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On the Fano variety of linear spaces contained in two odd-dimensional quadrics

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We describe the geometry of the $2m$–dimensional Fano manifold G parametrizing $(m-1)$–planes in a smooth complete intersection Z of two quadric hypersurfaces in the complex projective space $\mathbb{P}^{2m+2}$ for $m \geq 1$. We show that there are exactly 2^{2m+2} distinct isomorphisms in codimension one between G and the blow-up of $\mathbb{P}^{2m}$ at $2m+3$ general points, parametrized by the 2^{2m+2} distinct m–planes contained in Z, and describe these rational maps explicitly. We also describe the cones of nef, movable and effective divisors of G, as well as their dual cones of curves. Finally, we determine the automorphism group of G.

These results generalize to arbitrary even dimension the classical description of quartic del Pezzo surfaces ($m=1$).

14E30, 14J45; 14M15, 14N20, 14E05

1 Introduction

In this paper we describe the geometry of the $2m$–dimensional Fano manifold $G^{(2m)}$ parametrizing $(m-1)$–planes in a smooth complete intersection of two quadric hypersurfaces in the complex projective space $\mathbb{P}^{2m+2}$ for $m \geq 1$. The case $m=1$ is classical:

1.1 The surface $S = G^{(2)}$ is itself a smooth complete intersection of two quadric hypersurfaces in $\mathbb{P}^4$, and hence a quartic del Pezzo surface. It is well-known that $\rho(S) = 6$, and that the cone of effective curves of S is generated by the classes of its 16 lines. These 16 lines have a very special incidence relation: each line intersects properly exactly 5 lines. The del Pezzo surface S can also be described as the blow-up of $\mathbb{P}^2$ at 5 points in general linear position. In fact, there are 16 different ways to realize S as this blow-up: For every line $\ell \subset S$, there is a birational morphism $\pi_\ell\colon S \to \mathbb{P}^2$, unique up to projective transformation of $\mathbb{P}^2$, contracting the 5 lines incident to ℓ to points $p_1^\ell, \dots, p_5^\ell \in \mathbb{P}^2$ in general linear position. The image of ℓ under π_ℓ is the unique conic through the p_i, and the image of the other 10 lines are the 10 lines through 2 of the p_i. Moreover, for any two lines $\ell, \ell' \subset S$, the sets of points $\{p_1^\ell, \dots, p_5^\ell\}$ and $\{p_1^{\ell'}, \dots, p_5^{\ell'}\}$ are related by a projective transformation of $\mathbb{P}^2$.

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The automorphism group $\mathrm{Aut}(S)$ of S is also well understood (see for instance Dolgachev [12, Section 8.6.4]). In order to describe it, we view $\mathrm{Pic}(S)$ with the intersection product as a unimodular lattice. Its primitive sublattice $K_S^{\perp}$ is a D_5–lattice. We denote by $W(D_5)$ the Weyl group of automorphisms of this lattice. For any $\zeta \in \mathrm{Aut}(S)$, the induced isomorphism $\zeta^*\colon \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ preserves the intersection product and fixes K_S. This yields an inclusion of groups $\mathrm{Aut}(S) \hookrightarrow W(D_5) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, whose image contains the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^4$. Moreover, if S is general, then $\mathrm{Aut}(S) \cong (\mathbb{Z}/2\mathbb{Z})^4$.

We will show that the picture described in Section 1.1 above generalizes to arbitrary even dimension. We start by fixing some notation. Let m be a positive integer, set $n = 2m$ and fix $n+3$ distinct points in $\mathbb{P}^1$, up to order and projective equivalence,

$$(\lambda_1 : 1), \dots, (\lambda_{n+3} : 1) \in \mathbb{P}^1.$$

With this fixed data, we introduce the two main characters of this paper, $G^{(n)}$ and $X^{(n)}$:

1.2 ($G^{(n)}$) Let $Z^{(n)}$ be a smooth complete intersection of the two quadric hypersurfaces in $\mathbb{P}^{n+2}$

$$Q_1 : \sum_{i=1}^{n+3} x_i^2 = 0 \quad \text{and} \quad Q_2 : \sum_{i=1}^{n+3} \lambda_i x_i^2 = 0.$$

(Up to projective transformation of $\mathbb{P}^{n+2}$, any smooth complete intersection of two quadric hypersurfaces can be written in this way; see Section 2.) Then consider the subvariety $G^{(n)}$ of the Grassmannian $\mathrm{Gr}(m-1, \mathbb{P}^{n+2})$ parametrizing $(m-1)$–planes contained in $Z^{(n)}$. It is well known that $G^{(n)}$ is a smooth n–dimensional Fano variety with Picard number $\rho(G^{(n)}) = n+4$ (see Section 3 and references therein).

1.3 ($X^{(n)}$) Fix a Veronese embedding $\nu_n\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$, and set $p_i = \nu_n((\lambda_i : 1)) \in \mathbb{P}^n$. The points $p_1, \dots, p_{n+3}$ are in general linear position. (In fact, this gives a natural correspondence between sets of $n+3$ distinct points in $\mathbb{P}^1$, up to projective equivalence, and $n+3$ points in general linear position in $\mathbb{P}^n$, up to projective equivalence.) Let $X^{(n)}$ be the blow-up of $\mathbb{P}^n$ at the points $p_1, \dots, p_{n+3}$.

Our starting point is the following:

1.4 Theorem (Bauer [3], Casagrande [8]) *The varieties $G^{(n)}$ and $X^{(n)}$ are isomorphic in codimension* 1*.*

The proof of Theorem 1.4 makes use of moduli spaces of parabolic vector bundles. By [8], $G^{(n)}$ is isomorphic to the moduli space $\mathcal{M}^{(n)}$ of stable rank 2 parabolic vector bundles on $(\mathbb{P}^1, (\lambda_1 : 1), \dots, (\lambda_{n+3} : 1))$ of degree zero and weights $\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$. On the other hand, by [3] (see also Mukai [22, Theorem 12.56]), $X^{(n)}$ is isomorphic to the moduli space of stable rank 2 parabolic vector bundles on $(\mathbb{P}^1, (\lambda_1 : 1), \dots, (\lambda_{n+3} : 1))$ of degree zero and weights $\left(\frac{1}{n}, \dots, \frac{1}{n}\right)$, which is isomorphic to $\mathcal{M}^{(n)}$ in codimension 1.

This proof, however, does not give much information about the possible isomorphisms in codimension 1 between $G^{(n)}$ and $X^{(n)}$. We call an isomorphism in codimension 1 a *pseudoisomorphism*. In this paper we describe explicitly the birational maps $G^{(n)} \dashrightarrow \mathbb{P}^n$ inducing a pseudoisomorphism $G^{(n)} \dashrightarrow X^{(n)}$. As we shall see, up to automorphism of $\mathbb{P}^n$, there are exactly 2^{n+2} distinct such birational maps, parametrized by the 2^{n+2} linear copies of $\mathbb{P}^m$ contained in $Z^{(n)}$. In order to state this precisely, we need to recall some facts about $Z^{(n)}$ (see Section 2 and references therein).

The set $\mathcal{F}_m(Z^{(n)})$ of m–planes in $Z^{(n)}$ has cardinality 2^{n+2}. For each $i = 1, \dots, n+3$, consider the involution $\sigma_i\colon Z^{(n)} \to Z^{(n)}$ switching the sign of the coordinate x_i. The group generated by these involutions is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n+2}$, and acts on $\mathcal{F}_m(Z^{(n)})$ freely and transitively. For every subset $I \subseteq \{1, \dots, n+3\}$, we set $\sigma_I := \prod_{i \in I} \sigma_i = \prod_{j \in I^c} \sigma_j$. For every $M \in \mathcal{F}_m(Z^{(n)})$ and $I \subset \{1, \dots, n+3\}$ with $|I| \le m+1$, we have $\dim(M \cap \sigma_I(M)) = m - |I|$. Consider the incidence variety $\mathcal{I} := \{([L], p) \in G^{(n)} \times Z^{(n)} \mid p \in L\}$ and the associated diagram:

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\pi"'] \arrow[dr, "e"] & \\ G^{(n)} & & Z^{(n)} \end{tikzcd}$$

We show that for every m–plane $M \in \mathcal{F}_m(Z^{(n)})$, $E_M := \pi_*(e^*(M))$ is the class of a unique prime divisor on $G^{(n)}$, which we denote by the same symbol (see Proposition 5.5).

Now we can state our main result. See Theorem 5.7 for more details, including explicit descriptions of the linear systems on $G^{(n)}$ defining the birational maps $G^{(n)} \dashrightarrow \mathbb{P}^n$.

1.5 Theorem (Theorem 5.7 and Corollary 5.8) *Let $M \in \mathcal{F}_m(Z^{(n)})$, in the notation above. Up to a unique permutation of the p_i, there is a unique birational map $\rho_M\colon G^{(n)} \dashrightarrow \mathbb{P}^n$, inducing a pseudoisomorphism $G^{(n)} \dashrightarrow X^{(n)}$, with the following properties:*

- *The image of E_M under ρ_M is $\mathrm{Sec}_{m-1}(C)$, the $(m-1)^{st}$ secant variety of the unique rational normal curve C through $p_1, \dots, p_{n+3}$ in $\mathbb{P}^n$.*
- *The map ρ_M contracts $E_{\sigma_i(M)}$ to the point $p_i \in \mathbb{P}^n$.*
- *For each $I \subseteq \{1, \dots, n+3\}$ of even cardinality $|I| \le n$, the image of $E_{\sigma_I(M)}$ under ρ_M is the join of $\langle p_i \rangle_{i \in I}$ and $\mathrm{Sec}_{s-1}(C)$, where $s = \frac{1}{2}(n - |I|)$.*

Moreover, any pseudoisomorphism between $G^{(n)}$ and any blow-up $\widetilde{X}$ of $\mathbb{P}^n$ at $n+3$ points is of this form. In particular, $\widetilde{X} \cong X^{(n)}$.

As immediate corollaries of Theorem 1.5, we obtain the following:

1.6 Corollary *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{P}^n$ be subsets of $n+3$ distinct points and let $X_{\mathcal{P}_i}$ be the blow-up of $\mathbb{P}^n$ along $\mathcal{P}_i$ for $i = 1, 2$. Assume that the points in $\mathcal{P}_1$ are in general linear position. Then the following are equivalent:*

(i) $X_{\mathcal{P}_1} \cong X_{\mathcal{P}_2}$.

(ii) $X_{\mathcal{P}_1}$ *and* $X_{\mathcal{P}_2}$ *are pseudoisomorphic.*

(iii) $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are projectively equivalent (as unordered sets).*

1.7 Corollary *Let $\mathcal{S}_i = \{(\lambda_1^i : 1), \dots, (\lambda_{n+3}^i : 1)\} \subset \mathbb{P}^1$ for $i = 1, 2$ be subsets of $n+3$ distinct points. For each $i \in \{1, 2\}$, let $Z_{\mathcal{S}_i} \subset \mathbb{P}^{n+2}$ be the smooth complete intersection of the two quadrics*

$$Q_1 : \sum_{j=1}^{n+3} x_j^2 = 0 \quad \text{and} \quad Q_2^i : \sum_{j=1}^{n+3} \lambda_j^i x_j^2 = 0,$$

and let $G_{\mathcal{S}_i}$ be the variety of $(m-1)$–planes contained in $Z_{\mathcal{S}_i}$. Then the following are equivalent:

(i) $G_{\mathcal{S}_1} \cong G_{\mathcal{S}_2}$.

(ii) $G_{\mathcal{S}_1}$ *and* $G_{\mathcal{S}_2}$ *are pseudoisomorphic.*

(iii) $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are projectively equivalent (as unordered sets).*

Notice that Corollary 1.6 is a classical result, originally due to Coble (see Dolgachev and Ortland [14]). See also Biswas, Holla and Kumar [4] for a result related to Corollary 1.7, in terms of moduli spaces of rank 2 parabolic vector bundles on $\mathbb{P}^1$.

To prove Theorem 1.5, we determine the nef cone of $G^{(n)}$ explicitly, and then compare it with the Mori chamber decomposition of the effective cone of $X^{(n)}$ described by Mukai [23]. This decomposition encodes the nef cones of all varieties pseudoisomorphic to $X^{(n)}$. In order to determine the cone of effective curves and the nef cone of $G^{(n)}$, we generalize to arbitrary dimension a construction of Borcea [6] in dimension $n = 4$. We define isomorphisms

$$H^{2n-2}(G^{(n)}, \mathbb{Z}) \xrightarrow{\alpha} H^n(Z^{(n)}, \mathbb{Z}) \xrightarrow{\beta} H^2(G^{(n)}, \mathbb{Z})$$

such that $\beta(M) = E_M$ and $\alpha^{-1}(M)$ is the class of a line on the dual m–plane $M^* \subset G^{(n)}$ for every $M \in \mathcal{F}_m(Z^{(n)})$. These isomorphisms are dual with respect to the intersection products, ie $x \cdot \beta(y) = \alpha(x) \cdot y$ for every $x \in H^{2n-2}(G^{(n)}, \mathbb{Z})$ and $y \in H^n(Z^{(n)}, \mathbb{Z})$. They allow us to describe explicitly special cones of curves and divisors on $G^{(n)}$:

1.8 Theorem (Theorem 5.1 and Proposition 5.5) *Let $\mathcal{E} \subset H^n(Z,\mathbb{R})$ be the polyhedral cone generated by the classes $\{M\}_{M\in\mathcal{F}_m(Z)}$, and denote by $\mathcal{E}^\vee \subset H^n(Z,\mathbb{R})$ its dual cone. Then $\mathcal{E}^\vee \subset \mathcal{E}$, and the cones of nef and effective divisors of $G^{(n)}$ and their dual cones of effective and moving curves satisfy*

$$\mathrm{Nef}(G^{(n)}) = \beta(\mathcal{E}^\vee) \subset \beta(\mathcal{E}) = \mathrm{Eff}(G^{(n)}),$$
$$\mathrm{Mov}_1(G^{(n)}) = \alpha^{-1}(\mathcal{E}^\vee) \subset \alpha^{-1}(\mathcal{E}) = \mathrm{NE}(G^{(n)}).$$

We give a geometric description of the extremal rays and facets of these cones, and the associated contractions in Section 6. In Proposition 6.6 and its following subsection, we also describe the cone $\mathrm{Mov}^1(G^{(n)})$ of movable divisors of $G^{(n)}$, and give a geometric description of the curves corresponding to its facets.

We end this paper by determining the automorphism group of the Fano variety $G^{(n)}$, generalizing the description of the automorphism group of a quartic del Pezzo surface in Section 1.1. In what follows, we write $W(D_{n+3})$ for the Weyl group of automorphisms of a D_{n+3}–lattice, and we denote by the same symbol the involution of $G^{(n)}$ induced by the involution σ_i of $Z^{(n)}$.

1.9 Proposition (Proposition 7.1) *There is an inclusion of groups*

$$\mathrm{Aut}(G^{(n)}) \hookrightarrow W(D_{n+3}) \cong (\mathbb{Z}/2\mathbb{Z})^{n+2} \rtimes S_{n+3},$$

whose image contains the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^{n+2}$ generated by the involutions σ_i of $G^{(n)}$.

Moreover, if the points $(\lambda_1 : 1), \dots, (\lambda_{n+3} : 1) \in \mathbb{P}^1$ are general, then $\mathrm{Aut}(G^{(n)}) \cong (\mathbb{Z}/2\mathbb{Z})^{n+2}$.

This paper is organized as follows. Section 2 is dedicated to smooth complete intersections $Z \subset \mathbb{P}^{n+2}$ for $n = 2m$ of two quadric hypersurfaces in even-dimensional projective spaces. In particular, we investigate the set $\mathcal{F}_m(Z)$ of m–planes in Z, and the cone it spans in $H^n(Z,\mathbb{R})$. In Section 3, we address the Fano variety G of $(m-1)$–planes in Z. We construct the isomorphisms $H^{2n-2}(G,\mathbb{Z}) \xrightarrow{\alpha} H^n(Z,\mathbb{Z}) \xrightarrow{\beta} H^2(G,\mathbb{Z})$, and determine some extremal rays of the cone of effective curves of G. In Section 4, we consider the blow-up X of $\mathbb{P}^n$ at $n+3$ points in general linear position. We describe the Mori chamber decomposition of $\mathrm{Eff}(X)$, following Mukai [23] and Bauer [3]. From this we can write the nef cone of G in terms of a natural basis for $\mathcal{N}^1(X)$. In Section 5, we put together the results from the previous sections to prove Theorem 1.5. In Section 6, we study cones of curves and divisors in G, giving a geometric description of their facets and extremal rays. In Section 7, we describe the automorphism group of the Fano variety G.

Notation and conventions We always work over the field $\mathbb{C}$ of complex numbers.

Given a subvariety $Z \subset \mathbb{P}^n$ and a nonnegative integer $d < n$, we denote by $\mathcal{F}_d(Z)$ the closed subset of the Grassmannian $\mathrm{Gr}(d, \mathbb{P}^n)$ parametrizing d–planes contained in Z.

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2 Smooth complete intersections of two quadrics

In this section we describe the geometry of smooth complete intersections of two quadric hypersurfaces in even dimensional complex projective spaces. Many of the results are well known and can be found in Reid [25, Chapter 3] or Borcea [6, Section 1], to which we refer for details and proofs. See also the recent paper by Dolgachev and Duncan [13] for a study of these complete intersections over a field of characteristic 2.

Let $n = 2m \geq 2$ be an even integer, and let $Z = Q_1 \cap Q_2 \subset \mathbb{P}^{n+2}$ be a smooth complete intersection of two quadric hypersurfaces. Up to a projective transformation of $\mathbb{P}^{n+2}$, we can assume that the quadrics have equations

$$(2.1) \qquad Q_1 : \sum_{i=1}^{n+3} x_i^2 = 0, \quad Q_2 : \sum_{i=1}^{n+3} \lambda_i x_i^2 = 0,$$

with $\lambda_i \neq \lambda_j$ if $i \neq j$. Thus Z is determined by $n+3$ distinct points

$$(\lambda_1 : 1), \dots, (\lambda_{n+3} : 1) \in \mathbb{P}^1.$$

Acting on these points by permutations and projective automorphisms of $\mathbb{P}^1$ yields projectively isomorphic varieties $Z \subset \mathbb{P}^{n+2}$.

2.2 (involutions and double covers) For each $i = 1, \dots, n+3$, let $\sigma_i \colon Z \to Z$ be the involution switching the sign of the coordinate x_i. Then $\sigma_1, \dots, \sigma_{n+3}$ commute and have the unique relation $\sigma_1 \cdots \sigma_{n+3} = \mathrm{Id}_Z$, so they generate a subgroup W' of $\mathrm{Aut}(Z)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n+2}$. For every subset $I \subseteq \{1, \dots, n+3\}$, we set $\sigma_I := \prod_{i \in I} \sigma_i$. Notice that $\sigma_I = \sigma_{I^c}$.

For each $i = 1, \dots, n+3$, the projection from the i^{th} coordinate point in $\mathbb{P}^{n+2}$ yields a double cover $\pi_i \colon Z \to Q^n$, where $Q^n \subset \mathbb{P}^{n+1}$ is the smooth quadric having equation $\sum_{j \neq i} (\lambda_j - \lambda_i) x_j^2 = 0$ for projective coordinates $(x_1 : \cdots : \hat{x}_i : \cdots : x_{n+3})$ in $\mathbb{P}^{n+1}$. The involution associated to this double cover is σ_i.

2.3 (the set of m–planes in Z) Consider the set $\mathcal{F}_m(Z)$ of m–planes in Z. It is a finite set with cardinality 2^{n+2}. The group W' generated by the involutions σ_i acts on $\mathcal{F}_m(Z)$ freely and transitively.

For every $M \in \mathcal{F}_m(Z)$ and $I \subset \{1, \dots, n+3\}$ with $|I| \leq m+1$, we have

$$\dim(M \cap \sigma_I(M)) = m - |I|. \tag{2.4}$$

2.5 For each $i = 1, \dots, n+3$, the double cover $\pi_i \colon Z \to Q^n$ induces a map

$$\mathcal{F}_m(Z) \to \mathcal{F}_m(Q^n).$$

Recall that $\mathcal{F}_m(Q^n)$ has two connected components T^φ and T^ψ, and that two m–planes $\Lambda, \Lambda' \subset Q^n$ belong to the same connected component if and only if $\dim(\Lambda \cap \Lambda') \equiv m \bmod 2$ (see for instance Reid [25, Theorem 1.2(b)] or Harris [17, Theorem 22.14]).

Let $M \in \mathcal{F}_m(Z)$. We have $\pi_i(\sigma_i(M)) = \pi_i(M)$. On the other hand, if j is in $\{1, \dots, n+3\} \setminus \{i\}$, then M and $\sigma_j(M)$ intersect in codimension one, by (2.4), and the same holds for $\pi_i(M)$ and $\pi_i(\sigma_j(M))$. Therefore $\pi_i(M)$ and $\pi_i(\sigma_j(M))$ belong to different connected components of $\mathcal{F}_m(Q^n)$. In general, if $I \subseteq \{1, \dots, n+3\}$ does not contain i, then $\pi_i(M)$ and $\pi_i(\sigma_I(M))$ belong to the same connected component of $\mathcal{F}_m(Q^n)$ if and only if $|I|$ is even. This shows that the image of $\mathcal{F}_m(Z)$ in $\mathcal{F}_m(Q^n)$ consists of 2^{n+1} points, half in each connected component.

2.6 (the cohomology group $H^n(Z, \mathbb{Z})$) The cohomology group $H^n(Z, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{n+4}$, and is generated over $\mathbb{Z}$ by the classes of the m–planes in Z. Moreover $H^n(Z, \mathbb{Z})$ is a unimodular lattice with respect to the intersection form.

For every $M \in \mathcal{F}_m(Z)$ we denote by the same symbol M the corresponding fundamental class in $H^n(Z, \mathbb{Z})$. We denote by $\eta \in H^n(Z, \mathbb{Z})$ the class of a codimension-m linear section of $Z \subset \mathbb{P}^{n+2}$, so that

$$\eta^2 = 4 \qquad \text{and} \qquad \eta \cdot M = 1 \quad \text{for every } M \in \mathcal{F}_m(Z).$$

The sublattice $\eta^\perp$ (namely the primitive part $H^n(Z, \mathbb{Z})_0$) is a D_{n+3}–lattice. We denote by $W(D_{n+3})$ its Weyl group of automorphisms, which is generated by the reflections in the roots of $\eta^\perp$. It is the full group of automorphisms of the triple $(H^n(Z, \mathbb{Z}), \cdot, \eta)$, and it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n+2} \rtimes S_{n+3}$.

The group $W' \cong (\mathbb{Z}/2\mathbb{Z})^{n+2}$ generated by the involutions σ_i acts naturally and faithfully on $H^n(Z,\mathbb{Z})$. We still denote by σ_I the involution of $H^n(Z,\mathbb{Z})$ induced by $\sigma_I\colon Z \to Z$. So we view W' as a subgroup of $W(D_{n+3})$. It is a normal subgroup with quotient $W(D_{n+3})/W'$ isomorphic to the symmetric group S_{n+3}.

For every $M \in \mathcal{F}_m(Z)$ and $i,j \in \{1,\dots,n+3\}$ with $i \neq j$, we have

$$\eta = M + \sigma_i(M) + \sigma_j(M) + \sigma_{ij}(M). \tag{2.7}$$

2.8 Notation Fix $M_0 \in \mathcal{F}_m(Z)$. For every $i = 1,\dots,n+3$, we set $M_i := \sigma_i(M_0)$. More generally, for every subset $I \subseteq \{1,\dots,n+3\}$, we set $M_I := \sigma_I(M_0)$. Notice again that $M_I = M_{I^c}$. We also set

$$\varepsilon_i := M_0 + M_i - \tfrac{1}{2}\eta \in H^n(Z,\mathbb{R}) \quad \text{for every } i = 1,\dots,n+3. \tag{2.9}$$

Then $\{\eta,\varepsilon_1,\dots,\varepsilon_{n+3}\}$ is an orthogonal basis for $H^n(Z,\mathbb{R})$, which is useful for computations. We have

$$\eta^2 = 4 \qquad \text{and} \qquad \varepsilon_i^2 = (-1)^m \quad \text{for every } i = 1,\dots,n+3. \tag{2.10}$$

In particular, the intersection form on $H^n(Z,\mathbb{R})$ is positive definite when $n \equiv 0 \bmod 4$, and has signature $(1,n+3)$ when $n \equiv 2 \bmod 4$. Notice that this basis depends on the choice of M_0.

Let $G_0 \subset W(D_{n+3})$ be the stabilizer of M_0. Then $G_0 \cong S_{n+3}$ and G_0 acts by (the same) permutations both on $\{M_1,\dots,M_{n+3}\}$ and on $\{\varepsilon_1,\dots,\varepsilon_{n+3}\}$. We have $W(D_{n+3}) = W' \rtimes G_0$. Moreover, for every $I \subseteq \{1,\dots,n+3\}$ of even cardinality, we have

$$\sigma_I(\varepsilon_i) = \begin{cases} \varepsilon_i & \text{if } i \notin I, \\ -\varepsilon_i & \text{if } i \in I. \end{cases} \tag{2.11}$$

Thus we see the usual action of $W(D_{n+3})$ on the linear span of $\varepsilon_1,\dots,\varepsilon_{n+3}$ by permutation and even sign changes of $\varepsilon_1,\dots,\varepsilon_{n+3}$ (see for instance Humphreys [19, Section 12.1]).

We collect some identities in $H^n(Z,\mathbb{R})$ that we will use in later computations.

$$M_I = \frac{1}{4}\eta + \frac{(-1)^{|I|}}{2}\Bigg(\sum_{j\notin I}\varepsilon_j - \sum_{i\in I}\varepsilon_i\Bigg) \quad \text{for every } I \subseteq \{1,\dots,n+3\}, \tag{2.12}$$

$$M_I = \frac{1}{n+1}\Bigg((n+2-|I|)\Bigg(\frac{1}{2}\eta - \sum_{i\in I} M_i\Bigg) + (|I|-1)\sum_{j\in I^c} M_j\Bigg) \tag{2.13}$$

for every $I \subseteq \{1,\dots,n+3\}$ with even cardinality,

$$\varepsilon_i = \frac{1}{2(n+1)}\eta - \frac{1}{n+1}\sum_{j=1}^{n+3} M_j + M_i \quad \text{for every } i = 1, \dots, n+3. \tag{2.14}$$

Our next goal is to describe the polyhedral cone $\mathcal{E}$ in $H^n(Z,\mathbb{R})$ generated by the classes of m–planes in Z. As we shall see below, this is a cone over a $(n+3)$–dimensional *demihypercube*. Before we start discussing the cone $\mathcal{E}$, we gather some results about demihypercubes.

2.15 (the demihypercube) Let $N \geq 4$ be an integer. Write $(\alpha_1, \dots, \alpha_N)$ for coordinates in $\mathbb{R}^N$. The vertices of the hypercube $\left[-\frac{1}{2}, \frac{1}{2}\right]^N \subset \mathbb{R}^N$ are the points of the form $v_I = ((v_I)_1, \dots, (v_I)_N)$, where $I \subseteq \{1, \dots, N\}$, $(v_I)_i = \frac{1}{2}$ if $i \in I$, and $(v_I)_i = -\frac{1}{2}$ otherwise. The parity of the vertex v_I is the parity of $|I|$. For each subset $I \subseteq \{1, \dots, N\}$, define the degree 1 polynomial in the α_i

$$H_I := \sum_{j \notin I}\left(\tfrac{1}{2} + \alpha_j\right) + \sum_{i \in I}\left(\tfrac{1}{2} - \alpha_i\right). \tag{2.16}$$

Notice that, for any two subsets $I, J \subset \{1, \dots, N\}$,

$$H_I(v_J) = \#(I \smallsetminus J) + \#(J \smallsetminus I) \tag{2.17}$$

is the graph distance of v_I and v_J in the skeleton of the hypercube $\left[-\frac{1}{2}, \frac{1}{2}\right]^N$.

The *demihypercube* is the polytope $\Delta \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^N$ generated by the odd vertices of the hypercube. The polytope Δ has $2^{N-1} + 2N$ facets (see for instance Green [15, Lemma 2.3]). More precisely, the polytope Δ is defined in a minimal way by the set of inequalities

$$\Delta = \begin{cases} -\frac{1}{2} \leq \alpha_i \leq \frac{1}{2}, & i \in \{1, \dots, N\}, \\ H_I \geq 1, & |I| \text{ even.} \end{cases} \tag{2.18}$$

Notice that the facets of Δ supported on the hyperplanes $\left(\alpha_i = \pm\frac{1}{2}\right)$ are isomorphic to the $(N-1)$–dimensional demihypercube. In particular, they are not simplicial. On the other hand, the facet supported on the hyperplane $(H_I = 1)$ for $|I|$ even is the $(N-1)$–dimensional simplex generated by the N vertices of $\left[-\frac{1}{2}, \frac{1}{2}\right]^N$ at graph distance 1 to v_I.

The demihypercube can also be described as a weight polytope of the root system of type D_N; see Green [16, Example 8.5.13].

Now we go back to $H^n(Z,\mathbb{R})$ and consider the convex rational polyhedral cone

$$\mathcal{E} := \mathrm{Cone}(M)_{M \in \mathcal{F}_m(Z)} \subset H^n(Z,\mathbb{R}).$$

It is the cone over the $(n+3)$–dimensional polytope

$$\mathcal{E}_0 = \mathrm{Conv}(M)_{M \in \mathcal{F}_m(Z)}$$

obtained by intersecting $\mathcal{E}$ with the affine hyperplane $\mathcal{H} := \{\gamma \mid \gamma \cdot \eta = 1\}$. Note that the Weyl group $W(D_{n+3})$ preserves $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{E}_0$.

We fix $M_0 \in \mathcal{F}_m(Z)$ and consider the orthogonal basis $\{\eta, \varepsilon_1, \dots, \varepsilon_{n+3}\}$ for $H^n(Z, \mathbb{R})$ introduced in (2.9). Then $\frac{1}{4}\eta \in \mathcal{H}$ and $\{\varepsilon_1, \dots, \varepsilon_{n+3}\}$ is a basis for $\eta^\perp$, so that $\left(\frac{1}{4}\eta, \{\varepsilon_1, \dots, \varepsilon_{n+3}\}\right)$ induces affine coordinates

$$(\alpha_1, \dots, \alpha_{n+3}) \tag{2.19}$$

on the hyperplane $\mathcal{H} \cong \mathbb{R}^{n+3}$. With these coordinates, $\frac{1}{4}\eta$ is identified with the origin and, by (2.12), for every $I \subset \{1, \dots, n+3\}$ with $|I|$ even, M_I is identified with v_{I^c}. Thus the polytope $\mathcal{E}_0$ is identified with the demihypercube Δ described in Section 2.15, and $\mathcal{E}$ with the cone over Δ.

2.20 Example (the surface case) When $n = 2$, $Z \subset \mathbb{P}^4$ is a smooth quartic del Pezzo surface (see Section 1.1). The cone $\mathcal{E} \subset H^2(Z, \mathbb{R})$, generated by the classes of the 16 lines in Z, is the cone of effective curves of Z. In this case the polytope $\mathcal{E}_0$ is a 5–dimensional demihypercube, and coincides with the 5–dimensional Gosset polytope (see Dolgachev [12, Sections 8.2.5 and 8.2.6]). In higher dimensions, demihypercubes and Gosset polytopes are different polytopes.

Let us explicitly describe the facets of $\mathcal{E}$, or equivalently the generators of the dual cone $\mathcal{E}^\vee \subset H^n(Z, \mathbb{R})$. Let $(y, x_1, \dots, x_{n+3})$ be the coordinates on $H^n(Z, \mathbb{R}) \cong \mathbb{R}^{n+4}$ induced by the basis $\{\eta, \varepsilon_1, \dots, \varepsilon_{n+3}\}$. It follows from (2.18) that the cone $\mathcal{E}$ is defined in a minimal way by the set of inequalities

$$\mathcal{E} = \begin{cases} 2y + x_i \geq 0, & i \in \{1, \dots, n+3\}, \\ 2y - x_i \geq 0, & i \in \{1, \dots, n+3\}, \\ 2(n+1)y + \sum_{j \notin I} x_j - \sum_{i \in I} x_i \geq 0, & I \subset \{1, \dots, n+3\} \text{ even}. \end{cases} \tag{2.21}$$

This is equivalent to saying that the dual cone $\mathcal{E}^\vee \subset H^n(Z, \mathbb{R})$ is the convex polyhedral cone generated by the classes

$$\begin{cases} \frac{1}{2}\eta + \varepsilon_i \text{ and } \frac{1}{2}\eta - \varepsilon_i, & i \in \{1, \dots, n+3\}, \\ \frac{1}{2}(n+1)\eta + (-1)^m \sum_{j \notin I} \varepsilon_j - (-1)^m \sum_{i \in I} \varepsilon_i, & I \subset \{1, \dots, n+3\},\ |I| \text{ even}. \end{cases} \tag{2.22}$$

2.23 Remark Using (2.7), (2.9) and (2.12), we can write the generators (2.22) of $\mathcal{E}^\vee$ in terms of η and the M_I:

$$\begin{cases} \frac{1}{2}\eta+\varepsilon_i = M_0+M_i, \\ \frac{1}{2}\eta-\varepsilon_i = M_j+M_{ij} \quad \text{for any } j\neq i, \\ \frac{1}{2}(n+1)\eta+(-1)^m\sum_{j\notin I}\varepsilon_j-(-1)^m\sum_{i\in I}\varepsilon_i = 2\big(\lfloor\frac{1}{2}(m+1)\rfloor\eta+(-1)^m M_I\big). \end{cases}$$

Note in particular that $\mathcal{E}^\vee\subset\mathcal{E}$.

For $I\subseteq\{1,\dots,n+3\}\smallsetminus\{i\}$, it follows from (2.10) and (2.12) that

$$\tag{2.24} \begin{aligned} \big(\tfrac{1}{2}\eta+\varepsilon_i\big)\cdot M_I &= \begin{cases} 1 & \text{if } |I|\equiv m \bmod 2, \\ 0 & \text{otherwise,} \end{cases} \\ \big(\tfrac{1}{2}\eta-\varepsilon_i\big)\cdot M_I &= \begin{cases} 0 & \text{if } |I|\equiv m \bmod 2, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

This describes the generators of the (nonsimplicial) facets of $\mathcal{E}$, corresponding to the extremal rays of $\mathcal{E}^\vee$ generated by $\frac{1}{2}\eta\pm\varepsilon_i$.

For each $M\in\mathcal{F}_m(Z)$, set

$$\delta_M := \lfloor\tfrac{1}{2}(m+1)\rfloor\eta+(-1)^m M.$$

The facet of the cone $\mathcal{E}$ corresponding to the extremal ray of $\mathcal{E}^\vee$ generated by δ_M is simplicial, and given by

$$\mathrm{Cone}(\sigma_i(M))_{i\in\{1,\dots,n+3\}}.$$

Indeed, for $I\subseteq\{1,\dots,n+3\}$ with $|I|$ odd, one computes, using (2.12),

$$\delta_M\cdot\sigma_I(M)=\tfrac{1}{2}(|I|-1).$$

Let $(z,t_1,\dots,t_{n+3})$ be the coordinates induced by the basis $\{\eta,M_1,\dots,M_{n+3}\}$ on $H^n(Z,\mathbb{R})$. In the sequel we need equations for $\mathcal{E}^\vee$ in these coordinates. Let $I\subseteq\{1,\dots,n+3\}$ be such that $|I|\equiv m \bmod 2$. Using (2.12), one computes

$$\left(z\eta+\sum_{i=1}^{n+3}t_iM_i\right)\cdot M_I = 2z+(|I|-m)\sum_{i=1}^{n+3}t_i-2\sum_{i\in I}t_i.$$

So we get the following:

2.25 Lemma *An element $z\eta+\sum_{i=1}^{n+3}t_iM_i$ is in $\mathcal{E}^\vee$ if and only if*

$$\tag{2.26} 2z+(|I|-m)\sum_{i=1}^{n+3}t_i-2\sum_{i\in I}t_i\geq 0$$

for every $I\subseteq\{1,\dots,n+3\}$ such that $|I|\equiv m \bmod 2$.

We conclude this section with the following elementary description of the symmetry group of the cone $\mathcal{E}$:

2.27 Lemma *Let $f\colon H^n(Z,\mathbb{R}) \to H^n(Z,\mathbb{R})$ be a linear map. The following are equivalent:*

(i) $f(\mathcal{E}) = \mathcal{E}$ *and* $f(x)\cdot\eta = x\cdot\eta$ *for every* $x \in H^n(Z,\mathbb{R})$.

(ii) $f(\mathcal{E}^\vee) = \mathcal{E}^\vee$ *and* $f(\eta) = \eta$.

(iii) $f \in W(D_{n+3})$.

Proof The implications (iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii) are clear.

We prove (i) $\Longrightarrow$ (iii). Let f be an endomorphism of $H^n(Z,\mathbb{R})$ satisfying (i). Then f permutes the vertices of $\mathcal{E}_0$, and hence $f(\mathcal{F}_m(Z)) = \mathcal{F}_m(Z)$.

Recall Notation 2.8; let $M_0 \in \mathcal{F}_m(Z)$. By Remark 2.23, $\delta_{M_0} = \lfloor \frac{1}{2}(m+1) \rfloor \eta + (-1)^m M_0$ generates an extremal ray of $\mathcal{E}^\vee$, and the corresponding facet of $\mathcal{E}$ is simplicial, given by

$$\mathrm{Cone}(M_1,\dots,M_{n+3}).$$

Then $f(\mathrm{Cone}(M_1,\dots,M_{n+3}))$ must be another simplicial facet of $\mathcal{E}$, of the form

$$\mathrm{Cone}(\sigma_1(M_I),\dots,\sigma_{n+3}(M_I)) = \sigma_I(\mathrm{Cone}(M_1,\dots,M_{n+3}))$$

for some $I \subseteq \{1,\dots,n+3\}$. By composing f with the involution $\sigma_I \in W(D_{n+3})$, we may assume that f fixes the facet $\mathrm{Cone}(M_1,\dots,M_{n+3})$ of $\mathcal{E}$. In particular, f induces a permutation on the set $\{M_1,\dots,M_{n+3}\}$. Let $\omega \in W(D_{n+3})$ be the element in the stabilizer of M_0 inducing the same permutation as f on the set $\{M_1,\dots,M_{n+3}\}$. Then, by composing f with ω^{-1}, we may assume that f fixes each of $M_1,\dots,M_{n+3}$.

We also have $f(\mathcal{F}_m(Z) \smallsetminus \{M_1,\dots,M_{n+3}\}) = \mathcal{F}_m(Z) \smallsetminus \{M_1,\dots,M_{n+3}\}$, therefore f must fix the point

$$v := \sum_{M \in \mathcal{F}_m(Z) \smallsetminus \{M_1,\dots,M_{n+3}\}} M.$$

Since $\delta_{M_0}\cdot v > 0$, v is not contained in the linear span of $M_1,\dots,M_{n+3}$ (see Remark 2.23). This implies that $f = \mathrm{Id}_{H^n(Z,\mathbb{R})} \in W(D_{n+3})$.

Finally we prove (ii) $\Longrightarrow$ (iii). Let f be an endomorphism of $H^n(Z,\mathbb{R})$ satisfying (ii). Then the dual map $g := f^t\colon H^n(Z,\mathbb{R}) \to H^n(Z,\mathbb{R})$ satisfies (i), hence, by what precedes, $g \in W(D_{n+3})$. In particular g is orthogonal, and $f = g^t = g^{-1} \in W(D_{n+3})$. □

3 The Fano variety G of $(m-1)$–planes in $Q_1 \cap Q_2 \subset \mathbb{P}^{2m+2}$

Let $n = 2m \geq 2$ be an even integer, and let $Z = Q_1 \cap Q_2 \subset \mathbb{P}^{n+2}$ be a smooth complete intersection of two quadric hypersurfaces as in (2.1). In this section we consider the variety G of $(m-1)$–planes in Z:

$$G := \mathcal{F}_{m-1}(Z) = \{[L] \in \mathrm{Gr}(m-1, \mathbb{P}^{n+2}) \mid L \subset Z\}.$$

This is a smooth n–dimensional Fano variety that has been much studied. In particular, it is known that $\mathrm{Pic}(G) \cong H^2(G,\mathbb{Z}) \cong \mathbb{Z}^{n+4}$, $\mathcal{N}^1(G) \cong H^2(G,\mathbb{R})$ and $-K_G$ is the restriction of $\mathcal{O}(1)$ on $\mathrm{Gr}(m-1, \mathbb{P}^{n+2})$ (see Reid [25, Theorem 2.6], Borcea [5, Theorem 4.1 and Remark 4.3] and Jiang [20, Proposition 3.2]). Moreover G is rational, hence $H^{2n-2}(G,\mathbb{Z})$ is torsion-free (Artin and Mumford [2, Proposition 1]) and generated by fundamental classes of one-cycles (Soulé and Voisin [26, Lemma 1]). Thus we also have $H^{2n-2}(G,\mathbb{Z}) \cong \mathbb{Z}^{n+4}$ and $\mathcal{N}_1(G) \cong H^{2n-2}(G,\mathbb{R})$.

For each $M \in \mathcal{F}_m(Z)$ we set

$$M^* := \{[L] \in G \mid L \subset M\}. \tag{3.1}$$

It is an m–plane in G (under the Plücker embedding). Let $\ell_M \in H^{2n-2}(G,\mathbb{Z})$ be the class of a line in M^*. By (2.4), for every $M,\ M' \in \mathcal{F}_m(Z)$ we have

$$M^* \cap (M')^* \neq \varnothing \qquad \Longleftrightarrow \qquad M' = \sigma_i(M) \quad \text{for some } i = 1, \dots, n+3,$$

and $M^* \cap \sigma_i(M)^*$ is the point $[M \cap \sigma_i(M)] \in G$.

3.2 (the fibrations φ_i and ψ_i on G) We define $2(n+3)$ fibrations on G, generalizing a construction by Borcea in the case $n = 4$ [6, Section 3]. For each $i = 1, \dots, n+3$, the double cover $\pi_i\colon Z \to Q^n$ introduced in Section 2.2 induces a map

$$\Pi_i\colon G \to \mathcal{F}_{m-1}(Q^n).$$

Each $(m-1)$–plane in Q^n is contained in exactly one m–plane of each of the two families T^φ and T^ψ of m–planes in Q^n (see for instance Harris [17, Theorem 22.14]). This yields two morphisms

$$\mathcal{F}_{m-1}(Q^n) \to T^\varphi \subset \mathrm{Gr}(m, \mathbb{P}^{n+1}) \quad \text{and} \quad \mathcal{F}_{m-1}(Q^n) \to T^\psi \subset \mathrm{Gr}(m, \mathbb{P}^{n+1}).$$

By composing them with $\Pi_i\colon G \to \mathcal{F}_{m-1}(Q^n)$, we get two distinct morphisms

$$\bar{\varphi}_i,\ \bar{\psi}_i\colon G \to \mathrm{Gr}(m, \mathbb{P}^{n+1})$$

such that $\bar{\varphi}_i(G) \subseteq T^\varphi$ and $\bar{\psi}_i(G) \subseteq T^\psi$. Let

$$G \xrightarrow{\varphi_i} Y_{\varphi_i} \to \bar{\varphi}_i(G) \qquad \text{and} \qquad G \xrightarrow{\psi_i} Y_{\psi_i} \to \bar{\psi}_i(G)$$

be the Stein factorizations of $\bar{\varphi}_i$ and $\bar{\psi}_i$, respectively.

3.3 Lemma *The morphism $\varphi_i\colon G \to Y_{\varphi_i}$ has general fiber $\mathbb{P}^1$, and has exactly 2^n singular fibers, each isomorphic to a union of two copies of $\mathbb{P}^m$ meeting transversally at one point. More precisely, the singular fibers of φ_i are of the form $M^* \cup \sigma_i(M)^*$, with $M \in \mathcal{F}_m(Z)$ such that $[\pi_i(M)] \in T^\varphi$. An analogous statement holds for ψ_i.*

As a consequence, the cone $\mathrm{NE}(\varphi_i)$ is the convex cone generated by the classes ℓ_M for $M \in \mathcal{F}_m(Z)$ such that $[\pi_i(M)] \in T^\varphi$, and similarly for $\mathrm{NE}(\psi_i)$.

Proof For simplicity we assume in the proof that $m \geq 2$ and $n \geq 4$, the case $n = 2$ being classical.

Let $[\Lambda] \in T^\varphi \subset \mathrm{Gr}(m, \mathbb{P}^{n+1})$, and let $\Lambda' \subset \mathbb{P}^{n+2}$ be the $(m+1)$–plane through the i^{th} coordinate point that projects onto $\Lambda \subset \mathbb{P}^{n+1}$. Then Λ' is contained in a singular quadric of the pencil of quadrics through Z, so that $\Lambda' \cap Z = \Lambda' \cap Q_1$ is an m–dimensional quadric in Λ'. Hence $[\Lambda] \in \overline{\varphi}_i(G)$ if and only if $\Lambda' \cap Z$ contains an $(m-1)$–plane. This happens if and only if the quadric $\Lambda' \cap Z$ has rank at most 4.

If the m–dimensional quadric $\Lambda' \cap Z$ has rank 4, then it is the join of an $(m-3)$–plane with a smooth quadric surface $\cong \mathbb{P}^1 \times \mathbb{P}^1$. So it contains two distinct 1–dimensional families of $(m-1)$–planes, each parametrized by $\mathbb{P}^1$. Therefore $\overline{\varphi}_i^{-1}([\Lambda])$ is the disjoint union of two copies of $\mathbb{P}^1$, and this yields two smooth fibers of φ_i, each isomorphic to $\mathbb{P}^1$.

If $\Lambda' \cap Z$ has rank 3, then it is the join of an $(m-2)$–plane with a plane conic. So it contains a one-dimensional family of $(m-1)$–planes, parametrized by the conic. Thus in this case $\overline{\varphi}_i^{-1}([\Lambda])_{\mathrm{red}} \cong \mathbb{P}^1$, and this yields a fiber of φ_i with reduced structure isomorphic to $\mathbb{P}^1$.

If $\Lambda' \cap Z$ has rank 2, then it is the union of two m–planes intersecting in codimension one, both projecting onto Λ. Thus there exists $M \in \mathcal{F}_m(Z)$ such that $\Lambda = \pi_i(M)$, $\Lambda' \cap Z = M \cup \sigma_i(M)$ and $\overline{\varphi}_i^{-1}([\Lambda]) = M^* \cup \sigma_i(M)^*$. It follows from (2.4) that M^* and $\sigma_i(M)^*$ intersect in one point.

Finally, if $\Lambda' \cap Z$ has rank 1, then set-theoretically we should have $\Lambda' \cap Z = M$ for some $M \in \mathcal{F}_m(Z)$, and hence $\overline{\varphi}_i^{-1}([\pi_i(M)]) = M^*$, which is impossible because we have already seen that $\overline{\varphi}_i^{-1}([\pi_i(M)]) = M^* \cup \sigma_i(M)^*$.

Now set

$$U := Y_{\varphi_i} \smallsetminus \{\varphi_i(M^* \cup \sigma_i(M)^*) \mid M \in \mathcal{F}_m(Z) \text{ and } [\pi_i(M)] \in T^\varphi\}.$$

We have shown that φ_i has one-dimensional fibers over U, and since G is Fano, φ_i is a conic bundle over U. A general singular fiber should be reduced with two irreducible components. Since there are no such fibers, φ_i is smooth over U. $\square$

In Section 6.5 we will characterize the varieties Y_{φ_i} and Y_{ψ_i}.

Fix $M_0 \in \mathcal{F}_m(Z)$ such that $[\pi_i(M_0)] \in T^{\psi}$, and follow Notation 2.8. It follows from Section 2.5 that, for every $I \subseteq \{1,\dots,n+3\}$ such that $i \notin I$,

$$[\pi_i(M_I)] \in \begin{cases} T^{\varphi} & \text{if } |I| \text{ is odd,} \\ T^{\psi} & \text{if } |I| \text{ is even.} \end{cases}$$

So we get the following corollary of Lemma 3.3:

3.4 Corollary *We have*

$$\mathrm{NE}(\varphi_i) = \mathrm{Cone}(\ell_{M_I})_{|I|\text{odd}, i\notin I} \quad \textit{and} \quad \mathrm{NE}(\psi_i) = \mathrm{Cone}(\ell_{M_I})_{|I|\text{even}, i\notin I}.$$

The general fiber of φ_i has class $\ell_{M_j} + \ell_{M_{ij}}$ for $j \neq i$, and the general fiber of ψ_i has class $\ell_{M_0} + \ell_{M_i}$.

3.5 (the isomorphisms between $H^{2n-2}(G,\mathbb{Z})$, $H^n(Z,\mathbb{Z})$ and $H^2(G,\mathbb{Z})$) Recall that, by Poincaré duality, the intersection product gives a perfect pairing

$$H^2(G,\mathbb{Z}) \times H^{2n-2}(G,\mathbb{Z}) \to \mathbb{Z}.$$

We will define natural isomorphisms $H^{2n-2}(G,\mathbb{Z}) \cong H^n(Z,\mathbb{Z})$ and $H^2(G,\mathbb{Z}) \cong H^n(Z,\mathbb{Z})$, which behave well with respect to the intersection products. This construction is due to Borcea in the case $n = 4$ [6, Section 2]. Throughout this section, we use the same notation as in Section 2.

Consider the incidence variety

$$\mathcal{I} := \{([L], p) \in G \times Z \mid p \in L\}$$

and the associated diagram:

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\pi"'] \arrow[dr, "e"] & \\ G & & Z \end{tikzcd}$$

The morphism π is a $\mathbb{P}^{m-1}$–bundle, hence $\mathcal{I}$ is smooth, irreducible, of dimension $3m - 1 = \frac{3}{2}n - 1$. Consider the morphisms given by pull-backs and Gysin homomorphisms

$$\alpha := e_* \circ \pi^* \colon H^{2n-2}(G,\mathbb{Z}) \xrightarrow{\pi^*} H^{2n-2}(\mathcal{I},\mathbb{Z}) \xrightarrow{e_*} H^n(Z,\mathbb{Z}),$$
$$\beta := \pi_* \circ e^* \colon H^n(Z,\mathbb{Z}) \xrightarrow{e^*} H^n(\mathcal{I},\mathbb{Z}) \xrightarrow{\pi_*} H^2(G,\mathbb{Z}),$$

so that we have

(3.6) $$H^{2n-2}(G,\mathbb{Z}) \xrightarrow{\alpha} H^n(Z,\mathbb{Z}) \xrightarrow{\beta} H^2(G,\mathbb{Z}).$$

Note that $\alpha(\ell_M) = M$ for every $M \in \mathcal{F}_m(Z)$. We set $E_M := \beta(M) \in H^2(G,\mathbb{Z})$ for every $M \in \mathcal{F}_m(Z)$.

3.7 Proposition [6, Proposition 2.2] *Both α and β are isomorphisms, and they are dual to each other with respect to the intersection products. Namely,*

$$x \cdot \beta(y) = \alpha(x) \cdot y \quad \text{for every } x \in H^{2n-2}(G,\mathbb{Z}) \text{ and } y \in H^n(Z,\mathbb{Z}).$$

Proof Since $\alpha(\ell_M) = M$ and the classes $\{M\}_{M\in\mathcal{F}_m(Z)}$ generate $H^n(Z,\mathbb{Z})$, the homomorphism α is surjective. Then α must be an isomorphism, because $H^{2n-2}(G,\mathbb{Z})$ and $H^n(Z,\mathbb{Z})$ are free of the same rank.

It follows from properties of Poincaré duality that $\alpha^t = (e_* \circ \pi^*)^t = (\pi^*)^t \circ (e_*)^t = \pi_* \circ e^* = \beta$, so α is the transpose homomorphism of β. It follows that β must be an isomorphism too. □

3.8 Corollary *We have $\beta(\eta) = -K_G$.*

Proof Using Proposition 3.7, for every $M \in \mathcal{F}_m(Z)$ we have

$$1 = \eta \cdot M = \eta \cdot \alpha(\ell_M) = \beta(\eta) \cdot \ell_M = -K_G \cdot \ell_M.$$

Since α is an isomorphism, and the classes $\{M\}_{M\in\mathcal{F}_m(Z)}$ generate $H^n(Z,\mathbb{Z})$, the classes $\{\ell_M\}_{M\in\mathcal{F}_m(Z)}$ generate $H^{2n-2}(G,\mathbb{Z})$. This yields the statement. □

Consider the involution $\sigma_I\colon Z \to Z$ for $I \subseteq \{1,\dots,n+3\}$ defined in Section 2.2. It induces an involution of G, which we denote by the same symbol,

$$\sigma_I\colon G \to G, \quad [L] \mapsto [\sigma_I(L)].$$

Therefore the group $W' \cong (\mathbb{Z}/2\mathbb{Z})^{n+2}$ generated by the involutions σ_i acts on G, $H^2(G,\mathbb{Z})$ and $H^{2n-2}(G,\mathbb{Z})$. It also acts on the incidence variety $\mathcal{I}$ in such a way that both morphisms π and e are W'–equivariant. It follows that the isomorphisms α and β are W'–equivariant.

3.9 Proposition *For every $M \in \mathcal{F}_m(Z)$, ℓ_M generates an extremal ray of* $\mathrm{NE}(G)$.

Proof Fix $M_0 \in \mathcal{F}_m(Z)$ and $i \in \{1,\dots,n+3\}$ such that $[\pi_i(M_0)] \in T^{\psi}$, and follow Notation 2.8. By Corollary 3.4, we have

$$\alpha(\mathrm{NE}(\varphi_i)) = \mathrm{Cone}(M_I)_{|I|\mathrm{odd},\, i\notin I} \quad \text{and} \quad \alpha(\mathrm{NE}(\psi_i)) = \mathrm{Cone}(M_I)_{|I|\mathrm{even},\, i\notin I}.$$

By (2.24), these are facets of the cone $\mathcal{E} \subset H^n(Z,\mathbb{R})$, whose extremal rays are generated by the classes $M = \alpha(\ell_M)$ contained in these facets. Thus, for every $M \in \mathcal{F}_m(Z)$ the class ℓ_M generates an extremal ray of either $\mathrm{NE}(\varphi_i)$ or $\mathrm{NE}(\psi_i)$, and hence of $\mathrm{NE}(G)$. □

4 The blow-up X of $\mathbb{P}^n$ at $n+3$ points

Let $n \geq 3$ be an integer. Unless otherwise stated, in this section we do not assume that n is even. Let $\mathcal{P} = \{p_1, \dots, p_{n+3}\} \subset \mathbb{P}^n$ be a set of distinct points in general linear position, and denote by C the unique rational normal curve in $\mathbb{P}^n$ through these points. Let $X = X_{\mathcal{P}}$ be the blow-up of $\mathbb{P}^n$ at $p_1, \dots, p_{n+3}$. Notice that acting on $\mathcal{P} = \{p_1, \dots, p_{n+3}\}$ by permutations and projective automorphisms of $\mathbb{P}^n$ yields isomorphic varieties $X_{\mathcal{P}}$. The variety X and its birational geometry have been widely studied. We refer the reader to Dolgachev [10], Bauer [3], Mukai [21; 23], Castravet and Tevelev [9], Araujo and Massarenti [1] and Brambilla, Dumitrescu and Postinghel [7].

We have $\mathrm{Pic}(X) \cong H^2(X,\mathbb{Z})$ and $\mathcal{N}^1(X) \cong H^2(X,\mathbb{R})$. We denote by H the pullback to X of the hyperplane class in $\mathbb{P}^n$, and by E_i the exceptional divisor over the point p_i (as well as its class in $H^2(X,\mathbb{Z})$).

4.1 (special subvarieties of X) Given a subset $I \subset \{1, \dots, n+3\}$, with $|I| = d \leq n$, and an integer $0 \leq s \leq \frac{1}{2}(n-d)$, we consider the join

$$\mathrm{Join}(\langle p_i \rangle_{i \in I}, \mathrm{Sec}_{s-1}(C)) \subset \mathbb{P}^n.$$

(Here we write $\mathrm{Sec}_k(C)$ for the subvariety of $\mathbb{P}^n$ obtained as the closure of the union of all k–planes spanned by $k+1$ general points of C for $k \geq 0$; in particular $\mathrm{Sec}_0(C) = C$. We also set $\mathrm{Sec}_{-1}(C) = \varnothing$.)

This join has dimension equal to $d+2s-1$. We denote by $J_{I,s} \subset X$ the strict transform of $\mathrm{Join}(\langle p_i \rangle_{i \in I}, \mathrm{Sec}_{s-1}(C))$. When $d+2s = n$ (so that $|I^c| = n+3-3 = 2s+3$ is odd) we denote the divisor $J_{I,s}$ and its class in $H^2(X,\mathbb{Z})$ by E_I; in particular, for $n = 2m$ even, $E_\varnothing = J_{\varnothing,m}$ is the strict transform of $\mathrm{Sec}_{m-1}(C)$. For $I = \{i\}^c$, we set $E_I = E_i$. For every $I \subset \{1, \dots, n+3\}$ with $|I^c| = 2s+3$ odd and $s \geq 0$, we have the following identity in $H^2(X,\mathbb{Z})$:

$$E_I = (s+1)H - (s+1)\sum_{i \in I} E_i - s \sum_{j \notin I} E_j. \tag{4.2}$$

By Castravet and Tevelev [9, Theorem 1.2], each E_I generates an extremal ray of $\mathrm{Eff}(X)$, and all extremal rays are of this form. Moreover, by [9, Theorem 1.3] and Mukai [23], X is a Mori dream space (MDS for short). We refer to Hu and Keel [18] for the definition and basic properties of MDSs. Here we only recall an important feature of a MDS, the Mori chamber decomposition of its effective cone.

4.3 (the Mori chamber decomposition) Let Y be a projective, normal and $\mathbb{Q}$–factorial MDS. The effective cone $\mathrm{Eff}(Y)$ admits a fan structure, called *Mori chamber decomposition* and denoted by $\mathrm{MCD}(Y)$, which can be described as follows (see Hu and

Keel [18, Proposition 1.11(2)] and Okawa [24, Section 2.2]). There are finitely many birational contractions (ie birational maps whose inverses do not contract any divisor) from Y to projective, normal and $\mathbb{Q}$–factorial MDSs, denoted by $g_i\colon Y \dashrightarrow Y_i$. The set $\mathrm{Exc}(g_i)$ of classes of exceptional prime divisors of g_i has cardinality $\rho(Y)-\rho(Y_i)$. The maximal cones $\mathcal{C}_i$ of the fan $\mathrm{MCD}(Y)$ are of the form:

$$\mathcal{C}_i = \mathrm{Cone}\big(g_i^*(\mathrm{Nef}(Y_i)), \mathrm{Exc}(g_i)\big).$$

By abuse of notation, we often write $\mathrm{Nef}(Y_i) \subset \mathrm{Eff}(Y)$ for $g_i^*(\mathrm{Nef}(Y_i)) \subset \mathrm{Eff}(Y)$. If $\mathrm{Exc}(g_i) = \varnothing$, then we say that $g_i\colon Y \dashrightarrow Y_i$ is a small $\mathbb{Q}$–factorial modification of Y. The movable cone $\mathrm{Mov}(Y)$ of Y is the union

$$\mathrm{Mov}(Y) = \bigcup_{\mathrm{Exc}(g_i)=\varnothing} \mathcal{C}_i.$$

An arbitrary cone $\sigma \in \mathrm{MCD}(Y)$ is of the form

$$\sigma = \mathrm{Cone}\big(f^*(\mathrm{Nef}(W)), \mathcal{E}\big),$$

where $f\colon Y \dashrightarrow W$ is a dominant rational map to a normal projective variety, which factors as $Y \overset{g_i}{\dashrightarrow} Y_i \xrightarrow{f_i} W$ for some i, where $f_i\colon Y_i \to W$ is the contraction of an extremal face of $\mathrm{Nef}(Y_i)$, and $\mathcal{E} \subset \mathrm{Exc}(g_i)$.

Given an effective divisor D on Y, its class in $\mathcal{N}^1(Y)$ lies in the relative interior of some cone in $\mathrm{MCD}(Y)$, say $\mathrm{Cone}\big(f^*(\mathrm{Nef}(W)), \mathcal{E}\big)$. The map $f\colon Y \dashrightarrow W$ coincides with the map $\varphi_{|mD|}$ for $m \gg 1$ divisible enough. In this case, we write Y_D for the variety W.

Now we go back to X. Our next goal is to describe the Mori chamber decomposition of $\mathrm{Eff}(X)$, following Mukai [23] and Bauer [3] (see also Araujo and Massarenti [1, Section 3]).

Let us consider the coordinates $(y, x_1, \dots, x_{n+3})$ in $H^2(X,\mathbb{R})$ induced by the basis $(H, E_1, \dots, E_{n+3})$, and consider the affine hyperplane

$$\mathcal{H} = \Big((n+1)y + \sum x_i = 1\Big) \subset H^2(X,\mathbb{R}).$$

It contains all the generators E_I of $\mathrm{Eff}(X)$ described above, as well as $\frac{1}{4}(-K_X)$.

We now observe that the convex hull of the E_I in $\mathcal{H}$ is a demihypercube. To see this, we need suitable coordinates in $\mathcal{H}$. For $i = 1, \dots, n+3$, set

$$\widetilde{\varepsilon}_i := \frac{1}{2}\Big(H - \sum_{j\neq i} E_j + E_i\Big). \tag{4.4}$$

Then $\{\widetilde{\varepsilon}_1,\dots,\widetilde{\varepsilon}_{n+3}\}$ is a basis for the linear subspace $((n+1)y+\sum x_i=0)$, so that $\bigl(\frac{1}{4}(-K_X),\{\widetilde{\varepsilon}_1,\dots,\widetilde{\varepsilon}_{n+3}\}\bigr)$ induces affine coordinates $(\alpha_1,\dots,\alpha_{n+3})$ in $\mathcal{H}\cong\mathbb{R}^{n+3}$. The radial projection

$$H^2(X,\mathbb{R})\smallsetminus\Bigl((n+1)y+\sum x_i=0\Bigr)\to\mathcal{H}$$

is given in coordinates by

$$\alpha_i=\frac{y+x_i}{(n+1)y+\sum x_i}-\frac{1}{2}\quad\text{for } i=1,\dots,n+3. \tag{4.5}$$

In the coordinates α_i, $\frac{1}{4}(-K_X)$ is identified with the origin, and E_I with v_{I^c}, with the notation introduced in Section 2.15. Thus $\mathrm{Eff}(X)\cap\mathcal{H}$ is identified with the demihypercube $\Delta\subset\mathbb{R}^{n+3}$ described in Section 2.15,

$$\Delta=\begin{cases}-\frac{1}{2}\le\alpha_i\le\frac{1}{2}, & i\in\{1,\dots,N\},\\ H_I\ge 1, & |I|\ \text{even}.\end{cases}$$

Recall the degree 1 polynomials H_I introduced in (2.16), and consider the hyperplane arrangement

$$(H_I=k)_{I\subset\{1,\dots,n+3\},\,k\in\mathbb{N},\,2\le k\le(n+3)/2,\,|I|\not\equiv k \bmod 2}\,. \tag{4.6}$$

It defines a subdivision of Δ in polytopes, and a fan structure on $\mathrm{Eff}(X)$, given by the cones over these polytopes. By Mukai [23] and Bauer [3], this fan coincides with $\mathrm{MCD}(X)$. Moreover, one has the following description of the wall crossings (see [23, Propositions 2 and 3] and also [3, Section 2]):

(1) The intersection of $\mathrm{Mov}(X)$ with the hyperplane $\mathcal{H}$ is given by

$$\Delta_{\mathrm{Mov}}=\mathrm{Mov}(X)\cap\mathcal{H}=\begin{cases}-\frac{1}{2}\le\alpha_i\le\frac{1}{2}, & i\in\{1,\dots,n+3\},\\ H_I\ge 2, & |I|\ \text{odd}.\end{cases}$$

(2) All small $\mathbb{Q}$–factorial modifications of X are smooth.

(3) Let $\mathcal{C}$ be a maximal cone of $\mathrm{MCD}(X)$, contained in $\mathrm{Mov}(X)$, corresponding to a small $\mathbb{Q}$–factorial modification $\widetilde{X}$ of X. Let $\sigma\subset\partial\mathcal{C}$ be a wall such that $\sigma\subset\partial\,\mathrm{Mov}(X)$, and let $f\colon\widetilde{X}\to Y$ be the corresponding elementary contraction. Then $\sigma\cap\mathcal{H}\subset\Delta_{\mathrm{Mov}}$ is supported on a hyperplane of one of the following forms:

- (a) $\bigl(\alpha_i=-\frac{1}{2}\bigr)$ or $\bigl(\alpha_i=\frac{1}{2}\bigr)$.
- (b) $(H_I=2)$, with $|I|$ odd.

In case (a), $f\colon\widetilde{X}\to Y$ is a $\mathbb{P}^1$–bundle. In case (b), $f\colon\widetilde{X}\to Y$ is the blow-up of a smooth point, and the exceptional divisor of f is the strict transform in $\widetilde{X}$ of the divisor $E_{I^c}\subset X$.

(4) Let $\mathcal{C}$ and $\mathcal{C}'$ be two maximal cones of $\mathrm{MCD}(X)$, contained in $\mathrm{Mov}(X)$ and having a common facet. Let $f\colon X \dashrightarrow \widetilde{X}$ and $f'\colon X \dashrightarrow \widetilde{X}'$ be the corresponding small $\mathbb{Q}$–factorial modifications of X. The intersections of these cones with $\mathcal{H}$ are separated in Δ by a hyperplane of the form $(H_I = k)$, with $3 \le k \le \frac{1}{2}(n+3)$ and $|I| \not\equiv k \bmod 2$. Suppose that $\mathcal{C} \cap \mathcal{H} \subset (H_I \le k)$ and $\mathcal{C}' \cap \mathcal{H} \subset (H_I \ge k)$. Then the birational map $f' \circ f^{-1}\colon \widetilde{X} \dashrightarrow \widetilde{X}'$ flips a $\mathbb{P}^{k-2}$ into a $\mathbb{P}^{n+1-k}$.

4.7 Remark It is possible to give a more precise description of the flipping locus $\mathbb{P}^{k-2} \subset \widetilde{X}$ (or $\mathbb{P}^{n+1-k} \subset \widetilde{X}'$) in the situation described under (4) above (see [3, Proposition 2.6(iv) and Theorem 2.9]): Consider the nef cone of X and its section with $\mathcal{H}$,

$$\Delta_{\mathrm{Nef}} = \mathrm{Nef}(X) \cap \mathcal{H} = \begin{cases} H_{\{i\}} \ge 2, & i \in \{1, \dots, n+3\}, \\ H_{\{i,j\}} \le 3, & i, j \in \{1, \dots, n+3\},\ i \ne j. \end{cases}$$

Suppose that $\Delta_{\mathrm{Nef}} \subset (H_I \le k)$. Then the $\mathbb{P}^{k-2} \subset \widetilde{X}$ flipped by $f' \circ f^{-1}$ is the strict transform in $\widetilde{X}$ of the special variety $J_{I,s} \subset X$, where $s = \frac{1}{2}(k - |I| - 1) \ge 0$.

Suppose that $\Delta_{\mathrm{Nef}} \subset (H_I \ge k)$. Then the $\mathbb{P}^{n+1-k} \subset \widetilde{X}'$ flipped by $f \circ (f')^{-1}$ is the strict transform in $\widetilde{X}'$ of the special variety $J_{I^c,s'} \subset X$, where $s' = \frac{1}{2}(|I| - k - 1) \ge 0$.

4.8 Remark Recall from Section 2.15 the description of the facets of Δ. Each of the $2(n+3)$ facets of Δ supported on the hyperplanes $\left(\alpha_i = \pm\frac{1}{2}\right)$ intersects Δ_{Mov} along a facet, while the other facets of Δ, supported on the hyperplanes $(H_I = 1)$ for $|I|$ even, are disjoint from Δ_{Mov}. Let us describe the rational maps associated to the facets of Δ_{Mov} supported on the hyperplanes $\left(\alpha_i = \pm\frac{1}{2}\right)$.

Fix $i \in \{1, \dots, n+3\}$ and let $\mathcal{P}_i \subset \mathbb{P}^{n-1}$ be the image of the set $\mathcal{P} \smallsetminus \{p_i\}$ under the projection $\pi_{p_i}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ from p_i. Let $Y = (X_{\mathcal{P}_i})^{n-1}$ be the blow-up of $\mathbb{P}^{n-1}$ at the $n+2$ points in $\mathcal{P}_i$.

There is a small $\mathbb{Q}$–factorial modification $X \dashrightarrow X_i$ and a $\mathbb{P}^1$–bundle $X_i \to Y$ extending π_{p_i} (see [23, Example 1]). Let $\pi_i\colon X \dashrightarrow Y$ be the composite map. The general fiber of π_i is the strict transform in X of a general line in $\mathbb{P}^n$ through p_i. The hyperplane $(\pi_i)^* H^2(Y, \mathbb{R})$ has equation $y + x_i = 0$. Using (4.5), we see that $(\pi_i)^* H^2(Y, \mathbb{R}) \cap \mathcal{H}$ is the hyperplane $\left(\alpha_i = -\frac{1}{2}\right)$. Thus the cone $(\pi_i)^* \mathrm{Eff}(Y)$ is the cone over the polytope $\Delta \cap \left(\alpha_i = -\frac{1}{2}\right)$, which is an $(n+2)$–dimensional demihypercube.

Similarly, there is a map $\pi_i'\colon X \dashrightarrow Y$ whose general fiber is the strict transform in X of a general rational normal curve through the points p_λ for $\lambda \ne i$. Indeed, fix $j \ne i$ and let $\varphi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the standard Cremona transformation centered at the points p_λ for $\lambda \ne i, j$. This map sends rational normal curves through the points p_λ for $\lambda \ne i$ to lines through $\varphi(p_j)$. There is an automorphism of $\mathbb{P}^n$ fixing p_λ for $\lambda \ne i, j$, sending p_j to $\varphi(p_i)$ and sending p_i to $\varphi(p_j)$ (see Remark 7.2). By composing φ with the projection from $\varphi(p_j)$, we obtain a rational map $\pi'_{p_i}\colon \mathbb{P}^n \dashrightarrow Y$ whose general fiber is a general rational normal curve through the points p_λ for $\lambda \neq i$. This yields a $\mathbb{P}^1$–bundle $X'_i \to Y$ on a small $\mathbb{Q}$–factorial modification of X, and the desired map $\pi'_i\colon X \dashrightarrow Y$. As before, one checks that $(\pi_i)^* \operatorname{Eff}(Y)$ is the cone over the demihypercube $\Delta \cap \left(\alpha_i = \frac{1}{2}\right)$.

The center of the polytopes Δ_{Mov} and Δ is the origin $\overline{0} \in \mathbb{R}^{n+3}$, which corresponds to $\frac{1}{4}(-K_X)$. In particular, the divisor $-K_X$ is movable. We want to describe the Fano model $X^n_{\mathrm{Fano}} := X_{-K_X}$.

If n is odd, then $\overline{0}$ is a vertex in the subdivision of Δ and is contained in the intersection of the hyperplanes

$$\left(H_I = \tfrac{1}{2}(n+3)\right)_{|I| \not\equiv (n+3)/2 \mod 2}.$$

Thus $-K_X$ lies in a one-dimensional cone of the fan $\operatorname{MCD}(X)$, contained in the interior of $\operatorname{Mov}(X)$. Therefore X^n_{Fano} is non-$\mathbb{Q}$–factorial and has Picard number 1.

For the remainder of this section, we assume that $n = 2m \geq 2$ is even. Then $\overline{0}$ lies in the interior of a maximal polytope in the subdivision of Δ_{Mov}, namely the polytope defined by

$$\Delta_{\mathrm{Fano}} = (H_I \geq m+1)_{|I| \equiv m \mod 2}. \tag{4.9}$$

Then X^n_{Fano} is a small $\mathbb{Q}$–factorial modification of X, it is a smooth Fano manifold, and $\operatorname{Nef}(X^n_{\mathrm{Fano}}) \subset \operatorname{Eff}(X)$ is the cone over the polytope Δ_{Fano}.

4.10 Remark By Theorem 1.4, when $\mathcal{P}$ is the image of $\{(\lambda_1 : 1), \dots, (\lambda_{n+3} : 1)\} \subset \mathbb{P}^1$ under a Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$, X is pseudoisomorphic to the Fano variety G addressed in Section 3. This implies that X^n_{Fano} is isomorphic to G.

4.11 Using the properties of MDSs, and the description of $\operatorname{MCD}(X)$ above, we can deduce many properties of X^n_{Fano}:

- The Mori cone $\operatorname{NE}(X^n_{\mathrm{Fano}})$ admits exactly 2^{n+2} extremal rays, whose corresponding contractions all contract a $\mathbb{P}^m$ to a point.
- The variety X^n_{Fano} admits $2(n+3)$ distinct (nontrivial) contractions of fiber type. Indeed, the points in $\partial\Delta_{\mathrm{Mov}} \cap \Delta_{\mathrm{Fano}}$ are those of the form $\alpha = (\alpha_1, \dots, \alpha_{n+3})$, where $\alpha_i = -\frac{1}{2}$ or $\frac{1}{2}$ for some fixed i, and $\alpha_j = 0$ for $j \neq i$. These points all lie in $\partial\Delta$. We denote the corresponding contractions by ϕ_i and ϕ'_i, respectively.

4.12 Lemma *The morphisms ϕ_i and ϕ'_i are generic $\mathbb{P}^1$–bundles over $(X_{\mathcal{P}_i})^{n-1}_{\mathrm{Fano}}$, where $\mathcal{P}_i \subset \mathbb{P}^{n-1}$ is as in Remark 4.8. The general fiber of ϕ_i is the strict transform in X^n_{Fano} of a general line in $\mathbb{P}^n$ through p_i. The general fiber of ϕ'_i is the strict transform in X^n_{Fano} of a general rational normal curve in $\mathbb{P}^n$ through $\mathcal{P} \smallsetminus \{p_i\}$.*

Proof Let $\alpha=(\alpha_1,\dots,\alpha_{n+3})$, where $\alpha_i=-\frac{1}{2}$ and $\alpha_j=0$ for $j\neq i$, and consider the corresponding fibration $\phi_i\colon X^n_{\mathrm{Fano}}\to X_D$, where D is an effective divisor such that $\mathbb{R}_{\geq 0}[D]\cap\mathcal{H}=\alpha$.

Consider the map $\pi_i\colon X \dashrightarrow Y := (X_{\mathcal{P}_i})^{n-1}$ introduced in Remark 4.8, and recall that $(\pi_i)^*\operatorname{Eff}(Y)$ is the cone over the $(n+2)$–dimensional demihypercube $\Delta\cap\left(\alpha_i=-\frac{1}{2}\right)$. The center of this demihypercube is α, hence D is a positive multiple of $(\pi_i)^*(-K_Y)$. So the image X_D of ϕ_i is precisely the Fano model $(X_{\mathcal{P}_i})^{n-1}_{\mathrm{Fano}}$ of Y.

A similar argument shows the statement for ϕ'_i. □

4.13 Let $(z,t_1,\dots,t_{n+3})$ be new coordinates in $H^2(X,\mathbb{R})$, induced by the basis $\{-K_X,E_1,\dots,E_{n+3}\}$. These are related to $(y,x_1,\dots,x_{n+3})$ by $y=z(n+1)$ and $x_i=t_i-(n-1)z$. Using the defining inequalities for Δ_{Fano} in (4.9), and the expression for the radial projection onto $\mathcal{H}$ in (4.5), we conclude that $\operatorname{Nef}(X^n_{\mathrm{Fano}})\subset H^2(X,\mathbb{R})$ is defined by the inequalities

$$2z+(|I|-m)\sum_{i=1}^{n+3}t_i-2\sum_{i\in I}t_i\geq 0 \tag{4.14}$$

for every $I\subseteq\{1,\dots,n+3\}$ such that $|I|\equiv m \bmod 2$.

4.15 We end this section by describing the birational map $X\dashrightarrow X^n_{\mathrm{Fano}}$. First notice that to go from the interior of the polytope $\Delta_{\mathrm{Nef}}=\operatorname{Nef}(X)\cap\mathcal{H}$ to the interior of the polytope $\Delta_{\mathrm{Fano}}=\operatorname{Nef}(X^n_{\mathrm{Fano}})\cap\mathcal{H}$, we must cross the wall $(H_I=k)$ for every $I\subset\{1,\dots,n+3\}$ and $3\leq k\leq m+1$ such that $|I|\not\equiv k \bmod 2$ and $|I|\leq k-1$. By Remark 4.7 and [3, Theorem 2.9], we conclude that the rational map $X\dashrightarrow X^n_{\mathrm{Fano}}$ factors as

$$X=X_0\overset{\varphi_1}{\dashrightarrow}X_1\overset{\varphi_2}{\dashrightarrow}X_2\dashrightarrow\cdots\overset{\varphi_{m-1}}{\dashrightarrow}X_{m-1}=X^n_{\mathrm{Fano}},$$

where each $\varphi_i\colon X_{i-1}\dashrightarrow X_i$ flips the strict transforms in X_{i-1} of all special subvarieties $J_{I,s}\subset X$ of dimension i. These strict transforms are disjoint in X_{i-1} and each isomorphic to $\mathbb{P}^i$. The flipped locus on X_i is a disjoint union of copies of $\mathbb{P}^{n-1-i}$, one for each $J_{I,s}$ of dimension i. Notice that in general the map φ_i is not the flip of a small contraction: it is a pseudoisomorphism that can be factored as a sequence of flips.

In particular, we can describe the 2^{n+2} copies of $\mathbb{P}^m$ in X^n_{Fano} corresponding to the 2^{n+2} extremal rays of $\operatorname{NE}(X^n_{\mathrm{Fano}})$. These are the strict transforms of the special subvarieties $J_{I,s}\subset X$ of dimension m, and the flipped locus of the flips of the strict transforms of the special subvarieties $J_{I,s}\subset X$ of dimension $m-1$. These are, respectively,

$$\sum_{\substack{d=0\\ d\not\equiv m \bmod 2}}^{m+1} \binom{n+3}{d} \quad \text{for } m\text{–dimensional } J_{I,s},$$

$$\sum_{\substack{d=0\\ d\equiv m \bmod 2}}^{m} \binom{n+3}{d} \quad \text{for } (m-1)\text{–dimensional } J_{I,s}.$$

We can also describe the strict transforms in X^n_{Fano} of the divisors $\mathbb{P}^{n-1} \cong E_i \subset X$ under the rational map $X \dashrightarrow X^n_{\mathrm{Fano}}$. There are $n+3$ special points $q_1, \dots, q_{n+3} \subset E_i$: q_j is the intersection of E_i with the strict transform of the line through p_i and p_j when $j \neq i$, and q_i is the intersection of E_i with the strict transform of C. The points q_i all lie in a rational normal curve C' of degree $n-1$ in $E_i \cong \mathbb{P}^{n-1}$. Given a subset $I \subset \{1, \dots, n+3\}$, with $|I| \le n-1$, and an integer $0 \le s \le \frac{1}{2}(n-1-|I|)$, we denote by $J^i_{I,s}$ the join $\mathrm{Join}(\langle q_j \rangle_{j\in I}, \mathrm{Sec}_{s-1}(C')) \subset E_i$. One can check that

$$E_i \cap J_{I,s} = \begin{cases} J^i_{I\smallsetminus\{i\},s} & \text{if } i \in I, \\ \varnothing & \text{if } i \notin I \text{ and } s = 0, \\ J^i_{I\cup\{i\},s-1} & \text{if } i \notin I \text{ and } s \ge 1. \end{cases}$$

Therefore, the strict transform of E_i under φ_1 is the blow-up of $\mathbb{P}^{n-1}$ at the points $q_1, \dots, q_{n+3}$. For $2 \le j \le m-1$, the restriction of φ_j to the strict transform of E_i in X_{j-1} flips the strict transforms of every $J^i_{I,s}$ of dimension $j-1$.

4.16 When $n = 4$, the birational map $\varphi_1\colon X = X_0 \dashrightarrow X_1 = X^4_{\mathrm{Fano}}$ flips $J_{\{ij\},0}$ (strict transform of the line $\overline{p_i p_j} \subset \mathbb{P}^4$) for $1 \le i, j \le 7$, and $J_{\varnothing,1}$ (strict transform of $C \subset \mathbb{P}^4$); this yields 22 among the 64 special copies of $\mathbb{P}^2$ in X^4_{Fano}, corresponding to the 64 extremal rays of $\mathrm{NE}(X^4_{\mathrm{Fano}})$. The remaining ones are the strict transforms of the 7 surfaces $\mathrm{Join}(\langle p_i \rangle, C)$ and of the 35 planes $\langle p_i, p_j, p_h \rangle$ in $\mathbb{P}^4$.

Notice in particular that $E_i \subset X$ does not contain any special subvariety $J_{I,s}$, while the strict transform of E_i in X^4_{Fano} contains 7 special copies of $\mathbb{P}^2$, namely the flipped loci of the flips of $J_{\{ij\},0}$ for $j \neq i$ and of $J_{\varnothing,1}$.

5 Pseudoisomorphisms between G and X

Let m be a positive integer, and set $n = 2m$. Fix $n+3$ distinct points

$$(\lambda_1 : 1), \dots, (\lambda_{n+3} : 1) \in \mathbb{P}^1,$$

and let $p_1, \dots, p_{n+3} \in \mathbb{P}^n$ be their images under a Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$. Let Z, G and X be the varieties introduced in Sections 2, 3 and 4. We follow the notation introduced in those sections. In this section we determine the nef cone of G, and then we prove Theorem 1.5, which follows from Theorem 5.7 and Corollary 5.8. Our aim is to identify the line bundles on G whose linear systems define rational maps $G \dashrightarrow \mathbb{P}^n$ inducing a pseudoisomorphism $G \dashrightarrow X$. This is achieved by combining the description of $\mathrm{Nef}(G) \subset H^2(G, \mathbb{R})$ given by Theorem 5.1 and the description of $\mathrm{Nef}(X^n_{\mathrm{Fano}}) \subset H^2(X, \mathbb{R})$ in terms of the basis $\{-K_X, E_1, \dots, E_{n+3}\}$ for $H^2(X, \mathbb{R})$, which was obtained from the Mori chamber decomposition of $\mathrm{Eff}(X)$ in Section 4.

We first describe the cones $\mathrm{Nef}(G)$ and $\mathrm{NE}(G)$. For $n = 4$, this was proved by Borcea [6, Theorem 4.3].

5.1 Theorem *Let the notation be as above. Then*

$$\mathrm{NE}(G) = \mathrm{Cone}(\ell_M)_{M \in \mathcal{F}_m(G)} = \alpha^{-1}(\mathcal{E}) \quad \textit{and} \quad \mathrm{Nef}(G) = \beta(\mathcal{E}^\vee).$$

Proof By Proposition 3.9, the class ℓ_M generates an extremal ray of $\mathrm{NE}(G)$ for every $M \in \mathcal{F}_m(G)$. This yields 2^{n+2} distinct extremal rays of $\mathrm{NE}(G)$. On the other hand, $G \cong X_{\mathrm{Fano}}$ by Remark 4.10, and $\mathrm{NE}(X_{\mathrm{Fano}})$ has precisely 2^{n+2} extremal rays, as explained in Section 4.11. So we have

$$\mathrm{NE}(G) = \mathrm{Cone}(\ell_M)_{M \in \mathcal{F}_m(G)} = \alpha^{-1}(\mathcal{E}).$$

The equality $\mathrm{Nef}(G) = \beta(\mathcal{E}^\vee)$ follows from the duality between $\mathrm{Nef}(G)$ and $\mathrm{NE}(G)$ and from Proposition 3.7. □

Similarly, we will show in Proposition 5.5 that $\mathrm{Eff}(G) = \beta(\mathcal{E})$ and $\mathrm{Mov}_1(G) = \alpha^{-1}(\mathcal{E}^\vee)$. So the cones $\mathrm{NE}(G)$ and $\mathrm{Eff}(G)$ are isomorphic under $\beta \circ \alpha$, and the same holds for $\mathrm{Mov}_1(G)$ and $\mathrm{Nef}(G)$.

Recall from Section 3 that $E_M = \beta(M) \in H^2(G, \mathbb{Z})$ for every $M \in \mathcal{F}_m(Z)$. For each $M \in \mathcal{F}_m(Z)$, consider the linear map

$$h_M \colon H^2(X, \mathbb{R}) \to H^2(G, \mathbb{R})$$

defined by

$$h_M(-K_X) = -K_G \qquad \text{and} \qquad h_M(E_i) = E_{\sigma_i(M)} \quad \text{for every } i = 1, \dots, n+3.$$

One can check that h_M respects the integral points, namely that it is induced by an isomorphism $H^2(X, \mathbb{Z}) \to H^2(G, \mathbb{Z})$, and that $h_{\sigma_I(M)} = \sigma_I \circ h_M$ for every $I \subseteq \{1, \dots, n+3\}$.

We also set

$$\tilde{h}_M := \beta^{-1} \circ h_M \colon H^2(X, \mathbb{R}) \to H^n(Z, \mathbb{R}), \tag{5.2}$$

so that $\tilde{h}_M(-K_X) = \eta$ and $\tilde{h}_M(E_i) = \sigma_i(M)$ for every $i = 1, \dots, n+3$.

5.3 Lemma *For every $M \in \mathcal{F}_m(Z)$ and $I \subseteq \{1, \dots, n+3\}$ of even cardinality, we have*

$$h_M(E_I) = E_{\sigma_I(M)}, \quad h_M(\mathrm{Eff}(X)) = \beta(\mathcal{E}), \quad h_M(\mathrm{Nef}(X_{\mathrm{Fano}})) = \mathrm{Nef}(G).$$

Proof Let $I \subseteq \{1, \dots, n+3\}$ be such that $|I| = n-2s$ is even with $s \geq 0$. We can rewrite (4.2) as

$$E_I = \frac{1}{n+1}\left((s+1)(-K_X) - 2(s+1)\sum_{i \in I} E_i + (n-1-2s)\sum_{j \in I^c} E_j\right).$$

It follows from (2.13) that $\tilde{h}_M(E_I) = \sigma_I(M)$, and hence $h_M(E_I) = E_{\sigma_I(M)}$. This implies that $h_M(\mathrm{Eff}(X)) = \beta(\mathcal{E})$.

By comparing (4.14) and (2.26), we see that $\tilde{h}_M(\mathrm{Nef}(X^n_{\mathrm{Fano}})) = \mathcal{E}^\vee$. Therefore $h_M(\mathrm{Nef}(X^n_{\mathrm{Fano}})) = \beta(\mathcal{E}^\vee) = \mathrm{Nef}(G)$ by Theorem 5.1. □

5.4 Proposition *Let $\xi\colon G \dashrightarrow X$ be a pseudoisomorphism, and consider the induced linear map*

$$\xi^*\colon H^2(X, \mathbb{R}) \to H^2(G, \mathbb{R}).$$

Then, up to a unique permutation of $E_1, \dots, E_{n+3} \subset X$, there is a unique $M \in \mathcal{F}_m(Z)$ such that $\xi^* = h_M$.

Proof We have $\xi^*(-K_X) = -K_G$, and hence $\xi^*(\mathrm{Nef}(X^n_{\mathrm{Fano}})) = \mathrm{Nef}(G)$.

Recall Notation 2.8; fix $M_0 \in \mathcal{F}_m(Z)$. Consider $\xi^* \circ (h_{M_0})^{-1}\colon H^2(G, \mathbb{R}) \to H^2(G, \mathbb{R})$. By Lemma 5.3, this map fixes $-K_G$ and sends $\mathrm{Nef}(G)$ to itself. Using the isomorphism $\beta\colon H^n(Z, \mathbb{R}) \to H^2(G, \mathbb{R})$ and Theorem 5.1, we obtain a linear map $f\colon H^n(Z, \mathbb{R}) \to H^n(Z, \mathbb{R})$ such that $f(\eta) = \eta$ and $f(\mathcal{E}^\vee) = \mathcal{E}^\vee$:

$$\begin{tikzcd}
& H^2(X,\mathbb{R}) \arrow[dl, "h_{M_0}"'] \arrow[dr, "\xi^*"] & \\
H^2(G,\mathbb{R}) \arrow[rr, "\xi^*\circ(h_{M_0})^{-1}"] & & H^2(G,\mathbb{R}) \\
H^n(Z,\mathbb{R}) \arrow[u, "\beta"] \arrow[rr, "f"] & & H^n(Z,\mathbb{R}) \arrow[u, "\beta"']
\end{tikzcd}$$

By Lemma 2.27, we have $f \in W(D_{n+3})$.

Consider the stabilizer $G_0 \subset W(D_{n+3})$ of M_0, and recall that $W(D_{n+3}) = W' \rtimes G_0$ and $G_0 \cong S_{n+3}$. Thus there are uniquely defined $\omega \in G_0$, $\sigma_I \in W'$ and $\kappa \in S_{n+3}$

such that $f = \sigma_I \circ \omega$ and $\omega(M_i) = M_{\kappa(i)}$ for every $i = 1, \dots, n+3$. Since β is W'–equivariant, this means that

$$\xi^*(E_i) = \beta(f(M_i)) = \beta(\sigma_{\kappa(i)}(M_I)) = \sigma_{\kappa(i)}(\beta(M_I)) = \sigma_{\kappa(i)}(E_{M_I})$$

for every $i = 1, \dots, n+3$. Apply the permutation κ^{-1} to $E_1, \dots, E_{n+3} \subset X$. After this reordering, we get $f = \sigma_I \in W'$ and $\xi^* = \sigma_I \circ h_{M_0} = h_{M_I}$. □

From now on we order the divisors $E_1, \dots, E_{n+3} \subset X$, and correspondingly the points $p_1, \dots, p_{n+3} \in \mathbb{P}^n$, as in Proposition 5.4. At this point we can determine the cone of effective divisors and the cone of moving curves of G.

5.5 Proposition *For every $M \in \mathcal{F}_m(Z)$, there is a unique effective divisor in G with class $E_M \in H^2(G, \mathbb{Z})$. This is a fixed prime divisor, which we still denote by $E_M \subset G$. We have*

$$\mathrm{Eff}(G) = \beta(\mathcal{E}) = \mathrm{Cone}(E_M)_{M \in \mathcal{F}_m(Z)} \quad \textit{and} \quad \mathrm{Mov}_1(G) = \alpha^{-1}(\mathcal{E}^\vee).$$

Proof By Theorem 1.4, there exists a pseudoisomorphism $\xi\colon G \dashrightarrow X$. By Proposition 5.4 there exists $M \in \mathcal{F}_m(Z)$ such that $\xi^* = h_M$. In particular, for every $I \subset \{1, \dots, n+3\}$ with $|I|$ even, we have $\xi^*(E_I) = E_{\sigma_I(M)}$ by Lemma 5.3. Thus the strict transform in G of $E_I \subset X$ is a fixed prime divisor, and it is the unique effective divisor with class $E_{\sigma_I(M)}$. It also follows from Lemma 5.3 that

$$\mathrm{Eff}(G) = \xi^* \,\mathrm{Eff}(X) = \beta(\mathcal{E}) = \mathrm{Cone}(E_M)_{M \in \mathcal{F}_m(Z)}.$$

The equality $\mathrm{Mov}_1(G) = \alpha^{-1}(\mathcal{E}^\vee)$ follows from the duality $\mathrm{Mov}_1(G) = \mathrm{Eff}(G)^\vee$ and from Proposition 3.7. □

For each $M \in \mathcal{F}_m(Z)$, we set

(5.6)
$$\begin{aligned} H_M := h_M(H) &= \frac{1}{n+1}\left(-K_G + (n-1)\sum_{i=1}^{n+3} E_{\sigma_i(M)}\right) \\ &= m(-K_G) - (n-1)E_M \in H^2(G, \mathbb{Z}), \end{aligned}$$

where the last equality follows from (2.13) (taking $M = M_0$ and $I = \varnothing$), using the isomorphism $\beta\colon H^n(Z, \mathbb{R}) \to H^2(G, \mathbb{R})$.

5.7 Theorem *For every $M \in \mathcal{F}_m(Z)$, the divisor class H_M is movable, and its complete linear system defines a birational map*

$$\rho_M\colon G \dashrightarrow \mathbb{P}^n,$$

with exceptional divisors $E_{\sigma_1(M)}, \dots, E_{\sigma_{n+3}(M)}$*, inducing a pseudoisomorphism*

$$\xi_M \colon G \dashrightarrow X$$

whose induced map $\xi_M^* \colon H^2(X, \mathbb{R}) \to H^2(G, \mathbb{R})$ *coincides with* h_M*.*

For every $I \subseteq \{1, \dots, n+3\}$*,* $\rho_{\sigma_I(M)} = \rho_M \circ \sigma_I$ *and* $\xi_{\sigma_I(M)} = \xi_M \circ \sigma_I$*.*

Proof By Theorem 1.4, there exists a pseudoisomorphism $\xi \colon G \dashrightarrow X$. Let the map $\rho \colon G \dashrightarrow \mathbb{P}^n$ be the composition of ξ with the blow-up morphism $X \to \mathbb{P}^n$.

By Proposition 5.4, there exists $M_0 \in \mathcal{F}_m(Z)$ such that $\xi^* = h_{M_0}$. This implies that $\rho^*(\mathcal{O}_{\mathbb{P}^n}(1)) = H_{M_0}$. Hence the class H_{M_0} is movable, and $H^0(G, H_{M_0}) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. This proves the first statement for $M = M_0$, with $\rho_{M_0} = \rho$ and $\xi_{M_0} = \xi$.

Let $I \subseteq \{1, \dots, n+3\}$. We use Notation 2.8. The automorphism $\sigma_I \colon G \to G$ fixes $-K_G$ and maps E_{M_0} to E_{M_I}, hence it maps H_{M_0} to H_{M_I}. This yields the first statement for $M = M_I$, with $\rho_{M_I} = \rho \circ \sigma_I$ and $\xi_{M_I} = \xi \circ \sigma_I$.

The last statement is clear. □

5.8 Corollary *Let* $\widetilde{X}$ *be any blow-up of* $\mathbb{P}^n$ *at* $n+3$ *points. If* $\widetilde{X}$ *is pseudoisomorphic to* G*, then* $\widetilde{X}$ *is isomorphic to* X*.*

Proof Let $\tilde{\xi} \colon G \dashrightarrow \widetilde{X}$ be a pseudoisomorphism, and let $\tilde{\rho} \colon G \dashrightarrow \mathbb{P}^n$ be the composition of $\tilde{\xi}$ with the blow-up morphism $\widetilde{X} \to \mathbb{P}^n$. Then $\tilde{\rho}$ has $n+3$ exceptional prime divisors, whose classes must generate a simplicial facet of $\mathrm{Eff}(G)$. By Proposition 5.5 and the description of the facets of $\mathcal{E}$ in Remark 2.23, every simplicial facet of $\mathrm{Eff}(G)$ is generated by $E_{\sigma_1(M)}, \dots, E_{\sigma_{n+3}(M)}$ for some $M \in \mathcal{F}_m(Z)$. Since each $E_{\sigma_i(M)}$ is unique in its linear system, $\tilde{\rho} \colon G \dashrightarrow \mathbb{P}^n$ and $\rho_M \colon G \dashrightarrow \mathbb{P}^n$ have the same exceptional divisors. This means that $\tilde{\rho}$ and ρ_M coincide up to a projective transformation of $\mathbb{P}^n$, and therefore $\widetilde{X} \cong X$. □

5.9 Remark (comparing the intersection product in $H^n(Z, \mathbb{Z})$ with Dolgachev's pairing on $H^2(X, \mathbb{Z})$) In [10], Dolgachev defined a nondegenerate symmetric bilinear form $(\ ,\ )$ on $H^2(X, \mathbb{Z})$, by imposing that the basis $H, E_1, \dots, E_{n+3}$ is orthogonal,

$$(H, H) = n-1 \qquad \text{and} \qquad (E_i, E_i) = -1 \quad \text{for all } i = 1, \dots, n+3.$$

This pairing has signature $(1, n+3)$, and $(-K_X, -K_X) = 4(n-1)$. Consider $\tilde{\varepsilon}_i \in H^2(X, \mathbb{R})$, defined in (4.4),

$$\tilde{\varepsilon}_i := \frac{1}{2}\Bigl(H - \sum_{j \neq i} E_j + E_i\Bigr) \quad \text{for } i = 1, \dots, n+3.$$

Then we have

$$(-K_X, \widetilde{\varepsilon}_i) = 0 \qquad \text{and} \qquad (\widetilde{\varepsilon}_i, \widetilde{\varepsilon}_j) = -\delta_{ij} \quad \text{for every } i, j = 1, \dots, n+3,$$

thus $-K_X, \widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_{n+3}$ is another orthogonal basis for $H^2(X, \mathbb{R})$.

Fix $M_0 \in \mathcal{F}_m(Z)$, and consider the orthogonal basis $\eta, \varepsilon_1, \dots, \varepsilon_{n+3}$ for $H^n(Z, \mathbb{R})$ introduced in (2.9). Recall that $\eta^2 = 4$ and $\varepsilon_i^2 = (-1)^m$ for every $i = 1, \dots, n+3$. Consider the isomorphism introduced in (5.2),

$$\tilde{h}_{M_0}\colon H^2(X, \mathbb{R}) \to H^n(Z, \mathbb{R}).$$

From (5.6) and (2.14) we have $\tilde{h}_{M_0}(\widetilde{\varepsilon}_i) = \varepsilon_i$ for every $i = 1, \dots, n+3$. Therefore $\tilde{h}_{M_0}$ maps an orthogonal basis for Dolgachev's pairing in $H^2(X, \mathbb{R})$ to an orthogonal basis for the intersection product in $H^n(Z, \mathbb{R})$. In particular, $\tilde{h}_{M_0}$ sends the D_{n+3}–lattice $(-K_X)^\perp \subset H^2(X, \mathbb{Z})$ to the D_{n+3}–lattice $\eta^\perp \subset H^n(Z, \mathbb{Z})$, and the restriction of $\tilde{h}_{M_0}$ to these lattices is an isometry up to the sign $(-1)^{m-1}$. (Notice that $\tilde{h}_{M_0}$ is globally an isometry if and only if $n = 2$.) This also shows that $\tilde{h}_{M_0}$ is $W(D_{n+3})$–equivariant.

6 Cones of curves and divisors in G

Let the setup be as in Section 5. Recall that in Section 4 we considered the cones

$$\mathrm{Nef}(X^n_{\mathrm{Fano}}) \subset \mathrm{Mov}^1(X) \subset \mathrm{Eff}(X) \subset H^2(X, \mathbb{R}),$$

the affine hyperplane $\mathcal{H} \subset H^2(X, \mathbb{R})$ containing all the E_I, and the polytopes given by the intersections of these cones with $\mathcal{H}$,

$$\Delta_{\mathrm{Fano}} \subset \Delta_{\mathrm{Mov}} \subset \Delta \subset \mathcal{H} \cong \mathbb{R}^{n+3}.$$

From the linear inequalities defining these polytopes in $\mathbb{R}^{n+3}$ and the expression (4.5) of the radial projection onto $\mathcal{H}$, one can write explicitly the linear inequalities defining the cones $\mathrm{Nef}(X^n_{\mathrm{Fano}}) \cong \mathrm{Nef}(G)$, $\mathrm{Mov}^1(X) \cong \mathrm{Mov}^1(G)$ and $\mathrm{Eff}(X) \cong \mathrm{Eff}(G)$ with respect to the basis $H, E_1, \dots, E_{n+3}$ of $H^2(X, \mathbb{R})$. Inequalities defining $\mathrm{Mov}^1(X)$ and $\mathrm{Eff}(X)$ were obtained in a different way by Brambilla, Dumitrescu and Postinghel [7]. In this section, we reinterpret the facets and extremal rays of these cones in terms of special divisors and curves in G.

Recall from Section 2 that $\mathcal{E} \subset H^n(Z, \mathbb{R})$ is the cone over the demihypercube Δ with vertices $\{M\}_{M \in \mathcal{F}_m(Z)}$. Its dual cone $\mathcal{E}^\vee \subset \mathcal{E}$ has $2(n+3) + 2^{n+2}$ extremal rays, generated by the classes

$$\bigl\{M + \sigma_i(M) \mid M \in \mathcal{F}_m(Z),\ i \in \{1, \dots, n+3\}\bigr\} \\ \cup \bigl\{\delta_M = \bigl\lfloor \tfrac{1}{2}(m+1) \bigr\rfloor \eta + (-1)^m M\bigr\}_{M \in \mathcal{F}_m(Z)}.$$

For a fixed $i \in \{1, \dots, n+3\}$, there are two distinct classes $M + \sigma_i(M)$ as M varies in $\mathcal{F}_m(Z)$, and they form an orbit for the action of W' on $H^n(Z, \mathbb{Z})$. The stabilizer of this orbit is the subgroup $G_i := \{\sigma_I \mid i \notin I \text{ and } |I| \text{ is even}\}$. The group W' acts transitively and freely on the set $\{\delta_M\}_{M \in \mathcal{F}_m(Z)}$. The facet of $\mathcal{E}$ corresponding to each extremal ray of $\mathcal{E}^\vee$ was described in Remark 2.23:

- $(M + \sigma_i(M))^\perp \cap \mathcal{E}$ is the cone over the $(n+2)$–dimensional demihypercube with vertices $\{\sigma_I(M) \mid I \subset \{1, \dots, n+3\} \smallsetminus \{i\},\ |I| \not\equiv m \bmod 2\}$.
- $(\delta_M)^\perp \cap \mathcal{E}$ is a simplicial cone generated by the classes $\sigma_i(M)$, $i \in \{1, \dots, n+3\}$.

Now we turn to cones of curves and divisors in G. We showed in Theorem 5.1 and Proposition 5.5 that

$$\mathrm{Nef}(G) = \beta(\mathcal{E}^\vee) \subset \beta(\mathcal{E}) = \mathrm{Eff}(G),$$
$$\mathrm{Mov}_1(G) = \alpha^{-1}(\mathcal{E}^\vee) \subset \alpha^{-1}(\mathcal{E}) = \mathrm{NE}(G).$$

We give a geometric description of the facets and extremal rays of these cones in terms of special divisors and curves in G.

6.1 (Eff(G)) The cone Eff(G) has 2^{n+2} extremal rays, generated by the classes $\{E_M\}_{M \in \mathcal{F}_m(Z)}$. Each E_M is a fixed prime divisor. The group $W' \subset \mathrm{Aut}(G)$ acts transitively and freely on the set $\{E_M\}_{M \in \mathcal{F}_m(Z)}$. In particular, all these divisors are isomorphic, and they can be described as a small modification of the blow-up of $\mathbb{P}^{n-1}$ at $n+3$ points contained in a rational normal curve (see Section 4.15 for a precise description).

6.2 (the divisor E_M when $n = 4$) Set $n = 4$; in this case E_M is isomorphic to the blow-up of $\mathbb{P}^3$ at 7 points contained in a rational normal curve. To describe geometrically E_M inside G, consider the closed subset

$$\{[L] \in G \mid L \cap M \neq \varnothing\}.$$

Then this locus is not equidimensional, and E_M is its unique divisorial component.

Indeed, let us consider again the incidence diagram

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\pi"'] \arrow[dr, "e"] & \\ G & & Z \end{tikzcd}$$

as in 3.5, so that $\dim \mathcal{I} = 5$, π is a $\mathbb{P}^1$–bundle and $\{[L] \in G \mid L \cap M \neq \varnothing\} = \pi(e^{-1}(M))$. For the purposes of this subsection only, it is better to denote by $[M] \in H^4(Z, \mathbb{Z})$ the fundamental class of the plane $M \subset Z$.

It is not difficult to see that e is flat, so that $e^{-1}(M)$ is equidimensional of dimension 3, and $e^*([M]) = [e^{-1}(M)] \in H^4(\mathcal{I}, \mathbb{Z})$. Then $\beta([M]) = \pi_* e^*([M]) = [\pi_*(e^{-1}(M))]$. By Proposition 5.5, we have $E_M = \pi_*(e^{-1}(M))$, so that E_M is the unique divisorial component of $\pi(e^{-1}(M))$.

Now let us consider the planes $M^*, \sigma_1(M)^*, \dots, \sigma_7(M)^* \subset G$ (see (3.1)); they are all contained in $\pi(e^{-1}(M))$.

Let $i \in \{1, \dots, 7\}$. Recall that $\ell_{\sigma_i(M)} \subset \sigma_i(M)^*$ is a line and that $\ell_{\sigma_i(M)} = \alpha(\sigma_i(M))$. By Proposition 3.7, using for instance (2.12), we have

$$E_M \cdot \ell_{\sigma_i(M)} = M \cdot \sigma_i(M) = -1,$$

so that $\sigma_i(M)^* \subset E_M$. On the other hand E_M contains only 7 planes $(M')^*$ (see Section 4.16), therefore M^* cannot be contained in E_M. This shows that M^* is a 2–dimensional irreducible component of $\pi(e^{-1}(M))$.

6.3 $(\mathrm{NE}(G))$ The cone $\mathrm{NE}(G)$ has 2^{n+2} extremal rays, generated by the classes $\{\ell_M\}_{M \in \mathcal{F}_m(Z)}$, on which $W' \subset \mathrm{Aut}(G)$ acts transitively. The contraction of the extremal ray generated by ℓ_M contracts $M^* \cong \mathbb{P}^m$ to a point.

Fix $M \in \mathcal{F}_m(Z)$ and consider the pseudoisomorphism $\xi_M \colon G \dashrightarrow X$ from Theorem 5.7. This fixes an identification of G with X^n_{Fano}, which identifies each divisor $E_{\sigma_I(M)} \subset G$ with the strict transform of the divisor $E_I \subset X$. Let $I \subset \{1, \dots, n+3\}$ be such that $|I| \leq m+1$. It follows from the discussion in Section 4.15 that

- If $|I| \not\equiv m \bmod 2$, then $(\sigma_I(M))^* \subset G$ is the strict transform of $J_{I,s} \subset X$, where $s = \frac{1}{2}(m+1-|I|)$.
- If $|I| \equiv m \bmod 2$, then $(\sigma_I(M))^* \subset G$ is the flipped locus of the flip of the strict transform of $J_{I,s} \subset X$, where $s = \frac{1}{2}(m-|I|)$.

In particular, we see that $(M')^* \subset E_M$ if and only if $M' = \sigma_I(M)$ for some $I \subset \{1, \dots, n+3\}$ with $|I| \leq m-1$ and $|I| \not\equiv m \bmod 2$.

6.4 $(\mathrm{Nef}(G))$ The cone $\mathrm{Nef}(G)$ has $2^{n+2} + 2(n+3)$ extremal rays, generated by the classes

$$\{D_M = \beta(\delta_M)\}_{M \in \mathcal{F}_m(Z)} \cup \bigl\{E_M + E_{\sigma_i(M)} \mid M \in \mathcal{F}_m(Z),\ i = 1, \dots, n+3\bigr\}.$$

For fixed i, the morphisms associated to the extremal rays generated by $E_M + E_{\sigma_i(M)}$ and $E_{\sigma_j(M)} + E_{\sigma_{ij}(M)}$ for $j \neq i$ are the generic $\mathbb{P}^1$–bundles $\varphi_i \colon G \to Y_{\varphi_i}$ and $\psi_i \colon G \to Y_{\psi_i}$ described in Lemma 3.3. The morphism associated to the extremal ray generated by D_M is the composition of the (disjoint) small contractions of $\sigma_i(M)^* \subset G$ to a point for $i = 1, \dots, n+3$.

6.5 $(\mathrm{Mov}_1(G))$ The cone $\mathrm{Mov}_1(G)$ has $2(n+3)+2^{n+2}$ extremal rays, generated by the curve classes

$$\{\ell_M+\ell_{\sigma_i(M)} \mid M\in\mathcal{F}_m(Z),\ i=1,\dots,n+3\}\cup\{d_M \mid M\in\mathcal{F}_m(Z)\},$$

where

$$d_M:=\alpha^{-1}(\delta_M)=\lfloor\tfrac{1}{2}(m+1)\rfloor\alpha^{-1}(\eta)+(-1)^m\ell_M\in\mathcal{N}_1(G).$$

For a fixed $i\in\{1,\dots,n+3\}$, there are two distinct classes $\ell_M+\ell_{\sigma_i(M)}$ as M varies in $\mathcal{F}_m(Z)$, and they form an orbit for the action of W' on $\mathcal{N}_1(G)$. By Corollary 3.4, these are the classes of the fibers of the generic $\mathbb{P}^1$–bundles $\varphi_i\colon G\to Y_{\varphi_i}$ and $\psi_i\colon G\to Y_{\psi_i}$. Under the identification $G\cong X^n_{\mathrm{Fano}}$ induced by a pseudoisomorphism $G\dashrightarrow X$, these correspond to the generic $\mathbb{P}^1$–bundles $\phi_i,\phi_i'\colon X^n_{\mathrm{Fano}}\to(X_{\mathcal{P}_i})^{n-1}_{\mathrm{Fano}}$ described in Lemma 4.12. In particular, we see that $Y_{\varphi_i}\cong Y_{\psi_i}\cong(X_{\mathcal{P}_i})^{n-1}_{\mathrm{Fano}}$.

As for the class d_M, using Proposition 3.7 and Remark 2.23, one computes

$$-K_G\cdot d_M=\eta\cdot\delta_M=n+1,$$
$$E_{\sigma_i(M)}\cdot d_M=\sigma_i(M)\cdot\delta_M=0\quad\text{for every } i=1,\dots,n+3.$$

Therefore d_M is the class of the strict transform in G of a general line in $\mathbb{P}^n$ under the map $\rho_M\colon G\dashrightarrow\mathbb{P}^n$.

In order to complete the picture, next we describe equations for the movable cone $\mathrm{Mov}^1(G)\subset H^2(G,\mathbb{R})$ and give a geometric description of the extremal rays of the dual cone $\mathrm{Mov}^1(G)^\vee\subset\mathcal{N}_1(G)$. We do this for $n\geq 4$, since when $n=2$ we have $\mathrm{Mov}^1(G)=\mathrm{Nef}(G)$ and $\mathrm{Mov}^1(G)^\vee=\mathrm{NE}(G)$.

6.6 Proposition *Suppose $n\geq 4$. The cone $\mathrm{Mov}^1(G)^\vee\subset\mathcal{N}_1(G)$ has $2^{n+2}+2(n+3)$ extremal rays, generated by the classes*

$$\{e_M \mid M\in\mathcal{F}_m(Z)\}\cup\{\ell_M+\ell_{\sigma_i(M)} \mid M\in\mathcal{F}_m(Z), i=1,\dots,n+3\},$$

where $e_M:=\lfloor\frac{1}{2}(m)\rfloor\alpha^{-1}(\eta)+(-1)^{m-1}\ell_M$.

Proof Recall from Section 4 that the intersection of $\mathrm{Mov}^1(X)$ with the affine hyperplane $\mathcal{H}\subset H^2(X,\mathbb{R})$ is given by

$$\Delta_{\mathrm{Mov}}=\begin{cases}-\frac{1}{2}\leq\alpha_i\leq\frac{1}{2}, & i\in\{1,\dots,n+3\},\\ H_I\geq 2, & |I|\ \text{odd}.\end{cases}$$

So $\mathrm{Mov}^1(G)=\beta(\mathcal{M})$, where $\mathcal{M}$ is the cone over Δ_{Mov}, now viewed as a polytope in the hyperplane $\{\gamma\mid\gamma\cdot\eta=1\}\subset H^n(Z,\mathbb{R})$.

Notice that the facet $(H_I = 2) \cap \Delta_{\mathrm{Mov}}$ of Δ_{Mov} is the convex hull of the vertices v_J such that $\#(I \smallsetminus J) + \#(J \smallsetminus I) = 2$. This follows from (2.17). In the same way as done in Section 2 for $\mathcal{E}$, one can use the linear inequalities defining Δ_{Mov} to compute the linear inequalities defining $\mathcal{M}$, or equivalently the generators of the dual cone $\mathcal{M}^\vee$. These are

$$\bigl\{M + \sigma_i(M) \mid M \in \mathcal{F}_m(Z),\ i \in \{1, \dots, n+3\}\bigr\} \cup \{\eta_M\}_{M \in \mathcal{F}_m(Z)},$$

where $\eta_M = \lfloor \frac{m}{2} \rfloor \eta + (-1)^{m-1} M$ (notice that $e_M = \alpha(\eta_M)$). Indeed, one can check using (2.12) that

$$(6.7) \qquad \eta_M \cdot \sigma_{ij}(M) = 0 \quad \text{for all } i \neq j.$$

By the duality properties of α and β, we have $\mathrm{Mov}^1(G)^\vee = \alpha^{-1}(\mathcal{M}^\vee)$, and the result follows. □

6.8 The classes $\ell_M + \ell_{\sigma_i(M)}$ were described in Section 6.5 above. Now we want to describe the classes e_M.

Given $M \in \mathcal{F}_m(Z)$ and $i \in \{1, \dots, n+3\}$, set $M_0 = \sigma_i(M)$, and follow Notation 2.8, so that $M = M_i$. Consider the pseudoisomorphism $\xi_{M_0}\colon G \dashrightarrow X$ from Theorem 5.7, and note that the divisor $E_M \subset G$ is the strict transform of the divisor $E_i \subset X$ under ξ_{M_0}. By (6.7) above, we have that

$$E_{M_j} \cdot e_M = 0 \quad \text{for all } j \neq i.$$

Similarly one computes that $E_M \cdot e_M = -1$. We conclude that e_M is the class of the strict transform under $\xi_{M_0}^{-1}$ of a general line in $E_i \cong \mathbb{P}^{n-1}$.

6.9 Remark Set $c := \alpha^{-1}(\eta) \in \mathcal{N}_1(G)$. We have

$$-K_G \cdot c = 4 \qquad \text{and} \qquad E_M \cdot c = 1 \quad \text{for every } M \in \mathcal{F}_m(Z).$$

The class c is fixed by the action of $W(D_{n+3})$ and sits in the interior of the cone $\mathrm{Mov}_1(G) \subset \mathrm{NE}(G)$. Let $M \in \mathcal{F}_m(Z)$ and consider the rational map $\rho_M\colon G \dashrightarrow \mathbb{P}^n$ from Theorem 5.7. Then c is the class of the strict transform via ρ_M^{-1} of an elliptic curve of degree $n+1$ in $\mathbb{P}^n$ through $p_1, \dots, p_{n+3}$. There is a 4–dimensional family of such curves (see Dolgachev [11]).

6.10 Remark Brambilla, Dumitrescu and Postinghel [7] describe the effective cone $\mathrm{Eff}^1(X) \subset H^2(X, \mathbb{R})$ by 3 sets of linear inequalities (A_n), (B_n) and $(C_{n,t})$. Similarly, the movable cone $\mathrm{Mov}^1(X) \subset H^2(X, \mathbb{R})$ is described by 3 sets of linear inequalities (A_n), (B_n) and $(D_{n,t})$ (see [7, Theorems 5.1 and 5.3]). These are related to the extremal rays of $\mathrm{Mov}_1(G)$ and $\mathrm{Mov}^1(G)^\vee$ described in Section 6.5 and 6.8 as follows. A divisor class $D \in H^2(G,\mathbb{R})$ satisfies the inequalities (A_n) and (B_n) if and only if

$$D \cdot (\ell_M + \ell_{\sigma_i(M)}) \geq 0 \quad \text{for every } M \in \mathcal{F}_m(Z) \text{ and } i = 1, \dots, n+3.$$

It satisfies the inequalities $(C_{n,t})$ if and only if

$$D \cdot d_M \geq 0 \quad \text{for every } M \in \mathcal{F}_m(Z).$$

Finally, it satisfies the inequalities $(D_{n,t})$ if and only if

$$D \cdot e_M \geq 0 \quad \text{for every } M \in \mathcal{F}_m(Z).$$

6.11 $(\mathrm{MCD}(G))$ Consider the subdivision in polytopes of the demihypercube $\Delta \subset \mathcal{H} \subset H^n(Z,\mathbb{R})$ given by the hyperplane arrangement (4.6). By taking the cones over these polytopes and using the isomorphism $\beta\colon H^n(Z,\mathbb{R}) \to H^2(G,\mathbb{R})$, this subdivision yields the fan $\mathrm{MCD}(G)$.

Fix $M_0 \in \mathcal{F}_m(Z)$ and consider the orthogonal basis $\varepsilon_1, \dots, \varepsilon_{n+3}$ of $\eta^\perp \subset H^n(Z,\mathbb{R})$ introduced in (2.9) and the affine coordinates $\alpha_1, \dots, \alpha_{n+3}$ in the hyperplane $\mathcal{H} := \{\gamma \mid \gamma \cdot \eta = 1\}$ described in (2.19). The group W' fixes $\mathcal{H}$ and η, thus it acts linearly in the coordinates α_i. More precisely it follows from (2.11) that, if $I \subset \{1, \dots, n+3\}$ has even cardinality, then $\sigma_I(\alpha_1, \dots, \alpha_{n+3}) = (\alpha'_1, \dots, \alpha'_{n+3})$ with

$$\alpha'_i = \begin{cases} \alpha_i & \text{if } i \notin I, \\ -\alpha_i & \text{if } i \in I. \end{cases}$$

The group W' fixes both Δ and Δ_{Mov}, while the 2^{n+2} polytopes $\sigma_I(\Delta_{\mathrm{Nef}})$ are all distinct. The corresponding cones in $\mathrm{MCD}(G)$ are $\xi^*_{M_I}(\mathrm{Nef}(X)) = \sigma^*_I(\xi^*_{M_0}(\mathrm{Nef}(X)))$.

7 The automorphism group of G

Let the setup be as in Section 5. In this section we describe the automorphism group of the Fano variety G, generalizing the description of the automorphism group of a quartic del Pezzo surface in Section 1.1.

7.1 Proposition *There are inclusion of groups*

$$(\mathbb{Z}/2\mathbb{Z})^{n+2} \cong W' \subseteq \mathrm{Aut}(G) \subseteq W(D_{n+3}) \cong (\mathbb{Z}/2\mathbb{Z})^{n+2} \rtimes S_{n+3}.$$

Moreover, if the points $(\lambda_1 : 1), \dots, (\lambda_{n+3} : 1) \in \mathbb{P}^1$ are general, then $\mathrm{Aut}(G) = W' \cong (\mathbb{Z}/2\mathbb{Z})^{n+2}$.

Notice that in the general case we also have $\mathrm{Aut}(Z) = W'$ (see Reid [25, Lemma 3.1]), so that Z and G have the same automorphism group.

Proof Clearly we have $W' \subseteq \mathrm{Aut}(G)$.

For any $\zeta \in \mathrm{Aut}(G)$, the induced isomorphism $\zeta^*\colon H^2(G,\mathbb{R}) \to H^2(G,\mathbb{R})$ preserves $-K_G$ and $\mathrm{Eff}(G)$. As in the proof of Proposition 5.4, one shows that $\zeta^* \in W(D_{n+3})$. This yields a group homomorphism

$$\mathrm{Aut}(G) \to W(D_{n+3}).$$

Fix $M_0 \in \mathcal{F}_m(Z)$. Consider the stabilizer G_0 of M_0 in $W(D_{n+3})$, and recall that $W(D_{n+3}) = W' \rtimes G_0 \cong (\mathbb{Z}/2\mathbb{Z})^{n+2} \rtimes S_{n+3}$. So, given $\zeta \in \mathrm{Aut}(G)$, there are unique elements $\omega \in G_0$ and $\sigma_I \in W'$ such that $\zeta^* = \omega \circ \sigma_I$. Set $\tilde{\zeta} := \sigma_I \circ \zeta \in \mathrm{Aut}(G)$. Then $\tilde{\zeta}^* = \zeta^* \circ \sigma_I = \omega$, so $\tilde{\zeta}^*$ fixes E_{M_0}, and hence it also fixes H_{M_0}.

Consider the rational map $\rho_{M_0}\colon G \dashrightarrow \mathbb{P}^n$ induced by H_{M_0}, which contracts the divisors $E_{M_1}, \dots, E_{M_{n+3}}$ to the points $p_1, \dots, p_{n+3}$ (see Theorem 5.7). Then $\tilde{\zeta}^*(\rho^*_{M_0}(\mathcal{O}_{\mathbb{P}^n}(1))) = \rho^*_{M_0}(\mathcal{O}_{\mathbb{P}^n}(1)) = H_{M_0}$, so ρ_{M_0} and $\rho_{M_0} \circ \tilde{\zeta}$ differ by a projective transformation $f \in \mathrm{Aut}(\mathbb{P}^n)$ preserving the set of points $\{p_1, \dots, p_{n+3}\}$:

$$\begin{tikzcd} G \arrow[r, "\tilde{\zeta}"] \arrow[d, dashed, "\rho_{M_0}"'] & G \arrow[d, dashed, "\rho_{M_0}"] \\ \mathbb{P}^n \arrow[r, "f"] & \mathbb{P}^n \end{tikzcd}$$

In particular, if the points $p_1, \dots, p_{n+3}$ are general, then $f = \mathrm{Id}_{\mathbb{P}^n}$, and so $\zeta = \sigma_I$.

Suppose that $\zeta^* = \mathrm{Id}_{H^2(G,\mathbb{R})}$. Then $\tilde{\zeta} = \zeta$ and f must fix each p_i. Since $p_1, \dots, p_{n+3}$ are in general linear position, this implies that $f = \mathrm{Id}_{\mathbb{P}^n}$, and hence $\zeta = \tilde{\zeta} = \mathrm{Id}_G$. This shows that the homomorphism $\mathrm{Aut}(G) \to W(D_{n+3})$ is injective, yielding the statement. □

Every automorphism of X is induced by a projective transformation of $\mathbb{P}^n$ preserving the set $\{p_1, \dots, p_{n+3}\}$. This in turns corresponds to a projective transformation of $\mathbb{P}^1$ preserving the set of points $\{(\lambda_1 : 1), \dots, (\lambda_{n+3} : 1)\} \subset \mathbb{P}^1$. In particular, if $\lambda_1, \dots, \lambda_{n+3}$ are general, then $\mathrm{Aut}(X) = \{\mathrm{Id}_X\}$.

For any projective variety Y, we denote by $\mathrm{Bir}^0(Y)$ the group of *pseudoautomorphisms* of Y. These are birational maps $Y \dashrightarrow Y$ which are isomorphisms in codimension one.

Since X and G are pseudoisomorphic, we have $\mathrm{Bir}^0(X) \cong \mathrm{Bir}^0(G)$. On the other hand, since G is a Fano manifold, we have $\mathrm{Bir}^0(G) = \mathrm{Aut}(G)$. Indeed if $\zeta \in \mathrm{Bir}^0(G)$, then $\zeta^*(-K_G) = -K_G$. Since ζ is an isomorphism in codimension one and $-K_G$ is ample, ζ must be regular, and similarly for ζ^{-1}.

7.2 Remark (explicit description of pseudoautomorphisms of X) The action of W' on X by pseudoautomorphisms is described by Dolgachev in [11, Sections 4.4–4.6]. Up to a projective transformation, we may assume that $p_1, \dots, p_{n+1}$ are the coordinate points, $p_{n+2} = (1 : \dots : 1)$ and $p_{n+3} = (a_0 : \dots : a_{n+3})$. Since no $n+1$ of the points lie on a hyperplane, all the a_j are nonzero.

Consider the standard Cremona map centered at $p_1, \dots, p_{n+1}$,

$$s\colon (z_0 : \dots : z_n) \mapsto \left(\frac{1}{z_0} : \dots : \frac{1}{z_n}\right).$$

It is regular at p_{n+2} and p_{n+3}, which map to itself and $(1/a_0 : \dots : 1/a_n)$, respectively. The projective transformation

$$r\colon (z_0 : \dots : z_n) \mapsto (a_0 z_0 : \dots : a_n z_n)$$

fixes $p_1, \dots, p_{n+1}$, maps p_{n+2} to p_{n+3}, and maps $(1/a_0 : \dots : 1/a_n)$ to p_{n+2}. So the composition

$$f_{n+2,n+3} = r \circ s\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$$

induces a pseudoautomorphism $\omega_{n+2,n+3}\colon X \dashrightarrow X$.

Similarly, for every $i, j \in \{1, \dots, n+3\}$ with $i < j$, we can define a birational involution $f_{ij}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, which is not regular only at $\{p_1, \dots, p_{n+3}\} \smallsetminus \{p_i, p_j\}$ and exchanges p_i and p_j. This induces a pseudoautomorphism $\omega_{ij}\colon X \dashrightarrow X$.

One can check that ω_{ij}^* acts on $H^2(X, \mathbb{Z})$ as follows:

$$\omega_{ij}^*(-K_X) = -K_X, \quad \omega_{ij}^*(E_i) = E_j, \quad \omega_{ij}^*(E_j) = E_i$$

$$\omega_{ij}^*(H) = nH - (n-1)\left(\sum_{h=1}^{n+1} E_h - E_i - E_j\right)$$

$$\begin{aligned}\omega_{ij}^*(E_r) &= H - \sum_{h=1}^{n+3} E_h + E_i + E_j + E_r \\ &= \frac{1}{n+1}(-K_X) - \frac{2}{n+1}\sum_{h=1}^{n+3} E_h + E_i + E_j + E_r \quad \text{for } r \neq i, j.\end{aligned}$$

Consider the isomorphism $\tilde{h}_{M_0}\colon H^2(X, \mathbb{R}) \to H^n(Z, \mathbb{R})$ defined in (5.2), and the corresponding action of ω_{ij}^* on $H^n(Z, \mathbb{R})$. We have

$$\omega_{ij}^*(\eta) = \eta \quad \text{and} \quad \omega_{ij}^*(\varepsilon_r) = \begin{cases} -\varepsilon_r & \text{if } r = i, j, \\ \varepsilon_r & \text{if } r \neq i, j. \end{cases}$$

(The latter can be checked using (2.14).) Hence $\omega_{ij}^* = \sigma_{ij}$ and ω_{ij} is the pseudoautomorphism of X induced by $\sigma_{ij} \in W'$. In particular, the pseudoautomorphism of X induced by $\sigma_1 \in W'$ is $\omega_{23}\omega_{45}\cdots\omega_{n+2,n+3}$, and so on.

References

[1] **C Araujo**, **A Massarenti**, *Explicit log Fano structures on blow-ups of projective spaces*, Proc. Lond. Math. Soc. 113 (2016) 445–473 MR

[2] **M Artin**, **D Mumford**, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. 25 (1972) 75–95 MR

[3] **S Bauer**, *Parabolic bundles, elliptic surfaces and $\mathrm{SU}(2)$–representation spaces of genus zero Fuchsian groups*, Math. Ann. 290 (1991) 509–526 MR

[4] **I Biswas**, **Y I Holla**, **C Kumar**, *On moduli spaces of parabolic vector bundles of rank 2 over $\mathbb{CP}^1$*, Michigan Math. J. 59 (2010) 467–479 MR

[5] **C Borcea**, *Deforming varieties of k–planes of projective complete intersections*, Pacific J. Math. 143 (1990) 25–36 MR

[6] **C Borcea**, *Homogeneous vector bundles and families of Calabi–Yau threefolds, II*, from "Several complex variables and complex geometry, II" (E Bedford, J P D'Angelo, R E Greene, S G Krantz, editors), Proc. Sympos. Pure Math. 52, Amer. Math. Soc., Providence, RI (1991) 83–91 MR

[7] **M C Brambilla**, **O Dumitrescu**, **E Postinghel**, *On the effective cone of $\mathbb{P}^n$ blown-up at $n+3$ points*, Exp. Math. 25 (2016) 452–465 MR

[8] **C Casagrande**, *Rank 2 quasiparabolic vector bundles on $\mathbb{P}^1$ and the variety of linear subspaces contained in two odd-dimensional quadrics*, Math. Z. 280 (2015) 981–988 MR

[9] **A-M Castravet**, **J Tevelev**, *Hilbert's 14^{th} problem and Cox rings*, Compos. Math. 142 (2006) 1479–1498 MR

[10] **I V Dolgachev**, *Weyl groups and Cremona transformations*, from "Singularities, I" (P Orlik, editor), Proc. Sympos. Pure Math. 40, Amer. Math. Soc., Providence, RI (1983) 283–294 MR

[11] **I V Dolgachev**, *On certain families of elliptic curves in projective space*, Ann. Mat. Pura Appl. 183 (2004) 317–331 MR

[12] **I V Dolgachev**, *Classical algebraic geometry: a modern view*, Cambridge University Press (2012) MR

[13] **I Dolgachev**, **A Duncan**, *Regular pairs of quadratic forms on odd-dimensional spaces in characteristic 2*, preprint (2015) arXiv

[14] **I Dolgachev**, **D Ortland**, *Point sets in projective spaces and theta functions*, Astérisque 165, Soc. Math. France, Paris (1988) MR

[15] **R M Green**, *Homology representations arising from the half cube*, Adv. Math. 222 (2009) 216–239 MR

[16] **R M Green**, *Combinatorics of minuscule representations*, Cambridge Tracts in Mathematics 199, Cambridge University Press (2013) MR

[17] **J Harris**, *Algebraic geometry: a first course*, Graduate Texts in Mathematics 133, Springer (1992) MR

[18] **Y Hu**, **S Keel**, *Mori dream spaces and GIT*, Michigan Math. J. 48 (2000) 331–348 MR

[19] **J E Humphreys**, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics 9, Springer (1972) MR

[20] **Z Jiang**, *A Noether–Lefschetz theorem for varieties of r–planes in complete intersections*, Nagoya Math. J. 206 (2012) 39–66 MR

[21] **S Mukai**, *Counterexample to Hilbert's fourteenth problem for the 3–dimensional additive group*, preprint, RIMS, Kyoto University (2001) Available at http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1343.pdf

[22] **S Mukai**, *An introduction to invariants and moduli*, Cambridge Studies in Advanced Mathematics 81, Cambridge University Press (2003) MR

[23] **S Mukai**, *Finite generation of the Nagata invariant rings in A–D–E cases*, preprint, RIMS, Kyoto University (2005) Available at http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1502.pdf

[24] **S Okawa**, *On images of Mori dream spaces*, Math. Ann. 364 (2016) 1315–1342 MR

[25] **M Reid**, *The complete intersection of two or more quadrics*, PhD thesis, University of Cambridge (1972) Available at http://homepages.warwick.ac.uk/~masda/3folds/qu.pdf

[26] **C Soulé**, **C Voisin**, *Torsion cohomology classes and algebraic cycles on complex projective manifolds*, Adv. Math. 198 (2005) 107–127 MR

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