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Koszul duality patterns in Floer theory

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We study symplectic invariants of the open symplectic manifolds X_Γ obtained by plumbing cotangent bundles of 2–spheres according to a plumbing tree Γ. For any tree Γ, we calculate (DG) algebra models of the Fukaya category $\mathcal{F}(X_\Gamma)$ of closed exact Lagrangians in X_Γ and the wrapped Fukaya category $\mathcal{W}(X_\Gamma)$. When Γ is a Dynkin tree of type A_n or D_n (and conjecturally also for E_6, E_7, E_8), we prove that these models for the Fukaya category $\mathcal{F}(X_\Gamma)$ and $\mathcal{W}(X_\Gamma)$ are related by (derived) Koszul duality. As an application, we give explicit computations of symplectic cohomology of X_Γ for $\Gamma = A_n, D_n$, based on the Legendrian surgery formula of Bourgeois, Ekholm and Eliashberg.

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1 Introduction

Let us begin by recalling a simple example that we learned from Blumberg, Cohen and Teleman [14]. Consider a *simply connected* smooth compact manifold S and its cotangent bundle $M = T^*S$ with its canonical symplectic structure. The zero section S is a Lagrangian submanifold. We choose a basepoint $x \in S$ and consider the corresponding cotangent fiber $L = T^*_x S$. This is another Lagrangian submanifold, a noncompact one. Throughout, our Lagrangian submanifolds will be equipped with a *brane structure*. This means that they will be given an orientation, a spin structure (in particular, we assume here that S is spinnable) and they will be equipped with a grading in the sense of Seidel [59].

Fix a coefficient field $\mathbb{K}$. Lagrangian Floer theory gives us $\mathbb{Z}$–graded A_∞–algebras over $\mathbb{K}$

$$\mathscr{A} = \mathrm{CF}^*(S,S), \quad \mathscr{B} = \mathrm{CW}^*(L,L).$$

Indeed, S is an object of $\mathcal{F}(M)$, the Fukaya category of closed exact Lagrangian branes in the Liouville manifold M (see Seidel [61]). The endomorphisms of the object S in this category are given by the Fukaya–Floer A_∞–algebra $\mathrm{CF}^*(S,S)$. On the other hand, L is an object of $\mathcal{W}(M)$, the wrapped Fukaya category of M

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(see Abouzaid and Seidel [6]). The endomorphisms of the object L in this category are given by the wrapped Floer cochain complex $\mathrm{CW}^*(L,L)$, which again has an associated A_∞–structure (well-defined up to quasi-isomorphism).

Now, in this setting, by construction, there exists a full and faithful embedding

$$\mathcal{F}(M) \to \mathcal{W}(M)$$

since by definition $\mathcal{W}(M)$ allows certain noncompact Lagrangians in M with controlled behavior at infinity, in addition to the exact compact Lagrangians in M. Furthermore, it is a general fact (see Abouzaid [2]) that a cotangent fiber generates the wrapped Fukaya category in the derived sense. Hence, in particular, one has a Yoneda functor to the DG-category of A_∞–modules over $\mathscr{B}$,

$$\mathcal{Y}\colon \mathcal{F}(M) \to \mathscr{B}^{\mathrm{mod}},$$

which is a cohomologically full and faithful embedding. This sends an exact compact Lagrangian T to the (right) A_∞–module $\mathcal{Y}_T = \mathrm{CW}^*(L,T)$ over $\mathscr{B}$. As a consequence, one can compute $\mathscr{A}$ via its quasi-isomorphic image under $\mathcal{Y}$:

$$\mathscr{A} \simeq \hom_{\mathscr{B}}(\mathbb{K},\mathbb{K}), \tag{1}$$

where we write $\mathbb{K}$ for the right A_∞–module over $\mathscr{B}$ with underlying vector space $\mathbb{K}\cdot x = \mathrm{CW}^*(L,S)$. Equipping S and L with suitable brane structures, one can arrange that the degree $|x|$ is 0. The only nontrivial module map is the multiplication by the idempotent element in $\mathrm{CW}^0(L,L) = \mathbb{K}\cdot e$, which acts as the identity. The other products (including the higher products) are necessarily trivial. This can be seen from the fact that $\mathrm{CW}^*(L,L)$ is supported in nonpositive degrees (as we shall see below). Note that we are following the conventions of [61], where, for example, the A_∞–module maps are given by Floer products

$$\mathrm{CW}^*(L,S) \otimes \mathrm{CW}^*(L,L)^{\otimes k} \to \mathrm{CW}^*(L,S)[1-k], \quad k = 0, \dots.$$

Throughout, upwards shift of grading by n is written as $[-n]$.

On the other hand, $\mathrm{CW}^*(L,S)$ is also a (left) A_∞–module over $\mathrm{CF}^*(S,S)$, where A_∞–module maps are given by Floer products

$$\mathrm{CF}^*(S,S)^{\otimes k} \otimes \mathrm{CW}^*(L,S) \to \mathrm{CW}^*(L,S)[1-k], \quad k = 0, \dots.$$

To be in line with the conventions of [61], we prefer to view this as a right $\mathscr{A}^{\mathrm{op}}$–module (which entails slightly different sign conventions). In fact, in our setting, it turns out that $\mathscr{A}$ is quasi-isomorphic to $\mathscr{A}^{\mathrm{op}}$.

Somewhat more surprisingly, one can also compute $\mathscr{B}$ as

$$\mathscr{B}^{\mathrm{op}} \simeq \hom_{\mathscr{A}^{\mathrm{op}}}(\mathbb{K}, \mathbb{K}). \tag{2}$$

This is an instance of *Koszul duality*.

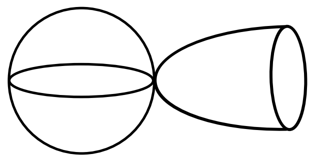

Figure 1: A picture of Koszul duality

To see this, we observe that both $\mathscr{A}$ and $\mathscr{B}$ have topological models due to Abouzaid [3; 4]. Indeed, there are A_∞–equivalences

$$\mathscr{A} \simeq C^*(S; \mathbb{K}) \quad \text{and} \quad \mathscr{B} \simeq C_{-*}(\Omega_x S; \mathbb{K}),$$

where $\Omega_x S$ is the based loop space of S at x. Notice the cohomological grading in place. In particular, $\mathscr{A}$ is supported in nonnegative degrees and $\mathscr{B}$ is supported in nonpositive degrees.

Now, (1) becomes an Eilenberg–Moore equivalence (of DGA's)

$$\mathrm{RHom}_{C_{-*}(\Omega_x S)}(\mathbb{K}, \mathbb{K}) \simeq C^*(S; \mathbb{K}),$$

and (2) is Adams' cobar equivalence (see Adams [8] and Jones and McCleary [44])

$$\mathrm{RHom}_{C^*(S)^{\mathrm{op}}}(\mathbb{K}, \mathbb{K}) \simeq C_{-*}(\Omega_x S)^{\mathrm{op}}$$

($^{\mathrm{op}}$ operations get removed from both sides if one considers $\mathbb{K}$ as a left $C^*(S)$–module).

This duality is relevant to us because it induces an isomorphism at the level of Hochschild cohomology. Namely, by a general result of Keller [47] (see also Félix, Menichi and Thomas [36]) one obtains an isomorphism of Gerstenhaber algebras (in fact, of B_∞–algebras at the chain level)

$$\mathrm{HH}^*(C^*(S), C^*(S)) \cong \mathrm{HH}^*(C_{-*}(\Omega_x S), C_{-*}(\Omega_x S)).$$

In view of Abouzaid's generation result [4], the right-hand side is in turn isomorphic to $\mathrm{HH}^*(\mathcal{W}(M))$ as a Gerstenhaber algebra. On the other hand, the work of Bourgeois, Ekholm and Eliashberg [17] can be interpreted, over a field $\mathbb{K}$ of characteristic 0, to give an isomorphism of Gerstenhaber algebras

$$\mathrm{HH}^*(\mathcal{W}(M)) \cong \mathrm{SH}^*(M).$$

The group on the right-hand side is called *symplectic cohomology*. Strictly speaking, the results of [17] relate symplectic and Hochschild *homologies*. However, in our computations, we will use an explicit DG-algebra as a model for $\mathcal{W}(M)$, which has an (open) Calabi–Yau property (in the sense of Ginzburg [39]) implying an isomorphism between Hochschild homology and cohomology. This allows us to use the cohomological statement above that we find more convenient. Symplectic (co)homology of a Liouville manifold is a symplectic invariant based on an extension of Hamiltonian Floer (co)homology to noncompact symplectic manifolds. It was introduced by Viterbo [70] in its current form. We recommend Seidel [60] for an excellent introduction to symplectic cohomology and the recent manuscript Abouzaid [5] for more. Briefly, this is a very interesting invariant of a Stein manifold that is sensitive to the underlying symplectic structure (cf Eliashberg and Gromov [31]). Symplectic cohomology is in general difficult to calculate explicitly. However, Bourgeois, Ekholm and Eliashberg [16; 17] recently outlined a proof of a surgery formula for symplectic (co)homology. Combining this with the very recent work of Ekholm and Ng [28], one obtains a purely combinatorial description of symplectic cohomology of any 4–dimensional Weinstein manifold. (In the absence of 1–handles and when the coefficient field is $\mathbb{Z}_2$, one had Chekanov [19] as a precursor to [28].) This combinatorial description is in general still highly complicated. It involves noncommutative and infinite-dimensional chain complexes.

In the above setting, assuming that $\mathscr{A} = C^*(S)$ is a formal DG-algebra, that is, it is quasi-isomorphic to its homology $A = H^*(S)$, we get a promising way of computing symplectic cohomology. Namely, one has

$$\mathrm{HH}^*(H^*(S), H^*(S)) = \mathrm{SH}^*(M).$$

By a famous result of Deligne, Griffiths, Morgan and Sullivan [25], the formality assumption holds if S is a Kähler manifold and $\mathbb{K}$ has characteristic 0. Note that as a consequence of formality of $C^*(S)$, one has a bigrading on $\mathrm{HH}^*(C^*(S), C^*(S))$; there is a cohomological grading r associated with the Hochschild cochain complex and there is an internal grading s coming from the grading on $H^*(S)$. To get an isomorphism to $\mathrm{SH}^*(M)$, where the grading is given by a Conley–Zehnder type index, one has to consider the grading of the total complex corresponding to $r+s$.

Let us note that one could arrive at the same conclusion by combining theorems of Viterbo [70] and Cohen and Jones [20].

In this paper, we give a generalization of the above (in dimension 4) to Liouville manifolds $M = X_\Gamma$ obtained via plumbings of T^*S^2 according to a plumbing tree Γ. We will work over semisimple rings k of the form

$$\mathrm{k} = \bigoplus_v \mathbb{K} e_v,$$

where $e_v^2 = e_v$ and $e_v e_w = 0$ for $v \neq w$ and the index set of the sum is the vertex set Γ_0 of Γ.

To wit, using Floer complexes over $\mathbb{K}$, we set

$$\mathscr{A}_\Gamma = \bigoplus_{v,w} \mathrm{CF}^*(S_v, S_w),$$

where the S_v are the Lagrangian spheres corresponding to the zero sections of the cotangent bundles T^*S^2 that we are plumbing, and similarly we have

$$\mathscr{B}_\Gamma = \bigoplus_{v,w} \mathrm{CW}^*(L_v, L_w),$$

where L_v is a cotangent fiber at a chosen basepoint on S_v (away from the plumbing region).

In fact, assuming $\operatorname{char} \mathbb{K} = 0$, it turns out that $\mathscr{A}_\Gamma$ tends to be a formal DG-algebra (we can prove this when Γ is of type A_n or D_n, and conjecture it for E_6, E_7, E_8), hence, in such cases, we may replace it with $A_\Gamma = H^*(\mathscr{A}_\Gamma)$. Furthermore, very early on, we will replace $\mathscr{B}_\Gamma$ with a quasi-isomorphic DG-algebra (see [17, Proposition 4.11 and Theorem 5.8]) which has a combinatorial description. Namely, we will use Chekanov's DG-algebra [19], with the cohomological grading, associated to a Legendrian link $\Lambda_\Gamma = \bigcup \Lambda_v$ giving a Legendrian surgery diagram for X_Γ where the components are indexed by vertices v of Γ and each component Λ_v is a Legendrian unknot in $\mathbb{R}^3$ (see Figure 3). In the context of [17], the homologically graded version of this is also called the Legendrian contact homology algebra.

At this point, one obtains an explicit description of the DG-algebra $\mathscr{B}_\Gamma$. A careful choice of the surgery diagram (with suitable decorations) enables us to observe that the DG-algebra $\mathscr{B}_\Gamma$ is a *deformation* of Ginzburg's (cohomologically graded) DG-algebra $\mathscr{G}_\Gamma$ associated with the tree Γ (see Theorem 8).[1] Note that Ginzburg [39] associates a CY3 DG-algebra to any graph Γ and a potential function. In this paper, Γ is a tree and the potential function vanishes. On the other hand, since we are plumbing copies of T^*S^2, our DG-algebras are CY2. This generalization of the construction of Ginzburg's DG-algebra in order to obtain a CY2 DG-algebra appears in Van den Bergh [12]. (See Definition 5 for the definition of $\mathscr{G}_\Gamma$.)

The observation that $\mathscr{B}_\Gamma$ is a deformation of the corresponding Ginzburg DG-algebra $\mathscr{G}_\Gamma$ enables us to use the vast literature on the study of Ginzburg's DG-algebras to study symplectic invariants of X_Γ. Now, our discussion branches into two according to whether the underlying tree Γ is of Dynkin type or not.

[1] An earlier version of this manuscript claimed an isomorphism between $\mathscr{B}_\Gamma$ and $\mathscr{G}_\Gamma$, due to our blindness to some higher energy curves. We are indebted to the referee for opening our eyes.

Dynkin case For Γ of type A_n or D_n, we prove the following theorem:

Theorem 1 *For $\Gamma = A_n$ and $\mathbb{K}$ arbitrary field, or $\Gamma = D_n$ and $\mathbb{K}$ a field with $\operatorname{char}\mathbb{K} \neq 2$, there is a quasi-isomorphism of DG-algebras*

$$\mathscr{B}_\Gamma \simeq \mathscr{G}_\Gamma.$$

For $\Gamma = A_n$, this result follows from a direct computation of $\mathscr{B}_\Gamma$. However, for $\Gamma = D_n$, direct computation only shows that $\mathscr{B}_\Gamma$ is a certain deformation of $\mathscr{G}_\Gamma$. We then appeal to standard deformation theory arguments to show that this deformation is trivial when $\operatorname{char}\mathbb{K} \neq 2$. In fact, we also prove that $\mathscr{B}_\Gamma$ and $\mathscr{G}_\Gamma$ are not quasi-isomorphic when $\Gamma = D_n$ and $\operatorname{char}\mathbb{K} = 2$ by showing that the relevant obstruction class in $\mathrm{HH}^2(\mathscr{G}_\Gamma)$ is nontrivial.

We conjecture that $\mathscr{B}_\Gamma \simeq \mathscr{G}_\Gamma$ for $\Gamma = E_6, E_7$ if $\operatorname{char}\mathbb{K} \neq 2, 3$ and for $\Gamma = E_8$ if $\operatorname{char}\mathbb{K} \neq 2, 3, 5$, but we leave the study of these exceptional cases to a future work.

Assuming for brevity $\operatorname{char}\mathbb{K} \neq 2$, and $\Gamma = A_n$ or D_n, we can now write $\mathscr{B}_\Gamma \simeq \mathscr{G}_\Gamma$. For Γ of type ADE, $\mathscr{G}_\Gamma$ turns out to be nonformal; see Hermes [41]. Its cohomology has locally finite grading. Indeed, for an (algebraically closed) field with characteristic 0, it was computed in [41] that

$$H^*(\mathscr{G}_\Gamma) \cong \Pi_\Gamma \rtimes_\nu \mathrm{k}[u]$$

as a k–algebra, where Π_Γ is the preprojective algebra associated with the tree Γ, $|u| = -1$, and the multiplication is twisted by the Nakayama automorphism ν of Π_Γ. This is an involution, which is induced by an involution of the underlying Dynkin graph (see Section 3).

Because $\mathscr{G}_\Gamma$ is not formal, it is not immediately clear how to compute Hochschild cohomology of $\mathscr{G}_\Gamma$. To help with this, we prove in Section 5 the following:

Theorem 2 *Let $\mathbb{K}$ be any field. For any tree Γ, the associative algebra A_Γ is Koszul dual to the DG-algebra $\mathscr{G}_\Gamma$, in the sense that there are DG-algebra isomorphisms*

$$\mathrm{RHom}_{\mathscr{G}_\Gamma}(\mathrm{k}, \mathrm{k}) \simeq A_\Gamma \quad \textit{and} \quad \mathrm{RHom}_{A_\Gamma^{\mathrm{op}}}(\mathrm{k}, \mathrm{k}) \simeq \mathscr{G}_\Gamma^{\mathrm{op}}.$$

Therefore, by Keller's result [47], we can use this to compute $\mathrm{SH}^*(X_\Gamma)$ as

$$\mathrm{SH}^*(X_\Gamma) \cong \mathrm{HH}^*(\mathscr{G}_\Gamma) \cong \mathrm{HH}^*(A_\Gamma).$$

Since A_Γ is a rather small finite-dimensional algebra over k, one can find a minimal projective resolution to compute the latter group. Indeed, Brenner, Butler and King [18] give a minimal periodic (graded) resolution for A_Γ. However, we will find a shortcut to compute $\mathrm{HH}^*(A_\Gamma)$ as a bigraded algebra for $\Gamma = A_n, D_n$ over a field $\mathbb{K}$ of arbitrary characteristic. An explicit presentation of $\mathrm{HH}^*(A_\Gamma)$ as a (graded) commutative $\mathbb{K}$–algebra is provided in Theorem 40 for A_n and in Theorem 44 for D_n.

As we noted above in the case $\Gamma = D_n$ and when $\operatorname{char}\mathbb{K} = 2$, $\mathscr{B}_\Gamma$ is indeed a nontrivial deformation of $\mathscr{G}_\Gamma$. In this case $\mathscr{A}_\Gamma$ is not formal and indeed $\mathscr{B}_\Gamma$ and $\mathscr{A}_\Gamma$ are Koszul dual in the above sense. Therefore, it appears that a natural statement (that applies in all characteristics) may be that $\mathscr{A}_\Gamma$ and $\mathscr{B}_\Gamma$ are Koszul dual when Γ is a Dynkin tree.

Non-Dynkin case In this case, we only know that $\mathscr{B}_\Gamma$ is a deformation of $\mathscr{G}_\Gamma$ and even at the formal level there are many nontrivial deformations of $\mathscr{G}_\Gamma$ since $\mathrm{HH}^2(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma)$ is big (see Theorem 30) and $\mathrm{HH}^3(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma) = 0$. Hence, it is not clear whether the deformation corresponding to $\mathscr{B}_\Gamma$ is trivial or not. On the other hand, as $\mathscr{B}_\Gamma$ (being a model for the wrapped Fukaya category of X_Γ) is also a Calabi–Yau (CY) algebra, one can see the deformation of $\mathscr{G}_\Gamma$ to $\mathscr{B}_\Gamma$ as a deformation of CY2-algebras. In characteristic 0, this allows one to conclude that the corresponding formal deformation is trivial as follows.

$\mathscr{G}_\Gamma$ is in a sense simpler for Γ non-Dynkin. Namely, in this case, the homology $H^*(\mathscr{G}_\Gamma)$ turns out to be concentrated in degree 0 and

$$H^0(\mathscr{G}_\Gamma) \cong \Pi_\Gamma$$

is the preprojective algebra associated with the tree Γ. For a non-Dynkin tree Γ, working over $\mathbb{K}$ of characteristic 0, Hermes [41] proved that $\mathscr{G}_\Gamma$ is formal, that is, $\mathscr{G}_\Gamma$ is quasi-isomorphic to its homology Π_Γ (see also Corollary 26 for another proof that works over any field). Furthermore, it is well-known that Π_Γ is Koszul in the classical sense (cf [54; 10]) over k. The quadratic dual to Π_Γ is given by the associative algebra $A_\Gamma = H^*(\mathscr{A}_\Gamma)$ — the zigzag algebra associated with the tree Γ [43].

The Gerstenhaber algebra structure of the Hochschild cohomology of the preprojective algebra Π_Γ in the non-Dynkin case has already been computed by Crawley-Boevey, Etingof and Ginzburg in [23] when $\mathbb{K}$ has characteristic zero, and by Schedler [57] in general. $\mathrm{HH}^*(\Pi_\Gamma) \neq 0$ only for $* = 0, 1, 2$. We give a brief review of these computations of $\mathrm{HH}^*(\Pi_\Gamma)$ for completeness; see Section 6.1 for a full description. Now, $\mathscr{B}_\Gamma$ can be seen as a deformation of the CY2 algebra Π_Γ. If we consider the corresponding formal deformation, then the associated Kodaira–Spencer class lives in $\mathrm{Ker}(\Delta\colon \mathrm{HH}^2(\Pi_\Gamma) \to \mathrm{HH}^1(\Pi_\Gamma))$, where Δ is the BV-operator (see for example [35]). Now, it can be observed from the description given in Section 6.1 that this kernel is trivial if $\operatorname{char}\mathbb{K} = 0$. This result does not hold in arbitrary characteristic; see Remark 15 (cf Remark 33) for a proof that this deformation is nontrivial over a field $\mathbb{K}$ of characteristic 2.

Finally, let us remark that the above argument only shows that the associated formal deformation is trivial. This does not mean that $\mathscr{B}_\Gamma$ is quasi-isomorphic to $\mathscr{G}_\Gamma$ — such a quasi-isomorphism holds only after a certain completion. As was shown in our subsequent work [33], $H^0(\mathscr{B}_\Gamma)$ is isomorphic to the multiplicative preprojective algebra associated with Γ, introduced by Crawley-Boevey and Shaw [24]. On the other hand $H^0(\mathscr{G}_\Gamma)$ is isomorphic to the additive preprojective algebra Π_Γ. It is known that additive and multiplicative preprojective algebras are isomorphic only when $\operatorname{char}\mathbb{K}=0$ and Γ is Dynkin, and in general, they are isomorphic when $\operatorname{char}\mathbb{K}=0$ only after completion, as follows from the above deformation theory argument.

In Section 2, we provide geometric preliminaries on plumbings of cotangent bundles. In Section 3, we give a computation of Legendrian contact homology of the Legendrian link Λ_Γ associated to a tree Γ, show that it is isomorphic to a deformation of the corresponding CY2 Ginzburg DG-algebra $\mathscr{G}_\Gamma$ (Theorem 8) and that this deformation is trivial for $\Gamma = A_n$ or D_n, when $\operatorname{char}\mathbb{K}\neq 2$ in the latter case (Theorem 13). Section 4 computes the Floer cohomology algebra $\mathscr{A}_\Gamma$ of the spheres in X_Γ. The main result appears in Section 5, where we show that $\mathscr{G}_\Gamma$ and $A_\Gamma = H^*(\mathscr{A}_\Gamma)$ are Koszul duals for any tree Γ. Finally, as an application of our main result, in Section 6, we explicitly compute Hochschild cohomology of $\mathscr{G}_\Gamma$, hence also of $\mathscr{B}_\Gamma$ for $\Gamma = A_n$ and D_n, assuming $\operatorname{char}\mathbb{K}\neq 2$ if $\Gamma = D_n$.

Convention Throughout, we adhere to the following conventions. $\mathbb{K}$ is a field, of arbitrary characteristic unless otherwise specified, and k is a semisimple ring, generated over $\mathbb{K}$ by finitely many mutually orthogonal idempotents. Letters $A, B, \dots$ denote associative algebras over k. All our modules are *right* modules and all our multiplications are read from *right to left*. Letters $\mathscr{A}, \mathscr{B}, \dots$ denote A_∞– or DG-algebras over k. We follow the sign conventions as given in Seidel [61, Chapter 1] and its sequel Seidel [63]. In particular, an A_∞–algebra $\mathscr{A}$ over k is a $\mathbb{Z}$–graded k–module with a collection of k–linear maps

$$\mu^d\colon \mathscr{A}^{\otimes d} \to \mathscr{A}[2-d] \quad \text{for } d \geq 1,$$

where $[2-d]$ means μ^d lowers the degree by $d-2$. These maps are required to satisfy the A_∞–relations

$$\sum_{m,n} (-1)^{|a_1|+\cdots+|a_n|-n} \mu^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots a_1) = 0.$$

A DG-algebra over k is an A_∞–algebra over k such that $\mu^d = 0$ for $d \geq 3$. In this case, we put

$$da = (-1)^{|a|}\mu^1(a), \quad a_2 a_1 = (-1)^{|a_1|}\mu^2(a_2, a_1). \tag{3}$$

With this convention the A_∞–equation for $d=2$ gives us the usual graded Leibniz rule

$$d(a_2a_1) = (da_2)a_1 + (-1)^{|a_2|}a_2(da_1).$$

$\mathscr{A}^{\mathrm{op}}$ denotes the opposite of an A_∞–algebra $\mathscr{A}$ and its operations are given by

$$\mu^d_{\mathscr{A}^{\mathrm{op}}}(a_d,\dots,a_1) = (-1)^{|a_1|+\dots+|a_d|-d}\mu^d_{\mathscr{A}}(a_1,\dots,a_d).$$

With the above conventions, a DG-algebra and its opposite are related via

$$d^{\mathrm{op}}(a) = (-1)^{|a|-1}da, \quad a_2a_1 = a_1a_2.$$

All our complexes are cohomological, ie the differential increases the grading by 1. It often happens that our complexes are bigraded. In this case, we denote these gradings by the pair (r,s), where r refers to a cohomological (or length) grading and s refers to an internal grading (the notation $|a|$ is used to denote the internal grading of a specific element). The grading $r+s$ is referred to as the total degree. If a second grading is not specified in the notation, for example as in $\mathrm{HH}^*(A_\Gamma)$, it is understood that the grading $*$ refers to the total degree.

The notation $\mathrm{HH}^*(A)$ is used to denote Hochschild cohomology of a graded $\mathbb{K}$–algebra A with coefficients in A. It is a bigraded algebra over $\mathbb{K}$. We write $\deg(x)$ for the total degree $r+s$ of a specific element. There are two binary $\mathbb{K}$–linear operations: an associative graded commutative product of bidegree $(0,0)$ and a Lie bracket of bidegree $(-1,0)$. These are called the cup product and Gerstenhaber bracket, respectively. The product is graded commutative:

$$xy = (-1)^{\deg(x)\deg(y)}yx.$$

The Gerstenhaber bracket is graded antisymmetric on $\mathrm{HH}^*(A)[1]$, that is,

$$[x,y] = -(-1)^{(\deg(x)-1)(\deg(y)-1)}[y,x].$$

Finally, Hochschild cohomology of a (formal) Calabi–Yau algebra can be equipped with a Batalin–Vilkovisky operator Δ of bidegree $(-1,0)$, and we have the following compatibility equation between these structures:

$$[x,y] = (-1)^{|x|}\Delta(xy) - (-1)^{|x|}\Delta(x)y - x\Delta(y).$$

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projection than the one given in Figure 4, which resulted in higher energy curves being immersed and elusive. We are indebted to the referee for communicating to us the existence of these higher-order contributions to the differential of $\mathscr{B}_\Gamma$.

2 Plumbing of cotangent bundles of 2–spheres

Let Γ be a finite tree. In the body of the paper, we will study Weinstein manifolds that are given by a plumbing of cotangent bundles of the 2–sphere according to the tree Γ. These are exact symplectic manifolds with a convexity condition at infinity. We briefly recall the construction of these manifolds (cf [3]).

Associated to each vertex of Γ, we prepare a copy of D^*S^2, the unit cotangent bundle of the 2–sphere with its canonical symplectic structure. Now, say we have an edge that connects the vertices v and w, and let us write D^*S_v and D^*S_w for the corresponding copies of T^*S^2 and choose basepoints $s_v \in S_v$ and $s_w \in S_w$. Near s_v and s_w one can find real coordinates p_1, p_2, q_1, q_2 where the coordinates q_1, q_2 correspond to variations on the base and the coordinates p_1, p_2 correspond to variations in the fiber direction. Furthermore, on these neighborhoods symplectic form can be identified with $dp \wedge dq$. We then glue D^*S_v and D^*S_w together near s_v and s_w via a symplectomorphism that sends (q, p) to $(p, -q)$.

This leads to a symplectic manifold which has a boundary with corners. One then smoothens the boundary and completes it to obtain a Weinstein manifold. The precise details of this construction are somewhat technical; we refer to [3, Section 2.3] (see also [37, Section 7.6]).

An alternative description of X_Γ can be given via *Legendrian surgery* à la Eliashberg [29] and Gompf [40], which we will take as primary.[2] In this description, we exhibit X_Γ as a surgery along a Legendrian link Λ on (S^3, ξ_{std}) such that the vertices v of Γ correspond to the components Λ_v of this link, which are Legendrian unknots. Two such Legendrian unknots are "plumbed together" if there is an edge in Γ between the corresponding vertices. To be precise, by choosing a vertex as the *root* of our tree, we put our tree Γ in a standard form as in Figure 2, and the corresponding Legendrian unknots in a standard form in $(\mathbb{R}^3, dz - y\,dx)$, which when projected to (x, z) (front projection) gives the surgery diagram as in Figure 3.

The surgery construction equips X_Γ with a Weinstein structure (in fact, a Stein structure) by extending the standard Weinstein structure on D^4 via attaching 2–handles [73]

[2] Both the plumbing and surgery constructions lead to homotopic Weinstein manifolds but we do not check this here. Throughout, we use the surgery construction and appeal to the plumbing picture only for differential topological aspects.

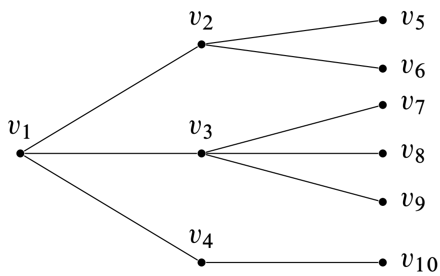

Figure 2: Standard form of Γ

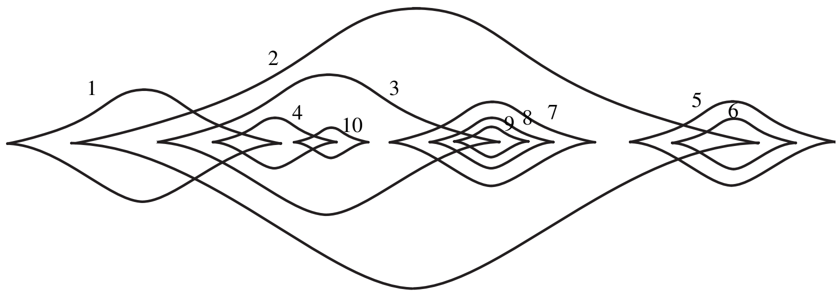

Figure 3: X_Γ as given by Legendrian surgery along Λ

along Legendrian unknots Λ_i. Each such Legendrian unknot bounds an embedded Lagrangian disk in D^4 and another capping disk given by the attaching disk of the corresponding Weinstein 2–handle. These fit together, as can be seen from the case of T^*S^2, to give the Lagrangian spheres S_v in X_Γ corresponding to the vertices of Γ, whereas the edges of Γ encode the intersection pattern of these spheres. The symplectic form ω on X_Γ is exact and it can be written as a primitive of a one-form θ for which the embedding of each sphere S_v is an exact Lagrangian submanifold of X_Γ. Both of these are easy facts since $H_2(X_\Gamma;\mathbb{Z})$ is generated by the Lagrangian spheres S_v and $H^1(S_v;\mathbb{Z}) = 0$.

Furthermore, the cocores of the 2–handles give noncompact (exact) Lagrangians L_v which are asymptotically Legendrian. The Lagrangian L_v intersects S_w only if $v = w$, in which case the intersection is transverse at a unique point x_v. In the plumbing description, the L_v correspond to the cotangent fibers $T^*_{x_v} S_v \subset T^*S_v$, where the x_v are basepoints on each S_v away from the plumbing regions.

In the next section, we will be concerned with Reeb chords between the components of Λ in $(\mathbb{R}^3, dz - y\,dx)$. The Reeb flow is in the direction of the vector field $\partial/\partial z$, hence it is more convenient for computations to consider the Lagrangian projection, ie the projection to (x, y) as in Figure 4. Then the crossings of the projection Λ are in one-to-one correspondence with Reeb chords from Λ to itself. There is some freedom in drawing the Lagrangian projection; we prefer the one given in Figure 4 as it makes enumeration of holomorphic curves manifest. (Notice that the diagram has the property that each component links at most one other component on its left. Clearly this is an artifact of the way we put our tree in a standard form and is not necessary.)

In Figure 4, besides a basepoint on each component, we also indicated an orientation on our Legendrian link Λ by putting an arrow on each component. This, in turn, induces orientations on the Lagrangian spheres S_v. Notice that

$$S_v \cdot S_w = +1$$

if v and w are adjacent vertices. This ensures that the Floer complex $\mathrm{CF}^*(S_v, S_w)$ is supported at an odd degree (see [59, Section 2d]).

We orient the noncompact Lagrangians L_v so that the algebraic intersection number $L_v \cdot S_v$ is given by

$$L_v \cdot S_v = -1.$$

As above, this ensures that the Floer complex $\mathrm{CF}^*(L_v, S_v)$ is supported at an even degree (which we will fix below to be 0 by picking suitable grading structures).

The classical topology of X_Γ is easy to study via the plumbing description, which shows that X_Γ deformation retracts onto a wedge of spheres formed by the union of

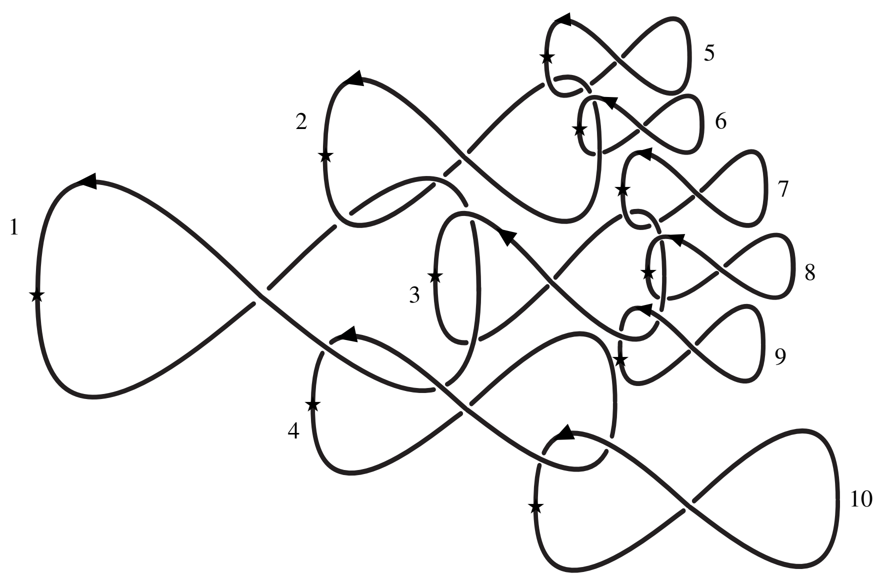

Figure 4: Lagrangian projection of Λ decorated with orientations and basepoints

the S_v. In particular, X_Γ is simply connected and the nonzero cohomology groups of X_Γ are given by

$$H^0(X_\Gamma;\mathbb{K})=\mathbb{K},\quad H^2(X_\Gamma;\mathbb{K})=\bigoplus_v \mathbb{K}\cdot[S_v]^\vee.$$

The noncompact end of X_Γ is a symplectization of a contact 3–manifold Y_Γ which is topologically a plumbing of circle bundles over S^2 with Euler number -2. By abuse of notation, we will write $\partial X_\Gamma = Y_\Gamma$.

To equip our Lagrangians with a brane structure, so as to have $\mathbb{Z}$–gradings, we need:

Lemma 3 $$c_1(X_\Gamma,\omega)=0.$$

Proof We have $\langle c_1(X_\Gamma),[S_v]\rangle = \mathrm{rot}(\Lambda_v)$ (see [40, Proposition 2.3]). Now, each Λ_v is an oriented Legendrian unknot in (S^3,ξ_{std}) and as such its rotation number can be computed to be $\mathrm{rot}(\Lambda_v)=0$. □

Therefore, the canonical bundle $\mathcal{K}=\Lambda^2_{\mathbb{C}}(T^*X_\Gamma)$ representing $-c_1(X_\Gamma)$ is trivial. To define $\mathbb{Z}$–gradings in various Floer type invariants, one needs to fix a trivialization of $\mathcal{K}^{\otimes 2}$. Of course, since $H^1(X_\Gamma)=0$, there is actually only one homotopy class of trivializations. We can induce a trivialization by picking a complexified volume form $\Omega\in\Lambda^2_{\mathbb{C}}(T^*X_\Gamma)$.

In this setup, a grading structure on a Lagrangian L can be thought of as a lift of the squared-phase map

$$\alpha_L\colon L\to S^1,\quad \alpha_L(x)=\frac{\Omega(T_xL)^2}{|\Omega(T_xL)^2|}$$

to a map $\tilde{\alpha}_L\colon L\to\mathbb{R}$. The fact that S_v and L_v are simply connected ensures that such a lift exists for our Lagrangians.

A grading structure allows one to associate an absolute Maslov index in $\mathbb{Z}$ to an intersection point $x\in S_v\cap S_w$ (see [59, Section 2d]). In our situation, all our Lagrangians S_v are simply connected, and if any two of them intersect they do so at a unique point. If x is the intersection point of S_v and S_w, then for any given $d\in\mathbb{Z}$ we can ensure that $x\in \mathrm{CF}^*(S_v,S_w)$ lies in degree d by shifting the grading structure on, say, S_w. When viewed as a generator of $\mathrm{CF}^*(S_w,S_v)$, the same intersection point would then be forced to have degree $2-d$ by Poincaré duality in Floer cohomology of compact Lagrangians (see [61, Section 12e]). Furthermore, since Γ is a tree, we can grade our Lagrangians inductively using the standard form of Γ as in Figure 2. Therefore we can grade all of our Lagrangians S_v at once such that for any pair of intersecting

Lagrangians S_v and S_w we are free to pick the gradings $(d, 2-d)$ as we would like. Collapsing a grading structure on a Lagrangian to a $\mathbb{Z}_2$–grading, we get an orientation of the underlying Lagrangian. To be compatible with the above choice of orientations for the Lagrangian spheres S_v, we will need to demand that the gradings d be odd. Throughout, a convenient choice will be to simply demand that $d = 1$, that is,

$$\mathrm{CF}^*(S_v, S_w) = \mathbb{K}[-1] \quad \text{if } v, w \text{ are adjacent.}$$

Having graded the Lagrangian spheres S_v for all v, we now pick grading structures for the noncompact Lagrangians L_v. As L_v is simply connected as well, we have the freedom to choose a grading such that

$$\mathrm{CF}^*(L_v, S_v) = \mathbb{K}[0].$$

This is compatible with our choice of orientations on L_v and S_v as given before.

These considerations fix the orientations and the grading data up to an overall shift (which does not change the degrees of intersection points) on our Lagrangians. (Note that there is a unique choice of Spin structures as our Lagrangians are simply connected.)

Somewhat more nontrivially, these choices force that if v and w are adjacent vertices, then we have the following.

Lemma 4 *For v and w adjacent vertices of the tree Γ, the shortest Reeb chord between L_v and L_w lies in the degree 0 part of $\mathrm{CW}^*(L_v, L_w)$. Furthermore, for any pair v, w, the complex $\mathrm{CW}^*(L_v, L_w)$ is supported in nonpositive degrees.*

Proof The first claim follows from a rigidity of a certain holomorphic square that contributes to the higher multiplication

$$\mu^3\colon \mathrm{HF}^0(L_v, S_v) \otimes \mathrm{HW}^0(L_w, L_v) \otimes \mathrm{HF}^2(S_w, L_w) \to \mathrm{HF}^1(S_w, S_v),$$

as explained in [7, Section 4.2]. The second claim is a consequence of the first by additivity properties of the Maslov grading (see [7, Lemma 4.11]). □

We do not use the above result in our computations below. We have stated and proved it as it helps motivate various grading choices (see also Remark 10). Let us also note that Theorem 23 below provides an indirect check of this lemma.

3 Ginzburg DG-algebra of Γ and Legendrian cohomology DG-algebra of Λ_Γ

3.1 Ginzburg DG-algebra of Γ

A *quiver* Q is a directed graph with a vertex set Q_0 and an arrow set Q_1. A rooted tree Γ in a standard form, as in Figure 2, gives rise to a quiver by orienting the edges so that they point *away* from the root. We will denote this quiver again by Γ unless otherwise specified. Recall that the path algebra $\mathbb{K}\Gamma$ of quiver Γ is defined as a vector space having all the paths in the quiver as basis (including, for each vertex v of the quiver Γ, a trivial path e_v of length 0), and multiplication is given by concatenation of paths. As mentioned before, throughout we concatenate paths from right to left, when we express them as a product.

The cohomologically graded 2–Calabi–Yau *Ginzburg DG-algebra* $\mathscr{G}_\Gamma$ of Γ (with zero potential) is defined as follows (see [39; 12; 41]).

Definition 5 Consider the extended quiver $\widehat{\Gamma}$ with vertices $\widehat{\Gamma}_0 = \Gamma_0$ and arrows $\widehat{\Gamma}_1$ consisting of

- the original arrows g in Γ_1 in bidegree $(1,-1)$;
- the opposite arrows g^* to g in Γ_1 in bidegree $(1,-1)$;
- loops h_v at the vertex $v \in \Gamma_0$ of bidegree $(1,-2)$.

We define $\mathscr{G}_\Gamma$ to be the DG-algebra over the semisimple ring $\mathrm{k} = \bigoplus_{v\in\Gamma_0} \mathbb{K}e_v$ given by the path algebra $\mathbb{K}\widehat{\Gamma}$ with the differential d of bidegree $(1,0)$ defined as a k–bimodule map by

$$dg = dg^* = 0 \quad \text{and} \quad dh = \sum_{g\in\Gamma_1} g^*g - gg^*,$$

where $h = \sum_{v\in\Gamma_0} h_v$.

In the notation (r,s) for bigraded complexes, r corresponds to the path-length grading and as usual we will call $r+s$ the total degree. In particular, the notation $H^*(\mathscr{G})$ will stand for the cohomology graded by the total degree. Note also that with respect to the total grading $\mathscr{G}_\Gamma$ is supported in nonpositive degrees.

The way we chose to orient the edges of Γ has only a minor effect on $\mathscr{G}_\Gamma$. Namely, different choices change the signs in the formula for the differential. Our choice is to ensure the consistency with the choice of orientations of the Lagrangians L_v, as we shall see in the next section. In particular, let Γ^{op} be the quiver obtained from Γ by reversing the orientation of all edges of Γ. Then the associated Ginzburg algebra gives $\mathscr{G}_\Gamma^{\mathrm{op}}$, the opposite of the Ginzburg algebra $\mathscr{G}_\Gamma$ associated to the original quiver Γ. In other words,

$$\mathscr{G}_{\Gamma^{\mathrm{op}}} = \mathscr{G}_\Gamma^{\mathrm{op}}.$$

Definition 6 The cohomology in total degree 0 of $\mathscr{G}_\Gamma$ is called the *preprojective algebra* $\Pi_\Gamma := H^0(\mathscr{G}_\Gamma)$. It is the quotient of the path algebra $\mathbb{K}D\Gamma$ by the ideal generated by

$$\sum_{g\in\Gamma_1} g^*g - gg^*,$$

where $D\Gamma$ denotes the double of Γ, obtained by adding the opposite arrow g^* for every $g\in\Gamma_1$.

It turns out that the nature of the DG-algebra $\mathscr{G}_\Gamma$ depends on whether Γ is of Dynkin type or not, as shown in the following theorem. It was first proven by Hermes [41] under the assumption that $\mathbb{K}$ is algebraically closed and characteristic 0. In Corollary 26, we give a proof of the first part of the theorem over an arbitrary field.

Theorem 7 (Hermes [41] and also Corollary 26) (1) *Suppose Γ is non-Dynkin. Then $H^*(\mathscr{G}_\Gamma) = \Pi_\Gamma$ is supported in degree 0 and is quasi-isomorphic to $\mathscr{G}_\Gamma$. In other words, $\mathscr{G}_\Gamma$ is formal.*

(2) *Suppose Γ is Dynkin and $\mathbb{K}$ is characteristic 0 and algebraically closed. Then*

$$H^*(\mathscr{G}_\Gamma) \cong \Pi_\Gamma \rtimes_\nu \mathrm{k}[u], \quad |u| = -1$$

as a k–algebra, where the multiplication is twisted by the Nakayama automorphism ν on Π_Γ. Furthermore, $\mathscr{G}_\Gamma$ is not formal and there is an A_∞–structure $(\mu^n)_{n\geq 2}$ on the twisted polynomial algebra $\Pi_\Gamma \rtimes_\nu \mathrm{k}[u]$ making it a minimal model of $\mathscr{G}_\Gamma$. Moreover, this A_∞–structure is u–equivariant, and $\mu^n = 0$ for $n \neq 2, 3$.

The *Nakayama automorphism* $\nu\colon \Pi_\Gamma \to \Pi_\Gamma$ in the above theorem refers to the automorphism defined by

$$\nu(g_{wv}) = \begin{cases} g_{\rho(w)\rho(v)} & \text{if } g_{wv}\in\Gamma \text{ or } g_{\rho(w)\rho(v)}\in\Gamma, \\ -g_{\rho(w)\rho(v)} & \text{if } g_{vw}, g_{\rho(v)\rho(w)}\in\Gamma, \end{cases}$$

where g_{wv} denotes the arrow from the vertex v to w in Π_Γ, and ρ denotes either the natural involution of the Dynkin graph (precisely when Γ is of type A_n, D_{2n+1} or E_6) or the identity. We will abuse the notation and always denote the arrow from v to w by g_{wv} regardless of where it is considered, in the quiver Γ, its double $D\Gamma$, the extended quiver $\widehat{\Gamma}$ or in the algebras $\mathscr{G}_\Gamma$ and Π_Γ, for that matter. In particular, $g_{vw} = g^*_{wv}$ if g_{wv} belongs to Γ. Note that ν has order at most 2 and it is the identity if and only if Γ is of type A_1 or it is of type D_{2n}, E_7 or E_8 and the base field $\mathbb{K}$ is of characteristic 2 (see [18, Definition 4.6]).

3.2 Legendrian cohomology DG-algebra of Λ_Γ

We recall the definition of the $\mathbb{Z}$–graded Chekanov–Eliashberg DG-algebra of the Legendrian link $\Lambda_\Gamma = \bigcup \Lambda_v$ following [17, Section 4], where it is denoted as $\mathrm{LHA}(\Lambda_\Gamma)$. It was originally introduced in [30; 19].

Let $\mathcal{R}$ denote the finite set of Reeb chords from Λ_Γ to itself. Recall from Section 2 that $\mathcal{R}$ is in bijection with the set of crossings in the Lagrangian projection of Λ_Γ (Figure 4). We endow the vector space $\mathbb{K}\langle\mathcal{R}\rangle$ with a k–bimodule structure by declaring

$$e_w \mathcal{R} e_v$$

to be the set of Reeb chords from Λ_w to Λ_v. As a k–module, $\mathrm{LHA}(\Lambda)$ is the tensor algebra over the semisimple ring k given by

$$\mathrm{LHA}_*(\Lambda_\Gamma) := \bigoplus_{i=0}^{\infty} \mathbb{K}\langle\mathcal{R}\rangle^{\otimes_{\mathrm{k}} i}.$$

After decorating Λ_Γ with extra data by orienting each component and picking a basepoint at each component as in Figure 4, the chords $c \in \mathcal{R}$ acquire a kind of Conley–Zehnder grading by $\mathbb{Z}$ which we denote by $|c|$. The subscript in the notation of $\mathrm{LHA}_*(\Lambda_\Gamma)$ denotes the induced grading on the tensor algebra. Elements $e_v \in \mathrm{k}$ have degree 0; however, in general there may also be Reeb chords which have degree 0. The differential $D\colon \mathrm{LHA}_*(\Lambda_\Gamma) \to \mathrm{LHA}_{*-1}(\Lambda_\Gamma)$ is defined as a map $D\colon \mathbb{K}\langle\mathcal{R}\rangle_* \to \mathrm{LHA}_{*-1}(\Lambda_\Gamma)$ and extended by the graded Leibniz rule to $\mathrm{LHA}_*(\Lambda)$.

Note that in general the differential is not compatible with the path-length grading corresponding to the index i in the definition of $\mathrm{LHA}_*(\Lambda)$.

As we follow the cohomological convention to be consistent with the literature on Fukaya categories, instead of $\mathrm{LHA}_*(\Lambda)$ we will use the cohomologically graded DG-algebra $\mathrm{LCA}^*(\Lambda)$. As a k–module, it is given by

$$\mathrm{LCA}^*(\Lambda_\Gamma) := \mathrm{LHA}_{-*}(\Lambda_\Gamma).$$

The differential $D\colon \mathrm{LCA}^*(\Lambda_\Gamma) \to \mathrm{LCA}^{*+1}(\Lambda_\Gamma)$ is just carried over from the one on $\mathrm{LHA}_*(\Lambda_\Gamma)$.

Let us describe the Legendrian cohomology DG-algebra of Λ_Γ more explicitly. The underlying algebra of $\mathrm{LCA}^*(\Lambda_\Gamma)$ is the tensor algebra of the k–bimodule $\mathbb{K}\langle\mathcal{R}\rangle$ generated by the Reeb chords (ie crossings in Figure 4):

$$\mathcal{R} = \{c_{wv}, c_{vw} : g_{wv} \in \Gamma_1\} \cup \{c_v : v \in \Gamma_0\},$$

where c_v is the Reeb chord at the unique self-crossing of the component Λ_v, and for every two adjacent vertices v and w of the tree Γ, c_{wv} corresponds to the unique Reeb chord from Λ_w to Λ_v, ie the chord at the unique crossing between Λ_v and Λ_w where Λ_w is the undercrossing component.

Notice the remarkable coincidence of the k–bimodule structure on $\mathrm{LCA}^*(\Lambda_\Gamma)$ and the k–bimodule structure on $\mathscr{G}_\Gamma$ from Definition 5. Next, we will see that the differentials do not agree in general. Nonetheless the Legendrian cohomology DG-algebra is isomorphic to a deformation of the Ginzburg algebra.

Theorem 8 *If Λ_Γ is the Legendrian link in the standard form associated to the tree Γ with Lagrangian projection in Figure 4 with the grading decoration as indicated, then there is an isomorphism between $(\mathrm{LCA}^*(\Lambda_\Gamma), D)$ and a deformation of $(\mathscr{G}_\Gamma, d)$ as DG-algebras. More precisely, there is a graded derivation $\mathfrak{d}\colon \mathscr{G}_\Gamma \to \mathscr{G}_\Gamma$ with homogeneous components $\mathfrak{d} = d_3 + d_5 + \cdots + d_{2m-1}$ for some $m \geq 1$, d_{2i-1} having bidegree $(2i-1, 2-2i)$, and there is an isomorphism of DG-algebras*

$$(\mathrm{LCA}^*(\Lambda_\Gamma), D) \simeq (\mathscr{G}_\Gamma, d + \mathfrak{d})$$

such that the Conley–Zehnder degree on the left-hand side agrees with the total degree on the right-hand side.

Proof **Generators** The natural one-to-one correspondence, ie $g_{wv} \leftrightarrow c_{wv}$, $h_v \leftrightarrow c_v$, between the arrow set $\widehat{\Gamma}_1$ of the extended quiver $\widehat{\Gamma}$ and the set $\mathcal{R}$ of Reeb chords provides the isomorphism of the underlying k–algebras, the path algebra $\mathbb{K}\widehat{\Gamma}$ and the tensor algebra of $\mathbb{K}\langle\mathcal{R}\rangle$. Note that the Reeb orientation of the chord c_{wv} is from Λ_w to Λ_v, whereas the arrow g_{wv} goes from the vertex v to w.

Gradings It suffices to identify the gradings of the generators. We first recall the definition for an arbitrary Legendrian link $\Lambda \subset (S^3, \xi_{\mathrm{std}})$.

According to the original combinatorial description [19], LCA has a $\mathbb{Z}/r\mathbb{Z}$–grading, where r is the gcd of the rotation numbers of the components. In our case, each component of Λ_Γ is an unknot with rotation number 0, providing a $\mathbb{Z}$–grading on $\mathrm{LCA}^*(\Lambda_\Gamma)$.

Let $z_\pm$ be the endpoints of a Reeb chord c of an oriented Legendrian link Λ equipped with basepoints on every component, z_+ being the one with the greater z–coordinate. Let $\gamma_\pm$ be the shortest paths in Λ, from $z_\pm$ to the basepoint of the corresponding component, in the direction of the orientation of Λ. The grading of c in LCA is defined to be $2r_- - 2r_+ + \frac{1}{2}$, where $r_\pm \in \mathbb{Q}$ is the number of counterclockwise rotations the tangent vector of $\gamma_\pm$ makes (in the xy–plane). It is straightforward to verify that the grading of every generator of the form c_v of $\mathrm{LCA}(\Lambda_\Gamma)$ is -1 and that of the form c_{wv} is 0.

Differential We briefly recall the definition of the differential of LCA for any Legendrian link in the standard contact S^3, and then compute the differentials on the set $\mathcal{R}$ of generators of $\mathrm{LCA}^*(\Lambda_\Gamma)$. The rest will be determined by the Leibniz rule.

To simplify the definition, we arrange that at every crossing of the Lagrangian projection, the understrand and the overstrand have slopes $+1$ and -1, respectively. We also use the same notation for a crossing in the Lagrangian projection as the corresponding Reeb chord.

First of all, each quadrant around a crossing in the Lagrangian projection is decorated with a Reeb sign. The right and left quadrants at a crossing have positive signs whereas the top and bottom quadrants have negative signs.

There is also a second set of signs, orientation signs, for these quadrants. Every quadrant has orientation sign $+1$ except for the bottom and right quadrants at an even-graded crossing, which are decorated with -1, as in Figure 5. In fact, the choice of orientation signs for a given diagram depends on an isotopy of the diagram near the crossing so that the strand with a positive slope goes under the strand with a negative slope, as in Figure 5. We indicated our choice in the upper left diagram of Figure 6. This affects the signs, but different choices give isomorphic DG-algebras (see [28, page 80]).

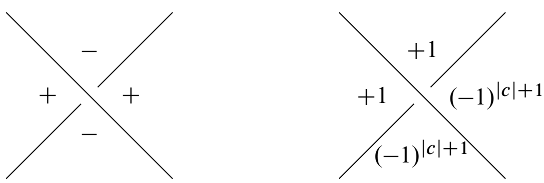

Figure 5: Reeb signs (left) and orientation signs (right) at a crossing c

On a generator, the differential is given by a count of immersed polygons and it is extended by the graded Leibniz rule. The polygons taken into account are in the xy–plane with boundary on the Lagrangian projection of the link and vertices at the crossings. It is also required that at all but one vertex of the polygon, the quadrant included in the polygon should have a negative *Reeb* sign. Suppose that Δ is such an immersed polygon whose positive vertex is at c and the negative vertices $c_1, c_2, \dots, c_m$ are in order as we traverse the boundary of Δ counterclockwise starting at c. Note that m may be 0 and the c_i are not necessarily distinct. If b is the total number of times the boundary of Δ passes through basepoints of the Legendrian link, the orientation sign ϵ_Δ is defined to be $(-1)^b$ times the product of the *orientation* signs at the vertices.

With this setup, we have

$$dc = \sum_\Delta \epsilon_\Delta c_m c_{m-1} \cdots c_1$$

for any generator c. Observe that the differential of a generator of the form c_{wv} vanishes since it has grading 0 and $\mathrm{LCA}^*(\Lambda_\Gamma)$ is nonpositively graded. Again for grading reasons, any negative vertex of an immersed polygon which contributes to the differential of a generator c_v is of type c_{uw}.

In the rest of the proof we will show that

$$D(c_v) = -\sum_{u:g_{vu}\in\Gamma_1} c_{vu}c_{uv} + \sum_{i\geq 1} \sum_{\substack{w_1,\dots,w_i \\ g_{w_j v}\in\Gamma_1 \\ w_1<\cdots<w_i}} c_{vw_1}c_{w_1v}\cdots c_{vw_i}c_{w_iv},$$

where the ordering in the last summation refers to the clockwise ordering of the components of Λ_Γ which are linked to v from the right in the Lagrangian projection in Figure 4, eg the natural ordering of the integers associated to components in Figure 4. Note that the second sum not only corresponds to higher-order terms in the length filtration, it also contributes terms of word-length 2 of the form $c_{vw_1}c_{w_1v}$. Indeed, all the terms of word-length 2 that appear in the image of $D(c_v)$ precisely correspond to $d(c_v)$ in $\mathscr{G}_\Gamma$. In particular, the first sum has at most one term as long as our Legendrian link is associated to a tree in the standard form.

We will prove that all the terms in the above differential are induced by *embedded* polygons as indicated in Figure 6, the relevant piece of the Lagrangian projection

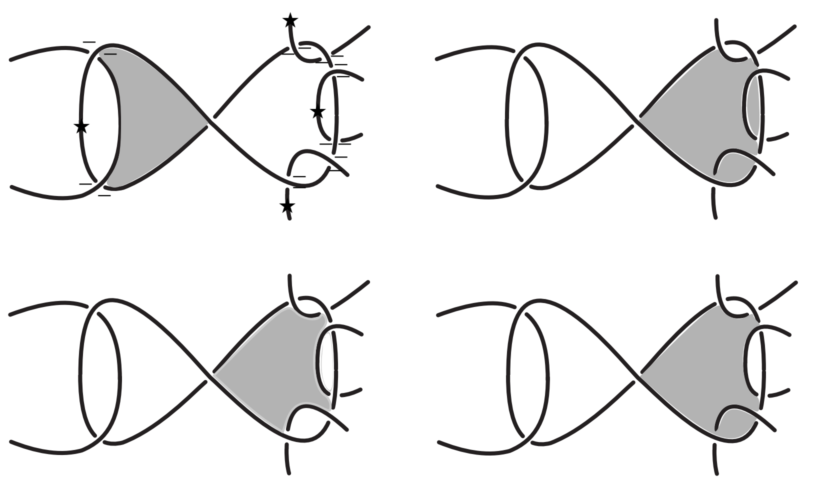

Figure 6: The polygons which correspond to the words in the differential $D(c_v)$: (from top left in clockwise order) a triangle (with a negative orientation sign), a triangle, a pentagon, and a heptagon (all with positive orientation signs). The quadrants with negative orientation signs and the basepoints are indicated in the upper left diagram.

given in Figure 4, together with the orientation signs at the crossings. There are also two unigons with a unique vertex at c_v, one to the left and the other to the right with canceling contributions to the differential $D(c_v)$ since they come with opposite signs.

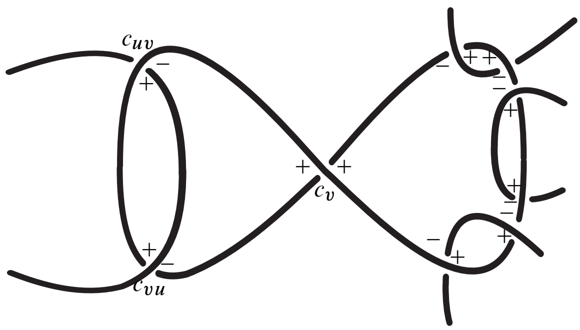

Figure 7: Reeb signs

We now prove that there are no other immersed polygons which contribute to the differential $D(c_v)$. To begin with, any such polygon has a (Reeb-) positive vertex at c_v (see Figure 7 for the Reeb signs at the relevant crossings). Start traversing its boundary in the counterclockwise direction assuming that the polygon includes the left quadrant at c_v. If it has a vertex other than c_v, ie if it is not the unigon canceled by a similar unigon to the right, then the only option for an initial negative vertex is at c_{uv} because of the configuration of the Reeb signs. Moreover, this vertex has to be followed (as we continue traversing the boundary) by a vertex at c_{vu} since otherwise the polygon would intersect the region outside the Lagrangian projection, which is prohibited. Similar considerations imply that a polygon which includes the right quadrant at c_v can only have vertices at the crossings of Λ_v with other components of Λ as shown in Figure 6 above so as not to intersect the noncompact region. □

Remark 9 A relation between Ginzburg's construction of CY3 DG-algebras associated with quivers (with potentials) and Fukaya categories of certain quasiprojective 3–folds also appears in the work of Smith [69].

Remark 10 Recall that $\mathrm{LCA}^*(\Lambda_\Gamma)$ is associated to the Legendrian attaching spheres Λ_v of Weinstein 2–handles. Stated results of [17] provide a dual picture given in terms of the wrapped Floer cohomology of the cocores L_v of these handles induced by cobordism maps associated to the handle attachments. Namely, there is a grading-preserving quasi-isomorphism of A_∞–algebras

$$\mathrm{LCA}^*(\Lambda_\Gamma) \simeq \bigoplus_{v,w} \mathrm{CW}^*(L_v, L_w).$$

A rigorous justification of the equivalence of these two dual pictures is not fully established at this time. However, a detailed sketch of proof based on the results of [17] has recently appeared in [27, Theorem 2]. We must emphasize that we do not make use of this correspondence anywhere in our computations. Rather, this appealing geometric picture serves us as a guide to find the correct algebraic statement to be proven rigorously.

3.2.1 Recourse to deformation theory of DG-algebras As a consequence of the explicit computation given above we can see the Legendrian cohomology DG-algebra $\mathrm{LCA}^*(\Lambda_\Gamma)$ as a deformation of the Ginzburg DG-algebra $\mathscr{G}_\Gamma$. Therefore, it is natural to check whether this deformation is trivial or not (up to equivalence). We recall here the basics of deformation theory of DG-algebras and exploit it to determine the relationship between our computation of $\mathrm{LCA}^*(\Lambda_\Gamma)$ and the Ginzburg DG-algebra $\mathscr{G}_\Gamma$. A classical reference for this material is [38]. A recent exposition close to our purpose appears in [65, Appendix A].

Unfortunately, these methods do not help directly as they apply in the setting of formal deformations (such as a deformation over $\mathrm{k}[\![t]\!]$) whereas here we have that $\mathrm{LCA}^*(\Lambda_\Gamma)$ is a global deformation of $\mathscr{G}_\Gamma$ (over $\mathrm{k}[t]$). Nonetheless, it is helpful to start at the formal level and observe that we can arrange for a globalization in certain cases.

There is a decreasing, exhaustive, bounded-above filtration on the complex $\mathrm{LCA}^*(\Lambda_\Gamma)$:

$$\mathcal{F}^0 := \mathrm{LCA}^*(\Lambda_\Gamma) \supset \mathcal{F}^1 := \bigoplus_{i=1}^{\infty} \mathbb{K}\langle\mathcal{R}\rangle^{\otimes_{\mathrm{k}}^i} \supset \cdots \supset \mathcal{F}^p := \bigoplus_{i=p}^{\infty} \mathbb{K}\langle\mathcal{R}\rangle^{\otimes_{\mathrm{k}}^i} \supset \cdots .$$

Let us write $(\mathrm{LCA}^*(\Lambda_\Gamma), D) = (\mathscr{G}_\Gamma, d_1 + d_2 + \cdots + d_m)$, for some finite m, where $d_i\colon \mathcal{F}^p \to \mathcal{F}^{p+i}$ is the i^{th} homogeneous piece of the differential. Observe that $d_1 = d$ can be identified as the differential in the Ginzburg DG-algebra. It follows from k–linearity of the differential that in fact d_i is identically zero for even i. Note also that since $\mathscr{G}_\Gamma$ is bigraded, this complex is doubly graded. Denoting the second grading by s, we have $s(d_{2i-1}) = 2-2i$.

Now, the first nontrivial d_i for $i > 1$ is possibly d_3. Because $D^2 = 0$, using the filtration, we deduce that

$$d_1 d_3 + d_3 d_1 = 0.$$

Recall that the reduced bar complex $(\hom_{\mathrm{k}}(\mathrm{T}\overline{\mathscr{G}_\Gamma}, \mathscr{G}_\Gamma), \delta = \delta_0 + \delta_1)$ can be used to compute Hochschild cohomology of $\mathscr{G}_\Gamma$. Here, we only need the explicit form of the Hochschild differential for elements $\phi \in \hom_{\mathrm{k}}(\overline{\mathscr{G}_\Gamma}, \mathscr{G}_\Gamma)$ (see formula in [61, Equation (1.8)], which we adapted using DG-algebra conventions given in the introduction).

For such ϕ, we have

$$(-1)^{|\phi|+|b|}(\delta_0\phi)(a \otimes_{\mathrm{k}} b) = a\phi(b) + (-1)^{|\phi||b|}\phi(a)b - \phi(ab),$$
$$(-1)^{|\phi|+|a|}(\delta_1\phi)(a) = d\phi(a) - \phi(da).$$

By definition, $\mathscr{G}_\Gamma$ is bigraded and its differential has bidegree $(1,0)$, so the Hochschild cochain complex $\mathrm{CC}^*(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma) = \hom_{\mathrm{k}}(\mathrm{T}\overline{\mathscr{G}_\Gamma}, \mathscr{G}_\Gamma)$ has three gradings: the cohomological degree, the degree induced by the total degree $r+s$ on $\mathscr{G}_\Gamma$ and the internal grading induced by the second grading s on $\mathscr{G}_\Gamma$. However, the Hochschild differential $\delta = \delta_0 + \delta_1$ is homogeneous (of degree 1) with respect to the sum of the first two gradings and it also preserves the internal degree, hence we get a bigrading on $\mathrm{HH}^*(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma)$, which we write as

$$\mathrm{HH}^*(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma) \cong \bigoplus_{r,s} \mathrm{HH}^r(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma[s]), \tag{4}$$

where r is the total degree (the sum of the cohomological degree and the degree induced by the total degree on $\mathscr{G}_\Gamma$) and s is the internal grading induced by the internal grading on $\mathscr{G}_\Gamma$.

Now, the fact that d_3 is a degree-1 derivation which anticommutes with d_1 means that the sign-modified map $\tilde{d}_3 \in \hom^1_{\mathrm{k}}(\overline{\mathscr{G}_\Gamma}, \mathscr{G}_\Gamma)$, defined by

$$\tilde{d}_3 a = (-1)^{|a|} d_3 a,$$

is closed under the Hochschild differential. This yields the first obstruction class of the deformation:

$$[\tilde{d}_3] \in \mathrm{HH}^2(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma[-2]).$$

If this class is trivial, choosing a trivializing class $\phi_2 \in \hom^0_{\mathrm{k}}(\overline{\mathscr{G}_\Gamma}, \mathscr{G}_\Gamma[-2])$, we get a map ϕ_2 for which we have

$$d_3 = d\phi_2 - \phi_2 d.$$

Note that ϕ_2 is induced by a map $\mathbb{K}\langle\mathcal{R}\rangle \to \mathbb{K}\langle\mathcal{R}\rangle^{\otimes_{\mathrm{k}} 3}$. Therefore, we can consider an *algebra* map

$$\Phi_2 = \mathrm{Id} + \phi_2 \colon \mathscr{G}_\Gamma \to \mathscr{G}_\Gamma$$

defined initially as a map on $\mathbb{K}\langle\mathcal{R}\rangle \to \mathscr{G}_\Gamma$ and then extended to an algebra map.

Then, we would like to define a new differential D' on $\mathscr{G}_\Gamma$ of the form

$$D' = d + d'_5 + \cdots$$

so that $\Phi_2\colon (\mathscr{G}_\Gamma, D') \to (\mathscr{G}_\Gamma, D)$ is a chain map (in addition to being an algebra map). The obvious candidate for D' is given by

$$D' = (\mathrm{Id} - \phi_2 + \phi_2^2 - \cdots) \circ D \circ (\mathrm{Id} + \phi_2).$$

However, the alternating sum $(\mathrm{Id} - \phi_2 + \phi_2^2 - \cdots)$ will in general be an infinite series; therefore, to make sense of this we need to consider the completion of $\mathscr{G}_\Gamma$ with respect to the length filtration $\mathcal{F}^\bullet$:

$$\widehat{\mathscr{G}_\Gamma} = \varprojlim_p \mathscr{G}_\Gamma / \mathcal{F}^p \mathscr{G}_\Gamma.$$

The differential D of $\mathrm{LCA}^*(\Lambda_\Gamma)$ extends naturally to $\widehat{\mathscr{G}_\Gamma}$. We write the resulting complex as

$$\widehat{\mathrm{LCA}}(\Lambda_\Gamma) = (\widehat{\mathscr{G}_\Gamma}, D)$$

Concretely, we can write the underlying k–bimodule as $\widehat{\mathrm{LCA}}(\Lambda_\Gamma) = \mathbb{K}\langle \mathcal{R} \rangle [\![t]\!]$, where t is a formal parameter in degree 0. In other words, we now allow formal power series in Reeb chords.

We can now proceed with the construction mentioned above. Notice that since ϕ_2 increases the length by 2, there is no convergence issue for the series $(\mathrm{Id} - \phi_2 + \phi_2^2 - \cdots)$ on $\widehat{\mathscr{G}_\Gamma}$. Therefore, we have a filtered DG-algebra map

$$\Phi_2\colon (\widehat{\mathscr{G}_\Gamma}, D') \to (\widehat{\mathscr{G}_\Gamma}, D)$$

which by construction is a chain map with an inverse, hence is in particular a quasi-isomorphism.

We can then focus on the complex $(\widehat{\mathscr{G}_\Gamma}, D' = d + d_5' + \cdots)$. As before, we have that d_5' is a derivation which anticommutes with d, hence the sign-twisted map $\widetilde{d}_5'$ leads to an obstruction class $[\widetilde{d}_5'] \in \mathrm{HH}^2(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma[-4])$. If this vanishes we can continue along and find a quasi-isomorphism of the form $\mathrm{Id} + \phi_4$. Iterating this argument infinitely many times (which we can do as each quasi-isomorphism increases the length), we obtain the following lemma (cf [65, Lemma A.5]).

Lemma 11 *Suppose that $\mathrm{HH}^2(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma[s]) = 0$ for all $s < 0$. Then there exists a quasi-isomorphism of completed DG-algebras*

$$(\widehat{\mathscr{G}_\Gamma}, d) \simeq (\widehat{\mathrm{LCA}}(\Lambda_\Gamma), D).$$

We next apply these ideas to the case where $\Gamma = D_n$ and show that all the obstructions vanish in this case. Furthermore, we prove that one can truncate the above quasi-isomorphism, eliminating the need for completions. Here, we make use of the results of Section 6.2.3, where $\mathrm{HH}^*(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma)$ is computed for $\Gamma = D_n$. We would like to point out that the computation given there is independent of the conclusions we are drawing here.

The following lemma is the key technical result that we will use to truncate the quasi-isomorphism given on completions by the above deformation theory argument.

Lemma 12 *Let $\mathcal{F}^\bullet$ denote the length filtration on $\mathrm{LCA}^*(\Lambda_{D_n})$. For each grading k, there exists a $p(k)$ such that for all $p \geq p(k)$ we have that*

$$\mathcal{F}^p H^k(\mathrm{LCA}(\Lambda_{D_n})) = \mathrm{Im}\big(H^k(\mathcal{F}^p\,\mathrm{LCA}(\Lambda_{D_n})) \to H^k(\mathrm{LCA}(\Lambda_{D_n}))\big) = 0.$$

In particular, for all k, the filtration on $H^k(\mathrm{LCA}(\Lambda_{D_n}))$ induced by $\mathcal{F}^\bullet$ is complete and Hausdorff.

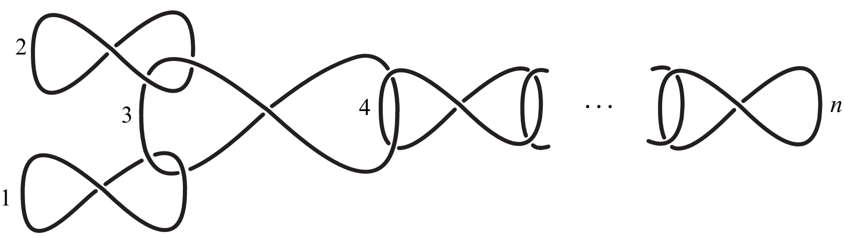

Figure 8: Lagrangian projection of a Legendrian link associated to the D_n tree

Proof Consider the Lagrangian projection in Figure 8. The proof of Theorem 8 gives us the following description of the differential on $(\mathrm{LCA}^*(\Lambda_{D_n}), D)$:

$$\begin{aligned}
Dc_1 &= c_{13}c_{31},\\
Dc_2 &= c_{23}c_{32},\\
Dc_3 &= -c_{31}c_{13} - c_{32}c_{23} + c_{34}c_{43} - c_{31}c_{13}c_{32}c_{23},\\
Dc_4 &= -c_{43}c_{34} + c_{45}c_{54},\\
&\;\vdots\\
Dc_{n-1} &= -c_{(n-1)(n-2)}c_{(n-2)(n-1)} + c_{(n-1)n}c_{n(n-1)},\\
Dc_n &= -c_{n(n-1)}c_{(n-1)n},
\end{aligned}$$

where the gradings are given by $|c_i| = -1$ and $|c_{ij}| = 0$. In particular, $H^*(\mathrm{LCA}(\Lambda_{D_n}))$ is supported in nonpositive degrees.

Notice that $D = d_1 + d_3$, where d_1 is the differential on the Ginzburg DG-algebra $\mathscr{G}_{D_n}$ and d_3 is zero on all the generators except c_3, and we have

$$d_3(c_3) = -c_{31}c_{13}c_{32}c_{23}.$$

We shall first establish the result for $H^0(\mathrm{LCA}(\Lambda_{D_n}))$ by direct computation. The goal here is to take any word in c_{ij} and prove that if the word is long enough, then it is actually null-homologous.

Note that we have a decomposition

$$H^0(\mathrm{LCA}(\Lambda_{D_n})) \cong \bigoplus_{i,j=1}^{n} e_i H^0(\mathrm{LCA}(\Lambda_{D_n}))e_j.$$

Letting $x = c_{31}c_{13}$, $y = c_{32}c_{23}$ and $z = c_{34}c_{43}$ we obtain

$$e_3 H^0(\mathrm{LCA}(\Lambda_{D_n}))e_3 \cong \mathbb{K}\langle x, y, z\rangle/(x^2, y^2, z^{n-2}, x + y + xy - z)$$

(cf [57, Proposition 11.3.2(i)]). Indeed, we have

$$x^2 = D(c_{31}c_1c_{13}), \quad y^2 = D(c_{32}c_2c_{23}), \quad x + y + xy - z = D(-c_3).$$

Next, observe that for $4 \le i \le n-1$, we have $c_{i(i-1)}c_{(i-1)i} = c_{i(i+1)}c_{(i+1)i} \in H^0(\mathrm{LCA}(\Lambda_{D_n}))$ since their difference is precisely Dc_i. Consequently, we get

$$\begin{aligned} z^{n-2} &= c_{34}(c_{43}c_{34})^{n-3}c_{43} = c_{34}(c_{45}c_{54})^{n-3}c_{43} = c_{34}c_{45}(c_{56}c_{65})^{n-4}c_{54}c_{43} = \cdots \\ &= c_{34}c_{45}\cdots c_{(n-1)n}c_{n(n-1)}c_{(n-1)n}c_{n(n-1)}\cdots c_{54}c_{43} \\ &= D(-c_{34}c_{45}\ldots c_{(n-1)n}c_nc_{n(n-1)}\ldots c_{54}c_{43}). \end{aligned}$$

Furthermore, any word in $e_3 H^0(\mathrm{LCA}(\Lambda_{D_n}))e_3$ is cohomologous to a word in x, y, z which is of the same length (note that the lengths of x, y and z are 2). Namely, whenever a word w has terms which goes along the long branch of the D_n tree, it has to return back at some point, hence it will include a subword of the form $c_{i(i+1)}c_{(i+1)i}$ which can be replaced with $c_{i(i-1)}c_{(i-1)i}$ applying the relation Dc_i. This can be repeated until we replace each subword that lies in the long branch by a power of z.

Arguing similarly, one can see why it suffices to consider $e_3 H^0(\mathrm{LCA}(\Lambda_{D_n}))e_3$ to prove the statement in the lemma for the zeroth cohomology. Indeed, the relations given by $Dc_4, Dc_5, \ldots, Dc_n$ can be used to show that any sufficiently long word in $\mathrm{LCA}^0(\Lambda_{D_n})$ can be replaced by a word which contains a sufficiently long subword in $e_3\,\mathrm{LCA}^0(\Lambda_{D_n})e_3$. More precisely, for any word $w \in \langle c_{ij} \mid i, j = 1, n\rangle$ we can write

$$w = \alpha v \beta + \langle \mathrm{Im}\, D\rangle$$

such that v lies in $e_3\,\mathrm{LCA}^0(\Lambda_{D_n})e_3$ and is sufficiently long. In fact, since we only use the preprojective relations, Dc_i for $i \ne 3$, one can show that the analogue of [57, Proposition 11.3.2(ii)] holds in this case.

We can simplify the presentation of $e_3 H^0(\mathrm{LCA}(\Lambda_{D_n}))e_3$ further by eliminating the z variable and write

$$e_3 H^0(\mathrm{LCA}(\Lambda_{D_n}))e_3 \cong \mathbb{K}\langle x, y\rangle/(x^2, y^2, (x+y+xy)^{n-2}).$$

Let us define two-sided ideals

$$I_n = (x^2, y^2, (x+y+xy)^{n-2}) \quad \text{and} \quad J_n = (x^2, y^2, (x+y)^{n-2})$$

in $\mathbb{K}\langle x, y\rangle$ and claim that they are equal for $n \geq 4$. Note that in $\mathbb{K}\langle x, y\rangle/J_n$ any word that is long enough is trivial; in particular, this is a finite-dimensional vector space. This is because the only words that are not killed by the relations $x^2 = y^2 = 0$ are words alternating in x and y, and sufficiently long such words are killed by $x(x+y)^{n-2}y$ and $y(x+y)^{n-2}x$. Therefore the result for $H^0(\mathrm{LCA}(\Lambda_{D_n}))$ follows from the claim $I_n = J_n$.

To prove this claim, first observe that $A = x+y$ and $B = x+y+xy$ satisfy

$$B^2 = (1+x)A^2(1+y) \in \mathbb{K}\langle x, y\rangle/(x^2, y^2).$$

Moreover, since $(1+x)(1-x) = 1 = (1+y)(1-y)$ the above identity leads to $A^2 = (1-x)B^2(1-y)$ and together they show $I_4 = J_4$. We similarly obtain $I_5 = J_5$, using the observation

$$B^3 = (1+x)A^3(1+x)(1+y) \in \mathbb{K}\langle x, y\rangle/(x^2, y^2).$$

The fact that A^2 is in the center of $\mathbb{K}\langle x, y\rangle/(x^2, y^2)$ implies

$$\begin{aligned} B^{2k} &= (B^2)^k = (1+x)A^{2k}(1+y)(1+x)\cdots(1+y), \\ B^{2k+1} &= B^3(B^2)^{k-1} = (1+x)A^{2k+1}(1+y)(1+x)\cdots(1+y), \end{aligned}$$

proving $I_n = J_n$ for every $n \geq 4$.

Alternatively, one can check that a noncommutative Gröbner basis (with respect to the lexicographical order) for both I_n and J_n is given by the collection of the following three elements:

$$\{x^2, y^2, xyxy\cdots + yxyx\cdots\}$$

where the lengths of the words in the last element are $n-2$.

This completes the proof of the lemma for $H^0(\mathrm{LCA}(\Lambda_{D_n}))$. It is much harder to directly compute $H^i(\mathrm{LCA}(\Lambda_{D_n}))$ for $i < 0$ and verify Hausdorffness of the length filtration. Fortunately, there is an alternative way to go about this, making use of a recent result of Dimitroglou Rizell [26] which in turn exploits the weak division algorithm in free noncommutative algebras due to P M Cohn [22]. This is a general result about Legendrian cohomology DG-algebras which states that the natural algebra homomorphism

$$H^*(\mathrm{LCA}(\Lambda_\Gamma)) \to \mathrm{LCA}^*(\Lambda_\Gamma)/\langle \mathrm{Im}\, D\rangle$$

induced by inclusion is injective, where $\langle \mathrm{Im}\, D\rangle$ denotes the two-sided ideal in the tensor algebra $\mathrm{LCA}^*(\Lambda_\Gamma)$ generated by the image of the differential. In view of this, it suffices to show that for each k there exists a $p(k)$ such that if w is a word in c_{ij} of length greater than $p(k)$ containing exactly k instances of c_i, then w is in $\langle \mathrm{Im}\, D\rangle$.

This is, however, quite straightforward given what we have already proven. Namely, in any such word, since the number of degree -1 generators, c_i, is precisely k as soon as the length is sufficiently large, we can find a sufficiently long subword consisting of degree 0 generators c_{ij} only. Now, we proved above that any sufficiently long word in the degree 0 generators c_{ij} is in the image of D. Thus, the result follows. □

Note that the corresponding result also holds true for $\mathscr{G}_{D_n}$ but this is much simpler. The cohomology $H^*(\mathscr{G}_{D_n})$ is a graded filtered algebra, where the filtered subalgebras $\mathcal{F}^p H^*(\mathscr{G}_{D_n})$ for $p \geq 0$ are induced by the length filtration on $\mathscr{G}_{D_n}$. We claim that this filtration on $H^*(\mathscr{G}_{D_n})$ is complete and Hausdorff. To see this, observe the image of the differential of $\mathscr{G}_{D_n}$ consists of homogeneous terms (with respect to length filtration), hence the filtration is Hausdorff. The filtration is complete because $H^*(\mathscr{G}_{D_n})$ is finite-dimensional at each degree. To see this, when $\mathbb{K}$ is algebraically closed and of characteristic 0, one can use the result by Hermes (see Theorem 7) that $H^i(\mathscr{G}_{D_n}) \cong \Pi_{D_n}$ for every $i \geq 0$, and the well-known fact that the preprojective algebra of a Dynkin quiver is finite-dimensional. Alternatively, for any field, $H^0(\mathscr{G}_{D_n}) = \Pi_{D_n}$ by definition, hence we can appeal to the argument given in the last part of the above lemma to conclude. (Note that the result of [26] requires an action filtration on the chain complex respected by the differential. This is automatic for $\mathrm{LCA}^*(\Lambda_\Gamma)$ as the relevant filtration is given by the geometric action functional. On the other hand, if the complex is supported in nonpositive (or nonnegative) degrees, then one can easily construct an action filtration of the required type inductively, hence the main result of [26] is applicable to $\mathscr{G}_\Gamma$ as well for any Γ.)

We are now ready to prove the main result of this section:

Theorem 13 *Let $\Gamma = A_n$ or D_n, and assume that* $\mathrm{char}\,\mathbb{K} \neq 2$ *if $\Gamma = D_n$. Then there exists a quasi-isomorphism*

$$\mathrm{LCA}^*(\Lambda_\Gamma) \simeq \mathscr{G}_\Gamma.$$

Furthermore, if $\mathrm{char}\,\mathbb{K} = 2$ *and $\Gamma = D_n$, then* $\mathrm{LCA}^*(\Lambda)$ *and $\mathscr{G}_\Gamma$ are not quasi-isomorphic.*

(We conjecture that $\mathrm{LCA}^*(\Lambda) \simeq \mathscr{G}_\Gamma$ for $\Gamma = E_6, E_7$ if $\operatorname{char}\mathbb{K} \neq 2, 3$ and for $\Gamma = E_8$ if $\operatorname{char}\mathbb{K} \neq 2, 3, 5$.)

Proof The case when $\Gamma = A_n$ is immediate since $\mathrm{LCA}^*(\Lambda_\Gamma)$ and $\mathscr{G}_\Gamma$ are identical in this case. So we will focus on the case $\Gamma = D_n$.

When $\operatorname{char}\mathbb{K} \neq 2$, we will construct a chain map $\Phi\colon \mathscr{G}_\Gamma \to \mathrm{LCA}^*(\Lambda_\Gamma)$ which is of the form

$$\Phi = \mathrm{Id} + \text{h.o.t.},$$

where h.o.t. stands for higher-order terms in terms of the length filtration $\mathcal{F}^\bullet$ on $\mathrm{LCA}^*(\Lambda_\Gamma)$.

In Section 6.2.3, we computed

$$\mathrm{HH}^*(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma) \cong \mathrm{HH}^*(A_\Gamma, A_\Gamma),$$

where A_Γ is the Koszul dual to $\mathscr{G}_\Gamma$ as proven in Theorem 23. Note that the isomorphism between the Hochschild cohomologies of $\mathscr{G}_\Gamma$ and A_Γ is a consequence of the Koszul duality given by Theorem 23, which also states that the Koszul duality functor sends the internal grading of $\mathscr{G}_\Gamma$ to those of A_Γ, implying that the internal gradings on their Hochschild cohomologies match as well. In particular, we have

$$\mathrm{HH}^2(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma[s]) \cong \mathrm{HH}^{2-s}(A_\Gamma, A_\Gamma[s]).$$

Let us warn the reader of a potentially confusing point in our notation. On the right-hand side, $r = 2 - s$ refers to the length grading in Hochschild cohomology, and s refers to the internal grading induced from the internal grading of the algebra A_Γ. This group is a summand of $\mathrm{HH}^2(A_\Gamma, A_\Gamma)$ where $2 = r + s$ is the total degree. On the other hand, $\mathrm{HH}^2(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma[s])$ is a summand of $\mathrm{HH}^2(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma)$ where s refers to the second grading on $\mathscr{G}_\Gamma$ (as was explained after (4)).

The computation given in Section 6.2.3 implies that for $\Gamma = D_n$ and when $\operatorname{char}\mathbb{K} \neq 2$, we have

$$\mathrm{HH}^2(\mathscr{G}_\Gamma, \mathscr{G}_\Gamma[s]) = 0 \quad \text{for } s < 0.$$

Therefore, from Lemma 11, we deduce that there exists a quasi-isomorphism

$$\Phi\colon \widehat{\mathscr{G}}_\Gamma \to \widehat{\mathrm{LCA}}^*(\Lambda_\Gamma).$$

Now, let N be an integer large enough that $\mathcal{F}^N H^0(\mathrm{LCA}(\Lambda_\Gamma)) = 0$; such an N exists, as we proved above in Lemma 12. We then consider the truncation of Φ at length N to define an algebra map between *uncompleted* algebras

$$\Phi^N\colon \mathscr{G}_\Gamma \to \mathrm{LCA}^*(\Lambda_\Gamma).$$

The apparent problem with Φ^N is that it is not a chain map, though it fails to be a chain map only at large length. So, we can correct it as follows. For each vertex v, let us find a chain α_v such that

$$D\Phi^N(h_v) - \Phi^N(dh_v) = D\alpha_v.$$

Note that the left-hand side is automatically D–closed since it lies in $\mathrm{LCA}^0(\Lambda_\Gamma)$.

We then define a new algebra map by setting

$$\Psi(h_v) := \Phi^N(h_v) + \alpha_v, \quad \Psi(g_{vw}) := \Phi^N(g_{vw}).$$

We now have a filtered chain map $\mathscr{G}_\Gamma \to \mathrm{LCA}^*(\Lambda_\Gamma)$ which respects the length filtrations on each side. Note that the E_2–pages of the associated spectral sequences are identical:

$$E_2^{p,q} \cong \mathcal{F}^p \mathscr{G}_\Gamma / \mathcal{F}^{p+2}, \mathscr{G}_\Gamma$$

with the differential induced from the differential on the Ginzburg DG-algebra. Furthermore, the length filtration is not only complete and Hausdorff on both sides by Lemma 12 and the discussion following its proof, but also easily seen to be weakly convergent. Therefore the spectral sequences converge *strongly* to $H^*(\mathscr{G}_{D_n})$ and $H^*(\mathrm{LCA}(\Lambda_{D_n}))$, respectively. Moreover, since

$$\Psi = \mathrm{Id} + \text{h.o.t.},$$

where h.o.t. refers to a higher-order term that sends $\mathcal{F}^\bullet$ to $\mathcal{F}^{\bullet+2}$, it induces an isomorphism on the E_2–page, therefore we conclude that it induces a quasi-isomorphism of chain complexes by [15, Theorem 2.6]. This completes the proof that $\mathrm{LCA}^*(\Lambda_{D_n})$ and $\mathscr{G}_{D_n}$ are quasi-isomorphic over a field of characteristic $\neq 2$.

Next suppose that $\mathbb{K}$ is a field of characteristic 2. Let us write $D = d + d_3$ for the differential on $\mathrm{LCA}^*(\Lambda_{D_n})$ where, in the notation of Lemma 12, we have

$$d_3(c_3) = -c_{31}c_{13}c_{32}c_{23}.$$

We want to show that there is no degree 0 derivation ϕ_2 which increases length by 2 and solves $d_3 = d\phi_2 - \phi_2 d$. For $\Gamma = D_4$, this is equivalent to the following set of *linear* equations:

$$\begin{aligned} 0 &= d\phi_2(c_1) - \phi_2(c_{13})c_{31} - c_{13}\phi_2(c_{31}), \\ 0 &= d\phi_2(c_2) - \phi_2(c_{23})c_{32} - c_{23}\phi_2(c_{32}), \\ -c_{31}c_{13}c_{32}c_{23} &= d\phi_2(c_3) + \phi_2(c_{31}c_{13} + c_{32}c_{23} - c_{34}c_{43}), \\ 0 &= d\phi_2(c_4) + \phi_2(c_{43})c_{34} + c_{43}\phi_2(c_{34}). \end{aligned}$$

(Although we are working over characteristic 2 here, we have kept the signs in their general form for reference.)

Since ϕ_2 is supposed to preserve the degree and increase the length by 2, there are only a few possibilities. The general form of the possibilities is as follows:

$$\begin{aligned}
\phi_2(c_1) &\in \mathbb{K}c_1c_{13}c_{31} \oplus \mathbb{K}c_{13}c_{31}c_1 \oplus \mathbb{K}c_{13}c_3c_{31},\\
\phi_2(c_2) &\in \mathbb{K}c_2c_{23}c_{32} \oplus \mathbb{K}c_{23}c_{32}c_2 \oplus \mathbb{K}c_{23}c_3c_{32},\\
\phi_2(c_3) &\in \mathbb{K}c_3c_{31}c_{13} \oplus \mathbb{K}c_{31}c_{13}c_3 \oplus \mathbb{K}c_3c_{32}c_{23} \oplus \mathbb{K}c_{32}c_{23}c_3 \oplus \mathbb{K}c_3c_{34}c_{43}\\
&\qquad \oplus \mathbb{K}c_{34}c_{43}c_3 \oplus \mathbb{K}c_{31}c_1c_{13} \oplus \mathbb{K}c_{32}c_2c_{23} \oplus \mathbb{K}c_{34}c_4c_{43},\\
\phi_2(c_4) &\in \mathbb{K}c_4c_{43}c_{34} \oplus \mathbb{K}c_{43}c_{34}c_4 \oplus \mathbb{K}c_{43}c_3c_{34},\\
\phi_2(c_{13}) &\in \mathbb{K}c_{13}c_{31}c_{13} \oplus \mathbb{K}c_{13}c_{32}c_{23} \oplus \mathbb{K}c_{13}c_{34}c_{43},\\
\phi_2(c_{31}) &\in \mathbb{K}c_{31}c_{13}c_{31} \oplus \mathbb{K}c_{32}c_{23}c_{31} \oplus \mathbb{K}c_{34}c_{43}c_{31},\\
\phi_2(c_{23}) &\in \mathbb{K}c_{23}c_{32}c_{23} \oplus \mathbb{K}c_{23}c_{31}c_{13} \oplus \mathbb{K}c_{23}c_{34}c_{43},\\
\phi_2(c_{32}) &\in \mathbb{K}c_{32}c_{23}c_{32} \oplus \mathbb{K}c_{31}c_{13}c_{32} \oplus \mathbb{K}c_{34}c_{43}c_{32},\\
\phi_2(c_{43}) &\in \mathbb{K}c_{43}c_{34}c_{43} \oplus \mathbb{K}c_{43}c_{31}c_{13} \oplus \mathbb{K}c_{43}c_{32}c_{23},\\
\phi_2(c_{34}) &\in \mathbb{K}c_{34}c_{43}c_{34} \oplus \mathbb{K}c_{31}c_{13}c_{34} \oplus \mathbb{K}c_{32}c_{23}c_{34}.
\end{aligned}$$

This leads to a system of 18 linear equations of 36 variables. It is straightforward, if tedious, to verify directly (or with the help of a computer) that none of the possibilities gives a solution when $\mathbb{K} = \mathbb{Z}_2$. This, in turn, implies that the class of $[\widetilde{d}_3]$ is nontrivial over any field $\mathbb{K}$ of characteristic 2 by the universal coefficient theorem.

This implies that there is a nonvanishing obstruction for constructing a chain map between $\mathscr{G}_{D_4}$ and $\mathrm{LCA}^*(\Lambda)$ over a field of characteristic 2 for D_4. In other words, the class $[\widetilde{d}_3] \in \mathrm{HH}^2(\mathscr{G}_{D_4}, \mathscr{G}_{D_4}[-2])$ is nontrivial. (Compare this with our computation of $\mathrm{HH}^2(\mathscr{G}_{D_4}, \mathscr{G}_{D_4}[-2])$ given later on in Table 4, where this group is shown to be nontrivial only in characteristic 2.) Now, the class of $[\widetilde{d}_3]$ for $\Gamma = D_n$ restricts to the class of $\Gamma = D_4$ under the restriction map. (Note that in general Hochschild cohomology does not have good functoriality properties; however, there is a full and faithful inclusion of the $\mathscr{G}_{D_4}$ to $\mathscr{G}_{D_n}$, and there is a restriction map on Hochschild cohomology in this case.) Hence, it cannot vanish for $\Gamma = D_n$ either. □

Remark 14 Over a field of characteristic $\neq 2$, and for $\Gamma = D_4$, we constructed an explicit chain map between $\mathscr{G}_{D_4}$ and $\mathrm{LCA}^*(\Lambda_{D_4})$ as a check on our arguments above. The complication in this also displays the effectiveness of the deformation theory argument given above. (Notice the factors of $\frac{1}{2}$, which are indeed necessary.) The map is given as follows:

$$
\begin{aligned}
h_1 &\mapsto c_1 - \tfrac{1}{2}(c_{13}c_{31}c_1 + c_{13}c_3c_{31} + c_1c_{13}c_{32}c_{23}c_{31}),\\
h_2 &\mapsto c_2 - \tfrac{1}{2}(c_{23}c_{32}c_2 + c_{23}c_3c_{32} + c_{23}c_{31}c_{13}c_{32}c_2)\\
&\qquad\qquad + \tfrac{1}{4}\big(c_{23}c_{34}c_{43}c_3c_{32} + c_{23}c_{34}c_4c_{43}c_{32}\\
&\qquad\qquad\qquad + c_{23}c_{34}c_{43}c_{32}c_2 + c_{23}c_{34}c_{43}c_{31}c_{13}c_{32}c_2\big),\\
h_3 &\mapsto c_3 - \tfrac{1}{4}\big(c_{31}c_{13}c_3c_{34}c_{43} + c_{31}c_1c_{13}c_{34}c_{43}\\
&\qquad\qquad + c_{31}c_{13}c_{34}c_4c_{43} + c_{31}c_1c_{13}c_{32}c_{23}c_{34}c_{43}\big),\\
h_4 &\mapsto c_4 - \tfrac{1}{2}\big(c_4c_{43}c_{34} + c_{43}c_3c_{34} - c_{43}c_3c_{32}c_{23}c_{34} - c_{43}c_{32}c_2c_{23}c_{34}\\
&\qquad\qquad - c_4c_{43}c_{32}c_{23}c_{34} - c_{43}c_{31}c_{13}c_{32}c_2c_{23}c_{34}\big),\\
g_{13} &\mapsto c_{13} + \tfrac{1}{2}(c_{13}c_{32}c_{23} - c_{13}c_{34}c_{43}),\\
g_{31} &\mapsto c_{31},\\
g_{23} &\mapsto c_{23} - \tfrac{1}{2}c_{23}c_{34}c_{43},\\
g_{32} &\mapsto c_{32} + \tfrac{1}{2}c_{31}c_{13}c_{32},\\
g_{34} &\mapsto c_{34} - \tfrac{1}{2}(c_{32}c_{23}c_{34} + c_{31}c_{13}c_{34}),\\
g_{43} &\mapsto c_{43}.
\end{aligned}
$$

Remark 15 One can deduce from the argument given in the last part of the proof of Theorem 13 that for any tree Γ which is not of type A_n, we have that $\mathscr{B}_\Gamma := \mathrm{LCA}^*(\Lambda_\Gamma)$ is a nontrivial deformation of $\mathscr{G}_\Gamma$ over a field of characteristic 2 since any such tree has a subtree of the form D_4 (see also Remark 33).

4 Floer cohomology algebra of the spheres in X_Γ

We next consider the A_∞–algebra over k given by the Floer cochain complexes:

$$\mathscr{A}_\Gamma := \bigoplus_{v,w} \mathrm{CF}^*(S_v, S_w).$$

Recall that the Lagrangian 2–spheres S_v and S_w intersect only if the vertices v and w are connected by an edge, in which case $S_v \cap S_w$ is a unique point. Recall also that we made choices of grading structures on the sphere S_v in Section 2 so that $\mathrm{CF}^*(S_v, S_w)$ is concentrated in degree 1 if v, w are adjacent vertices. On the other hand, the self-Floer cochain complex $\mathrm{CF}^*(S_v, S_v)$ is quasi-isomorphic to the singular chain complex $C^*(S_v)$ since S_v is an exact Lagrangian sphere in X_Γ. Therefore, we can take a model for $\mathscr{A}_\Gamma$ such that the differential on $\mathscr{A}_\Gamma$ necessarily vanishes for degree reasons.

Let us put $A_\Gamma = H^*(\mathscr{A}_\Gamma)$ for the corresponding associative algebra. We can think of $\mathscr{A}_\Gamma$ as a minimal A_∞–structure $(\mu^n)_{n\geq 2}$ on the associative algebra A_Γ. As before, by choosing a root, we make Γ into a directed graph such that oriented edges point away from the root. Let $D\Gamma$ denote the double of the quiver Γ, formed by introducing a new oriented edge a_{vw} from w to v for every oriented edge a_{wv} from v to w.

Proposition 16 *Suppose $\Gamma \neq A_1$. The graded* k*–algebra A_Γ is isomorphic to the* ***zigzag algebra*** *of Γ given by the path algebra $\mathbb{K}D\Gamma$ equipped with the path-length grading modulo the homogeneous ideal generated by the following elements:*

- $a_{uv}a_{vw}$ *such that $u \neq w$, where v is adjacent to both u, w.*
- $a_{vw}a_{wv} - a_{vu}a_{uv}$*, where v is adjacent to both u, w.*

If $\Gamma = A_1$, then $A_\Gamma \cong H^*(S^2) = \mathbb{K}[x]/(x^2)$ with $|x| = 2$.

Proof Note that S_v intersects S_w for $w \neq v$ if and only if v and w are adjacent vertices, in which case the intersection is transverse at a unique point. Furthermore, we have chosen the grading structures on the Lagrangians S_v so as to ensure that for v, w adjacent $\mathrm{CF}^*(S_v, S_w)$ is of rank 1 and concentrated in degree 1. We let a_{vw} be a generator for this 1–dimensional vector space. Finally, the algebra structure is determined by the general Poincaré duality property of Floer cohomology (see [61, Section 12e]). □

The algebra A_Γ only depends on the underlying tree Γ; different ways of orienting its edges results in the same algebra. We call the algebra A_Γ the *zigzag algebra* of Γ, following Khovanov and Huerfano [43], who studied properties of this algebra and its appearances in a variety of areas related to representation theory and categorification. On the other hand, the case where Γ is the A_n quiver appeared in an earlier paper of Seidel and Thomas [67] in the context of Floer cohomology (as it does here) and mirror symmetry. In the context of Koszul duality (see [54; 10]), the algebras A_Γ were studied much earlier by Martínez-Villa in [52]. This remarkable work is the first paper, as far as we know, which draws attention to the fact that A_Γ is a Koszul algebra if and only if Γ is not Dynkin or $\Gamma = A_1$.

We will next discuss formality of $\mathscr{A}_\Gamma$, ie the question of whether there is a quasi-isomorphism between $\mathscr{A}_\Gamma$ and $A_\Gamma = H^*(\mathscr{A}_\Gamma)$. In the case when Γ is the A_n quiver, the formality was proven by Seidel and Thomas [67, Lemma 4.21] based on the notion of *intrinsic formality*.

Definition 17 A graded algebra A is called intrinsically formal if any A_∞–algebra $\mathscr{A}$ with $H^*(\mathscr{A}) \cong A$ is quasi-isomorphic to A.

Furthermore, Seidel and Thomas give a useful method to recognize intrinsically formal algebras. Recall that for a graded algebra A, $\mathrm{HH}^*(A)$ has two gradings: the cohomological grading r and the grading s coming from the grading of the algebra A. To specify the decomposition into graded pieces, we write

$$\mathrm{HH}^*(A) = \bigoplus_{*=r+s} \mathrm{HH}^r(A, A[s]).$$

Notice that the superscript denotes the diagonal grading, as usual. It is also the grading that survives, if A is more generally a DG-algebra or an A_∞–algebra.

Theorem 18 (Kadeishvili [45]; see also Seidel and Thomas [67]) *Let A be an augmented graded algebra. If*

$$\mathrm{HH}^{2-s}(A, A[s]) = 0 \quad \textit{for all } s < 0,$$

then A is intrinsically formal.

As mentioned above, Seidel and Thomas proved intrinsic formality of A_Γ where Γ is the A_n quiver by showing the vanishing of $\mathrm{HH}^{2-s}(A_\Gamma, A_\Gamma[s])$ for $s < 0$. In a similar vein, we prove in Theorem 44 that A_Γ is intrinsically formal if Γ is the D_n quiver and the characteristic of the ground field is not 2.

We have the following conjecture for the remaining Dynkin types.

Conjecture 19 *Working over a ground field $\mathbb{K}$ of characteristic 0, let Γ be a tree of type E_6, E_7 or E_8. Then the corresponding zigzag algebra A_Γ is intrinsically formal.*

Unlike the A_n case, some restriction on the characteristic of $\mathbb{K}$ is necessary as we have checked that the zigzag algebras are not intrinsically formal in type D_n, $n \geq 4$, over characteristic 2, in type E_6 and E_7 over characteristic 2 or 3, and in the type E_8, over characteristic 2, 3 or 5. It is very likely that these are the only "bad" characteristics (cf [57]).

5 Koszul duality

By combining the work of Bourgeois, Ekholm and Eliashberg [17] with Abouzaid's generation criteria [1], one might suspect that the Lagrangians L_v split-generate the wrapped Fukaya category $\mathcal{W}(X_\Gamma)$. Now, there exists a full and faithful embedding

$$\mathcal{F}(X_\Gamma) \to \mathcal{W}(X_\Gamma)$$

of the exact Fukaya category of compact Lagrangians. Therefore, in view of Remark 10, we would conclude that there is a quasi-isomorphism of DG-algebras

$$\mathrm{RHom}_{\mathscr{B}_\Gamma}(\mathrm{k},\mathrm{k}) \simeq \mathscr{A}_\Gamma. \tag{5}$$

The right-hand side is in turn quasi-isomorphic to A_Γ if one checks that $\mathscr{A}_\Gamma$ is formal (for example this is known if Γ is of type A_n [67] and we prove it in Theorem 44 for type D_n over a field of characteristic $\neq 2$). We will provide an alternative independent approach via a purely algebraic argument based on Koszul duality theory for DG- or A_∞–algebras (see [51]) to stay within the algebraic framework of this paper (and avoid the technicalities that go into the discussion in Remark 10).

In fact, as we shall see below, Koszul duality theory allows us to work directly with $A_\Gamma = H^*(\mathscr{A}_\Gamma)$, hence in this way we bypass formality questions for $\mathscr{A}_\Gamma$.

We now give a brief review of Koszul duality, first in the case of associative algebras and then for A_∞–algebras.

5.1 Quadratic duality and Koszul algebras

To begin with, we review quadratic duality for associative algebras following [64, Section 2.1] which has an explicit discussion of signs in the context relevant here. The original reference is [54], and see also the excellent exposition in [10].

Let $\mathrm{k} = \bigoplus_v \mathbb{K} e_v$ be the commutative semisimple ring of orthogonal primitive idempotents over the base field $\mathbb{K}$, as before. Let V be a finite-dimensional graded $\mathbb{K}$–vector space with a k–bimodule structure. We write

$$T_{\mathrm{k}} V := \bigoplus_{i=0}^{\infty} V^{\otimes_{\mathrm{k}} i}$$

for the tensor algebra over k. A quadratic graded algebra A is an associative unital graded k–algebra that is a quotient

$$A := T_{\mathrm{k}} V / J$$

of $T_{\mathrm{k}} V$ by the two-sided ideal generated by a graded k–submodule $J \subset V \otimes_{\mathrm{k}} V$. In fact, this makes A into a bigraded algebra: it has an internal grading coming from the graded vector space V, denoted by s or $|x|$ if for a specific element, and a length grading coming from the tensor algebra, denoted by r. The reference [51] refers to s as Adams grading.

Let $V^\vee = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ be the linear dual of V viewed naturally as a k–bimodule, ie $e_i V^\vee e_j$ is the dual of $e_j V e_i$. Next, we consider the orthogonal dual $J^\perp \subset V^\vee \otimes_{\mathrm{k}} V^\vee$

with respect to the canonical pairing given by

$$V^\vee \otimes_{\mathrm{k}} V^\vee \otimes_{\mathrm{k}} V \otimes_{\mathrm{k}} V \to \mathrm{k}, \quad v_2^\vee \otimes_{\mathrm{k}} v_1^\vee \otimes_{\mathrm{k}} v_1 \otimes_{\mathrm{k}} v_2 \mapsto (-1)^{|v_2|} v_2^\vee(v_2) v_1^\vee(v_1).$$

The quadratic dual to A is defined as

$$A^! = T_{\mathrm{k}}(V^\vee[-1])/J^\perp[-2].$$

As does A, the graded quadratic algebra $A^!$ has two natural gradings: one internal grading coming from the internal grading of the vector space $V^\vee[-1]$, denoted by s or $|x^!|$ for a specific element, and the length grading coming from the tensor algebra, denoted by r.

The Koszul complex of a quadratic algebra is the graded right A–module $A^! \otimes_{\mathrm{k}} A$ with the differential[3]

$$x^! \otimes_{\mathrm{k}} x \to \sum\nolimits_i (-1)^{|x|} x^! a_i^\vee \otimes_{\mathrm{k}} a_i x, \tag{6}$$

where the sum is over a basis of $\{a_i\}$ of V, and $\{a_i^\vee\}$ is the dual basis in $V^\vee[-1]$. This should be thought of as an (r,s)–bigraded complex, where the grading r is the path-length grading in the $A^!$ factor and the total grading $r+s$ corresponds to the natural grading $|x^!| + |x|$. In particular, one has $|a_i^\vee| + |a_i| = 1$ for all i, hence the s grading is preserved by the differential.

A Koszul algebra A is a quadratic algebra for which the Koszul complex is acyclic (ie homology is isomorphic to $\mathrm{k}[0]$). Taking the dual by applying the left exact functor $\mathrm{Hom}_A(\,\cdot\,, A)$, we get a resolution of k as a graded right A^{op}–module (see [10, Section 2] for more details).

In fact, if A is Koszul, considering k as a simple module in the abelian category of graded *right* A^{op}–modules, one has a canonical isomorphism of bigraded rings

$$A^! \cong \mathrm{Ext}^*_{A^{\mathrm{op}}}(\mathrm{k}, \mathrm{k}).$$

Since A is bigraded, a priori $\mathrm{Ext}^*_{A^{\mathrm{op}}}(\mathrm{k}, \mathrm{k})$ is triply graded (by the cohomological degree and by the length and internal gradings, derived from the corresponding ones in A). One characterization of Koszulity is that the cohomological degree, which we denote by r, agrees with the grading induced by length. Finally, we denote the internal grading by s. With this understood, we have the graded identifications

$$A^!_{r,s} \cong \mathrm{Ext}^r_{A^{\mathrm{op}}}(\mathrm{k}, \mathrm{k}[s]).$$

[3] [10] prefers to use the graded left module $A \otimes_{\mathrm{k}} {}^\vee(A^!)$; the two graded modules are related by the right module isomorphism $A^! \otimes A \simeq \mathrm{Hom}_A(A \otimes_{\mathrm{k}} {}^\vee(A^!), A)$ and the sign $(-1)^{|x|}$ coming from this dualization.

If A is Koszul, then its Koszul dual $A^!$ is also Koszul and $(A^!)^! \cong A$.

Finally, for a Koszul algebra A, the Hochschild cohomology can be computed via the Koszul bimodule resolution of A. The resulting complex which computes Hochschild cohomology is

$$(A^! \otimes_{\mathrm{k}} A)_{\mathrm{diag}} = \bigoplus_v e_v A^! \otimes_{\mathrm{k}} A e_v \tag{7}$$

with the differential

$$x^! \otimes_{\mathrm{k}} x \to \sum_i (-1)^{|x|} x^! a_i^{\vee} \otimes_{\mathrm{k}} a_i x - (-1)^{(|a_i|+1)(|x|+|x^!|)} a_i^{\vee} x^! \otimes_k x a_i.$$

It is often the case, as in this paper, that V is generated either by odd elements or even elements; this simplifies the signs in the above formula. For Koszul algebras, the homology of this complex coincides with the bigraded Hochschild cohomology groups $\mathrm{HH}^r(A, A[s])$, where $r+s$ corresponds to the natural grading on $(A^! \otimes A)_{\mathrm{diag}}$, that is, an element $x^! \otimes_{\mathrm{k}} x$ has grading $|x^!| + |x|$. The length grading r corresponds to the path-length grading in the $A^!$ factor.

Example 20 Let $A_\Gamma = \mathbb{K}[x]/(x^2)$ with $|x| = 2$ be the zigzag algebra associated with a single vertex, ie Γ is of type A_1. It is easy to see that this is a Koszul algebra and we have $A_\Gamma^! = \mathbb{K}[x^\vee]$, the free algebra with $|x^\vee| = -1$. One can compute Hochschild cohomology using the Koszul bimodule complex. This has generators $(x^\vee)^i \otimes 1$ and $(x^\vee)^i \otimes x$ for $i \geq 0$. The differential can be computed as

$$d((x^\vee)^i \otimes 1) = (1 + (-1)^{i+1})(x^\vee)^{i+1} \otimes x,$$
$$d((x^\vee)^i \otimes x) = 0.$$

Therefore, whenever $\operatorname{char} \mathbb{K} = 2$, the differential vanishes, and as a consequence $\mathrm{HH}^*(A_\Gamma)$ has a basis $(x^\vee)^i \otimes 1$, for $i \geq 0$, in bigrading $(r, s) = (i, -2i)$ and $(x^\vee)^i \otimes x$, for $i \geq 0$, in bigrading $(r, s) = (i, 2-2i)$.

If $\operatorname{char} \mathbb{K} \neq 2$, then $\mathrm{HH}^*(A_\Gamma)$ has a basis $(x^\vee)^{2i} \otimes 1$, for $i \geq 0$, in bigrading $(r, s) = (2i, -4i)$ and $(x^\vee)^{2i+1} \otimes x$, for $i \geq 0$, in bigrading $(r, s) = (2i+1, -4i)$ and $1 \otimes x$ in bigrading $(0, 2)$.

In view of the discussion in the introduction, this result computes $\mathrm{SH}^*(T^*S^2)$ for $* = r + s$. For convenient access, we record a finite portion of this calculation in Table 1.

By Viterbo's isomorphism [70; 5], this computation also gives $H_{2-*}(\mathcal{L}S^2)$, where $\mathcal{L}S^2$ is the free loop space of S^2. This was previously computed as a ring by Cohen, Jones and Yan [21] over $\mathbb{Z}$ to be

$$H_{2-*}(\mathcal{L}S^2;\mathbb{Z}) \cong (\Lambda b \otimes \mathbb{Z}[a,v])/(a^2,ab,2av), \quad |a|=2,\ |b|=1,\ |v|=-2$$

using the fibration $\Omega_x S^2 \to \mathcal{L}S^2 \to S^2$. From this, one can deduce that

$$H_{2-*}(\mathcal{L}S^2;\mathbb{K}) \cong \Lambda a \otimes \mathbb{K}[u], \quad |a|=2,\ |u|=-1$$

if $\operatorname{char}\mathbb{K}=2$, and

$$H_{2-*}(\mathcal{L}S^2;\mathbb{K}) \cong (\Lambda b \otimes \mathbb{K}[a,v])/(a^2,ab,av), \quad |a|=2,\ |b|=1,\ |v|=-2r$$

if $\operatorname{char}\mathbb{K}\neq 2$, in agreement with our computation.

$r+s\downarrow \quad s\rightarrow$	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
2	1	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	x	0	0	0	0	0	0
−1	0	0	0	0	x	0	1	0	0	0	0
−2	0	0	0	0	0	0	1	0	x	0	0

Table 1: $\Gamma = A_1$; x is 1 if $\operatorname{char}\mathbb{K}=2$, 0 otherwise

5.2 Koszul duality for A_∞–algebras

We now review Koszul duality for A_∞–algebras. Our primary reference for this material is [51]. The discussion in [51] is about A_∞–algebras over a field $\mathbb{K}$, but as in classical Koszul duality, the proofs extend readily to A_∞–algebras over a semisimple ring k (see also [58]). The extension of Koszul duality theory to DG- or A_∞–algebras has appeared earlier (see eg [46]).

Suppose $A=\bigoplus_{i\geq 0} A_i$ is a positively graded associative algebra over $A_0=\mathrm{k}$. Then, as before, the complex

$$\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{k},\mathrm{k})$$

inherits a bigrading by cohomological and length gradings. However, it usually happens that at the level of homology these two gradings do not agree, that is, A is not Koszul as an associative algebra, and passing to the homology of this complex yields an associative algebra $\mathrm{Ext}^*_{A^{\mathrm{op}}}(\mathrm{k},\mathrm{k})$ from which one cannot recover A. In this case, the idea is that rather than passing to homology, one thinks of the DG-algebra $\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{k},\mathrm{k})$ as the DG-Koszul dual of A. To be able to carry this out, one is led to work with DG- or A_∞–algebras from the beginning. So, let $\mathscr{A}$ be a $\mathbb{Z}$–graded A_∞–algebra over k together with an augmentation $\epsilon\colon \mathscr{A} \to \mathrm{k}$, making k into a right A_∞–module over $\mathscr{A}^{\mathrm{op}}$. One defines

$$\mathscr{A}^! = \mathrm{RHom}_{\mathscr{A}^{\mathrm{op}}}(\mathrm{k},\mathrm{k}).$$

Note that the Yoneda image of k given by $\mathrm{RHom}_{\mathscr{A}^{\mathrm{op}}}(\mathscr{A}^{\mathrm{op}},\mathrm{k})$ makes k into a right $(\mathscr{A}^!)^{\mathrm{op}}$–module. Now, the obvious concern is whether $(\mathscr{A}^!)^!$ gets back to $\mathscr{A}$ (up to quasi-isomorphism). This is not quite the case in general; one recovers a certain completion of $\mathscr{A}$ (see [58] for a beautiful geometric description of this construction). However, suppose that $\mathscr{A}$ has an additional s grading (called Adams grading in [51]) which is required to be preserved by the A_∞–operations. Furthermore, assume that $\mathscr{A}$ is connected and locally finite with respect to this grading; this means that $\mathscr{A}$ is either nonnegatively or nonpositively graded and the s–homogeneous subspace of $\mathscr{A}$ is of finite dimension for each s (see [51, Definition 2.1]). Then it is true that $(\mathscr{A}^!)^!$ is quasi-isomorphic to $\mathscr{A}$. We state this as:

Theorem 21 (Lu, Palmieri, Wu and Zhang [51, Theorem 2.4][4]) *Suppose $\mathscr{A}$ is an augmented A_∞–algebra over the semisimple ring* k *with a bigrading for which μ^k has degree $(2-k,0)$ and suppose $\mathscr{A}$ is connected and locally finite with respect to the second grading. Let*

$$\mathscr{A}^! = \mathrm{RHom}_{\mathscr{A}^{\mathrm{op}}}(\mathrm{k},\mathrm{k})$$

be its Koszul dual as an A_∞–algebra. Then there is a quasi-isomorphism of A_∞–algebras

$$\mathscr{A} \simeq \mathrm{RHom}_{(\mathscr{A}^!)^{\mathrm{op}}}(\mathrm{k},\mathrm{k}).$$

Below, we will apply this result for $\mathscr{A} = A_\Gamma$ viewed as a formal A_∞–algebra.

Example 22 To see the importance of the connectedness and finiteness assumptions, let us consider $A = \mathbb{K}[x,x^{-1}]$ with x in bigrading $(0,0)$, the (trivially graded) algebra of Laurent polynomials. Consider the augmentation $\epsilon\colon A^{\mathrm{op}} \to \mathbb{K}$ given by mapping x to $1 \in \mathbb{K}$, which makes $\mathbb{K}$ into a right A–module. Then one can check that $A^! = \mathrm{RHom}_{A^{\mathrm{op}}}(\mathbb{K},\mathbb{K})$ is quasi-isomorphic to the exterior algebra $\mathbb{K}[x^!]/((x^!)^2)$ with $x^!$ in bigrading $(0,1)$. However, $\mathrm{RHom}_{(A^!)^{\mathrm{op}}}(\mathbb{K},\mathbb{K}) \cong \mathbb{K}[\![y]\!]$ gives the power series ring with y in bigrading $(0,0)$. Hence, dualizing twice does not get us back in this case.

[4] The proof of [51, Theorem 2.4] uses [51, Lemma 1.15] which omits a necessary hypothesis. Namely, in the notation of [51, Lemma 1.15], one should further assume $B^{\infty}_{\mathrm{aug}}R$ is locally finite. By [51, Lemma 2.2], this requirement holds under our hypothesis.

5.3 Koszul dual of $\mathscr{G}_\Gamma$

We next prove that the DG-algebras $\mathscr{G}_\Gamma$ and A_Γ (viewed as a formal A_∞–algebra) are related by Koszul duality. We remind the reader that we always work with right modules (as we follow the sign conventions from [61]).

We have the following analogue of [39, Proposition 2.9.5] in our setting:

Theorem 23 *Consider* $\mathrm{k} = A_\Gamma^{\mathrm{op}}/(A_\Gamma^{\mathrm{op}})_{>0}$ *as a right* A_Γ^{op}*–module. There is a DG-algebra isomorphism*

$$\mathrm{RHom}_{A_\Gamma^{\mathrm{op}}}(\mathrm{k},\mathrm{k}) \simeq \mathscr{G}_{\Gamma^{\mathrm{op}}}$$

such that the cohomological (resp. internal) grading on the left-hand side agrees with the path-length (resp. internal) grading on the right-hand side.

Proof First, let us clarify the multiplication on A_Γ^{op}, which we view as a formal A_∞–algebra. We identify the elements of A_Γ^{op} with the elements of A_Γ which are given by the symbols a_{vw} and $a_{vw}a_{wv}$ as before. Since $|a_{wv}| = 1$ for all w adjacent to v, the product is given by

$$\mu^2_{A_\Gamma^{\mathrm{op}}}(a_{wv}, a_{vw}) = (-1)^{|a_{wv}|+|a_{vw}|}\mu^2_{A_\Gamma}(a_{vw}, a_{wv}) = (-1)^{|a_{vw}|}a_{vw}a_{wv} = -a_{vw}a_{wv}$$

for w adjacent to v (see [61, Section (1a)] for signs used in defining the opposite of an A_∞–algebra).

We use the reduced bar resolution of k as a right A_Γ^{op}–module to calculate $\mathrm{RHom}_{A_\Gamma^{\mathrm{op}}}(\mathrm{k},\mathrm{k})$, which takes the form

$$\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{k},\mathrm{k}) \simeq \hom_{A^{\mathrm{op}}}((A \otimes_{\mathrm{k}} T\bar{A})^{\mathrm{op}}, \mathrm{k}),$$

where $A = A_\Gamma$, $\bar{A} = A_\Gamma/\mathrm{k}$, and $T\bar{A}$ is the tensor algebra of $\bar{A}_\Gamma$ over k.

The fact that $k = A_0$ allows us to identify $\bar{A}$ with the positive graded subalgebra $A_1 \oplus A_2$ of A. We follow the conventions in [61, Section (1j)] for the DG-algebra structure of $\hom_{A^{\mathrm{op}}}((A \otimes_{\mathrm{k}} T\bar{A})^{\mathrm{op}}, \mathrm{k})$. However, we view $\hom_{A^{\mathrm{op}}}((A \otimes_{\mathrm{k}} T\bar{A})^{\mathrm{op}}, \mathrm{k})$ as a DG-algebra rather than an A_∞–algebra with $\mu^k = 0$ for $k > 2$ since $\mathscr{G}_\Gamma$ is always viewed as a DG-algebra. The difference is in the signs, and this was explained in the introduction (see (3)).

More precisely, a generator $t \in \hom_{A^{\mathrm{op}}}((A \otimes_{\mathrm{k}} T\bar{A})^{\mathrm{op}}, \mathrm{k})$ of bidegree (r,s) is an A^{op}–module homomorphism $t\colon A \otimes_{\mathrm{k}} \bar{A}^{\otimes r} \to \mathrm{k}$ of internal degree $|t| = s$. Observe that any such t maps an element $(a_{r+1}, a_r, \dots, a_1)$ to 0 unless $a_{r+1} \in A_0$ because of the A^{op}–module structure of k.

The differential and the product on the DG-algebra $\hom_{A^{\mathrm{op}}}((A\otimes_{\mathrm{k}} T\overline{A})^{\mathrm{op}},\mathrm{k})$ are defined by

$$(dt)(e_v,a_{r+1},\dots,a_1) = \sum_{n=1}^{r} (-1)^{\dagger+|t|} t(e_v,a_{r+1},\dots a_{n+2},\mu^2_{A^{\mathrm{op}}}(a_{n+1},a_n),a_{n-1},\dots,a_1)$$

and if t_1 and t_2 are two generators of lengths r_1,r_2, then

$$(t_2\cdot t_1)(e_v,a_{r_2+r_1},\dots,a_1) = (-1)^{\ddagger+|t_1|} t_2(t_1(e_v,a_{r_2+r_1},\dots,a_{r_2+1}),a_{r_2},\dots,a_1),$$

where $\dagger = \sum_{i=n}^{r+1}(|a_i|-1)$ and $\ddagger = \sum_{i=r_2+1}^{r_2+r_1}(|a_i|-1)$.

We now construct a chain map

$$\Phi\colon \mathscr{G}_{\Gamma^{\mathrm{op}}} \to \hom_{A^{\mathrm{op}}}((A\otimes_{\mathrm{k}} T\overline{A})^{\mathrm{op}},\mathrm{k})$$

that respects the cohomological and internal gradings, first by defining it on the generators g_{wv} and h_v of the underlying tensor algebra of $\mathscr{G}_{\Gamma^{\mathrm{op}}}$, and then extending by mapping the product p_2p_1 of two elements p_2 and p_1 in $\mathscr{G}_{\Gamma^{\mathrm{op}}}$ to the homomorphism $\Phi(p_2)\cdot\Phi(p_1)\in \hom_{A^{\mathrm{op}}}((A\otimes_{\mathrm{k}} T\overline{A})^{\mathrm{op}},\mathrm{k})$.

Indeed, let us define $\Phi(g_{wv})$ and $\Phi(h_v)$ to be A–module homomorphisms each of which is nonzero only on a 1–dimensional subspace of $A\otimes_{\mathrm{k}} T\overline{A}$, given by

$$\Phi(g_{wv})\colon (e_v,a_{wv})\mapsto \epsilon_{wv}e_w \quad \text{and} \quad \Phi(h_v)\colon (e_v,a_{vw}a_{wv})\mapsto \epsilon_v e_v,$$

for any vertex w adjacent to v in Γ. Here the signs ϵ_{wv}, ϵ_v are determined as follows. For a vertex $v\in\Gamma_0$, we set $\epsilon_v=(-1)^{\delta_v}$, where δ_v is the distance from the root of Γ to the vertex v. If g_{wv} is an arrow in the quiver Γ^{op}, then define $\epsilon_{wv}=\epsilon_v$ and $\epsilon_{vw}=+1$. Note that $\epsilon_{wv}\epsilon_{vw}/\epsilon_v$ is $+1$ if and only if g_{wv} is an arrow in the quiver Γ^{op}.

Observe that the internal gradings are

$$|\Phi(g_{wv})| = -|a_{wv}| = -1 \quad \text{and} \quad |\Phi(h_v)| = -|a_{vw}a_{wv}| = -2,$$

respectively. Note also that Φ takes the path-length grading on $\mathscr{G}_\Gamma$ to the cohomological grading on $\hom_{A^{\mathrm{op}}}((A\otimes_{\mathrm{k}} T\overline{A})^{\mathrm{op}},\mathrm{k})$, hence Φ respects the bigraded structure of both sides.

The differentials on the DG-algebras $\mathscr{G}_{\Gamma^{\mathrm{op}}}$ and $\hom_{A^{\mathrm{op}}}((A\otimes_{\mathrm{k}} T\overline{A})^{\mathrm{op}},\mathrm{k})$ obey the graded Leibniz rule, hence it suffices to check that

$$d(\Phi(g_{wv})) = \Phi(dg_{wv}) = 0 \quad \text{and} \quad d(\Phi(h_v)) = \Phi(dh_v)$$

to verify that Φ is a DG-algebra homomorphism.

The first identity follows immediately since both g_{wv} and $\Phi(g_{wv})$ are in total degree 0 and the differential vanishes here. To check the second identity, observe that $d(\Phi(h_v))$ is nonzero only on the subspace of $A \otimes_{\mathrm{k}} T\overline{A}$ spanned by

$$\{(e_v, a_{wv}, a_{vw}) : w \text{ is adjacent to } v\},$$

and for every w adjacent to v,

$$\begin{aligned}(d(\Phi(h_v)))(e_v, a_{wv}, a_{vw}) &= (-1)^{|\Phi(h_v)|+(|a_{wv}|-1)+(|a_{vw}|-1)}\Phi(h_v)(e_v, -a_{vw}a_{wv}) \\ &= -\epsilon_v e_v.\end{aligned}$$

Note that the appearance of the extra sign here is precisely the point where the use of A_Γ^{op} rather than A_Γ takes effect.

On the other hand,

$$\Phi(dh_v) = \Phi\left(\sum_w \frac{\epsilon_{wv}\epsilon_{vw}}{\epsilon_v} g_{vw}g_{wv}\right) = \sum_w \frac{\epsilon_{wv}\epsilon_{vw}}{\epsilon_v}\Phi(g_{vw})\cdot\Phi(g_{wv}).$$

For each w adjacent to v, $\Phi(g_{vw})\cdot\Phi(g_{wv})$ is nonzero only on the subspace spanned by (e_v, a_{wv}, a_{vw}), and

$$\begin{aligned}&(\Phi(g_{vw})\cdot\Phi(g_{wv}))(e_v, a_{wv}, a_{vw}) \\ &\qquad = (-1)^{|\Phi(g_{wv})|+(|a_{wv}|-1)}\Phi(g_{vw})((\Phi(g_{wv})(e_v, a_{wv})), a_{vw}) \\ &\qquad = -\epsilon_{wv}\epsilon_{vw}e_v.\end{aligned}$$

Indeed, we also have an extra sign here, and hence the second identity holds.

To prove the bijectivity of Φ, consider a generator $(e_v, a_r, \dots, a_1)$ of $A \otimes_{\mathrm{k}} \overline{A}^{\otimes r}$. Note that such a generator is uniquely determined by the initial and terminal points of a_i considered as paths in A_Γ which in turn determine a unique path $g_r \cdots g_1$ of length r in $\mathscr{G}_\Gamma$, so that the initial and terminal points of each arrow g_i in the extended quiver $\widehat{\Gamma}$ match those of a_{r+1-i}. It is straightforward to check that

$$(\Phi(g_r \cdots g_1))(e_v, a_r, \dots, a_1) = \pm e_w,$$

where w is the terminal point of a_1. This proves that Φ is injective since the algebra underlying $\mathscr{G}_\Gamma$ is the path algebra generated by the arrows in $\widehat{\Gamma}$. Moreover, the observation that $\Phi(g_r \cdots g_1)$ is nonzero only on the subspace of $A \otimes_{\mathrm{k}} T\overline{A}$ spanned by $(e_v, a_r, \dots, a_1)$ shows that Φ is surjective as well. □

Remark 24 As can be seen from the proof of Theorem 23, we could arrange the definition of the DG-algebra isomorphism Φ so as to obtain an isomorphism

$$\mathrm{RHom}_{A_\Gamma}(\mathrm{k}, \mathrm{k}) \simeq \mathscr{G}_\Gamma,$$

where $\mathrm{k} = A_\Gamma/(A_\Gamma)_{>0}$ is viewed as a right A_Γ–module. This is because there happens to be an isomorphism of algebras between A_Γ and A_Γ^{op}. We have opted to use A_Γ^{op} to be consistent with the general framework of Koszul duality (see [10, Theorem 2.10.1]).

The following corollary is immediate from Theorem 23 and Theorem 21:

Corollary 25 *Consider* $\mathrm{k} = \mathscr{G}_\Gamma/(\mathscr{G}_\Gamma)_{r>0}$ *as a right* $\mathscr{G}_\Gamma$*–module, and* A_Γ *as a DG-algebra with trivial differential. There is a quasi-isomorphism of DG-algebras*

$$\mathrm{RHom}_{\mathscr{G}_\Gamma}(\mathrm{k},\mathrm{k}) \simeq A_\Gamma$$

such that the cohomological and internal gradings on the left-hand side coincide with each other and they agree with the path-length grading on the right-hand side.

Proof In view of Theorem 23 and Theorem 21, we only need to check the hypothesis in Theorem 21, but this is straightforward. Certainly, A_Γ is positively graded and the local finiteness condition holds since A_Γ is finite-dimensional (see [51, Definition 2.1]). □

Since A_Γ is known to be Koszul in the classical sense for non-Dynkin Γ, we easily get an alternative proof of the formality result mentioned in Theorem 7(1).

Corollary 26 *For* Γ *non-Dynkin,* $\mathscr{G}_\Gamma$ *is formal, that is, it is quasi-isomorphic to the preprojective algebra* $\Pi_\Gamma = H^0(\mathscr{G}_\Gamma)$*.*

Proof Recall that the differential on the complex $\mathrm{RHom}_{A_\Gamma^{\mathrm{op}}}(\mathrm{k},\mathrm{k})$ has bidegree $(1,0)$. Therefore, after applying the homological perturbation lemma, we obtain a minimal A_∞–structure on $\mathrm{Ext}^*_{A^{\mathrm{op}}}(\mathrm{k},\mathrm{k})$ such that μ^d has bidegree $(2-d,0)$. On the other hand, Koszulity of A_Γ means that the two gradings agree at the level of cohomology. Therefore, it is impossible to have a nontrivial μ^d for $d \neq 2$. □

Note that if Γ is a Dynkin-type graph, $\mathscr{G}_\Gamma$ is not quasi-isomorphic to the preprojective algebra Π_Γ. Our result above can be described as stating that $\mathscr{G}_\Gamma$ and A_Γ are A_∞–Koszul dual. This should be seen as the natural extension to all Γ of the classical Koszul duality between Π_Γ and A_Γ which only worked when Γ is non-Dynkin.

Finally, in view of the Theorem 23 and Corollary 25, we conclude from Keller's theorem [47] that there is an isomorphism of Hochschild cohomologies as Gerstenhaber algebras. Besides this isomorphism, the following theorem also uses the fact that $\mathrm{HH}_{2-*}(\mathscr{G}_\Gamma) \cong \mathrm{HH}^*(\mathscr{G}_\Gamma)$ by the Calabi–Yau property [39], together with [17] which applies over $\mathbb{K}$ of characteristic 0, and Theorem 13.

Theorem 27 *For any tree Γ, there is an isomorphism of Gerstenhaber algebras over $\mathbb{K}$*

$$\mathrm{HH}^*(\mathscr{G}_\Gamma) \cong \mathrm{HH}^*(A_\Gamma).$$

If Γ is of Dynkin type A_n or D_n (and conjecturally also for E_6, E_7, E_8) and $\mathbb{K}$ is of characteristic 0, then we have

$$\mathrm{SH}^*(X_\Gamma) \cong \mathrm{HH}^*(\mathscr{G}_\Gamma) \cong \mathrm{HH}^*(A_\Gamma).$$

Remark 28 Note that all of the Gerstenhaber algebras appearing in the above theorem are induced from a natural underlying Batalin–Vilkovisky (BV) algebra structure. In the case of symplectic cohomology, BV-algebra structure is given by a geometric construction reminiscent of the loop rotation in string topology and in the cases of $\mathscr{G}_\Gamma$ and A_Γ, it is induced by the underlying Calabi–Yau structure on these DG-algebras, which allows one to dualize the Connes differential B on Hochschild homology. However, the above theorem does not claim an isomorphism of the underlying Batalin–Vilkovisky structures. We believe that this can be achieved, however, it requires a finer investigation of Calabi–Yau structures. On the other hand, we explain in Remark 33 that for Γ non-Dynkin and non-extended Dynkin, we have an isomorphism of Batalin–Vilkovisky algebras between $\mathrm{HH}^*(\mathscr{G}_\Gamma)$ and $\mathrm{HH}^*(A_\Gamma)$ as it turns out that there is a unique way of equipping this Gerstenhaber algebra with a BV-algebra structure.

Remark 29 It is well-known that in the case when Γ is Dynkin, the exact Lagrangian spheres S_v split-generate the Fukaya category $\mathcal{F}(X_\Gamma)$ of compact exact Lagrangians — this follows for example by combining [59, Lemma 4.15] and [61, Corollary 5.8]. Furthermore, as mentioned in the beginning of Section 5, one expects that the noncompact Lagrangians L_v split-generate the wrapped Fukaya category. Hence, one could interpret the above result as showing that

$$\mathrm{HH}^*(\mathcal{F}(X_\Gamma)) \cong \mathrm{HH}^*(\mathcal{W}(X_\Gamma)).$$

On the other hand, it is by no means the case that $D^\pi\mathcal{F}(X_\Gamma)$ and $D^\pi\mathcal{W}(X_\Gamma)$ are equivalent as triangulated categories. (Here, we mean an equivalence between the Karoubi-completed triangulated closures of $\mathcal{F}(X_\Gamma)$ and $\mathcal{W}(X_\Gamma)$.) This is clear from the fact that the latter category has objects with infinite-dimensional endomorphisms (over $\mathbb{K}$) but every object in the former has finite-dimensional endomorphisms. More strikingly, the monotone Lagrangian tori studied in [49] give objects in $D^\pi\mathcal{W}(X_\Gamma)$ for $\Gamma = A_n$ with finite-dimensional endomorphisms and yet these do not belong to the category $D^\pi\mathcal{F}(X_\Gamma)$. One has to collapse the grading to $\mathbb{Z}_2$ in order to admit these objects in $\mathcal{F}(X_\Gamma)$.

In the next section, we compute $\mathrm{HH}^*(A_\Gamma)$ for all trees Γ except E_6, E_7, E_8.

6 Hochschild cohomology computations

6.1 Non-Dynkin case

In this section we assume that Γ is a non-Dynkin tree and describe the Hochschild cohomology $\mathrm{HH}^*(\mathscr{G}_\Gamma)$ of the associated Ginzburg DG-algebra. Note, however, that as explained in the introduction, when Γ is non-Dynkin, $\mathscr{B}_\Gamma$ is a nontrivial deformation of $\mathscr{G}_\Gamma$, and so this computation does not directly give enough information to compute $\mathrm{HH}^*(\mathscr{B}_\Gamma)$, and thus $\mathrm{SH}^*(X_\Gamma)$. However, at least away from characteristic 0, the computation of $\mathrm{HH}^*(\mathscr{G}_\Gamma) \cong \mathrm{HH}^*(A_\Gamma)$ is still of geometric significance as it controls the deformations of the compact Fukaya category $\mathcal{F}(X_\Gamma)$.

Recall that for non-Dynkin Γ, the cohomology $H^*(\mathscr{G}_\Gamma) \cong \Pi_\Gamma$ is supported in total degree 0 and moreover $\mathscr{G}_\Gamma$ is formal, ie it is quasi-isomorphic to the preprojective algebra Π_Γ. Therefore we have an isomorphism of Gerstenhaber algebras

$$\mathrm{HH}^*(\mathscr{G}_\Gamma) \cong \mathrm{HH}^*(\Pi_\Gamma),$$

where Π_Γ is to be considered as a trivially graded algebra. For any non-Dynkin quiver Γ, the Gerstenhaber structure of the Hochschild cohomology of $\Pi = \Pi_\Gamma$ is described in [57] (and previously in [23] when $\operatorname{char} \mathbb{K} = 0$). We do not have anything new to say here, we simply review some of the results of [23] and [57] to give a flavor of what's known. For an impressive amount of further information, see the comprehensive work of Schedler [57].

The Hochschild cohomology $\mathrm{HH}^*(\Pi_\Gamma)$ turns out to be trivial in every grading except for $0, 1$ and 2. A way to see this is to use the Koszul bimodule resolution given in (7). Recall that for Γ non-Dynkin, Π_Γ is Koszul in the classical sense with Koszul dual $A = A_\Gamma$. The latter has a decomposition into its graded pieces as $A = A_0 \oplus A_1 \oplus A_2$. Hence, the Koszul bimodule resolution takes the form

$$0 \to \bigoplus_v e_v \Pi e_v \to \bigoplus_v e_v A_1 \otimes_{\mathrm{k}} \Pi e_v \to \bigoplus_v e_v A_2 \otimes_{\mathrm{k}} \Pi e_v \to 0.$$

Moreover, it is well known that Π is Calabi–Yau of dimension 2 (see [39, Definition 3.2.3]), hence a duality result of Van den Bergh [11] applies and we have a canonical isomorphism

$$\mathrm{HH}^*(\Pi) \cong \mathrm{HH}_{2-*}(\Pi).$$

For the $\mathbb{K}$–vector space structure of the Hochschild cohomology let us recall some general facts (see eg [50]) which apply to any algebra (with trivial grading and differential). The zeroth cohomology $\mathrm{HH}^0(\Pi)$ is given by the center $Z(\Pi)$, and $\mathrm{HH}^1(\Pi)$ is given by *outer derivations* $\mathrm{Der}(\Pi)/\mathrm{Inn}(\Pi)$. Recall that a derivation is a linear map

$D\colon \Pi \to \Pi$ satisfying the Leibniz rule, and each $a \in \Pi$ defines an inner derivation by $D_a(b) = ab - ba$. The zeroth homology $\mathrm{HH}_0(\Pi)$ is isomorphic to $\Pi_{\mathrm{cyc}} := \Pi/[\Pi, \Pi]$, where $[\Pi, \Pi] \subset \Pi$ is the $\mathbb{K}$–linear subspace spanned by the commutators.

Theorem 30 [56, Corollary 10.1.2; cf 23, Theorem 8.4.1] *The $\mathbb{K}$–vector space structure of the Hochschild cohomology $\mathrm{HH}^*(\Pi)$ of the preprojective algebra associated to a non-Dynkin quiver is as follows.*

(1) *If Γ is extended Dynkin, then $\mathrm{HH}^0(\Pi) \cong Z(\Pi) \cong e_{v_0}\Pi e_{v_0}$, where v_0 is a vertex in Γ whose complement is Dynkin. Otherwise the center $Z(\Pi)$ is isomorphic to $\mathbb{K}$.*

(2) $\mathrm{HH}^1(\Pi) \cong \mathrm{Der}(\Pi)/\mathrm{Inn}(\Pi) \cong Z(\Pi) \oplus (F \otimes_{\mathbb{Z}} \mathbb{K}) \oplus \big(T \otimes_{\mathbb{Z}} \bigoplus_p \mathrm{Hom}_{\mathbb{Z}}(\mathbb{F}_p, \mathbb{K})\big)$, *where F and T are the free and torsion parts of $\Pi^{\mathbb{Z}}_{\mathrm{cyc}}$, respectively, and $\Pi^{\mathbb{Z}}$ is the preprojective $\mathbb{Z}$–algebra associated to Γ.*

(3) $\mathrm{HH}^2(\Pi) \cong \mathrm{HH}_0(\Pi) \cong \Pi_{\mathrm{cyc}}$.

Remark 31 In the extended Dynkin case, by the McKay correspondence $Z(\Pi)$ is isomorphic to the ring of invariant polynomials in $\mathbb{K}[x, y]$ under the action of the corresponding finite subgroup $G \subset \mathrm{SL}_2(\mathbb{K})$ as long as $\mathbb{K}$ has $|G|^{\mathrm{th}}$ roots of unity (see [56, Theorem 9.1.1]). Furthermore, in this case T is trivial and hence $\mathrm{HH}^*(\Pi)$ is determined by $Z(\Pi)$ and Π_{cyc}, unless the characteristic of $\mathbb{K}$ is a "bad prime" for Γ, ie 2 for $\widetilde{D}_n$, 2 or 3 for $\widetilde{E}_6$ and $\widetilde{E}_7$, and 2, 3 or 5 for $\widetilde{E}_8$ [57]. Note that the Hilbert series of $Z(\Pi)$ and Π_{cyc}, as algebras graded by path-length, are given in [34] and [57].

The quotient Π_{cyc} can be considered as a graded Lie algebra with the path-length grading and the Lie bracket induced by the *necklace Lie bracket* $\{\cdot,\cdot\}$ on Π, given by

$$\{p, q\} = \sum_{g_{wv} \in \Gamma_1} (\partial_{vw} q)(\partial_{wv} p) - (\partial_{wv} q)(\partial_{vw} p).$$

Here, for any path $p \in \Pi$ and adjoint pair (v, w) in Γ, $\partial_{wv} p$ is given as the sum

$$\sum\nolimits_i g_{i-1} \cdots g_1 g_l \cdots g_{i+1},$$

taken over all i for which the i^{th} arrow g_i in the path $p = g_l \cdots g_1$ is g_{wv}.

Note that the Lie bracket $[D, D'] = D \circ D' - D' \circ D$ on $\mathrm{Der}(\Pi)/\mathrm{Inn}(\Pi)$ coincides with the Gerstenhaber bracket on $\mathrm{HH}^1(\Pi)$ in favorable cases, eg if $\mathrm{char}\,\mathbb{K} = 0$ and Γ is not extended Dynkin.

The Lie brackets above are used to describe the (cup) product as well as the Gerstenhaber bracket on $\mathrm{HH}^*(\Pi)$ in [23], when $\mathrm{char}\,\mathbb{K} = 0$. We now recall the description of the

Gerstenhaber algebra structure of $\mathrm{HH}^*(\Pi)$ in [57], for arbitrary $\operatorname{char}\mathbb{K}$, using the BV operator Δ dual to the Connes differential (see eg [50]) on $\mathrm{HH}_*(\Pi)$. The *Euler derivation* eu on Π_{cyc} is defined as multiplication by l on each path of length l, and the derivation u, called *half Euler derivation* in [57], multiplies each path by the number of edges from Γ that it contains. Note that we have $\mathrm{eu} = 2u$ as elements of $\mathrm{HH}^1(\Pi)$. In other words, their difference is an inner derivation. The first summand of $\mathrm{HH}^1(\Pi)$ in Theorem 30 consists of multiples of u by $Z(\Pi)$.

Theorem 32 [56, Theorem 10.3.1] *As a BV-algebra,* $\mathrm{HH}^*(\Pi)$ *is determined by the following properties.*

(1) *The graded-commutative product*

$$\cup\colon \mathrm{HH}^i(\Pi)\otimes \mathrm{HH}^j(\Pi)\to \mathrm{HH}^{i+j}(\Pi)$$

is given as follows:

(a) *If* $\theta,\theta'\in \mathrm{Der}(\Pi)/\mathrm{Inn}(\Pi)\cong \mathrm{HH}^1(\Pi)$ *and* θ' *belongs to the* $F\otimes_{\mathbb{Z}}\mathbb{K}$ *summand of* $\mathrm{HH}^1(\Pi)$, *then* $\theta\cup\theta'$ *is obtained by considering* θ' *as an element of* Π_{cyc} *and applying the derivation* θ *to it.*

(b) *If none of* $\theta,\theta'\in \mathrm{HH}^1(\Pi)$ *belongs to the* $F\otimes_{\mathbb{Z}}\mathbb{K}$ *summand, then* $\theta\cup\theta'=0$.

(c) *If* $ij=0$, *then* $\cup$ *is given by multiplication in* Π.

(2) *The BV-operator*

$$\Delta\colon \mathrm{HH}^i(\Pi)\to \mathrm{HH}^{i-1}(\Pi)$$

dual to the Connes differential is given as follows.

(a) *We have*

$$\Delta(u)=1,\quad \Delta(z\cup\theta)=\theta(z)+z\Delta(\theta)$$

for every $z\in \mathrm{HH}^0(\Pi)\cong Z(\Pi)$, $\theta\in \mathrm{Der}(\Pi)/\mathrm{Inn}(\Pi)\cong \mathrm{HH}^1(\Pi)$. *The BV-operator vanishes on the* $\big(T\otimes_{\mathbb{Z}}\bigoplus_p \mathrm{Hom}_{\mathbb{Z}}(\mathbb{F}_p,\mathbb{K})\big)$ *summand of* $\mathrm{HH}^1(\Pi)$.

(b) *The operator* $\Delta\colon \mathrm{HH}^2(\Pi)\cong\Pi_{\mathrm{cyc}}\to \mathrm{Der}(\Pi)/\mathrm{Inn}(\Pi)\cong \mathrm{HH}^1(\Pi)$ *maps to the* $F\otimes_{\mathbb{Z}}\mathbb{K}$ *summand and it is given by*

$$\Delta(g_l\cdots g_1)=\sum_{i=1}^{l}\pm\partial_{g_i^*}(\cdot)g_{i-1}\cdots g_1 g_l\cdots g_{i+1},$$

where each g_i *is an arrow in the double of the quiver* Γ *and the sign is positive if and only if* $g_i\in\Gamma$.

Remark 33 A word of caution is in order. For Γ non-Dynkin, the BV-algebra structure on $\mathrm{HH}^*(\Pi_\Gamma)$ is induced by the 2–Calabi–Yau structure (in the sense of Ginzburg [39], also known as smooth Calabi–Yau structure) on the homologically smooth algebra Π_Γ. This means that we have an isomorphism of Π_Γ–bimodules

$$\Pi_\Gamma \simeq \mathrm{RHom}_{\Pi_\Gamma - \Pi_\Gamma}(\Pi_\Gamma, \Pi_\Gamma \otimes \Pi_\Gamma)[2],$$

where the bimodule structure on the right is with respect to the inner bimodule structure on $\Pi_\Gamma \otimes \Pi_\Gamma$ and RHom is taken with respect to the outer bimodule structure on $\Pi_\Gamma \otimes \Pi_\Gamma$. Two such 2–Calabi–Yau structures differ by an invertible element in $\mathrm{HH}^0(\Pi_\Gamma)$. The effect by such an invertible ϕ is to replace Δ by $\phi^{-1}\Delta\phi$ [66, Remark 4.8].

We can consider the Koszul dual notion. Namely, by Koszul duality, for Γ non-Dynkin, we have $\mathrm{HH}^*(\Pi_\Gamma) \cong \mathrm{HH}^*(A_\Gamma)$ and then the BV-algebra structure can be seen as naturally arising from a weak Calabi–Yau structure on A_Γ. Recall that a weak Calabi–Yau structure (also known as Frobenius structure or compact Calabi–Yau structure) of dimension 2 on the finite-dimensional algebra A_Γ is a quasi-isomorphism of A_Γ–bimodules

$$A_\Gamma \simeq A_\Gamma^\vee[-2],$$

where $A_\Gamma^\vee$ is the $\mathbb{K}$–linear dual of A_Γ. Two such Calabi–Yau structures again differ by an invertible element in $\mathrm{HH}^0(A_\Gamma)$.

In any case, if Γ is non-Dynkin and non-extended Dynkin, then by Theorem 30, $\mathrm{HH}^0(\Pi_\Gamma) \cong \mathrm{HH}^0(A_\Gamma) \cong \mathbb{K}$ is rank-1 generated by the identity, hence there exists (up to scaling) at most one (Ginzburg) Calabi–Yau structure on Π_Γ and at most one (weak) Calabi–Yau structure on A_Γ. These Calabi–Yau structures can either be constructed algebraically as in [39] or symplectically as a manifestation of Poincaré duality for the Fukaya category of compact Lagrangians or the open Calabi–Yau property of the wrapped Fukaya category.

Now, *suppose* $\mathscr{B}_\Gamma \simeq \mathscr{G}_\Gamma$. Then, since $\mathscr{G}_\Gamma$ is formal, we would have an isomorphism $\mathrm{SH}^*(X_\Gamma) \cong \mathrm{HH}^*(\mathscr{B}_\Gamma) \cong \mathrm{HH}^*(\Pi_\Gamma)$. Under this isomorphism, the natural BV-algebra structure on $\mathrm{SH}^*(X_\Gamma)$ given by the loop rotation operator $\Delta\colon \mathrm{SH}^*(X_\Gamma) \to \mathrm{SH}^{*-1}(X_\Gamma)$ has to coincide with the algebraically constructed BV-algebra structure on $\mathrm{HH}^*(\Pi_\Gamma)$ in the case that Γ is non-Dynkin and non-extended Dynkin.

On the other hand, combining the results from [53] and [5] one deduces that

$$\mathrm{SH}^*(T^*S^2) \cong \mathrm{HH}^*(C_{2-*}(\Omega S^2)) \cong \mathrm{HH}^*(C^*(S^2))$$

does not admit a *dilation* over a field of characteristic 2.[5] Recall that a dilation is an element $b \in \mathrm{SH}^1(X_\Gamma)$ such that

$$\Delta b = 1,$$

where $\Delta\colon \mathrm{SH}^*(X_\Gamma) \to \mathrm{SH}^{*-1}(X_\Gamma)$ is the BV-operator in symplectic cohomology. Furthermore, since T^*S^2 can be embedded as a Liouville subdomain of X_Γ, one has a restriction map, $\mathrm{SH}^*(X_\Gamma) \to \mathrm{SH}^*(T^*S^2)$ which is a map of BV-algebras. Therefore, a dilation on X_Γ can be restricted to a dilation on T^*S^2. On the other hand, we see from the above theorem that there is a class $u \in \mathrm{HH}^1(\Pi_\Gamma)$ that is sent to the identity by the BV-operator induced from the Calabi–Yau structure on Π_Γ. Hence, we arrive at a contradiction.

This is in agreement with Remark 15 where we have seen that $\mathscr{B}_\Gamma$ is a nontrivial deformation of $\mathscr{G}_\Gamma$ over a field of characteristic 2.

6.2 Dynkin case

In this section we compute the Hochschild cohomology of the zigzag algebra A_Γ associated with a *Dynkin* tree. If the underlying tree Γ is of type A_1, ie a single vertex, then $A_\Gamma = \mathbb{K}[x]/(x^2)$ with $|x| = 2$ and it is a Koszul algebra. Its Hochschild cohomology was computed in Example 20 above. Thus, hereafter we assume $\Gamma \neq A_1$. It turns out that if the underlying tree Γ is of Dynkin type but not a single vertex, then A_Γ is an almost-Koszul algebra (in the sense of [18]). In this situation, the Koszul complex leads to a construction of a minimal *periodic* resolution. We first review the basics of quadratic algebras and the associated Koszul complexes.

6.2.1 Zigzag algebra A_Γ as a trivial extension Recall that for any Γ, the zigzag algebra A_Γ is defined as the quotient of the path algebra $\mathbb{K}\mathrm{D}\Gamma$ of the double quiver $\mathrm{D}\Gamma$ by the ideal J generated by the elements

- $a_{uv}a_{vw}$ such that $u \neq w$, where v is adjacent to both u, w, and
- $a_{vw}a_{wv} - a_{vu}a_{uv}$ where v is adjacent to both u, w.

Clearly, this is an example of a quadratic algebra over k where V is the $\mathbb{K}$–vector space generated by the edges a_{wv} of $\mathrm{D}\Gamma$ and supported in grading 1. The path-length grading on $\mathbb{K}\mathrm{D}\Gamma$ descends to A_Γ where it is supported in degrees $0, 1$ and 2. It is straightforward to verify that:

Proposition 34 *For any tree Γ the quadratic dual $A_\Gamma^!$ of the zigzag algebra A_Γ is the preprojective algebra Π_Γ, when both are equipped with path-length grading.* □

[5] An independent verification of this fact based on a Morse–Bott computation of BV-operator on $\mathrm{SH}^*(T^*S^2)$ was communicated to us by P. Seidel.

As mentioned before, when Γ is a single vertex, or not a Dynkin-type tree, A_Γ is a Koszul algebra. For these cases, we have already computed $\mathrm{HH}^*(A_\Gamma)$ above (see Section 6.1 and Example 20). Henceforth, we will assume that Γ is *Dynkin*, but not a single vertex. These are the only cases when $A_\Gamma^! = \Pi_\Gamma$ is finite-dimensional.

Let us drop Γ from the notation for the moment and write

$$A = A_0 \oplus A_1 \oplus A_2 \quad \text{and} \quad \Pi = \Pi_0 \oplus \Pi_1 \oplus \cdots \oplus \Pi_{h-2}$$

for the graded pieces of A and Π. Here h stands for the Coxeter number of the Dynkin tree and it is equal to $n+1$, $2n-2$, 12, 18 and 30, for A_n, D_n, E_6, E_7 and E_8, respectively [18].

It turns out that, in this case, A_Γ is not Koszul and its Koszul complex (6) is not acyclic. Indeed, the Koszul complex is given by

$$0 \to A_\Gamma \to \Pi_1 \otimes_{\mathrm{k}} A_\Gamma \to \cdots \to \Pi_{h-2} \otimes_{\mathrm{k}} A_\Gamma \to 0 \tag{8}$$

and it fails to be exact at the right end but only there [18]. Nonetheless, in [18] the authors are able to modify the Koszul bimodule complex to obtain a $(2h-2)$–periodic complex that computes Hochschild cohomology of A_Γ. Indeed, the algebras A_Γ belong to a class of periodic algebras which are *almost Koszul*.

We will, however, now turn to a slightly different approach, which makes use of the fact that A_Γ is isomorphic to a *trivial extension algebra*.

Definition 35 Let B be a finite-dimensional algebra over the field $\mathbb{K}$. Let $B^\vee := \mathrm{Hom}_{\mathbb{K}}(B, \mathbb{K})$ be the linear dual of B, viewed naturally as a B–bimodule. The trivial extension algebra of B, denoted by $\mathcal{T}(B)$, is the vector space $B \oplus B^\vee$ equipped with the multiplication

$$(x, f) \cdot (y, g) = (xy, xg + fy).$$

If B is graded, to get a CY2 algebra, we grade $\mathcal{T}(B)$ so that $\mathcal{T}(B) = B \oplus B^\vee[-2]$.

Let $A^{\to} = \mathbb{K}\Gamma/J$ be the quotient of the path algebra of a quiver with respect to an arbitrary orientation of the edges modulo the ideal generated by paths of length 2. The following proposition appears in [43, Proposition 9] and results from an easy computation.

Proposition 36 *A_Γ is isomorphic to the trivial extension algebra $\mathcal{T}(A^{\to})$.* □

In particular, if we orient Γ so that each vertex is either a sink or a source, then there are no paths of length 2, hence A_Γ is a trivial extension algebra of the path algebra $\mathbb{K}\Gamma$ in the bipartite orientation.

Remark 37 There is a way to understand the above proposition in terms of symplectic topology. Namely, one can consider a Lefschetz fibration $f\colon \mathbb{C}^3 \to \mathbb{C}$, $(x,y,z) \mapsto f(x,y,z)$ given by perturbing the simple singularities

$$\begin{aligned} A_n: &\quad x^2+y^2+z^{n+1} &&\text{for } n \geq 1,\\ D_n: &\quad x^2+zy^2+z^{n-1} &&\text{for } n \geq 4,\\ E_6: &\quad x^2+y^3+z^4,\\ E_7: &\quad x^2+y^3+yz^3,\\ E_8: &\quad x^2+y^3+z^5. \end{aligned}$$

One can then identify the surface X_Γ with a regular fiber of these fibrations, ie the Milnor fiber of the singularity. The spheres S_v can be identified with the vanishing spheres and the corresponding thimbles generate the Fukaya–Seidel category of f by a famous result of Seidel [61]. For a suitable choice of grading structures and ordering of objects, the Floer endomorphism algebra $A^{\rightarrow}$ of these thimbles in the Fukaya–Seidel category of f coincides with the path algebra of $\mathbb{K}\Gamma$ modulo the ideal generated by length 2 paths. The algebra isomorphism

$$A_\Gamma = A^{\rightarrow} \oplus A^{\rightarrow}[-2]$$

follows from the general relationship between the Fukaya–Seidel category of a Lefschetz fibration and the Fukaya category of its fiber (see [62, Section 4]).

We next recall the following theorem about trivial extension algebras, which we will apply to path algebras of quivers whose underlying graph is a tree. Note that by a well-known result of Bernšteĭn, Gel'fand and Ponomarev [13], the path algebras $\mathbb{K}Q$ of quivers Q obtained by orienting edges of the same *tree* in different ways are derived equivalent algebras.

Theorem 38 (Rickard [55]) *Suppose C and D are derived equivalent algebras. Then their trivial extensions $\mathcal{T}(C)$ and $\mathcal{T}(D)$ are also derived equivalent. In particular, $\mathrm{HH}^*(\mathcal{T}(C))$ and $\mathrm{HH}^*(\mathcal{T}(D))$ are isomorphic as Gerstenhaber algebras.*

Our strategy will be to apply the above theorem to $\mathcal{T}(A^{\rightarrow}) = A_\Gamma$ to pass to another algebra whose Hochschild cohomology is previously computed. However, it is important to note that the above theorem is for trivially graded algebras. On the other hand, we need to compute $\mathrm{HH}^*(A_\Gamma)$ as a bigraded algebra. What's worse, since A_Γ has elements in both even and odd degrees, we cannot simply forget about the grading and reinstate it afterwards, as in a graded resolution, odd elements affect the signs.

We next explain how to deal with this tricky point. Namely, recall from Proposition 16 that A_Γ is the graded algebra obtained as

$$A_\Gamma = \bigoplus_{v,w} \mathrm{HF}^*(S_v, S_w).$$

On the other hand, given integers $\sigma_v \in \mathbb{Z}$ for every vertex v, we can define another graded algebra

$$\widetilde{A}_\Gamma = \bigoplus_{v,w} \mathrm{Hom}(S_v[\sigma_v], S_w[\sigma_w]) = \bigoplus_{v,w} \mathrm{HF}^*(S_v, S_w)[\sigma_w - \sigma_v],$$

where $S_v[n_v]$ denotes a graded object whose grading is shifted down by n_v. Clearly, A_Γ and $\widetilde{A}_\Gamma$ are graded Morita equivalent (in particular, derived equivalent). Therefore, the (graded) Hochschild cohomologies of A_Γ and $\widetilde{A}_\Gamma$ are canonically isomorphic (see for example [64, Section (1c)]). Hence, for the purpose of computing Hochschild cohomology of A_Γ, we can choose the shifts σ_v so that the shifted algebra is supported in even degrees. In fact, using the standard tree form of Γ as in Figure 2, we simply shift the object S_v up $S_v[-\delta_v]$, where δ_v is the distance from the root to the vertex v. In this way, any arrow in the double $\mathrm{D}\Gamma$ is in degree 0 or 2 according to whether it points towards or away from the root.

Summary To compute $\mathrm{HH}^*(A_\Gamma)$ as a graded Gerstenhaber algebra, we follow this procedure:

- First check that it is possible to shift gradings so that A_Γ is supported in even degrees.
- Forget the grading altogether and treat A_Γ as an ungraded algebra.
- Compute the algebra structure of the Hochschild cohomology of the ungraded algebra by relating it to previous computations using derived equivalences of ungraded algebras in Theorem 38. This algebra will have only the cohomological grading r.
- Finally, reinstate the s–grading on $\mathrm{HH}^*(A_\Gamma)$ by finding explicit (graded) cocycles for the generators of Hochschild cohomology as an algebra.

6.2.2 Type A Throughout this section, Γ is the Dynkin tree A_n, $n > 1$. We describe the Hochschild cohomology ring of the zigzag algebra A_Γ in detail. We follow the strategy outlined in the previous section. Namely, we first determine the Hochschild cohomology of A_Γ as an ungraded algebra. The result will be singly graded with the cohomological grading r. We then reinstate the s–grading by explicitly identifying generators.

As was mentioned in Proposition 36, A_Γ is isomorphic to the trivial extension algebra of the path algebra $\mathbb{K}Q$ of the quiver Q with the underlying tree $\Gamma = A_n$ and oriented with the bipartite orientation (see Figure 9). Furthermore, as explained above, the derived equivalence class of a path algebra of a quiver, and hence by Theorem 38, the derived equivalence class of trivial extensions of $\mathbb{K}Q$, does not depend on the choice of the orientation of the edges of the underlying tree.

Figure 9: A_n quiver in bipartite orientation

Let B_Γ be the trivial extension algebra of the path algebra of $\Gamma = A_n$ where the underlying quiver is now oriented in the linear orientation (see Figure 10).

Figure 10: A_n quiver in linear orientation

Let $\widetilde{A}_{n-1}$ be the extended Dynkin quiver of type A_{n-1}, namely the quiver with cyclic orientation whose underlying graph is a simple cycle with n vertices and n edges (see Figure 11), and let us denote the ideal generated by paths of length $\geq n+1$ by J_{n+1}.

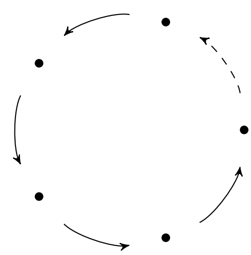

Figure 11: Cyclic quiver $\widetilde{A}_{n-1}$

The following well-known fact (cf [18]) can be verified by identifying $\mathbb{K}\Gamma$ with its image under the natural inclusion $\mathbb{K}\Gamma \to \mathbb{K}\widetilde{A}_{n-1}/J_{n+1}$, and observing that the subspace of $\mathbb{K}\widetilde{A}_{n-1}/J_{n+1}$ spanned by paths containing the unique arrow in the complement of Γ in $\widetilde{A}_{n-1}$ is canonically isomorphic to the linear dual of $\mathbb{K}\Gamma$ as a $\mathbb{K}\Gamma$–bimodule.

Lemma 39 *B_Γ is isomorphic to the truncated algebra $\mathbb{K}\widetilde{A}_{n-1}/J_{n+1}$.*

The derived equivalence between A_Γ and B_Γ implies an isomorphism between the Hochschild cohomology rings. On the other hand, the Hochschild cohomology of the (trivially graded) algebra B_Γ is studied in [42; 32; 9]. In particular, the algebra structure of $\mathrm{HH}^*(B_\Gamma)$ over a field of arbitrary characteristic was already known. Our contribution is to determine the internal s–grading coming from the grading of A_Γ. We have the following result:

Theorem 40 *As a (graded) commutative $\mathbb{K}$–algebra, the (r,s)–bigraded Hochschild cohomology algebra*

$$\mathrm{HH}^*(A_\Gamma) = \bigoplus_{r+s=*} \mathrm{HH}^r(A_\Gamma, A_\Gamma[s]),$$

of the graded $\Bbbk$*–algebra* A_Γ *is given by the following generators and relations.* (The subscripts of the generators, except for s_i, refer to total degrees.)

- *Suppose* $\operatorname{char}\mathbb{K} \nmid n+1$. *We have generators labeled along with their bidegrees* (r,s) *given by*

$$\begin{aligned} s_1,\dots,s_n &\quad (0,2),\\ t_1 &\quad (1,0),\\ t_0 &\quad (2,-2),\\ t_{-2} &\quad (2n,-2n-2) \end{aligned}$$

  and relations

$$s_i s_j = s_i t_j = t_1^2 = t_0^n = 0.$$

- *Suppose* $\operatorname{char}\mathbb{K} \mid n+1$. *We have generators labeled along with their bidegrees* (r,s) *given by*

$$\begin{aligned} s_1,\dots,s_n &\quad (0,2),\\ t_1 &\quad (1,0),\\ t_0 &\quad (2,-2),\\ u_{-1} &\quad (2n-1,-2n),\\ t_{-2} &\quad (2n,-2n-2) \end{aligned}$$

  and relations

$$\begin{aligned} s_i s_j = s_i t_1 = s_i t_0 = t_1^2 &= 0,\\ s_i u_{-1} &= t_1 t_0^{n-1},\\ s_i t_{-2} &= t_0^n,\\ t_0 u_{-1} &= t_1 t_{-2},\\ t_1 u_{-1} &= \alpha t_0^n,\\ u_{-1}^2 &= \beta t_0^{n-1} t_{-2}, \end{aligned}$$

  where $\alpha=\beta=1$ *if* $\operatorname{char}\mathbb{K}=2$ *and* $4 \nmid n+1$, *otherwise* $\alpha=\beta=0$.

Proof The presentation of $\mathrm{HH}^*(A_\Gamma)$ given above is adapted from the presentation of $\mathrm{HH}^*(B_\Gamma)$ as a $\mathbb{K}$–algebra graded by the cohomological grading, which was calculated in [42, Theorems 8.1 and 8.2] and [32, Theorem 5.19]. In view of the isomorphism between $\mathrm{HH}^*(A_\Gamma)$ and $\mathrm{HH}^*(B_\Gamma)$ as $\mathbb{K}$–algebras graded with respect to the cohomological r–gradings, it remains to determine the s–gradings. In particular, the rank of $\mathrm{HH}^r(B_\Gamma) \cong \bigoplus_s \mathrm{HH}^r(A_\Gamma, A_\Gamma[s])$ is given explicitly in [42; 32] for each r and it can be recovered from the presentations in the statement. We will make extensive use of this information in the following arguments.

In what follows, we describe generators as elements of the reduced bar-resolution

$$\mathrm{CC}^*(A, A) := \hom_{\mathrm{k}}(T\overline{A}, A), \tag{9}$$

where $A = A_\Gamma$ and $\overline{A} = A/\mathrm{k}$. The grading on A gives a decomposition

$$\mathrm{CC}^*(A, A) = \bigoplus_{*=r+s} \mathrm{CC}^r(A, A[s]),$$

where the Hochschild differential δ is of bidegree $(1, 0)$. We find explicit cocycles for $r = 0, 1, 2$ and show that the s–gradings of other generators are determined by the relations given above.

As a graded algebra, $A_\Gamma = A_0 \oplus A_1 \oplus A_2$, with components given by

$$A_0 = \bigoplus_{i=1}^{n} \mathbb{K} e_i, \quad A_1 = \bigoplus_{i=1}^{n-1} \mathbb{K} a_i \oplus \bigoplus_{i=1}^{n-1} \mathbb{K} b_i, \quad A_2 = \bigoplus_{i=1}^{n} \mathbb{K} s_i,$$

where $e_{i+1} a_i e_i = a_i$, $e_i b_i e_{i+1} = b_i$ and $s_{i+1} = a_i b_i = b_{i+1} a_{i+1}$.

The Hochschild differential δ in the complex (9) is given by the formula in [61, Equation (1.8)] (recall also the convention in (3)). We will only need the differentials on $\mathrm{CC}^r(A, A[s])$ for $r = 0, 1, 2$. These are given by

$$\begin{aligned}
\delta(c)(x_1) &= \mu^2(x_1, c) + (-1)^{(s-1)(|x_1|-1)} \mu^2(c, x_1), \\
\delta(c)(x_2, x_1) &= \mu^2(x_2, c(x_1)) + (-1)^{(s-1)(|x_1|-1)} \mu^2(c(x_2), x_1) \\
&\qquad + (-1)^s c(\mu^2(x_2, x_1)), \\
\delta(c)(x_3, x_2, x_1) &= \mu^2(x_3, c(x_2, x_1)) + (-1)^{(s-1)(|x_1|-1)} \mu^2(c(x_3, x_2), x_1) \\
&\qquad + (-1)^s c(x_3, \mu^2(x_2, x_1)) + (-1)^{s+|x_1|-1} c(\mu^2(x_3, x_2), x_1)
\end{aligned}$$

for $c \in \mathrm{CC}^0(A, A[s])$, for $c \in \mathrm{CC}^1(A, A[s])$, and for $c \in \mathrm{CC}^2(A, A[s])$, respectively.

$r = 0$ The 0–cocycles are given by central elements. The identity element

$$\sum_j e_j \in \mathrm{CC}^0(A, A[0])$$

and the elements

$$s_i \in \mathrm{CC}^0(A, A[2]) \quad \text{for } i = 1, \dots, n$$

give a basis of the center of A over $\mathbb{K}$.

$r = 1$ The 1–cocycles are given by derivations. We define a 1–cocycle $\tau_1 \in \mathrm{CC}^1(A, A[0])$ by

$$\tau_1(a_i) = -a_i, \quad \tau_1(b_i) = 0, \quad \tau_1(s_i) = s_i$$

for all $i = 1, \dots, n$. It is straightforward to check that τ_1 is a derivation but not an inner derivation, so it is a nontrivial element of $\bigoplus_s \mathrm{HH}^1(A, A[s])$, which is 1–dimensional for any $\mathbb{K}$. Therefore, any generator of this group, in particular t_1, must have the same s–grading as τ_1.

$r = 2$ We define a 2–cocycle $\tau_0 \in \mathrm{CC}^2(A, A[-2])$ given by

$$\begin{aligned}
\tau_0(a_i, b_i) &= (-1)^i e_{i+1}, \\
\tau_0(a_i, s_i) &= (-1)^{i+1} a_i, \\
\tau_0(s_i, b_i) &= (-1)^i b_i, \\
\tau_0(s_i, s_i) &= (-1)^{i+1} s_i
\end{aligned}$$

for all $i = 1, \dots, n$. Applying the Hochschild differential we get

$$\begin{aligned}
(\delta(\tau_0))(x_3, x_2, x_1) = (-1)^{|x_1|+|x_2|} x_3 \tau_0(x_2, x_1) - \tau_0(x_3, x_2) x_1 \\
+ (-1)^{|x_1|} \tau_0(x_3, x_2 x_1) - (-1)^{|x_1|+|x_2|} \tau_0(x_3 x_2, x_1).
\end{aligned}$$

It is straightforward (if tedious) to check that this expression vanishes identically on $\overline{A}^{\otimes 3}$. On the other hand, τ_0 cannot be a coboundary, since any $\kappa \in \mathrm{CC}^1(A, A[-2])$ has to be of the form

$$\kappa(s_i) = m_i e_i \quad \text{for some } m_i \in \mathbb{K}$$

and the Hochschild differential takes the form

$$(-1)^{|x_1|}(\delta(\kappa))(x_2, x_1) = x_2 \kappa(x_1) + \kappa(x_2 x_1) - (-1)^{|x_1|} \kappa(x_2) x_1,$$

which gives, in particular, that $\delta(\kappa)(s_i, s_i) = 0$ and $\delta(\kappa)(a_i, s_i) = m_i a_i$.

Hence, τ_0 cannot be of the form $\delta(\kappa)$ and therefore it represents a nontrivial element of the group $\bigoplus_s \mathrm{HH}^2(A, A[s])$. But we know that this group is 1–dimensional over any field $\mathbb{K}$, consequently any generator of this group over an arbitrary field $\mathbb{K}$ must have the same s–grading as τ_0.

It is harder to find explicit cocycles representing the elements u_{-1} and t_{-2} given in the statement of the theorem. Fortunately, for the purpose of determining the s–gradings we do not need explicit cocycles for these.

The element u_{-1} appears only if $\operatorname{char}\mathbb{K} \mid n+1$, and it satisfies the equation

$$s_i u_{-1} = t_1 t_0^{n-1}.$$

Since the s–gradings of s_i, t_1 and t_0 are $2, 0$ and -2, respectively, it follows that the projection u'_{-1} of u_{-1} to $\mathrm{HH}^{2n-1}(A, A[-2n])$ must be nonzero. A priori u_{-1} is not necessarily homogeneous with respect to the s–grading, but it has r–grading $2n-1$, and $\bigoplus_s \mathrm{HH}^{2n-1}(A, A[s])$ is 2–dimensional with generators u_{-1} and $t_1 t_0^{n-1}$. Therefore, u_{-1} has a decomposition $u'_{-1} + \lambda t_1 t_0^{n-1}$ into (r,s)–homogeneous elements for some $\lambda \in \mathbb{K}$. On the other hand, the relations in the statement of the theorem which involve u_{-1} are satisfied by u_{-1} if and only if they are satisfied by $u'_{-1} = u_{-1} - \lambda t_1 t_0^{n-1}$. Therefore, we may freely replace u_{-1} by u'_{-1} and hence assume that it is homogeneous with s–grading $-2n$.

Similarly, if $\operatorname{char}\mathbb{K} \mid n+1$, then $t_{-2} \in \bigoplus_s \mathrm{HH}^{2n}(A, A[s])$ appears in the relation

$$s_i t_{-2} = t_0^n$$

and $\bigoplus_s \mathrm{HH}^{2n}(A, A[s])$ is 2–dimensional with generators t_{-2} and t_0^n. As a consequence, t_{-2} has a decomposition $t_{-2} = t'_{-2} + \lambda t_0^n$ into (r,s)–homogeneous elements for some $\lambda \in \mathbb{K}$ and $t'_{-2} \neq 0$. The argument we used for u_{-1} applies here as well and we may assume that t_{-2} is homogeneous with s–grading $-2n-2$.

Finally, we need to determine the s–grading of t_{-2} over a field $\mathbb{K}$ for which $\operatorname{char}\mathbb{K} \nmid n+1$. Since A can be defined over $\mathbb{Z}$, its Hochschild cohomology groups can also be defined over $\mathbb{Z}$. Furthermore, since A has finite rank as a $\mathbb{Z}$–module, the bar-complex over $\mathbb{Z}$ is just a chain complex of finitely generated free abelian groups. So we can apply the universal coefficient theorem

$$(10)\quad 0 \to \bigoplus_s \mathrm{HH}^r_{\mathbb{Z}}(A, A[s]) \otimes \mathbb{K} \to \bigoplus_s \mathrm{HH}^r_{\mathbb{K}}(A \otimes \mathbb{K}, A[s] \otimes \mathbb{K}) \to \operatorname{Tor}\Big(\bigoplus_s \mathrm{HH}^{r+1}_{\mathbb{Z}}(A, A[s]), \mathbb{K}\Big) \to 0.$$

Now, it follows from the presentation given in the statement that the middle group for $r = 2n+1$ has rank 1 for any field $\mathbb{K}$ and we know that it is supported in internal degree

$s=-2n-2$ if $\operatorname{char}\mathbb{K} \mid n+1$. Therefore, we deduce from the universal coefficient theorem (by testing $\mathbb{K}=\mathbb{F}_p$ for infinitely many primes p) that

$$\bigoplus_s \mathrm{HH}^{2n+1}_{\mathbb{Z}}(A,A[s]) = \mathbb{Z}[2n+2],$$

hence, in particular,

$$\bigoplus_s \mathrm{HH}^{2n+1}_{\mathbb{K}}(A,A[s]) = \mathbb{K}[2n+2].$$

Finally, observe that the element

$$t_1 t_{-2} \in \bigoplus_s \mathrm{HH}^{2n+1}(A,A[s]) = \mathbb{K}[2n+2]$$

is a generator of the Hochschild cohomology group in grading $r=2n+1$ over an arbitrary field $\mathbb{K}$, and hence t_{-2} must have s–grading $-2n-2$ over an arbitrary field $\mathbb{K}$. □

Remark 41 Over the finite field $\mathbb{F}_3$ of characteristic 3, the group algebra $\mathbb{F}_3\mathfrak{S}_3$ of the symmetric group in three letters is isomorphic to the algebra A_Γ for $\Gamma=A_2$. A presentation for the Hochschild cohomology ring of this group algebra was given in [68, Theorem 7.1]. This agrees with the presentation given above.

As a consequence of Theorem 40 we conclude that the group $\bigoplus_{r+s=*}\mathrm{HH}^r(A_\Gamma,A_\Gamma[s])$ is nontrivial if and only if $*\le 2$. If $\operatorname{char}\mathbb{K}\nmid n+1$, the rank is n at each $*\le 2$, otherwise the rank is n for $*=2,1$ and $n+1$ for $*\le 0$.

Recall that we have proved in Theorem 27 that there is an isomorphism of Gerstenhaber algebras

$$\mathrm{SH}^*(X_\Gamma)\cong \mathrm{HH}^*(A_\Gamma)$$

over a field $\mathbb{K}$ of characteristic 0, where the Conley–Zehnder grading on the left corresponds to the total grading $r+s$ on the right. Having computed $\mathrm{HH}^*(A_\Gamma)$ as a bigraded algebra, we immediately get a description of the algebra structure of the symplectic cohomology. Let us also record its rank.

Corollary 42 *The symplectic cohomology group* $\mathrm{SH}^*(X_\Gamma)$ *over a field* $\mathbb{K}$ *of characteristic* 0 *is of rank* n *if* $*\le 2$ *and it is trivial otherwise.*

We have also performed computer-aided checks on our calculations. Tables 2 and 3 list the ranks (of a finite portion) for the cases A_2 and A_3.

$r+s\downarrow$ $s\rightarrow$	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
2	2	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	1	0	0	0	0	0	0
0	0	0	1	0	1	0	x	0	0	0	0
−1	0	0	0	0	0	0	x	0	1	0	1
−2	0	0	0	0	0	0	0	0	1	0	1

Table 2: $\Gamma = A_2$; x is 1 if $\operatorname{char}\mathbb{K} = 3$, 0 otherwise

$r+s\downarrow$ $s\rightarrow$	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
2	3	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	1	0	1	0	0	0	0
0	0	0	1	0	1	0	1	0	x	0	0
−1	0	0	0	0	0	0	0	0	x	0	1
−2	0	0	0	0	0	0	0	0	0	0	1

Table 3: $\Gamma = A_3$; x is 1 if $\operatorname{char}\mathbb{K} = 2$, 0 otherwise

6.2.3 Type *D* In this section we consider the case where Γ is the Dynkin tree D_n, $n \geq 4$. Most of the arguments in the previous section apply verbatim or with minor modifications. So we will focus on the differences and provide details as necessary.

Considering the quiver based on Γ with the orientation of the arrows given by Figure 12, we obtain the following result.

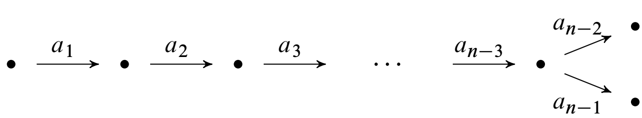

Figure 12: D_n quiver

Lemma 43 *The trivial extension algebra B_Γ of the path algebra $\mathbb{K}\Gamma$ is isomorphic to the quotient $\mathbb{K}Q/I$, where Q is the quiver given in Figure 13 and I is the ideal generated by the elements*

$$\beta_{n-1}\gamma_{n-1} - \beta_n\gamma_n, \quad \alpha_i \cdots \alpha_1\beta_n\gamma_n\alpha_{n-3}\cdots\alpha_i,$$
$$\gamma_n\alpha_{n-3}\cdots\alpha_1\beta_{n-1}, \quad \gamma_{n-1}\alpha_{n-3}\cdots\alpha_1\beta_n.$$

Proof Using the identifications $a_i \leftrightarrow \alpha_i$ for $1 \leq i \leq n-3$ and $a_j \leftrightarrow \gamma_{j+1}$ for $j = n-2$ and $n-1$, we can consider $\mathbb{K}\Gamma$ as a subalgebra of $\mathbb{K}Q/I$. Observe that $\mathbb{K}Q/I$ decomposes as a direct sum $\mathbb{K}\Gamma \oplus V$ and V is generated by β_{n-1} and β_n as a

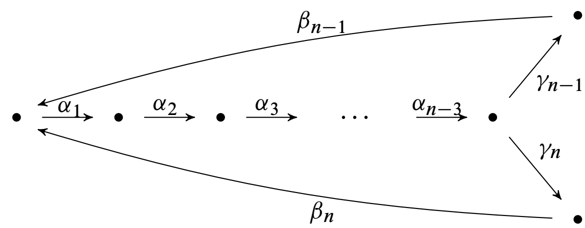

Figure 13: The quiver Q

$\mathbb{K}\Gamma$–bimodule. Moreover, as $\mathbb{K}\Gamma$–bimodules, V and the dual of $\mathbb{K}\Gamma$ are isomorphic via

$$\begin{aligned} \psi\colon V &\to (\mathbb{K}\Gamma)^\vee, \\ \beta_{n-1} &\mapsto (a_{n-2}a_{n-3}\cdots a_2a_1)^\vee, \\ \beta_n &\mapsto (a_{n-1}a_{n-3}\cdots a_2a_1)^\vee. \end{aligned}$$

It is straightforward to check that this map is a well-defined isomorphism.

In fact, $(\mathbb{K}\Gamma)^\vee$ can also be considered as a subalgebra of $\mathbb{K}Q/I$ by identifying the dual $p^\vee$ of a path $p \in \mathbb{K}\Gamma$ with the path $q \in \mathbb{K}Q/I$ such that

$$q \cdot p = \tau^t(\beta_n\gamma_n\alpha_{n-3}\cdots\alpha_1) = \tau^t(\beta_{n-1}\gamma_{n-1}\alpha_{n-3}\cdots\alpha_1) \in \mathbb{K}Q/I,$$

where τ denotes the simple rotation action on the cycles and t is the distance between the initial points of p and α_1. □

As a consequence of this lemma and the discussions in the previous section, there is an isomorphism between the Hochschild cohomology rings of the zigzag algebra A_Γ and B_Γ. On the other hand, the Hochschild cohomology of B_Γ as a trivially graded algebra was described in detail in [72; 71]. As in the case of $\Gamma = A_n$ (see Theorem 40), we determine the internal grading s induced by the zigzag algebra and obtain the following result. This extra information does not appear in [72; 71] and the determination of this grading is the main contribution given in the following theorem.

Theorem 44 *Let $\Gamma = D_n$, $n \geq 4$. The (r,s)–bigraded Hochschild cohomology algebra*

$$\mathrm{HH}^*(A_\Gamma) = \bigoplus_{r+s=*} \mathrm{HH}^r(A_\Gamma, A_\Gamma[s])$$

of the graded $\Bbbk$–algebra A_Γ is (graded) commutative and given by the following generators and relations. (The subscripts of the generators, except for the s_i, refer to total degrees.)

(1) *Suppose* $\operatorname{char}\mathbb{K} \neq 2$. *We have generators labeled along with their bidegrees* (r,s) *given by*

$$\begin{aligned} s_1,\dots,s_n &\quad (0,2),\\ t_1 &\quad (1,0),\\ r_1 &\quad (2n-3,-2n+4),\\ t_0 &\quad (4,-4),\\ r_0 &\quad (2n-4,-2n+4),\\ t_{-2} &\quad (4n-6,-4n+4) \end{aligned}$$

and relations

$$s_i s_j = s_i t_j = s_i r_j = t_1^2 = t_1 r_1 = r_1^2 = t_0^{n-1} = 0,$$

together with

	if n is even	if n is odd
$t_1 r_0 =$	$\left(\frac{n}{2}\right) t_1 t_0^{(n-2)/2} - (n-1) r_1$	$\left(\frac{n-1}{2}\right) r_1$
$2t_0 r_1 =$	$t_1 t_0^{n/2}$	0
$2r_1 r_0 =$	0	$t_1 t_0^{n-2}$
$2t_0 r_0 =$	$t_0^{n/2}$	0
$2r_0^2 =$	$\left(\frac{n}{2}\right) t_0^{n-2}$	$\left(\frac{n-1}{2}\right) t_0^{n-2}$

(2) *Suppose* $\operatorname{char}\mathbb{K} = 2$. *We have generators labeled along with their bidegrees* (r,s) *given by*

$$\begin{aligned} s_1,\dots,s_n &\quad (0,2),\\ t_1 &\quad (1,0),\\ u_1 &\quad (3,-2),\\ t_0 &\quad (4,-4),\\ r_0 &\quad (2n-4,-2n+4),\\ u_0 &\quad \left(4\left\lfloor \tfrac{n}{2}\right\rfloor, -4\left\lfloor \tfrac{n}{2}\right\rfloor\right),\\ u_{-1} &\quad \left(4\left\lfloor \tfrac{n-1}{2}\right\rfloor + 1, -4\left\lfloor \tfrac{n-1}{2}\right\rfloor - 2\right),\\ t_{-2} &\quad (4n-6,-4n+4) \end{aligned}$$

and relations

$$
\begin{gathered}
s_i s_j = s_i t_1 = s_i u_1 = s_i u_0 = 0, \\
t_1^2 = u_1^2 = u_0^2 = u_1 u_0 = 0, \\
t_0^{\lfloor \frac{n}{2} \rfloor} = u_1 t_0^{\lfloor \frac{n-1}{2} \rfloor} = 0, \\
r_0^2 = \lfloor \tfrac{n}{2} \rfloor u_0 t_0^{\lfloor \frac{n-3}{2} \rfloor}, \\
s_j t_0 = t_1 u_1
\end{gathered}
$$

together with

	if n is even	if n is odd
$u_{-1}^2 =$	t_{-2}	$t_{-2}t_0$
$u_1u_{-1} =$	u_0	u_0t_0
$t_0r_0 =$	u_1u_{-1}	t_1u_{-1}
$u_1r_0 =$	0	t_1u_0
$s_ju_{-1} =$	$\begin{cases} \left(\frac{n-2}{2}\right)t_1t_0^{(n-2)/2} + t_1r_0 & \text{if } j \le n-1, \\ \left(\frac{n}{2}\right)t_1t_0^{(n-2)/2} + t_1r_0 & \text{if } j = n \end{cases}$	u_1r_0
$s_jr_0 =$	$\begin{cases} t_1u_1t_0^{(n-4)/2} & \text{if } j \le n-1, \\ 0 & \text{if } j = n \end{cases}$	0
$u_{-1}r_0 =$		t_1t_{-2}
$t_1r_0 =$		$\left(\frac{n-1}{2}\right)u_1t_0^{(n-3)/2}$
$s_jt_{-2} =$		r_0u_0

Proof The presentation of the algebra structure of $\mathrm{HH}^*(B_\Gamma)$ in [71, Theorem 4] provides all the generators with their r–gradings and relations. The derived equivalence between A_Γ and B_Γ gives

$$\mathrm{HH}^r(B_\Gamma) \cong \bigoplus_s \mathrm{HH}^r(A_\Gamma, A_\Gamma[s]).$$

Therefore it suffices to determine the s–gradings of the generators in the statement. Extending the notation in Figure 12, we consider the decomposition of the graded algebra A_Γ into homogeneous $\mathbb{K}$–subspaces A_0, A_1 and A_2, spanned by

$$\{e_1, \dots, e_n\}, \quad \{a_1, b_1, \dots, a_{n-1}, b_{n-1}\} \quad \text{and} \quad \{s_1, \dots, s_n\},$$

respectively, where

$$e_{i+1}a_ie_i = a_i, \quad e_ib_ie_{i+1} = b_i, \quad e_na_{n-1}e_{n-2} = a_{n-1}, \quad e_{n-2}b_{n-1}e_n = b_{n-1},$$
$$s_1 = b_1a_1, \quad s_{i+1} = a_ib_i = b_{i+1}a_{i+1}, \quad s_{n-2} = a_{n-3}b_{n-3} = b_ja_j, \quad s_{j+1} = a_jb_j$$

for $1 \le i \le n-4$ and $j = n-2, n-1$.

As in the proof of Theorem 40, we will again use the reduced bar-resolution associated to $A = A_\Gamma$ and denote the Hochschild differential by δ. Consequently, the discussion for $r = 0, 1$ is exactly the same as in the proof of Theorem 40. We identify the s–gradings of $s_1, \dots, s_n$ and t_1 as in the statement.

For every nonnegative integer r, the dimension of $\bigoplus_s \mathrm{HH}^r(A, A[s]) \cong \mathrm{HH}^r(B_\Gamma)$ can be deduced from the presentation in the statement and it is explicitly given in [71, Theorem 3]. We will make extensive use of this information. To begin with, note that $\bigoplus_s \mathrm{HH}^2(A, A[s])$ is trivial over any field $\mathbb{K}$, and $\bigoplus_s \mathrm{HH}^3(A, A[s])$ is 1–dimensional if $\operatorname{char}\mathbb{K} = 2$ and trivial otherwise. Over a field $\mathbb{K}$ of characteristic 2, for $c \in \mathrm{CC}^3(A, A[s])$, the Hochschild differential δ is given by

$$\delta(c)(x_4, x_3, x_2, x_1) = x_4c(x_3, x_2, x_1) + c(x_4, x_3, x_2)x_1 + c(x_4x_3, x_2, x_1) + c(x_4, x_3x_2, x_1) + c(x_4, x_3, x_2x_1).$$

We claim that, if $\operatorname{char}\mathbb{K} = 2$, there is a cocycle $\upsilon_1 \in \mathrm{CC}^3(A, A[-2])$ which is not the coboundary of any $\kappa \in \mathrm{CC}^2(A, A[s])$. This and the fact that $\bigoplus_s \mathrm{HH}^3(A, A[s])$ is 1–dimensional imply that the s–grading of u_1 must be -2, the same as υ_1. To describe the graded homomorphism $\upsilon_1\colon \bar{A}^{\otimes 3} \to A[-2]$ uniquely, it suffices to list the generators of $\bar{A}^{\otimes 3}$ on which υ_1 is nonzero. It necessarily vanishes on any element of degree 5 or 6 in $\bar{A}^{\otimes 3}$ since A is supported in gradings between 0 and 2. We declare υ_1 to be nonzero exactly on those nontrivial elements $(x_3, x_2, x_1) \in \bar{A}^{\otimes 3}$ which satisfy one of the following conditions:

- One of x_1, x_2 and x_3 is of the form a_i and the other two is of the form b_i, possibly with different indices, and $(x_3, x_2, x_1) \neq (b_{n-1}, a_{n-1}, b_{n-2})$.
- Exactly one of x_1, x_2 and x_3 is of the form s_k, and the initial point of x_1 matches the terminal point of x_3.
- $(x_3, x_2, x_1) = (a_{n-2}, b_{n-1}, a_{n-1})$.

It is straightforward to check that υ_1 is a cocycle. To see that it is not a coboundary, suppose that $c \in \mathrm{CC}^2(A, A[-2])$. Then

$$\delta(\kappa)((b_2, a_2, s_2) + (a_2, s_2, b_2) + (s_2, b_2, a_2)) = b_2\kappa(a_2, s_2) + a_2\kappa(s_2, b_2) + \kappa(a_2, s_2)b_2 + \kappa(s_2, b_2)a_2$$

after cancellations. Observe that the right-hand side is either $s_2 + s_3$ or 0, depending on the values of $\kappa(a_2, s_2)$ and $\kappa(s_2, b_2)$. Since

$$\upsilon_1((b_2, a_2, s_2) + (a_2, s_2, b_2) + (s_2, b_2, a_2)) = s_3,$$

υ_1 cannot be a coboundary.

Next we determine the s–grading of t_0. Consider the case $\operatorname{char} \mathbb{K} = 2$. If $n = 4$, then $\bigoplus_s \mathrm{HH}^4(A, A[s])$ has generators t_0, r_0 and $t_1 u_1$. Note that any relation satisfied by t_0 and r_0 is also satisfied by $t_0 - \gamma t_1 u_1$ and $r_0 - \gamma t_1 u_1$, respectively, for any $\gamma \in \mathbb{K}$. Therefore, without loss of generality, we may assume that there are s–homogeneous generators t_0', r_0' and constants $\alpha, \beta \in \mathbb{K}$ such that

$$t_0 = t_0' + \alpha r_0' \quad \text{and} \quad r_0 = r_0' + \beta t_0'.$$

From the relations regarding $s_n r_0$ and $s_n t_0$ we obtain

$$0 = s_n r_0 = s_n r_0' + \beta s_n t_0' \quad \text{and} \quad 0 \neq u_1 t_1 = s_n t_0 = s_n t_0' + \alpha s_n r_0'.$$

Since the gradings of u_1, t_1 and s_n are established above, the second equation implies that at least one of t_0' and r_0' has s–grading -4; in fact they both do, as the following arguments show. If $s_n r_0' \neq 0$, then the first equation proves that r_0' and t_0' have the same s–grading, which is necessarily -4. So suppose $s_n r_0' = 0$. Now the second equation gives $s_n t_0' \neq 0$. Moreover, the first equation implies $\beta = 0$, which means $r_0 = r_0'$; in particular, r_0 is s–homogeneous. So we can use the relation $s_1 r_0 = t_1 u_1$ to establish the s–grading of r_0' as -4. On the other hand, under the assumption $s_n r_0' = 0$, the second equation becomes $s_n t_0' = u_1 t_1$, implying that t_0' has s–grading -4 as well. Therefore, regardless of the value of $s_n r_0'$, the s–gradings of t_0 and r_0 are both -4.

If $n > 4$ and $\operatorname{char} \mathbb{K} = 2$, then $\bigoplus_s \mathrm{HH}^4(A, A[s])$ has rank 2 with generators t_0 and $t_1 u_1$, hence we may assume that there is an s–homogeneous generator t_0' and $\alpha \in \mathbb{K}$ such that $t_0 = t_0' + \alpha t_1 u_1$. The relation $s_n t_0 = t_1 u_1$ implies that the s–grading of t_0' is -4. The s–grading of $t_1 u_1$ is -2 by previous computations. If n is even, then any relation in the statement holds for t_0 if and only if it holds for t_0'. Therefore, without loss of generality, we may assume that $t_0 = t_0'$ is s–homogeneous with grading -4, at least when n is even. The same conclusion holds for odd n as well, but we will not prove (nor use) it until Case 3 below.

Let us now consider the s–grading of t_0 when $\operatorname{char} \mathbb{K} \neq 2$. Regardless of whether $n = 4$ or not, the argument uses the universal coefficient theorem (10) as in the proof of Theorem 40. First of all, considering that $\bigoplus_s \mathrm{HH}^2(A, A[s])$ is trivial for any field $\mathbb{K}$ and using (10) for $r = 2$, we conclude that $\bigoplus_s \mathrm{HH}^3_{\mathbb{Z}}(A, A[s])$ has no torsion. Since $\bigoplus_s \mathrm{HH}^3(A, A[s])$ is trivial when $\operatorname{char} \mathbb{K} \neq 2$, applying the universal coefficient theorem (10) for $r = 3$ implies that $\bigoplus_s \mathrm{HH}^3_{\mathbb{Z}}(A, A[s])$ has no free component either, hence it is trivial. Moreover, the same exact sequence and the fact that for $\operatorname{char}\mathbb{K} = 2$, $\bigoplus_s \mathrm{HH}^3(A, A[s])$ is generated by u_1 whose s–grading is computed as -2 above, establish the torsion of $\bigoplus_s \mathrm{HH}^4_{\mathbb{Z}}(A, A[s])$ as $\mathbb{Z}_2[2]$.

The argument above for $\operatorname{char}\mathbb{K} = 2$ shows that $\bigoplus_s \mathrm{HH}^4(A, A[s]) \cong \mathbb{K}^d[4] \oplus \mathbb{K}[2]$, where $d = 2$ if $n = 4$ and $d = 1$ otherwise. Using the fact that $\bigoplus_s \mathrm{HH}^4(A, A[s])$ is d–dimensional for any field $\mathbb{K}$ with $\operatorname{char}\mathbb{K} \neq 2$, and applying the universal coefficient theorem (10) for $r = 4$ to infinitely many characteristics, we conclude that $\bigoplus_s \mathrm{HH}^4_{\mathbb{Z}}(A, A[s])$ is in fact $\mathbb{Z}^d[4] \oplus \mathbb{Z}_2[2]$. In particular, $\bigoplus_s \mathrm{HH}^4(A, A[s])$ is supported in s–grading -4 whenever $\operatorname{char}\mathbb{K} \neq 2$, and the s–grading of t_0 is -4 unless n is odd and $\operatorname{char}\mathbb{K} = 2$.

The rest of the argument varies slightly according to the parity of n and the characteristic of the base field.

Case 1 (n even and $\operatorname{char}\mathbb{K} = 2$)

We need to determine the s–gradings of the rest of the generators, namely u_{-1}, t_{-2}, u_0 and r_0. Since

$$\{u_{-1}, t_1 r_0, t_1 t_0^{(n-2)/2}\}$$

forms a basis of $\bigoplus_s \mathrm{HH}^{2n-3}(A, A[s])$,

$$u_{-1} = u'_{-1} + \alpha t_1 r_0 + \beta t_1 t_0^{(n-2)/2}$$

for some s–homogeneous $u'_{-1} \neq 0$ and some $\alpha, \beta \in \mathbb{K}$. Observe that any relation satisfied by u_{-1} is satisfied by u'_{-1} as well. Therefore, without loss of generality, we may assume that $u_{-1} = u'_{-1}$ and its s–grading is $-2n + 2$ as a result of the relation

$$s_n u_{-1} - s_1 u_{-1} = t_1 t_0^{(n-2)/2}.$$

Moreover, by the relations $u_0 = u_1 u_{-1}$ and $t_{-2} = u_{-1}^2$, both u_0 and t_{-2} are s–homogeneous with gradings $-2n$ and $-4n + 4$, respectively. Regarding r_0, note that

$$\{r_0, t_0^{(n-2)/2}, t_1 u_1 t_0^{(n-4)/2}\}$$

forms a basis of $\bigoplus_s \mathrm{HH}^{2n-4}(A, A[s])$. Hence

$$r_0 = r'_0 + \alpha t_0^{(n-2)/2} + \beta t_1 u_1 t_0^{(n-4)/2}$$

for some s–homogeneous $r'_0 \neq 0$ and some $\alpha, \beta \in \mathbb{K}$. It is straightforward to check that any relation satisfied by r_0 is also satisfied by $r_0 - \beta t_1 u_1 t_0^{(n-4)/2}$, so we may assume that $r_0 = r'_0 + \alpha t_0^{(n-2)/2}$. Moreover, the relation $u_0 = t_0 r_0 = t_0 r'_0$ implies that the s–grading of r_0' is $-2n+4$, the same as that of $t_0^{(n-2)/2}$. Therefore, r_0 is s–homogeneous with this grading as well.

Case 2 (n even and $\operatorname{char}\mathbb{K} \neq 2$)

We have a single argument for the s–grading of r_0 and r_1 which belong to 2–dimensional spaces $\bigoplus_s \mathrm{HH}^{2n-4}(A, A[s])$ and $\bigoplus_s \mathrm{HH}^{2n-3}(A, A[s])$, respectively. We take s–homogeneous elements $r_0' \neq 0$ and $r_1' \neq 0$ such that

$$r_0 = r_0' + \alpha t_0^{(n-2)/2} \quad \text{and} \quad r_1 = r_1' + \beta t_1 t_0^{(n-2)/2}.$$

Suppose that $\operatorname{char}\mathbb{K} \nmid n-1$. By way of contradiction, assume that r_0 is not s–homogeneous, ie $\alpha \neq 0$ and the s–grading of r_0' is not $-2n+4$. Then $t_1 r_0 = \left(\frac{n}{2}\right) t_1 t_0^{(n-2)/2} - (n-1) r_1$ implies that $-(n-1) r_1' = t_1 r_0'$ for grading reasons. Consequently, the s–gradings of r_0' and r_1' should match. Moreover, since $2t_0 r_1 = t_1 t_0^{n/2}$, and again for grading reasons, $\beta \neq 0$. But then, $\alpha\beta t_1 t_0^{n-2} \neq 0$ and its s–grading does not match with the s–grading of any other term in the product $r_1 r_0$ contradicting with $r_1 r_0 = 0$. Therefore r_0 is s–homogeneous, and so is r_1, in fact with the same s–grading, as a consequence of

$$t_1 r_0 = \left(\tfrac{1}{2}n\right) t_1 t_0^{(n-2)/2} - (n-1) r_1.$$

In order to account for the possibility that $\operatorname{char}\mathbb{K} \mid (n/2)$, instead of the relation above we use the relation $2t_0 r_0 = t_0^{n/2}$ to obtain the common s–grading of r_0 and r_1.

For a field $\mathbb{K}$ with $\operatorname{char}\mathbb{K} \neq 2$, both $\bigoplus_s \mathrm{HH}^{2n-3}(A, A[s])$ and $\bigoplus_s \mathrm{HH}^{2n-4}(A, A[s])$ are 2–dimensional, and moreover we just proved that when $\operatorname{char}\mathbb{K} \nmid n-1$, each of these spaces are supported in $s = -2n+4$. By using the universal coefficient theorem (10) for $r = 2n-4$ we conclude that, as long as $\operatorname{char}\mathbb{K} \neq 2$ (even if $\operatorname{char}\mathbb{K}$ divides $n-1$) both $\bigoplus_s \mathrm{HH}^{2n-3}(A, A[s])$ and $\bigoplus_s \mathrm{HH}^{2n-4}(A, A[s])$ are supported in $s = -2n+4$. In particular, the common s–grading of r_0 and r_1 is $-2n+4$.

The s–grading of the remaining generator t_{-2} is obtained by the following argument, which applies to odd n as well. First of all, t_{-2} is s–homogeneous as it belongs to the 1–dimensional space $\bigoplus_s \mathrm{HH}^{4n-6}(A, A[s])$. On the other hand, $\bigoplus_s \mathrm{HH}^{4n-5}(A, A[s])$ is 1–dimensional over any field $\mathbb{K}$ and it is generated by $t_1 t_{-2}$. Since we already have the s–grading of $t_1 t_{-2}$ for $\operatorname{char}\mathbb{K} = 2$ from the previous case, we obtain the s–grading of t_{-2} over any field using the universal coefficient theorem (10) for $r = 4n-5$.

Case 3 (n odd and $\operatorname{char}\mathbb{K} = 2$)

In this case, the s–grading of r_0 can be obtained by an argument which works regardless of $\operatorname{char}\mathbb{K}$. Over any $\mathbb{K}$, $\bigoplus_s \mathrm{HH}^{2n-4}(A, A[s])$ is 1–dimensional and generated by r_0, which is therefore s–homogeneous. Applying the universal coefficient theorem

(10) for $r = 2n-4$ and infinitely many different characteristics, we conclude that $\bigoplus_s \mathrm{HH}^{2n-4}_{\mathbb{Z}}(A, A[s]) \cong \mathbb{Z}$ and to establish the s–grading of this group, it suffices to use the relation $2r_0^2 = \binom{n-1}{2} t_0^{n-2}$ over a field of characteristic 0. In particular, r_0 has s–grading $-2n+4$ for any field $\mathbb{K}$.

The generator u_0 belongs to the 1–dimensional space $\bigoplus_s \mathrm{HH}^{2n-2}(A, A[s])$, hence it is s–homogeneous, and its s–grading is determined by the relation $u_1 r_0 = t_1 u_0$.

Next we consider u_{-1}. It belongs to $\bigoplus_s \mathrm{HH}^{2n-1}(A, A[s])$ which is generated by u_{-1} and $u_1 r_0$. So $u_{-1} = u'_{-1} + \alpha u_1 r_0$ for some $\alpha \in \mathbb{K}$ and s–homogeneous $u'_{-1} \neq 0$. Observe that any relation which involves u_{-1} is satisfied by u'_{-1} as well. Hence we may assume that u_{-1} is s–homogeneous. Its s–grading is obtained from

$$t_1 u_{-1} = t_0 r_0 = t'_0 r_0.$$

Note that we have not established the s–homogeneity of t_0 in this case yet, and that is why we had to refer to t'_0 in the relation above and use the fact that $t_0 r_0 - t'_0 r_0 = 0$ since it is a multiple of $t_1 u_1 r_0 = s_j t_0 r_0 = 0$.

Finally, we determine the s–gradings of t_0 and t_{-2} simultaneously. In the case we consider, they belong to 2–dimensional spaces

$$\bigoplus_s \mathrm{HH}^4(A, A[s]) \quad \text{and} \quad \bigoplus_s \mathrm{HH}^{4n-6}(A, A[s]),$$

with respective bases $\{t_0, t_1 u_1\}$ and $\{t_{-2}, r_0 u_0\}$. So there are s–homogeneous elements t'_0 and t'_{-2} with constants $\alpha, \beta \in \mathbb{K}$ such that

$$t_0 = t'_0 + \alpha t_1 u_1 \quad \text{and} \quad t_{-2} = t'_{-2} + \beta r_0 u_0 .$$

In fact, the s–gradings of t'_0 and t'_{-2} are -4 and $-4n+4$, respectively, since $s_j t'_0 = t_1 u_1$ and $s_j t'_{-2} = r_0 u_0$. It is straightforward to check that any relation in the statement, except for $u_{-1}^2 = t_{-2} t_0$, holds for t_0 and t_{-2} if and only if it holds for t'_0 and t'_{-2}. To check that the remaining relation holds, we use

$$u_{-1}^2 = t_{-2} t_0 = t'_{-2} t'_0 + \alpha t'_{-2} t_1 u_1 + \beta t'_0 r_0 u_0 + \alpha\beta t_1 u_1 r_0 u_0$$

and observe that the only term on the right-hand side of the above relation whose s–grading matches that of u_{-1}^2 is $t'_{-2} t'_0$. Therefore, without loss of generality, we may assume that $t_0 = t'_0$ and $t_{-2} = t'_{-2}$.

Case 4 (n is odd and $\operatorname{char} \mathbb{K} \neq 2$)

The s–gradings of t_{-2} and r_0 are already obtained in Cases 2 and 3 above.

The remaining generator r_1 is s–homogeneous since it belongs to the 1–dimensional space $\bigoplus_s \mathrm{HH}^{2n-3}(A, A[s])$ and its s–grading is determined by the relation $2r_1 r_0 = t_1 t_0^{n-2}$. □

Using Theorem 18, which is due to Seidel and Thomas, one gets the following consequence of the computation above.

Corollary 45 *If* $\operatorname{char}\mathbb{K} \neq 2$ *and* Γ *is of type* D_n, $n \geq 4$, *then the zigzag algebra* A_Γ *is intrinsically formal.*

One can write explicit bases for the relevant $\mathbb{K}$–vector subspaces of $\mathrm{HH}^*(A_\Gamma)$ as follows.

If $\operatorname{char}\mathbb{K} \neq 2$, then $\bigoplus_{r+s=2} \mathrm{HH}^r(A, A[s])$ is spanned by $\{s_1, \dots, s_n\}$, and for any nonnegative integer m and $i = 0, 1$, a basis of $\bigoplus_{r+s=i-2m} \mathrm{HH}^r(A, A[s])$ is given by

$$\{r_i t_{-2}^m, t_1^i t_0^k t_{-2}^m : 0 \leq k \leq n-2\}.$$

When $\operatorname{char}\mathbb{K} = 2$, the increase in the dimensions of these spaces is immediate from the statement of Theorem 44. The subspace $\bigoplus_{r+s=2} \mathrm{HH}^r(A, A[s])$ is spanned by

$$\left\{s_j, t_1 u_1 t_0^k = s_n t_0^{k+1} : 1 \leq j \leq n,\ 0 \leq k \leq \left\lfloor \tfrac{n-4}{2} \right\rfloor\right\},$$

and depending on the parity of n, $\bigoplus_{r+s=1} \mathrm{HH}^r(A, A[s])$ is spanned by

$$\left\{u_1 t_0^k, t_1 t_0^l, t_1 r_0 t_0^l : 0 \leq k \leq \tfrac{n-4}{2},\ 0 \leq l \leq \tfrac{n-2}{2}\right\}$$

if n is even, and by

$$\left\{u_1 t_0^l, t_1 t_0^l, t_1 u_0 t_0^l : 0 \leq l \leq \tfrac{n-3}{2}\right\}$$

if n is odd.

If n is even and m is nonnegative, then a basis of $\bigoplus_{r+s=-m} \mathrm{HH}^r(A, A[s])$ can be given as

$$\left\{t_0^l u_{-1}^m, r_0 t_0^l u_{-1}^m, t_1 t_0^l u_{-1}^{m+1}, r_0 t_1 t_0^l u_{-1}^{m+1} : 0 \leq l \leq \tfrac{n-2}{2}\right\}.$$

If n is odd and m is nonnegative, then

$$\bigoplus_{r+s=-2m} \mathrm{HH}^r(A, A[s]) \quad \text{and} \quad \bigoplus_{r+s=-2m-1} \mathrm{HH}^r(A, A[s])$$

are spanned by

$$\left\{t_0^l t_{-2}^m, r_0 t_0^l t_{-2}^m, u_0 t_0^l t_{-2}^m, u_0 r_0 t_0^l t_{-2}^m : 0 \leq l \leq \tfrac{n-3}{2}\right\}$$

and

$$\left\{u_{-1} t_0^l t_{-2}^m, u_{-1} r_0 t_0^l t_{-2}^m, u_{-1} u_0 t_0^l t_{-2}^m, u_{-1} u_0 r_0 t_0^l t_{-2}^m :\ 0 \leq l \leq \tfrac{n-3}{2}\right\},$$

respectively.

Therefore, the group $\bigoplus_{r+s=*} \mathrm{HH}^r(A_\Gamma, A_\Gamma[s])$ is nontrivial if and only if $* \leq 2$. If the ground field has characteristic 2, the rank is $n + \left\lfloor \frac{n-2}{2} \right\rfloor$ for $* = 2, 1$ and $4\left\lfloor \frac{n}{2} \right\rfloor$ for

$* \leq 0$. Otherwise the rank is n at each $* \leq 2$. Therefore, it follows from Theorem 27 that we have:

Corollary 46 *The symplectic cohomology group* $\mathrm{SH}^*(X_\Gamma)$ *over a field of characteristic* 0 *is of rank* n *if* $* \leq 2$ *and it is trivial otherwise.*

As before, for convenient access, we give tables listing the ranks of a truncated piece of our calculation. As mentioned in Section 6.2.1, A_Γ has a graded periodic resolution as a graded bimodule, from which it follows easily that for $\Gamma = D_n$, $n \geq 4$, the ranks of the Hochschild cohomology groups obeys the following periodicity:

$$\operatorname{rank} \mathrm{HH}^r(A, A[s]) = \operatorname{rank} \mathrm{HH}^{r+(4n-6)}(A, A[s-(4n-4)]) \quad \text{for } r > 0.$$

In this presentation, multiplication by the generator t_{-2} gives rise to this periodicity. The tables below give the truncation, which includes a fundamental domain of the period in the cases $\Gamma = D_4,\ D_5,\ D_6$. We have also performed computer-aided checks in these cases.

$r+s\downarrow \quad s\rightarrow$	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10
2	4	0	0	0	x	0	0	0	0	0	0	0	0
1	0	0	1	0	x	0	2	0	0	0	1	0	0
0	0	0	1	0	0	0	2	0	x	0	1	0	$2x$
−1	0	0	0	0	0	0	0	0	x	0	0	0	$2x$

Table 4: $\Gamma = D_4$; x is 1 if $\operatorname{char} \mathbb{K} = 2$, 0 otherwise

$r+s\downarrow \quad s\rightarrow$	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14
2	5	0	0	0	x	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	x	0	1	0	1	0	1	0	0	0	1	0	0
0	0	0	1	0	0	0	1	0	1	0	1	0	x	0	1	0	x
−1	0	0	0	0	0	0	0	0	0	0	0	0	x	0	0	0	x

Table 5: $\Gamma = D_5$; x is 1 if $\operatorname{char} \mathbb{K} = 2$, 0 otherwise

Remark 47 As a result of the computation for $\Gamma = D_n$, we have $\mathrm{HH}^2(A_\Gamma, A_\Gamma[s]) = 0$ for all s over any field $\mathbb{K}$. This rigidity has a useful implication in Floer theory: namely, if one has a D_n–configuration of Lagrangian spheres S_v in a symplectic 4–manifold M, then the Floer cohomology algebra $\bigoplus_{v,w} \mathrm{HF}^*_M(S_v, S_w)$ is isomorphic to A_Γ, ie it is independent of the symplectic manifold M. Furthermore, if $\operatorname{char} \mathbb{K} \neq 2$, intrinsic formality implies that in fact the A_∞–algebra $\bigoplus_{v,w} \mathrm{CF}^*_M(S_v, S_w)$ is quasi-isomorphic to A_Γ.

7 Conclusion

7.1 Comparison with geometric viewpoint

We would like to discuss the algebraic computations given in Section 6.2.2 in terms of the symplectic geometry of the Milnor fiber X_Γ. We shall omit some of the details, but the geometric setup that we are about to lay out is taken from [59]. Consider $\mathbb{C}^3$ with its standard symplectic form $d\alpha$, where

$$\alpha = -\tfrac{1}{4} d^c(|z_1|^2 + |z_2|^2 + |z_3|^2).$$

Let $p\colon \mathbb{C}^3 \to \mathbb{C}$ be the polynomial

$$p(z_1, z_2, z_3) = z_1^{n+1} + z_2^2 + z_3^2,$$

which has an isolated singularity at the origin of type A_n. Consider also the Hamiltonian function $H\colon \mathbb{C}^3 \to \mathbb{R}$ given by

$$H(z_1, z_2, z_3) = 2|z_1|^2 + (n+1)|z_2|^2 + (n+1)|z_3|^2.$$

Let ψ be a cutoff function such that $\psi(t^2) = 1$ for $t \le \frac{1}{3}$ and $\psi(t^2) = 0$ for $t \ge \frac{2}{3}$. For $u \in \mathbb{C}\backslash\{0\}$ with $0 < |u| < \epsilon$ for sufficiently small ϵ, we consider the Milnor fiber

$$\{z \in \mathbb{C}^3 : p(z) = \psi(H(z))u\}.$$

For sufficiently small ϵ, this is a symplectic submanifold of $\mathbb{C}^3$ and can be symplectically identified with X_Γ. For $r \ge \frac{2}{3}$, we let $L_r = F \cap \{H = r\}$ be the link of the singularity. In other words, for such r, we have

$$L_r = \{z \in \mathbb{C}^3 : 2|z_1|^2 + (n+1)|z_2|^2 + (n+1)|z_3|^2 = r,\ p(z) = 0\}.$$

$r+s \downarrow \quad s \rightarrow$	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
2	6	0	0	0	x	0	0	0	x	0	0
1	0	0	1	0	x	0	1	0	x	0	2
0	0	0	1	0	0	0	1	0	0	0	2
−1	0	0	0	0	0	0	0	0	0	0	0

$r+s \downarrow \quad s \rightarrow$	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18
2	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1	0	0
0	0	x	0	1	0	x	0	1	0	$2x$
−1	0	x	0	0	0	x	0	0	0	$2x$

Table 6: $\Gamma = D_6$; x is 1 if $\operatorname{char} \mathbb{K} = 2$, 0 otherwise

For $r > 0$, L_r inherits a contact structure $\alpha|_{L_r}$ and outside of a compact set X_Γ can be identified with the positive symplectization of L_r. The appealing feature of this setup is that the Reeb vector field R_r on L_r has a periodic flow given by

$$t \cdot (z_1, z_2, z_3) = (e^{4it/r} z_1, e^{2(n+1)it/r} z_2, e^{2(n+1)it/r} z_3).$$

Thus, all the Reeb orbits are along the circle direction of a Seifert fibered structure on the lens space $L_r \cong L(n+1, n)$. Furthermore, since the Reeb flow is explicit, we can actually write down all the orbits. Let us take $Y_\Gamma = L_1$ as our contact boundary. There are two types of simple orbits:

- Generic simple orbits of period $\frac{\pi}{2n+2} \operatorname{lcm}(2, n+1)$. These are orbits through points $(z_1, z_2, z_3) \in Y_\Gamma$ such that $z_1 \neq 0$. The N^{th} multiple cover of these orbits have Conley–Zehnder index $2N$ if n is odd, $4N$ if n is even.
- Exceptional simple orbits of period $\frac{\pi}{n+1}$. These are orbits through points $(0, z_2, z_3) \in Y_\Gamma$. The N^{th} multiple cover of this orbit has Conley–Zehnder index $2\lfloor \frac{2N}{n+1} \rfloor + 1$ except when $2N = M(n+1)$ for some $M \in \mathbb{Z}$, in which case the index is $2M$.

For each $N \in \mathbb{Z}_+$, we can consider N–fold multiple covers of generic simple orbits together with $(n+1)N$–fold $\big($resp. $\frac{(n+1)N}{2}$–fold$\big)$ for n even (resp. n odd) multiple covers of exceptional orbits as parametrized by the manifold $L(n+1, n)$ and the N–fold cover of exceptional orbits for each $N \in \mathbb{Z}_+$ not divisible by $n+1$ $\big($resp. $\frac{n+1}{2}\big)$ for n even (resp. n odd) as parametrized by $S^1 \sqcup S^1$. This leads to a standard Morse–Bott-type spectral sequence converging to $\mathrm{SH}^*(X_\Gamma)$ (see [60] and/or [48] for a more recent exposition). For example, for $n = 2$, the E_1 page is given by

$$(11) \quad E_1^{pq} = \begin{cases} H^q(X_\Gamma; \mathbb{K}) & \text{if } p = 0, \\ H^{q-p-2}((S^1 \sqcup S^1); \mathbb{K}) & \text{if } p = 2l+1 < 0, \\ H^{q-p}(L(3,2); \mathbb{K}) \oplus H^{q-p-2}((S^1 \sqcup S^1); \mathbb{K}) & \text{if } p = 2l < 0, \\ 0 & \text{if } p > 0. \end{cases}$$

The higher differentials come from contributions of holomorphic cylinders counted in the differential of symplectic cohomology. A finite truncation of the E_1 page of this spectral sequence is shown in Table 7.

Comparing this with our results from Section 6.2.2, which correspond to a calculation of the total complex at the E_∞ page of the spectral sequence, gives us information about the holomorphic cylinders contributing to the differential of symplectic cohomology. For example, if $\operatorname{char} \mathbb{K} = 3$, the spectral sequence has to be degenerate but otherwise there has to be a nontrivial differential. See also the appendix of [48] for a similar

$r+s\downarrow$ $s\rightarrow$	2	1	0	-1	-2	-3	-4
2	2	0	0	0	0	0	0
1	2	0	0	0	0	0	0
0	0	2	1	0	0	0	0
-1	0	3	0	0	0	0	0
-2	0	0	$x+2$	0	0	0	0
-3	0	0	2	x	0	0	0
-4	0	0	0	2	1	0	0
-5	0	0	0	3	0	0	0
-6	0	0	0	0	$x+2$	0	0
-7	0	0	0	0	2	x	0
-8	0	0	0	0	0	2	1

Table 7: E_1 page of the Morse–Bott spectral sequence for $\Gamma = A_2$; x is 1 if $\operatorname{char}\mathbb{K} = 3$, 0 otherwise.

spectral sequence obtained via another natural choice of a contact form on the lens space $L(n+1,n)$.

In conclusion, even though this geometric point of view leads to an appealing description of the generators of the chain complex, it seems harder to determine the cohomology this way, let alone its multiplicative structure. However, it is reassuring that the algebraic approach taken in this paper and the geometric picture just outlined are compatible.

7.2 Generalizations

In this paper, we have studied Legendrian links $\Lambda \subset (S^3, \xi_{\mathrm{std}})$ which are obtained by plumbing Legendrian unknots according to a plumbing tree Γ. One might wonder what Koszul duality has to say when Λ is a more general Legendrian submanifold. Of course, one can study this plumbing construction in higher dimensions. Both the Ginzburg DG-algebra and the zigzag algebra have analogues corresponding to higher-dimensional plumbings, and we expect that our calculations can be extended in a straightforward way.

Perhaps a more interesting direction to pursue is the following. One of our main observations was that the Legendrian cohomology DG-algebra of Λ admits a certain natural augmentation $\epsilon\colon \mathrm{LCA}^*(\Lambda) \to \mathrm{k}$ such that

$$\mathrm{RHom}_{\mathrm{LCA}^*(\Lambda)^{\mathrm{op}}}(\mathrm{k},\mathrm{k}) \tag{12}$$

is quasi-isomorphic to a finite-dimensional associative algebra A, whose Hochschild complex is isomorphic to that of $\mathrm{LCA}^*(\Lambda)$ by an A_∞–version of Koszul duality.

One could contemplate generalizing this construction to an arbitrary Legendrian link Λ whose $\mathrm{LCA}^*(\Lambda)$ admits an augmentation ϵ. In general, one cannot expect to have the connectedness and the finiteness conditions required in Theorem 21. Furthermore, in general, $\mathrm{LCA}^*(\Lambda)$ is not graded over $\mathbb{Z}$ but over $\mathbb{Z}/N$ for some $N > 0$. These pose important restrictions, analogous to the assumption of simple connectedness that appears in the classical story discussed in the introduction. One could partially extend Koszul duality to these more general situations if one takes completions with respect to the augmentation ideal.

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