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Convexity of the extended K-energy and the large time behavior of the weak Calabi flow

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Let (X,ω) be a compact connected Kähler manifold and denote by $(\mathcal{E}^p, d_p)$ the metric completion of the space of Kähler potentials $\mathcal{H}_\omega$ with respect to the L^p–type path length metric d_p. First, we show that the natural analytic extension of the (twisted) Mabuchi K-energy to $\mathcal{E}^p$ is a d_p–lsc functional that is convex along finite-energy geodesics. Second, following the program of J Streets, we use this to study the asymptotics of the weak (twisted) Calabi flow inside the CAT(0) metric space $(\mathcal{E}^2, d_2)$. This flow exists for all times and coincides with the usual smooth (twisted) Calabi flow whenever the latter exists. We show that the weak (twisted) Calabi flow either diverges with respect to the d_2–metric or it d_1–converges to some minimizer of the K-energy inside $\mathcal{E}^2$. This gives the first concrete result about the long-time convergence of this flow on general Kähler manifolds, partially confirming a conjecture of Donaldson. We investigate the possibility of constructing destabilizing geodesic rays asymptotic to diverging weak (twisted) Calabi trajectories, and give a result in the case when the twisting form is Kähler. Finally, when a cscK metric exists in $\mathcal{H}_\omega$, our results imply that the weak Calabi flow d_1–converges to such a metric.

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1 Introduction

Given a compact connected Kähler manifold (X,ω), we denote by $\mathcal{H}$ the space of smooth Kähler metrics in the cohomology class $[\omega]$. As follows from the $\partial\bar\partial$–lemma of Hodge theory, up to a constant, this space is in one-to-one correspondence with the space of Kähler potentials

$$\mathcal{H}_\omega = \{u \in C^\infty(X) : \omega_u := \omega + i\partial\bar\partial u > 0\}.$$

As $\mathcal{H}_\omega$ is an open subset of $C^\infty(X)$, it is a Fréchet manifold and it is possible to endow it with different L^p–type Finsler metrics for $p \geq 1$, via

$$\|\xi\|_{p,u} := \left(V^{-1}\int_X |\xi|^p \omega_u^n\right)^{1/p}, \quad \xi \in T_u\mathcal{H}_\omega = C^\infty(X). \tag{1}$$

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For $p = 2$ one recovers the Riemannian structure of Mabuchi which turns $\mathcal{H}$ into a Riemannian symmetric space of constant negative curvature (see Donaldson [38], Mabuchi [55] and Semmes [59]), but as will be explained below, the Finsler case $p = 1$ will also play a key role in the present paper.

One of the central questions of Kähler geometry, going back to Calabi, is to understand under what conditions $\mathcal{H}$ contains a constant scalar curvature Kähler (csc-K) metric. From a variational point of view this amounts to looking for critical points (minimizers) of Mabuchi's K-energy functional $\mathcal{K}\colon \mathcal{H}_\omega \to \mathbb{R}$ [38; 55], whose first variation is defined by the formula

$$\langle D\mathcal{K}(u), \delta u\rangle = V^{-1}\int_X \delta u(\overline{S} - S_{\omega_u})\omega_u^n,$$

where $V = \int_X \omega^n$ is the total volume and $\overline{S} = nV^{-1}\int_X \operatorname{Ric}\omega \wedge \omega^{n-1} = V^{-1}\int_X S_\omega \omega^n$ is the mean scalar curvature. According to a formula of Chen and Tian, the K-energy can be expressed explicitly in terms of the Kähler potential as

$$\mathcal{K}(u) := \operatorname{Ent}(\omega^n, \omega_u^n) + \overline{S}\,\mathrm{AM}(u) - n\,\mathrm{AM}_{\operatorname{Ric}\omega}(u), \tag{2}$$

where $\operatorname{Ent}(\omega^n, \omega_u^n) = V^{-1}\int_X \log(\omega_u^n/\omega^n)\omega_u^n$ is the entropy of the measure ω_u^n with respect to ω^n and $\mathrm{AM}, \mathrm{AM}_\gamma\colon \mathcal{H}_\omega \to \mathbb{R}$ are the Aubin–Mabuchi (also Aubin–Yau) energy and its "γ–contracted" version:

$$\mathrm{AM}(u) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X u\omega_u^j \wedge \omega^{n-j}, \quad \mathrm{AM}_\gamma(u) = \frac{1}{nV}\sum_{j=0}^{n-1}\int_X u\gamma \wedge \omega_u^j \wedge \omega^{n-1-j}.$$

As shown by Mabuchi, the K-energy is convex along geodesics in $\mathcal{H}_\omega$ when the geodesics are defined in terms of the corresponding L^2–Riemannian structure. However, a major technical stumbling block in this infinite-dimensional setting is that the Riemannian structure on $\mathcal{H}_\omega$ is not geodesically complete, and this is one of the reasons that we will be forced to work with various completions of $\mathcal{H}_\omega$, as discussed below.

In the finite-dimensional Riemannian setting, a time-honored approach to finding minimizers of convex functions is to follow their negative (downward) gradient flow. In the present infinite-dimensional Riemannian setting the negative gradient flow of the K-energy is precisely the Calabi flow $t \to c_t$:

$$\frac{d}{dt}c_t = S_{\omega_{c_t}} - \overline{S}.$$

Given arbitrary initial potential $c_0 \in \mathcal{H}_\omega$, short-time existence of the flow, assuming the initial potential is $C^{3,\alpha}$, is due to Chen and He [24], but long-time existence is still an open conjecture due to Calabi and Chen. In the case where $\dim X = 1$, long-time existence and convergence of the flow was first explored by Chruściel [28].

Fine [42] used finite-dimensional flows to approximate the Calabi flow. Under various restrictive conditions, convergence and existence theorems for the Calabi flow have been extensively studied. We refer the reader to Chen and He [25], Feng and Huang [40], He [48], Huang [49], Huang and Zheng [50], Li, Wang and Zheng [54], Székelyhidi [63] and Tosatti and Weinkove [65], to cite a few works from a very fast-growing literature.

The main motivation of our paper is the following conjecture of Donaldson on the long-time asymptotics and convergence of the Calabi flow, which, roughly stated, says:

Conjecture 1.1 [39] *Let $[0,\infty) \ni t \to c_t \in \mathcal{H}$ be a Calabi flow trajectory. Exactly one of the following alternatives holds:*

(i) *The curve $t \to c_t$ converges smoothly to some csc-K potential $c_\infty \in \mathcal{H}_\omega$ as $t \to \infty$.*

(ii) *The curve $t \to c_t$ diverges as $t \to \infty$ and encodes destabilizing information about the Kähler structure.*

We refer to Donaldson [39] for a precise statement and further details about this conjecture. To avoid the difficulties arising in PDE theory related to long-time existence, we recast the Calabi flow in the metric completion of $(\mathcal{H}_\omega, d_2)$ following Streets [61; 62], who applied the work of Mayer [56] and Bačák [3] concerning gradient flows of convex functionals on Hadamard spaces (ie $\mathrm{CAT}(0)$ spaces) to the setting of the "minimizing movement" Calabi flow. Before we can do this, however, we need to understand how the K-energy extends to certain spaces of singular potentials. The key new feature of our approach is that we take advantage of the fact that the corresponding abstract metric space (defined in terms of Cauchy sequences in [61]) can be realized concretely in terms of certain singular Kähler potentials, ie using pluripotential theory, which in particular allows us to improve on the abstract convergence result in [62].

Finite-energy spaces and extensions of the twisted K-energy In order to briefly introduce our setting, we denote by $\mathcal{E}^p$ the space of ω–psh functions on X which have finite energy with respect to the standard p–homogenous weight, as introduced by Guedj and Zeriahi [45]. As shown in Darvas and He [30], the abstract metric completion of the L^p–type Finsler metric (1) on $\mathcal{H}_\omega$ may be identified with the finite-energy space $\mathcal{E}^p$ equipped with a natural distance function that we will denote by d_p, which is comparable to an explicit energy-type expression (8). When $p = 2$, this identification was conjectured by Guedj in [44]. Furthermore, in the case $p = 1$, it yields a Finsler realization $(\mathcal{E}^1, d_1)$ of the strong topology on $\mathcal{E}^1$ introduced in Berman, Boucksom, Eyssidieux, Guedj and Zeriahi [9] (which can be seen as a higher-dimensional "nonlinear" generalization of the classical strong topology defined by the Dirichlet norm on a Riemann surface).

Moreover, as shown in Darvas [29] and Darvas and He [30], for any pair of potentials $u_0, u_1 \in \mathcal{E}^p$ one can construct a d_p–geodesic segment (in the metric sense) explicitly, as a decreasing pointwise limit of $C^{1,\bar{1}}$–weak geodesics (in the sense of Chen [20], ie as $C^{1,\bar{1}}$–solutions to certain complex Monge–Ampère equations). These d_p–geodesic segments will be referred to as *finite-energy geodesics* in the future and we direct the reader to Theorem 2.3 for more details. A recurrent theme in the present work is the interaction between the cases $p = 2$ and $p = 1$, which in particular will allow us to exploit the energy/entropy compactness theorem from [9] to get a convergence result for the Calabi flow with respect to the d_1–topology. This strengthens the general convergence result of [62], concerning the weak d_2–topology, which does not imply any convergence in the sense of pluripotential theory (Remark 5.4).

Our starting point is the observation that the K-energy functional $\mathcal{K}$ originally defined on $\mathcal{H}_\omega$ admits a natural "analytic extension" to the finite-energy space $\mathcal{E}^1$ (and hence by restriction to all spaces $\mathcal{E}^p$). This is simply the extension obtained by interpreting the entropy part (the first term) and the energy part (the second two terms) in formula (2) in the general sense of probability theory and pluripotential theory, respectively, essentially as in the Fano setting previously considered in Berman [7] and in [9]. As we will see, the energy part is d_1–continuous, whereas the entropy part is only d_1–lsc, and in the particular case of $C^{1,\bar{1}}$–potentials, this extension coincides with the one introduced by Chen [19]. We then go on to show that the restriction to $\mathcal{E}^p$ of the analytic extension coincides with the canonical "topological extension" of the K-energy, ie the greatest d_p–lsc extension from $\mathcal{H}_\omega$. In particular, applied to the case $p = 2$, which is the one relevant to the Calabi flow, this yields an analytic formula for Streets' extension of the K-energy.

The analytic extension formula allows us to establish the convexity of the extended K-energy along finite-energy geodesics, using an approximation argument and the $C^{1,\bar{1}}$–case recently settled in Berman and Berndtsson [8, Theorem 1.1] (originally conjectured by Chen).

Before we state our first theorem, recall that in various applications of Kähler geometry it is necessary to deal with the more general concept of twisted csc-K metrics and the corresponding twisted-K energy (see eg Chen [23], Chen, Paun and Zeng [26], Dervan [36], Fine [41] and Stoppa [60]). As it takes little extra effort, throughout this paper we work at this level of generality, with χ denoting a very general twisting form (3) and $\mathcal{K}_\chi$ the corresponding twisted K-energy (5). The relevant terminology will be recalled in Section 2.1.

Theorem 1.2 (Theorem 4.7) *Suppose (X, ω) is a compact connected Kähler manifold. The K-energy can be extended to a functional $\mathcal{K}\colon \mathcal{E}^1 \to (-\infty, \infty]$ using (2). The* *restricted functional* $\mathcal{K}|_{\mathcal{E}^p}$ *is the greatest* d_p*–lsc extension of* $\mathcal{K}|_{\mathcal{H}_\omega}$ *for any* $p \geq 1$*. Additionally,* $\mathcal{K}|_{\mathcal{E}^p}$ *is convex along the finite-energy geodesics of* $\mathcal{E}^p$*. If* $\chi = \beta + i\partial\bar{\partial} f$ *satisfies* (3)*, the corresponding result also holds for the twisted K-energy* $\mathcal{K}_\chi$*.*

An important ingredient in the proof of Theorem 1.2 is understanding approximation of potentials of $\mathcal{E}^p$ while also approximating entropy. In this direction, we note the following theorem. More precise results can be obtained using the flow techniques of Guedj and Zeriahi [46] and Nezza and Lu [57], and will be discussed elsewhere.

Theorem 1.3 (Theorem 3.2) *Suppose* $u \in \mathcal{E}^p$ *and* f *is a usc function on* X *satisfying* $e^{-f} \in L^1(X, \omega^n)$*. Then one can find* $u_k \in \mathcal{H}_\omega$ *with* $d_p(u_k, u) \to 0$ *and* $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_{u_k}) \to \mathrm{Ent}(e^{-f}\omega^n, \omega^n_u)$*.*

Finally, as a consequence of Theorem 1.2 we obtain that the space of finite χ–entropy potentials $\mathrm{Ent}_\chi(X, \omega)$ is geodesically closed, and if $\mathrm{Ric}\,\omega \geq \beta$ then the twisted entropy is convex along finite-energy geodesics, giving the Kähler analog of a central result of Lott, Sturm and Villani in optimal transport theory; see Villani [66]. For details on notation and a detailed discussion on relationship with the literature, we refer to Section 4.4.

Theorem 1.4 (Theorem 4.10) *If* $\chi = \beta + i\partial\bar{\partial} f$ *satisfies* (3)*, then* $(\mathrm{Ent}_\chi(X, \omega), d_1)$ *is a geodesic sub-metric space of* $(\mathcal{E}^1(X, \omega), d_1)$*. Additionally, if* $\mathrm{Ric}\,\omega \geq \beta$ *then the map* $\mathrm{Ent}_\chi(X, \omega) \ni u \to \mathrm{Ent}(e^{-f}\omega^n, \omega^n_u) \in \mathbb{R}$ *is convex along finite-energy geodesics.*

Convergence and large-time behavior of the weak twisted Calabi flow As advertised above, using Theorem 1.2, we can run the weak twisted Calabi flow $[0, \infty) \ni t \to c_t \in \mathcal{E}^2$ for any starting point $c_0 \in \mathcal{E}^2$. Indeed, $(\mathcal{E}^2, d_2)$ is a $\mathrm{CAT}(0)$-space and the extended functional $\mathcal{K}_\chi$ is convex along d_2–geodesics, hence we are in the setting of Mayer [56], as detailed in Section 2.5. This yields a flow of (possibly singular) Kähler potentials which is uniquely determined by the corresponding normalized Monge–Ampère measures, which in turn yields a flow of probability measures which is regularizing in the sense that the entropy immediately becomes finite and in particular the measures have an L^1–density for positive times.

When χ is smooth and X is a Riemann surface, the smooth twisted Calabi flow was recently explored by Pook [58]. To provide consistency, we will show that the weak twisted Calabi flow agrees with the smooth version whenever the latter exists (Proposition 6.1), generalizing a result of Streets [62] in the case where $\chi = 0$. Providing additional consistency, as an application of Theorem 1.2, in Section 6 we show that Streets' (a priori different) minimizing movement Calabi flow coincides with our weak Calabi flow.

Generalizing twisted csc-K metrics, by $\mathcal{M}^p_\chi$ we denote the minimizers of the extended K-energy on $\mathcal{E}^p$:

$$\mathcal{M}^p_\chi = \bigl\{u \in \mathcal{E}^p : \mathcal{K}_\chi(u) = \inf_{v \in \mathcal{E}^p} \mathcal{K}_\chi(v)\bigr\}.$$

In the case $\chi = 0$ we will simply use $\mathcal{M}^p := \mathcal{M}^p_0$. Concerning the convergence and blow-up behavior of the weak twisted Calabi flow, we prove the following concrete result:

Theorem 1.5 (Theorem 6.3) *Suppose (X, ω) is a compact connected Kähler manifold and $\chi = \beta + i\partial\bar\partial f$ satisfies* (3). *The following statements are equivalent:*

(i) *$\mathcal{M}^2_\chi$ is nonempty.*

(ii) *For any weak twisted Calabi flow trajectory $t \to c_t$, there exists $c_\infty \in \mathcal{M}^2_\chi$ such that $d_1(c_t, c_\infty) \to 0$ and $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_{c_t}) \to \mathrm{Ent}(e^{-f}\omega^n, \omega^n_{c_\infty})$.*

(iii) *Any weak twisted Calabi flow trajectory $t \to c_t$ is d_2–bounded.*

(iv) *There exists a weak twisted Calabi flow trajectory $t \to c_t$ and $t_j \to \infty$ for which the sequence $\{c_{t_j}\}_j$ is d_2–bounded.*

• By the consistency result discussed above, the previous theorem in particular applies to the smooth Calabi flow (when it exists) and it should be stressed that the result and its elaborations discussed below are new also in this smooth case. In particular, it generalizes results of the first author on the smooth Calabi flow on Fano manifolds without nontrivial holomorphic vector fields; see Berman [7]. One new feature of our result is that the latter assumption, which guarantees the uniqueness of csc-K metrics, is not needed. This means that the limit c_∞ is not uniquely determined by X and will, in general, depend on the initial data c_0.

• By Darvas and He [30, Theorem 5] and part (ii) of the above theorem, if a csc-K potential exists in $\mathcal{H}_\omega$ then the weak Calabi flow $t \to c_t$ converges pointwise ae to some potential $c_\infty \in \mathcal{M}^2$, and the measures $\omega^n_{c_t}$ converge weakly and in entropy to $\omega^n_{c_\infty}$. In the Fano case it additionally follows that c_∞ is csc-K. However, due to progress on the regularity Conjecture 1.8 discussed in the companion paper Berman, Darvas and Lu [11], this result also holds on general Kähler manifolds as well, making further progress on Donaldson's conjecture (see Theorem 1.10 and Theorem 1.11).

• Finally, in light of Theorem 1.6, we mention that Proposition 2.11(ii) strengthens the corresponding convergence result of Streets in [62]. Given a CAT(0) metric space (M, d), it is possible to introduce a notion of weak d–convergence, generalizing the concept of weak convergence on Hilbert spaces (Section 2.4). In general, little concrete is known about this type of convergence; see Kirk and Panyanak [51]. Streets, however, observed that one can adapt the result of Bačák [3] to our setting, ie whenever $\mathcal{M}^2$ is nonempty, each weak Calabi flow trajectory converges d_2–weakly to an element of $\mathcal{M}^2$ [62]. Though weak d_2 convergence does not even imply weak L^1 convergence of the potentials (Remark 5.4), we use this idea in the proof of the above theorem together with the following result, which sheds light on the relationship between all the different topologies involved:

Theorem 1.6 (Theorem 5.3) *Suppose $\{u_k\}_k \subset \mathcal{E}^2$ is d_2–bounded and $u \in \mathcal{E}^2$. Then $d_1(u_k, u) \to 0$ if and only if $\|u_j - u\|_{L^1(X)} \to 0$ and u_k converges to u d_2–weakly.*

The conjectural picture of Donaldson Before we proceed, let us note a last corollary of Theorem 1.5, a consequence of the equivalence between (i) and (iv):

Corollary 1.7 *Suppose that (X, ω) is a compact connected Kähler manifold and that $[0,\infty) \ni t \to c_t \in \mathcal{E}^2$ is a weak twisted Calabi flow trajectory. Exactly one of the following holds:*

(i) *The curve $t \to c_t$ d_1–converges to some $c_\infty \in \mathcal{M}^2_\chi$.*

(ii) *$d_2(c_0, c_t) \to \infty$ as $t \to \infty$.*

Though this corollary is in line with Donaldson's conjectural picture, one would like to understand how a diverging Calabi flow trajectory "destabilizes" the Kähler structure, as proposed in Conjecture 1.1. In this direction we recall the following concept from Darvas and He [31]: suppose (M, d) is a geodesic metric space and $[0,\infty) \ni t \to \gamma_t \in M$ is a continuous curve. We say that the unit speed d–geodesic ray $[0,\infty) \ni t \to g_t \in M$ is *d–weakly asymptotic* to the curve $t \to \gamma_t$ if there exists $t_j \to \infty$ and unit speed d–geodesic segments $[0, d(\gamma_0, \gamma_{t_j})] \ni t \to g^j_t \in M$ connecting γ_0 and γ_{t_j} such that $\lim_{j\to\infty} d(g^j_t, g_t) = 0$ for $t \in [0,\infty)$.

Clearly, to have a geodesic ray weakly asymptotic to $t \to \gamma_t$, we need $t \to d(\gamma_0, \gamma_t)$ to be unbounded. By the above corollary, this condition makes diverging weak Calabi flow trajectories $t \to c_t$ perfect candidates for this construction. However, more needs to be known about $t \to c_t$ before we can proceed. In Darvas and Rubinstein [32, Conjecture 2.8] it was pointed out that an important roadblock in resolving Tian's properness conjecture for csc-K metrics is a conjecture about regularity of minimizers of $\mathcal{K}$. The twisted version of this conjecture should also hold:

Conjecture 1.8 (Darvas and Rubinstein [32]) *Suppose (X, ω) is a compact connected Kähler manifold and χ is smooth. Then $\mathcal{M}^1_\chi \subset \mathcal{H}_\omega$, ie $\mathcal{M}^1_\chi$ contains only smooth twisted csc-K potentials.*

We note that this conjecture generalizes an earlier conjecture of Chen [22, Conjecture 6.3] about $C^{1,1}$ minimizers of $\mathcal{K}$. When (X,ω) is Fano, Conjecture 1.8 was proved in Berman [7] and Berman, Boucksom, Eyssidieux, Guedj and Zeriahi [9]. The next result partially confirms Donaldson's conjecture in the Fano case and also in the case when χ is a Kähler form.

Theorem 1.9 (Theorem 6.5) *Suppose (X,ω) is a compact connected Kähler manifold, $\chi \geq 0$ is smooth and Conjecture 1.8 holds. Let $[0,\infty) \ni t \to c_t \in \mathcal{E}^2$ be a weak twisted Calabi flow trajectory. Exactly one of the following holds:*

- (i) *The curve $t \to c_t$ d_1–converges to a smooth twisted csc-K potential c_∞.*
- (ii) *$d_1(c_0,c_t) \to \infty$ as $t \to \infty$ and the curve $t \to c_t$ is d_1–weakly asymptotic to a finite-energy geodesic $[0,\infty) \ni t \to u_t \in \mathcal{E}^1$ along which $\mathcal{K}_\chi$ decreases.*

If $\chi > 0$, then, independently of Conjecture 1.8, exactly one of the following holds:

- (i′) *The curve $t \to c_t$ d_1–converges to a unique minimizer in $\mathcal{E}^1$ of $\mathcal{K}_\chi$.*
- (ii′) *$d_1(c_0,c_t) \to \infty$ as $t \to \infty$ and the curve $t \to c_t$ is d_1–weakly asymptotic to a finite-energy geodesic $[0,\infty) \ni t \to u_t \in \mathcal{E}^1$ along which $\mathcal{K}_\chi$ strictly decreases.*

Though stated differently, when (X,ω) is Fano and $\chi = 0$ the analog of this result for the Kähler–Ricci flow has been obtained in [31, Theorem 2]. There we have smooth convergence in (i) and along the geodesic ray of (ii) the potentials are bounded, all thanks to the Perelman estimates available for the Kähler–Ricci flow. It would be interesting to compare the above theorem to the results in Chen and Sun [27], where the authors construct in a specific situation a geodesic ray asymptotic to the Calabi flow and are able to draw geometric conclusions based on this.

Concluding remarks and additional results Based on geometric considerations, and the analogous picture in case of the Kähler–Ricci flow (see Guedj and Zeriahi [46]), it is natural to speculate that for any starting point $c_0 \in \mathcal{E}^2$, the weak Calabi flow $t \to c_t$ is instantly smooth, ie $c_t \in \mathcal{H}_\omega$ for $t > 0$ (see also Chen [23, Conjecture 3.5]). Such a result would instantly give the $\mathcal{E}^2$ version of Conjecture 1.8, that $\mathcal{E}^2$–minimizers of $\mathcal{K}$ are smooth csc-K metrics. Indeed, by the general result of Mayer [56], the weak Calabi flow $t \to c_t$ starting at a minimizer $c_0 \in \mathcal{E}^2$ has to be stationary. If $t \to c_t$ was instantly smooth, then we could conclude that $c_0 \in \mathcal{H}_\omega$.

In the companion paper [11] we make progress on Conjecture 1.8 using different techniques from the ones presented in this paper:

Theorem 1.10 *Suppose (X,ω) is a Kähler manifold and $\mathcal{H}_\omega$ contains a csc-K potential. Then $\mathcal{M}^1$ contains only smooth csc-K potentials.*

The consequences of this theorem related to K-stability and energy properness will be discussed in [11]. As $\mathcal{M}^2 \subset \mathcal{M}^1$, here we just mention the following consequence of this result and Theorem 1.5(ii), making further progress on Conjecture 1.1 (see also Streets [62, Remark 1.10]):

Theorem 1.11 *Suppose (X,ω) is a Kähler manifold and $\mathcal{H}_\omega$ contains a csc-K potential u. Then any weak Calabi flow trajectory $t \to c_t$ d_1–converges to a smooth csc-K potential $c_\infty \in \mathcal{H}_\omega$. In addition, the densities $\omega^n_{c_t}/\omega^n$ converge in L^1 to the density $\omega^n_{c_\infty}/\omega^n$.*

Organization of the paper In the first part of Section 2 we recall recent results on complex Monge–Ampère theory which we will use in this paper. In the second part we briefly recall Mayer's theory of gradient flows on nonpositively curved metric spaces. The approximation of finite-energy ω–plurisubharmonic functions with convergent entropy is presented in Section 3. The twisted Mabuchi energy is studied in Section 4. The weak d_2 topology is explored in Section 5, while the last section is devoted to the weak twisted Calabi flow.

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2 Preliminaries

2.1 The twisted K-energy

Suppose χ is a closed positive $(1,1)$–current and β is a smooth closed $(1,1)$–form in the same cohomology class as χ. In most applications of Kähler geometry, the twisting current χ can be smooth, but in order to treat the case of smooth and singular canonical metrics (eg conical csc-K metrics) together, it is natural to ask for the following more general restriction on χ:

$$\chi = \beta + i\partial\bar\partial f, \quad \text{where } f \in \mathrm{PSH}(X,\beta) \text{ with } e^{-f} \in L^1(X,\omega^n). \tag{3}$$

We observe that the integrability condition $e^{-f} \in L^1(X,\omega^n)$ implies that $e^{-f} \in L^p(X,\omega^n)$ for some $p>1$, as follows from the openness conjecture, recently proved by Berndtsson [13] (see also [43]). We note that some of our results, in particular Theorem 1.2 above, hold for more general χ. However, it is unlikely that greater generality will have applications, and we leave it to the reader to find optimal conditions for χ in our theorems.

The twisted K-energy $\mathcal{K}_\chi\colon \mathcal{H}_\omega \to \mathbb{R}$ can now be defined as

$$\mathcal{K}_\chi(u) = \mathrm{Ent}(e^{-f}\omega^n, \omega_u^n) + \bar{S}_\chi \,\mathrm{AM}(u) - n\,\mathrm{AM}_{\mathrm{Ric}\,\omega-\beta}(u) - \int_X f\omega^n, \tag{4}$$

where $\bar{S}_\chi = nV^{-1}\int_X(\mathrm{Ric}_\omega - \chi)\wedge\omega^{n-1}$. Notice that for $\beta = 0$, $f = 0$ we get back the usual K-energy (2). Using the identity $n\,\mathrm{AM}_\chi(u) = n\,\mathrm{AM}_\beta(u) + \int f\omega_u^n - \int f\omega^n$ one can give an alternative formula for $\mathcal{K}_\chi$, perhaps more familiar from the literature:

$$\mathcal{K}_\chi(u) = \mathrm{Ent}(\omega^n, \omega_u^n) + \bar{S}_\chi \,\mathrm{AM}(u) - n\,\mathrm{AM}_{\mathrm{Ric}\,\omega-\chi}(u). \tag{5}$$

The virtue of this formula is that it shows that $\mathcal{K}_\chi$ is independent of the choice of β and f. As will be made clear shortly, when trying to extend $\mathcal{K}_\chi$, our original definition is more advantageous, however. Note that when χ is smooth, the first-order variation of $\mathcal{K}_\chi$ is given by the formula

$$\langle D\mathcal{K}_\chi(u), \delta v\rangle = V^{-1}\int_X \delta v(\bar{S}_\chi - S_{\omega_u} + \mathrm{Tr}^{\omega_u}\chi)\omega_u^n.$$

Hence, the critical points of this functional are the twisted csc-K potentials, as these satisfy $\bar{S}_\chi - S_{\omega_u} + \mathrm{Tr}^{\omega_u}\chi = 0$. The smooth twisted Calabi flow is defined analogously.

2.2 The complete geodesic metric spaces $(\mathcal{E}^p, d_p)$

In this section we summarize results from [29; 30; 17; 9] needed the most in this paper. Formula (1) introduces L^p–type weak Finsler metrics on the Fréchet manifold $\mathcal{H}_\omega$. A curve $[0,1] \ni t \to \alpha_t \in \mathcal{H}_\omega$ is called smooth if $\alpha(t,z) = \alpha_t(z) \in C^\infty([0,1]\times M)$. The L^p–length of a smooth curve $t \to \alpha_t$ is given by

$$l_p(\alpha) := \int_0^1 \|\dot\alpha_t\|_{p,\alpha_t}\,dt.$$

Definition 2.1 The path length pseudo-distance of $(\mathcal{H}_\omega, d_p)$ is defined by

$$d_p(u_0,u_1) := \inf\bigl\{l_p(\alpha)\colon [0,1]\ni t\to\alpha_t\in\mathcal{H}_\omega \text{ is a smooth curve with } \alpha_0 = u_0,\ \alpha_1 = u_1\bigr\}.$$

It turns out that d_p is an honest metric [30, Theorem 3.5]. To state the result, consider $[0,1]\times\mathbb{R}\times X$ as a complex manifold of dimension $n+1$, and let $\pi_2\colon [0,1]\times\mathbb{R}\times X \to X$ be the natural projection.

Theorem 2.2 *$(\mathcal{H}_\omega, d_p)$ is a metric space. Moreover, for any $t \in [0,1]$,*

$$d_p(u_0,u_1) = \|\dot u_t\|_{p,u_t} \ge 0,$$

where $\dot{u}_t = du_t/dt$ is the "tangent" at time t of $t \to u_t$, the $\mathbb{R}$–invariant solution of the Monge–Ampère equation

$$\begin{cases} \varphi \in \mathrm{PSH}(\pi_2^\star\omega, [0,1]\times\mathbb{R}\times X), \\ (\pi_2^\star\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^{n+1} = 0, \\ \varphi|_{\{i\}\times\mathbb{R}} = u_i, \quad i = 0,1. \end{cases} \tag{6}$$

Some comments are in order. By the main result of [20] (see also [14]), the equation (6) has a unique $\mathbb{R}$–invariant solution for which $u(t,x) = u_t(x)$ has bounded Laplacian in $[0,1]\times\mathbb{R}\times X$. We can look at this solution as a curve

$$[0,1] \ni t \to u_t \in \mathcal{H}_\omega^\Delta = \{u \in \mathrm{PSH}(X,\omega) : \Delta_\omega u \in L^\infty(X)\}.$$

We call this curve the *weak geodesic* connecting $u_0, u_1 \in \mathcal{H}_\omega$. Recall that

$$\mathrm{PSH}(X,\omega) = \{\varphi \in L^1(X,\omega^n) : \varphi \text{ is usc and } \omega_\varphi \geq 0\}.$$

Given $\varphi_k \in \mathrm{PSH}(X,\omega)$, $k = 1,\dots,n$, one can introduce the following *nonpluripolar product* [17], generalizing the Bedford–Taylor product [6] concerning the case with bounded potentials:

$$\omega_{\varphi_1}\wedge\omega_{\varphi_2}\wedge\cdots\wedge\omega_{\varphi_n} := \lim_{j\to-\infty} \mathbf{1}_{\bigcap_k\{\varphi_k > j\}}\omega_{\max(\varphi_1,j)}\wedge\omega_{\max(\varphi_2,j)}\wedge\cdots\wedge\omega_{\max(\varphi_n,j)}. \tag{7}$$

The measures $\omega_{\max(\varphi_1,j)}\wedge\cdots\wedge\omega_{\max(\varphi_n,j)}$ are defined by the work of Bedford and Taylor [6] since $\max\{\varphi, j\}$ is bounded. Restricted to $\bigcap_k\{\varphi_k > j\}$, these measures are increasing, hence the above limit is well defined [45; 17] and $\int_X \omega_{\varphi_1}\wedge\cdots\wedge\omega_{\varphi_n} \leq \int_X \omega^n$.

Following Guedj and Zeriahi [45, Definition 1.1] we introduce the class of potentials with "full volume", $\mathcal{E}(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\omega) : \int_X \omega_\varphi^n = \int_X \omega^n\}$, and the corresponding finite-energy classes

$$\mathcal{E}^p := \left\{\varphi \in \mathcal{E}(X,\omega) : \int_X |\varphi|^p \omega_\varphi^n < \infty\right\}.$$

The next result characterizes the d_p–metric completion of $\mathcal{H}_\omega$:

Theorem 2.3 [30, Theorem 2] *The metric completion of $(\mathcal{H}_\omega, d_p)$ equals $(\mathcal{E}^p, d_p)$, where*

$$d_p(u_0,u_1) := \lim_{k\to\infty} d_p(u_0^k, u_1^k),$$

for any smooth decreasing sequences $\{u_i^k\}_{k\in\mathbb{N}} \subset \mathcal{H}_\omega$ converging pointwise to $u_i \in \mathcal{E}^p$, $i = 0,1$. Moreover, for each $t \in (0,1)$, define

$$u_t := \lim_{k\to\infty} u_t^k,$$

where u_t^k is the weak geodesic connecting u_0^k and u_1^k. Then $u_t \in \mathcal{E}^p$, the curve $[0,1] \ni t \to u_t \in \mathcal{E}^p$ is well-defined independently of the choices of approximating sequences, and this curve is a d_p–geodesic.

In the rest of the paper we will call the d_p–geodesics constructed in this theorem *finite-energy geodesics*. As mentioned in [30], for arbitrary p and u_0, u_1, the finite-energy geodesic joining these potentials may not be unique as a d_p–geodesic.

By [34; 15] it is always possible to find approximating sequences $\{u_0^k\}_k, \{u_1^k\}_k$ as in the above theorem. We now recall [30, Theorem 3], giving a concrete characterization of the growth of all d_p metrics:

Theorem 2.4 *There exists $C > 1$ such that, for all $u, v \in \mathcal{E}^p$,*

$$C^{-1} d_p(u,v) \le \left(\int_X |u-v|^p \omega_u^n\right)^{1/p} + \left(\int_X |u-v|^p \omega_v^n\right)^{1/p} \le C d_p(u,v). \tag{8}$$

The inequalities in (8) have an important consequence: $|\sup_X u| \le C d_p(u,0)$ for all $u \in \mathcal{E}^p$. Also, when $p = 1$, d_1–convergence is equivalent to convergence with respect to the quasidistance $I(u,v) = \int_X (u-v)(\omega_v^n - \omega_u^n)$ introduced in [9], as shown in [30, Theorem 5.5].

Monotonic sequences behave well with respect to all d_p–metrics [30, Proposition 4.9]:

Proposition 2.5 *Suppose $u_k, u \in \mathcal{E}^p$. If $\{u_k\}_k$ is monotone decreasing/increasing and converges to u ae then $d_p(u_k, u) \to 0$.*

Given $u_0, u_1, \dots, u_k \in \mathrm{PSH}(X,\omega)$, by $P(u_0, u_1, \dots, u_k) \in \mathrm{PSH}(X,\omega)$ we define the upper envelope

$$P(u_0, u_1, \dots, u_k) = \sup\{v \in \mathrm{PSH}(X,\omega) \text{ such that } v \le u_0, \dots, v \le u_k\}.$$

According to the next proposition it is possible to sandwich a subsequence of any d_p–convergent sequence between two monotone sequences converging to the same limit.

Proposition 2.6 *Suppose $u_k, u \in \mathcal{E}^p$. If $d_p(u_k, u) \to 0$ then there exists a subsequence $k_j \to \infty$ and $\{w_{k_j}\}_j \subset \mathcal{E}^p$ decreasing, $\{v_{k_j}\}_j \subset \mathcal{E}^p$ increasing with $v_{k_j} \le u_{k_j} \le w_{k_j}$ and $d_p(w_{k_j}, u), d_p(v_{k_j}, u) \to 0$.*

Proof By (8) there exists $C > 0$ such that $|\sup_X u_j| \le C$ for $j \ge 1$. We introduce the sequence

$$w_k = \mathrm{usc}(\sup_{j \ge k} u_j).$$

As $u_k \leq w_k \leq C$, by [45] it follows that $w_k \in \mathcal{E}^p$. As $d_p(u_k, u) \to 0$, we have that $u_k \to u$ pointwise ae, hence w_k decreases to u. Proposition 2.5 then gives $d_p(w_k, u) \to 0$.

Now we construct the increasing sequence v_{k_j}. To do this, first take a subsequence u_{k_j} of u_k satisfying $d_p(u_{k_j}, u) \leq 2^{-j}$. As follows from the proof of [30, Theorem 4.17] and [29, Theorem 9.2], the following limit exists:

$$v_{k_j} = P(u_{k_j}, u_{k_{j+1}}, u_{k_{j+2}}, \ldots) := \lim_{h \to \infty} P(u_{k_j}, u_{k_{j+1}}, \ldots, u_{k_{j+h}}).$$

Additionally, $\{v_{k_j}\}_j \subset \mathcal{E}^p$ and v_{k_j} increases ae to u. The previous proposition now gives $d_p(u, v_{k_j}) \to 0$. □

Though stated differently, the next proposition is essentially contained in [17]:

Proposition 2.7 *Suppose $p \geq 1$, $\{u_j\}_j \subset \mathcal{E}^p$ is a d_p–bounded sequence and $u \in$ $\mathrm{PSH}(X, \omega)$ with $\|u_j - u\|_{L^1(X, \omega^n)} \to 0$. Then $u \in \mathcal{E}^p$.*

Proof Boundedness with respect to d_p implies that $|\sup_X u_j| \leq B$ for some $B \in \mathbb{R}$ (8). For simplicity assume that $B = 0$. The following sequence converges ae to u:

$$w_k = \mathrm{usc}(\sup_{j \geq k} u_j) \leq 0.$$

This sequence is additionally decreasing, and because $u_k \leq w_k \leq 0$, we have that $w_k \in \mathcal{E}^p$. If we could argue that $\{w_k\}_k$ is uniformly d_p–bounded then we would be finished by [30, Lemma 4.16]. But d_p–boundedness follows from (8). Indeed,

$$\int_X |w_k|^p \omega^n \leq \int_X |u_k|^p \omega^n \quad \text{and} \quad \int_X |w_k|^p \omega^n_{w_k} \leq C(p) \int_X |u_k|^p \omega^n_{u_k}$$

by [45, Lemma 3.5], hence by (8) the quantity $d_p(0, w_k)$ is uniformly bounded. □

Given two Borel measures μ, ν on X, if ν is not subordinate to μ, then by definition $\mathrm{Ent}(\mu, \nu) = \infty$. On the other hand, if ν is subordinate to μ then $\mathrm{Ent}(\mu, \nu) = \int_X \log(f)\nu$, where f is the Radon–Nikodym density of ν with respect to μ. The entropy functional $\mu \to \mathrm{Ent}(\mu, \nu)$ is lsc with respect to weak convergence of measures [35]. Related to entropy, we recall the following crucial compactness result [9, Theorem 2.17]:

Theorem 2.8 *Let $p > 1$ and suppose $\mu = f\omega^n$ is a probability measure with $f \in$ $L^p(X, \omega^n)$. Suppose there exists $C > 0$ such that $\{u_k\}_k \subset \mathcal{E}^1$ satisfies*

$$|\sup_X u_k| < C, \quad \mathrm{Ent}(\mu, \omega^n_{u_k}) < C.$$

Then $\{u_k\}_k$ contains a d_1–convergent subsequence.

2.3 The complex Monge–Ampère equation in $\mathcal{E}^p$

We summarize in this section basic results concerning solutions of degenerate complex Monge–Ampère equations that are needed in this paper.

A subset $E \subset X$ is called pluripolar if it is contained in the singular set of a function $\varphi \in \mathrm{PSH}(X,\omega)$, ie $E \subset \{\varphi = -\infty\}$. Let μ be a positive measure on X with total mass $\mu(X) = \int_X \omega^n$. We consider the equation

$$\omega_\varphi^n = \mu. \tag{9}$$

It was proved in [45, Theorem A] that when μ does not charge pluripolar sets the equation (9) has a solution $\varphi \in \mathcal{E}(X,\omega)$. The solution turns out to be unique up to an additive constant [37]. For each $\varepsilon > 0$, the same variational approach as in the proof of Theorem C on page 222 of [10] (see also [47, Corollary 11.9]) applied to the functional

$$F_\varepsilon(u) := \mathrm{AM}(u) - \frac{1}{\varepsilon}\log\int_X e^{\varepsilon u}\,d\mu, \quad u \in \mathcal{E}^1,$$

shows that there exists a solution $\varphi_\varepsilon \in \mathcal{E}^1$ to the equation

$$\omega_{\varphi_\varepsilon}^n = e^{\varepsilon\varphi_\varepsilon}\mu. \tag{10}$$

The solution is uniquely determined as follows from the comparison principle (see [12, Proposition 4.1]). The following version of the comparison principle will be useful later.

Lemma 2.9 *Let $\varepsilon > 0$. Assume that $\varphi \in \mathcal{E}(X,\omega)$ is a solution of* (10), *while $\psi \in \mathcal{E}(X,\omega)$ is a subsolution, ie $\omega_\psi^n \geq e^{\varepsilon\psi}\mu$. Then $\varphi \geq \psi$ on X.*

This result might be well known to experts in Monge–Ampère theory. As a courtesy to the reader we give a proof below.

Proof By the comparison principle for the class $\mathcal{E}(X,\omega)$ (see [45, Theorem 1.5]) we have

$$\int_{\{\varphi<\psi\}} \omega_\psi^n \leq \int_{\{\varphi<\psi\}} \omega_\varphi^n.$$

As φ is a solution and ψ is a subsolution to (10) we also have

$$\int_{\{\varphi<\psi\}} e^{\varepsilon\psi}\,d\mu \leq \int_{\{\varphi<\psi\}} \omega_\psi^n \leq \int_{\{\varphi<\psi\}} \omega_\varphi^n = \int_{\{\varphi<\psi\}} e^{\varepsilon\varphi}\,d\mu \leq \int_{\{\varphi<\psi\}} e^{\varepsilon\psi}\,d\mu.$$

It follows that all inequalities above are equalities, hence $\varphi \geq \psi$ μ–almost everywhere on X. By Dinew's domination principle [16, Proposition 5.9] we get $\varphi \geq \psi$ everywhere on X. □

One might wonder whether the solution of (9) arises as a limit of solutions of (10) as $\varepsilon \to 0$. The following result answers this affirmatively.

Lemma 2.10 *Let $p \geq 1$. Assume that $\mu = \omega_\varphi^n$ with $\varphi \in \mathcal{E}^p$ and $\int_X \varphi \, d\mu = 0$. For each $\varepsilon > 0$, let $\varphi_\varepsilon \in \mathcal{E}^1$ be the unique solution to* (10). *Then in fact $\varphi_\varepsilon \in \mathcal{E}^p$ and $d_p(\varphi_\varepsilon, \varphi) \to 0$ as $\varepsilon \to 0$.*

Proof As $\varphi - \sup_X \varphi$ is a subsolution of (10), it follows from Lemma 2.9 that $\varphi_\varepsilon \geq \varphi - \sup_X \varphi$ for all $\varepsilon > 0$, hence $\varphi_\varepsilon \in \mathcal{E}^p$. We claim that φ_ε is uniformly bounded from above for $\varepsilon \in [0, 1]$. Assume on the contrary that we can extract a subsequence denoted by $\varphi_j = \varphi_{\varepsilon_j}$ such that $\sup_X \varphi_j \to \infty$. The sequence $\psi_j := \varphi_j - \sup_X \varphi_j$ stays in a compact set in $L^1(X, \omega^n)$, hence a subsequence (still denoted by φ_j) converges to some $\psi \in \mathrm{PSH}(X, \omega)$. It then follows that $\varphi_j = \psi_j + \sup_X \varphi_j$ converges uniformly to ∞. In the other hand, by Jensen's inequality (for simplicity we may assume that $\mu(X) = 1$) we have

$$\int_X \varphi_j \, d\mu \leq 0.$$

Since φ_j is bounded from below by $\varphi - \sup_X \varphi$, which is integrable with respect to $d\mu$, the above inequality contradicts the fact that φ_j converges uniformly to ∞. Hence the claim follows.

Now the family φ_ε stays in a compact set of $L^1(X, \omega^n)$. As $\varepsilon \to 0$ each cluster point φ_0 satisfies

$$\omega_{\varphi_0}^n \geq \Big(\liminf_{\varepsilon \to 0} e^{\varepsilon \varphi_\varepsilon}\Big)\mu = \mu,$$

as follows from [17, Corollary 2.21]. As the two measures have the same total mass, one obtains equality. That $\varphi_0 = \varphi$ follows from uniqueness of complex Monge–Ampère measures [37] and the identity

$$0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log \int_X e^{\varepsilon \varphi_\varepsilon} \, d\mu = \int_X \varphi_0 \, d\mu.$$

Finally, the last statement can be addressed using the identity

$$\int_X |\varphi_\varepsilon - \varphi|^p (\omega_{\varphi_\varepsilon}^n + \omega_\varphi^n) = \int_X |\varphi_\varepsilon - \varphi|^p (e^{\varepsilon \varphi_\varepsilon} + 1)\omega_\varphi^n.$$

Using this, (8) and the fact that $\sup_X \varphi_\varepsilon$ is bounded from above, by the dominated convergence theorem we conclude that $d_p(\varphi_\varepsilon, \varphi) \to 0$. □

2.4 Weak convergence in a CAT(0) space

Let us recall that a geodesic metric space (M, d) is a metric space for which any two points can be connected with a geodesic. By a geodesic connecting two points $a, b \in M$ we understand a curve $\alpha \colon [0, 1] \to M$ such that $\alpha(0) = a$, $\alpha(1) = b$ and

$$d(\alpha(t_1), \alpha(t_2)) = |t_1 - t_2| d(a, b)$$

for any $t_1, t_2 \in [0,1]$. Furthermore, a geodesic metric space (M,d) is nonpositively curved (in the sense of Alexandrov) or $\mathrm{CAT}(0)$ if for any distinct points $q, r \in M$ there exists a geodesic $\gamma\colon [0,1] \to M$ joining q and r such that for any $s \in \{\gamma\}$ and $p \in M$ the inequality

$$d(p,s)^2 \le \lambda d(p,r)^2 + (1-\lambda) d(p,q)^2 - \lambda(1-\lambda) d(q,r)^2$$

is satisfied, where $\lambda = d(q,s)/d(q,r)$. A basic property of $\mathrm{CAT}(0)$ spaces is that geodesic segments joining different points are unique. For more about these spaces we refer to [18].

Let $\{x_n\}_n$ be a bounded sequence in a $\mathrm{CAT}(0)$ metric space (M,d). For $x \in M$, we set

$$r(x, \{x_n\}_n) = \limsup d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}_n$ is given by $r(\{x_n\}_n) = \inf\{r(x,\{x_n\}_n) : x \in M\}$, and the *asymptotic center* $A(\{x_n\}_n)$ of $\{x_n\}_n$ is the set

$$A(\{x_n\}_n) = \{x \in M : r(x,\{x_n\}_n) = r(\{x_n\}_n)\}.$$

It is well known (see eg [62, Lemma 4.3]) that, in a $\mathrm{CAT}(0)$ space, $A(\{x_n\}_n)$ consists of exactly one point. A sequence $\{x_n\}_n$ *converges d–weakly* to $x \in M$ if x is the asymptotic center of all subsequences of $\{x_n\}_n$.

For a more detailed account of weak d–convergence we refer to [51], and for results related to the Calabi flow to [62, Section 4]. If (M,d) is a Hilbert space then weak d–convergence is the same as weak convergence in the sense of Hilbert spaces. With this in mind, the contents of the next result may seem less surprising:

Proposition 2.11 *Suppose (M,d) is a $\mathrm{CAT}(0)$ space. The following hold:*

(i) [51, Proposition 3.5] *If $\{x_n\}_n$ is a d–bounded sequence then it has a weak d–convergent subsequence.*

(ii) [51, Proposition 3.2] *Suppose $C \subset M$ is a geodesically convex closed set and $\{x_n\}_n \subset C$ converges d–weakly to $x \in M$. Then $x \in C$.*

2.5 General weak gradient flows

Let G be a d–lsc function on a complete metric space (M,d). In this generality there are, as explained in [1], various notions of weak gradient flows c_t for G, emanating from an initial point c_0 in M. A natural approximation scheme (the so-called minimizing movement) for obtaining such a candidate $t \to c_t$ was introduced by De Giorgi [33]. It can be seen as a variational formulation of the (backward) Euler scheme: given

$t \in [0, \infty)$ and a positive integer m, one first defines a discrete version c_t^m of c_t as the m^{th} step in the following (m–dependent) iteration with initial data $c_t^{m,0} = c_0$: given $c_t^{m,j} \in M$, the next step $c_t^{m,j+1}$ is obtained by minimizing on M the functional

$$v \to \frac{1}{2} d(v, c_t^{m,j})^2 + \frac{t}{m} G(v). \tag{11}$$

If such a minimizer always exists then the corresponding minimizing movement c_t is defined as the large m limit of $c_t^m = c_t^{m,m}$, if the limit exists in (M, d). As shown by Mayer [56], if (M, d) is a CAT(0) metric space and G is convex this procedure indeed produces a unique limit c_t with a number of useful properties.

Theorem 2.12 [56, Theorem 1.13] *If (M, d) is CAT(0), G is a d–lsc convex function on (M, d), then for any initial point c_0 with $G(c_0) < \infty$ the corresponding minimizing movement $t \to c_t$ exists and defines a contractive continuous semigroup (which is locally Lipschitz continuous on $[0, \infty)$).*

Moreover, as shown in [56], the curve $t \to c_t$ can be thought of as the curve of steepest descent with respect to G in the sense that

$$-\frac{d}{dt}(G(c_t)) = |(\partial G)(c_t)| \left| \frac{dc_t}{dt} \right|, \quad \left| \frac{dc_t}{dt} \right| = |(\partial G)(c_t)| \tag{12}$$

for almost every t, where $|(\partial G)(y)|$ is the local upper gradient of G at y and $|dc_t/dt|$ is the metric derivative of $t \to c_t$ at t (in the sense of [1]):

$$|(\partial G)(y)| := \limsup_{z \to y} \frac{(G(y) - G(z))^+}{d(y, z)}, \quad \left| \frac{dc_t}{dt} \right| = \lim_{s \to t} \left| \frac{d(c_s, c_t)}{s - t} \right|.$$

In the case when (M, d) is a finite-dimensional Riemannian manifold and G is smooth, the relations (12) are equivalent to the usual gradient flow formulation for G. In the terminology of [1] the relations (12) imply that the minimizing movement $t \to c_t$ provided by Mayer's theorem is a curve of *maximal slope* with respect to the upper gradient $|(\partial G)|$ (see [1, Definition 1.3.2]). Moreover, by [1, Theorem 4.0.4] the curve $t \to c_t$ is the unique solution of the *evolution variational inequality*

$$\frac{1}{2} \frac{d}{dt} d^2(c_t, v) \le G(v) - G(c_t) \quad \text{for ae } t > 0 \text{ and all } v \text{ such that } G(v) < \infty \tag{13}$$

among all locally absolutely continuous curves in (M, d) such that $\lim_{t \to 0} c_t = c_0$. Among other things, this inequality shows that

$$\lim_{t \to \infty} G(c_t) = \inf_{y \in M} G(y).$$

Remark 2.13 A necessary condition for the solvability of the minimization steps (11) is to have $G(c_0) < \infty$. An approximation argument using contractivity of the minimizing movement yields that it is possible to uniquely define $t \to c_t$ for any c_0 in the d–closure of the set $\{G < \infty\}$. This slightly more general movement satisfies all the above-mentioned properties and additionally $G(c_t) < \infty$ for any $t > 0$ (for more details see [1]).

Lastly, we recall a theorem of Bačák, central in our later developments:

Theorem 2.14 [3, Theorem 1.5] *Given a* CAT(0) *space* (M, d) *and a* d*–lsc convex function* $G\colon M \to (-\infty, \infty]$*, assume that* G *attains its minimum on* M*. Then any minimizing movement trajectory* $t \to c_t$ *weakly* d*–converges to some minimizer of* G *as* $t \to \infty$*.*

3 Approximation in d_p with convergent entropy

The approximation results in this section will be used in the proof of Theorem 1.2. Our main tools will come from Sections 2.1–2.3. We begin with the simplified situation of approximation in $\mathcal{E}^1$:

Lemma 3.1 *Suppose* f *is usc on* X *with* $e^{-f} \in L^1(X, \omega^n)$*. Given* $u \in \mathcal{E}^1$*, there exists* $u_k \in \mathcal{H}_\omega$ *such that* $d_1(u_k, u) \to 0$ *and* $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_{u_k}) \to \mathrm{Ent}(e^{-f}\omega^n, \omega^n_u)$*.*

Proof If $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_u) = \infty$ then any sequence $u_k \in \mathcal{H}_\omega$ with $d_1(u_k, u) \to 0$ satisfies the requirements, as the entropy is d_1–lsc. Indeed, this follows from the classical fact that the entropy is lsc with respect to weak convergence of measures [35], and d_1–convergence implies weak convergence of the complex Monge–Ampère measures [30].

We can suppose that $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_u) < \infty$. Let $g = \omega^n_u/\omega^n \geq 0$ be the density function of ω^n_u. We will show that there exist positive functions $g_k \in C^\infty(X)$ such that $|g - g_k|_{L^1} \to 0$ and

$$\int_M g_k \log \frac{g_k}{e^{-f}} \omega^n \to \int_M g \log \frac{g}{e^{-f}} \omega^n = \mathrm{Ent}(e^{-f}\omega^n, \omega^n_u).$$

First introduce $h_k = \min\{k, g\}$, $k \in \mathbb{N}$. As $\phi(t) = t \log t$, $t > 0$, is bounded from below by $-e^{-1}$ and increasing for $t > 1$, we get

$$-e^{-1}e^{-f} \leq h_k \log \frac{h_k}{e^{-f}} \leq \max\left\{0, g \log \frac{g}{e^{-f}}\right\}.$$

Clearly $|h_k - g|_{L^1} \to 0$, and as $e^{-f} \in L^1(X, \omega^n)$ and $g \log(g/e^{-f}) \in L^1(X, \omega^n)$, the Lebesgue dominated convergence theorem gives that

$$\int_M h_k \log \frac{h_k}{e^{-f}} \omega^n \to \int_M g \log \frac{g}{e^{-f}} \omega^n = \mathrm{Ent}(e^{-f}\omega^n, \omega_u^n).$$

Using the density of $C^\infty(M)$ in $L^1(M)$, by another application of the dominated convergence theorem, we find a positive sequence $g_k \in C^\infty(X)$ such that $|g_k - h_k|_{L^1} \leq 1/k$ and

$$\left| \int_M h_k \log \frac{h_k}{e^{-f}} \omega^n - \int_M g_k \log \frac{g_k}{e^{-f}} \omega^n \right| \leq \frac{1}{k}.$$

Using the Calabi–Yau theorem we find potentials $v_k \in \mathcal{H}_\omega$ with $\sup_M v_k = 0$ and $\omega^n_{v_k} = g_k \omega^n / \int_M g_k \omega^n$. Theorem 2.8 now guarantees that (after possibly passing to a subsequence) $d_1(v_k, h) \to 0$ for some $h \in \mathcal{E}^1(X)$. But [30, Theorem 5(i)] implies the equality of measures $\omega_h^n = \omega_u^n$. Finally, by the uniqueness theorem [45, Theorem B] we get that h and u can differ by at most a constant. Hence, after possibly adding a constant, we can suppose that $d_1(v_k, u) \to 0$. □

The key point in this proof is that a bound on the entropy implies compactness in $(\mathcal{E}^1, d_1)$. There are examples showing that the d_2 version of this compactness result does not hold in general. Therefore, to approximate functions in $(\mathcal{E}^p, d_p)$, $p > 1$ with convergent entropy, a new approach is necessary:

Theorem 3.2 *Suppose $\varphi \in \mathcal{E}^p$, $p \geq 1$ and f is usc on X with $e^{-f} \in L^1(X, \omega^n)$. Then there exists $\varphi_j \in \mathcal{H}_\omega$ such that $d_p(\varphi_j, \varphi) \to 0$ and*

$$\mathrm{Ent}(e^{-f}\omega^n, \omega^n_{\varphi_j}) \to \mathrm{Ent}(e^{-f}\omega^n, \omega^n_\varphi).$$

Proof We divide the approximation procedure into three steps.

Step 1 Assume that $u \in \mathcal{E}^p$ has finite twisted entropy $\mathrm{Ent}(e^{-f}\omega^n, \omega_u^n) < \infty$ and

$$\omega_u^n = e^g \omega^n$$

for some measurable function g. We also normalize u so that $\int_X u\omega_u^n = 0$. For each $\varepsilon > 0$ let $u_\varepsilon \in \mathcal{E}^p(X, \omega)$ be the unique solution to

$$\omega^n_{u_\varepsilon} = e^{\varepsilon u_\varepsilon + g}\omega^n.$$

Then we claim that $d_p(u_\varepsilon, u) \to 0$ and $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_{u_\varepsilon}) \to \mathrm{Ent}(e^{-f}\omega^n, \omega^n_u)$ as $\varepsilon \to 0$.

Indeed, from Lemma 2.10, u_ε is uniformly bounded from above for $\varepsilon \in [0, 1]$, and converges in d_p to u as $\varepsilon \to 0$. Also, by the comparison principle (Lemma 2.9), $u_\varepsilon \geq u - \sup_X u$. As in the proof of Lemma 3.1 we can show using the dominated convergence theorem that $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_{u_\varepsilon})$ converges to $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_u)$ as $\varepsilon \to 0$.

Step 2 Let g be a measurable function such that $\int_X e^g \omega^n < \infty$. Assume that $u \in \mathcal{E}^p$ has finite twisted entropy $\mathrm{Ent}(e^{-f}\omega^n, \omega_u^n) < \infty$ and

$$\omega_u^n = e^{\varepsilon u + g}\omega^n$$

for some $\varepsilon > 0$. Consider $g_k := \min(g, k)$, $k \in \mathbb{N}$. Let $u_k \in \mathrm{PSH}(X, \omega) \cap \mathcal{C}^0(X)$ be the unique solution to

$$\omega_{u_k}^n = e^{\varepsilon u_k + g_k}\omega^n.$$

The fact that u_k is continuous follows from Kołodziej's $\mathcal{C}^0$ estimate [52]. By the comparison principle u_k is decreasing and converges to u as $k \to \infty$. It follows from Proposition 2.5 that $d_p(u_k, u) \to 0$ as $k \to \infty$. Again, the proof of Lemma 3.1 shows that $\mathrm{Ent}(e^{-f}\omega^n, \omega_{u_k}^n)$ converges to $\mathrm{Ent}(e^{-f}\omega^n, \omega_u^n)$ as $k \to \infty$.

Step 3 Assume that g is bounded from above, $u \in \mathcal{E}^p$ has finite twisted entropy $\mathrm{Ent}(e^{-f}\omega^n, \omega_u^n) < \infty$ and

$$\omega_u^n = e^{\varepsilon u + g}\omega^n \tag{14}$$

for some $\varepsilon > 0$. Let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence of smooth functions, uniformly bounded above, such that e^{g_k} converges to e^g in $L^2(X, \omega^n)$ and $\int_X e^{g_k}\omega^n = \int_X e^g \omega^n$. Let $u_k \in \mathcal{H}_\omega$ be the unique smooth solution to

$$\omega_{u_k}^n = e^{\varepsilon u_k + g_k}\omega^n. \tag{15}$$

The fact that u_k is smooth on X is well known (see [2] or [64; 47, Chapter 14] for other proofs).

We claim that $\sup_X u_k$ is bounded above. Indeed, by an argument similar to that of Lemma 2.10, suppose that for some subsequence (again denoted by u_k) we have that $\sup_X u_k \to \infty$. Then a subsequence of $v_k := u_k - \sup_X u_k$ (again denoted by v_k) L^1–converges to some $v \in \mathrm{PSH}(X, \omega)$. As all L^p topologies are equivalent on $\mathrm{PSH}(X, \omega)$, we actually have $v_k \to_{L^p} v$ for any $p \geq 1$. However, using Jensen's inequality and (15), we obtain that

$$\int_X u_k e^{g_k}\omega^n = \int_X v_k e^{g_k}\omega^n + \sup_X u_k \int_X e^g \omega^n$$

is uniformly bounded above. We have that $v_k \to_{L^2} v$ and $e^{g_k} \to_{L^2} e^g$, hence using Hölder's inequality we obtain $\int_X v_k e^{g_k}\omega^n \to \int_X v e^g \omega^n \neq \infty$. As $\sup_X u_k \to \infty$ we arrive at a contradiction with the upper bound on $\int_X u_k e^{g_k}\omega^n$, finishing the proof of the claim.

As $\sup_X u_k$ is bounded above, using Kołodziej's estimates [53] for (15), we obtain a uniform upper bound on $\|u_k\|_{C^{0,\alpha}}$ for some $\alpha > 0$. After perhaps choosing a subsequence, $\{u_k\}_k$ will converge uniformly to some $v \in C^{0,\alpha} \cap \mathrm{PSH}(X,\omega)$, ultimately giving $d_p(u_k, v) \to 0$. As both the left- and right-hand sides of (15) converge, we get that $\omega_v^n = e^{\varepsilon v + g}\omega^n$, hence by uniqueness of solutions to (14) (Lemma 2.9) we get $v = u$. By a repeated use of the dominated convergence theorem, the corresponding twisted entropies also converge.

Now, we come back to the proof of Theorem 3.2. If $\mathrm{Ent}(e^{-f}\omega^n, \omega_\varphi^n) = \infty$ then any decreasing sequence $\varphi_j \in \mathcal{H}_\omega$ which converges pointwise to φ satisfies our requirement since the entropy is lsc with respect to weak convergence of measures. We can thus assume that $\mathrm{Ent}(e^{-f}\omega^n, \omega_\varphi^n) < \infty$. Then we can write $\omega_\varphi^n = e^g \omega^n$. We can also assume that $\int_X \varphi \omega_\varphi^n = 0$. Fix $\delta > 0$ arbitrarily small. Denoting $\varphi_0 = \varphi$, by the three steps above we can find $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{E}^p$, with $\varphi_3 \in \mathcal{H}_\omega$, such that

$$d_p(\varphi_j, \varphi_{j+1}) \le \delta \quad \text{and} \quad \left|\mathrm{Ent}(e^{-f}\omega^n, \omega_{\varphi_j}^n) - \mathrm{Ent}(e^{-f}\omega^n, \omega_{\varphi_{j+1}}^n)\right| \le \delta, \quad j = 0, 1, 2.$$

From this the result follows. □

4 Extension of the twisted K-energy

The main goal of this section is to prove Theorem 1.2. Before we can attempt a proof, we need to understand the d_1–continuity properties of each functional appearing in the right-hand side of (4). Some of the preliminary results below are well known, but as a courtesy to the reader we give a detailed account.

4.1 The AM functional

The Aubin–Mabuchi functional is given by the formula (see [55, Theorem 2.3])

$$\mathrm{AM}(u) := \frac{V^{-1}}{n+1} \sum_{j=0}^{n} \int_X u\, \omega^j \wedge \omega_u^{n-j}, \quad u \in \mathcal{H}_\omega. \tag{16}$$

A series of integrations by parts gives

$$\mathrm{AM}(v) - \mathrm{AM}(u) = \frac{V^{-1}}{n+1} \int_X (v-u) \sum_{k=0}^{n} \omega_u^{n-k} \wedge \omega_v^k, \quad u, v \in \mathcal{H}_\omega. \tag{17}$$

Among other things, this formula shows that

$$u \le v \implies \mathrm{AM}(u) \le \mathrm{AM}(v),$$

and by computing $\lim_{t\to 0}(\mathrm{AM}(v_t) - \mathrm{AM}(v))/t$ we arrive at the first-order variation of AM:

$$\langle D\,\mathrm{AM}(v), \delta v\rangle = V^{-1} \int_X \delta v \omega_v^n, \quad v \in \mathcal{H}_\omega, \delta v \in C^\infty(X). \tag{18}$$

Suppose $u \in \mathcal{E}^1$ and let $u_j \in \mathcal{H}_\omega$ be pointwise decreasing to u. Using Proposition 2.5 we have $d_1(u, u_j) \to 0$. We hope to extend AM to $\mathcal{E}^1$ via

$$\mathrm{AM}(u) = \lim_j \mathrm{AM}(u_j). \tag{19}$$

As it turns out, this choice of extension is justified by the following precise result:

Proposition 4.1 *The map* AM: $\mathcal{H}_\omega \to \mathbb{R}$ *is* d_1*–Lipschitz continuous. Thus,* (19) *gives* d_1*–Lipschitz extension of* AM *to* $\mathcal{E}^1$.

Proof First we argue that $|\mathrm{AM}(u_0) - \mathrm{AM}(u_1)| \leq d_1(u_0, u_1)$ for $u_0, u_1 \in \mathcal{H}_\omega$. Let $[0,1] \ni t \to \gamma_t \in \mathcal{H}_\omega$ be a smooth curve connecting u_0 and u_1. By (18) we can write

$$|\mathrm{AM}(u_1) - \mathrm{AM}(u_0)| = \left| V^{-1} \int_0^1 \int_X \dot{\gamma}_t \omega_{\gamma_t}^n \, dt \right| \leq V^{-1} \int_0^1 \int_X |\dot{\gamma}_t| \omega_{\gamma_t}^n \, dt = l(\gamma).$$

Taking the infimum over all smooth curves connecting u_0 and u_1, we obtain that

$$|\mathrm{AM}(u_1) - \mathrm{AM}(u_0)| \leq d_1(u_0, u_1).$$

The density of $\mathcal{H}_\omega$ in $\mathcal{E}^1$ implies that AM extends to $\mathcal{E}^1$ using the formula (19). The extension has to be d_1–Lipschitz continuous. □

Before we proceed, we note that the "abstract" d_1–continuous extension AM: $\mathcal{E}^1 \to \mathbb{R}$ given by the above result is the same as the "concrete" one given by the expression of (16) after replacing the smooth products $\omega^j \wedge \omega_u^{n-j}$ with the nonpluripolar products from (7), as done in [17]. Moving on, we give a kind of "domination principle" for the extended Aubin–Mabuchi energy on $\mathcal{E}^1$:

Proposition 4.2 *Suppose* $\phi, \psi \in \mathcal{E}^1$ *with* $\phi \geq \psi$. *If* $\mathrm{AM}(\phi) = \mathrm{AM}(\psi)$, *then* $\phi = \psi$.

Proof Suppose $\phi_k, \psi_k \in \mathcal{H}_\omega$ are sequences pointwise decreasing to ϕ and ψ, respectively, with $\phi_k \geq \psi_k$. Then (17) gives that

$$0 \leq \frac{1}{(n+1)V} \int_X (\phi_k - \psi_k) \omega_{\psi_k}^n \leq \mathrm{AM}(\phi_k) - \mathrm{AM}(\psi_k).$$

Using the previous proposition and [30, Lemma 5.2] with $\chi(t) = |t|$, $v_k = \phi_k$, $u_k = \psi_k$, $w_k = \psi_k$, we may take the limit in this estimate to obtain

$$0 \leq \frac{1}{(n+1)V} \int_X (\phi - \psi) \omega_\psi^n \leq \mathrm{AM}(\phi) - \mathrm{AM}(\psi) = 0,$$

hence $\psi \geq \phi$ ae with respect to ω_ψ^n. The domination principle of the class $\mathcal{E}$ [16, Proposition 5.9] gives now that $\psi \geq \phi$ globally on X, hence $\psi = \phi$. □

The last result of this subsection points out that the family of finite-energy geodesics inside $\mathcal{E}^p$ is in fact "endpoint-stable". We note that in the case $p = 2$ this follows from the fact that $(\mathcal{E}^2, d_2)$ is CAT(0) [18].

Proposition 4.3 *Suppose $[0,1] \ni t \to u^j_t \in \mathcal{E}^p$ is a sequence of finite-energy geodesic segments such that $d_p(u^j_0, u_0), d_p(u^j_1, u_1) \to 0$. Then $d_p(u^j_t, u_t) \to 0$ for all $t \in [0,1]$, where $[0,1] \ni t \to u_t \in \mathcal{E}^p$ is the finite-energy geodesic segment connecting u_0 and u_1.*

Proof Let $t \in [0,1]$. Notice that we only have to show that any subsequence of $\{u^j_t\}_j$ contains a subsubsequence d_p–converging to u_t.

Let $\{u^{j_k}_t\}_k$ be an arbitrary subsequence of $\{u^j_t\}_j$. Let j_{k_l} be a subsequence of j_k with the following property: for $i = 0, 1$, there exist a monotone increasing sequence $\{v^{j_{k_l}}_i\}_l$ and a monotone decreasing sequence $\{w^{j_{k_l}}_i\}_l$ such that

$$v^{j_{k_l}}_i \le u^{j_{k_l}}_i \le w^{j_{k_l}}_i \quad \text{for all } j_{k_l} \qquad \text{and} \qquad v^{j_{k_l}}_i, w^{j_{k_l}}_i \to_{d_p} u_i.$$

This is possible to arrange according to Proposition 2.6.

By $[0,1] \ni t \to v^{j_{k_l}}_t \in \mathcal{E}^p$ and $[0,1] \ni t \to w^{j_{k_l}}_t \in \mathcal{E}^p$ we denote finite-energy geodesics connecting $v^{j_{k_l}}_0$ to $v^{j_{k_l}}_1$ and $w^{j_{k_l}}_0$ to $w^{j_{k_l}}_1$, respectively. By the maximum principle of finite-energy geodesics we can write

$$v_t := \operatorname{usc}\Big(\lim_l v^{j_{k_l}}_t\Big) \le u_t \le w_t := \lim_l w^{j_{k_l}}_t.$$

As AM is d_p–continuous it follows that

$$\lim_l \mathrm{AM}(v^{j_{k_l}}_i) = \mathrm{AM}(u_i) = \lim_l \mathrm{AM}(w^{j_{k_l}}_i) \quad \text{for } i = 0, 1.$$

As AM is also linear along finite-energy geodesics we get

$$\mathrm{AM}(v_t) = \mathrm{AM}(u_t) = \mathrm{AM}(w_t) \quad \text{for any } t \in [0,1].$$

Proposition 4.2 gives that $v_t = u_t = w_t$, hence

$$d_p(v^{j_{k_l}}_t, u_t) \to 0 \quad \text{and} \quad d_p(w^{j_{k_l}}_t, u_t) \to 0.$$

Using $v^{j_{k_l}}_t \le u^{j_{k_l}}_t \le w^{j_{k_l}}_t$, [30, Lemma 4.2] gives that

$$d_p(v^{j_{k_l}}_t, u^{j_{k_l}}_t) \le d_p(v^{j_{k_l}}_t, w^{j_{k_l}}_t) \to 0,$$

hence $d_p(u^{j_{k_l}}_t, u_t) \to 0$, as desired. □

4.2 The AM_γ functional

For the moment we fix a closed $(1,1)$–current γ on X, not necessarily positive. Recall from the introduction that the functional AM_γ is defined as follows:

$$\mathrm{AM}_\gamma(u) := \frac{1}{nV}\sum_{j=0}^{n-1}\int_X u\,\gamma\wedge\omega^j\wedge\omega_u^{n-1-j}, \quad u\in\mathcal{H}_\omega. \tag{20}$$

Similarly to AM, integrating by parts gives

$$\mathrm{AM}_\gamma(v)-\mathrm{AM}_\gamma(u) = \frac{1}{nV}\int_X (v-u)\sum_{k=0}^{n-1}\gamma\wedge\omega_u^{n-k-1}\wedge\omega_v^k. \tag{21}$$

When $\gamma\ge 0$ this last formula gives

$$u\le v \implies \mathrm{AM}_\gamma(u)\le\mathrm{AM}_\gamma(v).$$

By computing $\lim_{t\to 0}(\mathrm{AM}_\gamma(v_t)-\mathrm{AM}_\gamma(v))/t$ we arrive at the first-order variation of AM_γ:

$$\langle D\,\mathrm{AM}_\gamma(v),\delta v\rangle = V^{-1}\int_X \delta v\gamma\wedge\omega_v^{n-1}, \quad v\in\mathcal{H}_\omega, \delta v\in C^\infty(X). \tag{22}$$

Extension of AM_γ to $\mathcal{E}^1$ when γ is smooth For this paragraph suppose γ is smooth. Suppose $u\in\mathcal{E}^1$ and let $u_j\in\mathcal{H}_\omega$ be pointwise decreasing to u. Using Proposition 2.5 we have $d_1(u,u_j)\to 0$. We hope to extend AM_γ to $\mathcal{E}^1$ via

$$\mathrm{AM}_\gamma(u) = \lim_j \mathrm{AM}_\gamma(u_j). \tag{23}$$

As it turns out, this extension is rigorous as we have the following precise result:

Proposition 4.4 *Formula (23) gives a d_1–continuous functional $\mathrm{AM}_\gamma\colon\mathcal{E}^1\to\mathbb{R}$. Additionally, AM_γ thus extended is bounded on d_1–bounded subsets of $\mathcal{E}^1$.*

Proof We argue that for any $R>0$ there exists $f_R\colon\mathbb{R}\to\mathbb{R}$ continuous with $f_R(0)=0$ such that

$$|\mathrm{AM}_\gamma(u_0)-\mathrm{AM}_\gamma(u_1)|\le f_R(d_1(u_0,u_1)) \tag{24}$$

for any $u_0,u_1\in\mathcal{H}_\omega\cap\{v : d_1(0,v)\le R\}$. We have $-C\omega\le\gamma\le C\omega$ for some $C>1$. Using (21) and the observation $\omega_{(u_0+u_1)/4}=\frac12\omega+\frac14\omega_{u_0}+\frac14\omega_{u_1}$ it follows that

$$|\mathrm{AM}_\gamma(u_0)-\mathrm{AM}_\gamma(u_1)|\le C\int_X |u_0-u_1|\omega^n_{(u_0+u_1)/4}.$$

By [30, Corollary 5.7] and its proof, for each $R > 0$ there exists a continuous function $f_R\colon \mathbb{R} \to \mathbb{R}$ with $f_R(0) = 0$ such that

$$\int_X |v - w|\omega_h^n \leq f_R(d_1(v, w))$$

for any $v, w, h \in \mathcal{E}^1 \cap \{v : d_1(0, v) \leq R\}$. Using this last fact, to argue that (24) holds, it is enough to show that $d_1\big(0, \frac{1}{4}(u_0 + u_1)\big)$ is bounded in terms of $d_1(0, u_0)$ and $d_1(0, u_1)$. We recall [30, Lemma 5.3], which says that there exists $D > 1$ such that $d_1\big(a, \frac{1}{2}(a + b)\big) \leq D d_1(a, b)$ for any $a, b \in \mathcal{E}^1$. Using this several times along with the triangle inequality, we can write

$$\begin{aligned} d_1\big(0, \tfrac{1}{4}(u_0 + u_1)\big) &\leq C d_1\big(0, \tfrac{1}{2}(u_0 + u_1)\big) \leq C\big(d_1(0, u_0) + d_1\big(u_0, \tfrac{1}{2}(u_0 + u_1)\big)\big) \\ &\leq C^2(d_1(0, u_0) + d_1(u_0, u_1)) \leq 2C^2(d_1(0, u_0) + d_1(0, u_1)), \end{aligned}$$

finishing the proof. □

As in the case of AM, the "abstract" d_1–continuous extension $\mathrm{AM}_\gamma\colon \mathcal{E}^1 \to \mathbb{R}$ given by the above result is identical to the one given by the "concrete" expression of (20) after replacing the smooth products $\gamma \wedge \omega^j \wedge \omega_u^{n-j-1}$ with nonpluripolar products similar to (7).

Convexity and extension of AM_χ to $\mathcal{H}_\omega^\Delta$ when χ satisfies (3) Suppose that $\chi = \beta + i\partial\bar\partial f$ is a $(1,1)$–current satisfying (3). Observe that it is not possible to extend AM_χ to $\mathcal{H}_\omega^\Delta$ using the techniques of the previous paragraph directly. Instead, using integration by parts, we notice that, given $u \in \mathcal{H}_\omega$, we have an alternative formula for $\mathrm{AM}_\chi(u)$:

$$\begin{aligned} \text{(25)} \qquad \mathrm{AM}_\chi(u) &= \frac{1}{nV}\sum_{j=0}^{n-1}\int_X u\, \beta \wedge \omega^j \wedge \omega_u^{n-1-j} + \frac{1}{nV}\int_X f(\omega_u^n - \omega^n) \\ &= \mathrm{AM}_\beta(u) + \frac{1}{nV}\int_X f(\omega_u^n - \omega^n). \end{aligned}$$

As β is smooth, AM_β extends d_1–continuously to $\mathcal{H}_\omega^\Delta$ by the previous paragraph. The map $u \to \int_X f\omega_u^n$ clearly makes sense and is finite for all $u \in \mathcal{H}_\omega^\Delta$, hence using (25) it is possible to extend AM_χ to $\mathcal{H}_\omega^\Delta$. Though not needed, it can be further shown that this extension is independent of the choice of β and f.

Given $u_0, u_1 \in \mathcal{H}_\omega$, for the weak geodesic $[0,1] \ni t \to u_t \in \mathcal{H}_\omega^\Delta$ connecting u_0 and u_1 we would like to show that $t \to \mathrm{AM}_\chi(t)$ is convex. When χ is smooth this follows from the result of [21]. It turns out that for more general χ the same proof gives an analogous result:

Proposition 4.5 *Suppose $\chi = \beta + i\partial\bar\partial f \geq 0$ satisfies* (3)*. Equation* (25) *gives an extension $\mathrm{AM}_\chi\colon \mathcal{H}_\omega^\Delta \to \mathbb{R}$ for which $t \to \mathrm{AM}_\chi(u_t)$ is convex for any weak geodesic segment $[0,1] \ni t \to u_t \in \mathcal{H}_\omega^\Delta$.*

Proof Suppose $t_1 \geq t_0$. When χ is smooth, it is well known that for $[0,1] \ni t \to v_t \in \mathcal{H}_\omega$ smooth subgeodesic (ie $\pi^*\omega + i\partial\bar\partial v \geq 0$) we actually have

$$\frac{d}{dt}\Big|_{t=t_1} \mathrm{AM}_\chi(v_t) - \frac{d}{dt}\Big|_{t=t_0} \mathrm{AM}_\chi(v_t) = \int_{S_{t_0,t_1}\times X} \pi^*\chi \wedge (\pi^*\omega + i\partial\bar\partial v)^n,$$

where $S_{t_0,t_1} \subset \mathbb{C}$ is the strip $\{t_0 \leq \operatorname{Re} z \leq t_1\}$. Hence, $t \to \mathrm{AM}_\chi(v_t)$ is convex. We claim that the same proof goes through for any positive closed current $\chi = \beta + i\partial\bar\partial f$ as well.

When dealing with a weak geodesic $[0,1] \ni t \to u_t \in \mathcal{H}_\omega^\Delta$, it is possible to approximate it uniformly with a decreasing sequence of smooth subgeodesics $t \to u_t^\varepsilon$ called ε–geodesics (see [20]). All measures $\omega^n_{u_t^\varepsilon} = g_t^\varepsilon \omega^n$ have uniformly bounded density g_t^ε, and converge weakly to $\omega^n_{u_t}$. Hence, by the dominated convergence theorem we can write

$$\lim_{\varepsilon\to 0} \int_X f\omega^n_{u_t^\varepsilon} = \int_X f\omega^n_{u_t} \quad \text{and} \quad \lim_{\varepsilon\to 0} \mathrm{AM}_\beta(u_t^\varepsilon) = \mathrm{AM}_\beta(u_t),$$

where in the last limit we have used the continuity property of the mixed Monge–Ampère operator (see [4; 5] for the original statement and [45] for the corresponding theory on compact Kähler manifolds). Hence, after repeatedly taking limit in (25), it follows that $t \to \lim_{\varepsilon\to 0} \mathrm{AM}_\chi(u_t^\varepsilon) = \mathrm{AM}_\chi(u_t)$ is convex. □

Finally, we note the following useful inequality for AM_γ.

Lemma 4.6 *Let $\psi \in \mathcal{E}^1$ and set $\theta = \omega_\psi$. For any $u, v \in \mathcal{E}^1$ we have*

$$\frac{1}{V}\int_X (u-v)\omega_u^{n-1} \wedge \theta \leq \mathrm{AM}_\theta(u) - \mathrm{AM}_\theta(v) \leq \frac{1}{V}\int_X (u-v)\omega_v^{n-1} \wedge \theta.$$

For AM *we have similar inequalities*

$$\frac{1}{V}\int_X (u-v)\omega_u^n \leq \mathrm{AM}(u) - \mathrm{AM}(v) \leq \frac{1}{V}\int_X (u-v)\omega_v^n.$$

Proof Using (17) and (21) the desired inequalities simply follow from the fact that

$$\int_X (u-v) i\partial\bar\partial(u-v) \wedge T \leq 0$$

for any $T = \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_{n-1}}$ with $\varphi_j \in \mathcal{E}^1$ for all j. □

4.3 The twisted K-energy

For the remainder of the paper suppose $\chi = \beta + i\partial\bar\partial f$ satisfies (3) unless specified otherwise. Recall that the twisted K-energy $\mathcal{K}_\chi\colon \mathcal{H}_\omega \to \mathbb{R}$ is defined as

$$\mathcal{K}_\chi = \operatorname{Ent}(e^{-f}\omega^n, \omega_u^n) + \bar{S}_\chi \operatorname{AM}(u) - n\operatorname{AM}_{\operatorname{Ric}\omega-\beta}(u) - \int_X f\omega^n.$$

When f is smooth, recall the following formula for the variation of the entropy:

$$\langle D\operatorname{Ent}(e^{-f}\omega^n, \omega_v^n), \delta v\rangle = nV^{-1}\int_X \delta v(\operatorname{Ric}\omega - \operatorname{Ric}\omega_v + i\partial\bar\partial f)\wedge\omega_v^{n-1}.$$

When χ is smooth, putting the above formula, (18) and (22) together we obtain

$$\begin{aligned}\langle D\mathcal{K}_\chi(v), \delta v\rangle &= \frac{n}{V}\int_X \delta v(\bar{S}_\chi\omega_v - \operatorname{Ric}\omega_v + \chi)\wedge\omega_v^{n-1}\\ &= V^{-1}\int_X \delta v(\bar{S}_\chi - S_{\omega_v} + \operatorname{Tr}^{\omega_u}\chi)\omega_v^n.\end{aligned}$$

We arrive at the main theorem of this section:

Theorem 4.7 *Suppose (X,ω) is a compact connected Kähler manifold and $\chi = \beta + i\partial\bar\partial f$ satisfies* (3)*. The twisted K-energy can be extended to a functional $\mathcal{K}_\chi\colon \mathcal{E}^1 \to \mathbb{R}\cup\{\infty\}$ using the formula*

$$\mathcal{K}_\chi(u) = \operatorname{Ent}(e^{-f}\omega^n, \omega_u^n) + \bar{S}_\chi \operatorname{AM}(u) - n\operatorname{AM}_{\operatorname{Ric}\omega-\beta}(u) - \int_X f\omega^n. \tag{26}$$

Thus extended, $\mathcal{K}_\chi|_{\mathcal{E}^p}$ is the greatest d_p–lsc extension of $\mathcal{K}_\chi|_{\mathcal{H}_\omega}$ for any $p \geq 1$. Additionally, $\mathcal{K}_\chi|_{\mathcal{E}^p}$ is convex along the finite-energy geodesics of $\mathcal{E}^p$.

Proof First we argue that the expression given by (26) does give a d_1–lsc function on $\mathcal{E}^1$. Indeed, by Propositions 4.1 and 4.4 the functionals AM and $\operatorname{AM}_{\operatorname{Ric}\omega-\beta}$ admit a d_1–continuous extension to $\mathcal{E}^1$. Lastly, as d_1–convergence of potentials implies weak convergence of the corresponding complex Monge–Ampère measures, it follows that the correspondence $u \to \operatorname{Ent}(e^{-f}\omega^n, \omega_u^n)$ is d_1–lsc. When restricted to $\mathcal{E}^p$, (26) is additionally d_p–lsc, because d_p–convergence dominates d_1–convergence for any $p > 1$.

We now show that, thus extended, $\mathcal{K}_\chi|_{\mathcal{E}^p}$ is indeed the greatest d_p–lsc extension of $\mathcal{K}_\chi|_{\mathcal{H}_\omega}$. For this we only have to argue that for any $u \in \mathcal{E}^p$ there exists $\{u_j\} \subset \mathcal{H}_\omega$ such that $d_p(u_j, u) \to 0$ and

$$\mathcal{K}_\chi(u) = \lim_j \mathcal{K}_\chi(u_j).$$

As $\operatorname{AM}(\cdot)$, $\operatorname{AM}_{\operatorname{Ric}-\beta}(\cdot)$ are d_p–continuous, this is exactly the content of Theorem 3.2.

Since finite-energy geodesics of $\mathcal{E}^p$ are also finite-energy geodesics in $\mathcal{E}^1$, it remains to show that for any finite-energy geodesic $[0,1] \ni t \to u_t \in \mathcal{E}^1$ the curve $t \to \mathcal{K}_\chi(u_t)$ is convex and continuous.

Suppose $t_0, t_1 \in [0,1]$ with $t_0 \le t_1$. As $\mathcal{K}_\chi$ was extended in the greatest d_1–lsc manner, we can find $u^k_{t_0}, u^k_{t_0} \in \mathcal{H}_\omega$ with $d_1(u^k_{t_0}, u_{t_0}) \to 0$, $d_1(u^k_{t_1}, u_{t_1}) \to 0$ and

$$\mathcal{K}_\chi(u_{t_0}) = \lim_k \mathcal{K}_\chi(u^k_{t_0}), \quad \mathcal{K}_\chi(u_{t_1}) = \lim_k \mathcal{K}_\chi(u^k_{t_1}).$$

Let $[t_0, t_1] \ni t \to u^k_t \in \mathcal{H}^\Delta_\omega$ be the weak geodesics connecting $u^k_{t_0}$ and $u^k_{t_1}$. By Proposition 4.3 we get that $d_1(u^k_t, u_t) \to 0$ for any $t \in [t_0, t_1]$. Note that for $u \in \mathcal{H}^\Delta_\omega$ we can write

$$\mathcal{K}_\chi(u) = \mathrm{Ent}(\omega^n, \omega^n_u) + \bar{S}_\chi \,\mathrm{AM}(u) - n\,\mathrm{AM}_{\mathrm{Ric}\,\omega}(u) + \left(n\,\mathrm{AM}_\beta(u) + \frac{1}{V}\int_X f\omega^n_u\right).$$

Using this, Proposition 4.5, [8, Theorem 1.1] and the linearity of AM along finite-energy geodesics, it follows that $t \to \mathcal{K}_\chi(u^k_t)$ is convex on $[0,1]$. As $\mathcal{K}_\chi\colon \mathcal{E}^1 \to \mathbb{R} \cup \{\infty\}$ is d_1–lsc, it follows that

$$\begin{aligned}\mathcal{K}_\chi(u_t) \le \liminf_k \mathcal{K}_\chi(u^k_t) &\le \frac{t-t_0}{t_1-t_0}\lim_k \mathcal{K}_\chi(u^k_{t_0}) + \frac{t_1-t}{t_1-t_0}\lim_k \mathcal{K}_\chi(u^k_{t_1}) \\ &\le \frac{t-t_0}{t_1-t_0}\mathcal{K}_\chi(u_{t_0}) + \frac{t_1-t}{t_1-t_0}\mathcal{K}_\chi(u_{t_1}),\end{aligned}$$

hence $[0,1] \ni t \to \mathcal{K}_\chi(u_t) \in (-\infty, \infty]$ is convex. As $\mathcal{K}_\chi$ is d_1–lsc it follows additionally that $t \to \mathcal{K}_\chi(u_t)$ is continuous up to the boundary of $[0,1]$. □

Finally, we bring Theorem 2.8 into a form that will be most convenient to use in our later developments:

Corollary 4.8 *Suppose $\chi = \beta + i\partial\bar\partial f$ satisfies* (3) *and $\{u_k\}_k \subset \mathcal{E}^1$ is a sequence for which*

$$d_1(0, u_k) < C, \quad \mathcal{K}_\chi(u_k) < C.$$

Then $\{u_k\}_k$ contains a d_1–convergent subsequence.

Proof By (8) it follows that $|\sup_X u_k| < C$. From (26) and Propositions 4.1 and 4.4 we get that $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_{u_k})$ is also uniformly bounded. Now we can invoke Theorem 2.8 to finish the argument. □

4.4 Convexity in the finite entropy space.

Suppose $\chi = \beta + i\partial\bar\partial f$ satisfies (3). Denote by $\mathrm{Ent}_\chi(X, \omega)$ the space of finite-entropy potentials:

$$\mathrm{Ent}_\chi(X,\omega) = \{u \in \mathcal{E}(X,\omega) : \mathrm{Ent}(e^{-f}\omega^n, \omega^n_u) < \infty\}.$$

Observe that $\mathrm{Ent}_\chi(X,\omega)$ is independent of the choice of β and f. Also, we show that $\mathrm{Ent}_\chi(X,\omega)$ is contained in the finite-energy space $\mathcal{E}^1$:

Lemma 4.9 *Suppose $\chi=\beta+i\partial\bar\partial f$ satisfies* (3). *Then* $\mathrm{Ent}_\chi(X,\omega)\subset\mathcal{E}^1$.

Proof Suppose $u\in\mathrm{Ent}_\chi(X,\omega)$ with $\omega_u^n=h\omega^n$. The functions $\phi,\psi\colon[0,\infty)\to[0,\infty)$ given by $\phi(t)=(t+1)\log(t+1)-t$ and $\psi(t)=e^t-t-1$ are convex conjugates of each other, implying that $ab\le\phi(a)+\psi(b)$. Using this, we can write

$$\begin{aligned}\int_X|u|\omega_u^n&=\int_X|u|(he^f)e^{-f}\omega^n\\&\le\int_X(e^{|u|}-|u|-1)e^{-f}\omega^n+\int_X((he^f+1)\log(he^f+1)-h)e^{-f}\omega^n.\end{aligned}$$

To finish the proof it is enough to argue that both terms in this last expression are bounded. For the first term, suppose $1/p+1/q=1$. Using Young's inequality, we arrive at

$$\int_X(e^{|u|}-|u|-1)e^{-f}\omega^n\le\frac1q\int_X(e^{|u|}-|u|-1)^q\omega^n+\frac1p\int_Xe^{-pf}\omega^n.$$

As u has zero Lelong numbers [45, Corollary 1.8], the first integral is finite by Skoda's theorem. For an appropriate p the second integral is bounded, as $e^{-f}\in L^p(X,\omega^n)$ for some $p>1$.

For the second term, observe that $\phi(t)\le 2t\log t$ for t big enough, hence we can write

$$\begin{aligned}\int_X((he^f+1)\log(he^f+1)-h)e^{-f}\omega^n&\le2\int_Xh\log(he^f)\omega^n+C\\&=2V\,\mathrm{Ent}(e^{-f}\omega^n,\omega_u^n)+C.\end{aligned}$$ □

As a consequence of Theorem 4.7 we obtain that $\mathrm{Ent}_\chi(X,\omega)\subset\mathcal{E}^1$ is to some extent "geodesically convex":

Theorem 4.10 *Suppose $\chi=\beta+i\partial\bar\partial f$ satisfies* (3). *Then $(\mathrm{Ent}_\chi(X,\omega),d_1)$ is a geodesic sub-metric space of $(\mathcal{E}^1(X,\omega),d_1)$. Additionally, if $\mathrm{Ric}\,\omega\ge\beta$ then the map $\mathrm{Ent}_\chi(X,\omega)\ni u\to\mathrm{Ent}(e^{-f}\omega^n,\omega_u^n)\in\mathbb{R}$ is convex along finite-energy geodesics.*

Proof Suppose $u_0,u_1\in\mathrm{Ent}_\chi(X,\omega)$. Let $[0,1]\ni t\to u_t\in\mathcal{E}^1$ be the finite-energy geodesic connecting u_0 and u_1. By Theorem 4.7 it follows that $t\to\mathcal{K}_\chi(u_t)$ is convex on $[0,1]$, hence $\mathcal{K}_\chi(u_t)$ is finite for all $t\in[0,1]$. Using the finiteness of AM and $\mathrm{AM}_{\mathrm{Ric}\,\omega-\beta}$, this necessarily gives that $\mathrm{Ent}(e^{-f}\omega^n,\omega_{u_t}^n)$ is also finite for all $t\in[0,1]$.

For the last statement, notice that $t\to n\,\mathrm{AM}_{\mathrm{Ric}\,\omega-\beta}(u_t)-\bar S_\chi\,\mathrm{AM}(u_t)$ is convex, as follows from Proposition 4.5. As $t\to\mathcal{K}_\chi(u_t)$ is also convex, from (26) it follows that $t\to\mathrm{Ent}(e^{-f}\omega^n,\omega_{u_t}^n)$ is also convex. □

In the case $\beta = 0$, this convexity result can be seen as the complex version of one of the central results of the theory of optimal transport of measure, which says that, if g_0 is a given Riemannian metric on a compact real manifold X with nonnegative Ricci curvature and whose normalized volume form is denoted by μ_0, then the relative entropy function $\mu \to \mathrm{Ent}(\mu_0, \mu)$ is convex along curves $t \to \mu_t$ defined by McCann's displacement interpolation (which may be formulated in terms of optimal transport maps). The latter curves can be seen as weak geodesics for Otto's Riemannian metric on the space of all normalized volume forms on X. More precisely, the curves $t \to \mu_t$ are the geodesics in the metric space $(\mathcal{P}(M), d_{W_2})$ defined by the space $\mathcal{P}(M)$ of all probability measures on X equipped with the Wasserstein 2–metric, which can be viewed as a completion of Otto's Riemannian structure [66]. Hence, the role of Otto's Riemannian metric is in the present complex setting played by Mabuchi's Riemannian metric.

4.5 Uniqueness of twisted K-energy minimizers

In this subsection we suppose χ is a Kähler form. We are going to prove that there is at most one minimizer in $\mathcal{E}^1$ of the twisted K-energy $\mathcal{K}_\chi$. We need the following result, which may be of independent interest.

Lemma 4.11 *Let $\varphi_0, \varphi_1 \in \mathcal{E}^1$ and let $[0,1] \ni t \to \varphi_t$ be the finite-energy geodesic connecting φ_0 and φ_1. Suppose that $\omega_{\varphi_t}^n$ is absolutely continuous with respect to ω^n for every $t \in [0,1]$. Then for almost every $t \in (0,1)$ we have*

$$\mathrm{AM}(\varphi_1) - \mathrm{AM}(\varphi_0) = \frac{1}{V}\int_X \dot{\varphi}_t^+ \omega_{\varphi_t}^n = \frac{1}{V}\int_X \dot{\varphi}_t^- \omega_{\varphi_t}^n, \tag{27}$$

where, for fixed $x \in X$, $\dot{\varphi}_t^+(x)$ and $\dot{\varphi}_t^-(x)$ are the right and left derivatives of $\varphi(\,\cdot\,, x)$, respectively.

Proof For simplicity we assume that $V = 1$. Fix two real numbers a, b such that $0 < a < b < 1$. We first observe that for $t \in (a,b)$ and $h > 0$ small enough, by convexity we have

$$\frac{\varphi_t - \varphi_0}{t} \le \frac{\varphi_{t+h} - \varphi_t}{h} \le \frac{\varphi_1 - \varphi_t}{1-t}.$$

It thus follows that both $\dot{\varphi}_t^+$ and $\dot{\varphi}_t^-$ are integrable with respect to $\omega_{\varphi_t}^n$. From Lemma 4.6 we obtain

$$\mathrm{AM}(\varphi_{t+h}) - \mathrm{AM}(\varphi_t) \le \int_X (\varphi_{t+h} - \varphi_t)\omega_{\varphi_t}^n.$$

Since AM is linear along the weak geodesic φ_t, by dividing the above inequality by h and letting $h \to 0$ we obtain

$$\int_X \dot{\varphi}_t^- \omega_{\varphi_t}^n \le \mathrm{AM}(\varphi_1) - \mathrm{AM}(\varphi_0) \le \int_X \dot{\varphi}_t^+ \omega_{\varphi_t}^n. \tag{28}$$

For each $x \in X$ the function $t \to \varphi_t(x)$ is convex, hence differentiable almost everywhere in $[0,1]$. It follows that the set

$$\{(x,t) \in X \times [a,b] : \dot{\varphi}_t^-(x) < \dot{\varphi}_t^+(x)\}$$

has zero measure (where the measure here is the product of ω^n and dt). Let $f(t,x)$ be the density of the Monge–Ampère measure $(\omega + i\partial\bar{\partial}\varphi_t)^n$. We then have

$$\int_{X\times[a,b]} \dot{\varphi}_t^- f(t,x)\omega^n\,dt = \int_{X\times[a,b]} \dot{\varphi}_t^+ f(t,x)\omega^n\,dt. \tag{29}$$

Now, by Fubini's theorem, (28) and (29) we see that the inequalities in (28) become equalities for almost every t in $[a,b]$, completing the proof. □

Theorem 4.12 *Let α be a Kähler form. Let $\varphi_0, \varphi_1 \in \mathcal{E}^1$ and let φ_t be the finite-energy geodesic connecting φ_0 and φ_1. Suppose that $\omega_{\varphi_t}^n$ is subordinate to ω^n for any $t \in [0,1]$. If AM_α is linear along $t \to \varphi_t$ then $\varphi_1 - \varphi_0$ is constant.*

Proof We can assume that $\mathrm{AM}(\varphi_0) = \mathrm{AM}(\varphi_1)$ and we normalize ω so that $V = 1$. We claim that AM_β is also linear along φ_t, where β is any Kähler form. Indeed, multiplying β by some small positive constant, we can assume that $\gamma := \alpha - \beta > 0$. It follows from Proposition 4.5 that both $t \to \mathrm{AM}_\gamma(\varphi_t)$ and $t \to \mathrm{AM}_\beta(\varphi_t)$ are convex. Because $\mathrm{AM}_\alpha = \mathrm{AM}_\beta + \mathrm{AM}_\gamma$ is linear along φ_t, it follows that in fact $t \to \mathrm{AM}_\beta(\varphi_t)$ is linear as well. By approximation it follows that $\mathrm{AM}_{\omega_\psi}$ is linear along φ_t for any $\psi \in \mathcal{E}^1$.

Fix $s \in (0,1)$ such that (27) holds in Lemma 4.11. For $h > 0$ small enough we have

$$\begin{aligned}
\int_X \frac{\varphi_{s+h} - \varphi_s}{h}\omega_{\varphi_s}^n &\geq \frac{\mathrm{AM}_{\omega_{\varphi_s}}(\varphi_{s+h}) - \mathrm{AM}_{\omega_{\varphi_s}}(\varphi_s)}{h} \\
&= -\frac{1}{n}\int_X \frac{\varphi_{s+h} - \varphi_s}{h}\omega_{\varphi_{s+h}}^n \\
&\geq -\frac{1}{n}\int_X \frac{\varphi_{s+h} - \varphi_s}{h}\omega_{\varphi_s}^n.
\end{aligned}$$

In the first line we have used Lemma 4.6. In the second line we have used the assumption that AM is constant along $s \to \varphi_s$. In the last line we have used again Lemma 4.6. Now, letting $h \to 0$ and using Lemma 4.11 we see that the right derivative of $l \to \mathrm{AM}_{\omega_{\varphi_s}}(\varphi_l)$ at s is zero. Thus $l \to \mathrm{AM}_{\omega_{\varphi_s}}(\varphi_l)$ is in fact constant. This combined with $l \to \mathrm{AM}(\varphi_l)$ being constant imply that

$$\begin{aligned}
0 &= (n+1)\big(\mathrm{AM}(\varphi_1) - \mathrm{AM}(\varphi_s)\big) - n\big(\mathrm{AM}_{\omega_{\varphi_s}}(\varphi_1) - \mathrm{AM}_{\omega_{\varphi_s}}(\varphi_s)\big) \\
&= \int_X (\varphi_1 - \varphi_s)\omega_{\varphi_1}^n.
\end{aligned} \tag{30}$$

A computation similar to the one in Lemma 4.6 gives that all terms in the expression of $\mathrm{AM}(\varphi_1) - \mathrm{AM}(\varphi_s)$ from (17) are greater than $\int_X (\varphi_1 - \varphi_s)\omega^n_{\varphi_1}$. Using this, (30) and $\mathrm{AM}(\varphi_1) - \mathrm{AM}(\varphi_s) = 0$ we obtain $\int_X (\varphi_1 - \varphi_s)\omega^n_{\varphi_s} = 0$. Together with (30) this gives

$$I(\varphi_1, \varphi_s) = \int_X (\varphi_1 - \varphi_s)(\omega^n_{\varphi_s} - \omega^n_{\varphi_1}) = 0.$$

Hence, by the results in [9, Section 2.1], the difference $\varphi_s - \varphi_1$ is constant. In fact $\varphi_s = \varphi_1$, as we have assumed that the Aubin–Mabuchi energy is constant along the geodesic $l \to \varphi_l$. Now, φ_1 can be replaced by φ_0 in (30), and the same arguments as above show that $\varphi_0 = \varphi_s$, ultimately giving $\varphi_0 = \varphi_1$. □

We are now ready to prove the uniqueness result.

Theorem 4.13 *Assume that χ is a Kähler form. If φ_0 and φ_1 are minimizers in $\mathcal{E}^1$ of the twisted Mabuchi energy $\mathcal{K}_\chi$ then $\varphi_1 - \varphi_0$ is constant.*

Proof Let $t \to \varphi_t$ be the finite-energy geodesic connecting φ_0 and φ_1. By the convexity of $\mathcal{K}_\chi$ it follows that $\mathcal{K}_\chi$ is linear along $t \to \varphi_t$. Since $t \to \mathrm{AM}(\varphi_t), \mathcal{K}_\chi(\varphi_t)$ are linear and $t \to \mathrm{AM}_\chi(\varphi_t), \mathcal{K}(\varphi_t)$ are convex, the decomposition

$$\mathcal{K}_\chi = \mathcal{K} + (\bar{S}_\chi - \bar{S})\,\mathrm{AM} + n\,\mathrm{AM}_\chi$$

then reveals that AM_χ is also linear along $t \to \varphi_t$ and $\omega^n_{\varphi_t}$ is subordinate to ω^n. The result now follows from Theorem 4.12. □

Remark 4.14 When χ is a Kähler form, using this last theorem, it can be seen that the conditions (A1)–(A4) and (P1)–(P7) are verified in [32, Theorem 3.4] for the data $(\mathcal{E}^1, d_1, \mathcal{K}_\chi, \{\mathrm{Id}\})$ to give that a minimizer of $\mathcal{K}_\chi$ exists in $\mathcal{E}^1$ if and only if there exist $C, D > 0$ such that

$$\mathcal{K}_\chi(u) \geq C d_1(0, u) - D, \quad u \in \mathcal{H}_\omega.$$

This verifies a weak version of [23, Conjecture 1.21] going back to [22, Conjecture 6.1]. For related partial results, see also [36].

5 Relating d_1–convergence to weak d_2–convergence

Before we get into the details of our particular situation, we start with a pedagogical example: suppose (M, μ) is a measure space with finite volume. By $(L^p(M, \mu), \|\cdot\|_p)$ we denote the usual L^p spaces on M. From Hölder's inequality it follows that on $L^2(M, \mu)$ the $\|\cdot\|_2$ norm dominates the $\|\cdot\|_1$ norm. Our focus however is on the weak-L^2–topology. As it turns out, the L^1–topology dominates the weak-L^2–topology. The simple explanation for this is that L^1–balls inside $L^2(M,\mu)$ are closed convex sets, and it is a classical fact that weak L^2–limits do not exit closed convex sets. Though much simplified, as it turns out, this idea generalizes to the setting of the metric spaces $(\mathcal{E}^p, d_p)$. As we show below, the d_1–metric balls have a certain convexity property that will make these sets d_2–convex and closed inside $\mathcal{E}^2$. This will imply that d_1–convergence dominates weak d_2–convergence. In the next section, coupled with Theorem 2.14, this fact will have implications for the convergence of the weak twisted Calabi flow.

As advocated in [29; 30], a proper understanding of the "rooftop" envelopes $P(u_0,u_1)$ gives insight into the geometry of the spaces $(\mathcal{E}^p, d_p)$. Furthering this relationship, we state the following proposition:

Proposition 5.1 *Suppose* $[0,1] \ni t \to u_t, v_t \in \mathcal{E}^1$ *are finite-energy geodesics. Then the map* $t \to \mathrm{AM}(P(u_t,v_t))$ *is concave. Consequently, the map* $t \to d_1(u_t,v_t)$ *is convex.*

The significance of this result comes from the fact that the d_1 metric, unlike the d_2 metric, is not $\mathrm{CAT}(0)$. Indeed, by the results of [30], the geodesic segments with fixed endpoints inside $(\mathcal{E}^1, d_1)$ are not even unique. On the other hand, by the above proposition, the d_1 metric structure has some geometric convexity that can be exploited.

Proof Let $a,b \in [0,1]$. As shown in [29, Theorem 3], we have $P(u_a,v_a), P(u_b,v_b) \in \mathcal{E}^1(X,\omega)$. Let $[0,1] \ni t \to w_t \in \mathcal{E}^1(X,\omega)$ be a finite-energy geodesic connecting $w_0 = P(u_a,v_a)$ and $w_1 = P(u_b,v_b)$. By the maximum principle of finite-energy geodesics we have $w_t \le u_{ta+(1-t)b}, v_{ta+(1-t)b}$, hence also $w_t \le P(u_{ta+(1-t)b}, v_{ta+(1-t)b})$. By the monotonicity of the Aubin–Mabuchi energy and since $t \to \mathrm{AM}(w_t)$ is linear we obtain

$$\begin{aligned} t\,\mathrm{AM}(P(u_a,v_a)) + (1-t)\,\mathrm{AM}(P(u_b,v_b)) &= \mathrm{AM}(w_t) \\ &\le \mathrm{AM}(P(u_{ta+(1-t)b}, v_{ta+(1-t)b})). \end{aligned}$$

The last statement of the proposition follows from the linearity of AM along finite-energy geodesics, the concavity we just established and the explicit formula for d_1 given in [30, Corollary 4.14], according to which

$$d_1(u_t,v_t) = \mathrm{AM}(u_t) + \mathrm{AM}(v_t) - 2\,\mathrm{AM}(P(u_t,v_t)). \qquad \square$$

The geodesic convexity and closedness of d_1–balls inside $\mathcal{E}^2$ is an immediate consequence:

Corollary 5.2 *For any $\rho > 0$ and $u \in \mathcal{E}^2(X,\omega)$, the set*

$$B_\rho(u) = \{v \in \mathcal{E}^2(X,\omega) : d_1(v,u) \le \rho\}$$

is d_2–closed and d_2–convex, ie for any $v_0, v_1 \in B_\rho(u)$ the finite-energy geodesic $[0,1] \ni t \to v_t \in \mathcal{E}^2$ connecting v_0 and v_1 is contained in $B_\rho(u)$.

Proof Closedness with respect to d_2 follows from the fact that d_2 dominates d_1. Let $[0,1] \ni t \to v_t \in \mathcal{E}^2$ be a finite-energy geodesic with $v_0, v_1 \in B_\rho(u)$. By definition, since $\mathcal{E}^2 \subset \mathcal{E}^1$, the curve $t \to v_t$ is a finite-energy geodesic inside $\mathcal{E}^1$ as well. By the previous proposition $t \to d_1(u, v_t)$ is convex, hence $d_1(u, v_t) \le \rho$. □

The main result of this subsection is the following:

Theorem 5.3 *Suppose $\{u_k\}_k \subset \mathcal{E}^2$ is d_2–bounded and $u \in \mathcal{E}^2$. Then $d_1(u_k, u) \to 0$ if and only if $\|u_j - u\|_{L^1(X)} \to 0$ and u_k converges to u d_2–weakly.*

Proof Assume first that $d_1(u_k, u) \to 0$. From [30, Theorem 5(ii)] it follows that $\|u_j - u\|_{L^1(X)} \to 0$. As recalled in Proposition 2.11, any subsequence of $\{u_k\}_k$ contains a d_2–weakly convergent subsubsequence u_{k_l}, converging d_2–weakly to some $v \in \mathcal{E}^2$. We show that $v = u$. Indeed, for any $j \in \mathbb{N}$ the set $B_{1/j}(u)$ is d_2–closed and d_2–convex by the previous corollary, and for large enough k_l we have $u_{k_l} \in B_{1/j}(u)$. As recalled in Proposition 2.11, it follows now that $v \in B_{1/j}(u)$ for all j, hence $v = u$.

For the reverse direction, as d_2–boundedness gives that $\mathrm{AM}(u_j)$ is uniformly bounded, by [30, Proposition 5.9] it suffices to show that any convergent subsequence of $\mathrm{AM}(u_j)$ converges to $\mathrm{AM}(u)$. Assume that u_{j_k} is such a subsequence and set $c = \lim_k \mathrm{AM}(u_{j_k})$. By definition, u_{j_k} still converges d_2–weakly to u. For each $\varepsilon > 0$, consider the set

$$E_\varepsilon := \{\phi \in \mathcal{E}^2 : c - \varepsilon \le \mathrm{AM}(\phi) \le c + \varepsilon\}.$$

Since d_2 dominates d_1 and AM is d_1–continuous and linear along finite-energy geodesics, it follows that E_ε is d_2–closed and d_2–convex. By Proposition 2.11, it follows that $u \in E_\varepsilon$. Letting $\varepsilon \to 0$ we get $\mathrm{AM}(u) = c$, finishing the proof. □

Remark 5.4 Using (the proof of) this last result, it is possible to construct a d_2–bounded sequence $u_j \in \mathcal{E}^2$ converging d_2–weakly to some $u \in \mathcal{E}^2$, but for which $\|u_j - u\|_{L^1(X)} \not\to 0$. Indeed, one can construct a d_2–bounded sequence $u_j \in \mathcal{E}^2$ such that $\|u_j - v\|_{L^1(X)} \to 0$ for some $v \in \mathcal{E}^2$ but $\omega^n_{u_j}$ does not converge weakly to ω^n_v, in particular $\mathrm{AM}(u_j)$ cannot converge to $\mathrm{AM}(v)$. By Proposition 2.11 we can extract a subsequence, again denoted by u_j, such that u_j converges d_2–weakly to some $u \in \mathcal{E}^2$. By the last step in the proof of the previous theorem AM is weak d_2–continuous, hence $\mathrm{AM}(u_j) \to \mathrm{AM}(u)$, but we cannot have $u = v$ as $\mathrm{AM}(u) \neq \mathrm{AM}(v)$.

6 The weak twisted Calabi flow

As shown in [29], the metric completion $(\mathcal{E}^2, d_2) = \overline{(\mathcal{H}, d_2)}$ is a CAT(0) space. Suppose χ satisfies (3). By Theorem 4.7, the extended $\mathcal{K}_\chi$ is d_2–lsc and convex on $\mathcal{E}^2$. By Theorem 2.12 and Remark 2.13, the weak gradient flow $t \to c_t$ of $\mathcal{K}_\chi$ emanating from any $c_0 \in \mathcal{E}^2$ is well defined and uniquely determined by the evolution variational inequality (13).

When χ is smooth, the *smooth twisted Calabi flow* is just a simple generalization of the usual smooth Calabi flow:

$$\frac{d}{dt} c_t = S_{\omega_{c_t}} - \bar{S}_\chi - \mathrm{Tr}^{\omega_{c_t}} \chi.$$

Comparison with Streets' setting In [61] another (a priori different) extension $\bar{\mathcal{K}}$ of the Mabuchi functional $\mathcal{M}$ on $\mathcal{H}$ to the completion $\overline{(\mathcal{H}, d_2)} = (\mathcal{E}^2, d_2)$ was considered, defined by

$$\bar{\mathcal{K}}(\bar{u}) := \liminf_{d(u_j, \bar{u}) \to 0} \mathcal{K}(u_j),$$

where the infimum is taken over all sequences u_j in $\mathcal{H}$ converging to $\bar{u}$ in $\overline{(\mathcal{H}, d_2)}$. It is shown in [61] that the functional $\bar{\mathcal{K}}$ thus defined is d_2–lsc on $\overline{(\mathcal{H}, d_2)}$, and then the author proceeds to study the gradient flow of $\bar{\mathcal{K}}$, dubbed the *minimizing movement Calabi flow*. By Theorem 4.7 we actually have $\bar{\mathcal{K}} = \mathcal{K}$, thus our finite-energy Calabi flow coincides with the minimizing movement Calabi flow considered in [61]. One of the advantages of our consideration is that computations in $\mathcal{E}^2$ are explicit and avoid the difficulties of using Cauchy sequences.

We show that the weak version of the twisted Calabi flow agrees with the smooth version as long as the latter exists. The following result was proved by Streets in the case $\chi = 0$ using different methods.

Proposition 6.1 *Suppose $\chi \geq 0$ is a smooth closed $(1,1)$–form. Given any initial point $c_0 \in \mathcal{H}_\omega$, the corresponding weak twisted Calabi flow $t \to c_t$ coincides with the smooth twisted Calabi flow, as long as the latter exists.*

Proof By the uniqueness property in [1, Theorem 4.0.4] for curves $t \to c_t$ satisfying the *evolution variational inequality* (13) (which is shown by differentiating $d(c_t^1, c_t^2)$ for two different solutions $t \to c_t^1$ and $t \to c_t^2$) it is enough to show that a solution $t \to h_t$ to the ordinary twisted Calabi flow with starting point $h_0 = c_0$ satisfies the inequality (13).

Suppose $v \in \mathcal{H}_\omega$ is arbitrary, fix a time $t = t_0$ and let $[0,1] \ni s \to u_s \in \mathcal{H}_\omega^\Delta$ be the weak geodesic connecting $u_0 = h_{t_0}$ and $u_1 = v$. From [8, Lemma 3.5] we get the following "slope inequality":

$$\mathcal{K}_\chi(v)-\mathcal{K}_\chi(h_{t_0})\geq\int_X(\bar{S}-S_{\omega_{h_{t_0}}}+\mathrm{Tr}^{\omega_{h_{t_0}}}\chi)\frac{du_s}{ds}\Big|_{s=0}\omega^n_{h_{t_0}}.$$

Now, by the definition of the twisted Calabi flow the right-hand side above may be written as minus the scalar product $\int_X(dh_t/dt)_{|t=t_0}(du_s/ds)_{|s=0}\omega^n_{h_{t_0}}$. Since $v\in\mathcal{H}_\omega$, the latter scalar product coincides with the derivative at $t=t_0$ of the function $t\to\frac{1}{2}d_2^2(h_t,v)$ (by [20, Theorem 6], or rather by a formula appearing in the proof of the latter theorem). This concludes the proof in the case when $v\in\mathcal{H}_\omega$.

We handle the general case: suppose $v\in\mathcal{E}^2$ and $\mathcal{K}_\chi(v)<\infty$. Notice that it is enough to show the following "integral" version of (13) (with $G=\mathcal{K}_\chi$):

$$\frac{1}{2}(d_2^2(c_{t_1},v)-d_2^2(c_{t_0},v))\leq(t_1-t_0)\mathcal{K}_\chi(v)-\int_{t_0}^{t_1}\mathcal{K}_\chi(c_t)\,dt \tag{31}$$

for any $t_0,t_1\in[0,\infty)$, $t_0\leq t_1$. Indeed, the left-hand side is locally Lipschitz, whereas $t\to\mathcal{K}_\chi(c_t)$ is smooth, hence we may divide both sides by t_1-t_0 and take the limit $t_1\to t_0$ to obtain (13). By Theorem 3.2 there exists a sequence $v_j\in\mathcal{H}_\omega$ that d_2–converges to v such that $\mathcal{K}_\chi(v_j)$ converges to $\mathcal{K}_\chi(v)$. After integrating, by the first part of the proof estimate (31) holds for v_j in place of v. Letting $j\to\infty$, we obtain (31) for v as well. □

Lemma 6.2 *The functional* AM *is constant along any weak twisted Calabi flow trajectory* $t\to c_t$.

Proof For a *smooth* Calabi flow this follows directly from differentiating along the flow, but here we have to proceed in a different manner. We can assume that $\mathrm{AM}(c_0)=0$, as $\mathcal{K}_\chi$ is invariant under adding constants. On the other hand, for any $u,v\in\mathcal{E}^2$, $d_2(u-\mathrm{AM}(u),v-\mathrm{AM}(v))\leq d_2(u,v)$. Thus the variational construction of the weak Calabi flow (see Section 2.5) gives "minimizing movement" c_t^m with $\mathrm{AM}(c_t^m)=0$ for all m. Since AM is continuous with respect to d_2 it follows that $\mathrm{AM}(c_t)=0$ for all t. □

Now we arrive at the main result of this section:

Theorem 6.3 *Suppose* (X,ω) *is a compact connected Kähler manifold and* $\chi=\beta+i\partial\bar\partial f$ *satisfies* (3). *The following statements are equivalent:*

(i) $\mathcal{M}^2_\chi\neq\varnothing$.

(ii) *For any weak twisted Calabi flow trajectory* $t\to c_t$ *there exists* $c_\infty\in\mathcal{M}^2_\chi$ *such that* $d_1(c_t,c_\infty)\to 0$ *and* $\mathrm{Ent}(e^{-f}\omega^n,\omega^n_{c_t})\to\mathrm{Ent}(e^{-f}\omega^n,\omega^n_{c_\infty})$.

(iii) *Any weak twisted Calabi flow trajectory $t \to c_t$ is d_2–bounded.*

(iv) *There exists a weak twisted Calabi flow trajectory $t \to c_t$ and $t_j \to \infty$ for which the sequence $\{c_{t_j}\}_j$ is d_2–bounded.*

Proof We start with the implication (i) $\implies$ (ii). Let $t \to c_t$ be a weak twisted Calabi flow trajectory. Let $v \in \mathcal{M}^2_\chi$. From (13) it follows that $d_2(v, c_t) \le d_2(v, c_0)$, hence $t \to c_t$ is a d_2–bounded curve.

As observed in [62], Theorem 2.14 guarantees the existence of $c_\infty \in \mathcal{M}^2_\chi$ such that $c_t \to c_\infty$ d_2–weakly. But $\{c_t\}_t$ is bounded in the d_2 metric and also $\mathcal{K}_\chi(c_t)$ is bounded. By Corollary 4.8 it follows that $\{c_t\}_t$ is d_1–relatively compact, ie each subsequence has a d_1–convergent subsubsequence. By Theorem 5.3 we must have $d_1(c_t, c_\infty) \to 0$.

In the definition of $\mathcal{K}_\chi$ all terms are d_1–continuous except for the entropy term. Since c_∞ is a minimizer, lower semicontinuity gives $\lim_{t\to\infty} \mathcal{K}_\chi(c_t) = \mathcal{K}_\chi(c_\infty)$. All this additionally implies $\mathrm{Ent}(e^{-f}\omega^n, \omega^n_{c_t}) \to \mathrm{Ent}(e^{-f}\omega^n, \omega^n_{c_\infty})$.

The implications (ii) $\implies$ (iii) $\implies$ (iv) are trivial. We finish the proof by arguing that (iv) $\implies$ (i). Let $t \to c_t$ be a weak twisted Calabi flow trajectory and $\{c_{t_j}\}_j$ be a d_2–bounded sequence with $t_j \to \infty$. From Proposition 2.7 and Corollary 4.8 it follows that there exists $c_\infty \in \mathcal{E}^2$ such that $d_1(c_{t_j}, c_\infty) \to 0$, and by the lower semicontinuity of $\mathcal{K}_\chi$, we get that in fact $c_\infty \in \mathcal{M}^2_\chi$. □

In Theorem 6.3(ii) one would like to have convergence with respect to d_2. The next result confirms this in the case when the flow is bounded from below by some potential:

Proposition 6.4 *Suppose (X, ω) is a compact connected Kähler manifold and that χ satisfies* (3). *Let $t \to c_t$ be a weak twisted Calabi flow trajectory. If there exists $\psi \in \mathcal{E}^2$ such that $c_t \ge \psi$ for all t, then c_t converges in d_2 to a minimizer of $\mathcal{K}_\chi$.*

Proof Without loss of generality we can assume that $\psi \le 0$. By hypothesis we have in particular that $\psi \in \mathcal{E}^1$ and $\mathrm{AM}(\psi)$ is finite.

We first claim that $d_2(c_t, 0)$ is uniformly bounded in t. Indeed, by [30, Corollary 4.14] the d_1–distance $d_1(c_t, 0)$ can be expressed as

$$d_1(c_t, 0) = \mathrm{AM}(c_t) + \mathrm{AM}(0) - 2\,\mathrm{AM}(P(c_t, 0)).$$

Since $\psi \le P(c_t, 0) \le 0$ it follows from monotonicity of AM that $\mathrm{AM}(P(c_t, 0))$ is uniformly bounded in t. As $\mathrm{AM}(c_t)$ is constant, it follows that $d_1(c_t, 0)$ is uniformly bounded. This together with [30, Corollary 4] implies that $\sup_X c_t$ is bounded. Finally, applying [30, Theorem 3] finishes the proof of the claim.

By Theorem 6.3 we know that $t \to c_t$ converges in d_1 to some $u \in \mathcal{E}^2$, a minimizer of $\mathcal{K}_\chi$. As $c_t \geq \psi$, by the dominated convergence theorem and Theorem 2.4 we only have to prove that $\int_X (c_t - c)^2 \omega_{c_t}^n \to 0$. For a fixed $s > 0$ we have

$$\int_{\{|c_t - c| \leq s\}} (c_t - c)^2 \omega_{c_t}^n \leq s \int_X |c_t - c| \omega_{c_t}^n \to 0 \quad \text{as } t \to \infty,$$

since $d_1(c_t, c) \to 0$. Thus it suffices to show that

$$\sup_{t>0} \int_{\{|c_t - c| > s\}} (c_t - c)^2 \omega_{c_t}^n \to 0 \tag{32}$$

as $s \to \infty$. Since $d^2(c, c_t)$ is bounded, by Theorem 2.4 one can find a positive constant C_1 such that $\sup_X c_t \leq C_1$ for all $t > 0$. By the comparison principle in $\mathcal{E}$ (see [45]) one has

$$\int_{\{c_t - c > s\}} \omega_{c_t}^n \leq \int_{\{c_t - c > s\}} \omega_c^n \leq \int_{\{c < C_1 - s\}} \omega_c^n,$$

which yields

$$\int_s^\infty \omega_{c_t}^n (c_t - c > r) r \, dr \leq \int_s^\infty \omega_c^n (c < C_1 - r) r \, dr.$$

The right-hand side converges to 0 as $s \to \infty$ because $c \in \mathcal{E}^2$. Therefore, to prove (32) it remains to show that

$$\sup_{t>0} \int_s^\infty \omega_{c_t}^n (c_t - c < -r) r \, dr \to 0. \tag{33}$$

Since $\sup_X c_t$ is bounded from above and $c_t \geq \psi$, we can find $C_2 > 0$ such that

$$\{c_t - c < -r\} \subset \bigl\{\psi \leq C_2 + \tfrac{1}{2}(c_t - r)\bigr\}.$$

Using $\omega_{c_t}^n \leq 2^n \omega_{c_t/2}^n$ and the comparison principle, we arrive at

$$\begin{aligned} \int_s^\infty \omega_{c_t}^n (c_t - c < -r) r \, dr &\leq \int_s^\infty \omega_{c_t}^n \bigl(\psi < C_2 + \tfrac{1}{2}(c_t - r)\bigr) r \, dr \\ &\leq \int_s^\infty \omega_\psi^n \bigl(\psi < C_3 - \tfrac{1}{2} r\bigr) r \, dr, \end{aligned}$$

where $C_3 = C_2 + \frac{1}{2} C_1$. The last term converges to 0 as $s \to \infty$ because $\psi \in \mathcal{E}^2$. This proves (33) and completes the proof. □

Finally, we prove a result about geodesic rays weakly asymptotic to diverging weak Calabi flow trajectories.

Theorem 6.5 *Suppose (X,ω) is a compact connected Kähler manifold, $\chi \geq 0$ is smooth and Conjecture 1.8 holds. Let $[0,\infty) \ni t \to c_t \in \mathcal{E}^2$ be a weak twisted Calabi flow trajectory. Exactly one of the following holds:*

(i) *The curve $t \to c_t$ d_1–converges to a smooth twisted csc-K potential c_∞.*

(ii) *$d_1(c_0, c_t) \to \infty$ as $t \to \infty$ and the curve $t \to c_t$ is d_1–weakly asymptotic to a finite-energy geodesic $[0,\infty) \ni t \to u_t \in \mathcal{E}^1$ along which $\mathcal{K}_\chi$ decreases.*

If $\chi > 0$, then, independently of Conjecture 1.8, exactly one of the following holds:

(i′) *The curve $t \to c_t$ d_1–converges to a unique minimizer in $\mathcal{E}^1$ of $\mathcal{K}_\chi$.*

(ii′) *$d_1(c_0, c_t) \to \infty$ as $t \to \infty$ and the curve $t \to c_t$ is d_1–weakly asymptotic to a finite-energy geodesic $[0,\infty) \ni t \to u_t \in \mathcal{E}^1$ along which $\mathcal{K}_\chi$ strictly decreases.*

Proof Suppose (i) holds. Then $t \to c_t$ is d_2–bounded hence also d_1–bounded, hence it is impossible for (ii) to hold.

Now suppose (i) does not hold. By Corollary 4.8 we must have $d_t := d_1(c_0, c_t) \to \infty$, otherwise there would exist $c_\infty \in \mathcal{M}^1_\chi$ smooth twisted csc-K; in particular, $c_\infty \in \mathcal{M}^2_\chi$. By Theorem 1.5 this would imply that (i) holds, a contradiction.

Let $[0, d_t] \ni l \to u^t_l \in \mathcal{E}^1$ be the d_1–unit finite-energy geodesic connecting c_0 and c_t. By convexity of $l \to \mathcal{K}_\chi(u^t_l)$ it follows that

$$\frac{\mathcal{K}_\chi(u^t_l) - \mathcal{K}_\chi(c_0)}{l} = \frac{\mathcal{K}_\chi(u^t_l) - \mathcal{K}_\chi(u^t_0)}{l} \leq \frac{\mathcal{K}_\chi(u^t_{d_t}) - \mathcal{K}_\chi(u^t_0)}{d_1(c_0, c_t)} = \frac{\mathcal{K}_\chi(c_t) - \mathcal{K}_\chi(c_0)}{d_1(c_0, c_t)} \leq 0,$$

hence $\{\mathcal{K}_\chi(u^t_l)\}_{t \in [0,\infty)}$ is uniformly bounded. As $d_1(u^t_l, u^t_0) = l$, we can apply Corollary 4.8 to find a subsequence d_1–converging to some $u_l \in \mathcal{E}^1$. Using a Cantor process, we can arrange for a subsequence t_k such that for all $l \in \mathbb{Q}$ there exists $u_l \in \mathcal{E}^1$ such that $d_1(u^{t_k}_l, u_l) \to 0$ as $k \to \infty$ for each l. As we are dealing with the limit of d_1–unit speed geodesic segments, we will clearly have

$$d_1(u_{l_1}, u_{l_2}) = |l_1 - l_2|, \quad l_1, l_2 \in \mathbb{Q}_+.$$

Using equicontinuity, in the complete metric space $\mathcal{E}^1$ we can extend the curve $\mathbb{Q}_+ \ni l \to u_l \in \mathcal{E}^1$ to a d_1–geodesic ray $[0,\infty) \ni l \to u_l \in \mathcal{E}^1$, satisfying $d_1(u^{t_k}_l, u_l) \to 0$ for all $l \in [0,\infty)$.

Using Proposition 4.3 we additionally obtain that $l \to u_l$ is in fact a finite-energy geodesic. Because all functions $l \to \mathcal{K}_\chi(u^{t_k}_l)$ are uniformly bounded above and $\mathcal{K}_\chi$ is d_1–lsc, it necessarily follows that $l \to \mathcal{K}_\chi(u_l)$ is also bounded above. Convexity and boundedness now give that $l \to \mathcal{K}_\chi(u_l)$ is actually decreasing.

Lastly, we focus on the case when $\chi > 0$ is Kähler. In Theorem 4.13 we have proved that a minimizer of the twisted Mabuchi functional is unique if exists. Also, when (i′) holds then by Remark 4.14 the curve $t \to c_t$ is d_1–bounded, hence it is impossible for (ii′) to hold.

We assume that (i′) does not hold. Let $t \to c_t$ be a weak twisted Calabi flow trajectory. We can assume that c_t is d_1–divergent, otherwise Theorem 2.8 would imply existence of a minimizer in $\mathcal{E}^1$. By the same argument as above, we can construct a weakly asymptotic finite-energy geodesic ray $t \to u_t$ along which $\mathcal{K}_\chi$ is decreasing. We claim that in fact $\mathcal{K}_\chi$ is strictly decreasing along $t \to u_t$. Indeed, if it were not the case, by convexity of $t \to \mathcal{K}_\chi(u_t)$ we would obtain that $t \to \mathcal{K}_\chi(u_t)$ is constant for t greater than some $t_0 > 0$. By Lemma 6.2 AM is constant along $t \to c_t$, hence also along $t \to u_t$. As both $t \to \mathcal{K}(u_t)$, $\mathrm{AM}_\chi(u_t)$ are convex, we obtain that $t \to \mathrm{AM}_\chi(u_t)$ is in fact linear and Theorem 4.12 then reveals that u_t is stationary after t_0, contradicting the d_1–divergence of the ray $t \to u_t$. □

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