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On 5–manifolds with free fundamental group and simple boundary links in S^5

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We classify compact oriented 5–manifolds with free fundamental group and π_2 a torsion-free abelian group in terms of the second homotopy group considered as a π_1–module, the cup product on the second cohomology of the universal covering, and the second Stiefel–Whitney class of the universal covering. We apply this to the classification of simple boundary links of 3–spheres in S^5. Using this we give a complete algebraic picture of closed 5–manifolds with free fundamental group and trivial second homology group.

57R65; 57R40

1 Introduction

There is a close relationship between classical links and closed 3–manifolds since all 3–manifolds are obtained by surgeries on links and Kirby calculus determines when two links give the same 3–manifold. We consider a special case of such a relation in dimension 5. The special condition on the side of links is that we only consider *simple boundary links* L of a disjoint union of 3–spheres in S^5, which means that the fundamental group of the complement is freely generated by the meridians of the link components. As in dimension 3 we can perform surgery on the link L to obtain a closed smooth manifold $M(L)$. It is easy to see that the fundamental group of $M(L)$ is a free group and $H_2(M(L);\mathbb{Z})=0$. In addition, the second homotopy group is that of the complement X of the link and this is torsion-free as an abelian group. One can ask which 5–manifolds are obtained this way and for the classification of the links and the determination of the fibers of the map from links to 5–manifolds given by surgery.

We answer this question by giving a classification of a more general class of closed 5–manifolds, namely we classify all 5–manifolds M with $\pi_1(M)$ a free group and $\pi_2(M)$ torsion-free as an abelian group, in terms of an invariant we call *generalized Milnor pairing*, since it is a generalization of the Milnor pairing for knots. We also consider compact manifolds with boundary the disjoint union of copies of $S^1\times S^3$ and free fundamental group that is freely generated by the circles in the boundary, and, as

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before, $\pi_2(M)$ is torsion-free as an abelian group. We also define a topological version of the generalized Milnor pairing, called topological generalized Milnor pairing, and prove a corresponding result for topological manifolds.

A second well-known class of examples are fibered 5–manifolds M over the circle with simply connected fiber. These are in the image of the surgery construction above if and only if we have a fibered knot and $H_2(M;\mathbb{Z}) = 0$. But in general fibered 5–manifolds over the circle have nontrivial second homology. Thus our more general class of manifolds also occurs naturally. See Remark 1.4 and the appendix for more on this class of manifolds.

To give a feeling for the generalized Milnor pairing, we define it in a special case, where M is spin. Then it is represented by the triple

$$\big(\pi_1(M),\, \pi_2(M),\, b_M\colon \pi_2(M)^* \times \pi_2(M)^* \to \big(H^1(B\pi_1 M;\mathbb{Q}[\pi_1 M])\big)^*\big),$$

where b_M is given by the cup product. For details we refer to Section 2. Now we formulate our main result.

Theorem 1.1 *Let M_0 and M_1 be two smooth (or topological), compact, oriented 5–manifolds with free fundamental group of rank n and torsion-free π_2, with empty boundary or boundary consisting of n copies of $S^1 \times S^3$ such that the circles in the boundary generate $\pi_1(M_i)$. Then there is an orientation-preserving diffeomorphism (homeomorphism) between M_0 and M_1 if and only if there is an isomorphism between their (topological) generalized Milnor pairings.*

We actually prove a stronger result (Theorem 2.4) about the realization of isomorphisms between the generalized Milnor pairings.

Levine [11] has classified 3–dimensional simple knots in S^5 in terms of S–equivalence classes of Seifert matrices and Liang [12] has extended this to higher-dimensional simple boundary links in terms of l–equivalence classes of Seifert matrices. The general case of 3–dimensional simple boundary links in S^5 seems to be open. Our classification result implies that Liang's result extends to dimension 3. Also, by extending Liang's argument to higher dimension we can characterize the Seifert matrices occurring from links. We call the corresponding conditions *unimodularity conditions*. Thus we obtain a complete algebraic picture of simple boundary links in S^5.

Theorem 1.2 *The l–equivalence classes of Seifert matrices of simple boundary links of 3–spheres in S^5 determine the isotopy type of the link. Moreover, the l–equivalence classes of Seifert matrices give a bijection from the set of isotopy classes of simple boundary links of 3–spheres in S^5 to the set of l–equivalence classes of square integral matrices D satisfying the unimodularity conditions.*

We would also like to give an algebraic picture of our closed 5–manifolds. In general we don't know which values the generalized Milnor pairing takes. But if we require that $H_2(M;\mathbb{Z})=0$, these manifolds are all results of surgeries on links and we can use the realization of the link invariants to give a complete answer.

Let D be an $m\times m$ integral matrix satisfying the unimodularity conditions; then there is associated to D a $\mathbb{Z}[F_n]$–module map $\varphi_D\colon (\mathbb{Z}[F_n])^m \to (\mathbb{Z}[F_n])^m$ and a generalized Milnor pairing

$$\big(F_n,\ \operatorname{coker}\varphi_D,\ b_D\colon (\operatorname{coker}\varphi_D)^*\times(\operatorname{coker}\varphi_D)^*\to \big(H^1(BF_n;\mathbb{Q}[F_n])\big)^*\big)$$

We will give a detailed description of this in Section 4.

Theorem 1.3 *There is a bijection between the diffeomorphism classes of closed oriented 5–manifolds M with $\pi_1(M)$ a free group of rank n and $H_2(M;\mathbb{Z})=0$, and the isomorphism classes of generalized Milnor pairings $(F_n,\operatorname{coker}\varphi_D,b_D)$ for all matrices D (with various sizes m) fulfilling the unimodularity conditions.*

We will give more details of the generalized Milnor paring in Section 2, and prove the main classification theorem in Section 3. The discussion of 3–links and their relation with 5–manifolds will be the contents of Section 4.

Remark 1.4 A special case of Theorem 2.4 is when $\pi_1(M)\cong\mathbb{Z}$ and $\pi_2(M)$ is a finitely generated abelian group. In this case we can show that $\pi_2(M)$ is torsion-free and the bilinear form on $\pi_2(M)$ is unimodular, $w_2(\widetilde{M})$ is determined by the bilinear form on $\pi_2(M)$, and the realization problem of the invariants can be solved. This gives a complete classification of closed 5–manifolds with $\pi_1=\mathbb{Z}$ and π_2 a finitely generated abelian group. As an application, this reproves the fibration theorems in dimension 5 in the topological and smooth category given by Hsu [7], Weinberger [20] and Shaneson [17], respectively. See more details in the appendix.

Remark 1.5 The notions of *Borel manifolds* and *strongly Borel manifolds* were coined by Kreck and Lück [10, Definition 0.2]. A manifold M is called a Borel manifold if for any homotopy equivalence $f\colon N\to M$ there exists a homeomorphism $h\colon N\to M$ such that f and h induce the same map on the fundamental groups up to conjugation. It is called strongly Borel if all homotopy equivalences are homotopic to a homeomorphism. If M^5 is a closed oriented spin topological 5–manifold with free fundamental group and torsion-free π_2, then it is Borel. Since for any homotopy equivalence $f\colon N^5\xrightarrow{\simeq} M^5$, f induces an isomorphism between the topological generalized Milnor pairings (in this case the Kirby–Siebenmann invariant is determined by the bilinear form b_M; see the proof of Theorem 2.4), the statement follows from Theorem 1.1. On the other hand, for a closed oriented topological 5–manifold M^5 with free fundamental group, a computation of the topological structure set of M using the surgery exact sequence gives $\mathscr{S}^{\mathrm{TOP}}(M^5) = H^2(M;\mathbb{Z}/2)$. Therefore, by [10, Theorem 1.1], M is strongly Borel if and only if $H_2(M;\mathbb{Z}/2) = 0$.

One often hears the statement that the classification of high-dimensional manifolds is completely understood. What people mean is that with the s–cobordism theorem one has a criterion of when two manifolds are diffeomorphic and with surgery theory one has a reduction of the problem of finding an s–cobordism to problems in homotopy theory (unstable and stable) and algebra (surgery obstruction groups), and the analysis of certain maps relating the homotopy theory and the algebra. But this doesn't mean that even for some very explicit manifolds, like for example complete intersections, the procedure can be carried out successfully. Given the complications of the homotopy groups of spheres, in higher dimensions the problems get harder and harder. But in comparatively low dimensions (say up to 8) one has a chance, which doesn't mean that it is routine. Most results in that dimension range concern simply connected manifolds. In this paper we make a first step towards a classification of 5–manifolds with fundamental group the free group F_n. This class is particular interesting, since such manifolds occur on the one hand as total spaces of bundles over the circle and on the other hand as fundamental groups of links of 3–spheres in S^5. We classify both in the smooth and topological category. It might be interesting to note that the topological classification of 4–manifolds with fundamental group the free group F_n is completely open for $n > 1$. The question of whether the group F_n is good in the sense of Freedman and Quinn [6] is the key question for topological 4–manifolds. If this is the case then one can use similar methods as in the present paper to attack the classification of 4–manifolds with fundamental group F_n.

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2 The generalized Milnor pairing and the statement of the main theorem

Now we describe the generalized Milnor pairing which we use to classify our manifolds. First we give the general algebraic definition. A *generalized Milnor pairing* is a quadruple (π_1, π_2, b, w_2) consisting of the following:

(1) π_1 a free group of rank n; let $\Lambda = \mathbb{Z}[\pi_1]$ be the integral group ring and $\Lambda_{\mathbb{Q}} = \mathbb{Q}[\pi_1]$ be the rational group ring.

(2) π_2 a finitely generated Λ–module, which is torsion-free as an abelian group.

(3) $b\colon \pi_2^* \times \pi_2^* \to (H^1(B\pi_1; \Lambda_{\mathbb{Q}}))^*$ a symmetric Λ–equivariant pairing, where $*$ stands for the $\mathbb{Q}$–dual $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q})$, and by Λ–equivariant we mean that b is a Λ–module map under the diagonal action of Λ on $\pi_2^* \times \pi_2^*$ and the natural Λ–module structure on $(H^1(B\pi_1; \Lambda_{\mathbb{Q}}))^*$.

(4) $w_2 \in \mathrm{Hom}(\pi_2, \mathbb{Z}/2)$.

An isomorphism $(\alpha, \beta)\colon (\pi_1, \pi_2, b, w_2) \to (\pi_1', \pi_2', b', w_2')$ between generalized Milnor pairings consists of

(1) an isomorphism $\alpha\colon \pi_1 \to \pi_1'$;

(2) an isomorphism $\beta\colon \pi_2 \to \pi_2'$, which is compatible with the Λ– and Λ'–module structure and the pairings b and b', and maps w_2' to w_2.

Let M^5 be a smooth closed oriented 5–manifold with $\pi_1(M) \cong F_n$ and $\pi_2(M)$ a torsion-free abelian group; we associate a generalized Milnor pairing $\varphi(M) = (\pi_1(M), \pi_2(M), b_M, w_2(\widetilde{M}))$ to M as follows. Let $\widetilde{M}$ be the universal cover of M. By Poincaré duality we have an isomorphism $H_4(\widetilde{M}; \mathbb{Q}) = H_4(M; \Lambda) \otimes \mathbb{Q} \cong H^1(M; \Lambda_{\mathbb{Q}})$ and the latter group is isomorphic to $H^1(B\pi_1(M); \Lambda_{\mathbb{Q}})$, because M has a CW–structure $M \simeq \bigvee_n S^1 \vee \bigvee S^2 \cup e^3 \cup \cdots$ [19, Proposition 3.3]. Next we use the Kronecker isomorphism to identify $H^4(\widetilde{M}; \mathbb{Q})$ with $H_4(\widetilde{M}; \mathbb{Q})^*$, where $*$ stands for the $\mathbb{Q}$–dual, and the isomorphism above to obtain an isomorphism $H^4(\widetilde{M}; \mathbb{Q}) \cong (H^1(B\pi_1(M); \Lambda_{\mathbb{Q}}))^*$. The cup product and this identification together define a symmetric Λ–equivariant form

$$H^2(\widetilde{M}; \mathbb{Q}) \times H^2(\widetilde{M}; \mathbb{Q}) \to (H^1(B\pi_1 M; \Lambda_{\mathbb{Q}}))^*.$$

Using the Kronecker isomorphism and the Hurewicz isomorphism we obtain a symmetric Λ–equivariant form

$$b_M\colon \pi_2(M)^* \times \pi_2(M)^* \to (H^1(B\pi_1 M; \Lambda_{\mathbb{Q}}))^*,$$

where $*$ is again the vector space of homomorphisms to $\mathbb{Q}$, We will discuss more about this bilinear form in the beginning of Section 3.

To this we add the second Stiefel Whitney class

$$w_2(\widetilde{M}) \in \mathrm{Hom}(H_2(\widetilde{M}; \mathbb{Z}), \mathbb{Z}/2) = \mathrm{Hom}(\pi_2(M), \mathbb{Z}/2)$$

to obtain our invariant and get the quadruple

$$\varphi(M) = (\pi_1(M), \pi_2(M), b_M, w_2(\widetilde{M}));$$

we call this the *generalized Milnor pairing of* M. The group of self-isomorphisms of $\varphi(M)$ is denoted by $\mathrm{Aut}(\varphi(M))$.

Remark 2.1 In the case where only spin manifolds are concerned, $w_2(\widetilde{M})$ is always 0, and the generalized Milnor pairing is actually a triple $\varphi(M) = (\pi_1(M), \pi_2(M), b_M)$. This is the case in Theorem 1.3.

Remark 2.2 It's easy to see from the Leray–Serre spectral sequence of the fibration $\widetilde{M} \xrightarrow{p} M \to \bigvee_n S^1$ that $p^*\colon H^2(M;\mathbb{Z}/2) \to H^2(\widetilde{M};\mathbb{Z}/2)$ is injective. Therefore $w_2(M)$ and $w_2(\widetilde{M})$ determine each other.

We also classify a special case of compact oriented manifolds M with boundary which is relevant for classifying links in S^5. The boundary has to be a disjoint union of n copies of $S^1 \times S^3$ and we require that the circles in the boundary components generate the fundamental group F_n of M. Here we replace $H_4(\widetilde{M};\mathbb{Q})$ by $H_4(\widetilde{M},\partial\widetilde{M};\mathbb{Q})$ and we note that $H^2(\widetilde{M};\mathbb{Q}) \cong H^2(\widetilde{M},\partial\widetilde{M};\mathbb{Q})$, so that the definition of b_M makes sense. With this modification we can consider the quadruple defining $\varphi(M)$ as before. But we have to observe that the identification of the fundamental groups of M and M' is now given by an identification of the boundary components.

Remark 2.3 When X is the complement of a simple 3–knot, we have a bilinear paring $b\colon H^2(\widetilde{X};\mathbb{Q}) \times H^2(\widetilde{X};\mathbb{Q}) \to \mathbb{Q}$, which is the Milnor paring [15]; see also [13].

We also classify the corresponding topological manifolds. Here we add a fifth term to our invariant, the Kirby–Siebenmann invariant

$$KS(M) \in H^4(M;\mathbb{Z}/2) \cong \pi_1(M)/[\pi_1(M),\pi_1(M)] \otimes \mathbb{Z}/2.$$

We call the quintuple $(\pi_1(M), \pi_2(M), b_M, w_2(\widetilde{M}), KS(M))$ the *topological generalized Milnor pairing* of the topological manifold M. Of course in the definition of an isomorphism (α,β) between two topological generalized Milnor pairings we require that the isomorphism $\alpha\colon \pi_1(M) \to \pi_1(M')$ respects the Kirby–Siebenmann invariant, too.

Now we restate the classification theorem of the manifolds under consideration and add the realization statement for induced maps.

Theorem 2.4 *Let M_0 and M_1 be two smooth (or topological), closed, oriented 5–manifolds with free fundamental group of rank n and torsion-free π_2. Then M_0 and M_1 are oriented-diffeomorphic (-homeomorphic) if and only if their (topological) generalized Milnor pairings are isomorphic. Any isomorphism between the (topological) generalized Milnor parings can be realized by an orientation-preserving diffeomorphism (homeomorphism) from M_0 to M_1.*

If M_0 and M_1 are compact with boundary consisting of n copies of $S^1 \times S^3$ such that the circles in the boundary generate $\pi_1(M_i)$, then M_0 and M_1 are oriented-diffeomorphic (-homeomorphic) if and only if there exists an isomorphism (α, β) between their (topological) generalized Milnor pairings, where α is induced by identifying the boundary components. Any such isomorphism can be realized by an orientation-preserving diffeomorphism (homeomorphism).

The isomorphism α above actually sends free generators x_i of $\pi_1(M_0)$ to conjugates of free generators x_i' of $\pi_1(M_1)$, which are represented by different arcs in the interior to a basepoint.

Remark 2.5 In the definition of the invariant $\varphi(M)$ we use the cup product on the cohomology with rational coefficients. Usually one loses information when passing from integral coefficients to rational coefficients. But in our situation the rational cohomology contains essentially more information than the integral cohomology. This can be illuminated by the following example.

Example Let

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix};$$

then $A + A'$ (where A' is the transpose of A) is unimodular and has signature 0. Therefore by [11, Theorem 2] there is a simple 3–knot $K \subset S^5$ with Seifert matrix S–equivalent to A. The Alexander polynomial of K is $\Delta_K(t) = \det(A - tA') = 2t^2 + 5t + 2$. Let X be the complement of K; then, by [4, Theorem 1.5], $H_2(\widetilde{X}) \cong \mathbb{Z}\big[\frac{1}{2}\big] \oplus \mathbb{Z}\big[\frac{1}{2}\big]$. Let M^5 be the result of surgery on K; then $\pi_1(M) \cong \mathbb{Z}$ and $\pi_2(M) \cong H_2(\widetilde{M}) \cong H_2(\widetilde{X}) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}\big[\frac{1}{2}\big]$. We see that $H^2(\widetilde{M};\mathbb{Z}) = 0$ but $H^2(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}^2$.

3 Proof of Theorem 2.4

Before giving the proof of the main theorem we first rephrase the bilinear form b_M in a more explicit form. Fix an identification $\pi_1(M) \xrightarrow{\cong} F_n$ and consider the classifying map of the fundamental group $f\colon M \to BF_n = \bigvee_{i=1}^n S_i^1$. From the Leray–Serre spectral sequence (with twisted coefficients, which we denote by an underline) of the fibration $\widetilde{M} \to M \to \bigvee_{i=1}^n S_i^1$, we get an isomorphism $H_5(M) \to H_1\big(\bigvee_n S^1; \underline{H_4(\widetilde{M})}\big)$. Note that

$$H_1\Big(\bigvee_n S^1; \underline{H_4(\widetilde{M})}\Big) = \mathrm{Ker}\Big(\bigoplus_n H_4(\widetilde{M}) \xrightarrow{d} H_4(\widetilde{M})\Big),$$

where $d(x_1, \dots, x_n) = \sum_i (g_i - 1)x_i$ with $g_1, \dots, g_n$ the corresponding generators of F_n. This leads to an injection $H_5(M) \to \bigoplus_n H_4(\widetilde{M})$. Denote the image of the fundamental class $[M]$ by $(\sigma_1, \dots, \sigma_n)$. Now denote by $I_i(M)$ the symmetric bilinear form $H^2(\widetilde{M};\mathbb{Q}) \times H^2(\widetilde{M};\mathbb{Q}) \to \mathbb{Q}$ given by $I_i(\alpha,\beta) = \langle \alpha \cup \beta, \sigma_i\rangle$. From the relation $\sum_i (g_i - 1)\sigma_i = 0$ we see that the bilinear forms satisfy the relation $\sum_i I_i(\alpha,\beta) = \sum_i I_i(g_i^*\alpha, g_i^*\beta)$.

Geometrically, we choose regular values $q_i \in S^1_i$ and let $F_i = f^{-1}(q_i)$. Let E be the complement of an open tubular neighborhood of $\bigcup_i F_i$; then E has boundary $\partial E = \bigcup_i F_i^{\pm}$, where $F_i^{\pm}$ are the positive and negative boundary components of the tubular neighborhood of F_i. We obtain $\widetilde{M}$ by gluing infinitely many copies of E under the deck transformation, ie $\widetilde{M} = \bigcup_{g \in F_n} E_g$. Let $\overline{M}_i \to M$ be the $\mathbb{Z}$–covering of M corresponding to $M \to \bigvee_{i=1}^n S^1_i \to S^1_i$; then it's easy to see that the Leray–Serre spectral sequence of this covering gives an isomorphism $H_5(M) \xrightarrow{\cong} H_4(\overline{M}_i)$, with $[M] \mapsto [F_i^-]$. Furthermore the commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & \overline{M}_i \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

induces

$$\begin{array}{ccc} & & \bigoplus_n H_4(\widetilde{M}) \\ & \nearrow & \downarrow {\scriptstyle \text{projection to the } i^{\text{th}} \text{ component}} \\ H_5(M) & & H_4(\widetilde{M})_i \\ & \searrow & \downarrow \\ & & H_4(\overline{M}_i) \end{array}$$

From this we see that each σ_i is represented by F_i^- in $E \subset \widetilde{M}$.

By [19, Proposition 3.3] we know that M has a CW–structure of the form $M \simeq \bigvee_{i=1}^n S^1_i \vee \bigvee S^2 \cup e^3 \cup \cdots$. Therefore we have isomorphisms $H_4(\widetilde{M}) \cong H^1_c(\widetilde{M}) \cong H^1\big(\bigvee_n S^1, \Lambda\big)$, where Λ denotes the group ring $\mathbb{Z}[F_n]$. Thus we have a surjection $\Lambda^n \to H_4(\widetilde{M})$. Let e_i be the standard basis of Λ^n; then e_i is mapped to σ_i. Therefore $\sigma_1, \dots, \sigma_n$ form a set of generators of the Λ–module $H_4(\widetilde{M})$. For any $\alpha, \beta \in H^2(\widetilde{M};\mathbb{Q})$ and $x \in H_4(\widetilde{M})$, we may assume that $x = \sum_i \lambda_i\sigma_i$, with $\lambda_i = \sum_g a_g^{(i)} \cdot g \in \Lambda$. Then $\langle \alpha \cup \beta, x\rangle = \big\langle \alpha \cup \beta, \sum_i \lambda_i \sigma_i\big\rangle = \sum_{i,g} a_g^{(i)} \langle g^{-1}\alpha \cup g^{-1}\beta, \sigma_i\rangle = \sum_{i,g} a_g^{(i)} I_i(g^{-1}\alpha, g^{-1}\beta)$. Thus we have shown:

Lemma 3.1 *The sequence of bilinear forms $(I_1, \dots, I_n)$ contains the same information as the bilinear pairing b_M together with an identification of $\pi_1(M)$ with the free group F_n.*

Next we relate the signature of forms I_i to the signatures of the fiber F_i^4.

Lemma 3.2 *The bilinear form $I_i\colon H^2(\widetilde{M};\mathbb{Q})\times H^2(\widetilde{M};\mathbb{Q})\to\mathbb{Q}$ has the same signature as the intersection form of F_i^4.*

Proof We use homology and cohomology with $\mathbb{Q}$–coefficients.

Let E be the exterior of an open tubular neighborhood of $\bigcup_i F_i$. Then the universal cover $\widetilde{M}$ is $\widetilde{M}=\bigcup_{g\in F_n} E_g$, where each E_g is a copy of E. Since $H_2(F_i)$ is finite-dimensional, there exists a connected compact submanifold $M_0\subset\widetilde{M}$ which is a union of finitely many of the E_g and $F_i\subset M_0$ such that any $x\in\operatorname{Ker}(H_2(F_i)\to H_2(\widetilde{M}))$ is in $\operatorname{Ker}(H_2(F_i)\to H_2(M_0))$. Therefore

$$\operatorname{Ker}(H_2(F_i)\to H_2(\widetilde{M}))=\operatorname{Ker}(H_2(F_i)\to H_2(M_0)).$$

Dually on cohomology, we have

$$\operatorname{Im}(H^2(\widetilde{M})\to H^2(F_i))=\operatorname{Im}(H^2(M_0)\to H^2(F_i)).$$

The boundary ∂M_0 has a component F_0 which is the image of F_i under a deck transformation by $g\in\pi_1(M)$. There is a commutative diagram

$$\begin{array}{ccc} H^2(M_0) & \longrightarrow & H^2(F_i) \\ & \searrow & \big\downarrow g^* \\ & & H^2(F_0) \end{array}$$

where g^* is an isometry. So we have

$$\begin{aligned} H^2(\widetilde{M})/\mathrm{rad}(I_i) &= \operatorname{Im}(H^2(\widetilde{M})\to H^2(F_i))/\mathrm{rad} \\ &= \operatorname{Im}(H^2(M_0)\to H^2(F_i))/\mathrm{rad} \\ &\cong \operatorname{Im}(H^2(M_0)\to H^2(F_0))/\mathrm{rad}. \end{aligned}$$

Note that $\operatorname{Ker}(H_2(\partial M_0)\to H_2(M_0))$ is a Lagrangian in $H_2(\partial M_0)$. Therefore

$$\operatorname{Ker}(H_2(F_0)\to H_2(M_0))=\operatorname{Ker}(H_2(\partial M_0)\to H_2(M_0))\cap H_2(F_0)$$

is isotropic. A standard argument in linear algebra shows that dually on cohomology, $\operatorname{Im}(H^2(M_0)\to H^2(F_0))$ has a complement which is isotropic. Let's denote it by K; it generates a hyperbolic form $H(K)$ in $H^2(F_0)$ and we have

$$\operatorname{Im}(H^2(M_0)\to H^2(F_0))/\mathrm{rad}\oplus H(K)=H^2(F_0).$$

Therefore $\operatorname{sign}(I_i)=\operatorname{sign}(H^2(F_0))$. □

The proof of Theorem 2.4 is based on modified surgery theory. We refer to [9] for the details of this machinery for classifying manifolds. For the convenience of the reader we summarize the basic concepts and the main theorem we apply. The basic idea is to weaken the normal homotopy type, which is the first basic invariant of a manifold M in classical surgery, to the normal k–type. This is roughly given by the k–skeleton of M together with the restriction of the normal bundle. Since the k–skeleton is not well-defined we pass to Postnikov towers instead or, better, Moore–Postnikov decompositions. The normal bundle is equivalent to the normal Gauss map $\nu\colon M \to BO$. The *normal k–type* is the k^{th} stage of the Moore–Postnikov tower of $\overline{\nu}$, which is a fibration $p\colon B_k(M) \to BO$ which is completely characterized by the property that there is a lift $\overline{\nu}\colon M \to B_k(M)$ of ν which induces an isomorphism on homotopy groups up to degree k and is surjective in degree $k+1$. Note that if k is larger than the dimension of M the normal k–type is equivalent to the normal homotopy type, thus modified surgery generalizes classical surgery. Such a lift is called a *normal k–smoothing*.

Given two normal k–smoothings $(M, \overline{\nu}_M)$ and $(M', \overline{\nu}_{M'})$ in the same fibration B_k, the first step is to decide whether these normal k–smoothings are bordant. This means that there is a coboundary W together with a lift of the normal Gauss map $\overline{\nu}_W$ (but this is not highly connected). The main theorem of modified surgery is that if $k \geq \frac{1}{2}\dim M - 1$, then there is a surgery obstruction in a monoid $l_{\dim M+1}(\pi_1(M), w_1(M))$, from which one can decide whether W is B_k–bordant to an s–cobordism.

Now we return to our situation of 5–manifolds. We will work with the normal 2–type of M. Then the obstruction is actually in the classical Wall group $L_5(\pi_1(M), w_1(M))$. We prepare the proof with a construction of the normal 2–type (cf [9, Proposition 2]) of a smooth manifold M (of arbitrary dimension), which might be of separate interest elsewhere. Let $u\colon M \to P$ be the second-stage Postnikov tower of M; there are unique cohomology classes $w_i \in H^i(P; \mathbb{Z}/2)$ for $i = 1, 2$ such that $u^*(w_i) = w_i(M)$. Let $w_1 \times w_2\colon P \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$ be the classifying map of these classes, and $w_1(EO) \times w_2(EO)\colon BO \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$ be the classifying map of the universal Stiefel–Whitney classes. Consider the following pullback square:

$$\begin{array}{ccc}
B(\pi_1(M), \pi_2(M), k_1, w_1(M), w_2(M)) & \xrightarrow{\;h\;} & P \\
\big\downarrow p & & \big\downarrow w_1 \times w_2 \\
BO & \xrightarrow{\;w_1(EO)\times w_2(EO)\;} & K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)
\end{array}$$

There is a lift $\overline{\nu}\colon M \to B(\pi_1(M), \pi_2(M), k_1, w_1(M), w_2(M))$ of the normal Gauss map $\nu\colon M \to BO$ of M, which a 3–equivalence, and p is 3–coconnected. Thus we have shown:

Lemma 3.3 *The fibration*

$$p\colon B(\pi_1(M), \pi_2(M), k_1, w_1(M), w_2(M)) \to BO$$

is the normal 2–type of M. There is a corresponding construction in the topological category, if one replaces BO by $B\mathrm{Top}$.

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4 We begin with the smooth category. In our situation, the second-stage Postnikov tower P of M is a fibration over $\bigvee_{i=1}^n S_i^1$ with fiber $K = K(\pi_2(M), 2)$ and monodromy given by the $\pi_1(M)$–module structure of $\pi_2(M)$. We denote the normal 2–type by $p\colon B \to BO$ and recall that by the lemma above it is determined by $\pi_1(M)$, $\pi_2(M)$ as a $\mathbb{Z}[\pi_1(M)]$–module, and $w_2(M)$.

Now we compute the bordism group $\Omega_5(B, p)$. Note that $\Omega_5(B, p) = \pi_5^S(M(p))$. We consider the fibration $\widetilde{B} \to B \to \bigvee_{i=1}^n S_i^1$; the Wang sequence of the generalized homology theory π_*^S is

$$\cdots \to \Omega_5(\widetilde{B}, \widetilde{p}) \to \Omega_5(B, p) \to \bigoplus_n \Omega_4(\widetilde{B}, \widetilde{p}) \to \cdots,$$

where $\widetilde{B}$ is the pullback

$$\begin{array}{ccc} \widetilde{B} & \longrightarrow & K \\ \Big\downarrow \widetilde{p} & & \Big\downarrow \mathrm{const}\times w_2 \\ BO & \longrightarrow & K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \end{array}$$

where $w_2 \in H^2(K;\mathbb{Z}/2)$ is the image of $w_2 \in H^2(P;\mathbb{Z}/2)$ under the injection $H^2(P;\mathbb{Z}/2) \to H^2(K;\mathbb{Z}/2)$. From this we have $\Omega_n(\widetilde{B}, \widetilde{p}) = \Omega_n^{\mathrm{spin}}(K;\eta)$, where the latter group is the bordism group of $f\colon M \to K$ together with a spin structure on $f^*\eta \oplus \nu M$, where η is a complex line bundle over K such that $w_2(\eta) = w_2 \in H^2(K;\mathbb{Z}/2)$.

Now $\pi_2(M)$ is the direct limit of its finitely generated subgroups, and by assumption $\pi_2(M)$ is a torsion-free abelian group, hence it is a direct limit of finitely generated free abelian groups $\varinjlim G_\alpha$. Therefore K is a direct limit of spaces $K = \varinjlim K(G_\alpha, 2)$. In general there is an Atiyah–Hirzebruch spectral sequence computing $\Omega_n^{\mathrm{spin}}(X;\eta)$ with E_2–terms $H_p(X;\Omega_q^{\mathrm{spin}})$, and the differential d_2 is dual to $\mathrm{Sq}^2 + w_2(\eta)\cdot$; see [18]. An easy computation with this spectral sequence shows that $\Omega_5^{\mathrm{spin}}(K(G_\alpha,2);\eta) = 0$ for a finitely generated free abelian group G_α, and henceforth $\Omega_5^{\mathrm{spin}}(K;\eta) = \varinjlim \Omega_5^{\mathrm{spin}}(K(G_\alpha,2);\eta) = 0$.

Therefore we have an injection $\Omega_5(B,p) \to \bigoplus_n \Omega_4^{\mathrm{spin}}(K;\eta)$. There is a commutative diagram

$$\begin{array}{ccc} \Omega_5(B,p) & \longrightarrow & \bigoplus_n \Omega_4^{\mathrm{spin}}(K;\eta) \\ \downarrow & & \downarrow \\ H_5(P) & \longrightarrow & \bigoplus_n H_4(K) \end{array}$$

with the horizontal arrows injective and the vertical arrows the edge homomorphisms. Following the definition of the boundary map in the Mayer–Vietoris sequence of the bordism theory, we see that a bordism class $[f\colon M \to B]$ is mapped to

$$([h \circ f\colon F_1 \to K], \dots, [h \circ f\colon F_n \to K]) \in \bigoplus_n \Omega_4^{\mathrm{spin}}(K;\eta),$$

where $h\colon B \to P$ is the map in the pullback square, $\pi\colon P \to \bigvee_n S^1$ is the projection map, and $F_i = (\pi \circ h \circ f)^{-1}(q_i)$ is the preimage of a regular value $q_i \in S_i^1$. A direct calculation with the Atiyah–Hirzebruch spectral sequence shows that a bordism class

$$[\varphi\colon N^4 \to K] \in \Omega_4^{\mathrm{spin}}(K(G_\alpha,2);\eta)$$

is determined by $\mathrm{sign}(N)$ and $\varphi_*[N] \in H_4(K(G_\alpha,2))$. Passing to the limit we see that a bordism class $[\varphi\colon N^4 \to K] \in \Omega_4^{\mathrm{spin}}(K;\eta)$ is determined by $\mathrm{sign}(N)$ and $\varphi_*[N] \in H_4(K)$. Now $H_4(K) = \varinjlim H_4(K(G_\alpha,2))$ is a direct limit of free abelian groups, hence is torsion-free, therefore $\varphi_*[N]$ is determined by its image in $H_4(K;\mathbb{Q})$, which is further determined by the evaluation with elements in $H^4(K;\mathbb{Q})$. Note that $H^4(K;\mathbb{Q}) = H^4(K(\pi_2(M)\otimes\mathbb{Q},2);\mathbb{Q})$, where $\pi_2(M)\otimes\mathbb{Q}$ is a $\mathbb{Q}$–vector space. From this it's easy to see that the cup product map

$$H^2(K;\mathbb{Q}) \otimes H^2(K;\mathbb{Q}) \xrightarrow{\cup} H^4(K;\mathbb{Q})$$

is surjective, therefore $\varphi_*[N] \in H_4(K;\mathbb{Q})$ is determined by $\langle \varphi^*\alpha \cup \varphi^*\beta, [N]\rangle$ for $\alpha,\beta \in H^2(K;\mathbb{Q})$.

For a normal 2–smoothing $\bar\nu\colon M \to B$, let $f\colon M \xrightarrow{\bar\nu} B \xrightarrow{h} P$ be the composition; we have a commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\tilde f} & K \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & P \end{array}$$

and $f = \tilde f \circ i\colon F_i \to K$ for $F_i \subset \widetilde{M}$. Notice that $\tilde f^*\colon H^2(K;\mathbb{Q}) \to H^2(\widetilde{M};\mathbb{Q})$ is an isomorphism, therefore the evaluation $\langle f^*\alpha \cup f^*\beta, [F_i]\rangle = \langle \tilde f^*\alpha \cup \tilde f^*\beta, i_*[F_i]\rangle = \langle \tilde f^*\alpha \cup \tilde f^*\beta, \sigma_i\rangle$ is exactly the bilinear form $I_i\colon H^2(\widetilde{M};\mathbb{Q}) \otimes H^2(\widetilde{M};\mathbb{Q}) \to \mathbb{Q}$.

By Lemma 3.2, $\mathrm{sign}(F_i)$ equals the signature of the bilinear form I_i. This shows that the bordism class $[M,\overline{\nu}]$ is determined by the bilinear forms I_i for $i=1,\dots,n$.

Now, given two manifolds M and M' with isomorphic algebraic invariants and — depending on an ordering of the boundary components in the bounded case — equal boundary as in Theorem 2.4, they have the same normal 2–type (B,p). We identify the boundaries (one of the manifolds with opposite orientation) to obtain a closed manifold and use the normal 2–smoothings $\overline{\nu}\colon M\to B$ and $\overline{\nu}'\colon M'\to B$ to obtain an element in $\Omega_5(B,p)$. We note here that we controlled the restriction of the normal 2–smoothings to the boundary by requiring that the identification of the boundary components be compatible with the identification of the fundamental groups. By the consideration above this is the zero element if our invariant φ agrees for M_0 and M_1 with the normal 2–smoothings chosen so that the invariants agree.

Let W be a B–null-bordism of the glued manifold, then there is an obstruction $\theta(W)\in l_6(F_n)$. If this is elementary, then W is B–bordant rel boundary to an s–cobordism [9, Theorem 3]. In our situation with $\pi_1(M)\cong F_n$, the Whitehead group $\mathrm{Wh}(F_n)=\bigoplus \mathrm{Wh}(\mathbb{Z})=0$, and so we won't have to consider the preferred bases. Furthermore by the remark on [9, page 730] the obstruction sits in the ordinary L–group $L_6(F_n)$. This group is isomorphic to $\mathbb{Z}/2$ and the obstruction is detected by the Arf-invariant [2, Theorem 16]. Since there is a simply connected closed 6–manifold with Arf-invariant 1 we can change W by disjoint sum with this, if necessary, to show that $\theta(W)=0\in L_6(F_n)$. This implies that $\theta(W)$ is elementary and finishes the proof in the smooth case.

The proof of the topological case is similar, since the modified surgery method also applies to topological manifolds (cf [9]). The only difference is that an element $[\varphi\colon F^4\to K]\in\Omega_4^{\mathrm{TopSpin}}(K;\eta)$ is determined by the image of the fundamental class $\varphi_*[F]\in H_4(K)$, the signature $\mathrm{sign}(F)$ and the Kirby–Siebenmann invariant $KS(F)$. Each F_i has trivial normal bundle in M, therefore under the isomorphism $H^4(M;\mathbb{Z}/2)\xrightarrow{\cong}\bigoplus_{i=1}^n H^4(F_i;\mathbb{Z}/2)$, $KS(M)$ is mapped to

$$(KS(F_1),\dots,KS(F_n)).$$

The rest is the same as in the smooth case. □

4 Proofs of Theorems 1.2 and 1.3

The Seifert matrix of a boundary link is defined as follows (cf [12]): choose Seifert manifolds F_i of the link L; then there are linking forms

$$\theta_{ij}\colon H_q(F_i)\otimes H_q(F_j)\to\mathbb{Z},\quad (\alpha,\beta)\mapsto L(z_1,z_2),$$

defined by linking numbers between z_1, representing α, and z_2, representing $i_+\beta$. With respect to a basis of the torsion-free part of $H_q(F_i)$ the linking forms θ_{ij} are represented by a matrix A_{ij}; then the Seifert matrix $D=(A_{ij})$ of L is an integral square matrix formed by the blocks A_{ij}, and D is $(-1)^q$–symmetric. Different choices of Seifert manifolds will lead to different Seifert matrices, but they are related by a sequence of "algebraic moves" and are *l–equivalent*. The l–equivalence class of the Seifert matrix D is a well-defined invariant of L [12, Theorem 1].

Given a square integral matrix $D=(A_{ij})$, consisting of matrices blocks A_{ij}, the *unimodularity condition* of D requires that $A_{ii}+A'_{ii}$ for $i=1,\dots,n$ and $D+D'$ are unimodular. It's shown in [12] that there is a boundary simple $(2q-1)$–link L whose Seifert matrix is $D=(A_{ij})$ when $q\geq 3$ [12, Theorem 1].

Given a link $f\colon \bigcup_{i=1}^n S^3 \hookrightarrow S^5$ we note that up to isotopy there is a unique tubular neighborhood U of Image(f). We denote the complement of the interior of this tubular neighborhood by X_f and use the tubular neighborhood to identify ∂X_f with $\bigcup_n (S^1\times S^3)$.

If two links $f\colon \bigcup_{i=1}^n S^3 \hookrightarrow S^5$ and $f'\colon \bigcup_{i=1}^n S^3 \hookrightarrow S^5$ are isotopic, the isotopy extension theorem implies that the identification $\partial X_f \to \partial X_{f'}$ extends to a diffeomorphism $X_f\to X_{f'}$. In turn, if there is an orientation-preserving diffeomorphism $g\colon X_f\to X_{f'}$ extending the identification on the boundary, then we can extend this by the identification on the tubular neighborhoods to an orientation-preserving diffeomorphism $\hat{g}\colon S^5\to S^5$ mapping the first link to the second. Now we use the fact that $\pi_0(\mathrm{Diff}^+(S^5))$ is isomorphic to the group of homotopy 6–spheres (using the h–cobordism theorem and Cerf's theorem [3] that pseudoisotopy implies isotopy) and that the group of 6–dimensional homotopy spheres is trivial [8]. Thus the two links are isotopic.

Now note that the link complement X has free fundamental group of rank n, generated by the circles in the boundary components. Furthermore, from Farber [5, Theorem 5.7] we know that π_2 of the complement of a simple boundary link is torsion-free. Thus Theorem 2.4 applies to complements of simple boundary 3–links in S^5.

The meridians give rise to an identification $\pi_1(X_f)\xrightarrow{\cong} F_n$; under this identification, by the reinterpretation of the invariants in the beginning of Section 3, we have an invariant

$$\psi(X_f)=(\pi_2(X_f),\ b_i\colon \pi_2(X_f)^*\times\pi_2(X_f)^*\to\mathbb{Q},\ i=1,\dots,n).$$

Here we consider $\pi_2(X_f)$ as an F_n–module and $*$ stands for the $\mathbb{Q}$–dual. The link complement X_f is a Spin–manifold, thus Theorem 2.4 implies that this invariant determines the oriented diffeomorphism type mod boundary, meaning that the identification on the boundary extends to an orientation-preserving diffeomorphism between the whole manifolds. Thus we have proved the following:

Lemma 4.1 *Two simple boundary 3–links $f\colon \bigcup_n S^3 \hookrightarrow S^5$ and $f'\colon \bigcup_n S^3 \hookrightarrow S^5$ are isotopic if and only if under certain identifications of $\pi_1(X_f)$ and $\pi_1(X_{f'})$ with F_n coming from enumerating the link components, $\psi(X_f)$ and $\psi(X_{f'})$ are isomorphic.*

Proof of Theorem 1.2 By Lemma 4.1, to prove that the l–equivalence class of the Seifert matrices determines the isotopy type of the link, we need to show that the l–equivalence class of the Seifert matrices determines $\psi(X_f)$. Let F_i be Seifert manifolds of a link given by an embedding f. Let X_f be the complement of the tubular neighborhood of the link; then the universal cover $\widetilde{X}_f$ is obtained by gluing infinitely many copies of Y via the deck transformation, where Y is obtained from X_f by cutting up along the Seifert manifolds. We identify $\pi_1(X_f)$ with F_n by sending the meridian (with the induced orientation from that of S^5 and S^3) of the i^{th} component of the link to the i^{th} standard generator t_i of F_n. The Mayer–Vietoris sequence computing $H_2(\widetilde{X}_f)$ is

$$\bigoplus_{i=1}^{n} H_2(F_i)\otimes_{\mathbb{Z}}\mathbb{Z}[F_n]\xrightarrow{\varphi} H_2(Y)\otimes_{\mathbb{Z}}\mathbb{Z}[F_n]\to H_2(\widetilde{X}_f)\to 0,$$

where, under the basis of $H_2(F_i)$ and the Alexander dual basis of $H_2(Y)$, φ is given by $(A_{ij}-t_iA'_{ij})$. Therefore the $\mathbb{Z}[F_n]$–module $H_2(\widetilde{X}_f)$ is determined by $D=(A_{ij})$. Also we see that the map $H_2(F_i)\to H_2(\widetilde{X}_f)$ is determined by $D=(A_{ij})$, hence the dual map $H^2(\widetilde{X}_f;\mathbb{Q})\to H^2(F_i;\mathbb{Q})$. And the intersection form of F_i is given by $A_{ii}+A'_{ii}$. It's easy to see from the definition that the bilinear pairing b_i is given by the composition of $H^2(\widetilde{X}_f;\mathbb{Q})\to H^2(F_i;\mathbb{Q})$ with the intersection form on $H^2(F_i;\mathbb{Q})$. Therefore the bilinear form b_i is determined by the Seifert matrix $D=(A_{ij})$.

Given two simple boundary 3–links L_0 and L_1, with l–equivalent Seifert matrices $D_0=(A^{(0)}_{ij})$ and $D_1=(A^{(1)}_{ij})$, then by [12, Lemma 1] we may choose Seifert manifolds $\{F^0_i\}$ and $\{F^1_i\}$ of L_0 and L_1, respectively, such that the corresponding Seifert matrices are equal. Then, by the above discussion, L_0 and L_1 are equivalent.

Using a stabilization trick introduced by Levine in the case of knots, we can extend the construction of links with given Seifert matrix in [12] to the case $q=2$. The construction goes as follows.

Firstly, by [11, Lemma 16] we may find embeddings $F^4_i\subset B^5_i\subset S^5$ with $\partial F_i=S^3$ a simple 3–knot whose Seifert matrix A_i is S–equivalent to A_{ii}. After stabilization by connected sum with copies of $S^2\times S^2$, these Seifert manifolds F^4_i are diffeomorphic to connected sums of $S^2\times S^2$ and the Kummer surface with a 4–ball B^4 deleted. These smooth 4–manifolds all have a handle decomposition of the form $F_i = D^4 \cup h_1 \cup \cdots \cup h_k$ where the h_i are 2–handles (see eg [14]). Then by the same argument in the proof of [12, Theorem 1] we can show that the new Seifert matrix D', which is l–equivalent to D, is the Seifert matrix of a boundary simple 3–link L. □

Now we describe the Milnor pairing associated to an $m \times m$ integral matrix $D = (A_{ij})$ satisfying the unimodularity conditions. Let $\varphi_D \colon (\mathbb{Z}[F_n])^m \to (\mathbb{Z}[F_n])^m$ be the $\mathbb{Z}[F_n]$–module map given by the matrix $(A_{ij} - t_i A'_{ij})$. Assume the square matrix A_{ii} has dimension m_i; then $A_{ii} + A'_{ii}$ defines a symmetric bilinear form I_i on $\mathbb{Z}^{m_i}$. Let ι_i be the composition

$$\iota_i \colon \mathbb{Z}^{m_i} \xrightarrow{\bigoplus_j A_{ij}} \bigoplus_j \mathbb{Z}^{m_j} \to \bigoplus_j \mathbb{Z}^{m_j} \otimes_{\mathbb{Z}} \mathbb{Z}[F_n] = (\mathbb{Z}[F_n])^m \to \operatorname{coker} \varphi_D$$

The $\mathbb{Q}$–dual of ι_i is $\iota_i^* \colon (\operatorname{coker} \varphi_D)^* \to \mathbb{Q}^{m_i}$. Let $C_1 = (\mathbb{Z}[F_n])^n \xrightarrow{d_1} C_0 = \mathbb{Z}[F_n]$ be the standard chain complex computing $H_*(BF_n; \mathbb{Z}[F_n])$, $\{e_i \mid i = 1, \dots, n\}$ be the standard basis of $(\mathbb{Z}[F_n])^n$, $\{e_i^* \mid i = 1, \dots, n\}$ be the dual basis, and $[e_i^*] \in H^1(BF_n; \mathbb{Z}[F_n])$ be the corresponding cohomology class. Then the bilinear form

$$b_D \colon (\operatorname{coker} \varphi_D)^* \times (\operatorname{coker} \varphi_D)^* \to (H^1(BF_n; \mathbb{Q}[F_n]))^*$$

is given by $\langle b_D(u, v), [e_i^*] \rangle = I_i(\iota_i^*(u), \iota_i^*(v))$. (See Lemma 3.1.)

Proof of Theorem 1.3 There is a surjective map from the set of isotopy classes of simple boundary n–components links $L \subset S^5$ to the set of diffeomorphism classes of smooth oriented closed 5–manifolds M^5 with free fundamental group of rank n and $H_2(M; \mathbb{Z}) = 0$. This is given by surgery: given a link L, we may do surgery on L and obtain a 5–manifold M with $H_2(M; \mathbb{Z}) = 0$. If L is simple boundary, then it's easy to see that $\pi_1(M)$ is isomorphic to F_n. The meridians of the link components form an embedding $\bigcup_n S^1 \subset M$, and these circles generate $\pi_1(M)$. On the other hand, given such an M^5 we may choose an embedding $\bigcup_n S^1 \subset M^5$ such that the circle generate $\pi_1(M)$. Then we do surgery on this embedding and obtain S^5; the complementary spheres $\bigcup_n S^3 \subset S^5$ form a link L. Clearly this is a simple boundary link.

By comparing the definitions, we see that the generalized Milnor pairing $\varphi(M)$ of M is the same as the generalized Milnor pairing $\psi(X_f)$ of the link complement defined before Lemma 4.1. In the proof of Theorem 1.2 we have shown how the generalized Milnor pairing $\psi(X_f)$ is determined by the Seifert matrix $D = (A_{ij})$. This is exactly $(F_n, \operatorname{coker} \varphi_D, b_D)$, which was described before the proof of Theorem 1.3. By Theorem 1.2, all such matrices satisfying the unimodular conditions are realized by simple boundary links. This finishes the proof. □

Appendix

In this appendix we show some basic properties of the class of manifolds mentioned in Remark 1.4, ie oriented closed 5–manifolds M with $\pi_1(M) \cong \mathbb{Z}$ and $\pi_2(M)$ a finitely generated abelian group.

Lemma A.1 *Let M^5 be a 5–manifold with $\pi_1(M) = \mathbb{Z}$ and $\pi_2(M)$ a finitely generated abelian group; then all higher homotopy groups $\pi_i(M)$ for $i \geq 2$ are finitely generated abelian groups.*

Proof By Serre's mod $\mathscr{C}$ theory [16], we only need to show that $H_i(\widetilde{M})$ for $i \geq 3$ are finitely generated abelian groups. The only problem is $H_3(\widetilde{M})$. We have $H_3(\widetilde{M}) = H_3(M;\Lambda) \cong H^2(M;\Lambda)$, where $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t,t^{-1}]$ is the group ring. By [19, Proposition 3.3], the CW–structure of M has the form

$$M = S^1 \vee \Big(\bigvee S^2\Big) \cup \cdots .$$

Therefore the cellular chain complex $C_*(M;\Lambda)$ has the form

$$\cdots \to C_3 \xrightarrow{d} C_2 \xrightarrow{0} C_1 \to C_0.$$

From the exact sequence $C_3 \xrightarrow{d} C_2 \to \operatorname{coker} d \to 0$ we have the dual exact sequence $0 \to (\operatorname{coker} d)^* \to C_2^* \xrightarrow{d^*} C_3^*$, hence $H^2(M;\Lambda) = \ker d^* = (\operatorname{coker} d)^*$. Now $\operatorname{coker} d = H_2(M;\Lambda) = \pi_2(M)$ is a finitely generated abelian group; the proof is done given the following lemma. □

Lemma A.2 *If a Λ–module G is a finitely generated abelian group, then*

$$\operatorname{Hom}_\Lambda(G,\Lambda) = 0.$$

Proof The torsion subgroup T is a sub-Λ–module and the exact sequence

$$0 \to T \to G \to G/T \to 0$$

induces an exact sequence

$$0 \to \operatorname{Hom}_\Lambda(G/T,\Lambda) \to \operatorname{Hom}_\Lambda(G,\Lambda) \to \operatorname{Hom}_\Lambda(T,\Lambda),$$

therefore $\operatorname{Hom}_\Lambda(G/T,\Lambda) \cong \operatorname{Hom}_\Lambda(G,\Lambda)$ since $\operatorname{Hom}_\Lambda(T,\Lambda) = 0$. Therefore we may assume that G is a finitely generated free abelian group.

Let $x_1,\dots,x_n$ be a basis of G; a Λ–module structure on G is given by $A \in \mathrm{GL}_n(\mathbb{Z})$, which specifies the action of the generator t on the basis. A Λ–homomorphism $G \to \Lambda$ is given by the images $v_1,\dots,v_n \in \Lambda$ of $x_1,\dots,x_n$. The n–tuple $v = (v_1,\dots,v_n)$ should satisfy the equation $(tI - A)v = 0$. Clearly $\det(tI - A) \neq 0$, thus the equation has no nonzero solution in the quotient field (Λ is an integral domain), hence also has no nonzero solution in Λ. Therefore $\mathrm{Hom}_\Lambda(G,\Lambda)=0$. □

Now let M^5 be a closed orientable 5–manifold with $\pi_1(M)=\mathbb{Z}$ and $\pi_2(M)$ a finitely generated abelian group. Fix an orientation of M and a generator t of $\pi_1(M)$; these choices determine a generator (a fundamental class) $\sigma_M\in H_4(\widetilde{M})=\mathbb{Z}$. Then, on the finitely generated free abelian group $H^2(\widetilde{M})$, a symmetric bilinear form $H^2(\widetilde{M})\times H^2(\widetilde{M})\to\mathbb{Z}$ is defined by $(\alpha,\beta)\mapsto\langle\alpha\cup\beta,\sigma_M\rangle$. By the following proposition, we see that this bilinear form is unimodular and $\pi_2(M)$ is a free abelian group. Thus this bilinear form induces a symmetric bilinear form on $\pi_2(M)=\pi_2(\widetilde{M})=H_2(\widetilde{M})=H^2(\widetilde{M})^*$, denoted by $I(M)$.

Proposition A.3 *Let M^5 be an orientable 5–manifold with $\pi_1(M)=\mathbb{Z}$ and $\pi_2(M)$ a finitely generated abelian group. Then we have the following:*

(1) *$\pi_2(M)$ is torsion-free.*

(2) *The symmetric bilinear form $I(M)$ is unimodular; $I(M)$ is even if and only if $w_2(M)=0$.*

(3) *$\langle p_1(M),\sigma_M\rangle=3\cdot\mathrm{sign}(I(M))$, where $\sigma_M\in H_4(\widetilde{M})$ is the generator determined by the orientation of M and the generator t of $\pi_1(M)$.*

Proof Consider $M\times\mathbb{C}\mathrm{P}^2$. By Lemma A.1 and Browder and Levine's fibration theorem [1], we know that this manifold is a fiber bundle over S^1 with simply connected fiber F^8. Therefore $\widetilde{M}\times\mathbb{C}\mathrm{P}^2$ is homotopy equivalent to F.

(1) By the Künneth formula and Poincaré duality, we have

$$H^3(\widetilde{M})\cong H^7(\widetilde{M}\times\mathbb{C}\mathrm{P}^2)\cong H^7(F)\cong H_1(F)=0.$$

This proves that $\mathrm{tors}\,\pi_2(M)=\mathrm{tors}\,H_2(\widetilde{M})=\mathrm{tors}\,H^3(\widetilde{M})=0$.

(2) On $H^4(\widetilde{M}\times\mathbb{C}\mathrm{P}^2)$ there is defined a symmetric bilinear form $I(M\times\mathbb{C}\mathrm{P}^2)$, which is isometric to the tensor product of $I(M)$ and the intersection form of $\mathbb{C}\mathrm{P}^2$ plus a hyperbolic form. On the other hand, the bilinear form $I(M\times\mathbb{C}\mathrm{P}^2)$ is isometric to the intersection form of F, which is unimodular by Poincaré duality. Therefore the bilinear form $I(M)$ is unimodular.

From the discussion above we see that $I(M)$ is even if and only if the Wu class $v_4(F)$ is zero. The Wu classes and Stiefel–Whitney classes of M and F are related as follows. Let $i\colon F\to M\times\mathbb{C}\mathrm{P}^2$ be the inclusion of the fiber; then $TF\oplus\underline{\mathbb{R}}=i^*T(M\times\mathbb{C}\mathrm{P}^2)$. We have

$$w_2(M)=v_2(M),\quad w_3(M)=\mathrm{Sq}^1w_2(M),\quad w_4(M)=w_2(M)^2,$$
$$v_2(F)=w_2(F)=i^*(w_2(M)+w_2(\mathbb{C}\mathrm{P}^2)),\quad w_3(F)=\mathrm{Sq}^1w_2(F)+v_3(F);$$

on the other hand, $w_3(F) = i^* w_3(M)$, from which we have

$$v_3(F) = i^*(\mathrm{Sq}^1 w_2(M) + w_3(M)).$$

By the Wu formula

$$w_4(F) = v_2(F)^2 + \mathrm{Sq}^1 v_3(F) + v_4(F);$$

on the other hand,

$$w_4(F) = i^*(w_4(M) + w_2(M) w_2(\mathbb{C}\mathrm{P}^2) + w_4(\mathbb{C}\mathrm{P}^2)).$$

Comparing these two equations we have

$$v_4(F) = i^*(w_2(M) w_2(\mathbb{C}\mathrm{P}^2)).$$

But $H^3(F;\mathbb{Z}_2) \cong H^3(\widetilde{M} \times \mathbb{C}\mathrm{P}^2;\mathbb{Z}_2) \cong H^3(\widetilde{M};\mathbb{Z}_2) = 0$ (the last identity is a consequence of the fact that $H_2(\widetilde{M})$ is free and $H_3(\widetilde{M}) = 0$; see Lemma A.1). From the Wang sequence we see that $i^*\colon H^4(M \times \mathbb{C}\mathrm{P}^2;\mathbb{Z}_2) \to H^4(F;\mathbb{Z}_2)$ is injective. Thus $v_4(F) = 0$ if and only if $w_2(M) = 0$.

(3) Since $I(M)$ and $I(M \times \mathbb{C}\mathrm{P}^2)$ differ by a hyperbolic form, we have

$$\mathrm{sign}(I(M)) = \mathrm{sign}(I(M \times \mathbb{C}\mathrm{P}^2)) = \mathrm{sign}(F) = \tfrac{1}{45}\langle 7p_2(F) - p_1(F)^2, [F]\rangle,$$

where the last identity is the Hirzebruch index formula. Since F has trivial normal bundle in $M \times \mathbb{C}\mathrm{P}^2$, we have

$$p_1(F) = i^* p_1(M \times \mathbb{C}\mathrm{P}^2) = i^*(p_1(M) + p_1(\mathbb{C}\mathrm{P}^2)),$$
$$p_2(F) = i^* p_2(M \times \mathbb{C}\mathrm{P}^2) = i^*(p_1(M) p_1(\mathbb{C}\mathrm{P}^2)).$$

A straightforward calculation shows that $3\,\mathrm{sign}(I(M)) = \langle p_1(M), \sigma_M\rangle$. □

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