Geometry & Topology 21 (2017) 3213–3230

The chromatic splitting conjecture at $n = p = 2$

AGNÈS BEAUDRY

We show that the strongest form of Hopkins' chromatic splitting conjecture, as stated by Hovey, cannot hold at chromatic level $n = 2$ at the prime $p = 2$. More precisely, for $V(0)$, the mod 2 Moore spectrum, we prove that $\pi_k L_1 L_{K(2)} V(0)$ is not zero when k is congruent to -3 modulo 8. We explain how this contradicts the decomposition of $L_1 L_{K(2)} S$ predicted by the chromatic splitting conjecture.

55P60, 55Q45

1 Introduction

Fix a prime p. Let S be the p–local sphere spectrum, and $L_n S$ be the Bousfield localization of S at the Johnson–Wilson spectrum $E(n)$. Let $K(n)$ be Morava K-theory. There is a homotopy pullback square called the chromatic fracture square:

$$\begin{array}{ccc} L_n S & \longrightarrow & L_{K(n)} S \\ \big\downarrow & & \big\downarrow \\ L_{n-1} S & \xrightarrow{\iota} & L_{n-1} L_{K(n)} S \end{array}$$

Let F_n be the fiber of the map $L_n S \to L_{K(n)} S$. Note that F_n is weakly equivalent to the fiber of ι. It was shown by Hovey [12, Lemma 4.1] that F_n is weakly equivalent to the function spectrum $F(L_{n-1} S, L_n S)$. Hopkins' chromatic splitting conjecture, as stated by Hovey [12, Conjecture 4.2], stipulates that ι is the inclusion of a wedge summand, so that

$$L_{n-1} L_{K(n)} S \simeq L_{n-1} S \vee \Sigma F_n. \tag{1-1}$$

We will call this the *weak* form of the chromatic splitting conjecture. However, [12, Conjecture 4.2] also gives an explicit decomposition of ΣF_n as a wedge of suspensions of spectra of the form $L_i S_p$ for $0 \le i < n$. We will call this the *strong* form of the chromatic splitting conjecture.

The conjectured decomposition comes from the connection between the $K(n)$–local category and the cohomology of a certain group called the Morava stabilizer group $\mathbb{G}_n$.

Published: 31 August 2017 DOI: 10.2140/gt.2017.21.3213

Let $\mathbb{S}_n$ be the group of automorphisms of the formal group law of $K(n)$ over $\mathbb{F}_{p^n}$. Then $\mathbb{G}_n$ is the extension of $\mathbb{S}_n$ by the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Let $\mathbb{W}$ be the Witt vectors on $\mathbb{F}_{p^n}$. There is a spectral sequence

$$H^s(\mathbb{G}_n, (E_n)_t) \Longrightarrow \pi_{t-s} L_{K(n)} S. \tag{1-2}$$

Note that $\mathbb{W}$ sits naturally in $(E_n)_0 \cong \mathbb{W}[\![u_1, \dots, u_{n-1}]\!]$. The inclusion induces a map

$$H^*(\mathbb{G}_n, \mathbb{W}) \to H^*(\mathbb{G}_n, (E_n)_0). \tag{1-3}$$

Morava proves in [16, Remark 2.2.5], using the work of Lazard, that

$$H^*(\mathbb{G}_n, \mathbb{W}) \otimes \mathbb{Q}_p \cong E(e_1, \dots, e_n)$$

for classes e_i of degree $2i-1$. Therefore, $H^*(\mathbb{G}_n, \mathbb{W})$ contains an exterior algebra $E(x_1, \dots, x_n)$ for appropriate integral multiples x_i of the generators e_i. The chromatic splitting conjecture stipulates that, for some choice of $x_1, \dots, x_n$, the exterior algebra $E(x_1, \dots, x_n)$ injects into $H^*(\mathbb{G}_n, (E_n)_0)$ under the map (1-3), and that the nonzero products $x_{i_1} \cdots x_{i_j}$ survive in (1-2) to nontrivial elements in $\pi_{-2(\sum i_k)+j} L_{K(n)} S$. Further, it states that there is a factorization

$$\begin{array}{ccc} S_p^{-2(\sum i_k)+j} & \longrightarrow & L_{n-\max(i_k)} S_p^{-2(\sum i_k)+j} \\ \big\downarrow & & \vdots \\ L_{K(n)} S & \longrightarrow & \Sigma F_n \end{array}$$

where S_p^m is the p–completion of S^m, and that these maps decompose ΣF_n as

$$\Sigma F_n \simeq \bigvee_{\substack{1 \le j \le n \\ 1 \le i_1 < \cdots < i_j \le n}} L_{n-\max(i_k)} S_p^{-2(\sum i_k)+j}. \tag{1-4}$$

The chromatic splitting conjecture has been shown for $n \le 2$ and for all primes p, except in the case $n = p = 2$. For $n = 1$, it follows immediately from a computation of $\pi_* L_1 S_p$; see Ravenel [19, Theorems 8.10 and 8.15]. At $n = 2$ and $p \ge 5$, it is due to Hopkins, and follows from Shimomura and Yabe's computations [23]. The proof can be found in Behrens' account of their work [4, Remark 7.8]. At $n = 2$ and $p = 3$, the conjecture was proved recently by Goerss, Henn and Mahowald [9].

In this paper, we show that the chromatic splitting conjecture as stated above cannot hold for $n = p = 2$. More precisely, we show that [12, Conjecture 4.2(iv)] fails in this case. At $n = 2$, (1-1) and (1-4) imply that

$$L_1 L_{K(2)} S \simeq L_1 S_p \vee L_1 S_p^{-1} \vee L_0 S_p^{-3} \vee L_0 S_p^{-4}. \tag{1-5}$$

We show that the right-hand side of (1-5) has too few homotopy groups for the equivalence to hold. However, our results do not contradict the possibility that ι is the inclusion of a wedge summand. Giving an alternative description for the fiber in this case is work in progress.

That our methods could disprove (1-5) was first suggested to the author by Paul Goerss. He and Mark Mahowald had been studying the computations of Shimomura and Wang [22] and Shimomura [21] and noticed that these suggest that the right-hand side of (1-5) is too small.

Statement of the results Let $V(0)$ be the cofiber of multiplication by p on S. Note that for any p–local spectrum X, there is a cofiber sequence

$$X \xrightarrow{p} X \to X \wedge V(0).$$

Since Bousfield localization of spectra preserves exact triangles, it follows that

$$L_E V(0) \simeq L_E S \wedge V(0)$$

for any spectrum E. This has the following consequence.

Proposition 1.1 *The strong form of the chromatic splitting conjecture at $n = 2$ implies that $L_1 L_{K(2)} V(0) \simeq L_1 V(0) \vee L_1 \Sigma^{-1} V(0)$.*

We now fix our attention to the case when $p = 2$. Since $L_0 V(0)$ is contractible, it follows from the chromatic fracture square that $L_1 V(0) \simeq L_{K(1)} V(0)$. Computing $\pi_* L_{K(1)} V(0)$ is a routine exercise using the spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}_1, (E_1)_* V(0)) \Longrightarrow \pi_{t-s} L_{K(1)} V(0). \tag{1-6}$$

The E_∞–term is given in Figure 1. At $p = 2$, we have that $V(0)$ is not a ring spectrum. This manifests itself by the fact that $\pi_* L_{K(1)} V(0)$ is not a ring. In fact,

$$\pi_* L_{K(1)} V(0) = \big(\mathbb{Z}_2[\eta, \beta^{\pm 1}, \zeta_1]/(2\eta, \eta^3, \zeta_1^2)\big)\{e_0, v_1 e_0\}/(2e_0, 2v_1 e_0 - \eta^2 e_0),$$

where $\eta \in \pi_1$ is the Hopf map, $\beta \in \pi_8$ is the v_1–self-map detected by v_1^4, and $\zeta_1 \in \pi_{-1}$ is detected by a generator of $H^1(\mathbb{G}_1, \mathbb{Z}_2) \cong H^1(\mathbb{Z}_2^\times, \mathbb{Z}_2)$. The element $e_0 \in \pi_0$ represents the inclusion of the bottom cell $S^0 \hookrightarrow V(0)$, and $v_1 e_0 \in \pi_2$ is a lift of $\Sigma\eta$ to the top cell:

$$\begin{array}{ccccccc} & & & & S^2 & & \\ & & & \overset{v_1 e_0}{\swarrow} & \downarrow \Sigma\eta & & \\ S^0 & \xrightarrow{e_0} & V(0) & \longrightarrow & S^1 & \xrightarrow{2} & S^1 \end{array}$$

The following result is a consequence of Proposition 1.1.

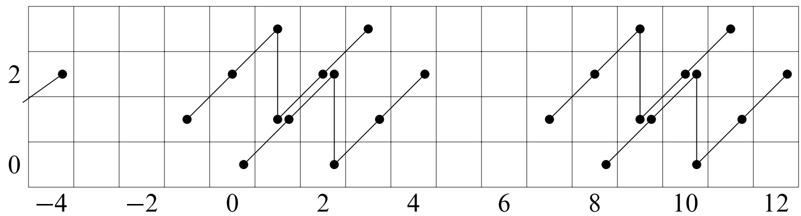

Figure 1: The E_∞–term of (1-6) computing $\pi_* L_{K(1)}V(0)$. Vertical lines denote extensions by multiplication by 2, and lines of slope one denote multiplication by η.

Corollary 1.2 *The chromatic splitting conjecture implies that $\pi_k L_1 L_{K(2)}V(0)$ is zero when $k \equiv -3$ modulo 8.*

However, in this paper, we prove the following result.

Theorem 1.3 *There are nontrivial homotopy classes $\beta^t x$ in $\pi_{8t-3}L_1L_{K(2)}V(0)$ and $\zeta_2\beta^t x$ in $\pi_{8t-4}L_1L_{K(2)}V(0)$.*

This has the following immediate consequence.

Theorem 1.4 *The homotopy group $\pi_k L_1 L_{K(2)}V(0)$ is nonzero when $k \equiv -3$ modulo 8. Therefore, the decomposition (1-5) of the chromatic splitting conjecture does not hold when $n=2$ and $p=2$.*

The broad strokes of the proof of Theorem 1.3 when $t=0$ are as follows. Let $G_{24} \cong Q_8 \rtimes C_3$ be a representative of the unique conjugacy class of maximal finite subgroups of $\mathbb{S}_2$. Let C_6 be a subgroup of G_{24} of order 6. Let $\mathbb{S}_2^1$ be the norm one subgroup so that $\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2$ (see Section 2). It follows from the duality resolution techniques of Goerss, Henn, Mahowald and Rezk and the work of Bobkova [6] that, for any X, there is a spectral sequence

$$E_1^{p,t} = \pi_t(\mathscr{E}_p \wedge X) \implies \pi_{t-p}(E_2^{h\mathbb{S}_2^1} \wedge X),$$

where $\mathscr{E}_p$ are spectra such that $\mathscr{E}_0 \simeq E_2^{hG_{24}}$, $\mathscr{E}_p \simeq E_2^{hC_6}$ if $p=1,2$ and $(E_2)_*\mathscr{E}_3 \cong (E_2)_*E_2^{hG_{24}}$ as Morava modules. Localizing at $E(1)$, we obtain a spectral sequence

$$E_1^{p,t} = \pi_t L_1(\mathscr{E}_p \wedge X) \implies \pi_{t-p} L_1(E_2^{h\mathbb{S}_2^1} \wedge X). \tag{1-7}$$

We use this spectral sequence to show that $\pi_{-3}L_1(E_2^{h\mathbb{S}_2^1} \wedge V(0)) \cong \mathbb{F}_4$, in Lemma 4.1 and Proposition 4.2. After taking Galois invariants, we obtain a nonzero element x in $\pi_{-3}L_1(E_2^{h\mathbb{G}_2^1} \wedge V(0))$. In the cofiber sequence

$$L_1L_{K(2)}V(0) \to L_1(E_2^{h\mathbb{G}_2^1} \wedge V(0)) \to L_1(E_2^{h\mathbb{G}_2^1} \wedge V(0)),$$

which is obtained from the cofiber sequence $L_{K(2)}S \to E_2^{h\mathbb{G}_2^1} \to E_2^{h\mathbb{G}_2^1}$ by smashing with $V(0)$ and localizing at $E(1)$; this class gives rise to nonzero elements $x \in \pi_{-3}L_1L_{K(2)}V(0)$ and $\zeta_2 x \in \pi_{-4}L_1L_{K(2)}V(0)$.

Warning 1.5 We use the notation ζ_2 to denote the homotopy class defined by

$$\begin{array}{ccccccc} & & & & S^0 & & \\ & & & & \downarrow{\scriptstyle 1} & \searrow^{\zeta_2} & \\ L_{K(2)}S & \longrightarrow & E_2^{h\mathbb{G}_2^1} & \longrightarrow & E_2^{h\mathbb{G}_2^1} & \longrightarrow & \Sigma L_{K(2)}S \end{array}$$

Experts will notice that this clashes with Ravenel [17, Lemma 2.1], but this is the natural generalization of what is now commonly denoted by ζ_n at odd primes.

Organization of the paper In Section 2, we specialize to the case $n = 2$ and $p = 2$ and describe the duality resolution spectral sequence and its $E(1)$–localization. In Section 3, we compute the E_1–page of this spectral sequence for $V(0)$. In Section 4, we prove Theorem 1.3.

Acknowledgements I thank Paul Goerss, Hans-Werner Henn and Peter May for their constant help and support. I thank Tobias Barthel, Daniel G Davis and Douglas Ravenel for helping me sort out some of the details for the proofs of Section 2. I also thank Mark Behrens, Irina Bobkova, Michael Hopkins, Jack Morava, Niko Naumann and Zhouli Xu for helpful conversations. Finally, I thank Mark Mahowald for the insight he shared with all of us throughout his life.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1612020.

2 The $E(1)$–local duality resolution spectral sequence

We take the point of view that, at height 2, the Honda formal group law may be replaced by the formal group law of a supersingular elliptic curve. This was carefully explained in [3, Section 1]. (The reader who wants to ignore this subtlety may take $\mathbb{S}_{\mathcal{C}}$, $\mathbb{G}_{\mathcal{C}}$ and $E_{\mathcal{C}}$ to mean $\mathbb{S}_2$, $\mathbb{G}_2$ and E_2, respectively.)

Let $\mathbb{S}_{\mathcal{C}}$ be the group of automorphisms of the formal group law of the supersingular elliptic curve

$$\mathcal{C}: \quad y^2 + y = x^3$$

of height two over $\mathbb{F}_4$; see [3, Section 3] for the comparison. It admits an action of the Galois group $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Define

$$\mathbb{G}_{\mathcal{C}} = \mathbb{S}_{\mathcal{C}} \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

Let $E_{\mathcal{C}}$ be the spectrum which classifies the deformations of the formal group law of $\mathcal{C}$ over $\mathbb{F}_4$ as described, for example, in Rezk [20]. It can be chosen to be a complex oriented ring spectrum with

$$(E_{\mathcal{C}})_* = \mathbb{W}[\![u_1]\!][u^{\pm 1}]$$

for $|u_1| = 0$, $|u| = -2$, whose formal group law is the formal group law of the curve

$$\mathcal{C}_U: \quad y^2 + 3u_1xy + (u_1^3 - 1)y = x^3. \tag{2-1}$$

It admits an action of $\mathbb{G}_{\mathcal{C}}$, and for any finite spectrum X,

$$L_{K(2)}X \simeq E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}} \wedge X \simeq (E_{\mathcal{C}} \wedge X)^{h\mathbb{G}_{\mathcal{C}}};$$

see Behrens and Davis [5, page 5]. The group of automorphisms $\mathrm{Aut}(\mathcal{C})$ of $\mathcal{C}$ is of order 24 and injects into $\mathbb{S}_{\mathcal{C}}$. We let G_{24} denote the image of $\mathrm{Aut}(\mathcal{C})$. We note that

$$G_{24} \cong Q_8 \rtimes C_3,$$

where Q_8 is a quaternion subgroup and C_3 a cyclic group of order 3. The group $\mathbb{S}_{\mathcal{C}}$ contains a central subgroup of order 2, which we denote by C_2. We define

$$C_6 = C_2 \times C_3.$$

There is a surjective homomorphism $N\colon \mathbb{S}_{\mathcal{C}} \to \mathbb{Z}_2^\times/(\pm 1) \cong \mathbb{Z}_2$, which we call the *norm*. It is constructed using the determinant of a representation $\rho\colon \mathbb{S}_{\mathcal{C}} \to GL_2(\mathbb{W})$; see [3, Section 3]. Further, it can be extended to $\mathbb{G}_{\mathcal{C}}$. We let $\mathbb{S}_{\mathcal{C}}^1$ and $\mathbb{G}_{\mathcal{C}}^1$ be the kernels of the norms, and note that the elements of finite order in $\mathbb{S}_{\mathcal{C}}$ and $\mathbb{G}_{\mathcal{C}}$ are contained in $\mathbb{S}_{\mathcal{C}}^1$ and $\mathbb{G}_{\mathcal{C}}^1$ respectively. Further,

$$\mathbb{S}_{\mathcal{C}} \cong \mathbb{S}_{\mathcal{C}}^1 \rtimes \mathbb{Z}_2 \quad \text{and} \quad \mathbb{G}_{\mathcal{C}} \cong \mathbb{G}_{\mathcal{C}}^1 \rtimes \mathbb{Z}_2. \tag{2-2}$$

The formal group law $F_{\mathcal{C}_U}$ of $\mathcal{C}_U$, is not 2–typical. Nonetheless, it is strictly isomorphic to a 2–typical formal group law classified by a map $BP_* \to (E_{\mathcal{C}})_*$. Further, $[2]_{F_{\mathcal{C}_U}}(x) \equiv u_1u^{-1}x^2$ modulo $(2, x^4)$; see [3, Section 6.1] for details on $F_{\mathcal{C}_U}$. The strict isomorphism between $F_{\mathcal{C}_U}$ and its 2–typification preserves this identity. Hence, v_1 is mapped to u_1u^{-1} modulo (2). Since we are working primarily modulo (2), we abuse notation and let $v_1 = u_1u^{-1} \in (E_{\mathcal{C}})_2$.

We will need the following result, which can be found in Henn [11, Theorem 13] and is also discussed in greater detail in Bobkova [6]. We restate it here using our notation for convenience.

Theorem 2.1 (Goerss, Henn, Mahowald, Rezk and Bobkova) *There is a resolution of spectra in the $K(2)$–local category given by*

$$\begin{array}{ccccccccc} E_{\mathcal{C}}^{h\mathbb{S}_{\mathcal{C}}^1} & \longrightarrow & E_{\mathcal{C}}^{hG_{24}} & \longrightarrow & E_{\mathcal{C}}^{hC_6} & \longrightarrow & E_{\mathcal{C}}^{hC_6} & \longrightarrow & \mathscr{E}_3 \\ & & \| & & \| & & \| & & \| \\ & & \mathscr{E}_0 & \longrightarrow & \mathscr{E}_1 & \longrightarrow & \mathscr{E}_2 & \longrightarrow & \mathscr{E}_3 \end{array}$$

where $(E_{\mathcal{C}})_*\mathscr{E}_3 \cong (E_{\mathcal{C}})_* E_{\mathcal{C}}^{hG_{24}}$ as Morava modules. Further, for any spectrum X, the resolution gives rise to a tower of fibrations spectral sequence

$$E_1^{p,t} = \pi_t(\mathscr{E}_p \wedge X) \xRightarrow{SS_1} \pi_{t-p}(E_{\mathcal{C}}^{h\mathbb{S}_{\mathcal{C}}^1} \wedge X) \tag{2-3}$$

with differentials $d_r\colon E_r^{p,t} \to E_r^{p+r,t+r-1}$.

We call the resolution of Theorem 2.1 the *duality resolution*. Let π generate $\mathbb{Z}_2$ in the decompositions (2-2), and let $G'_{24} = \pi G_{24}\pi^{-1}$. Recall from [3] or [2] that there is also an *algebraic* duality resolution:

$$\begin{array}{ccccccccccc} 0 \to \mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}^1/G'_{24}]\!] & \to & \mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}^1/C_6]\!] & \to & \mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}^1/C_6]\!] & \to & \mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}^1/G_{24}]\!] & \to \mathbb{Z}_2 \to 0 \\ \| & & \| & & \| & & \| & \\ \mathscr{C}_3 & \longrightarrow & \mathscr{C}_2 & \longrightarrow & \mathscr{C}_1 & \longrightarrow & \mathscr{C}_0 & \end{array} \tag{2-4}$$

Now, let X be a finite spectrum. Resolving (2-4) into a double complex of projective $\mathbb{S}_{\mathcal{C}}^1$–modules and applying the functor $\mathrm{Hom}_{\mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}^1]\!]}(-, (E_{\mathcal{C}})_t X)$ gives rise to a spectral sequence

$$E_1^{p,q,t} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}^1]\!]}(\mathscr{C}_p, (E_{\mathcal{C}})_t X) \xRightarrow{SS_2} H^{p+q}(\mathbb{S}_{\mathcal{C}}^1, (E_{\mathcal{C}})_t X) \tag{2-5}$$

with differentials $d_r\colon E_r^{p,q,t} \to E_r^{p+r,q-r+1,t}$. Further, in each fixed degree p, there are spectral sequences

$$E_1^{p,q,t} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}^1]\!]}(\mathscr{C}_p, (E_{\mathcal{C}})_t X) \xRightarrow{SS_3} \pi_{t-q}(\mathscr{E}_p \wedge X) \tag{2-6}$$

with differentials $d_r\colon E_r^{p,q,t} \to E_r^{p,q+r,t+r-1}$. Finally, there is also a spectral sequence

$$E_2^{s,t} = H^s(\mathbb{S}_{\mathcal{C}}^1, (E_{\mathcal{C}})_t X) \xRightarrow{SS_4} \pi_{t-s}(E_{\mathcal{C}}^{h\mathbb{S}_{\mathcal{C}}^1} \wedge X) \tag{2-7}$$

with differentials $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. Thus, for X finite, we obtain a diagram of spectral sequences:

$$\begin{array}{ccc} \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]}(\mathscr{C}_p, (E_{\mathcal{C}})_t X) & \overset{SS_2}{\Longrightarrow} & H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_t X) \\ {\scriptstyle SS_3}\Big\Downarrow & & \Big\Downarrow{\scriptstyle SS_4} \\ \pi_{t-q}(\mathscr{E}_p \wedge X) & \underset{SS_1}{\Longrightarrow} & \pi_{t-(p+q)}(E_{\mathcal{C}}^{h\mathbb{S}^1_{\mathcal{C}}} \wedge X) \end{array} \tag{2-8}$$

Remark 2.2 For elements of Adams–Novikov filtration $s = 0$ in $E_1^{p,t}(SS_1)$, the differentials d_1 are related to the d_1–differentials in the algebraic duality resolution spectral sequence SS_2 in the following way. If X is finite, as in [10, Proposition 2.4 and (2.7)], for G a closed subgroup of $\mathbb{G}_{\mathcal{C}}$, there are isomorphisms of Morava modules

$$(E_{\mathcal{C}})_t(E_{\mathcal{C}}^{hG} \wedge X) \cong \mathrm{Hom}^c(\mathbb{G}_{\mathcal{C}}/G, (E_{\mathcal{C}})_t X) \cong \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2[\![\mathbb{G}_{\mathcal{C}}/G]\!], (E_{\mathcal{C}})_t X). \tag{2-9}$$

Let

$$E_1(SS_1)^{p,t} \cong \pi_t(\mathscr{E}_p \wedge X) \xrightarrow{h} H^0(\mathbb{G}_{\mathcal{C}}, (E_{\mathcal{C}})_t(\mathscr{E}_p \wedge X)) \cong E_1^{p,0,t}(SS_2)$$

be the edge homomorphism for the spectral sequence

$$H^s(\mathbb{G}_{\mathcal{C}}, (E_{\mathcal{C}})_t(\mathscr{E}_p \wedge X)) \implies \pi_{t-s}(\mathscr{E}_p \wedge X).$$

The spectral sequence SS_1 is constructed so that the following diagram commutes:

$$\begin{array}{ccc} E_1^{p,t}(SS_1) & \xrightarrow{h} & E_1^{p,0,t}(SS_2) \\ {\scriptstyle d_1}\big\downarrow & & \big\downarrow{\scriptstyle d_1} \\ E_1^{p+1,t}(SS_1) & \xrightarrow{h} & E_1^{p+1,0,t}(SS_2) \end{array}$$

When both horizontal maps h are injective, one can deduce information in SS_1 from information in SS_2.

For the statement of the next result, recall that for any closed subgroup F of $\mathbb{G}_{\mathcal{C}}$ and finite spectrum X, there is a spectral sequence

$$E_2^{s,t}(F, X) = H^s(F, (E_{\mathcal{C}})_t X) \implies \pi_{t-s}(E_{\mathcal{C}}^{hF} \wedge X). \tag{2-10}$$

The author learned the proof of the following result from Paul Goerss.

Lemma 2.3 *Let S a closed subgroup of $\mathbb{S}_{\mathcal{C}}$ which is invariant under the action of $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Let $G \cong S \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ be the corresponding closed subgroup of $\mathbb{G}_{\mathcal{C}}$. Then for any finite X and any $2 \le r \le \infty$,*

$$E_r^{s,t}(S, X) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} E_r^{s,t}(G, X),$$

and the differentials of the spectral sequence $E_r^{s,t}(S, X)$ are $\mathbb{W}$–linear.

Proof The action of $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ on $(E_{\mathcal{C}})_*X$ is semilinear over $\mathbb{W}$, so there is an isomorphism $E_2^{*,*}(S,X) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} E_2^{*,*}(G,X)$. Now consider, $E_r^{s,t}(\mathbb{S}_{\mathcal{C}}, S^0)$. We have $E_2^{0,0}(\mathbb{S}_{\mathcal{C}}, S^0) \cong \mathbb{W}$ and the subring $\mathbb{Z}_2$ of $\mathbb{W}$ consists of permanent cycles. The spectral sequence $E_r^{*,*}(\mathbb{S}_{\mathcal{C}}, S^0)$ is multiplicative, so the differentials $d_r\colon E_r^{0,0} \to E_r^{r,r-1}$ are $\mathbb{Z}_2$–derivations. Since $\mathbb{W}$ is an étale extension of $\mathbb{Z}_2$, for any r, the $\mathbb{Z}_2$–derivations from $\mathbb{W}$ to the $\mathbb{W}$–module $E_r^{r,r-1}$ are zero. Hence, $E_2^{0,0}(\mathbb{S}_{\mathcal{C}}, S^0) \cong \mathbb{W}$ consists of permanent cycles and the differentials are $\mathbb{W}$–linear. Since the spectral sequence $E_r^{*,*}(S,X)$ is one of modules over $E_r^{*,*}(\mathbb{S}_{\mathcal{C}}, S^0)$, the differentials of $E_r^{*,*}(S,X)$ are also $\mathbb{W}$–linear, and the result follows. □

In what follows, we will use the following remark.

Remark 2.4 Let X be a finite spectrum and F be a closed subgroup of $\mathbb{G}_{\mathcal{C}}$. As noted by Devinatz in the proof of [7, Lemma 3.5], it follows from the fact that $E_{\mathcal{C}}^{hF}$ is $(K_{\mathcal{C}})_*$–local $E_{\mathcal{C}}$–nilpotent, (see Devinatz and Hopkins [8, Proposition A.3]) that the descent spectral sequence (2-10) has a horizontal vanishing line.

Now, recall that the telescope conjecture holds at height $n=1$. This was proved at odd primes by Miller [15] and at $p=2$ by Mahowald [14]. In particular, we have the following result.

Theorem 2.5 (Mahowald and Miller) *Let Y admit a v_1–self-map $v_1^k\colon \Sigma^{2k}Y \to Y$. Then*

$$L_1Y \simeq L_{K(1)}Y \simeq v_1^{-1}Y,$$

where

$$v_1^{-1}Y := \mathrm{colim}\big(\cdots \xrightarrow{v_1^k} \Sigma^{2k}Y \xrightarrow{v_1^k} Y \xrightarrow{v_1^k} \cdots\big).$$

Proposition 2.6 *For any finite type-1 spectrum X, with self map $v_1^k\colon \Sigma^{2k}X \to X$, there is a diagram of strongly convergent spectral sequences:*

$$\begin{array}{ccc}
v_1^{-1}\operatorname{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]}(\mathscr{C}_p, (E_{\mathcal{C}})_tX) & \xRightarrow{\;L_1SS_2\;} & v_1^{-1}H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_tX) \\
{\scriptstyle L_1SS_3}\Big\Downarrow & & \Big\Downarrow{\scriptstyle L_1SS_4} \\
\pi_{t-q}L_1(\mathscr{E}_p \wedge X) & \xRightarrow[\;L_1SS_1\;]{} & \pi_{t-(p+q)}L_1(E_{\mathcal{C}}^{h\mathbb{S}^1_{\mathcal{C}}} \wedge X)
\end{array}$$

Proof The spectral sequence L_1SS_2 is obtained from SS_2 by inverting the element $v_1^k \in (E_{\mathcal{C}})_{2k}X$, and L_1SS_1 is obtained by the applying L_1 to the tower of fibrations which gives rise to SS_1. The spectral sequences L_1SS_3 and L_1SS_4 are obtained by inverting the algebraic element v_1^k in the spectral sequences SS_3 or SS_4, and using the fact that

$$v_1^{-1}\pi_*(\mathscr{E}_p \wedge X) \cong \pi_*L_1(\mathscr{E}_p \wedge X).$$

With regards to the strong convergence of the four spectral sequences, note that localization with respect to v_1 is exact. Therefore, the localized spectral sequences will converge strongly if they have horizontal vanishing lines at the E_∞–term. The spectral sequences SS_1 and SS_2 have a vanishing line at $p=4$ for all $r \geq 1$. As noted in Remark 2.4, the descent spectral sequences SS_3 and SS_4 have horizontal vanishing lines. Therefore, the spectral sequences L_1SS_i exist and converge. □

Remark 2.7 As in Remark 2.2, the differentials d_1 in L_1SS_1 and L_1SS_2 commute with the edge homomorphisms

$$E_1(L_1SS_1)^{p,t} \cong \pi_t L_1(\mathscr{E}_p \wedge X) \xrightarrow{h} v_1^{-1}H^0(\mathbb{G}_{\mathcal{C}}, (E_{\mathcal{C}})_t(\mathscr{E}_p \wedge X)) \cong E_1^{p,0,t}(L_1SS_2).$$

Remark 2.8 For X as in Proposition 2.6, the element $v_1^{2k} \in (E_{\mathcal{C}})_{2k}X$ can be chosen to be Galois invariant. Therefore, the results of Lemma 2.3 also hold for the localized spectral sequences. That is, let

$$v_1^{-1}E_2^{s,t}(F,X) \cong v_1^{-1}H^s(F,(E_{\mathcal{C}})_tX) \implies \pi_{t-s}L_1(E_{\mathcal{C}}^{hF} \wedge X).$$

Then for S and G as in Lemma 2.3, we have

$$v_1^{-1}E_r^{s,t}(S,X) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} v_1^{-1}E_r^{s,t}(G,X)$$

for $2 \leq r \leq \infty$, and the differentials are $\mathbb{W}$–linear.

3 The homotopy of $L_1(E_{\mathcal{C}}^{hG_{24}} \wedge V(0))$ and $L_1(E_{\mathcal{C}}^{hC_6} \wedge V(0))$

The spectrum $V(0)$ has a self map

$$\beta\colon \Sigma^8V(0) \xrightarrow{v_1^4} V(0),$$

and in this section, we give the E_1–term for

$$E_1^{p,q}(L_1SS_1) = \pi_qL_1(\mathscr{E}_p \wedge V(0)) \xRightarrow{L_1SS_1} \pi_{q-p}L_1(E_{\mathcal{C}}^{h\mathbb{S}_{\mathcal{C}}^1} \wedge V(0)).$$

In order to do so, we must compute $\pi_*L_1(E_{\mathcal{C}}^{hG_{24}} \wedge V(0))$ and $\pi_*L_1(E_{\mathcal{C}}^{hC_6} \wedge V(0))$. We do this using the descent spectral sequences

$$v_1^{-1}H^s(G,(E_{\mathcal{C}})_tV(0)) \implies \pi_{t-s}L_1(E_{\mathcal{C}}^{hG} \wedge V(0)).$$

Notation 3.1 We use the following conventions. First,

$$v_1 = u_1u^{-1}, \quad v_2 = u^{-3} \quad \text{and} \quad j_0 = u_1^3.$$

The element Δ is the discriminant of $\mathcal{C}_U$, and hence is given by

$$\Delta = 27v_2(v_1^3 - v_2)^3 \equiv v_2(v_1^3+v_2)^3 \mod (2),$$

and

$$c_4 = 9v_1^4 + 72v_1v_2 \equiv v_1^4 \mod (2).$$

The j–invariant is

$$j = c_4^3\Delta^{-1} \equiv v_1^{12}\Delta^{-1} \mod (2).$$

These identities can be computed using Silverman [24, Section III.1]; see also [3, Section 4.2]. We abuse notation and let

$$\eta = \delta(v_1),$$

where δ is the Bockstein associated to

$$0 \to (E_{\mathcal{C}})_*/2 \xrightarrow{2} (E_{\mathcal{C}})_*/4 \to (E_{\mathcal{C}})_*/2 \to 0.$$

This is justified by the fact that $\delta(v_1)$ detects the homotopy class η (see [3, Section 4.1]).

The v_1–torsion-free elements of $H^*(G_{24},(E_{\mathcal{C}})_*V(0))$ generate a submodule isomorphic to

$$\mathbb{F}_4[\![j]\!][v_1,\eta,\Delta^{\pm 1},k]/(\eta^4 - v_1^4k,\ j\Delta - v_1^{12})$$

for elements of degrees (s,t), where s is the cohomological grading, t is the internal grading, and

$$|v_1| = (0,2), \quad |\eta| = (1,2), \quad |\Delta| = (0,24), \quad |k| = (4,0), \quad |j| = (0,0);$$

see Section 4.2 or the appendix of [3]. On the other hand, $H^*(C_6,(E_{\mathcal{C}})_*V(0))$ is v_1–torsion-free and is isomorphic to

$$\mathbb{F}_4[\![j_0]\!][v_1,\eta,v_2^{\pm 1},h]/(\eta - v_1h,\ j_0v_2 - v_1^3),$$

where $|v_2| = (0,6)$, $|h| = (1,0)$ and $|j_0| = (0,0)$; see Section 4.2 of [3].

The next proposition is an immediate consequence of these results. In its statement, we let $\mathbb{F}_4((x))$ denote the Laurent series on x.

Proposition 3.2 *There are isomorphisms*

$$v_1^{-1}H^*(G_{24},(E_{\mathcal{C}})_*V(0)) \cong \mathbb{F}_4((j))[v_1^{\pm 1},\eta]$$

and

$$v_1^{-1}H^*(C_6;(E_{\mathcal{C}})_*V(0)) \cong \mathbb{F}_4((j_0))[v_1^{\pm 1},\eta].$$

The degrees (s,t) are given by $|v_1| = (0,2)$, $|\eta| = (1,2)$, $|j| = (0,0)$ and $|j_0| = (0,0)$. The restriction associated to the inclusion of C_6 in G_{24} maps j to $j_0^4(1+j_0)^{-3}$.

Proof This follows from [3, Section 4.2] after inverting v_1. □

To compute the differentials, we will use the following observation.

Remark 3.3 There is a class α_3 in $\mathrm{Ext}^{1,6}_{BP_*BP}(BP_*, BP_*)$ (see Ravenel [18, page 430]) such that $d_3(\alpha_3) = \eta^4$. Further, α_3 reduces to ηv_1^2 in $\mathrm{Ext}^{1,6}_{BP_*BP}(BP_*, BP_*V(0))$, so $\eta d_3(v_1^2) = \eta^4$.

In general, for a 2–local BP–algebra spectrum E, the E–Adams spectral sequence for any spectrum X is a module over $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. There is a universal d_3–differential $d_3(\alpha_3 z) = \eta^4 z + \alpha_3 d_3(z)$. Further, if 2 annihilates $E_*(X)$, this reduces to $d_3(\eta v_1^2 z) = \eta^4 z + \eta v_1^2 d_3(z)$. If there is no η–torsion on the E_3–term as in our examples below, this gives a universal differential $d_3(v_1^2 z) = \eta^3 z + v_1^2 d_3(z)$.

Lemma 3.4 *Let G be a closed subgroup of $\mathbb{G}_{\mathcal{C}}$. Let X be a $K(2)$–local spectrum such that $(E_{\mathcal{C}})_*X \cong (E_{\mathcal{C}})_*E_{\mathcal{C}}^{hG}$. Then the $K(2)$–local, $E_{\mathcal{C}}$–Adams spectral sequence computing π_*X has E_2–term isomorphic to $H^*(G, (E_{\mathcal{C}})_*)$.*

Proof We first prove that the E_2–term is isomorphic to $H^*(\mathbb{G}_{\mathcal{C}}, (E_{\mathcal{C}})_*X)$. This can be deduced directly from Barthel and Heard [1, Theorem 4.3]. Nonetheless, we sketch the proof here. The assumption on $(E_{\mathcal{C}})_*X$ implies that it is profree as an $(E_{\mathcal{C}})_*$–module. An inductive argument using [13, Proposition 8.4] and [10, Proposition 2.4] shows that

$$\pi_*L_{K(2)}(E_{\mathcal{C}}^{\wedge k} \wedge X) \cong \mathrm{Hom}^c(\mathbb{G}_{\mathcal{C}}^{k-1}, (E_{\mathcal{C}})_*X),$$

which allows us to identify the E_2–term as $H^*(\mathbb{G}_{\mathcal{C}}, (E_{\mathcal{C}})_*X)$. Now, using the fact that $(E_{\mathcal{C}})_*X \cong (E_{\mathcal{C}})_*E_{\mathcal{C}}^{hG}$ as Morava modules, (2-9) and Shapiro's lemma imply that $H^*(\mathbb{G}_{\mathcal{C}}, (E_{\mathcal{C}})_*X) \cong H^*(G, (E_{\mathcal{C}})_*)$. □

Lemma 3.5 *Let X be a $K(2)$–local spectrum such that $(E_{\mathcal{C}})_*X \cong (E_{\mathcal{C}})_*E_{\mathcal{C}}^{hG_{24}}$ as Morava modules. Then the $K(2)$–local, $E_{\mathcal{C}}$–Adams spectral sequence computing $\pi_*(X \wedge V(0))$ has E_2–term isomorphic to $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$. Further, in this spectral sequence, the elements Δ^k and $v_1\Delta^k$ are d_3–cycles for all k.*

Proof The identification of the E_2–term follows from Lemma 3.4 and the five lemma. There are no d_2–differentials, so all elements survive to the E_3–term. Let $\epsilon = 0, 1$. It follows from [2, Theorem 4.2.2], that $d_3(v_1^{\epsilon}\Delta^k) = v_1^{10+\epsilon}\eta^3 p(j)\Delta^{k-1}$ for $p(j) \in \mathbb{F}_4[\![j]\!]$. Suppose that $p(j)$ is not zero. Then $p(j) = j^r p_0(j)$ for $r \geq 0$ and $p_0(j) \in \mathbb{F}_4[\![j]\!]$ such that $p_0(j) \equiv \ell$ modulo (j) for some $\ell \in \mathbb{F}_4^{\times}$. Using the fact that the differentials are η– and v_1^4–linear (since $X \wedge V(0)$ has a v_1^4–self map), Remark 3.3 and the identity $j = v_1^{12}\Delta^{-1}$, we have

$$\begin{aligned} 0 &= d_3(v_1^{10+\epsilon}\eta^3 p(j)\Delta^{k-1}) \\ &= v_1^{12r+8}\eta^3 d_3(v_1^{2+\epsilon} p_0(j)\Delta^{k-r-1}) \\ &= v_1^{12r+8+\epsilon}\eta^6 p_0(j)\Delta^{k-r-1} + v_1^{12r+10}\eta^3 d_3(v_1^{\epsilon} p_0(j)\Delta^{k-r-1}). \end{aligned}$$

Again, by [2, Theorem 4.2.2], $H^3(G_{24}, (E_{\mathcal{C}})_t V(0))$ is $\mathbb{F}_4[v_1, \eta]$–torsion-free in degrees $t \equiv 6 + 2\epsilon$ modulo (24), so we can conclude that

$$\eta^3 p_0(j)\Delta^{k-r-1} = v_1^{2-\epsilon} d_3(v_1^{\epsilon} p_0(j)\Delta^{k-r-1}).$$

Since $\epsilon = 0$ or 1, the right-hand side is divisible by v_1, while the left-hand side is not, a contradiction. Therefore, we must have $p(j) = 0$. □

In the next two propositions, we let

$$R(-) = \mathbb{W}((-))[\beta^{\pm 1}, \eta]/(2\eta, \eta^3).$$

Proposition 3.6 *Let X be as in Lemma 3.5. The $E(1)$–localization of the $K(2)$–local, $E_{\mathcal{C}}$–Adams spectral sequence*

$$E_2^{s,t} = v_1^{-1} H^s\big(\mathbb{G}_{\mathcal{C}}, (E_{\mathcal{C}})_t(X \wedge V(0))\big) \Longrightarrow \pi_{t-s} L_1(X \wedge V(0))$$

satisfies

$$E_\infty^{s,t} \cong R(j)\{x, v_1 x\}/(2\cdot x, 2v_1 x)$$

for x in $(0,0)$ and $v_1 x \in (0,2)$. Further, $\pi_{8t} L_1(X \wedge V(0)) \cong \mathbb{F}_4((j))\{\beta^t\}$ and the edge homomorphisms

$$h\colon \pi_{8t} L_1(X \wedge V(0)) \to v_1^{-1} H^0(G_{24}, (E_{\mathcal{C}})_{8t} V(0))$$

are isomorphisms.

Proof By Lemma 3.5 and naturality, $E_2^{s,t}$ is isomorphic to $v_1^{-1} H^s(G_{24}, (E_{\mathcal{C}})_t V(0))$ and $j^k = v_1^{12k}\Delta^{-k}$ and $v_1 j^k$ are d_3–cycles. By Remark 3.3, there are differentials $d_3(v_1^2 j^k) = \eta^3 j^k$ and $d_3(v_1^3 j^k) = v_1\eta^3 j^k$. This, together with the fact that the differentials are v_1^4–linear, determines all d_3–differentials. The E_4–term has a horizontal vanishing line at $s = 3$. Therefore, there cannot be any higher differentials. Letting x be the element detected by $1 \in H^0(G_{24}, (E_{\mathcal{C}})_0 V(0))$, $v_1 x$ the element detected by $v_1 \in H^0(G_{24}, (E_{\mathcal{C}})_2 V(0))$ and β^t the element detected by v_1^{4t}, we obtain the desired description of the E_∞–term. For degree reasons, $\pi_{8t} L_1(X \wedge V(0)) \cong \mathbb{F}_4((j))\{\beta^t\}$. That the edge homomorphisms are isomorphisms in degrees $8t$ follows since $v_1^{-1} H^0(G_{24}, (E_{\mathcal{C}})_{8t} V(0)) \cong \mathbb{F}_4((j))\{v_1^{4t}\}$ and $h(j^k\beta^t) = j^k v_1^{4t}$. □

Remark 3.7 When $X = V(0)$, the class x can be described as the composite $S^0 \to L_1 E_{\mathcal{C}}^{hG_{24}} \xrightarrow{1\wedge e_0} L_1(E_{\mathcal{C}}^{hG_{24}} \wedge V(0))$, where the first map is the unit and e_0 is the

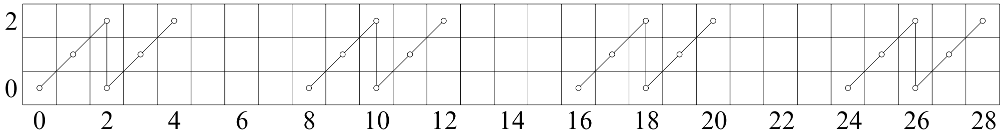

Figure 2: This picture is both an illustration of the homotopy groups $\pi_*L_1(E_{\mathcal{C}}^{hG_{24}} \wedge V(0))$ and of the homotopy groups $\pi_*L_1(E_{\mathcal{C}}^{hC_6} \wedge V(0))$. For the former, a $\circ$ denotes a copy of $\mathbb{F}_4((j))$, and for the latter, it denotes a copy of $\mathbb{F}_4((j_0))$.

inclusion of the bottom cell. In $\pi_*V(0)_{(2)}$, there is a relation $2v_1e_0 = \eta^2 e_0$ for v_1e_0 detected by $v_1 \in BP_2V(0)$ in the Adams–Novikov spectral sequence. This then implies that $2v_1x = \eta^2 x$ in $\pi_*L_1(E_{\mathcal{C}}^{hG_{24}} \wedge V(0))$, so

$$\pi_*L_1(E_{\mathcal{C}}^{hG_{24}} \wedge V(0)) \cong R(j)\{x, v_1x\}/(2\cdot x, 2v_1x - \eta^2 x).$$

With some work, one can show that the relation $2v_1x = \eta^2 x$ holds for arbitrary X satisfying the condition of Lemma 3.5. However, this fact is not needed here.

Proposition 3.8 *There is an isomorphism*

$$\pi_*L_1(E_{\mathcal{C}}^{hC_6} \wedge V(0)) \cong R(j_0)\{y, v_1y\}/(2\cdot y, 2v_1y - \eta^2 y)$$

for y in $(0,0)$ and $v_1y \in (0,2)$; see Figure 2. Hence, $\pi_*L_1(E_2^{hC_6} \wedge V(0))$ is 8–periodic with periodicity generator β. Further, the edge homomorphisms

$$h\colon \pi_{8t}L_1(E_{\mathcal{C}}^{hC_6} \wedge V(0)) \to v_1^{-1}H^0(C_6, (E_{\mathcal{C}})_{8t}V(0))$$

are isomorphisms.

Proof We prove that j_0^k is a d_3–cycle for all integers k. Then an argument similar to that of Proposition 3.6 finishes the computation of the E_∞–term, where we let y be the element detected by $1 \in H^0(C_6, (E_{\mathcal{C}})_0V(0))$ and v_1y be the element detected by $v_1y \in H^0(C_6, (E_{\mathcal{C}})_2V(0))$. The extension is obtained as in Remark 3.7.

The spectral sequence $H^*(C_6, (E_{\mathcal{C}})_*) \Rightarrow \pi_*E_{\mathcal{C}}^{hC_6}$ is multiplicative; hence, in this spectral sequence, all elements of the form a^2 are d_3–cycles. Note that j_0 lifts to an invariant in $H^0(C_6, (E_{\mathcal{C}})_0)$. This implies that $d_3(j_0^{2r}) = 0$ and $d_3(j_0^{2r+1}) = j_0^{2r}d_3(j_0)$. Hence, it suffices to prove that j_0 is a d_3–cycle. The restriction induced by the inclusion of C_6 in G_{24}, maps j to $j_0^4(1+j_0)^{-3}$. By naturality, the element $d_3(j_0^4(1+j_0)^{-3}) = 0$. However,

$$d_3(j_0^4(1+j_0)^{-3}) = j_0^4(1+j_0)^{-4}d_3(1+j_0) = j_0^4(1+j_0)^{-4}d_3(j_0),$$

which implies that $d_3(j_0) = 0$. □

4 Some elements in $\pi_* L_1 L_{K(2)} V(0)$

We now turn to examining the spectral sequence

$$E_1^{p,q}(L_1 SS_1) = \pi_q L_1(\mathscr{E}_p \wedge V(0)) \xrightarrow{L_1 SS_1} \pi_{q-p} L_1(E_{\mathcal{C}}^{h\mathbb{S}_{\mathcal{C}}^1} \wedge V(0)).$$

The idea is to use knowledge of the differentials in the spectral sequence

$$E_1^{p,q,t}(L_1 SS_2) = v_1^{-1} \operatorname{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}^1]\!]}(\mathscr{C}_p, (E_{\mathcal{C}})_t V(0)) \xrightarrow{L_1 SS_2} v_1^{-1} H^{p+q}(\mathbb{S}_{\mathcal{C}}^1 . (E_{\mathcal{C}})_t V(0))$$

to deduce information about the differentials of $L_1 SS_1$.

Lemma 4.1 *In the spectral sequence $L_1 SS_1$, we have $E_2^{3,8t} \cong \mathbb{F}_4\{\beta^t\}$.*

Proof From Section 3, we have that

$$E_1^{p,8t} \cong \begin{cases} \mathbb{F}_4((j))\{\beta^t\}, & p = 0, 3, \\ \mathbb{F}_4((j_0))\{\beta^t\}, & p = 1, 2. \end{cases}$$

From Remark 2.7 and the fact that the edge homomorphisms are isomorphisms in these degrees, we obtain a commutative diagram

$$\begin{array}{ccccccc} E_1^{0,8t}(L_1 SS_1) & \xrightarrow{d_1} & E_1^{1,8t}(L_1 SS_1) & \xrightarrow{d_1} & E_1^{2,8t}(L_1 SS_1) & \xrightarrow{d_1} & E_1^{3,8t}(L_1 SS_1) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ E_1^{0,0,8t}(L_1 SS_2) & \xrightarrow{d_1} & E_1^{1,0,8t}(L_1 SS_2) & \xrightarrow{d_1} & E_1^{2,0,8t}(L_1 SS_2) & \xrightarrow{d_1} & E_1^{3,0,8t}(L_1 SS_2) \end{array}$$

where β^{4t} maps to v_1^{4t}. Theorem 1.2.1 and Corollary 1.2.3 of [3] give a computation of the spectral sequence $L_1 SS_2$. In particular, it follows immediately from these results that

$$E_2^{3,0,8t}(L_1 SS_2) \cong \mathbb{F}_4((j))\{v_1^{4t}\}/(j) \cong \mathbb{F}_4\{v_1^{4t}\}.$$

The claim follows. □

Proposition 4.2 *If $k \equiv -3$ modulo 8, then $\pi_k L_1(E_{\mathcal{C}}^{h\mathbb{S}_{\mathcal{C}}^1} \wedge V(0)) \cong \mathbb{F}_4$.*

Proof We use the spectral sequence $E_r^{p,q} = E_r^{p,q}(L_1 SS_1)$. From Proposition 3.6 applied to $X = \mathscr{E}_0$ and $X = \mathscr{E}_3$, and from Proposition 3.8, it follows that for $r = 1$, 2 or 3 and for any p,

$$E_1^{p,8t-r} = \pi_{8t-r} L_1(\mathscr{E}_p \wedge V(0)) = 0.$$

By Lemma 4.1, $E_2^{3,8t} \cong \mathbb{F}_4\{\beta^{8t}\}$, which proves the claim. □

Proposition 4.3 *If $k \equiv -3$ modulo 8, then $\pi_k L_1(E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}^1} \wedge V(0)) \cong \mathbb{F}_2$.*

Proof It follows from Remark 2.8 that

$$v_1^{-1} E_\infty^{*,*}(\mathbb{S}_{\mathcal{C}}^1, V(0)) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} v_1^{-1} E_\infty^{*,*}(\mathbb{G}_{\mathcal{C}}^1, V(0)).$$

Since $\pi_k L_1(E_{\mathcal{C}}^{h\mathbb{S}_{\mathcal{C}}^1} \wedge V(0)) \cong \mathbb{F}_4$, there is a unique $s_0 \geq 0$ such that $E_\infty^{s_0,k+s_0}(\mathbb{S}_{\mathcal{C}}^1, V(0))$ is nonzero, and $E_\infty^{s_0,k+s_0}(\mathbb{S}_{\mathcal{C}}^1, V(0)) \cong \mathbb{F}_4$. Therefore, $E_\infty^{s,k+s}(\mathbb{G}_{\mathcal{C}}^1, V(0)) = 0$ if $s \neq s_0$ and $E_\infty^{s_0,k+s_0}(\mathbb{G}_{\mathcal{C}}^1, V(0)) \cong \mathbb{F}_2$. □

Definition 4.4 Define the class $x \in \pi_{-3} L_1(E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}^1} \wedge V(0))$ to be the nonzero element.

Recall that

$$\mathbb{G}_{\mathcal{C}} \cong \mathbb{G}_{\mathcal{C}}^1 \rtimes \mathbb{Z}_2.$$

Let π be a topological generator of the subgroup $\mathbb{Z}_2$ in $\mathbb{G}_{\mathcal{C}}$. There is a cofiber sequence

$$L_{K(2)} S \to E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}^1} \xrightarrow{\pi-1} E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}^1}. \tag{4-1}$$

We can now prove our main result.

Proof of Theorem 1.3 Since $L_{K(2)} S \wedge V(0) \simeq L_{K(2)} V(0)$ and localization preserves exact triangles, the fiber sequence (4-1) gives rise to a fiber sequence

$$L_1 L_{K(2)} V(0) \to L_1(E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}^1} \wedge V(0)) \xrightarrow{\pi-1} L_1(E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}^1} \wedge V(0)). \tag{4-2}$$

Since π acts by automorphisms and the only automorphism of $\mathbb{F}_2$ is the identity, the map $\pi - 1$ acts trivially on $\pi_{8t-3} L_1(E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}^1} \wedge V(0))$. Therefore, in the long exact sequence on homotopy groups, the class $\beta^t x$ is in the kernel of $\pi - 1$, and the image of $\beta^t x$ under the map $L_1(E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}^1} \wedge V(0)) \to \Sigma L_1 L_{K(2)} V(0)$ is nonzero. We denote it by $\zeta_2 \beta^t x$. □

References

[1] **T Barthel**, **D Heard**, *The E_2–term of the $K(n)$–local E_n–Adams spectral sequence*, Topology Appl. 206 (2016) 190–214 MR

[2] **A Beaudry**, *The algebraic duality resolution at $p = 2$*, Algebr. Geom. Topol. 15 (2015) 3653–3705 MR

[3] **A Beaudry**, *Towards the homotopy of the $K(2)$–local Moore spectrum at $p = 2$*, Adv. Math. 306 (2017) 722–788 MR

[4] **M Behrens**, *The homotopy groups of $S_{E(2)}$ at $p \geq 5$ revisited*, Adv. Math. 230 (2012) 458–492 MR

[5] **M Behrens**, **D G Davis**, *The homotopy fixed point spectra of profinite Galois extensions*, Trans. Amer. Math. Soc. 362 (2010) 4983–5042 MR

[6] **I Bobkova**, *Resolutions in the $K(2)$–local category at the prime 2*, PhD thesis, Northwestern University (2014) MR Available at `https://search.proquest.com/docview/1558126694`

[7] **E S Devinatz**, *A Lyndon–Hochschild–Serre spectral sequence for certain homotopy fixed point spectra*, Trans. Amer. Math. Soc. 357 (2005) 129–150 MR

[8] **E S Devinatz**, **M J Hopkins**, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology 43 (2004) 1–47 MR

[9] **P G Goerss**, **H-W Henn**, **M Mahowald**, *The rational homotopy of the $K(2)$–local sphere and the chromatic splitting conjecture for the prime 3 and level 2*, Doc. Math. 19 (2014) 1271–1290 MR

[10] **P Goerss**, **H-W Henn**, **M Mahowald**, **C Rezk**, *A resolution of the $K(2)$–local sphere at the prime 3*, Ann. of Math. 162 (2005) 777–822 MR

[11] **H-W Henn**, *On finite resolutions of $K(n)$–local spheres*, from "Elliptic cohomology" (H R Miller, D C Ravenel, editors), London Math. Soc. Lecture Note Ser. 342, Cambridge Univ. Press (2007) 122–169 MR

[12] **M Hovey**, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, from "The Čech centennial" (M Cenkl, H Miller, editors), Contemp. Math. 181, Amer. Math. Soc., Providence, RI (1995) 225–250 MR

[13] **M Hovey**, **N P Strickland**, *Morava K–theories and localisation*, Mem. Amer. Math. Soc. 666, Amer. Math. Soc., Providence, RI (1999) MR

[14] **M Mahowald**, *The image of J in the EHP sequence*, Ann. of Math. 116 (1982) 65–112 MR

[15] **H R Miller**, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space*, J. Pure Appl. Algebra 20 (1981) 287–312 MR

[16] **J Morava**, *Noetherian localisations of categories of cobordism comodules*, Ann. of Math. 121 (1985) 1–39 MR

[17] **D C Ravenel**, *The cohomology of the Morava stabilizer algebras*, Math. Z. 152 (1977) 287–297 MR

[18] **D C Ravenel**, *A novice's guide to the Adams–Novikov spectral sequence*, from "Geometric applications of homotopy theory, II" (M G Barratt, M E Mahowald, editors), Lecture Notes in Math. 658, Springer (1978) 404–475 MR

[19] **D C Ravenel**, *Localization and periodicity in homotopy theory*, from "Homotopy theory" (E Rees, J D S Jones, editors), London Math. Soc. Lecture Note Ser. 117, Cambridge Univ. Press (1987) 175–194 MR

[20] **C Rezk**, *Notes on the Hopkins–Miller theorem*, from "Homotopy theory via algebraic geometry and group representations" (M Mahowald, S Priddy, editors), Contemp. Math. 220, Amer. Math. Soc., Providence, RI (1998) 313–366 MR

[21] **K Shimomura**, *The Adams–Novikov E_2–term for computing $\pi_*(L_2V(0))$ at the prime 2*, Topology Appl. 96 (1999) 133–152 MR

[22] **K Shimomura**, **X Wang**, *The Adams–Novikov E_2–term for $\pi_*(L_2S^0)$ at the prime 2*, Math. Z. 241 (2002) 271–311 MR

[23] **K Shimomura**, **A Yabe**, *The homotopy groups $\pi_*(L_2S^0)$*, Topology 34 (1995) 261–289 MR

[24] **J H Silverman**, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics 106, Springer (1986) MR

Department of Mathematics, University of Colorado
Boulder, CO, United States

agnes.beaudry@colorado.edu

Proposed: Mark Behrens
Seconded: Stefan Schwede, Ralph Cohen

Received: 3 April 2015
Revised: 7 December 2016