I \cap \mathrm{cl}(\widetilde{V}_{FI}) \cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}}(p) \neq \varnothing\} = \min\{F : p \in \overline{\pi_{\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}}(Z_F)}\}$.

Proof We use the notation in Lemma 3.2.9. The components of $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^{\nu}}(Z_I, Z_J)$ consisting of morphisms of type (B) are taken by $s \times t \colon \mathrm{Mor}_{\widehat{\boldsymbol{Z}}^{\nu}}(Z_I, Z_J) \to Z_I \times Z_J \subset \mathrm{Obj}_{\widehat{\boldsymbol{Z}}^{\nu}} \times \mathrm{Obj}_{\widehat{\boldsymbol{Z}}^{\nu}}$ to the set of pairs

$$\bigl\{(\rho_{IJ}(y), y) \mid y \in Z_J \cap \widetilde{V}_{IJ} \cap \pi_{\mathcal{K}}^{-1}(V_I)\bigr\} \subset Z_I \times Z_J,$$

where we simplify notation by writing y instead of (J, y), and similarly for the source. If $(\rho_{IJ}(y_n), y_n) \to (x_\infty, y_\infty) \in Z_I \times Z_J$ is a convergent sequence of such points with limit $(x_\infty, y_\infty) \in Z_I \times Z_J$, then $y_\infty \in Z_J \cap \widetilde{U}_{IJ}$ since $y_n \in Z_J \cap \widetilde{V}_{IJ} \subset \widetilde{U}_{IJ}$ and $\widetilde{U}_{IJ}$ is closed in U_J, which implies that $\rho_{IJ}(y_\infty)$ is defined. We then must have $x_\infty = \rho_{IJ}(y_\infty)$ by the continuity of ρ_{IJ}. Thus

$$y_\infty \in \rho_{IJ}^{-1}(Z_I) \cap Z_J \subset \rho_{IJ}^{-1}(V_I) \cap V_J = \widetilde{V}_{IJ}.$$

Hence $y_\infty \in Z_J \cap \widetilde{V}_{IJ}$, so that $(I, J, y_\infty, \mathrm{id}) \in \mathrm{Mor}_{\widehat{\boldsymbol{Z}}^{\nu}}(Z_I, Z_J)$. Therefore the graph of this set of morphisms is closed in $Z_I \times Z_J$.

However the set of morphisms of type (A) from $Z_I \to Z_I$ is not closed in general; instead it has closure[8]

$$\overline{\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^{\nu}}(Z_I, Z_I)} := \bigcup_{F \subsetneq I} \bigl\{(I, I, y, \alpha) \mid y \in \mathrm{cl}(\widetilde{V}_{FI}) \cap Z_I,\ \alpha \in \Gamma_{I \setminus F}\bigr\}.$$

Notice that, as in the proof of Lemma 3.2.9(i), this set $\overline{\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^{\nu}}(Z_I, Z_I)}$ is invariant under compositions (and inverses) because the intersection properties of the sets in a reduction apply to their closures: $\pi_{\mathcal{K}}(\overline{V_{F_1}}) \cap \pi_{\mathcal{K}}(\overline{V_{F_2}}) \neq \varnothing \implies F_1 \subset F_2$ or $F_2 \subset F_1$. Next, observe that because $\rho_{IJ} \colon \widetilde{U}_{IJ} \to U_{IJ}$ is a local diffeomorphism, the map ρ_{IJ} induces a local diffeomorphism from $\widetilde{V}_{IJ} \cap \mathrm{cl}(\widetilde{V}_{FJ}) \cap Z_J$ into $U_{IJ} \cap \mathrm{cl}(\widetilde{V}_{FI}) \cap Z_I$.

[8] While we usually denote the closure of a set A by $\bar{A}$, for sets such as $\widetilde{V}_{IJ}$ that involve a tilde we will write $\mathrm{cl}(\widetilde{V}_{IJ})$.

Similarly, because $\mathrm{cl}(\widetilde{V}_{FJ}) \subset \widetilde{U}_{FJ}$ whenever $F \subset J$, the group $\Gamma_{I\setminus F}$ acts freely on $\mathrm{cl}(\widetilde{V}_{FJ})$, and, if $F \subset I$, commutes with the action of ρ_{IJ} as in diagram (3.2.7). Therefore the closure of $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_I, Z_J)$ when $I \subset J$ is given as follows:

$$\begin{aligned}(3.2.8)\quad \mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I, Z_J) &= \bigcup_{F\subset I} \bigl(Z_J \cap \widetilde{V}_{IJ} \cap \mathrm{cl}(\widetilde{V}_{FJ})\bigr) \times \Gamma_{I\setminus F} \\ &= \bigl\{(I, J, y, \alpha) \bigm| \exists F \subset I,\ \alpha \in \Gamma_{I\setminus F} \\ &\qquad\qquad \text{such that } y \in \mathrm{cl}(\widetilde{V}_{FJ}) \cap \widetilde{V}_{IJ} \cap Z_J\bigr\}.\end{aligned}$$

The arguments in Lemma 3.2.10 apply to show that this set of morphisms, together with inverses, are closed under composition and are uniquely determined by their source and target. Thus $\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}$ is a nonsingular groupoid. Its realization $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is Hausdorff as it is the quotient of the separable, locally compact metric space $\mathrm{Obj}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}$ by a relation with closed graph; see [1, Chapter I, Section 10, Example 19] or [13, Lemma 3.2.4]. Moreover, the space $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ can be identified with the maximal Hausdorff quotient of $|\widehat{\boldsymbol{Z}}^\nu|$ because any continuous map from $\mathrm{Obj}_{\widehat{\boldsymbol{Z}}^\nu}/\!\sim$ to a Hausdorff space Y must factor through the closure of the relation $\sim$ induced by the morphisms in $\widehat{\boldsymbol{Z}}^\nu$, and hence descends to $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$. This proves (i).

To see that $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)$ is open in $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ we must show that each intersection

$$Z_J \cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I))$$

is open. Since $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I) \cap \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_J) \neq \varnothing$ only if $I \subset J$ or $J \subset I$, it suffices to consider these two cases. Now

$$Z_J \cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I))$$

consists of all elements in Z_J that are targets of morphisms with source in Z_I. Therefore if $I \subsetneq J$, then

$$Z_J \cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)) = Z_J \cap \widetilde{V}_{IJ},$$

which is open by Lemma 3.2.9(i). On the other hand, if $J \subset I$ then because the set $\rho_{JI}(\widetilde{V}_{JI})$ is Γ_J–invariant, we have

$$Z_J \cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)) = Z_J \cap \rho_{JI}(\widetilde{V}_{JI}),$$

which is open by Lemma 3.2.9(ii). Thus $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)$ is open. It follows that the quotient topology on $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ has a countable basis because each Z_I does. We also have that $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is regular. Indeed, by [15, Lemma 31.1], we only need to check that each point $p \in |\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ with neighborhood $W \subset |\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ has a smaller neighborhood $W_1 \subset W$ such that $\overline{W_1} \subset W$, and this is an immediate consequence of the regularity and local compactness of the sets Z_I and the openness of the sets $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)$. Therefore $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is metrizable by the Urysohn metrization theorem. This proves (ii).

To prove (iii), note first that for each $x\in Z_I$ the subsets $F\in\mathcal{I}_{\mathcal{K}}$ such that $x\in \mathrm{cl}(\widetilde{V}_{FI})$ are nested and hence have a minimal element F_x. The precompactness of V_I in U_I implies that $x\in \mathrm{cl}(\widetilde{V}_{FI})\subset \widetilde{U}_{F_x I}$ so that its orbit under $\Gamma_{I\setminus F_x}$ is free. Moreover, because $F_x\subset F$ for every F for which $x\in\mathrm{cl}(\widetilde{V}_{FI})$, this orbit $\Gamma_{I\setminus F_x}(x)$ contains the targets of all the morphisms in $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}$ with source (I,x). This proves the formula $|\Gamma_{I\setminus F_x}| = \#\{x\in Z_I \mid \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(I,x)=p\}$.

It remains to check that F_x, which we defined as

$$\min\{F : Z_I\cap \mathrm{cl}(\widetilde{V}_{FI})\cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(p)\neq\varnothing\},$$

also equals $F'_x := \min\{F : p\in \overline{\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_F)}\}$. But if

$$Z_I\cap \mathrm{cl}(\widetilde{V}_{FI})\cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(p)\neq\varnothing$$

there is a sequence of elements $x_k\in Z_I\cap\widetilde{V}_{FI}$ with limit $x_\infty\in \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(p)$, implying by the continuity of $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}$ that, with $x'_k:=\rho_{FI}(x_k)$, the sequence

$$\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(x'_k) = \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(x_k)$$

converges to p. Hence $p\in\overline{\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_F)}$, which implies $F'_x\subset F_x$. Conversely, if

$$p\in \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)\cap \overline{\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_F)},$$

then since $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)$ is open in $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ there is a sequence p_k of elements in

$$\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)\cap \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_F)$$

that converges to $p\in\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)$. By (3.2.1), this lifts to a sequence $x_k\in\widetilde{V}_{FI}\subset Z_I$, and the sequence of images $\pi_{\mathcal{K}}(\iota_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(x_k))$ in $|\mathcal{V}|\subset|\mathcal{K}|$ converges to $|\iota_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}|(p)$, where $\iota_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}$ is as in (ii). But the composite

$$\pi_{\mathcal{K}}\circ\iota_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}\colon V_I\to\pi_{\mathcal{K}}(V_I)\cong V_I/\Gamma_I$$

simply quotients out by the action of Γ_I on V_I. Since the projection $V\to V_I/\Gamma_I$ is proper by Lemma 2.1.5(i), the sequence (x_k) must have a convergent subsequence with limit $x_\infty\in V_I$. But then by uniqueness of limits in the Hausdorff space $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$, $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(x_\infty)=\lim_{k\to\infty}\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(x_k)=p$. Therefore

$$x_\infty\in\mathrm{cl}(\widetilde{V}_{FI})\cap\pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(p).$$

Hence by the minimality of F_x we must have $F_x\subset F'_x$. This completes the proof. □

Proof of Theorem 3.2.8 Let us first consider the case when ν is an admissible, transverse, precompact perturbation of an oriented tame Kuranishi atlas $\mathcal{K}$ with respect to nested reductions $\mathcal{C} \sqsubset \mathcal{V} \sqsubset \mathrm{Obj}_{\mathcal{B}_{\mathcal{K}}}$. Then Lemma 3.2.9 shows that $\boldsymbol{Z}^\nu$ can be completed to an oriented, nonsingular étale groupoid $\widehat{\boldsymbol{Z}}^\nu$. Moreover, by Lemma 3.2.10 the maximal Hausdorff quotient $|\widehat{\boldsymbol{Z}}^\nu|_{\mathcal{H}}$ can be identified with the realization $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ of the groupoid $\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}$. To complete the proof of the first part of the theorem it remains to show that $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is compact, and that $(\widehat{\boldsymbol{Z}}^\nu, \Lambda^\nu)$ has the structure of a wnb groupoid as in Definition A.4.

Because $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is metrizable by Lemma 3.2.9(ii), it suffices to prove that $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is sequentially compact. Further, we saw in (3.2.5) that the precompactness condition for ν can be written without explicit mention of the isotropy groups Γ_I. Hence the proof of the sequential compactness of the zero set given in [13, Theorem 5.2.2] carries through, without change, to the current situation.

We next check that the weighting function Λ^ν is well defined, and compatible with a local branching structure as required by Definition A.4. To see that it is well defined, suppose that $p \in \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I) \cap \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_J)$. As usual we may suppose that $I \subset J$, so that $p = \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(y)$ for some $y \in \widetilde{V}_{IJ} \subset Z_J$. Let the minimal set F such that $p \in \overline{\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_F)}$ be denoted by F_p. Then there are $|\Gamma_{J \setminus F_p}|$ distinct elements in Z_J that map to p. Hence $\Lambda(p) = |\Gamma_{J\setminus F_p}|/|\Gamma_J|$, and we must check that this agrees with the calculation provided by replacing J by I. But if $x = \rho_{IJ}(y)$, then because $F_p \subset I$ does not depend on I, J we have $\Gamma_{J\setminus F_p} = \Gamma_{I\setminus F_p} \times \Gamma_{J\setminus I}$. Hence $|\Gamma_{J\setminus F_p}|/|\Gamma_J| = |\Gamma_{I\setminus F_p}|/|\Gamma_I|$. Thus Λ^ν is well defined.

Finally we describe the local branches at $p \in |\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$. Given $p \in |\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$, choose a minimal I such that $p \in \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)$, and a minimal $F_p \subset I$ such that $p \in \overline{\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_{F_p})}$. Then $F_p \subset I$, and there is $x \in Z_I \cap \mathrm{cl}(\widetilde{V}_{F_p I})$ such that $p = \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(x)$. As $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)$ is open in $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$, we may choose an open neighborhood $N \subset \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_I)$ of p whose closure $\overline{N}$ is disjoint from all sets $\overline{\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_F)}$ with $F \subsetneq F_p$. We saw in Lemma 3.2.10(iii) that $Z_I \cap (\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}})^{-1}(p) = \Gamma_{I\setminus F_p}(x)$. Hence, by shrinking N further if necessary, we may suppose that there is an precompact open neighborhood B_x of x in Z_I such that

- $\bigcup_{\gamma \in \Gamma_{I\setminus F_p}} \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(\gamma B_x) = N$;
- the closure $\overline{B_x}$ in Z_J is disjoint from its images under the action of $\Gamma_{I\setminus F_p}$.

Then choose the local branches to be the disjoint subsets $(\gamma B_x)_{\gamma \in \Gamma_{I\setminus F_p}}$ of Z_I, each with weight $1/|\Gamma_I|$. Notice that

$$\text{(3.2.9)} \qquad \bigcup_{\gamma \in \Gamma_{I\setminus F_p}} \gamma \overline{B_x} = Z_I \cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(\overline{N}) \quad \text{and} \quad \bigcup_{\gamma \in \Gamma_I \setminus F_p} \gamma B_x = Z_I \cap \pi^{-1}_{\widehat{\boldsymbol{Z}}^\nu}(N).$$

Here, the first claim holds because, by the minimality of F_p and the choice of B_x,

$$\overline{B_x} \cap \mathrm{cl}(\widetilde{V}_{FI}) \neq \varnothing \implies F_p \subset F,$$

so that the only morphisms in $\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}$ with source in $\bigcup_{\gamma \in \Gamma_{I \setminus F_p}} \gamma \overline{B_x}$ and target in Z_I are given by the action of an element in $\Gamma_{I\setminus F_p}$ and hence also have target in this set. The second claim holds similarly.

We must check that the three conditions in Definition A.4 hold.

- The covering property states that

$$\left(\pi^{\mathcal{H}}_{|\widehat{\boldsymbol{Z}}^\nu|}\right)^{-1}(N) = \bigcup_{\gamma \in \Gamma_{I\setminus F_p}} |\gamma B_x| \subset |\boldsymbol{Z}|.$$

If this were false there would be a point $y \in Z_J$ for some J such that there is a morphism in $\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}$ from (J, y) to a point $(I, x') \in \gamma B_x$ for some $\gamma \in \Gamma_{I\setminus F_p}$, but there is no such morphism in $\widehat{\boldsymbol{Z}}^\nu$. By construction, the morphisms in $\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}$ from Z_J to Z_I are composites of morphisms of type (B) from Z_J to Z_I (which lie in $\widehat{\boldsymbol{Z}}^\nu$) with morphisms in the closure of $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_I, Z_I)$. Therefore it suffices to consider the case $J = I$, and $y \notin \bigcup_{\gamma\in\Gamma_{I\setminus F_p}} \gamma B_x$. But (3.2.9) implies that the only elements of $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^\nu}(Z_I, Z_I)$ with target in γB_x must have source in some set αB_x. Therefore such y does not exist.

- For local regularity, we must check that for each γ the projection $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}\colon \gamma B_x \to |\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is a homeomorphism onto a relatively closed subset of N. But (3.2.9) implies that this map extends to an injective, continuous map f with compact domain $\gamma \overline{B_x}$. Hence f is a homeomorphism onto its image because compact subsets of the Hausdorff space $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ are closed. Further, $\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(\gamma B_x) = N \cap \pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(\gamma \overline{B_x})$ is closed in N because it is the intersection of a compact set with N.

- Finally, note that Λ^ν equals the branching function specified in Definition A.4; indeed, the number of branches through $q \in N$ is just the number of preimages of $q \in N$ in $\bigcup_{\gamma\in\Gamma_{I\setminus F_p}} \gamma B_x$, and we saw in Lemma 3.2.10(iii) that this is $|\Gamma_{I\setminus F_q}|$, where $F_q \supset F_p$ is the minimal set F such that $q \in \overline{\pi_{\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}}(Z_F)}$.

This completes the proof that $(\widehat{\boldsymbol{Z}}^\nu, \Lambda^\nu)$ is a compact wnb groupoid. It has a fundamental class by Proposition A.7, and hence defines a cycle in $\mathcal{C}$ as claimed.

The same arguments apply when $\mathcal{K}$ is a Kuranishi cobordism. In particular, $|\widehat{\boldsymbol{Z}}^\nu_{\mathcal{H}}|$ is compact so that, by Lemma 3.2.9(iii), $(\widehat{\boldsymbol{Z}}^\nu, \Lambda^\nu)$ is a wnb cobordism groupoid, and the boundary restrictions have the required properties by Lemma 3.1.14(iv). □

We end this section by some elementary examples of this construction: the fundamental class and the Euler class of an orbifold represented by Kuranishi atlases.

Example 3.2.11 Consider the orbifold case, ie a Kuranishi atlas $\mathcal{K}$ on X with trivial obstruction spaces so that $\mathfrak{s}_{\mathcal{K}}$ and ν are identically zero and $\iota_{\mathcal{K}}\colon X \to |\mathcal{K}|$ is a homeomorphism. In this case the zero set $\boldsymbol{Z}$ should represent the fundamental class of the oriented orbifold. We suppose that $X = M/\Gamma$ is the quotient of a compact oriented smooth manifold M by the action of a finite group Γ, and that $\mathcal{K}$ is the atlas with a single chart with domain M and $E = \{0\}$. Then $\boldsymbol{Z} = |\boldsymbol{B}_{\mathcal{K}}|^{\setminus\Gamma}$ is the category with objects M and only identity morphisms, because there are no pairs $I, J \in \mathcal{I}_{\mathcal{K}}$ such that $\varnothing \neq I \subsetneq J$. Therefore $\boldsymbol{Z} = \boldsymbol{Z}_{\mathcal{H}}$ has realization $|\boldsymbol{Z}_{\mathcal{H}}| = M$ and the weighting function $\Lambda\colon M \to \mathbb{Q}$ is given by $\Lambda(x) = 1/|\Gamma|$. If the action of Γ is effective on every open subset of M, then the pushforward of Λ by $\iota_{\boldsymbol{Z}_{\mathcal{H}}}\colon M \to X$, which is defined by

$$(\iota_{\boldsymbol{Z}_{\mathcal{H}}})_*\Lambda(p) \;:=\; \sum_{x\in(\iota_{\boldsymbol{Z}_{\mathcal{H}}})^{-1}(p)} \Lambda(x),$$

takes the value 1 at every smooth point (ie point with trivial stabilizer) of the orbifold M/Γ. On the other hand, if Γ acts by the identity so that the action is totally noneffective, then $\iota_{\boldsymbol{Z}_{\mathcal{H}}}\colon M \xrightarrow{\cong} X$ is the identity map and the weighting function $X \to \mathbb{Q}^+$ takes the constant value $1/|\Gamma|$.

Note that if we construct a fundamental class on $|\boldsymbol{Z}|_{\mathcal{H}}$ by the method of Proposition A.7 then our choice of weights gives a class that is consistent with standard conventions. For example, in dimension $d = 0$ the branched manifold $Z = |\boldsymbol{Z}|_{\mathcal{H}}$ is a finite collection of points $\{p_1, \dots, p_k\}$, one for each equivalence class in $\mathrm{Obj}_{\boldsymbol{Z}}$, where the point p_i corresponding to an equivalence class with stabilizer Γ^i has weight $1/|\Gamma^i|$. If each point is positively oriented, then the "number of elements" in $|\boldsymbol{Z}|_{\mathcal{H}}$ is the sum $\sum_{i=1}^k 1/|\Gamma^i|$, which gives the Euler characteristic of the groupoid; cf [17]. Other more substantive examples such as that of the football of Example 2.3.11 are discussed in [11, Example 4.6].

Example 3.2.12 Examples of Kuranishi atlases with nontrivial obstruction spaces can be seen in the calculation of the Euler class of the tangent bundle of S^2 and of the football orbifold using Kuranishi atlases.

(i) To build a Kuranishi atlas that models $\mathrm{T}S^2$, cover S^2 by two discs D_1, D_2 whose intersection $D_1 \cap D_2 =: D_{12} =: A$ is an annulus, and for $i = 1, 2$ define $\boldsymbol{K}_i := (U_i := D_i,\ E_i := \mathbb{C},\ s_i := 0,\ \psi_i := \mathrm{id})$. For $i = 1, 2$ choose trivializations $\tau_i\colon D_i \times \mathbb{C} \to \mathrm{T}S^2|_{D_i}$, $(x, e) \mapsto \tau_{i,x}(e)$ and then define the transition chart

$$\boldsymbol{K}_{12} := (U_{12} \subset E_1 \times E_2 \times A,\ E_1 \times E_2,\ s_{12} = \mathrm{pr}_{E_1\times E_2},\ \psi_{12} = \mathrm{pr}_A|_{0\times 0\times A}),$$

where

$$U_{12} := \{(e_1, e_2, x) \mid x \in A,\ \tau_{1,x}(e_1) + \tau_{2,x}(e_2) = 0\}.$$

The coordinate changes $\widehat{\Psi}_{i,12}$ are given by taking $U_{i,12} = A$ and $\psi_{i,12}(x) = (0,0,x)$. To justify this choice of Kuranishi atlas note that one can construct a commutative diagram

$$\begin{tikzcd} |\boldsymbol{E}_{\mathcal{K}}| \arrow[r] \arrow[d] & \mathrm{T}S^2 \arrow[d] \\ |\boldsymbol{B}_{\mathcal{K}}| \arrow[r] & S^2 \end{tikzcd}$$

that restricts over $U_{12} \times E_{12}$ to the map

$$((e_1, e_2, x), e_1', e_2') \mapsto \tau_{1,x}(e_1 + e_1') + \tau_{2,x}(e_2 + e_2') \in \mathrm{T}S^2|_A.$$

This construction is generalized to other (orbi)bundles in [10].

Next, in order to calculate the Euler class we identify A with $[0,1] \times S^1$ and consider the corresponding trivialization $\mathrm{T}S^2|_A = A \times \mathbb{R}_t \times \mathbb{R}_\theta$, where $t \in [0,1]$ and $\theta \in S^1$ are coordinates. Then for $i = 1,2$ there is a section $\nu_i \colon U_i \to E_i$ with one transverse zero such that $\tau_{i,x}(\nu_i(x)) = (x, 1, 0) \in \mathrm{T}S^2|_A$ for $x \in A$. (Take suitably modified versions of the sections $\nu_1(z) = z$ and $\nu_2(z) = -z$, where $D_i \subset \mathbb{C}$.)

Choose a reduction of the footprint covering with $V_{12} = (\varepsilon, 1-\varepsilon) \times S^1$ for some $\varepsilon \in \bigl(0, \frac{1}{4}\bigr)$ and so that $\widetilde{V}_{1,12} = (0,0) \times \bigl(\varepsilon, \frac{1}{4}\bigr] \times S^1 \subset U_{12}$ and $\widetilde{V}_{2,12} = (0,0) \times \bigl(\frac{3}{4}, 1-\varepsilon\bigr) \times S^1$, and choose a cutoff function $\beta \colon [0,1] \to [0,1]$ that equals 1 in $\bigl[0, \frac{1}{4}\bigr]$ and 0 in $\bigl[\frac{3}{4}, 1\bigr]$. Then the map $\nu_{12} \colon V_{12} \to E_1 \times E_2$ given by

$$\nu_{12}(e_1, e_2, x) \;=\; \bigl(\beta(x)\nu_1(x), (1-\beta(x))\nu_2(x)\bigr) \;\in\; E_1 \times E_2$$

defines an admissible perturbation section that restricts to ν_i on $V_{i,12} \subset (0,0) \times A$ for $i = 1,2$. Moreover $s_{12} + \nu_{12}$ does not vanish at any point $(e_1, e_2, x) \in V_{12}$ because the equation $\tau_{1,x}(e_1) + \tau_{2,x}(e_2) = 0$ together with

$$0 = \tau_{1,x}(e_1) + \beta(x)(1,0) = \tau_{2,x}(e_2) + (1-\beta(x))(1,0) \in x \times \mathbb{R}_t \times \mathbb{R}_\theta \in \mathrm{T}S^2|_A$$

imply that the vector $(1,0)$ is zero, a contradiction. Hence the perturbed zero set $\boldsymbol{Z}^\nu$ consists of two points, each with weight one.

(ii) It is easy to adjust this example to the tangent bundle of the "football" discussed in Example 2.3.11. In this case, the zero of the section $s_i + \nu_i$ would count with weight $1/|\Gamma_i|$ so that the Euler class is $\frac{1}{2} + \frac{1}{3}$. For further details of this and other related examples see [10, Section 5].

3.3 Construction of the virtual moduli cycle and fundamental class

The next step in the Kuranishi regularization Theorem A is to construct admissible, transverse, precompact perturbations ν that are unique up to interpolation by admissible, transverse, precompact cobordism perturbations. This — quite complicated — construction is developed in complete detail in [14] in such a way that it applies directly to our present setting, in which the Kuranishi atlas $\mathcal{K}$ has nontrivial isotropy groups, but the reduced and pruned category $\mathcal{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ is nonsingular, ie the remaining isotropy groups act freely. While we defer most of the proofs to [14], we will give full technical statements of the existence and uniqueness of perturbations, so that our constructions of VMC/VFC can be compared directly to other approaches, without reference to [14]. Based on this, Definition 3.3.4 and Theorem 3.3.5 then define the virtual moduli cycle (VMC) as a cobordism class of closed oriented weighted branched manifolds and construct the virtual fundamental class (VFC) as Čech homology class.

For the construction of (cobordism) perturbations we will consider a metric tame Kuranishi atlas (or cobordism) $(\mathcal{K}, d)$. That is, we fix the following data:

- $\mathcal{K}$ is a tame Kuranishi atlas on a compact metrizable space X in the sense of Definitions 2.3.1 and 2.5.1, or it is a tame Kuranishi cobordism on a compact collared cobordism Y in the sense of Definitions 2.4.2 and 2.5.1.
- d is an admissible metric on $|\mathcal{K}|$ in the sense of Definition 2.3.10.
- If $(\mathcal{K}, d)$ is a metric, tame Kuranishi cobordism on Y, then the boundary restrictions $(\mathcal{K}^\alpha, d^\alpha) := (\partial^\alpha \mathcal{K}, d|_{|\partial^\alpha \mathcal{K}|})$ are metric, tame Kuranishi atlases on $\partial^\alpha Y$ for $\alpha = 0, 1$.

For easy reference we list some consequences of this setting and notation conventions.

- The associated intermediate Kuranishi atlas $\underline{\mathcal{K}}$ is a tame topological Kuranishi atlas (resp. cobordism) by Lemma 2.3.4 (resp. Remark 2.4.3(ii)), which has the same realization $|\underline{\mathcal{K}}| = |\mathcal{K}|$, equipped with the quotient topology.
- d is a bounded metric on the set $|\underline{\mathcal{K}}|$ such that for each $I \in \mathcal{I}_{\mathcal{K}}$ the pullback metric $\underline{d}_I := (\pi_{\underline{\mathcal{K}}}|_{\underline{U}_I})^* d$ on $\underline{U}_I$ induces the quotient topology on the intermediate domain $\underline{U}_I = U_I / \Gamma_I$. By construction, these also induce Γ_I–invariant pseudometrics $d_I := (\pi_{\underline{\mathcal{K}}}|_{U_I})^* d = \pi_I^* \underline{d}_I$ on the Kuranishi domains U_I of $\mathcal{K}$. Moreover, [13, Lemma 3.1.8] shows that these (pseudo)metrics are compatible with coordinate changes. We denote the δ–balls around subsets $Q \subset |\underline{\mathcal{K}}|$, $R \subset \underline{U}_I$ and $S \subset U_I$ for $\delta > 0$ by, respectively,

$$B_\delta(Q) := \{w \in |\mathcal{K}| \mid \exists q \in Q \text{ such that } d(w,q) < \delta\},$$
$$B^I_\delta(R) := \{x \in \underline{U}_I \mid \exists r \in R \text{ such that } \underline{d}_I(x,r) < \delta\},$$
$$\widehat{B}^I_\delta(S) := \{y \in U_I \mid \exists s \in S \text{ such that } d_I(y,s) < \delta\},$$

and note that balls in the pseudometric are Γ_I–invariant preimages of balls in $\underline{U}_I$,

$$\widehat{B}^I_\delta(S) = \pi_I^{-1}(B^I_\delta(\underline{S})) \quad \text{and} \quad B_\delta(\pi_\mathcal{K}(S)) = B_\delta(\pi_{\underline{\mathcal{K}}}(\underline{S})). \tag{3.3.1}$$

- While the metric topology on $|\underline{\mathcal{K}}|$ is generally not compatible with the quotient topology, we know from [13, Lemma 3.1.8] that the identity map $|\mathcal{K}| \to (|\mathcal{K}|, d)$ is continuous, and thus $|\mathcal{K}|$ is a Hausdorff topology in which the metric δ–balls are open, and thus neighborhoods.

Given this setting, our goal is to construct admissible, precompact, transverse (cobordism) perturbations of the section $\mathfrak{s}_\mathcal{K}|_\mathcal{V}^{\setminus\Gamma}$ over a pruned domain category $\boldsymbol{B}_\mathcal{K}|_\mathcal{V}^{\setminus\Gamma}$; see Definition 3.2.4 and Lemma 3.2.3. For that purpose we will also need to fix nested (cobordism) reductions $\mathcal{C} \sqsubset \mathcal{V}$ of $\mathcal{K}$. These induce the following crucial data, on which the iterative construction of perturbations depends. The claims here are all consequences of [13, Theorem 5.1.6(iii)] and [14, Lemma 7.3.4, Proposition 7.3.10] applied to $\underline{\mathcal{K}}$ together with (3.3.1) and properness of the projections $\pi_I \colon U_I \to \underline{U}_I$ established in Lemma 2.1.5(i).

- Given a reduction $\mathcal{V}$ of $\mathcal{K}$, there exists $\delta_\mathcal{V} \in \left(0, \frac{1}{4}\right]$ such that for any $\delta < \delta_\mathcal{V}$,

$$\begin{gathered} \widehat{B}^I_{2\delta}(V_I) \sqsubset U_I \quad \forall I \in \mathcal{I}_\mathcal{K}, \\ B_{2\delta}(\pi_\mathcal{K}(V_I)) \cap B_{2\delta}(\pi_\mathcal{K}(V_J)) \neq \varnothing \quad \Longrightarrow \quad I \subset J \text{ or } J \subset I. \end{gathered}$$

  This gives rise to a continuum of nested reductions $V_I \sqsubset \cdots V_I^{k''} \sqsubset V_I^{k'} \cdots \sqsubset V_I^0$ for $k'' > k' > 0$, which for $k \geq 0$ are given by

$$V_I^k := \widehat{B}^I_{2^{-k}\delta}(V_I) = \pi_I^{-1}(\underline{V}_I^k) \sqsubset U_I \quad \text{with} \quad \underline{V}_I^k := B^I_{2^{-k}\delta}(\underline{V}_I).$$

- For suitable $k \geq 0$, the iteration will construct ν_J by extension of the pullbacks $\rho_{IJ}^*\nu_I$, which are defined for $I \subsetneq J$ on $N_{JI}^k := V_J^k \cap \pi_\mathcal{K}^{-1}(\pi_\mathcal{K}(V_I^k))$, also given as

$$N_{JI}^k = V_J^k \cap \rho_{IJ}^{-1}(V_I^k) = \pi_J^{-1}(\underline{N}_{JI}^k) \quad \text{with} \quad \underline{N}_{JI}^k := \underline{V}_J^k \cap \underline{\phi}_{IJ}(\underline{V}_I^k \cap \underline{U}_{IJ}).$$

- We need to make a choice of *equivariant norms on the obstruction spaces* as follows. For each basic chart $i \in \{1, \dots, N\}$ we choose a Γ_i–invariant norm $\|\cdot\|$ on E_i. Then the Γ_J–invariant norm on E_J for each $J \in \mathcal{I}_\mathcal{K}$ is given by

$$\|e\| := \left\| \sum_{i\in J} \widehat{\phi}_{iJ}(e_i) \right\| := \max_{i\in J} \|e_i\| \quad \forall e = \sum_{i\in J} \widehat{\phi}_{iJ}(e_i) \in E_J.$$

- While the sections $s_I\colon U_I \to E_I$ only induce continuous maps $\underline{s}_I\colon \underline{U}_I \to E_I/\Gamma_I$ to the quotient of obstruction spaces, equivariance of the norms guarantees that the norm of sections descends to a continuous function $\|\underline{s}_I\|\colon \underline{U}_I \to [0,\infty)$ given by $x \mapsto \|s_I(y)\|$ for any $y \in \pi_I^{-1}(x)$. These functions provide (rather nontransverse) topological Kuranishi charts over the intermediate domain with the same footprint: $\underline{\psi}_I$ maps $\|\underline{s}_I\|^{-1}(0) = s_I^{-1}(0)/\Gamma_I$ homeomorphically to F_I.
- Given equivariant norms $\|\cdot\|$, nested reductions $\mathcal{C} \sqsubset \mathcal{V}$ and $0 < \delta < \delta_{\mathcal{V}}$, we have

$$\begin{aligned}\sigma(\mathcal{V},\mathcal{C},\|\cdot\|,\delta) &:= \min_{J\in\mathcal{I}_{\mathcal{K}}} \inf\Big\{ \|s_J(x)\| \;\Big|\; x \in \overline{V_J^{|J|}} \setminus \Big(\widetilde{C}_J \cup \bigcup_{I\subsetneq J} \widehat{B}^J_{\eta_{|J|-\frac12}}\big(N_{JI}^{|J|-\frac14}\big)\Big)\Big\} \\ &= \min_{J\in\mathcal{I}_{\mathcal{K}}} \inf\Big\{ \|\underline{s}_J\|(y) \;\Big|\; y \in \overline{\underline{V}_J^{|J|}} \setminus \Big(\widetilde{\underline{C}}_J \cup \bigcup_{I\subsetneq J} B^J_{\eta_{|J|-\frac12}}\big(\underline{N}_{JI}^{|J|-\frac14}\big)\Big)\Big\} \\ &> 0,\end{aligned}$$

  where $\eta_{k-\frac12} := 2^{-k+\frac12}(1-2^{-\frac14})\delta$ and

$$\widetilde{C}_J := \bigcup_{K\supset J} \rho_{JK}(C_K) \;=\; \pi_J^{-1}(\widetilde{\underline{C}}_J), \quad \text{with} \quad \widetilde{\underline{C}}_J := \bigcup_{K\supset J} \underline{\phi}_{JK}^{-1}(\underline{C}_K),$$

  is a set containing $s_J^{-1}(0) = \pi_J^{-1}(\|\underline{s}_J\|^{-1}(0))$.
- In the case of a metric tame Kuranishi cobordism $(\mathcal{K}, d)$ with equivariant norms $\|\cdot\|$ and nested cobordism reductions $\mathcal{C} \sqsubset \mathcal{V}$, let $\varepsilon > 0$ be the smallest of the collar widths of $\mathcal{K}$, d, $\mathcal{C}$ and $\mathcal{V}$. Then for $0<\delta<\min\{\varepsilon,\delta_{\mathcal{V}}\}$, we obtain positive numbers

$$\sigma'(\mathcal{V},\mathcal{C},\|\cdot\|,\delta) := \min_{J\in\mathcal{I}_{\mathcal{K}}} \inf\Big\{\|s_J(x)\| \;\Big|\; x \in \overline{V_J^{|J|+1}} \setminus \Big(\widetilde{C}_J \cup \bigcup_{I\subsetneq J} \widehat{B}^J_{\eta_{|J|+\frac12}}\big(N_{JI}^{|J|+\frac34}\big)\Big)\Big\},$$

$$\sigma_{\mathrm{rel}}(\mathcal{V},\mathcal{C},\|\cdot\|,\delta) := \min\big(\{\sigma'(\mathcal{V},\mathcal{C},\|\cdot\|,\delta)\} \cup \{\sigma(\partial^\alpha\mathcal{V},\partial^\alpha\mathcal{C},\partial^\alpha\|\cdot\|,\delta) \mid \alpha=0,1\}\big).$$

  Here $\partial^\alpha\|\cdot\|$ denotes the collection of equivariant norms on E_I for $I \in \mathcal{I}_{\partial^\alpha\mathcal{K}} \subset \mathcal{I}_{\mathcal{K}}$.

The constants $\delta_{\mathcal{V}}$ and $\sigma(\mathcal{V},\mathcal{C},\|\cdot\|,\delta)$ defined here will control the permitted support and norm of the perturbation ν for a Kuranishi atlas. In particular, $\delta_{\mathcal{V}}$ measures the separation between the components $V_I \neq V_J$ of the reduction $\mathcal{V}$, while $\sigma(\mathcal{V},\mathcal{C},\|\cdot\|,\delta)$ measures the minimal norm of $\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ on the complement of an open neighborhood of the set $\pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(\mathcal{C}))$, in which all perturbed zero sets will need to be contained. We will construct perturbations $\nu = (\nu_I\colon V_I \to E_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ by an iteration which constructs and controls each ν_I over the larger set $V_I^{|I|}$. Here the domains are determined by a choice of $0 < \delta < \delta_{\mathcal{V}}$, and we ensure that the perturbed zero sets are contained in

$\pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(\mathcal{C}))$ by bounding the perturbations $\|\nu_I\| < \sigma$ by some $0 < \sigma < \sigma(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta)$. In order to prove uniqueness of the VMC, we moreover have to interpolate between any two such perturbations. This requires the adjusted bound $\sigma_{\rm rel}(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta)$ on the norm of cobordism perturbations for the following reason. The construction of a cobordism perturbation with prescribed boundary values is achieved by an iteration on the domains $V_I^{|I|+1}$ instead of $V_I^{|I|}$, which guarantees that the boundary values—which got constructed in iterations over $\partial^\alpha V_I^{|I|}$—are given on sufficiently large boundary collars. In view of this, it is also necessary to keep track of the refined properties arising from the iterative construction of a perturbation by the following notion of $(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta, \sigma)$–adapted, as well as a stronger notion which guarantees extensions to Kuranishi concordances.

Definition 3.3.1 Given nested reductions $\mathcal{C} \sqsubset \mathcal{V}$ of a metric tame Kuranishi atlas $(\mathcal{K}, d)$, a choice of equivariant norms $\|\cdot\|$ on the obstruction spaces, and constants $0 < \delta < \delta_{\mathcal{V}}$ and $0 < \sigma \le \sigma(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta)$, we say that a perturbation ν of $\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ is $(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta, \sigma)$*–adapted* if the sections $\nu_I\colon V_I \to E_I$ extend to sections over $V_I^{|I|}$ (also denoted ν_I) so that the following conditions hold for every $k = 1, \dots, M_{\mathcal{K}} := \max_{I \in \mathcal{I}_{\mathcal{K}}} |I|$:

(a) The perturbations are compatible in the sense that for $H \subsetneq I$ with $|I| \le k$,
$$\nu_I|_{\rho_{HI}^{-1}(V_H^k) \cap V_I^k} = \widehat{\phi}_{HI} \circ \nu_H \circ \rho_{HI}|_{\rho_{HI}^{-1}(V_H^k) \cap V_I^k}.$$

(b) The perturbed sections are transverse; that is, $(s_I|_{V_I^k} + \nu_I) \pitchfork 0$ for each $|I| \le k$.

(c) The perturbations are strongly admissible; that is, for all $H \subsetneq I$ and $|I| \le k$ we have $\nu_I(\widehat{B}^I_{\eta_k}(N^k_{IH})) \subset \widehat{\phi}_{HI}(E_H)$.

(d) The perturbed zero sets are controlled by $\pi_{\mathcal{K}}\big((s_I|_{V_I^k} + \nu_I)^{-1}(0)\big) \subset \pi_{\mathcal{K}}(\mathcal{C})$ for $|I| \le k$.

e) The perturbations are small; that is, $\sup_{x \in V_I^k} \|\nu_I(x)\| < \sigma$ for $|I| \le k$.

Also, we say that a perturbation ν is *strongly* $(\mathcal{V}, \mathcal{C})$*–adapted* if it is a $(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta, \sigma)$–adapted perturbation of $\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ for some choice of equivariant norms $\|\cdot\|$ and constants $0 < \delta < \delta_{\mathcal{V}}$, and using the product metric on $[0,1] \times |\mathcal{K}|$ we have
$$\begin{aligned} 0 < \sigma &\le \sigma_{\rm rel}([0,1] \times \mathcal{V}, [0,1] \times \mathcal{C}, \|\cdot\|, \delta) \\ &= \min\big\{\sigma(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta), \sigma'([0,1] \times \mathcal{V}, [0,1] \times \mathcal{C}, \|\cdot\|, \delta)\big\}, \end{aligned}$$

Remark 3.3.2 (i) Adapted perturbations are automatically admissible, precompact and transverse in the sense of Definition 3.2.4. Indeed, these properties are guaranteed by the inclusions $V_I \subset V_I^k$ and the fact that strong admissibility $\nu_I(x) \in \operatorname{im} \widehat{\phi}_{HI}$ for

$x \in \widehat{B}^I_{\eta_k}(N^k_{IH})$ for $H \subsetneq I$ implies admissibility $\operatorname{im} \mathrm{d}_y \nu_I \subset \operatorname{im} \widehat{\phi}_{HI}$ for $y \in \widetilde{V}_{HI} = V_I \cap \rho^{-1}_{HI}(V_H) \subset V^k_I \cap \rho^{-1}_{IJ}(V^k_I) = N^k_{IH}$.

(ii) The admissibility condition is crucial for the transfer of transversality as follows. Let ν be an admissible perturbation, and let $z \in V_I$ and $w \in V_J$ so that $\pi_{\mathcal{K}}(z) = \pi_{\mathcal{K}}(w) \in |\mathcal{K}|$. Then z is a transverse zero of $s_I|_{V_I} + \nu_I$ if and only if w is a transverse zero of $s_J|_{V_J} + \nu_J$.

Indeed, by the reduction property we can assume without loss of generality that $I \subset J$ and thus $z = \rho_{IJ}(w)$. Since ρ_{IJ} is a regular covering, we can pick a local inverse $\phi_I J$ so that $w = \phi_{IJ}(z)$. Then the proof of [14, Lemma 7.2.4] directly applies, using the index condition in terms of ϕ_{IJ}.

(iii) Any $(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta, \sigma)$–adapted perturbation for fixed $\mathcal{V}$, $\mathcal{C}$, $\|\cdot\|$, δ and sufficiently small $\sigma > 0$ is in fact strongly adapted. Indeed, given the product structure of all sets and maps involved in the definition of σ', we can rewrite the condition on $\sigma > 0$ in the definition of strong adaptivity as $\sigma < \|s_J(x)\|$ for all $x \in \overline{V^k_J} \setminus \big(\widetilde{C}_J \cup \bigcup_{I \subsetneq J} \widehat{B}^J_{\eta_{k-\frac{1}{2}}}(N^{k-\frac{1}{4}}_{JI})\big)$, $J \in \mathcal{I}_{\mathcal{K}}$ and $k \in \{|J|, |J|+1\}$. $\diamond$

By the above remark, the following in particular proves the existence of admissible, precompact, transverse perturbations as well as strongly adapted perturbations.

Proposition 3.3.3 (i) *Let $(\mathcal{K}, d)$ be a metric tame Kuranishi atlas with nested reductions $\mathcal{C} \sqsubset \mathcal{V}$ and equivariant norms $\|\cdot\|$ on the obstruction spaces. Then for any $0 < \delta < \delta_{\mathcal{V}}$ and $0 < \sigma \le \sigma(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta)$, there exists a $(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta, \sigma)$–adapted perturbation ν of $\mathfrak{s}_{\mathcal{K}}|^{\setminus\Gamma}_{\mathcal{V}}$.*

(ii) *Let $(\mathcal{K}, d)$ be a metric tame Kuranishi cobordism with nested cobordism reductions $\mathcal{C} \sqsubset \mathcal{V}$, equivariant norms $\|\cdot\|$ on the obstruction spaces, and minimal collar width $\varepsilon > 0$ of $(\mathcal{K}, d)$ and the reductions $\mathcal{C}, \mathcal{V}$. Then, given $0 < \delta < \min\{\varepsilon, \delta_{\mathcal{V}}\}$, $0 < \sigma \le \sigma_{\mathrm{rel}}(\delta, \mathcal{V}, \mathcal{C})$, and perturbations ν^α of $\mathfrak{s}_{\partial^\alpha \mathcal{K}}|^{\setminus\Gamma}_{\partial^\alpha \mathcal{V}}$ for $\alpha = 0, 1$ that are $(\partial^\alpha \mathcal{V}, \partial^\alpha \mathcal{C}, \delta, \sigma)$–adapted, there exists an admissible, precompact, transverse cobordism perturbation ν of $\mathfrak{s}_{\mathcal{K}}|^{\setminus\Gamma}_{\mathcal{V}}$ with $\pi_{\mathcal{K}}\big((\mathfrak{s}_{\mathcal{K}}|^{\setminus\Gamma}_{\mathcal{V}} + \nu)^{-1}(0)\big) \subset \pi_{\mathcal{K}}(\mathcal{C})$ and $\nu|_{\partial^\alpha \mathcal{V}} = \nu^\alpha$ for $\alpha = 0, 1$.*

(iii) *In the case of a product cobordism $[0,1] \times \mathcal{K}$ with product metric and nested product reductions $[0,1] \times \mathcal{C} \sqsubset [0,1] \times \mathcal{V}$, (ii) holds for $0 < \delta < \delta_{[0,1] \times \mathcal{V}}$ without restriction from the collar width.*

Proof As explained in [14, Remark 7.3.2], the iterative constructions in [14, Propositions 7.3.7 and 7.3.10] generalize directly to our setup based on the pruned domain category $\mathcal{B}_{\mathcal{K}}|^{\setminus\Gamma}_{\mathcal{V}}$. We indicated the necessary adjustments in a series of footnotes in the proofs of [14]. Beyond the above setting and notations, this requires the following two systematic changes.

Firstly, all relationships between (or definitions/constructions of) subsets of $\mathrm{Obj}_{\mathcal{B}_{\mathcal{K}}} = \bigcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I$ in [14] should be replaced by two statements: one for subsets of $\mathrm{Obj}_{\mathcal{B}_{\underline{\mathcal{K}}}} = \bigcup_{I\in\mathcal{I}_{\mathcal{K}}} \underline{U}_I$ in the intermediate atlas $\underline{\mathcal{K}}$, and one for subsets in the pruned domain category $\mathcal{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ with B_δ replaced by $\widehat{B}_\delta$. These two statements will always be equivalent via the projection π_I. Statements can then be checked by working in the intermediate category, but they will be applied on the level of the pruned domain category. Here it is crucial to know that the projections $\pi_I\colon U_I \to \underline{U}_I$ are continuous (by definition of the quotient topology) and proper by Lemma 2.1.5(i).

Secondly, our goal of constructing a precompact, transverse, admissible (cobordism) perturbation $\nu\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}$ is essentially the same as that of Definitions 7.2.1, 7.2.5 and 7.2.6 in [14]. Writing it in terms of the maps $\nu = (\nu_I\colon V_I \to E_I)_{I\in\mathcal{I}_{\mathcal{K}}}$, the only difference is that the compatibility conditions in [14, Equation (7.2.1)],

$$\nu_J\big|_{N_{JI}} = \widehat{\phi}_{IJ}\circ\nu_I\circ\phi_{IJ}^{-1}\big|_{N_{JI}} \quad\text{on}\quad N_{JI} := V_J\cap\phi_{IJ}(V_I\cap U_{IJ})$$

for all $I \subsetneq J$, are replaced by

$$\nu_J\big|_{\widetilde{V}_{IJ}} = \widehat{\phi}_{IJ}\circ\nu_I\circ\rho_{IJ}\big|_{\widetilde{V}_{IJ}} \quad\text{on}\quad \widetilde{V}_{IJ} := V_J\cap\rho_{IJ}^{-1}(V_I),$$

and the precompactness conditions in [14, Equation (7.2.5)],

$$(s_J|_{V_J}+\nu_J)^{-1}(0) \subset \bigcup_{H\supset J}\phi_{JH}^{-1}(C_H)\cup\bigcup_{H\subsetneq J}\phi_{HJ}(C_H)$$

for all $J\in\mathcal{I}_{\mathcal{K}}$, are replaced by (3.2.5) above,

$$(s_J|_{V_J}+\nu_J)^{-1}(0) \subset \bigcup_{H\supset J}\rho_{JH}(C_H)\cup\bigcup_{H\subsetneq J}\rho_{HJ}^{-1}(C_H).$$

Here our setup guarantees that $\rho_{IJ}\colon \widetilde{V}_{IJ}\to V_I\cap\rho_{IJ}(V_J)\subset U_{IJ}$ is a regular covering (ie local diffeomorphism with fibers given by the free action of a finite group $\Gamma_{J\setminus I}\cong\Gamma_J/\Gamma_I$) analogous to $\phi_{IJ}^{-1}\colon N_{IJ}\to V_I\cap\phi_{IJ}^{-1}(V_J)\subset U_{IJ}$ in [14], which is a regular covering with trivial fibers. Thus to adapt the proofs of [14] one should replace ϕ_{IJ} with ρ_{IJ}^{-1} and identify $N_{IJ}=\widetilde{V}_{IJ}$. □

Finally, we make the additional choice of an orientation of the Kuranishi atlases or cobordisms in the sense of Definition 3.1.10 to prove Theorem A from the introduction.

Definition 3.3.4 Let $(\mathcal{K},\sigma)$ be an oriented weak Kuranishi atlas of dimension D on a compact, metrizable space X. Then its *virtual moduli cycle* $\mathcal{Z}^{\mathcal{K}} := [(|\boldsymbol{Z}^\nu|_{\mathcal{H}},\Lambda^\nu)]$ is the cobordism class of weighted branched manifolds (without boundary) of dimension D given by the choices of a preshrunk tame shrinking $\mathcal{K}_{\rm sh}$ of $\mathcal{K}$, an admissible metric on $|\mathcal{K}_{\rm sh}|$, nested reductions $\mathcal{C} \sqsubset \mathcal{V}$ of $\mathcal{K}_{\rm sh}$ and a strongly $(\mathcal{V},\mathcal{C})$–adapted perturbation ν.

Moreover, the *virtual fundamental class*

$$[X]^{\rm vir}_{\mathcal{K}} := |\psi_{\mathcal{K}_{\rm sh}}|_*(\varprojlim[\iota^{\nu_k}]) \in \check{H}_D(X;\mathbb{Q})$$

is constructed as follows:

- Choose a preshrunk tame shrinking $\mathcal{K}_{\rm sh}$ of $\mathcal{K}$, an admissible metric on $|\mathcal{K}_{\rm sh}|$ and a nested sequence of open sets $\mathcal{W}_{k+1} \subset \mathcal{W}_k \subset (|\mathcal{K}_{\rm sh}|, d)$ with $\bigcap_{k\in\mathbb{N}} \mathcal{W}_k = |\mathfrak{s}_{\mathcal{K}_{\rm sh}}^{-1}(0)|$. (These exist by Theorem 2.5.3, and taking for instance $\mathcal{W}_k = B_{\frac{1}{k}}(|\mathfrak{s}_{\mathcal{K}_{\rm sh}}^{-1}(0)|.)$ Then equip $\mathcal{K}_{\rm sh}$ with the orientation induced from $\mathcal{K}$ by Lemma 3.1.12.
- For each $k \in \mathbb{N}$ choose a $(\mathcal{V}_k,\mathcal{C}_k)$–adapted perturbation ν_k of $\mathfrak{s}_{\mathcal{K}_{\rm sh}}|_{\mathcal{V}_k}^{\setminus\Gamma}$ for some nested reductions $\mathcal{C}_k \sqsubset \mathcal{V}_k$ with $\pi_{\mathcal{K}_{\rm sh}}(\mathcal{C}_k) \subset \mathcal{W}_k$. (These exist by Remark 3.2.2 and Proposition 3.3.3.)
- Denote by $[|\iota^{\nu_k}|_{\mathcal{H}}] \in \check{H}_D(\mathcal{W}_k;\mathbb{Q})$ the Čech homology classes induced by the maps

$$|\iota^{\nu_k}|_{\mathcal{H}}\colon (|\boldsymbol{Z}^{\nu_k}|_{\mathcal{H}}, \Lambda^{\nu_k}) \hookrightarrow \mathcal{W}_k \subset (|\mathcal{K}_{\rm sh}|, d),$$

take their inverse limit under pushforward with the inclusions $\mathcal{W}_{k+1} \hookrightarrow \mathcal{W}_k$, and finally take the pushforward under the homeomorphism $|\psi_{\mathcal{K}_{\rm sh}}| = \iota_{\mathcal{K}_{\rm sh}}^{-1}\colon |\mathfrak{s}_{\mathcal{K}_{\rm sh}}^{-1}(0)| \to X$ from Lemma 2.3.9(iii).

Note here that every weighted branched manifold (Y,Λ_Y) has a fundamental class $[Y] \in H_d(Y);\mathbb{Q})$ by Proposition A.7. This was constructed in [8] as an element of rational singular homology, and by the discussion after [13, Remark 8.2.4] gives a well-defined element in rational Čech homology. Thus the above construction makes sense. Further, Lemma 2.3.9(iii) identifies the quotient topology on $|\mathfrak{s}_{\mathcal{K}_{\rm sh}}^{-1}(0)|$ with the relative topology induced by the embedding $|\mathfrak{s}_{\mathcal{K}_{\rm sh}}^{-1}(0)| \hookrightarrow |\mathcal{K}_{\rm sh}|$. The latter is also identified with the metric topology given by restriction of d, due to the nesting uniqueness of Hausdorff topologies and the fact that the identity map $|\mathcal{K}| \to (|\mathcal{K}|, d)$ is continuous; see [13, Lemma 3.1.8, Remark 3.1.15]. Hence there is no ambiguity of topologies in the isomorphism explained in [14, Remark 8.2.4] and used in the definition of $[X]^{\rm vir}_{\mathcal{K}}$,

$$\check{H}_D(|\mathfrak{s}_{\mathcal{K}}^{-1}(0)|;\mathbb{Q}) \xrightarrow{\cong} \varprojlim \check{H}_D(\mathcal{W}_k;\mathbb{Q}).$$

Finally, we can prove our main theorem: the VMC/VFC are well defined and are invariants of the oriented weak Kuranishi cobordism class. The proof uses the same line of argument as [14, Theorems 8.2.2 and 8.2.5], just replacing manifolds with weighted branched manifolds. We summarize and unify these arguments here for ease of reference.

Theorem 3.3.5 (i) *The virtual moduli cycle $\mathcal{Z}^{\mathcal{K}}$ and the virtual fundamental class $[X]^{\mathrm{vir}}_{\mathcal{K}}$ are well defined and independent of the cobordism class of oriented weak Kuranishi atlases on a fixed compact, metrizable space X.*

(ii) *Let $\mathcal{K}$ be an oriented weak Kuranishi cobordism, and choose strongly adapted perturbations ν^α in the definition of $\mathcal{Z}^{\partial^\alpha \mathcal{K}} = [(|\boldsymbol{Z}^{\nu^\alpha}|_{\mathcal{H}}, \Lambda^{\nu^\alpha})]$ for $\alpha = 0, 1$. Then the perturbed zero sets $(|\boldsymbol{Z}^{\nu^0}|_{\mathcal{H}}, \Lambda^{\nu^0}) \sim (|\boldsymbol{Z}^{\nu^1}|_{\mathcal{H}}, \Lambda^{\nu^1})$ are cobordant as weighted branched manifolds, and thus $\mathcal{Z}^{\partial^0\mathcal{K}} = \mathcal{Z}^{\partial^1\mathcal{K}}$.*

(iii) *Let $\mathcal{K}$ be an oriented weak Kuranishi cobordism of dimension $D+1$ on a compact, metrizable collared cobordism $(Y, \iota^0_Y, \iota^1_Y)$. Then the virtual fundamental classes $[\partial^\alpha Y]^{\mathrm{vir}}_{\partial^\alpha\mathcal{K}} \simeq [\partial^1 Y]^{\mathrm{vir}}_{\partial^1\mathcal{K}}$ of the boundary restrictions are homologous in Y,*

$$(\iota^0_Y)_*\big([\partial^0 Y]^{\mathrm{vir}}_{\partial^0\mathcal{K}}\big) = (\iota^1_Y)_*\big([\partial^1 Y]^{\mathrm{vir}}_{\partial^1\mathcal{K}}\big) \in \check{H}_D(Y;\mathbb{Q}).$$

Proof First note that all the necessary choices of data exist, as noted in Definition 3.3.4. Given such choices, Step 1 below constructs a representative of the virtual moduli cycle, and Step 5 constructs the virtual fundamental class. To prove independence of those choices in (i), we use transitivity of the cobordism relation for compact weighted branched manifolds to prove increasing independence of choices in Steps 1–5. Parts (ii) and (iii) are then proven in Step 6. In the following, all Kuranishi atlases will be of dimension D, and all cobordisms of dimension $D+1$.

Step 1 *Fix an oriented, metric, tame Kuranishi atlas $(\mathcal{K}, d)$, nested reductions $\mathcal{C} \sqsubset \mathcal{V}$, equivariant norms $\|\cdot\|$, and constants δ, σ such that $0 < \delta < \delta_{\mathcal{V}}$ and $0 < \sigma \le \sigma_{\mathrm{rel}}([0,1]\times\mathcal{V}, [0,1]\times\mathcal{C}, \|\cdot\|, \delta)$. Then each $(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta, \sigma)$–adapted perturbation ν induces a D–dimensional weighted branched manifold $\mathcal{Z}^\nu := (|\boldsymbol{Z}^\nu|_{\mathcal{H}}, \Lambda^\nu)$ and a cycle $|\iota^\nu|_{\mathcal{H}}\colon \mathcal{Z}^\nu \to |\mathcal{C}|$, whose respective cobordism class and Čech homology class $[|\iota^\nu|_{\mathcal{H}}] \in \check{H}_D(|\mathcal{C}|;\mathbb{Q})$ are independent of the choice of ν.*

The regularity of the perturbed zero sets is proven in Theorem 3.2.8. To prove independence of the choice of ν, we consider two $(\mathcal{V}, \mathcal{C}, \|\cdot\|, \delta, \sigma)$–adapted perturbations ν^0 and ν^1. Then Proposition 3.3.3(iii) provides an admissible, precompact, transverse cobordism perturbation ν^{01} of $\mathfrak{s}_{[0,1]\times\mathcal{K}}|^{\setminus\Gamma}_{[0,1]\times\mathcal{V}}$ with boundary restrictions $\nu^{01}|_{\{\alpha\}\times\mathcal{V}} = \nu^\alpha$ for $\alpha = 0, 1$. Moreover, by Lemma 3.1.12(iii) the orientation of $\mathcal{K}$ induces an orientation of $[0,1]\times\mathcal{K}$, whose restriction to the boundaries $\partial^\alpha([0,1]\times\mathcal{K}) = \mathcal{K}$ equals the given orientation on $\mathcal{K}$. Now Theorem 3.2.8 implies that $\mathcal{Z} := (|\boldsymbol{Z}^{\nu^{01}}|, \Lambda^{\nu^{01}})$ is a cobordism from $\partial^0\mathcal{Z} = (|\boldsymbol{Z}^{\nu^0}|, \Lambda^{\nu^0})$ to $\partial^1\mathcal{Z} = (|\boldsymbol{Z}^{\nu^1}|, \Lambda^{\nu^1})$ and induces a cycle $|\iota^{\nu^{01}}|_{\mathcal{H}}\colon \mathcal{Z} \to [0,1]\times|\mathcal{C}|$. Finally, the boundary restrictions of this cycle prove the equality $[|\iota^{\nu^0}|_{\mathcal{H}}] = [|\iota^{\nu^1}|_{\mathcal{H}}]$ in $\check{H}_D(|\mathcal{C}|;\mathbb{Q})$; see [14, Equation (8.2.6)] for the detailed homological argument.

Step 2 *Fix an oriented, metric, tame Kuranishi atlas $(\mathcal{K},d)$ and nested reductions $\mathcal{C}\sqsubset\mathcal{V}$. Then the cobordism class of $\mathcal{Z}^\nu$, as well as $[|\iota^\nu|_\mathcal{H}]\in\check{H}_D(|\mathcal{C}|;\mathbb{Q})$, are independent of the choice of strongly $(\mathcal{V},\mathcal{C})$–adapted perturbation ν.*

To prove this we consider two strongly $(\mathcal{V},\mathcal{C})$–adapted perturbations ν^α for $\alpha=0,1$. Thus ν^α is $(\mathcal{V},\mathcal{C},\|\cdot\|^\alpha,\delta^\alpha,\sigma^\alpha)$–adapted for some choices of equivariant norms $\|\cdot\|^\alpha$ and constants $0<\delta^\alpha<\delta_\mathcal{V}$ and $0<\sigma^\alpha\le\sigma_{\mathrm{rel}}([0,1]\times\mathcal{V},[0,1]\times\mathcal{C},\|\cdot\|^\alpha,\delta^\alpha)$. We note that $\delta:=\max(\delta^0,\delta^1)<\delta_\mathcal{V}=\delta_{[0,1]\times\mathcal{V}}$, pick equivariant norms $\|\cdot\|$ on $\mathcal{K}$ such that $\|\cdot\|^\alpha\le\|\cdot\|$ for $\alpha=0,1$, and choose

$$\sigma\le\min\{\sigma^0,\sigma^1,\sigma_{\mathrm{rel}}([0,1]\times\mathcal{V},[0,1]\times\mathcal{C},\|\cdot\|,\delta)\}.$$

Then Proposition 3.3.3(iii) provides an admissible, precompact, transverse cobordism perturbation ν^{01} of $\mathfrak{s}_{[0,1]\times\mathcal{K}}|^{\setminus\Gamma}_{[0,1]\times\mathcal{V}}$, whose restrictions $\widetilde{\nu}^\alpha:=\nu^{01}|_{\{\alpha\}\times\mathcal{V}}$ for $\alpha=0,1$ are $(\mathcal{V},\mathcal{C},\|\cdot\|,\delta,\sigma)$–adapted perturbations of $\mathfrak{s}_\mathcal{K}|^{\setminus\Gamma}_\mathcal{V}$. Since we have that $\delta^\alpha\le\delta$, $\|\nu^{01}|_{\{\alpha\}\times\mathcal{V}}\|^\alpha\le\|\nu^{01}|_{\{\alpha\}\times\mathcal{V}}\|<\sigma$ and $\sigma\le\sigma^\alpha\le\sigma_{\mathrm{rel}}([0,1]\times\mathcal{V},[0,1]\times\mathcal{C},\|\cdot\|,\delta^\alpha)$, they are also $(\mathcal{V},\mathcal{C},\|\cdot\|,\delta^\alpha,\sigma^\alpha)$–adapted. Then, as in Step 1, the perturbed zero set of ν^{01} is a cobordism from $\mathcal{Z}^{\widetilde{\nu}^0}$ to $\mathcal{Z}^{\widetilde{\nu}^1}$, and the induced cycle in $[0,1]\times|\mathcal{C}|$ shows $[|\iota^{\widetilde{\nu}^0}|_\mathcal{H}]=[|\iota^{\widetilde{\nu}^1}|_\mathcal{H}]$ in $\check{H}_D(|\mathcal{C}|;\mathbb{Q})$.

Moreover, for fixed $\alpha\in\{0,1\}$ both the restriction $\widetilde{\nu}^\alpha=\nu^{01}|_{\{\alpha\}\times\mathcal{V}}$ and the given perturbation ν^α are $(\mathcal{V},\mathcal{C},\|\cdot\|,\delta^\alpha,\sigma^\alpha)$–adapted, so that Step 1 provides cobordisms $\mathcal{Z}^{\nu^\alpha}\sim\mathcal{Z}^{\widetilde{\nu}^\alpha}$ and identities $[|\iota^{\nu^\alpha}|_\mathcal{H}]=[|\iota^{\widetilde{\nu}^\alpha}|_\mathcal{H}]$ in $\check{H}_D(|\mathcal{C}|;\mathbb{Q})$. By transitivity of the cobordism relation this proves $\mathcal{Z}^{\nu^0}\sim\mathcal{Z}^{\nu^1}$ as claimed, and also $[|\iota^{\nu^0}|_\mathcal{H}]=[|\iota^{\nu^1}|_\mathcal{H}]\in\check{H}_D(|\mathcal{C}|;\mathbb{Q})$.

Step 3 *For a fixed oriented, metric, tame Kuranishi atlas $(\mathcal{K},d)$, the oriented cobordism class $\mathcal{A}^{(\mathcal{K},d)}$ of weighted branched manifolds $\mathcal{Z}^\nu$ is independent of the choice of strongly adapted perturbation ν. Moreover, given any open neighborhood $\mathcal{W}\subset(|\mathcal{K}|,d)$ of $|\mathfrak{s}_\mathcal{K}^{-1}(0)|$, the class $A^{(\mathcal{K},d)}_\mathcal{W}:=[|\iota^\nu|_\mathcal{H}\colon\mathcal{Z}^\nu\to\mathcal{W}]\in\check{H}_D(\mathcal{W};\mathbb{Q})$ is independent of the choice of strongly $(\mathcal{V},\mathcal{C})$–adapted perturbation ν for nested reductions $\mathcal{C}\sqsubset\mathcal{V}$ with $\pi_\mathcal{K}(\mathcal{C})\subset\mathcal{W}$.*

To prove this we consider two strongly $(\mathcal{V}^\alpha,\mathcal{C}^\alpha)$–adapted perturbations ν^α with respect to nested reductions $\mathcal{C}^\alpha\sqsubset\mathcal{V}^\alpha$ with $\pi_\mathcal{K}(\mathcal{C})\subset\mathcal{W}$, equivariant norms $\|\cdot\|^\alpha$ and admissible metrics d^α for $\alpha=0,1$. Remark 3.2.2 provides a nested cobordism reduction $\mathcal{C}\sqsubset\mathcal{V}$ of $[0,1]\times\mathcal{K}$ with $\partial^\alpha\mathcal{C}=\mathcal{C}^\alpha$, $\partial^\alpha\mathcal{V}=\mathcal{V}^\alpha$ and $\pi_{[0,1]\times\mathcal{C}}\subset[0,1]\times\mathcal{W}$. Now pick equivariant norms $\|\cdot\|$ on $\mathcal{K}$ such that $\|\cdot\|^\alpha\le\|\cdot\|$ for $\alpha=0,1$, and choose $0<\delta<\delta_\mathcal{V}$ smaller than the collar width of d, $\mathcal{V}$, and $\mathcal{C}$. Then, for any $0<\sigma\le\sigma_{\mathrm{rel}}(\mathcal{V},\mathcal{C},\|\cdot\|,\delta)$, Proposition 3.3.3(ii) provides an admissible, precompact, transverse cobordism perturbation ν^{01} of $\mathfrak{s}_{[0,1]\times\mathcal{K}}|^{\setminus\Gamma}_\mathcal{V}$ whose boundary restrictions $\widetilde{\nu}^\alpha:=\nu^{01}|_{\partial^\alpha\mathcal{V}}$ for $\alpha=0,1$ are $(\mathcal{V}^\alpha,\mathcal{C}^\alpha,\|\cdot\|,\delta,\sigma)$–adapted perturbations of $\mathfrak{s}_\mathcal{K}|^{\setminus\Gamma}_{\mathcal{V}^\alpha}$.

As before, $\mathcal{Z}^{\nu^{01}}$ is an oriented cobordism from $\mathcal{Z}^{\widetilde{\nu}^0}$ to $\mathcal{Z}^{\widetilde{\nu}^1}$ and induces a cycle in $[0,1]\times\mathcal{W}$ that shows $[|\iota^{\widetilde{\nu}^0}|_{\mathcal{H}}] = [|\iota^{\widetilde{\nu}^1}|_{\mathcal{H}}]$ in $\check{H}_D(\mathcal{W};\mathbb{Q})$. Moreover, we can pick $\sigma \le \sigma_{\mathrm{rel}}([0,1]\times\mathcal{V}^\alpha, [0,1]\times\mathcal{C}^\alpha, \|\cdot\|^\alpha, \delta)$ for $\alpha = 0,1$, so that each $\nu^{01}|_{\partial^\alpha\mathcal{V}}$ is also strongly $(\mathcal{V}^\alpha,\mathcal{C}^\alpha)$–adapted. Then the claim follows by transitivity as in Step 2.

Step 4 *Let $(\mathcal{K},d)$ be an oriented, metric, tame Kuranishi atlas, and let $\mathcal{W}_k\subset(|\mathcal{K}|,d)$ be a nested sequence of open sets with $\bigcap_{k\in\mathbb{N}}\mathcal{W}_k = |\mathfrak{s}_{\mathcal{K}}^{-1}(0)|$ as in Definition 3.3.4. Then the Čech homology class*

$$A^{(\mathcal{K},d)} := \varprojlim A^{(\mathcal{K},d)}_{\mathcal{W}_k} \in \check{H}_D(|\mathfrak{s}_{\mathcal{K}}^{-1}(0)|;\mathbb{Q})$$

is well defined and independent of the choice of nested sequence $(\mathcal{W}_k)_{k\in\mathbb{N}}$.

The pushforward $\check{H}_D(\mathcal{W}_{k+1};\mathbb{Q})\to\check{H}_D(\mathcal{W}_k;\mathbb{Q})$ by the inclusion $\mathcal{I}_{k+1}\colon \mathcal{W}_{k+1}\to\mathcal{W}_k$ maps $A^{(\mathcal{K},d)}_{\mathcal{W}_{k+1}} = [|\iota^{\nu_{k+1}}|_{\mathcal{H}}]$ to $A^{(\mathcal{K},d)}_{\mathcal{W}_k}$ since any strongly adapted perturbation ν_{k+1} with respect to nested reductions $\mathcal{C}_{k+1}\sqsubset\mathcal{V}_{k+1}$ with $\pi_{\mathcal{K}}(\mathcal{C}_{k+1})\subset\mathcal{W}_{k+1}$ can also be used as strongly adapted perturbation for $A^{(\mathcal{K},d)}_{\mathcal{W}_k}$. This shows that the homology classes $A^{(\mathcal{K},d)}_{\mathcal{W}_k}$ form an inverse system and thus have a well-defined inverse limit. To see that this limit is independent of the choice of nested sequence, note that the intersection $\mathcal{W}_k := \mathcal{W}^0_k\cap\mathcal{W}^1_k$ of any two such sequences $(\mathcal{W}^\alpha_k)_{k\in\mathbb{N}}$ is another nested sequence of open sets with $\bigcap_{k\in\mathbb{N}}\mathcal{W}_k = |\mathfrak{s}_{\mathcal{K}}^{-1}(0)|$. Now choose a sequence of strongly adapted perturbations ν_k with respect to nested reductions $\mathcal{C}_k\sqsubset\mathcal{V}_k$ with $\pi_{\mathcal{K}}(\mathcal{C}_k)\subset\mathcal{W}_k$, then these also fit the requirements for the larger open sets $\mathcal{W}^\alpha_k$ and hence the inclusions $\mathcal{W}_k\hookrightarrow\mathcal{W}^\alpha_k$ push $[|\iota^{\nu_k}|_{\mathcal{H}}]\in H_D(\mathcal{W}_k;\mathbb{Q})$ forward to $[|\iota^{\nu_k}|_{\mathcal{H}}]\in H_D(\mathcal{W}^\alpha_k;\mathbb{Q})$. Hence, by the definition of the inverse limit, we have equality

$$\varprojlim A^{(\mathcal{K},d)}_{\mathcal{W}^0_k} = \varprojlim A^{(\mathcal{K},d)}_{\mathcal{W}_k} = \varprojlim A^{(\mathcal{K},d)}_{\mathcal{W}^0_k} \in \check{H}_D(|\mathfrak{s}_{\mathcal{K}}^{-1}(0)|;\mathbb{Q}).$$

Step 5 *Given an oriented weak Kuranishi atlas $\mathcal{K}$, the cobordism class $\mathcal{Z}^{\mathcal{K}} := \mathcal{A}^{(\mathcal{K}_{\mathrm{sh}},d)}$ of weighted branched manifolds in Step 3 and the pullback $[X]^{\mathrm{vir}}_{\mathcal{K}} := |\psi_{\mathcal{K}_{\mathrm{sh}}}|_* A^{(\mathcal{K},d)} \in \check{H}_D(X;\mathbb{Q})$ of the Čech homology classes in Step 4 are independent of the choice of tame shrinking $\mathcal{K}_{\mathrm{sh}}$ of $\mathcal{K}$ and admissible metric d on $|\mathcal{K}_{\mathrm{sh}}|$.*

Here the pushforward under $|\psi_{\mathcal{K}_{\mathrm{sh}}}|$ is well defined since this is a homeomorphism by Lemma 2.3.9(iii). Given different choices $(\mathcal{K}^\alpha_{\mathrm{sh}}, d^\alpha)$ of metric tame shrinkings of $\mathcal{K}$ and strongly adapted perturbations ν^α and $(\nu^\alpha_k)_{k\in\mathbb{N}}$) that define $\mathcal{A}^{(\mathcal{K}^\alpha_{\mathrm{sh}},d^\alpha)}\sim\mathcal{Z}^{\nu^\alpha}$ and

$$A^{(\mathcal{K}^\alpha_{\mathrm{sh}},d^\alpha)} = \varprojlim [|\iota^{\nu^\alpha_k}|_{\mathcal{H}}] \in \check{H}_D(|\mathfrak{s}_{\mathcal{K}^\alpha_{\mathrm{sh}}}^{-1}(0)|;\mathbb{Q})$$

respectively, we can apply Step 6 below to the cobordism $[0,1]\times\mathcal{K}$ to obtain a weighted branched cobordism from $\mathcal{Z}^{\nu^0}$ to $\mathcal{Z}^{\nu^1}$ and the identity

$$I^0_*([X]^{\mathrm{vir}}_{\mathcal{K}^0_{\mathrm{sh}}}) = I^1_*([X]^{\mathrm{vir}}_{\mathcal{K}^1_{\mathrm{sh}}}) \in \check{H}_D([0,1]\times X;\mathbb{Q})$$

with the natural boundary embeddings $I^\alpha\colon X \to \{\alpha\}\times X \subset [0,1]\times X = Y$. Further, $I^0_* = I^1_*\colon \check{H}_D(X;\mathbb{Q}) \to \check{H}_D([0,1]\times X;\mathbb{Q})$ are the same isomorphisms, because the two maps I^0, I^1 are both homotopy equivalences and homotopic to each other. Hence we obtain the identity $[X]^{\mathrm{vir}}_{\mathcal{K}^0_{\mathrm{sh}}} = [X]^{\mathrm{vir}}_{\mathcal{K}^1_{\mathrm{sh}}}$ in $\check{H}_D(X;\mathbb{Q})$, which proves Step 5.

Step 6 *Let $\mathcal{K}$ be an oriented weak Kuranishi cobordism over a compact collared cobordism Y. For $\alpha = 0,1$ fix choices of preshrunk tame shrinkings $\mathcal{K}^\alpha_{\mathrm{sh}}$ of $\partial^\alpha\mathcal{K}$, and admissible metrics d^α on $|\partial^\alpha\mathcal{K}|$. Then, for any choice of strongly adapted perturbations ν^α on $\mathcal{K}^\alpha_{\mathrm{sh}}$, there is a weighted branched cobordism $\mathcal{Z}^{\nu^{01}}$ from $\mathcal{Z}^{\nu^0}$ to $\mathcal{Z}^{\nu^1}$. Moreover, the VFCs of the boundary components push forward by the embeddings $\iota^\alpha_Y\colon \{\alpha\}\times\partial^\alpha Y \to Y$ to the same Čech homology class in Y,*

$$(\iota^0_Y)_*([\partial^0 Y]^{\mathrm{vir}}_{\partial^0\mathcal{K}}) = (\iota^1_Y)_*([\partial^1 Y]^{\mathrm{vir}}_{\partial^1\mathcal{K}}) \in \check{H}_D(Y;\mathbb{Q}).$$

First, use Theorem 2.5.3 to find a preshrunk tame shrinking $\mathcal{K}_{\mathrm{sh}}$ of $\mathcal{K}$ with $\partial^\alpha\mathcal{K}_{\mathrm{sh}} = \mathcal{K}^\alpha_{\mathrm{sh}}$, and an admissible metric d on $|\mathcal{K}_{\mathrm{sh}}|$ with boundary restrictions $d|_{|\partial^\alpha\mathcal{K}_{\mathrm{sh}}|} = d^\alpha$. If we equip $\mathcal{K}_{\mathrm{sh}}$ with the orientation induced by $\mathcal{K}$, then by Lemma 3.1.12 the induced boundary orientation on $\partial^\alpha\mathcal{K}_{\mathrm{sh}} = \mathcal{K}^\alpha_{\mathrm{sh}}$ agrees with that induced by shrinking from $\partial^\alpha\mathcal{K}$. Next, Remark 3.2.2 provides nested cobordism reductions $\mathcal{C}\sqsubset\mathcal{V}$ of $\mathcal{K}_{\mathrm{sh}}$ and we may choose equivariant norms $\|\cdot\|$ on $\mathcal{K}_{\mathrm{sh}}$. Then Proposition 3.3.3 with

$$\sigma = \min\Bigl\{\sigma_{\mathrm{rel}}(\mathcal{V},\mathcal{C},\|\cdot\|,\delta),\ \min_{\alpha=0,1}\sigma_{\mathrm{rel}}([0,1]\times\partial^\alpha\mathcal{V},[0,1]\times\partial^\alpha\mathcal{C},\partial^\alpha\|\cdot\|,\delta)\Bigr\}$$

yields an admissible, precompact, transverse cobordism perturbation ν^{01} of $\mathfrak{s}_{\mathcal{K}_{\mathrm{sh}}}|^{\setminus\Gamma}_{\mathcal{V}}$, whose restrictions $\widetilde{\nu}^\alpha := \nu^{01}|_{\partial^\alpha\mathcal{V}}$ for $\alpha = 0,1$ are $(\partial^\alpha\mathcal{V},\partial^\alpha\mathcal{C},\partial^\alpha\|\cdot\|,\delta,\sigma)$–adapted perturbations of $\mathfrak{s}_{\mathcal{K}^\alpha_{\mathrm{sh}}}|^{\setminus\Gamma}_{\partial^\alpha\mathcal{V}}$. In particular, these are strongly adapted by the choice of σ, and $\mathcal{Z}^{\nu^{01}}$ is a cobordism from $\mathcal{Z}_{\widetilde{\nu}^0}$ to $\mathcal{Z}_{\widetilde{\nu}^1}$. Invariance of the VMC under oriented weak Kuranishi cobordism then follows from Step 3 by transitivity of weighted branched cobordism.

To prove the identity between VFCs, we first construct a sequence of nested cobordism reductions $\mathcal{C}_k\sqsubset\mathcal{V}$ of $\mathcal{K}_{\mathrm{sh}}$ by

$$\mathcal{C}_k := \mathcal{C}\cap\pi^{-1}_{\mathcal{K}_{\mathrm{sh}}}(\mathcal{W}_k)\sqsubset\mathcal{V} \quad\text{with}\quad \mathcal{W}_k := B_{\frac{1}{k}}(\iota_{\mathcal{K}_{\mathrm{sh}}}(Y))\subset|\mathcal{K}_{\mathrm{sh}}|,$$

in addition discarding components $C_k\cap V_I$ that have empty intersection with $s_I^{-1}(0)$. With that, Proposition 3.3.3 provides admissible, precompact, transverse cobordism perturbations ν_k with $|(\mathfrak{s}_{\mathcal{K}_{sh}}|^{\setminus\Gamma}_{\mathcal{V}}+\nu_k)^{-1}(0)|\subset\mathcal{W}_k$, and with boundary restrictions $\nu^\alpha_k := \nu_k|_{\partial^\alpha\mathcal{V}}$ that are strongly adapted perturbations of $(\mathcal{K}^\alpha_{\mathrm{sh}}, d^\alpha)$ for $\alpha = 0,1$. Since these boundary restrictions satisfy the requirements of Step 4, they define the Čech homology classes $A^{(\mathcal{K}^\alpha_{\mathrm{sh}},d^\alpha)} = \varprojlim[|\iota^{\nu^\alpha_k}|_{\mathcal{H}}]\in\check{H}_D(|\mathfrak{s}^{-1}_{\mathcal{K}^\alpha_{\mathrm{sh}}}(0)|;\mathbb{Q})$.

On the other hand, pushforward with the topological embeddings $J^\alpha\colon(|\mathcal{K}^\alpha_{\mathrm{sh}}|,d^\alpha)\to(|\mathcal{K}_{\mathrm{sh}}|,d)$ also yields Čech homology classes $J^\alpha_*[|\iota^{\nu^\alpha_k}|_{\mathcal{H}}]$ that form two inverse systems in $H_D(|\mathcal{K}_{\rm sh}|;\mathbb{Q})$. Now the cycles $\iota^{\nu_k}\colon |\mathcal{Z}^{\nu_k}| \to \mathcal{W}_k$ given by Theorem 3.2.8 give rise to identities $J^0_*[|\iota^{\nu^0_k}|_{\mathcal{H}}] = J^1_*[|\iota^{\nu^1_k}|_{\mathcal{H}}]$ in $\check{H}_D(\mathcal{W}_k;\mathbb{Q})$, and taking the inverse limit, which commutes with pushforward, we obtain $J^0_*(\varprojlim [|\iota^{\nu^0_k}|_{\mathcal{H}}]) = J^1_*(\varprojlim [|\iota^{\nu^1_k}|_{\mathcal{H}}])$ in $\check{H}_D(|\mathfrak{s}^{-1}_{\mathcal{K}_{\rm sh}}(0)|;\mathbb{Q})$. Further pushforward with $|\psi_{\mathcal{K}_{\rm sh}}|$ turns this into an equality in $\check{H}_D(Y;\mathbb{Q})$. Finally, we use the identities

$$|\psi_{\mathcal{K}_{\rm sh}}| \circ J^\alpha\big|_{\iota_{\mathcal{K}^\alpha_{\rm sh}}(\partial^\alpha Y)} = \iota^\alpha_Y \circ |\psi_{\mathcal{K}^\alpha_{\rm sh}}|$$

to obtain, in $\check{H}_D(Y;\mathbb{Q})$,

$$(|\psi_{\mathcal{K}_{\rm sh}}| \circ J^\alpha)_*(\varprojlim [i^{\nu^\alpha_k}]) = (\iota^\alpha_Y)_*\big(|\psi_{\mathcal{K}^\alpha_{\rm sh}}|_*(\varprojlim [i^{\nu^\alpha_k}])\big) = (\iota^\alpha_Y)_*[\partial^\alpha Y]^{\rm vir}_{\mathcal{K}^\alpha_{\rm sh}}.$$

This proves Step 6 since the left-hand side was shown to be independent of $\alpha = 0, 1$. □

Appendix: Groupoids and weighted branched manifolds

The purpose of this appendix is to review the definition and properties of weighted branched manifolds from [8], and slightly generalize these notions to a cobordism theory. This will be based on the following language of groupoids.

An *étale groupoid* $\boldsymbol{G}$ is a small category whose sets of objects $\mathrm{Obj}_{\boldsymbol{G}}$ and morphisms $\mathrm{Mor}_{\boldsymbol{G}}$ are equipped with the structure of a smooth manifold of a fixed finite dimension such that

- all morphisms are invertible;
- all structural maps[9] are local diffeomorphisms.

All groupoids considered in this appendix are étale. Moreover, a groupoid is called

- *proper* if the source and target map $s \times t\colon \mathrm{Mor}_{\boldsymbol{G}} \to \mathrm{Obj}_{\boldsymbol{G}} \times \mathrm{Obj}_{\boldsymbol{G}}$ is proper (ie preimages of compact sets are compact);
- *nonsingular* if there is at most one morphism between any two of its objects;
- *oriented* if its spaces of objects and morphisms are oriented manifolds and if all structural maps preserve these orientations;
- *d–dimensional* if $\mathrm{Obj}_{\boldsymbol{G}}$ and $\mathrm{Mor}_{\boldsymbol{G}}$ are d–dimensional manifolds;
- *compact* if its realization $|\boldsymbol{G}|$ is compact.

[9] The structure maps of a category are source and target maps $s, t\colon \mathrm{Mor}_{\boldsymbol{G}} \to \mathrm{Obj}_{\boldsymbol{G}}$, identity map $\mathrm{id}\colon \mathrm{Obj}_{\boldsymbol{G}} \to \mathrm{Mor}_{\boldsymbol{G}}$, and composition map $\mathrm{comp}\colon \mathrm{Mor}_{\boldsymbol{G}}\, {}_t\!\times_s \mathrm{Mor}_{\boldsymbol{G}} \to \mathrm{Mor}_{\boldsymbol{G}}$. If source and target are local diffeomorphisms, then the fiber product in the domain of composition is transverse and hence inherits a smooth structure. A groupoid has the additional structure map $\mathrm{inv}\colon \mathrm{Mor}_{\boldsymbol{G}} \to \mathrm{Mor}_{\boldsymbol{G}}$ given by the unique inverses.

Étale proper groupoids are often called *ep groupoids*. It is well known that in the current finite-dimensional context the properness assumption is equivalent to the condition that the realization $|\boldsymbol{G}|$ is Hausdorff.[10] Here the realization $|\boldsymbol{G}|$ of $\boldsymbol{G}$ is the quotient of the space of objects by the equivalence relation given by the morphisms, ie $x \sim y \iff \mathrm{Mor}_{\boldsymbol{G}}(x,y) \neq \varnothing$. It is equipped with the quotient topology, and the natural projection is denoted $\pi_{\boldsymbol{G}}\colon \mathrm{Obj}_{\boldsymbol{G}} \to |\boldsymbol{G}|$. In general, the realization $|\boldsymbol{G}|$ of an ep groupoid is an orbifold. It is a manifold if the groupoid is nonsingular, and an orientation of the groupoid induces an orientation of $|\boldsymbol{G}|$.

Two kinds of groupoids appear in this paper: Theorem 3.2.8 shows that the zero set of a transverse section defines a wnb groupoid (which is étale but generally not proper, and equipped with an additional weighting function; see Definition A.4). On the other hand, each Kuranishi chart $\boldsymbol{K}_I$ comprises two ep groupoids $\boldsymbol{G}_{(U_I,\Gamma_I)}$ and $\boldsymbol{G}_{(U_I\times E_I,\Gamma_I)}$, which arise from group quotients as follows.

Example A.1 (i) A group quotient (U,Γ) in the sense of Definition 2.1.1 defines an ep groupoid $\boldsymbol{G}_{(U,\Gamma)}$ with $\mathrm{Obj}_{\boldsymbol{G}} = U$, $\mathrm{Mor}_{\boldsymbol{G}} = U \times \Gamma$, $(s\times t)(u,\gamma) = (u,\gamma u)$, $\mathrm{id}(u) = (u,\mathrm{id})$, $\mathrm{comp}((u,\gamma),(\gamma u,\delta)) = (u,\delta\gamma)$, $\mathrm{inv}(u,\gamma) = (u,\gamma^{-1})$, and realization $|\boldsymbol{G}| = U/\Gamma = \underline{U}$. In particular, properness is proven in Lemma 2.1.5(i). This groupoid is nonsingular if and only if the action of Γ is free. It is oriented if U is oriented and the action of each $\gamma\in\Gamma$ preserves the orientation.

(ii) The category $\boldsymbol{B}_{\mathcal{K}}$ defined by a Kuranishi atlas with trivial obstruction spaces on a compact space X is not a groupoid, because when $I \subsetneq J$ the morphisms from U_I to U_J are not invertible. However, it is shown in [10] that $\boldsymbol{B}_{\mathcal{K}}$ may be completed to an ep groupoid with the same realization (namely, X itself) by adding appropriate inverses and composites to its set of morphisms. $\Diamond$

When we take restrictions of Kuranishi charts in the sense of Definition 2.2.6, this is reflected in the associated groupoids by an analogous notion:

- If $\boldsymbol{G}$ is an étale groupoid and $\underline{V} \subset |\boldsymbol{G}|$ is open, we define the *restriction* $\boldsymbol{G}|_{\underline{V}}$ to be the full subcategory of $\boldsymbol{G}$ with objects $\pi_{\boldsymbol{G}}^{-1}(\underline{V})$.

To discuss the theory of Kuranishi cobordisms in terms of groupoids, we need the following notions. Here we use the notation $A^0_\varepsilon := [0,\varepsilon)$ and $A^1_\varepsilon := (1-\varepsilon,1]$ for neighborhoods of $0,1\in[0,1]$ of size $\varepsilon>0$ as in [13].

- If $\boldsymbol{G}$ is a groupoid and $A\subset\mathbb{R}$ is an interval we define the *product groupoid* $A\times\boldsymbol{G}$ to be the groupoid with objects $A\times\mathrm{Obj}_{\boldsymbol{G}}$ and morphisms $A\times\mathrm{Mor}_{\boldsymbol{G}}$, and with all structural maps given by products with id_A.

[10] To see that proper groupoids have Hausdorff realization one can argue that the equivalence relation has closed graph and then use [1, Chapter I, Section 10, Exercise 19] or [13, Lemma 3.2.4].

- A *cobordism groupoid* is a triple $(\boldsymbol{G}, \iota^0_{\boldsymbol{G}}, \iota^1_{\boldsymbol{G}})$ consisting of a compact proper groupoid $\boldsymbol{G}$ and collaring functors $\iota^\alpha_{\boldsymbol{G}}\colon A^\alpha_\varepsilon \times \partial^\alpha \boldsymbol{G} \to \boldsymbol{G}$ for $\alpha = 0, 1$. Here $\boldsymbol{G}$ is required to be "étale with boundary" in the sense that its object and morphism spaces are manifolds with boundary. Moreover, these boundaries form a strictly full[11] subcategory $\partial \boldsymbol{G}$ of $\boldsymbol{G}$ that splits, $\partial(\mathrm{Obj}_{\boldsymbol{G}}) = \mathrm{Obj}_{\partial^0 \boldsymbol{G}} \sqcup \mathrm{Obj}_{\partial^1 \boldsymbol{G}}$, $\partial(\mathrm{Mor}_{\boldsymbol{G}}) = \mathrm{Mor}_{\partial^0 \boldsymbol{G}} \sqcup \mathrm{Mor}_{\partial^1 \boldsymbol{G}}$, into the disjoint union of two ep groupoids $\partial^0 \boldsymbol{G}$ and $\partial^1 \boldsymbol{G}$. Finally, the functors $\iota^\alpha_{\boldsymbol{G}}\colon A^\alpha_\varepsilon \times \partial^\alpha \boldsymbol{G} \to \boldsymbol{G}$ are defined for some $\varepsilon > 0$ and required to be tubular neighborhood diffeomorphisms on both the sets of objects and morphisms. In particular, $\iota^\alpha_{\boldsymbol{G}}(\alpha, \cdot)$ is the identification between $\partial^\alpha \boldsymbol{G}$ and the full subcategories formed by the boundary components of $\boldsymbol{G}$.
- An *oriented cobordism groupoid* is a cobordism groupoid $(\boldsymbol{G}, \iota^0_{\boldsymbol{G}}, \iota^1_{\boldsymbol{G}})$ such that both $\boldsymbol{G}$ and its boundary groupoids $\partial^0 \boldsymbol{G}, \partial^1 \boldsymbol{G}$ are oriented. Moreover the collaring functors are required to consist of orientation-preserving maps $\iota^\alpha_{\boldsymbol{G}}\colon A^\alpha_\varepsilon \times \mathrm{Obj}_{\partial^\alpha \boldsymbol{G}} \to \mathrm{Obj}_{\boldsymbol{G}}$ and $\iota^\alpha_{\boldsymbol{G}}\colon A^\alpha_\varepsilon \times \mathrm{Mor}_{\partial^\alpha \boldsymbol{G}} \to \mathrm{Mor}_{\boldsymbol{G}}$ for $\alpha = 0, 1$, where products are oriented as in Remark 3.1.11.

Lemma A.2 *Any topological space Y has a unique* ***maximal Hausdorff quotient*** *$Y_{\mathcal{H}}$, that is, a quotient of Y which is Hausdorff and satisfies the universal property: any continuous map from Y to a Hausdorff space factors through the quotient map $\pi_{\mathcal{H}}\colon Y \to Y_{\mathcal{H}}$.*

Proof To construct the maximal Hausdorff quotient let A be the set of all equivalence relations $\sim$ on Y for which the quotient topology on $Y/\sim$ is Hausdorff. This is a set since every relation $\sim$ on Y is represented by a subset of $Y \times Y$. Then the space $Y_A := \prod_{\sim \in A} Y/\sim$ is a product of Hausdorff spaces, hence Hausdorff. The map $\pi\colon Y \to Y_A$, $y \mapsto \prod_{\sim \in A} [y]_\sim$ is continuous by the definition of quotient topologies. Now the image $Y_{\mathcal{H}} := \pi(Y) \subset Y_A$ with the relative topology is Hausdorff, and π induces a continuous surjection $\pi_{\mathcal{H}}\colon Y \to Y_{\mathcal{H}}$.

To check that $\pi_{\mathcal{H}}\colon Y \to Y_{\mathcal{H}}$ satisfies the universal property, consider a continuous map $f\colon Y \to Z$ to a Hausdorff space Z. This induces an equivalence relation $\sim_f$ on Y given by $x \sim_f y \iff f(x) = f(y)$, whose quotient space $Y/\sim_f$ we equip with the quotient topology. Then $f\colon Y \to Z$ factors as

$$Y \xrightarrow{\pi_f} Y/\sim_f \xrightarrow{\iota_f} Z,$$

where $\iota_f\colon [y] \mapsto f(y)$ is continuous by definition of the quotient topology. Since ι_f is also injective, this implies that $Y/\sim_f$ is Hausdorff. Therefore, $Y/\sim_f$ is one of the

[11] A subcategory is strictly full if it contains all morphisms that have source or target in its objects.

factors of Y_A, so that $f\colon Y \to Z$ factors as the following sequence of continuous maps

$$Y \xrightarrow{\pi_{\mathcal{H}}} Y_{\mathcal{H}} \xrightarrow{\mathrm{pr}_f} Y/{\sim_f} \xrightarrow{\iota_f} Z,$$

where pr_f denotes the restriction to $Y_{\mathcal{H}}$ of the projection from Y_A to its factor $Y/{\sim_f}$.

To see that $Y_{\mathcal{H}}$ is in fact a quotient of Y, we will identify $Y_{\mathcal{H}} = \pi(Y)$ with the quotient $Y/{\sim_\pi}$ that is induced by the surjection $\pi_{\mathcal{H}}\colon Y \to Y_{\mathcal{H}}$. In this case the injection $\iota_\pi\colon Y/{\sim_\pi} \to Y_{\mathcal{H}}$ is in fact a continuous bijection by continuity and surjectivity of $\pi_{\mathcal{H}}$. In particular, this implies that $Y/{\sim_\pi}$ is Hausdorff, so that we have a continuous map $\mathrm{pr}_\pi\colon Y_{\mathcal{H}} \to Y/{\sim_\pi}$ by restriction of the projection $Y_A \to Y/{\sim_\pi}$ as above. It is inverse to ι_π because for $[y] \in Y/{\sim_\pi}$, we have

$$\mathrm{pr}_\pi(\iota_\pi([y])) = \mathrm{pr}_\pi(\pi_{\mathcal{H}}(y)) = \mathrm{pr}_\pi(\cdots \times [y] \times \cdots) = [y].$$

This identifies $Y_{\mathcal{H}} \cong Y/{\sim_\pi}$ as topological spaces and thus finishes the proof that a topological space $Y_{\mathcal{H}}$ with the above properties exists.

To prove uniqueness, consider another Hausdorff quotient $\mathrm{pr}\colon Y \to Y/{\sim}$ that satisfies the universal property. Then pr factors,

$$Y \xrightarrow{\pi_{\mathcal{H}}} Y_{\mathcal{H}} \xrightarrow{a} Y/{\sim},$$

and by the universal property $\pi_{\mathcal{H}}\colon Y \to Y_{\mathcal{H}}$ factors,

$$Y \xrightarrow{\mathrm{pr}} Y/{\sim} \xrightarrow{b} Y_{\mathcal{H}}.$$

Then a is surjective since pr is. Moreover, a is injective, because otherwise there would be two points $y_1, y_2 \in Y$ with $\pi_{\mathcal{H}}(y_1) \neq \pi_{\mathcal{H}}(y_2)$ but $\mathrm{pr}(y_1) = a(\pi(y_1)) = a(\pi(y_2)) = \mathrm{pr}(y_2)$, so that $\pi(y_1) = b(\mathrm{pr}(y_1)) = b(\mathrm{pr}(y_2)) = \pi(y_2)$, a contradiction. A similar argument shows that b is bijective. Moreover, the composite $b^{-1}a\colon Y_{\mathcal{H}} \to Y_{\mathcal{H}}$ has the property that $b^{-1}a \circ \pi_{\mathcal{H}} = \pi_{\mathcal{H}}$. Since $\pi_{\mathcal{H}}$ is surjective this implies $b^{-1}a = \mathrm{id}$, and similarly $a^{-1}b = \mathrm{id}$. Finally, note that because both $Y_{\mathcal{H}}$ and $Y/{\sim}$ have the quotient topology, a and b are continuous, and hence homeomorphisms. □

In the following we write $|\boldsymbol{G}|$ for the realization $\mathrm{Obj}_{\boldsymbol{G}}/{\sim}$ of an étale groupoid $\boldsymbol{G}$, and $|\boldsymbol{G}|_{\mathcal{H}}$ for its maximal Hausdorff quotient. We denote the natural maps by

$$\pi_{\boldsymbol{G}}\colon \mathrm{Obj}_{\boldsymbol{G}} \to |\boldsymbol{G}|, \quad \pi^{\mathcal{H}}_{|\boldsymbol{G}|}\colon |\boldsymbol{G}| \longrightarrow |\boldsymbol{G}|_{\mathcal{H}}, \quad \pi^{\mathcal{H}}_{\boldsymbol{G}} := \pi^{\mathcal{H}}_{|\boldsymbol{G}|} \circ \pi_{\boldsymbol{G}}\colon \mathrm{Obj}_{\boldsymbol{G}} \to |\boldsymbol{G}|_{\mathcal{H}}.$$

Moreover, for $U \subset \mathrm{Obj}_{\boldsymbol{G}}$ we write $|U| := \pi_{\boldsymbol{G}}(U) \subset |\boldsymbol{G}|$ and $|U|_{\mathcal{H}} := \pi_{\mathcal{H}}(U) \subset |\boldsymbol{G}|_{\mathcal{H}}$.

Lemma A.3 *Let $\boldsymbol{G}$ be an étale groupoid.*

(i) *Any smooth functor $F\colon \boldsymbol{G}\to\boldsymbol{G}'$ induces a continuous map $|F|_{\mathcal{H}}\colon |\boldsymbol{G}|_{\mathcal{H}}\to|\boldsymbol{G}'|_{\mathcal{H}}$.*

(ii) *If $A\subset\mathbb{R}$ is any interval, we may identify $|A\times\boldsymbol{G}|$ with $A\times|\boldsymbol{G}|$, and $|A\times\boldsymbol{G}|_{\mathcal{H}}$ with $A\times|\boldsymbol{G}|_{\mathcal{H}}$. More precisely, there are commutative diagrams*

$$\begin{tikzcd}
\mathrm{Obj}_{A\times\boldsymbol{G}} \arrow[r, "\mathrm{pr}_A\times\mathrm{pr}_{\boldsymbol{G}}"] \arrow[d, "\pi_{A\times\boldsymbol{G}}"'] & A\times\mathrm{Obj}_{\boldsymbol{G}} \arrow[d, "\mathrm{id}_A\times\pi_{\boldsymbol{G}}"] \\
|A\times\boldsymbol{G}| \arrow[r, "|\mathrm{pr}_A|\times|\mathrm{pr}_{\boldsymbol{G}}|"] & A\times|\boldsymbol{G}|
\end{tikzcd}
\qquad
\begin{tikzcd}
|A\times\boldsymbol{G}| \arrow[r, "|\mathrm{pr}_A|\times|\mathrm{pr}_{\boldsymbol{G}}|"] \arrow[d, "\pi^{\mathcal{H}}_{|A\times\boldsymbol{G}|}"'] & A\times|\boldsymbol{G}| \arrow[d, "\mathrm{id}_A\times\pi^{\mathcal{H}}_{|\boldsymbol{G}|}"] \\
|A\times\boldsymbol{G}|_{\mathcal{H}} \arrow[r, "|\mathrm{pr}_A|_{\mathcal{H}}\times|\mathrm{pr}_{\boldsymbol{G}}|_{\mathcal{H}}"] & A\times|\boldsymbol{G}|_{\mathcal{H}}
\end{tikzcd}$$

where the horizontal maps are homeomorphisms. Here $\mathrm{pr}_{\boldsymbol{A}}\colon A\times\boldsymbol{G}\to\boldsymbol{A}$ and $\mathrm{pr}_{\boldsymbol{G}}\colon A\times\boldsymbol{G}\to\boldsymbol{G}$ are the two projection functors from the product groupoid to its factors and $\boldsymbol{A}$ is the groupoid with objects A and only identity morphisms so that $A=|\boldsymbol{A}|=|\boldsymbol{A}|_{\mathcal{H}}$.

Proof Any smooth functor $F\colon \boldsymbol{G}\to\boldsymbol{G}'$ induces a continuous map $|\boldsymbol{G}|\xrightarrow{|F|}|\boldsymbol{G}'|$. Then by Lemma A.2 applied to $|\boldsymbol{G}|$, the composite

$$|\boldsymbol{G}|\xrightarrow{|F|}|\boldsymbol{G}'|\xrightarrow{\pi^{\mathcal{H}}_{|\boldsymbol{G}'|}}|\boldsymbol{G}'|_{\mathcal{H}}$$

factors uniquely through the quotient map $|\boldsymbol{G}|\xrightarrow{\pi^{\mathcal{H}}_{|\boldsymbol{G}|}}|\boldsymbol{G}|_{\mathcal{H}}$. The resulting continuous map $|F|_{\mathcal{H}}\colon |\boldsymbol{G}|_{\mathcal{H}}\to|\boldsymbol{G}'|_{\mathcal{H}}$ is uniquely determined by $\pi^{\mathcal{H}}_{|\boldsymbol{G}'|}\circ|F|=|F|_{\mathcal{H}}\circ\pi^{\mathcal{H}}_{|\boldsymbol{G}|}$. This proves (i).

To prove (ii), first consider the diagram on the left. The bottom horizontal map is bijective because $\mathrm{Mor}_{A\times\boldsymbol{G}}=A\times\mathrm{Mor}_{\boldsymbol{G}}$, and continuous by definition of the product topology. Finally, it is a homeomorphism because A is locally compact; cf [15, Exercise 29.11]. In the diagram on the right we define the bottom horizontal arrow using the product of the maps induced as in (i) by the two functors $\mathrm{pr}_{\boldsymbol{A}}$ and $\mathrm{pr}_{\boldsymbol{G}}$. Hence it is continuous. Since the diagram commutes and we have already seen that the top horizontal map is a homeomorphism, it remains to check this for the bottom map. But this holds because the uniqueness property of the maximal Hausdorff quotient implies that for any homeomorphism $\phi\colon Y\to Y'$, the unique continuous map $\phi_{\mathcal{H}}\colon Y_{\mathcal{H}}\to Y'_{\mathcal{H}}$ such that $Y\xrightarrow{\phi}Y'\xrightarrow{\pi_{Y'}}Y'_{\mathcal{H}}$ equals $Y\xrightarrow{\pi_{\mathcal{H}}}Y_{\mathcal{H}}\xrightarrow{\phi_{\mathcal{H}}}Y'_{\mathcal{H}}$ must be a homeomorphism. □

The smooth structure on a weighted branched manifold will be given by a homeomorphism to the realization of an étale groupoid with the following weighting structure.

Definition A.4 [8, Definition 3.2] A *weighted nonsingular branched groupoid* (or *wnb groupoid* for short) of dimension d is a pair $(\boldsymbol{G},\Lambda)$ consisting of an oriented, nonsingular, étale groupoid $\boldsymbol{G}$ of dimension d, together with a rational weighting function $\Lambda\colon |\boldsymbol{G}|_{\mathcal{H}} \to \mathbb{Q}^+ := \mathbb{Q} \cap (0,\infty)$ that satisfies the following compatibility conditions. For each $p \in |\boldsymbol{G}|_{\mathcal{H}}$ there is an open neighborhood $N \subset |\boldsymbol{G}|_{\mathcal{H}}$ of p, a collection $U_1,\dots,U_\ell$ of disjoint open subsets of $(\pi^{\mathcal{H}}_{\boldsymbol{G}})^{-1}(N) \subset \mathrm{Obj}_{\boldsymbol{G}}$ (called *local branches*), and a set of positive rational weights $m_1,\dots,m_\ell$ such that the following properties hold:

Covering $\qquad (\pi^{\mathcal{H}}_{|\boldsymbol{G}|})^{-1}(N) = |U_1| \cup \dots \cup |U_\ell| \subset |\boldsymbol{G}|.$

Local regularity For each $i = 1,\dots,\ell$ the projection $\pi^{\mathcal{H}}_{\boldsymbol{G}}|_{U_i}\colon U_i \to |\boldsymbol{G}|_{\mathcal{H}}$ is a homeomorphism onto a relatively closed subset of N.

Weighting For all $q \in N$, the number $\Lambda(q)$ is the sum of the weights of the local branches whose image contains q:

$$\Lambda(q) = \sum_{i\colon q \in |U_i|_{\mathcal{H}}} m_i.$$

A *wnb cobordism groupoid* is a tuple $(\boldsymbol{G},\iota^0_{\boldsymbol{G}},\iota^1_{\boldsymbol{G}},\Lambda)$ in which $(\boldsymbol{G},\iota^0_{\boldsymbol{G}},\iota^1_{\boldsymbol{G}})$ is an oriented, nonsingular, étale cobordism groupoid of dimension d, and $\Lambda\colon |\boldsymbol{G}|_{\mathcal{H}} \to \mathbb{Q}^+$ is a weighting function as above with the additional property that Λ and the local branches $U_1,\dots,U_\ell$ are of product form in the collars.

In particular, this means that each boundary groupoid $\partial^\alpha \boldsymbol{G}$ is equipped with a weighting function Λ^α as above such that the following diagram commutes:

$$\begin{tikzcd}
A^\alpha_\varepsilon \times |\partial^\alpha \boldsymbol{G}|_{\mathcal{H}} \arrow[r, "|\iota^\alpha_{\boldsymbol{G}}|_{\mathcal{H}}"] \arrow[d, "\mathrm{id}_{A^\alpha_\varepsilon} \times \Lambda^\alpha"'] & |\boldsymbol{G}|_{\mathcal{H}} \arrow[d, "\Lambda"] \\
\mathbb{Q}^+ \arrow[r, "\mathrm{id}"] & \mathbb{Q}^+
\end{tikzcd}$$

where $|\iota^\alpha_{\boldsymbol{G}}|_{\mathcal{H}}$ is induced by the collaring functor $\iota^\alpha_{\boldsymbol{G}}\colon A^\alpha_\varepsilon \times \partial^\alpha \boldsymbol{G} \to \boldsymbol{G}$ and we identify $|A^\alpha_\varepsilon \times \partial^\alpha \boldsymbol{G}|_{\mathcal{H}}$ with $A^\alpha_\varepsilon \times |\partial^\alpha \boldsymbol{G}|_{\mathcal{H}}$ as in Lemma A.3 with orientation as specified in Definition 3.1.10.

Now we can formulate the notions of weighted branched manifold and cobordism.

Definition A.5 A *weighted branched manifold/cobordism* of dimension d is a pair (Z,Λ_Z) consisting of a topological space Z together with a function $\Lambda_Z\colon Z \to \mathbb{Q}^+$ and an equivalence class[12] of wnb (cobordism) d–dimensional groupoids $(\boldsymbol{G},\Lambda_{\boldsymbol{G}})$ and homeomorphisms $f\colon |\boldsymbol{G}|_{\mathcal{H}} \to Z$ that induce the function $\Lambda_Z = \Lambda_{\boldsymbol{G}} \circ f^{-1}$.

[12] The precise notion of equivalence is given in [8, Definition 3.12]. In particular it ensures that the induced function $\Lambda_Z := \Lambda_{\boldsymbol{G}} \circ f^{-1}$ and the dimension of $\mathrm{Obj}_{\boldsymbol{G}}$ is the same for equivalent structures $(\boldsymbol{G},\Lambda_{\boldsymbol{G}},f)$. Moreover, if $(\boldsymbol{G},\iota^0_{\boldsymbol{G}},\iota^1_{\boldsymbol{G}})$ is a cobordism groupoid, then the images $f(|\partial^\alpha \boldsymbol{G}|_{\mathcal{H}}) := \partial^\alpha Z \subset Z$ of the two boundary components are well defined.

For a weighted branched cobordism $(Z, \Lambda_Z, [\boldsymbol{G}, \iota^0_{\boldsymbol{G}}, \iota^1_{\boldsymbol{G}}, \Lambda_{\boldsymbol{G}}, f])$, the induced *boundary components* $\partial^\alpha Z := f\big(|\iota^\alpha_{\boldsymbol{G}}|_{\mathcal{H}}(|\partial^\alpha \boldsymbol{G}|_{\mathcal{H}})\big) \subset Z$ for $\alpha = 0, 1$ are equipped with the weighted branched manifold structures $[(\partial^\alpha \boldsymbol{G}, \Lambda^\alpha_{\boldsymbol{G}}), f|_{|\partial^\alpha \boldsymbol{G}|_{\mathcal{H}}}]$.

The underlying space Z of a weighted branched manifold or cobordism is always Hausdorff due to the homeomorphism $Z \cong |\boldsymbol{G}|_{\mathcal{H}}$ to a Hausdorff quotient. Moreover, since cobordism groupoids are compact by definition, the underlying space Z of a weighted branched cobordism is always compact.

It is shown in [8, Proposition 3.5] that the weighting function $\Lambda\colon |\boldsymbol{G}|_{\mathcal{H}} \to (0, \infty)$ is locally constant on the complement of the *branch locus* $\mathrm{Br}(\boldsymbol{G}) \subset |\boldsymbol{G}|_{\mathcal{H}}$. (This is defined to be the set of points in $|\boldsymbol{G}|_{\mathcal{H}}$ over which $|\pi|^{\mathcal{H}}_{|\boldsymbol{G}|}\colon |\boldsymbol{G}| \to |\boldsymbol{G}|_{\mathcal{H}}$ is not injective, and is closed and nowhere dense.) Further, every point in $|\boldsymbol{G}|_{\mathcal{H}} \setminus \mathrm{Br}(\boldsymbol{G})$ has a neighborhood that is homeomorphic via $\pi^{\mathcal{H}}_{|\boldsymbol{G}|}$ to an open subset in a local branch and so has the structure of a smooth oriented manifold.

Example A.6 (i) Any compact oriented smooth manifold/cobordism may be considered as a weighted branched manifold/cobordism with weighting function $\Lambda_Z \equiv 1$ and empty branch locus.

(ii) A compact weighted branched manifold of dimension 0 also necessarily has empty branch locus and consists of a finite set of points $\{p_1, \dots, p_k\}$, each with a positive rational weight $m(p_i) \in \mathbb{Q}^+$ and orientation $\mathfrak{o}(p_i) \in \{\pm\}$. Any representing groupoid $\boldsymbol{G}$ has as object space $\mathrm{Obj}_{\boldsymbol{G}}$ a set with the discrete topology, which is equipped with an orientation function $\mathfrak{o}\colon \mathrm{Obj}_{\boldsymbol{G}} \to \{\pm\}$. The morphism space $\mathrm{Mor}_{\boldsymbol{G}}$ is also a discrete set and, because we assume that $\boldsymbol{G}$ is oriented, defines an equivalence relation on $\mathrm{Obj}_{\boldsymbol{G}}$ such that $x \sim y \Longrightarrow \mathfrak{o}(x) = \mathfrak{o}(y)$. Moreover, because $|\boldsymbol{G}|$ is Hausdorff, we can identify $|\boldsymbol{G}| = |\boldsymbol{G}|_{\mathcal{H}}$ and hence conclude that $\mathrm{Obj}_{\boldsymbol{G}}$ consists of precisely k classes of points that are equivalent under $\mathrm{Mor}_{\boldsymbol{G}}$ and project to $p_1, \dots, p_k$ in $Z \cong |\boldsymbol{G}|_{\mathcal{H}}$.

(iii) For the prototypical example of a 1–dimensional weighted branched cobordism $(|\boldsymbol{G}|_{\mathcal{H}}, \Lambda)$, take $\mathrm{Obj}(\boldsymbol{G}) = I \sqcup I'$ equal to two copies of the interval $I = I' = [0, 1]$, with nonidentity morphisms from $x \in I$ to $x \in I'$ for $x \in \big[0, \tfrac{1}{2}\big)$ and their inverses, where we suppose that I is oriented in the standard way. Then the realization and its Hausdorff quotient are

$$\begin{aligned}
|\boldsymbol{G}| &= I \sqcup I' / \big\{(I, x) \sim (I', x) \text{ if and only if } x \in \big[0, \tfrac{1}{2}\big)\big\}, \\
|\boldsymbol{G}|_{\mathcal{H}} &= I \sqcup I' / \big\{(I, x) \sim (I', x) \text{ if and only if } x \in \big[0, \tfrac{1}{2}\big]\big\},
\end{aligned}$$

and the branch locus is a single point $\mathrm{Br}(\boldsymbol{G}) = \big\{\big[I, \tfrac{1}{2}\big] = \big[I', \tfrac{1}{2}\big]\big\} \subset |\boldsymbol{G}|_{\mathcal{H}}$. The choice of weights $m, m' > 0$ on the two local branches I and I' determines the weighting function $\Lambda\colon |\boldsymbol{G}|_{\mathcal{H}} \to (0,\infty)$ as

$$\Lambda([I,x]) = \begin{cases} m+m' & \text{if } x \in \left[0,\tfrac{1}{2}\right], \\ m & \text{if } \ x \in \left(\tfrac{1}{2},1\right], \end{cases}$$

$$\Lambda([I',x]) = \begin{cases} m+m' & \text{if } x \in \left[0,\tfrac{1}{2}\right], \\ m' & \text{if } x \in \left(\tfrac{1}{2},1\right]. \end{cases}$$

For example, giving each branch I, I' the weight $m = m' = \frac{1}{2}$, together with an appropriate choice of collar functors $\iota^{\alpha}_{\boldsymbol{G}}$, yields a weighted branched cobordism $(|\boldsymbol{G}|_{\mathcal{H}}, \iota^0_{\boldsymbol{G}}, \iota^1_{\boldsymbol{G}}, \Lambda)$ with $|\partial^0 \boldsymbol{G}|_{\mathcal{H}} = \{[I,0] = [I',0]\}$, which is a single point with weight 1, and $|\partial^1 \boldsymbol{G}|_{\mathcal{H}} = \{[I,1],[I',1]\}$, which consists of two points with weight $\frac{1}{2}$, all with positive orientation because as explained in Remark 3.1.11, the induced orientation on the boundary $\partial^{\alpha}\boldsymbol{G}$ of a cobordism is completed to an orientation of the collar by adding as the first component the positive unit vector along A^{α}_{ε}.

Another choice of collar functors for the same weighted groupoid $(\boldsymbol{G},\Lambda)$ might give rise to a different partition of the boundary into incoming $\partial^0\boldsymbol{G}$ and outgoing $\partial^1\boldsymbol{G}$, for example yielding a weighted branched cobordism with $|\partial^0\boldsymbol{G}|_{\mathcal{H}} = \{[I,0]{=}[I',0],[I,1]\}$ consisting of two points with weights and orientations $(1,+)$ and $\left(\frac{1}{2},-\right)$, and with $|\partial^1\boldsymbol{G}|_{\mathcal{H}} = \{[I',1]\}$ consisting of one point with weight $\left(\frac{1}{2},+\right)$.

(iv) In the situation of Theorem 3.2.8, the nonsingular étale groupoid $\widehat{\boldsymbol{Z}}^{\nu}$ with $\mathrm{Obj}_{\widehat{\boldsymbol{Z}}^{\nu}} = (\mathfrak{s}_{\mathcal{K}}|_{\mathcal{V}}+\nu)^{-1}(0)$ has a maximal Hausdorff quotient $|\widehat{\boldsymbol{Z}}^{\nu}|_{\mathcal{H}} = |\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}|$ that, as we show in Lemma 3.2.10, is given by the realization of the groupoid $\widehat{\boldsymbol{Z}}^{\nu}_{\mathcal{H}}$ obtained as in (iii) above by closing the set of morphisms $\mathrm{Mor}_{\widehat{\boldsymbol{Z}}^{\nu}} \subset \mathrm{Obj}_{\widehat{\boldsymbol{Z}}^{\nu}} \times \mathrm{Obj}_{\widehat{\boldsymbol{Z}}^{\nu}}$. Therefore, in this case we can give a completely explicit description of $|\boldsymbol{Z}|_{\mathcal{H}}$ and its weighting function $\Lambda_{\boldsymbol{Z}}$; see the proof of Theorem 3.2.8. $\diamond$

The following is a version of some parts of [8, Proposition 3.25], which more generally defines a notion of integration over weighted branched manifolds and cobordisms.

Proposition A.7 *Any compact d–dimensional weighted branched manifold (Y,Λ_Y) induces a* ***fundamental class*** *$[Y] \in H_d(Y;\mathbb{Q})$, and any d–dimensional weighted branched cobordism (Z,Λ_Z) with boundary $\partial Z := \partial^0 Z \cup \partial^1 Z$ induces a* ***fundamental class*** *$[Z] \in H_d(Z,\partial Z;\mathbb{Q})$, whose image under the boundary map*

$$\partial\colon H_d(Z,\partial Z;\mathbb{Q}) \to H_{d-1}(\partial Z;\mathbb{Q}) \cong H_{d-1}(\partial^0 Z;\mathbb{Q}) + H_{d-1}(\partial^1 Z;\mathbb{Q})$$

is $\partial[Z] = [\partial^1 Z] - [\partial^0 Z]$.

Proof If (Y,Λ_Y) has a weighted branched manifold structure $(\boldsymbol{G},\Lambda_{\boldsymbol{G}})$ with well-behaved (eg piecewise smooth) branch locus, then one can triangulate $|\boldsymbol{G}|_{\mathcal{H}} \cong Y$

so that the branch locus lies in the codimension 1 skeleton. We may then define a singular cycle on Y by using the local weights m_i to assign a rational weight to each top-dimensional simplex. As explained in Remark 3.1.11, in the case of a cobordism Z the induced orientation on the boundary component $\partial^\alpha Z$ is completed to the orientation of the collar by adding the unit positive vector along the collar as the first component. In the case of $\partial^0 Z$ this yields an orientation of $\partial^0 Z$ that is the opposite of the standard way of orienting a boundary component by adding the outward pointing normal, a fact that is reflected in the minus sign in the formula $\partial[Z] = [\partial^1 Z] - [\partial^0 Z]$. For more details and the general case, see [8]. □

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