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Maximal representations, non-Archimedean Siegel spaces, and buildings

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Let $\mathbb{F}$ be a real closed field. We define the notion of a maximal framing for a representation of the fundamental group of a surface with values in $\mathrm{Sp}(2n,\mathbb{F})$. We show that ultralimits of maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$ admit such a framing, and that all maximal framed representations satisfy a suitable generalization of the classical collar lemma. In particular, this establishes a collar lemma for all maximal representations into $\mathrm{Sp}(2n,\mathbb{R})$. We then describe a procedure to get from representations in $\mathrm{Sp}(2n,\mathbb{F})$ interesting actions on affine buildings, and in the case of representations admitting a maximal framing, we describe the structure of the elements of the group acting with zero translation length.

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1 Introduction

Let Σ be a connected, orientable surface of genus g with $p \geq 0$ punctures and negative Euler characteristic, and let V be a symplectic vector space over $\mathbb{R}$. A current theme in higher Teichmüller theory is to which extent classical hyperbolic geometry and some fundamental structures on the Teichmüller space of Σ carry over to the geometry and the moduli space of maximal representations of $\Gamma = \pi_1(\Sigma)$ into $\mathrm{Sp}(V)$ or Hitchin representations into $\mathrm{SL}(V)$. For instance, compactifications of spaces of representations of Γ have been introduced and studied by Alessandrini [1], Le [14] and Parreau [22]. In the context of Hitchin representations, asymptotic properties of diverging sequences were studied by Collier and Li [7], Katzarkov, Noll, Pandit and Simpson [10], Loftin [18], Mazzeo, Swoboda, Weiss and Witt [19], Parreau [23] and Zhang [29; 30].

The purpose of this paper is to study the action on an asymptotic cone of the symmetric space $\mathcal{X}$ associated to $\mathrm{Sp}(V)$ defined by a sequence $(\rho_k)_{k\in\mathbb{N}}$ of maximal representations $\rho_k\colon \Gamma \to \mathrm{Sp}(V)$. More precisely, we fix a nonprincipal ultrafilter ω on $\mathbb{N}$ and

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let $(x_k)_{k\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$ be a sequence of basepoints. We say that a sequence of scales $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ is adapted to $(\rho_k, x_k)_{k\in\mathbb{N}}$ if

$$\lim_{\omega} \frac{D_S(\rho_k)(x_k)}{\lambda_k} < \infty.$$

Here for a representation ρ and a finite generating set S for Γ, we define $D_S(\rho)(x) = \max_{\gamma\in S} d(\rho(\gamma)x, x)$, where d denotes the Riemannian distance on $\mathcal{X}$. Observe that the above property is independent of the choice of the finite generating set S.

In this situation, we obtain an action ${}^{\omega}\rho_\lambda\colon \Gamma \to \mathrm{Iso}({}^{\omega}\mathcal{X}_\lambda)$ by isometries on the asymptotic cone ${}^{\omega}\mathcal{X}_\lambda$ of the sequence $(\mathcal{X}, x_k, d/\lambda_k)$. The space ${}^{\omega}\mathcal{X}_\lambda$ is not only CAT(0)-complete, but when the limit $\lim_\omega \lambda_k$ is infinite, it is an affine building associated to the algebraic group $\mathrm{Sp}(V)$ over a specific field (more on this below); see Kleiner and Leeb [11], Kramer and Tent [13], Parreau [20] and Thornton [28]. Depending on the choice of scales, the representation ${}^{\omega}\rho_\lambda$ might have a global fixed point, but as it turns out, if the representations ρ_k are maximal, the limiting action is always faithful. Our main result gives then the underlying geometric structure of the set of elements γ in Γ whose translation length $L({}^{\omega}\rho_\lambda(\gamma))$ in ${}^{\omega}\mathcal{X}_\lambda$ is zero; notice that for an isometry of an affine building, having zero translation length is equivalent to having a fixed point.

For convenience, we fix once and for all a complete hyperbolic metric on Σ of finite area, and identify Γ with a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. In order to state the main result, we recall that a decomposition $\Sigma = \bigcup_{v\in\mathcal{V}} \Sigma_v$ into subsurfaces with geodesic boundary gives rise to a presentation of Γ as fundamental group of a graph of groups with vertex set $\mathcal{V}$ and vertex groups $\pi_1(\Sigma_v)$. The group Γ acts on the associated Bass–Serre tree $\mathcal{T}$ and, in particular, on its vertex set $\widetilde{\mathcal{V}}$; observe that for $v \in \mathcal{V}$ and $w \in \widetilde{\mathcal{V}}$ lying above v, the stabilizer Γ_w of w in Γ is isomorphic to $\pi_1(\Sigma_v)$.

Theorem 1.1 *Let $\rho_k\colon \Gamma \to \mathrm{Sp}(V)$ be a sequence of maximal representations, $(\lambda_k)_{k\geq 1}$ an adapted sequence of scales and ${}^{\omega}\rho_\lambda$ the action of Γ on the asymptotic cone ${}^{\omega}\mathcal{X}_\lambda$. Then ${}^{\omega}\rho_\lambda$ is faithful. Moreover, there is a decomposition $\Sigma = \bigcup_{v\in\mathcal{V}} \Sigma_v$ of Σ into subsurfaces with geodesic boundary such that:*

(1) *for every $\gamma \in \Gamma$ whose corresponding closed geodesic is not contained in any subsurface, $L({}^{\omega}\rho_\lambda(\gamma)) > 0$;*

(2) *for every $v \in \mathcal{V}$, there is the following dichotomy:*

  (PT) *for every $w \in \widetilde{\mathcal{V}}$ lying above v, and any $\gamma \in \Gamma_w$ which is not boundary parallel, ${}^{\omega}\rho_\lambda(\gamma)$ has positive translation length;*

  (FP) *for every $w \in \widetilde{\mathcal{V}}$ lying above v, the stabilizer Γ_w has a common fixed point $b_w \in {}^{\omega}\mathcal{X}_\lambda$.*

A natural question is, given a sequence of maximal representations, how the choice of basepoints and scales influences the action of Γ on the asymptotic cone and, in particular, the decomposition given in Theorem 1.1. Turning to this issue, recall that for a maximal representation $\rho\colon \Gamma \to \mathrm{Sp}(V)$, the displacement function $x \mapsto D_S(\rho)(x)$ with respect to a generating set $S \subset \Gamma$ achieves its minimum $\mu_S(\rho)$ in a compact region of the symmetric space $\mathcal{X}$. Given a sequence $(\rho_k)_{k\in\mathbb{N}}$ of maximal representations, we have $\lim_\omega \mu_S(\rho_k) < \infty$ if and only if, up to modifying the sequence on a set of ω–measure zero, $(\rho_k)_{k\in\mathbb{N}}$ is contained in a compact subset of the character variety of maximal representations.

Assume thus that $\lim_\omega \mu_S(\rho_k) = \infty$. Choosing a sequence $(x_k)_{k\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$ of basepoints such that $D_S(\rho_k)(x_k) = \mu_S(\rho_k)$, the sequence of scales $(\mu_k := \mu_S(\rho_k))_{k\in\mathbb{N}}$ is obviously adapted to the sequence $(\rho_k, x_k)_{k\in\mathbb{N}}$, and the resulting Γ–action ${}^\omega\rho_\mu$ on ${}^\omega\mathcal{X}_\mu$ has no global fixed point. We show then (see Proposition 10.6) that if $(y_k)_{k\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$ is a sequence of basepoints and $(\lambda_k)_{k\in\mathbb{N}}$ is an adapted sequence of scales such that ${}^\omega\rho_\lambda$ has no global fixed point, then ${}^\omega\mathcal{X}_\lambda$ equals ${}^\omega\mathcal{X}_\mu$ with homothetic distance function, and the actions ${}^\omega\rho_\lambda$ and ${}^\omega\rho_\mu$ coincide. In particular, the decomposition of Σ into subsurfaces given by Theorem 1.1 is uniquely determined by the sequence $(\rho_k)_{k\in\mathbb{N}}$.

We say that a subsurface is of type (PT) (resp. (FP)) if the first (resp. the second) possibility in Theorem 1.1(2) holds. One can show that any decomposition of the surface Σ and any assignment of type (PT) or (FP) to the subsurfaces can be realized by the limiting action for an appropriate sequence $(\rho_k)_{k\in\mathbb{N}}$. On the other hand, Theorem 1.1 suggests that in a generic limiting action without a global fixed point, no element of Γ should have zero translation length. We plan on analyzing the properties of such representations in future work.

In case there is a subsurface of type (FP), the restriction $\rho_k|_{\Gamma_w}\colon \Gamma_w \to \mathrm{Sp}(V)$ is a sequence of maximal representations to which the preceding discussion applies; that is, either up to ω–measure zero the sequence is relatively compact in the character variety of Γ_w, or there is an essentially unique choice of basepoints and scales such that the limiting action does not have a global fixed point. Since, at each step, the topological complexity of the surface decreases, this procedure stops after finitely many iterations and can be seen as an asymptotic expansion of the initial sequence $(\rho_k)_{k\in\mathbb{N}}$.

When each subsurface in the decomposition of Theorem 1.1 is of type (FP), we can use the fixed points b_w to construct a map from the Bass–Serre tree $\mathcal{T}$ to the asymptotic cone:

Theorem 1.2 *Assume that for any subsurface of the decomposition, possibility (FP) holds. Then there is a ${}^\omega\rho_\lambda$–equivariant quasi-isometric embedding $\mathcal{T} \to {}^\omega\mathcal{X}_\lambda$.*

In the case of a vector space of dimension 2, maximal representations correspond to holonomies of hyperbolizations; in this case, the second possibility in Theorem 1.1(2) occurs, for example, for sequences of hyperbolizations obtained by pinching a multicurve. In this case, the image of the quasi-isometric embedding of Theorem 1.2 is a simplicial subtree of the asymptotic cone ${}^{\omega}\mathcal{X}_{\lambda}$. In higher rank, it is possible to construct examples in which the image of the Bass–Serre tree is not totally geodesic in the affine building ${}^{\omega}\mathcal{X}_{\lambda}$.

We finish our discussion about ultralimits of maximal representations mentioning two interesting geometric properties of maximal representations that can be deduced from our work. Let S be a connected generating set, namely a generating set for Γ such that the union of the closed geodesics representing the elements of S is a connected subset of Σ, and let $L_S(\rho)$ denote the maximal displacement of an element in the generating set S:

$$L_S(\rho) = \max_{\gamma \in S} L(\rho(\gamma)).$$

Corollary 1.3 *Let $S \subset \Gamma$ be a connected generating set for Γ. Then there is a constant C depending only on S and $2n = \dim V$ such that for any maximal representation $\rho\colon \Gamma \to \mathrm{Sp}(V)$, we have*

$$(\ln 2)\sqrt{n} \le L_S(\rho) \le \mu_S(\rho) \le C L_S(\rho).$$

We say that two diverging sequences of real numbers $(\lambda_k)_{k\in\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$ have the same growth rate according to the ultrafilter ω if $\lim_{\omega} \lambda_k/\mu_k$ is finite and nonzero.

Corollary 1.4 *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of maximal representations of the fundamental group Γ of a surface of genus g with p punctures. Then, varying $\gamma \in \Gamma$, there are at most $8g-8+4p$ distinct growth rate classes among the sequences $L(\rho_k(\gamma))_{k\in\mathbb{N}}$.*

1.1 Real closed fields

The building structure on ${}^{\omega}\mathcal{X}_{\lambda}$ alluded to previously comes about as follows. Assume that the sequence of scales $(\lambda_k)_{k\in\mathbb{N}}$ is unbounded. Then $\sigma = (e^{-\lambda_k})_{k\in\mathbb{N}}$ is an infinitesimal in the field $\mathbb{R}_{\omega}$ of the hyperreals, and the building ${}^{\omega}\mathcal{X}_{\lambda}$ is associated to $\mathrm{Sp}(V \otimes \mathbb{R}_{\omega,\sigma})$ [20; 28]. Here $\mathbb{R}_{\omega,\sigma}$ is the valuation field introduced by Robinson [24]. The characterizing properties of the representations arising as ultralimits of maximal representations make sense in the more general context of symplectic groups over arbitrary real closed fields.[1] When $V_{\mathbb{F}}$ is a symplectic vector space over a real closed field $\mathbb{F}$, the Kashiwara cocycle classifies the orbits of $\mathrm{Sp}(V_{\mathbb{F}})$ on triples $\mathcal{L}(V_{\mathbb{F}})^{(3)}$ of pairwise transverse Lagrangians and can be used to select maximal triples (see Section 2.3 for a precise definition of maximal triples). The general objects of our study are representations which admit a maximal framing:

[1] We refer to Kaplansky [9] for general facts about linear algebra over real closed fields.

Definition 1.5 A representation $\rho\colon \Gamma \to \mathrm{Sp}(V_{\mathbb{F}})$ *admits a maximal framing* if there exist a Γ–invariant subset $S \subset \partial\mathbb{H}^2$ including the fixed points of hyperbolic elements of Γ, and an equivariant map $\phi\colon S \to \mathcal{L}(V_{\mathbb{F}})$, such that for every positively oriented triple (x,y,z) in S^3, the image $(\phi(x),\phi(y),\phi(z))$ is maximal. If we want to emphasize the domain of definition, we will refer to a *maximal S–framing*.

Remark 1.6 For the conclusion of Theorem 1.8 (see below) to hold, the existence of a maximal S–framing for S the set of fixed points of hyperbolic elements is sufficient. However, the fact that the reduction (see Theorem 1.7) of a maximal S–framed representation admits a maximal S–framing will be used in subsequent papers where we study the structure of the real spectrum compactification of maximal representation varieties.

If $\mathbb{F} = \mathbb{R}$, any maximal representation admits a maximal framing (see Burger, Iozzi and Wienhard [6, Theorem 8]), and we show in Corollary 10.4 that this is also true for all ultralimits of maximal representations. Even more, the class of representations admitting a maximal framing is closed under the natural reduction process we are now going to describe. Let $\mathcal{O} \subset \mathbb{F}$ be an order convex local subring.[2] Its quotient by the maximal ideal, denoted by $\mathbb{F}_{\mathcal{O}}$, is real closed as well. Assume now that there exists a symplectic basis of $V_{\mathbb{F}}$ such that $\rho(\Gamma) \subset \mathrm{Sp}(2n,\mathcal{O})$. We can then consider the composition $\rho_{\mathcal{O}}$ of ρ with the quotient homomorphism $\mathrm{Sp}(2n,\mathcal{O}) \to \mathrm{Sp}(2n,\mathbb{F}_{\mathcal{O}})$:

Theorem 1.7 *Assume that $\rho\colon \Gamma \to \mathrm{Sp}(V_{\mathbb{F}})$ admits a maximal S–framing. Then the reduction $\rho_{\mathcal{O}}\colon \Gamma \to \mathrm{Sp}(V_{\mathbb{F}_{\mathcal{O}}})$ admits a maximal S–framing as well.*

Theorem 1.7 allows us in general to obtain well controlled actions on affine buildings. Indeed, for each infinitesimal $\sigma > 0$, the set of elements of $\mathbb{F}$ comparable with σ,

$$\mathcal{O}_\sigma = \{x \in \mathbb{F} : |x| \le \sigma^{-k} \text{ for some } k \in \mathbb{Z}\},$$

forms an order convex subring of $\mathbb{F}$. We denote by $\mathbb{F}_\sigma$ its residue field, which inherits from $\mathcal{O}_\sigma$ an order compatible valuation. As a consequence, to any reductive algebraic group over $\mathbb{F}_\sigma$ is associated an affine Bruhat–Tits building [2]. Since Γ is finitely generated, for each representation $\rho\colon \Gamma \to \mathrm{Sp}(V_{\mathbb{F}})$ and every choice of a basis, it is possible to choose an infinitesimal σ such that $\rho(\Gamma) \subset \mathrm{Sp}(2n,\mathcal{O}_\sigma)$. By passing to the quotient $\rho_\sigma\colon \Gamma \to \mathrm{Sp}(2n,\mathbb{F}_\sigma)$, we get an action on the affine building associated to $\mathrm{Sp}(2n,\mathbb{F}_\sigma)$. The main result for maximal framed representations over real closed fields with valuation is:

[2] The definition of an order convex subring is recalled in Section 5.

Theorem 1.8 *Let $\rho\colon \Gamma \to \mathrm{Sp}(V_{\mathbb{L}})$ be a maximal framed representation, where $\mathbb{L}$ is real closed with order compatible valuation, and let $\mathcal{B}$ be the Bruhat–Tits affine building associated to $\mathrm{Sp}(V_{\mathbb{L}})$. Then the action of Γ on $\mathcal{B}$ satisfies the conclusions of Theorem 1.1.*

When $\mathbb{L}$ is a real closed field with order compatible valuation, we denote by $\mathcal{U}$ the order convex valuation ring with residue field $\mathbb{L}_{\mathcal{U}}$. We already mentioned that the action on the affine building associated to a representation $\rho\colon \Gamma \to \mathrm{Sp}(V_{\mathbb{L}})$ might have a global fixed point. However, when this is the case, it is possible to find a symplectic basis of $V_{\mathbb{L}}$ such that $\rho(\Gamma) \subset \mathrm{Sp}(2n, \mathcal{U})$, and if ρ admits a maximal framing, then it follows from Theorem 1.7 that the reduction $\rho_{\mathcal{U}}\colon \Gamma \to \mathrm{Sp}(2n, \mathbb{L}_{\mathcal{U}})$ has the same property. In particular, this can be used to study the restriction of the representation ρ to the subsurfaces defined in Theorem 1.8.

As a consequence of Theorem 1.8, we get a concrete way of checking if a representation ρ admitting a maximal framing has a global fixed point: if S is a connected generating set for Γ, then ρ has a global fixed point if and only if each element of S has a fixed point (see Corollary 7.6 for a precise formulation of this result and some further comments).

1.2 Tools

We now turn to a short description of the key tools we develop in this paper. In the context of his approach to the compactification of the Teichmüller space [3], Brumfiel studied non-Archimedean hyperbolic planes [4]: for any ordered field $\mathbb{F}$, he associates to $\mathrm{PSL}(2, \mathbb{F})$ a nonstandard hyperbolic plane $\mathbb{H}\mathbb{F}^2$, and for fields with valuation, he introduces a pseudodistance on $\mathbb{H}\mathbb{F}^2$ whose Hausdorff quotient is the $\mathbb{R}$–tree associated to $\mathrm{PSL}(2, \mathbb{F})$. Inspired by Brumfiel's work (see also [13]), we associate to a symplectic group $\mathrm{Sp}(2n, \mathbb{F})$ over a real closed field $\mathbb{F}$ the space

$$\mathcal{X}_{\mathbb{F}} = \{X + iY \mid X, Y \in \mathrm{Sym}(n, \mathbb{F}),\ Y \text{ positive definite}\},$$

where $\mathrm{Sym}(n, \mathbb{F})$ denotes the vector space of symmetric $n \times n$ matrices with coefficients in $\mathbb{F}$. The group $\mathrm{Sp}(2n, \mathbb{F})$ acts on $\mathcal{X}_{\mathbb{F}}$ by fractional linear transformations, and the $\mathrm{Sp}(2n, \mathbb{F})$–space $\mathcal{X}_{\mathbb{F}}$ can be thought of as a nonstandard version of the Siegel upper half-space. Using a matrix-valued cross-ratio, we define, for any two transverse Lagrangians $a, b \in \mathcal{L}(\mathbb{F}^{2n})$, the $\mathbb{F}$*–tube* $\mathcal{Y}_{a,b}$ which is the nonstandard symmetric space associated to the stabilizer in $\mathrm{Sp}(2n, \mathbb{F})$ of the pair (a, b), a group isomorphic to $\mathrm{GL}(n, \mathbb{F})$. In the case of the hyperbolic plane, the $\mathbb{F}$–tubes are just the Euclidean half-circles joining the ideal points a, b. Given a representation $\rho\colon \Gamma \to \mathrm{Sp}(2n, \mathbb{F})$ admitting a maximal framing $\phi\colon S \to \mathcal{L}(\mathbb{F}^{2n})$, we can associate to every hyperbolic element $\gamma \in \Gamma$

the $\mathbb{F}$–tube $\mathcal{Y}_\gamma = \mathcal{Y}_{\phi(\gamma^-),\phi(\gamma^+)}$, where γ^-, γ^+ are the fixed points of γ in $\partial\mathbb{H}^2$. One key property that we exploit is that the intersection pattern of the axes of hyperbolic elements in Γ is reflected in the intersection pattern of the corresponding $\mathbb{F}$–tubes. When the field $\mathbb{F}$ has an order compatible valuation, there is a natural $\mathbb{R}_{\geq 0}$–valued pseudodistance on $\mathcal{X}_\mathbb{F}$, and the relation between cross-ratios and this pseudodistance allows us to quantify the intersection pattern of the $\mathbb{F}$–tubes. Finally, we exploit that the Hausdorff quotient of $\mathcal{X}_\mathbb{F}$ can be identified with the set of vertices of the affine Bruhat–Tits building associated to $\mathrm{Sp}(2n, \mathbb{F})$.

1.3 Collar lemma

We finish this introduction discussing another geometric property of representations admitting a maximal framing, which is at the basis of most of the results we discussed so far. Recall that, since any element $g \in \mathrm{Sp}(V)$ is conjugate to ${}^t g^{-1}$, the set of eigenvalues of a symplectic element is closed with respect to inverse: if λ is an eigenvalue of g, the same is true for λ^{-1}. With a slight abuse of terminology, we say that two hyperbolic elements $\gamma, \eta \in \Gamma < \mathrm{PSL}(2, \mathbb{R})$ intersect if their axes do.

Theorem 1.9 (collar lemma) *Let $\mathbb{F}$ be a real closed field, and let $\rho\colon \Gamma \to \mathrm{Sp}(V_\mathbb{F})$ be a representation admitting a maximal framing. Then if $\gamma \in \Gamma$ is hyperbolic, $\rho(\gamma)$ has no eigenvalue of absolute value 1. Let $|\lambda_1(\gamma)| \geq \cdots \geq |\lambda_n(\gamma)| > 1$ be the eigenvalues of absolute value larger than 1. If the hyperbolic elements γ, η in Γ intersect, then*

$$|\lambda_1(\gamma)|^{2n} \geq \frac{1}{|\lambda_n(\eta)|^2 - 1}, \tag{1}$$

$$\left(\prod_{i=1}^{n} |\lambda_i(\gamma)|^{2/n} - 1\right)\left(\prod_{i=1}^{n} |\lambda_i(\eta)|^{2/n} - 1\right) \geq 1. \tag{2}$$

Here $|\cdot|$ denotes the $\mathbb{F}$–valued absolute value on $\mathbb{F}[i]$, and we count the eigenvalues with their multiplicity as roots of the characteristic polynomial. We immediately get from Theorem 1.9(2):

Corollary 1.10 *Under the same assumptions of Theorem 1.9, we have:*

(1) *If γ is self-intersecting, then $|\lambda_1(\gamma)| \geq \sqrt{2}$.*

(2) *If γ satisfies $|\lambda_1(\gamma)| < \sqrt{2}$, then γ is simple and any η intersecting γ satisfies $|\lambda_1(\eta)| > \sqrt{2}$. In particular, there are at most $(3g - 3 + p)$ conjugacy classes of hyperbolic elements γ with $|\lambda_1(\gamma)| < \sqrt{2}$.*

(3) *There exists $\epsilon > 0$ in $\mathbb{F}$ with $|\lambda_1(\gamma)| > 1 + \epsilon$ for any hyperbolic $\gamma \in \Gamma$.*

As an application of the collar lemma, we establish a uniform discontinuity property of the $\Gamma = \pi_1(\Sigma)$ action on the non-Archimedean Siegel space $\mathcal{X}_{\mathbb{F}}$ by a maximal S–framed representation in the case where Σ has no boundary. Recall here that $\mathbb{F}$ has a natural topology given by the order, and so does $\mathcal{X}_{\mathbb{F}}$ as an open subset of $M(n, \mathbb{F}[i])$. Given an open subset $\mathcal{U} \subset \mathcal{X}_{\mathbb{F}} \times \mathcal{X}_{\mathbb{F}}$ containing the diagonal and $x \in \mathcal{X}_{\mathbb{F}}$, we let $\mathcal{U}_x$ denote the open neighborhood consisting of all $y \in \mathcal{X}_{\mathbb{F}}$ with $(x, y) \in \mathcal{U}$.

Corollary 1.11 *Let $\Gamma = \pi_1(\Sigma)$ where Σ has no boundary, and let $\rho\colon \Gamma \to \mathrm{Sp}(2n, \mathbb{F})$ be a representation admitting a maximal framing. Then there is an open neighborhood of the diagonal $\mathcal{U} \subset \mathcal{X}_{\mathbb{F}} \times \mathcal{X}_{\mathbb{F}}$ which is invariant for the diagonal $\mathrm{Sp}(2n, \mathbb{F})$–action and such that for every $x \in \mathcal{X}_{\mathbb{F}}$,*

$$\rho(\gamma)\mathcal{U}_x \cap \mathcal{U}_x = \varnothing \quad \text{for all } \gamma \in \Gamma \setminus \{e\}.$$

We finish the introduction drawing some consequences of the collar lemma in the case of classical maximal representations. It was established by Siegel in [26, Theorem 3] that, under suitable normalizations, the translation length of an isometry $g \in \mathrm{Sp}(2n, \mathbb{R})$ on the symmetric space $\mathcal{X}_{\mathbb{R}}$ is

$$L(g) = 2\sqrt{\sum_{i=1}^{n} \ln^2 |\lambda_i(g)|}.$$

Using this formula, we get, from Theorem 1.9(2) and the Cauchy–Schwarz inequality, the following:

Corollary 1.12 *Let $\rho\colon \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation. If γ and η intersect, then*

$$(e^{L(\rho(\gamma))/\sqrt{n}} - 1)(e^{L(\rho(\eta))/\sqrt{n}} - 1) \geq 1.$$

In particular, if γ is not simple then $L(\rho(\gamma)) \geq \log(2)\sqrt{n}$.

Using that $e^x - 1 \leq 2x$ for $0 \leq x \leq 1$, we get that, if $L(\rho(\eta)) \leq \sqrt{n}$, then

$$\frac{L(\rho(\gamma))}{\sqrt{n}} \geq \ln\left(\frac{\sqrt{n}}{2L(\rho(\eta))}\right),$$

which exhibits the same asymptotic growth relation as in the Teichmüller setting. However, it is worth remarking that, as opposed to the classical collar lemma, Corollary 1.12 is not just a consequence of the Margulis Lemma: in our setting, the sets of minimal displacement of the isometries $\rho(\gamma)$ and $\rho(\eta)$ do not necessarily intersect. A similar version of the collar lemma in the framework of Hitchin representations has been recently established by Lee and Zhang [15]; see Remark 3.5 for a comparison with our results.

Outline of the paper In Section 2, we define three different models for the nonstandard symmetric space, and we study the action of $\mathrm{Sp}(V)$ on n–tuples of transverse Lagrangians. Section 3 is devoted to the proof of the collar lemma, Theorem 3.3, for representations admitting a maximal framing. The matrix-valued cross-ratio and the $\mathbb{F}$–tubes are introduced and studied in Section 4. In Section 5, we focus on order convex subrings and describe how to obtain representations over the residue field. The main result of the section is Theorem 5.9 (Theorem 1.7 in the introduction), whose proof also exploits the geometric input coming from the collar lemma. In Section 6, we restrict to fields with valuations and use the cross-ratio to describe the projection from the nonstandard symmetric space to the affine Bruhat–Tits building. In Section 7, we initiate our study of elements with zero translation length: to each such element, we associate a pair of canonical fixed points (Proposition 7.1) and give sufficient conditions for these points to coincide (Proposition 7.3). The proof of the decomposition Theorem 1.8 (Theorem 8.1) occupies Section 8, while Theorem 1.2 is proven in Section 9. In the last section of the paper, we discuss the relation between ultralimits of maximal representations and representations in symplectic groups over the Robinson field $\mathbb{R}_{\omega,\sigma}$. This allows us to deduce Theorem 1.1 from the more general Theorem 1.8 and, in the case of closed surfaces, to completely characterize representations in $\mathrm{Sp}(2n,\mathbb{R}_{\omega,\sigma})$ which admit a maximal framing (Theorem 10.5).

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2 Symplectic geometry over real closed fields

2.1 Basic objects

Let V be a $2n$–dimensional vector space over a field $\mathbb{F}$, endowed with a symplectic form $\langle\cdot,\cdot\rangle$. The symplectic group $\mathrm{Sp}(V)$ is the subgroup of elements of $\mathrm{GL}(V)$ preserving the form $\langle\cdot,\cdot\rangle$. Recall that a Lagrangian subspace is a maximal isotropic subspace of V; they form a subset of the Grassmannian $\mathrm{Gr}_n(V)$ of n–dimensional subspaces of V, denoted by $\mathcal{L}(V)$. Whenever a Lagrangian l is fixed, we denote by $\mathcal{L}(V)^l$ the set of Lagrangians transverse to l, and by $\mathcal{Q}(l)$ the vector space of quadratic forms on l.

Given a, b in $\mathcal{L}(V)$ transverse, we recall the construction of an affine chart

$$j_{a,b}\colon \mathcal{Q}(a) \to \mathcal{L}(V)^b.$$

For each element f in $\mathcal{Q}(a)$, we denote by $b_f\colon a \times a \to \mathbb{F}$ the associated symmetric bilinear form. Since a and b are transverse, the symplectic pairing induces an isomorphism of b with the dual of a. We denote by $T_f\colon a \to b$ the unique linear map satisfying

$$\langle v, T_f(w)\rangle = b_f(v, w), \quad \text{for } v, w \in a.$$

The subspace of V defined by

$$j_{a,b}(f) := \{v + T_f(v) \mid v \in a\}$$

is a Lagrangian subspace transverse to b.

Conversely, if l is transverse to b, any vector v in a can be written uniquely as a combination of a vector in b and a vector in l. This allows us to define a linear map $T^l_{a,b}\colon a \to b$ by requiring that $v + T^l_{a,b}(v) \in l$. In turn, we can use $T^l_{a,b}$ to define the quadratic form $Q_{a,l,b}$ on a:

$$Q_{a,l,b}(v) = \langle v, T^l_{a,b}(v)\rangle, \quad v \in a,$$

which satisfies $j_{a,b}(Q_{a,l,b}) = l$.

In the theory of maximal representations, positive-definite quadratic forms play a prominent role. If q_1, q_2 are quadratic forms we will write $q_1 \gg 0$ to indicate that q_1 is positive definite, and $q_1 \gg q_2$ to indicate that the difference $q_1 - q_2$ is positive definite.

2.2 Three models of the Siegel space

The symmetric space associated to the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ was extensively studied by Siegel [26] and is often referred to as the Siegel space. We now show that the three most studied models for the Siegel space can be defined over arbitrary ordered fields, are always equivariantly isomorphic, and give rise to interesting geometries.

We fix an ordered field $\mathbb{F}$. Clearly the polynomial $f(x) = x^2 + 1$ is irreducible in $\mathbb{F}[x]$. We denote by $i \in \overline{\mathbb{F}}$ a root of the polynomial f and by $\mathbb{K}$ the splitting field of f, the degree two extension $\mathbb{K} = \mathbb{F}[i]$. If V is a $2n$–dimensional vector space over $\mathbb{F}$ endowed with a symplectic form $\langle \cdot, \cdot \rangle$, we denote by $V_{\mathbb{K}}$ the "complexification" $V_{\mathbb{K}} = V \otimes \mathbb{K}$ and by $\langle \cdot, \cdot \rangle_{\mathbb{K}}\colon V^2_{\mathbb{K}} \to \mathbb{K}$ the $\mathbb{K}$–linear extension to $V_{\mathbb{K}}$ of $\langle \cdot, \cdot \rangle$.

The first model of the Siegel space consists of *compatible complex structures* on V:

$$\mathbb{X}_V = \{J \in \mathrm{GL}(V) \mid J^2 = -\mathrm{Id},\ \langle J\cdot,\cdot\rangle \text{ is a scalar product}\}.$$

The set $\mathbb{X}_V$ is a semialgebraic subset of $\mathrm{End}(V)$ on which the symplectic group $\mathrm{Sp}(V)$ acts by conjugation. For $J \in \mathbb{X}_V$, we will denote by $(\cdot,\cdot)_J := \langle J\cdot,\cdot\rangle$ the corresponding scalar product.

The second model of the Siegel space corresponds to the image of the *Borel embedding*; see [5, Section 2.1.1; 25]. As in the real case, we realize $\mathbb{X}_V$ as a semialgebraic subset $\mathcal{T}_V$ of $\mathcal{L}(V_{\mathbb{K}})$. Indeed, if $J \in \mathrm{GL}(V)$ is an element of $\mathbb{X}_V$, the complexification $J \otimes \mathbb{I}_{\mathbb{K}}$ is diagonalizable over $\mathbb{K}$. It is easy to verify that the eigenspaces $L_J^{\pm}$ of $J \otimes \mathbb{I}_{\mathbb{K}}$ with respect to the eigenvalues $\pm i$ are elements of $\mathcal{L}(V_{\mathbb{K}})$. If we denote by $\sigma\colon V_{\mathbb{K}} \to V_{\mathbb{K}}$ the complex conjugation with respect to the real form V, we get that $\sigma(L_J^{\pm}) = L_J^{\mp}$. The image $\mathcal{T}_V$ of the Borel embedding can be characterized as the set

$$\mathcal{T}_V = \{L \in \mathcal{L}(V_{\mathbb{K}}) \mid i\langle\cdot,\sigma(\cdot)\rangle_{\mathbb{K}}|_{L\times L} \text{ is positive definite}\}.$$

The group $\mathrm{Sp}(V)$ acts by extension of scalars on $V_{\mathbb{K}}$, preserves the symplectic form $\langle\cdot,\cdot\rangle_{\mathbb{K}}$ and commutes with the complex conjugation σ; thus it acts on $\mathcal{T}_V$.

Lemma 2.1 *The algebraic map*

$$\mathbb{X}_V \to \mathcal{T}_V, \quad J \mapsto L_J^{+},$$

induces an $\mathrm{Sp}(V)$*–equivariant bijection.*

Proof If $v = x + iy$ is an eigenvector for the endomorphism $J \otimes \mathbb{I}_{\mathbb{K}}$ of eigenvalue i, it follows that $y = -Jx$. In particular, the restriction of $i\langle\cdot,\sigma(\cdot)\rangle_{\mathbb{K}}$ satisfies

$$\begin{aligned} i\langle v,\sigma(v)\rangle_{\mathbb{K}} &= i\langle x, iJx\rangle_{\mathbb{K}} + i\langle -iJx, x\rangle_{\mathbb{K}} \\ &= 2\langle Jx, x\rangle, \end{aligned}$$

and this implies that the image of $\mathbb{X}_V$ is contained in $\mathcal{T}_V$.

Conversely, if $L \in \mathcal{L}(V_{\mathbb{K}})$ is such that $i\langle\cdot,\sigma(\cdot)\rangle_{\mathbb{K}}|_{L\times L}$ is positive definite, L is transverse to $\sigma(L)$ since the restriction of the aforementioned Hermitian form to $\sigma(L)$ is negative definite. We denote by J_L the endomorphism of $V_{\mathbb{K}}$ defined by imposing that $J_L(v) = iv$ for each v in L and $J_L(v') = -iv'$ for each $v' \in \sigma(L)$.

Since any element w of V can be written uniquely as $w = v + \sigma(v)$ for some $v \in L$, and in particular, $J_L w = iv - i\sigma(v) = iv + \sigma(iv) \in V$, the endomorphism J_L preserves the real structure V. Let $J := J_L|_V$. Since the Hermitian form $i\langle\cdot,\sigma(\cdot)\rangle_{\mathbb{K}}|_{L\times L}$ is by assumption positive definite, the quadratic form $\langle J\cdot,\cdot\rangle$ is positive definite.

If J is a point in $\mathbb{X}_V$ and g belongs to $\mathrm{Sp}(V)$, then since g commutes with σ, we get that $gL_J^{\pm}$ is the $\pm i$–eigenspace of $gJg^{-1}\otimes\mathbb{I}_{\mathbb{K}}$. It follows that the map $J\mapsto L_J^+$ is $\mathrm{Sp}(V)$–equivariant. □

The third and most concrete model for the Siegel space is the *upper half-space* $\mathcal{X}_{\mathbb{F}}$, a specific set of $\mathbb{K}$–valued symmetric matrices:

$$\mathcal{X}_{\mathbb{F}}=\{X+iY\mid X\in\mathrm{Sym}(n,\mathbb{F}),\ Y\in\mathrm{Sym}^+(n,\mathbb{F})\}.$$

Here $\mathrm{Sym}(n,\mathbb{F})$ denotes the vector space of symmetric $n\times n$ matrices with coefficients in $\mathbb{F}$ and $\mathrm{Sym}^+(n,\mathbb{F})$ denotes the properly convex cone in $\mathrm{Sym}(n,\mathbb{F})$ consisting of positive-definite symmetric matrices.

In order to establish a bijection between $\mathcal{T}_V$ and $\mathcal{X}_{\mathbb{F}}$, we fix a Lagrangian l_∞ in $\mathcal{L}(V)$, a complex structure $J\in\mathbb{X}_V$ and a basis $e_1,\dots,e_n$ of l_∞ which is orthonormal for $(\cdot,\cdot)_J$. The matrix representing the symplectic form with respect to the basis

$$\mathcal{B}=\{e_1,\dots,e_n,-Je_1,\dots,-Je_n\}$$

of V is $\left(\begin{smallmatrix}0&\mathrm{Id}\\-\mathrm{Id}&0\end{smallmatrix}\right)$. Moreover, using the basis $\mathcal{B}$, we can associate to any $2n\times n$ matrix M of maximal rank the n–dimensional subspace of V spanned by the columns of M. We use this to give an explicit identification of $\mathrm{Sym}(n,\mathbb{K})$ with the affine chart of $\mathcal{L}(V_{\mathbb{K}})$ which consists of subspaces transverse to l_∞:

$$\iota\colon \mathrm{Sym}(n,\mathbb{K})\to\mathcal{L}(V_{\mathbb{K}}),\quad Z\mapsto\begin{pmatrix}Z\\\mathrm{Id}\end{pmatrix}.$$

It is easy to verify that if we use the basis $\{-Je_1,\dots,-Je_n\}$ to identify the space $\mathrm{Sym}(n,\mathbb{K})$ with $\mathcal{Q}(Jl_\infty)$, we get that the map ι corresponds to the map j_{Jl_∞,l_∞} described in Section 2.1.

Since the restriction of $i\langle\cdot,\sigma(\cdot)\rangle_{\mathbb{K}}$ to l_∞ is identically zero, every element l of $\mathcal{T}_V$ belongs to the image of ι, and it is easy to verify that the restriction of $i\langle\cdot,\sigma(\cdot)\rangle_{\mathbb{K}}$ to $\iota(X+iY)$ can be represented by the matrix $2Y$. In particular, ι restricts to a bijection between $\mathcal{T}_V$ and $\mathcal{X}_{\mathbb{F}}$. Notice that the restriction of ι to the subset of $\mathbb{F}$–valued symmetric matrices has image in $\mathcal{L}(V)$ and gives a parametrization of the affine chart of $\mathcal{L}(V)$ consisting of Lagrangians transverse to l_∞.

It follows from the identification between $\mathbb{X}_V$ and $\mathcal{X}_{\mathbb{F}}$ that the symplectic group $\mathrm{Sp}(2n,\mathbb{F})$ acts on $\mathcal{X}_{\mathbb{F}}$ by fractional linear transformations:

$$\begin{pmatrix}A&B\\C&D\end{pmatrix}\cdot Z=(AZ+B)(CZ+D)^{-1}.$$

It will be useful in the following to record that, with our choice for a basis of the symplectic form, an element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ belongs to $\mathrm{Sp}(2n, \mathbb{F})$ if and only if

$$^{t}AD - {}^{t}CB = \mathrm{Id}, \quad {}^{t}AC = {}^{t}CA, \quad \text{and} \quad {}^{t}BD = {}^{t}DB.$$

In order to achieve transitivity of the symplectic group on the Siegel upper half-space, we need to restrict to real closed fields:

Definition 2.2 A *real closed field* is an ordered field $\mathbb{F}$ in which every positive element is a square and such that every polynomial in one variable over $\mathbb{F}$ factors into linear and quadratic factors.

Lemma 2.3 *If $\mathbb{F}$ is a real closed field, the symplectic group $\mathrm{Sp}(2n, \mathbb{F})$ acts transitively on $\mathcal{X}_{\mathbb{F}}$.*

Proof Since $\mathbb{F}$ is, by assumption, real closed, every symmetric matrix is diagonalizable by an orthogonal matrix, and as soon as it is positive definite, it admits a unique positive square root [9, Sections 2–4]. Let now $X + iY$ be a point in $\mathcal{X}_{\mathbb{F}}$ and let S be the square root of Y. We have

$$X + iY = \begin{pmatrix} S & XS^{-1} \\ 0 & S^{-1} \end{pmatrix} \cdot i\,\mathrm{Id}. \qquad \square$$

2.3 Action on $\mathbb{F}$–Lagrangians

We now want to understand the action of $\mathrm{Sp}(V)$ on n–tuples of pairwise transverse Lagrangians. We denote this set by $\mathcal{L}(V)^{(n)}$:

$$\mathcal{L}(V)^{(n)} = \{(l_1, \dots, l_n) \in \mathcal{L}(V)^n \mid l_i \pitchfork l_j\}.$$

It is a general fact that, for any field $\mathbb{F}$, the symplectic group acts transitively on pairs of transverse Lagrangians.

Lemma 2.4 *The symplectic group $\mathrm{Sp}(V)$ acts transitively on $\mathcal{L}(V)^{(2)}$.*

Recall from Section 2.1 that, whenever two transverse Lagrangians a, b are fixed, we have an identification $j_{a,b}\colon \mathcal{Q}(a) \cong \mathcal{L}(V)^b$, and we denote by $Q_{a,l,b}$ the inverse image $j_{a,b}^{-1}(l)$. Clearly for any element g in $\mathrm{Sp}(V)$, the quadratic forms Q_{l_1,l_2,l_3} and Q_{gl_1,gl_2,gl_3} are equivalent. As it turns out, the equivalence class of the quadratic form Q_{l_1,l_2,l_3} is a complete invariant of the triple (l_1, l_2, l_3) up to the symplectic group action:

Proposition 2.5 *The triples $(l_1, l_2, l_3), (m_1, m_2, m_3)$ in $\mathcal{L}(V)^{(3)}$ are equivalent modulo the symplectic group action if and only if the quadratic forms Q_{l_1,l_2,l_3} and Q_{m_1,m_2,m_3} are equivalent.*

Proof Since $\mathrm{Sp}(V)$ is transitive on pairs of transverse subspaces, we can assume that $l_1 = m_1 = a$ and $l_3 = m_3 = b$. The result now follows from the fact that the stabilizer in $\mathrm{Sp}(V)$ of the pair a, b is $\mathrm{GL}(n, \mathbb{F})$ acting on $\mathcal{Q}(a)$ by congruence. □

In particular, Sylvester's theorem allows us to count the number of $\mathrm{Sp}(V)$–orbits when the field $\mathbb{F}$ is real closed: since in this case, the signature $\mathrm{sign}(Q)$ is a complete invariant of a quadratic form Q up to equivalence (see [9, Theorem 9]), we have

Corollary 2.6 *Let $\mathbb{F}$ be a real closed field, and let V be a symplectic $\mathbb{F}$–vector space of dimension $2n$. Then there are $n+1$ orbits of $\mathrm{Sp}(V)$ in $\mathcal{L}(V)^{(3)}$.*

A fundamental tool in the study of Lagrangian subspaces is the Kashiwara cocycle, which, at least when $\mathbb{F} = \mathbb{R}$, is also known as the Maslov cocycle:

Definition 2.7 The Kashiwara cocycle is the function

$$\tau\colon \mathcal{L}^3(V) \to \mathbb{Z}, \quad (l_1, l_2, l_3) \mapsto \mathrm{sign}(Q),$$

where Q is the quadratic form on the direct sum $l_1 \oplus l_2 \oplus l_3$ defined by

$$Q(x_1 + x_2 + x_3) = \langle x_1, x_2 \rangle + \langle x_2, x_3 \rangle + \langle x_3, x_1 \rangle.$$

The following properties of the Kashiwara cocycle are well known:

Proposition 2.8 (see [17, Section 1.5]) *Let $(V, \langle\cdot,\cdot\rangle)$ be a $2n$–dimensional symplectic vector space over a real closed field.*

(1) *τ is alternating and invariant for the diagonal action of $\mathrm{Sp}(V)$ on $\mathcal{L}(V)^3$.*

(2) *τ has values in $\{-n, -n+1, \dots, n\}$. On triples consisting of pairwise transverse Lagrangians, it only achieves the values $\{-n, -n+2, \dots, n\}$. If $|\tau(l_1, l_2, l_3)| = n$, then the l_i are pairwise transverse.*

(3) *If (l_1, l_2, l_3) are pairwise transverse, then*

$$\tau(l_1, l_2, l_3) = \mathrm{sign}(Q_{l_1,l_2,l_3}).$$

(4) *τ is a cocycle: for each 4–tuple (l_1, l_2, l_3, l_4), we have*

$$\tau(l_2, l_3, l_4) - \tau(l_1, l_3, l_4) + \tau(l_1, l_2, l_4) - \tau(l_1, l_2, l_3) = 0.$$

The second and the third statement in Proposition 2.8 justify the following definition:

Definition 2.9 A triple $(l_1, l_2, l_3) \in \mathcal{L}(V)^{(3)}$ is *maximal* if Q_{l_1,l_2,l_3} is positive definite. More generally, an n–tuple $(l_1, \ldots, l_n)$ is *maximal* if Q_{l_i,l_j,l_k} is positive definite for any ordered triple of indices $i < j < k$.

Let S^1 denote the unit circle in $\mathbb{C}$ endowed with its canonical orientation as boundary of the unit disc. Given a pair a, b of distinct points, $((a, b))$ denotes the connected component of $S^1 \setminus \{a, b\}$ consisting of the points crossed by a positively oriented C^1–path joining a to b. More generally, if $a, b \in \mathcal{L}(V)$ are transverse, we denote by $((a, b))$ the subset of $\mathcal{L}(V)$ consisting of points c such that the triple (a, c, b) is maximal:

$$((a, b)) = \{c \in \mathcal{L}(V) \mid (a, c, b) \in \mathcal{L}(V)^{(3)} \text{ is maximal}\}.$$

The key property of maximal triples that will be exploited throughout the paper is that they correspond to positive-definite quadratic forms:

Lemma 2.10 (1) *A triple (a, l, b) is maximal if and only if $l = j_{a,b}(q)$ for a positive-definite quadratic form $q \in \mathcal{Q}(a)$.*

(2) *A 4–tuple (l_1, l_2, l_3, l_4) is maximal if and only if it holds that $Q_{l_1,l_2,l_4} \gg 0$ and $Q_{l_1,l_3,l_4} - Q_{l_1,l_2,l_4} \gg 0$.*

Proof This follows from Proposition 2.8 together with the observation that the unipotent radical of the stabilizer in $\mathrm{Sp}(V)$ of b is isomorphic to $\mathrm{Sym}(n, \mathbb{F})$ and acts on $\mathcal{Q}(a)$ by translation. □

We finish this subsection by analyzing the $\mathrm{Sp}(V)$–orbits in $\mathcal{L}(V)^{(4)}$. Using the objects and notation introduced in Section 2.2, we fix a Lagrangian subspace l_∞, a complex structure J and a symplectic basis $\mathcal{B}$ of V of the form

$$\mathcal{B} = \{e_1, \ldots, e_n, -Je_1, \ldots, -Je_n\}.$$

Moreover, when this does not cause confusion, we suppress $\iota\colon \mathrm{Sym}(n, \mathbb{F}) \to \mathcal{L}(V)$ and simply represent an element in $\mathcal{L}(V)^{l_\infty}$ by an $\mathbb{F}$–valued symmetric matrix.

Proposition 2.11 *Let $\mathbb{F}$ be a real closed field, and let $(l_1, l_2, l_3, l_4) \in \mathcal{L}(V)^{(4)}$ be a maximal 4–tuple. Then there exist a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ satisfying $d_1 \geq \cdots \geq d_n > 0$, and an element $g_1 \in \mathrm{Sp}(V)$ such that*

$$g_1(l_1, l_2, l_3, l_4) = (-\mathrm{Id}, 0, D, l_\infty).$$

Moreover, there exists $g_2 \in \mathrm{Sp}(V)$ such that $g_2(l_1, l_2, l_3, l_4) = (-\mathrm{Id}, \Lambda, 0, l_\infty)$, where Λ is diagonal with eigenvalues $-1 < \lambda_i = -d_i/(1 + d_i) < 0$.

Proof Since $\mathrm{Sp}(V)$ is transitive on maximal triples of Lagrangians and the triple $(-\mathrm{Id}, 0, l_\infty)$ is maximal, we have an element $g_1' \in \mathrm{Sp}(V)$ such that $g_1'(l_1, l_2, l_3, l_4) = (-\mathrm{Id}, 0, Z, l_\infty)$ for some positive-definite matrix Z.

It is easy to verify that the stabilizer of the triple $(-\mathrm{Id}, 0, l_\infty)$ in $\mathrm{Sp}(2n, \mathbb{F})$ consists of matrices that have the form

$$\mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{F})}(-\mathrm{Id}, 0, l_\infty) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid A \in O(n) \right\}$$

with respect to the basis $\mathcal{B}$ and acts by congruence. This allows us to conclude: since $\mathbb{F}$ is real closed, every positive-definite matrix Z is orthogonally congruent to a diagonal matrix, namely there exists $A \in O(n)$ with $AZA^{-1} = D$, where $D = \mathrm{diag}(d_1, \dots, d_n)$ with $d_1 \geq \cdots \geq d_n$; see [9, Theorem 48]. Then

$$g_1 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} g_1'$$

satisfies the first assertion. For the second assertion, it is enough to take

$$g_2 = \begin{pmatrix} (\mathrm{Id} + D)^{-1/2} & -D(\mathrm{Id} + D)^{-1/2} \\ 0 & (\mathrm{Id} + D)^{1/2} \end{pmatrix} g_1. \qquad \square$$

An important role in the rest of the paper will be played by *Shilov hyperbolic* elements of $\mathrm{Sp}(V)$. We denote by $|\cdot|\colon \mathbb{K} \to \mathbb{F}^{\geq 0}$ the absolute value $|a + ib| = \sqrt{a^2 + b^2}$.

Definition 2.12 An element $g \in \mathrm{Sp}(V)$ is *Shilov hyperbolic* if there exists a g–invariant decomposition $V = L_g^+ \oplus L_g^-$, with $L_g^\pm \in \mathcal{L}(V)$, such that all eigenvalues of the restriction of g to L_g^- have absolute value strictly smaller than one and all eigenvalues of the restriction of g to L_g^+ have absolute value strictly bigger than one. In this case, we denote by M_g the restriction of g to L_g^+.

Remark 2.13 When V is a real vector space, the set of Lagrangians $\mathcal{L}(V)$ is the Shilov boundary of the symmetric space $\mathcal{T}_V$. Moreover, if $g \in \mathrm{Sp}(V)$ is Shilov hyperbolic, then there exists a Zariski open subset of $\mathcal{L}(V)$, the set of points transverse to L_g^-, which is contracted by g to L_g^+.

3 Representations admitting a maximal framing: the collar lemma

Let Σ be an oriented surface of negative Euler characteristic, genus g and p punctures. As mentioned in the introduction, we endow Σ with a complete hyperbolic metric of finite area and identify it with $\Gamma \backslash \mathbb{H}^2$ where $\mathbb{H}^2$ is the Poincaré upper half-plane.

We now turn to the study of representations $\rho\colon \Gamma \to \mathrm{Sp}(V)$ where V is a symplectic space over a real closed field $\mathbb{F}$. Recall from the introduction that we denote by $S \subseteq \partial\mathbb{H}^2$ any Γ–invariant subset containing all the fixed points of hyperbolic elements in Γ.

Definition 3.1 We say that the representation $\rho\colon \Gamma \to \mathrm{Sp}(V)$ admits a maximal S–framing if there exists an equivariant map $\phi\colon S \to \mathcal{L}(V)$ such that, whenever the triple (x, y, z) in S^3 is positively oriented, the triple of Lagrangians $(\phi(x), \phi(y), \phi(z))$ is maximal.

Remark 3.2 It is a fundamental result [6, Theorem 8] that if $\mathbb{F} = \mathbb{R}$, then any maximal representation admits a maximal framing. In addition, one can take $S = \partial\mathbb{H}^2$ and ϕ either left or right continuous.

In this section, we prove a generalization of the classical collar lemma of hyperbolic geometry to the context of representations which admit a maximal framing. In the case where $\mathbb{F}$ is the field of ordinary reals $\mathbb{R}$, this establishes a collar lemma for all maximal representations and gives a quantitative form of the fact due to Strubel [27] that for every hyperbolic element γ in Γ, the image $\rho(\gamma)$ is Shilov hyperbolic.

Theorem 3.3 (collar lemma) *If $\rho\colon \Gamma \to \mathrm{Sp}(V)$ is a representation admitting a maximal framing, then for every hyperbolic element γ, the image $\rho(\gamma)$ is Shilov hyperbolic. Let a, b be elements of Γ which intersect, and denote by $|\alpha_1| \geq \cdots \geq |\alpha_n| > 1$ the eigenvalues of the restriction of $\rho(a)$ to the attractive invariant Lagrangian $L^+_{\rho(a)}$, and analogously for $|\beta_1| \geq \cdots \geq |\beta_n| > 1$ and $\rho(b)$. Then:*

(1) $(\det M^{2/n}_{\rho(a)} - 1)(\det M^{2/n}_{\rho(b)} - 1) \geq 1$;

(2) $|\beta_1|^{2n} \geq \dfrac{1}{|\alpha_n|^2 - 1}$.

We isolate a useful lemma which is used many times in the proof:

Lemma 3.4 *Let $M \in \mathrm{GL}(n, \mathbb{F})$. Denote by $0 < \tau_n \leq \cdots \leq \tau_1$ the eigenvalues of M^tM and by $|\mu_n| \leq \cdots \leq |\mu_1|$ the absolute values of the eigenvalues of M. Then $\tau_n \leq |\mu_n|^2$ and $\tau_1 \geq |\mu_1|^2$.*

Proof If $S = M^tM$, then $S \gg 0$, and if $(\cdot, \cdot)$ denotes the standard scalar product, we have

$$\tau_n = \min_{v \neq 0} \frac{(Sv, v)}{(v, v)}.$$

Since $(Sv, v) = ({}^tMv, {}^tMv)$, we get $\tau_n \leq ({}^tMv, {}^tMv)/(v, v)$ for every nonzero v. If now μ_n belongs to $\mathbb{F}$, we get the statement applying this inequality to a corresponding eigenvector of ${}^tM$. If instead $\mu_n \in \mathbb{K} \setminus \mathbb{F}$, then there is a two-dimensional subspace $E \cong \mathbb{F}^2$ in $\mathbb{F}^n$ which is invariant under ${}^tM$ and where this latter matrix acts like $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ for some $a, b \in \mathbb{F}$ with $a^2 + b^2 = |\mu_n|^2$. Then for $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in E$, we have

$$\left({}^tM\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), {}^tM\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = (ax + by)^2 + (-bx + ay)^2 = (a^2 + b^2)(x^2 + y^2),$$

which implies the first assertion in the lemma. The second inequality follows from applying the first inequality to ${}^tM^{-1}$ and observing that

$$\max_{v \neq 0} \frac{(Sv, v)}{(v, v)} = \left(\min_{v \neq 0} \frac{(v, v)}{({}^tMv, {}^tMv)}\right)^{-1} = \left(\min_{v \neq 0} \frac{({}^tM^{-1}v, {}^tM^{-1}v)}{(v, v)}\right)^{-1}. \qquad \square$$

Proof of Theorem 3.3 Given two hyperbolic elements $a, b \in \Gamma$, we denote by $\mathrm{ax}(a)$ and $\mathrm{ax}(b)$ the axes of a and b, and by a^+ and b^+ (resp. a^- and b^-) the attractive (resp. repulsive) fixed points of a and b in $\partial\mathbb{H}^2$.

We can assume, without loss of generality, that a and b translate as represented by Figure 1 (left) and that the points $(a^-, b^-, ab^-, a^+, ab^+, ba^+, b^+, ba^-)$ are cyclically positively ordered; see [15, Lemma 2.2].

Let $\phi\colon S \to \mathcal{L}(V)$ be the maximal framing for ρ. Then the six points

$$\big(\phi(b^-),\ \phi(a^+),\ \rho(a)\phi(b^+),\ \rho(b)\phi(a^+),\ \phi(b^+),\ \phi(a^-)\big)$$

in $\mathcal{L}(V)^6$ form a maximal 6–tuple. This implies that they are pairwise transverse and every ordered subtriple forms a maximal triple.

We are going to perform our computations in the upper half-space model. As in Section 2.2, fix a symplectic basis $\{e_1, \ldots, e_n, -Je_1, \ldots, -Je_n\}$ of V, set $l_\infty = \langle e_1, \ldots, e_n \rangle$ and parametrize the set of Lagrangians transverse to l_∞ by symmetric

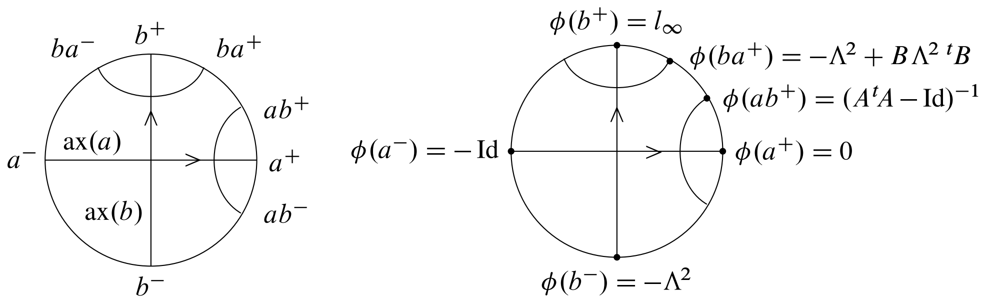


Figure 1: The points involved in the proof of Theorem 3.3

matrices. In view of Proposition 2.11, we may, modulo conjugating ρ, assume that the 4–tuple $(\phi(a^-),\phi(b^-),\phi(a^+),\phi(b^+))$ is equal to $(-\mathrm{Id},-\Lambda^2,0,l_\infty)$, where Λ is diagonal with eigenvalues $0<\lambda_i<1$. Since $\rho(a)$ fixes 0 and $-\mathrm{Id}$, and $\rho(b)$ fixes $-\Lambda^2$ and l_∞, we have

$$\rho(a)=\begin{pmatrix} {}^tA^{-1} & 0\\ -{}^tA^{-1}+A & A\end{pmatrix}\quad\text{and}\quad \rho(b)=\begin{pmatrix} B & B\Lambda^2-\Lambda^2\,{}^tB^{-1}\\ 0 & {}^tB^{-1}\end{pmatrix}$$

for some matrices A,B. Let $\alpha_1,\dots,\alpha_n$ (resp. $\beta_1,\dots,\beta_n$) denote the eigenvalues of A (resp. B) counted with multiplicity and ordered so that $|\alpha_i|\ge|\alpha_{i+1}|$ (resp. $|\beta_i|\ge|\beta_{i+1}|$).

An easy computation gives

$$\begin{aligned}\rho(b)\phi(a^+)&=\rho(b)\cdot 0 &&= -\Lambda^2+B\Lambda^2\,{}^tB,\\ \rho(a)\phi(b^+)&=\rho(a)\cdot l_\infty &&=(A\,{}^tA-\mathrm{Id})^{-1}.\end{aligned}$$

We summarize this information in Figure 1 (right) for the reader's convenience.

The maximality of the triple

$$(\phi(a^+),\phi(ab^+),\phi(b^+))=(0,(A\,{}^tA-\mathrm{Id})^{-1},l_\infty)$$

implies that the quadratic form represented by $(A\,{}^tA-\mathrm{Id})^{-1}$ is positive definite, and in particular, all the eigenvalues of $A\,{}^tA$ are bigger than one. Thus if we denote by $\tau_1\ge\dots\ge\tau_n>1$ the eigenvalues of $A\,{}^tA$, it follows from Lemma 3.4 that $1<\tau_n\le|\alpha_n|^2$, and hence we get that the eigenvalues of A satisfy $1<|\alpha_n|\le\dots\le|\alpha_1|$; in particular, $\rho(a)$ is Shilov hyperbolic.

We now exploit the maximality of the triple

$$(\phi(a^+),\phi(ba^+),\phi(b^+))=(0,B\Lambda^2\,{}^tB-\Lambda^2,l_\infty),$$

which is equivalent to the fact that the quadratic form

$$\Lambda((\Lambda^{-1}B\Lambda)^t(\Lambda^{-1}B\Lambda)-\mathrm{Id})\Lambda=B\Lambda^2\,{}^tB-\Lambda^2$$

is positive definite. Denoting by C the matrix $\Lambda^{-1}B\Lambda$, we get that all the eigenvalues of $C\,{}^tC$ are bigger than 1. Let $1<\sigma_n\le\dots\le\sigma_1$ denote the eigenvalues of $C\,{}^tC$. From Lemma 3.4, we get that the eigenvalues of B satisfy $1<|\beta_n|\le\dots\le|\beta_1|$. This implies that $\rho(b)$ is Shilov hyperbolic as well. Moreover, we have

$$\sigma_1\le\det(C\,{}^tC)=\det(C)^2\le|\beta_1|^{2n}.$$

Last we exploit the maximality of the quadruple

$$(\phi(a^+),\phi(ab^+),\phi(ba^+),\phi(b^+))=(0,(A\,{}^tA-\mathrm{Id})^{-1},B\Lambda^2\,{}^tB-\Lambda^2,l_\infty),$$

which is equivalent to the property that

$$\Lambda\big((\Lambda^{-1}B\Lambda)^t(\Lambda^{-1}B\Lambda)-\mathrm{Id}\big)\Lambda-(A^tA-\mathrm{Id})^{-1}\gg 0; \tag{3}$$

see Lemma 2.10(2).

Taking into account that $1<\sigma_n\le\cdots\le\sigma_1$, we obtain that if $x_n\le\cdots\le x_1$ are the eigenvalues of

$$X=\Lambda\big((\Lambda^{-1}B\Lambda)^t(\Lambda^{-1}B\Lambda)-\mathrm{Id}\big)\Lambda=\Lambda(C^tC-\mathrm{Id})\Lambda,$$

then (3) implies

$$x_i\ge\frac{1}{\tau_{n+1-i}-1},\quad\text{for all } 1\le i\le n. \tag{4}$$

Next we claim that $x_i<(\sigma_i-1)$. Indeed, by the minmax theorem, we have

$$\begin{aligned}x_k&=\min_{\dim W=n+1-k}\ \max_{v\in W\setminus\{0\}}\frac{(\Lambda(C^tC-\mathrm{Id})\Lambda v,v)}{\|v\|^2}\\&=\min_{\dim W=n+1-k}\ \max_{v\in W\setminus\{0\}}\left(\frac{((C^tC-\mathrm{Id})\Lambda v,\Lambda v)}{\|\Lambda v\|^2}\frac{\|\Lambda v\|^2}{\|v\|^2}\right)\\&\le(\sigma_k-1)\max_{v\in\mathbb{F}^n\setminus\{0\}}\frac{\|\Lambda v\|^2}{\|v\|^2}=(\sigma_k-1)\lambda_1^2<\sigma_k-1,\end{aligned}$$

where the last inequality takes into account that $\lambda_1<1$.

Setting $i=1$ in the above inequalities, we obtain $\sigma_1-1\ge 1/(\tau_n-1)$ which, together with the inequalities previously obtained, namely that $|\beta_1|^{2n}\ge\sigma_1$ and $\tau_n\le|\alpha_n|^2$, shows assertion (2).

We establish now the inequality (1). Since $x_i<\sigma_i-1$, we get

$$(\det B)^2=\prod_{i=1}^n\sigma_i>\prod_{i=1}^n(1+x_i),$$

and we deduce from (4) that

$$\prod_{i=1}^n(1+x_i)\ge\prod_{i=1}^n\frac{\tau_i}{(\tau_i-1)}.$$

Since over any real closed field $\mathbb{F}$, one has $\prod_{i=1}^n(a_i^n-1)\le(a_1a_2\cdots a_n-1)^n$ for any $a_1,\dots,a_n>1$ (see Proposition A.1), we deduce, choosing $a_i=\tau_i^{1/n}$, that

$$\left(\prod_{i=1}^n\frac{\tau_i}{(\tau_i-1)}\right)^{1/n}\ge\frac{(\tau_1\cdots\tau_n)^{1/n}}{(\tau_1\cdots\tau_n)^{1/n}-1}.$$

Using $\tau_1\cdots\tau_n=(\det A)^2$, this establishes the first inequality. □

Remark 3.5 In the specific case of a maximal representation with values in $\mathrm{Sp}(2n,\mathbb{R})$ and which in addition belongs to the Hitchin component, assertion (2) is a weaker version of the collar lemma for Hitchin representations proven by Lee and Zhang [15]: their result implies, under these hypotheses, that

$$\beta_1^2 \geq \frac{\alpha_n^2}{(\alpha_n^2-1)}.$$

This is Proposition 2.12(1) in their paper.

3.1 Proper discontinuity on the non-Archimedean Siegel space

We now turn to the topological property of maximal S–framed representations stated in Corollary 1.11; this will follow from the following fact of independent interest.

Proposition 3.6 *There exists a continuous,* $\mathrm{Sp}(V)$*–invariant, multiplicative distance function*

$$D\colon \mathbb{X}_V \times \mathbb{X}_V \to \mathbb{F}_{\geq 1}$$

with the following property: for any $g \in \mathrm{Sp}(V)$ *and* $J \in \mathbb{X}_V$*, we have*

$$D(gJ,J) \geq |\lambda_1(g)|^2,$$

where $|\lambda_1(g)|$ *is the maximum modulus of an eigenvalue of* g.

More precisely, the properties of D alluded to in Proposition 3.6 are

(MD1) $D(J_1,J_2) \geq 1$, with equality if and only if $J_1 = J_2$;

(MD2) $D(J_1,J_2) = D(J_2,J_1)$ for all J_1, J_2;

(MD3) $D(J_1,J_2) \leq D(J_1,J_3)D(J_3,J_2)$ for all J_1, J_2, J_3.

We begin with three observations concerning positive-definite forms Q_1, Q_2 on an $\mathbb{F}$–vector space W. We have that

$$\left|\frac{Q_2}{Q_1}\right| := \max_{x\neq 0} \frac{Q_2(x)}{Q_1(x)} \tag{5}$$

exists and coincides with the largest eigenvalue of the symmetric endomorphism S representing Q_2 with respect to Q_1. If moreover Q_1, Q_2 have the same determinant, that is $\det(S) = 1$, then

$$\left|\frac{Q_2}{Q_1}\right| \geq 1, \quad \text{with equality if and only if } Q_2 = Q_1. \tag{6}$$

The third observation is that if Q is positive definite and $g \in \mathrm{Sp}(V)$, then

$$\max_{x \neq 0} \frac{Q(gx)}{Q(x)} \geq |\lambda_1(g)|^2, \tag{7}$$

as follows immediately from Lemma 3.4 upon taking an orthonormal basis for Q.

Let $\mathbb{X}_V$ be the model of the Siegel space given by the set of compatible complex structures on V (see Section 2.2); given $J \in \mathbb{X}_V$, we let $Q_J(x) := \langle Jx, x \rangle$. Define, for $J_1, J_2 \in \mathbb{X}_V$,

$$D(J_1, J_2) := \max\left(\left| \frac{Q_{J_2}}{Q_{J_1}} \right|, \left| \frac{Q_{J_1}}{Q_{J_2}} \right| \right) \in \mathbb{F}_{\geq 1}.$$

Then D is well defined and continuous by (5); it verifies (MD1) as follows from (6). The $\mathrm{Sp}(V)$–invariance, as well as properties (MD2) and (MD3), are formal verifications. The inequality in Proposition 3.6 is then a direct consequence of (7).

Proof of Corollary 1.11 It follows from Corollary 1.10 and the assumption that Σ has no boundary that there exists $\epsilon > 0$ in $\mathbb{F}$ such that $|\lambda_1(\rho(\gamma))| \geq 1 + \epsilon$ for all $\gamma \in \Gamma \setminus \{e\}$. As a result, we have (Proposition 3.6)

$$D(\rho(\gamma)J, J) \geq (1+\epsilon)^2$$

for all $J \in \mathbb{X}_V$ and $\gamma \in \Gamma \setminus \{e\}$. It follows then from the fact that D is a continuous, $\mathrm{Sp}(V)$–invariant, multiplicative distance that

$$\mathcal{U} = \{(J_1, J_2) \in \mathbb{X}_V \times \mathbb{X}_V \mid D(J_1, J_2) < (1+\epsilon)\}$$

fulfills all the properties of Corollary 1.11. □

4 Cross-ratios and the geometry of $\mathbb{F}$–tubes

4.1 Cross-ratios

We now introduce a useful tool to study the geometry of the Siegel space. Let V be a $2n$–dimensional vector space over a field $\mathbb{L}$. Observe that if a, b are n–dimensional subspaces which are transverse ($a \pitchfork b$), then we have a direct sum decomposition $V = a \oplus b$, and thus we can define the projection $p_a^{/\!/b}\colon V \to a$ onto a parallel to b. Let now (l_1, l_2, l_3, l_4) be a quadruple in $\mathrm{Gr}_n(V)$ with the property that $l_1 \pitchfork l_2$ and $l_3 \pitchfork l_4$.

Definition 4.1 The cross-ratio of (l_1, l_2, l_3, l_4) is the endomorphism of l_1 defined by

$$R(l_1, l_2, l_3, l_4) = p_{l_1}^{/\!/l_2} \circ p_{l_4}^{/\!/l_3}|_{l_1}.$$

The cross-ratio has the following equivariance property: for all $g \in \mathrm{GL}(V)$, we have

$$R(gl_1, gl_2, gl_3, gl_4) = gR(l_1, l_2, l_3, l_4)g^{-1}.$$

It will be useful, in the following, to have an explicit expression for R once a basis $\mathcal{B} = \{e_1, \dots, e_{2n}\}$ of V is fixed. Recall that, as in Section 2.2, the choice of the basis $\mathcal{B}$ allows us to represent an element m of $\mathrm{Gr}_n(V)$ with a $2n \times n$ matrix M of maximal rank: the columns of the matrix M are understood to be the coordinates, with respect to $\mathcal{B}$, of a basis of m. With this notation, we have the following:

Lemma 4.2 *Let us assume that the columns of the matrix $\binom{X_i}{\mathrm{Id}_n}$ form a basis $\mathcal{B}_i$ of the n–dimensional vector space l_i. Then the expression for $R(l_1, l_2, l_3, l_4)$ with respect to the basis $\mathcal{B}_1$ of l_1 is given by*

$$R(l_1, l_2, l_3, l_4) = (X_1 - X_2)^{-1}(X_4 - X_2)(X_4 - X_3)^{-1}(X_1 - X_3).$$

Proof The matrix representing the linear map $p_{l_4}^{/\!/ l_3}|_{l_1}$ with respect to the bases $\mathcal{B}_1$ of l_1 and $\mathcal{B}_4$ of l_4 is the unique $A \in \mathrm{GL}(n, \mathbb{L})$ such that

$$\begin{pmatrix} X_1 \\ \mathrm{Id}_n \end{pmatrix} = \begin{pmatrix} X_4 \\ \mathrm{Id}_n \end{pmatrix} A + \begin{pmatrix} X_3 \\ \mathrm{Id}_n \end{pmatrix} (\mathrm{Id} - A).$$

Solving for A, we obtain

$$A = (X_4 - X_3)^{-1}(X_1 - X_3).$$

Notice that $X_4 - X_3$ is invertible since, by assumption, l_3 and l_4 are transverse.

Similarly, we get that the matrix representing the restriction of the linear map $p_{l_1}^{/\!/ l_2}$ to l_4 with respect to the bases $\mathcal{B}_4$ of l_4 and $\mathcal{B}_1$ of l_1 is given by

$$B = (X_1 - X_2)^{-1}(X_4 - X_2).$$

Since, by definition, the endomorphism $R(l_1, l_2, l_3, l_4)$ is the composition of $p_{l_1}^{/\!/ l_2}$ and $p_{l_4}^{/\!/ l_3}|_{l_1}$, and $p_{l_4}^{/\!/ l_3}|_{l_1}$ has image contained in l_4, we get that

$$R(l_1, l_2, l_3, l_4) = BA,$$

which gives the desired result. □

Let us now fix a basis $\mathcal{B}$ of V, set, as usual, $l_\infty = \langle e_1, \dots, e_n \rangle$ and represent with a matrix $M \in M(n, \mathbb{L})$ the subspace spanned by the columns of $\binom{M}{\mathrm{Id}}$. Here $M(n, \mathbb{L})$ is the set of $n \times n$ matrices. By a similar computation, we have

Lemma 4.3 *Assume 0, Z, X, l_∞ are pairwise transverse. Then*

$$R(0, Z, X, l_\infty) = Z^{-1}X.$$

It will be useful to understand how the cross-ratio varies with respect to permutations of the factors. In particular, we need to be able to compare endomorphisms of different vector spaces. Given two vector spaces l_1, l_2 of the same dimension, we say that two endomorphism $R_1 \in \mathrm{End}(l_1)$ and $R_2 \in \mathrm{End}(l_2)$ are *conjugate* if there exists an isomorphism $g\colon l_1 \to l_2$ such that $gR_1g^{-1} = R_2$. In this case, we write $R_1 \cong R_2$.

Lemma 4.4 *Assume that the subspaces l_i are pairwise transverse. Then*

(1) $R(l_1, l_2, l_4, l_3) = \mathrm{Id} - R(l_1, l_2, l_3, l_4)$;

(2) $R(l_4, l_1, l_2, l_3) \cong (\mathrm{Id} - R(l_1, l_2, l_3, l_4)^{-1})^{-1}$;

(3) $R(l_2, l_3, l_1, l_4) \cong R(l_1, l_4, l_2, l_3) = (\mathrm{Id} - R(l_1, l_2, l_3, l_4))^{-1}$.

Proof (1) By definition, we have

$$p_{l_1}^{/\!/l_2} \circ p_{l_4}^{/\!/l_3}|_{l_1} + p_{l_1}^{/\!/l_2} \circ p_{l_3}^{/\!/l_4}|_{l_1} = p_{l_1}^{/\!/l_2} \circ (p_{l_4}^{/\!/l_3} + p_{l_3}^{/\!/l_4})|_{l_1} = p_{l_1}^{/\!/l_2} \circ \mathrm{Id}\,|_{l_1} = \mathrm{Id}_{l_1}.$$

(2) Up to the $\mathrm{GL}(V)$ action, we can assume that $l_1 = 0$, $l_2 = Z$, $l_3 = X$ and $l_4 = l_\infty$. In particular, $R(0, Z, X, l_\infty) = Z^{-1}X$. In order to compute $R(l_\infty, 0, Z, X)$, we compute $p_{l_\infty}^{/\!/0}|_X = X$ and $p_X^{/\!/Z}|_{l_\infty} = (X - Z)^{-1}$.

(3) Similarly, one gets that $p_Z^{/\!/X}|_{l_\infty} = (Z - X)^{-1}$. The second equality follows from the fact that $p_0^{/\!/l_\infty}|_X = \mathrm{Id}$ and $p_X^{/\!/Z}|_0 = (\mathrm{Id} - Z^{-1}X)^{-1}$. □

4.2 $\mathbb{F}$–tubes

Let $(V, \langle\cdot,\cdot\rangle)$ be a symplectic vector space over a real closed field $\mathbb{F}$. Recall from Section 2.2 that $\mathbb{K}$ denotes the quadratic extension $\mathbb{F}[i]$, that $\sigma\colon \mathcal{L}(V_\mathbb{K}) \to \mathcal{L}(V_\mathbb{K})$ is induced by the complex conjugation with respect to the real structure V of $V_\mathbb{K}$ and that $\mathcal{T}_V$ is the model of the Siegel space contained in $\mathcal{L}(V_\mathbb{K})$. For any pair of transverse Lagrangians (a, b) in $\mathcal{L}(V)^{(2)}$, we introduce here an algebraic subset $\mathcal{Y}_{a,b}$ of the Siegel space $\mathcal{T}_V$ that is determined by the pair (a, b) and whose dimension is half the dimension of $\mathcal{T}_V$. We call such subsets $\mathbb{F}$–tubes. In the case when $\mathbb{F} = \mathbb{R}$, the subsets $\mathcal{Y}_{a,b}$ are Lagrangian submanifolds of the same rank as $\mathcal{X}_\mathbb{R}$; the $\mathbb{F}$–tube $\mathcal{Y}_{a,b}$ can be seen as the higher-rank generalization of a geodesic of the Poincaré model which is more suited to our purposes.

With the notation of Section 2.1, we define

$$\mathcal{Y}_{a,b} = \{l \in \mathcal{T}_V \mid R(a, l, \sigma(l), b) = -\mathrm{Id}\}.$$

Notice that requiring that an endomorphism of a vector space is equal to $-\mathrm{Id}$ does not depend on the choice of a basis. From the equivariance property of the cross-ratio and the fact that the symplectic group commutes with the complex conjugation σ, we deduce that

$$g\mathcal{Y}_{a,b} = \mathcal{Y}_{ga,gb} \quad \text{for any } g \in \mathrm{Sp}(V). \tag{8}$$

Our first goal is to give equations for $\mathcal{Y}_{a,b}$ in the Siegel upper half-space for some specific choice of the pair (a,b). In the sequel, if Z denotes a matrix with coefficients in $\mathbb{K}$, denote by $\overline{Z}$ the matrix obtained applying complex conjugation in $\mathbb{K}$ to all coefficients of Z. If Z is symmetric, this is the same as applying the complex conjugation σ to the corresponding Lagrangian.

Lemma 4.5 *The* $\mathbb{F}$*–tube with endpoints* $0, l_\infty$ *is*

$$\mathcal{Y}_{0,l_\infty} = \{iY \mid Y \in \mathrm{Sym}^+(n,\mathbb{F})\}.$$

Proof It follows from Lemma 4.3 that $R(0, Z, \sigma(Z), l_\infty) = Z^{-1}\overline{Z}$. Clearly we have $Z^{-1}\overline{Z} = -\mathrm{Id}$ if and only if $\overline{Z} = -Z$, and this concludes the proof. □

An immediate consequence of Lemma 4.5 and the equivariance property (8) is that if $\mathbb{F}$ is a real closed field, the stabilizer of $\mathcal{Y}_{a,b}$ is isomorphic to $\mathrm{GL}(n,\mathbb{F})$, and it acts transitively on $\mathcal{Y}_{a,b}$.

It will also be useful to have explicit expression for the set $\mathcal{Y}_{a,b}$ when a and b are transverse to l_∞. This has a particularly nice expression when $a = \langle e_1 - e_{n+1}, \dots, e_n - e_{2n}\rangle$ and $b = \langle e_1 + e_{n+1}, \dots, e_n + e_{2n}\rangle$:

Lemma 4.6 *If* $a, b \in \mathcal{L}(V)$ *correspond to the matrices* $-\mathrm{Id}$ *and* Id*, then*

$$\begin{aligned}\mathcal{Y}_{-\mathrm{Id},\mathrm{Id}} &= U(n) \cap \mathcal{X}_\mathbb{F} \\ &= \{X + iY \in \mathcal{X}_\mathbb{F} \mid YX = XY,\ X^2 + Y^2 = \mathrm{Id}\}.\end{aligned}$$

Proof Lemma 4.2 implies

$$R(-\mathrm{Id}, Z, \sigma(Z), \mathrm{Id}) = (-\mathrm{Id} - Z)^{-1}(\mathrm{Id} - Z)(\mathrm{Id} - \overline{Z})^{-1}(-\mathrm{Id} - \overline{Z}).$$

Since $\mathrm{Id} + \overline{Z}$ and $(\mathrm{Id} - \overline{Z})^{-1}$ commute, the equality $R(-\mathrm{Id}, Z, \sigma(Z), \mathrm{Id}) = -\mathrm{Id}$ reads

$$(\mathrm{Id} - Z)(\mathrm{Id} + \overline{Z}) = -(\mathrm{Id} + Z)(\mathrm{Id} - \overline{Z}),$$

which implies

$$\mathrm{Id} - Z + \overline{Z} - Z\overline{Z} = -\mathrm{Id} + \overline{Z} - Z + Z\overline{Z},$$

and hence, $Z\overline{Z} = Z^*Z = \mathrm{Id}$. □

As a consequence of the explicit parametrization of the sets $\mathcal{Y}_{0,l_\infty}$ and $\mathcal{Y}_{-\mathrm{Id},\mathrm{Id}}$, we obtain:

Proposition 4.7 *Assume that $\mathbb{F}$ is a real closed field. Let $(a,b,c,d)\in\mathcal{L}(V)^{(4)}$ be a maximal 4–tuple. The $\mathbb{F}$–tubes $\mathcal{Y}_{a,c}$ and $\mathcal{Y}_{b,d}$ meet exactly in one point.*

Proof Up to the symplectic group action, we can assume $(a,b,c,d)=(-\mathrm{Id},0,D,l_\infty)$ for some diagonal matrix $D=\mathrm{diag}(d_1,\dots,d_n)$ with $d_i>0$; see Proposition 2.11. Let y be a point in $\mathcal{Y}_{0,l_\infty}\cap\mathcal{Y}_{-\mathrm{Id},D}$. Since y belongs to $\mathcal{Y}_{0,l_\infty}$, we know that y has expression $y=iY$ for some positive-definite matrix Y. From the definition of $\mathcal{Y}_{-\mathrm{Id},D}$, we get

$$(-\mathrm{Id}-iY)^{-1}(D-iY)(D+iY)^{-1}(-\mathrm{Id}+iY)=-\mathrm{Id}.$$

This is equivalent to

$$(D-iY)(D+iY)^{-1}=(\mathrm{Id}+iY)(-\mathrm{Id}+iY)^{-1},$$

which in turn, using that $(\mathrm{Id}+iY)$ and $(-\mathrm{Id}+iY)^{-1}$ commute, is equivalent to

$$(-\mathrm{Id}+iY)(D-iY)=(\mathrm{Id}+iY)(D+iY).$$

This last equation reads $Y^2=D$, which has a unique positive solution. □

Remark 4.8 If the ordered field $\mathbb{F}$ is not real closed, one can similarly get that, if (a,b,c,d) is maximal, the $\mathbb{F}$–tubes $\mathcal{Y}_{a,c}$ and $\mathcal{Y}_{b,d}$ meet in at most one point.

4.3 Reflection with respect to $\mathcal{Y}_{a,b}$

In this subsection, we introduce a notion of orthogonality for $\mathbb{F}$–tubes and establish that the set of $\mathbb{F}$–tubes orthogonal to a fixed one foliate the space $\mathcal{T}_V$. Our main tool will be the characterization of $\mathcal{Y}_{a,b}$ as the fixed point set of an involution $\sigma_{a,b}$ which we now define. Let a,b be transverse Lagrangians in $\mathcal{L}(V)$. We consider the real form $V_{a,b}$ of $V_\mathbb{K}$ given by

$$V_{a,b}=\langle v+iw \mid v\in a,\ w\in b\rangle,$$

and denote by $\sigma_{a,b}$ the complex conjugation of $V_\mathbb{K}$ fixing $V_{a,b}$. The following properties of $\sigma_{a,b}$ can be checked easily:

Lemma 4.9 (1) *$\sigma_{a,b}$ is $\mathbb{K}$–antilinear;*

(2) *$\sigma_{a,b}\sigma=\sigma\sigma_{a,b}$, and in particular, $\sigma_{a,b}$ preserves V;*

(3) *$\langle\sigma_{a,b}(\cdot),\sigma_{a,b}(\cdot)\rangle_\mathbb{K}=-\overline{\langle\cdot,\cdot\rangle}_\mathbb{K}$;*

(4) *$g\sigma_{a,b}=\sigma_{ga,gb}\,g$, for every g in $\mathrm{Sp}(V)$.*

As a consequence of the first two facts of Lemma 4.9, we get that $\sigma_{a,b}$ induces a map on $\mathrm{Gr}_n(V)$ that, with a slight abuse of notation, will be also denoted by $\sigma_{a,b}$. The third fact of Lemma 4.9 implies that $\sigma_{a,b}$ restricts to a map

$$\sigma_{a,b}\colon \mathcal{L}(V_{\mathbb{K}}) \to \mathcal{L}(V_{\mathbb{K}}),$$

which preserves the subspaces we are interested in:

Lemma 4.10 *The involution $\sigma_{a,b}$ preserves the subspaces $\mathcal{T}_V$ and $\mathcal{L}(V)$ of $\mathcal{L}(V_{\mathbb{K}})$. It commutes with the cross-ratio.*

Proof Since the $\mathbb{F}$–linear map $\sigma_{a,b}$ preserves V, the induced map on $\mathcal{L}(V_{\mathbb{K}})$ preserves the subspace $\mathcal{L}(V)$. The fact that $\sigma_{a,b}$ induces a map of $\mathcal{T}_V$ can be seen from the following computation which uses Lemma 4.9(3): for every $v, w \in V_{\mathbb{K}}$,

$$\begin{aligned} i\langle \sigma_{a,b}(v), \sigma_{a,b}(w)\rangle_{\mathbb{K}} &= -i\,\overline{\langle v, w\rangle}_{\mathbb{K}} \\ &= \overline{i\,\langle v, w\rangle}_{\mathbb{K}}. \end{aligned}$$

In particular, the restriction of $i\langle\,\cdot\,,\sigma(\,\cdot\,)\rangle_{\mathbb{K}}$ to a Lagrangian $l \in \mathcal{L}(V_{\mathbb{K}})$ is positive definite if and only if its restriction to $\sigma_{a,b}(l)$ is.

For any pair $a, b \in \mathcal{L}(V)^{(2)}$ and for any 4–tuple (l_1, l_2, l_3, l_4) in the domain of the definition of R, we have

$$\sigma_{a,b} R(l_1, l_2, l_3, l_4)\sigma_{a,b} = R(\sigma_{a,b}(l_1), \sigma_{a,b}(l_2), \sigma_{a,b}(l_3), \sigma_{a,b}(l_4));$$

this follows from the equivariance property of the cross-ratio and that $\sigma_{a,b}^2 = \mathrm{Id}$. □

It is easy to check from the very definition of σ_{0,l_∞} that for any $Z \in \mathcal{X}_{\mathbb{F}}$, we have $\sigma_{0,l_\infty}(Z) = -\overline{Z}$. In particular, $\mathcal{Y}_{0,l_\infty} = \mathcal{T}_V \cap \mathrm{Fix}(\sigma_{0,l_\infty})$. An immediate corollary of the transitivity of the symplectic group action on $\mathcal{L}(V)^{(2)}$ is the following:

Corollary 4.11 *For any pair (a, b), we have $\mathcal{Y}_{a,b} = \mathcal{T}_V \cap \mathrm{Fix}(\sigma_{a,b})$.*

Another useful characterization of the $\mathbb{F}$–tubes is the following:

Lemma 4.12 *In the model $\mathbb{X}_V$,*

$$\mathcal{Y}_{a,b} = \{J \in \mathbb{X}_V \mid a \text{ and } b \text{ are orthogonal for } \langle J\,\cdot\,,\cdot\,\rangle\}.$$

Proof In the notation of Section 2, let $J \in \mathbb{X}_V$. Then $\sigma_{a,b}(L_J^+) = L_J^+$ if and only if $\sigma_{a,b}(L_J^-) = L_J^-$. Hence, since $\sigma_{a,b}$ is $\mathbb{K}$–antilinear, we deduce $\sigma_{a,b}(J \otimes \mathbb{I}_{\mathbb{K}}) = -(J \otimes \mathbb{I}_{\mathbb{K}})\sigma_{a,b}$, which, by restriction to $V = a \oplus b$, is equivalent to $\sigma_{a,b} J = -J\sigma_{a,b}$. The latter is equivalent to $J(a) = b$; that is, a and b are orthogonal with respect to $\langle J\,\cdot\,,\cdot\,\rangle$. □

The restriction of $\sigma_{a,b}$ to the subset of $\mathcal{L}(V)$ consisting of points that are transverse to a and b can also be characterized in term of the cross-ratio:

Proposition 4.13 *For each $c \in \mathcal{L}(V)$ transverse to a and b, we have that $\sigma_{a,b}(c)$ is the unique point satisfying*

$$R(a, c, \sigma_{a,b}(c), b) = -\operatorname{Id}.$$

Proof Up to the symplectic group action, we can assume that $a = 0$ and $b = l_\infty$. Since c is transverse to l_∞, it can be represented by a symmetric matrix S with coefficients in $\mathbb{F}$. The formula of Lemma 4.3 implies that $R(0, S, \sigma_{0,l_\infty}(S), l_\infty) = S^{-1}\sigma_{0,l_\infty}(S)$, and hence the unique point satisfying $R(0, S, \sigma_{0,l_\infty}(S), l_\infty) = -\operatorname{Id}$ is $-S$. □

When $\mathbb{F} = \mathbb{R}$ and the 4–tuple (a, b, c, d) is maximal, the two $\mathbb{R}$–tubes $\mathcal{Y}_{a,c}$ and $\mathcal{Y}_{b,d}$ are orthogonal as totally geodesic submanifolds of the Riemannian manifold $\mathcal{X}_\mathbb{R}$ precisely when $R(a, b, c, d) = 2\operatorname{Id}$. For arbitrary real closed fields, we take this property as a definition of orthogonality.

Definition 4.14 Let (a, b, c, d) be maximal. Two $\mathbb{F}$–tubes $\mathcal{Y}_{a,c}$ and $\mathcal{Y}_{b,d}$ are *orthogonal* if $R(a, b, c, d) = 2\operatorname{Id}$. In this case, we write $\mathcal{Y}_{a,c} \perp \mathcal{Y}_{b,d}$.

Notice that the orthogonality relation is symmetric since $R(d, a, b, c)$ is conjugate to $(\operatorname{Id} - R(a, b, c, d)^{-1})^{-1}$; see Lemma 4.4(2). The following lemma is a consequence of the property of the cross-ratio established in Lemma 4.4(1) and the characterization of the involution $\sigma_{a,b}$ in terms of the cross-ratio given in Proposition 4.13:

Lemma 4.15 *Let (a, b, c, d) be a maximal quadruple in $\mathcal{L}(V)^{(4)}$. The following are equivalent:*

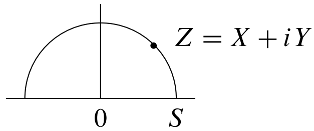
(1) $\mathcal{Y}_{a,c} \perp \mathcal{Y}_{b,d}$;
(2) $d = \sigma_{a,c}(b)$;
(3) $c = \sigma_{b,d}(a)$.

We now turn to an important geometric feature of the Siegel upper half-space, namely that the $\mathbb{F}$–tubes orthogonal to any fixed $\mathbb{F}$–tube foliate the whole space. We first verify this in a special case:

Proposition 4.16 *Assume that $\mathbb{F}$ is real closed. For any $Z = X + iY \in \mathcal{T}_V$, there exists a unique S in $\mathcal{L}(V)^{l_\infty}$ such that $(0, S, l_\infty)$ is maximal and $Z \in \mathcal{Y}_{-S,S}$. Moreover,*

$$S = Y^{1/2}\sqrt{\operatorname{Id} + (Y^{-1/2}XY^{-1/2})^2}\, Y^{1/2}.$$

Proof Given $Z = X + iY$, we look for a positive-definite matrix S with $Z \in \mathcal{Y}_{-S,S}$; see the following picture:



Denoting by $a(S^{-1/2})$ the element of $\mathrm{Sp}(2n, \mathbb{F})$ represented by the matrix $\begin{pmatrix} S^{-1/2} & 0 \\ 0 & S^{1/2} \end{pmatrix}$, we have $a(S^{-1/2})\mathcal{Y}_{-S,S} = \mathcal{Y}_{-\mathrm{Id},\mathrm{Id}}$. The condition $a(S^{-1/2})Z \in \mathcal{Y}_{-\mathrm{Id},\mathrm{Id}}$ leads, in view of the equations of Lemma 4.6, to

$$\begin{cases} (S^{-1/2}XS^{-1/2})(S^{-1/2}YS^{-1/2}) = (S^{-1/2}YS^{-1/2})(S^{-1/2}XS^{-1/2}), \\ (S^{-1/2}XS^{-1/2})^2 + (S^{-1/2}YS^{-1/2})^2 = \mathrm{Id}. \end{cases}$$

From the first equation, observing that Y is invertible, we get

$$XS^{-1} = YS^{-1}XY^{-1}.$$

Substituting this last equality in the second equation, and defining the matrix $V := Y^{-1/2}SY^{-1/2}$, we get

$$V^{-1}((Y^{-1/2}XY^{-1/2})^2 + \mathrm{Id}) = V,$$

which implies

$$V = \sqrt{\mathrm{Id} + (Y^{-1/2}XY^{-1/2})^2} \quad \text{and} \quad S = Y^{1/2}\sqrt{\mathrm{Id} + (Y^{-1/2}XY^{-1/2})^2}\, Y^{1/2}.$$

This shows the formula and implies uniqueness. □

Since all $\mathbb{F}$–tubes are $\mathrm{Sp}(V)$–conjugate, we obtain:

Corollary 4.17 *For any transverse pair $(a, b) \in \mathcal{L}(V)^{(2)}$ and any $z \in \mathcal{T}_V$, there exists a unique $c \in \mathcal{L}(V)$ such that (a, c, b) is maximal and z belongs to $\mathcal{Y}_{c,\sigma_{a,b}(c)}$.*

Corollary 4.17 allows us to define the orthogonal projection

$$\mathrm{pr}_{\mathcal{Y}_{a,b}}\colon \mathcal{T}_V \cup ((a, b)) \cup ((b, a)) \to \mathcal{Y}_{a,b}$$

as follows:

(1) if $c \in ((a, b)) \cup ((b, a))$, then we set $\mathrm{pr}_{\mathcal{Y}_{a,b}}(c) = \mathcal{Y}_{c,\sigma_{a,b}(c)} \cap \mathcal{Y}_{a,b}$;

(2) if $Z \in \mathcal{T}_V$, then we set $\mathrm{pr}_{\mathcal{Y}_{a,b}}(Z) = \mathcal{Y}_{c,\sigma_{a,b}(c)} \cap \mathcal{Y}_{a,b}$, where c is the unique Lagrangian in $\mathcal{L}(V)$ such that (a, c, b) is maximal and $Z \in \mathcal{Y}_{c,\sigma_{a,b}(c)}$.

It is easy to check that, when restricted to its set of definition in $\mathcal{L}(V)$, the orthogonal projection respects cross-ratios:

Lemma 4.18 *Let (a, b) be a pair of transverse Lagrangians, and let x, y be points in $((a, b))$. Then we have*

$$R(a, x, y, b) = R(a,\ \mathrm{pr}_{\mathcal{Y}_{a,b}}(x),\ \mathrm{pr}_{\mathcal{Y}_{a,b}}(y),\ b).$$

Proof Up to the symplectic group action, we can assume that $a=0$ and $b=l_\infty$. In that case, the result follows from the explicit formula for the cross-ratio and for the orthogonal projection. □

5 Reduction modulo an order convex subring

5.1 Order convex subrings

Let $\mathbb{F}$ be a real closed, non-Archimedean field. We denote by $\mathcal{O}<\mathbb{F}$ an *order convex* subring. This means that $\mathcal{O}$ is a subring with the additional property that for every positive element x in $\mathbb{F}$, if there exists y in $\mathcal{O}$ with $0<x<y$, then x belongs to $\mathcal{O}$ as well. It is easy to verify that, in this case, $\mathcal{O}$ is a local ring whose maximal ideal $\mathcal{I}$ is given by

$$\mathcal{I}=\{x\in\mathcal{O}\mid x^{-1}\notin\mathcal{O}\}.$$

We will denote by $\mathbb{F}_\mathcal{O}$ the quotient field $\mathbb{F}_\mathcal{O}:=\mathcal{O}/\mathcal{I}$. The field $\mathbb{F}_\mathcal{O}$ is real closed as well. The following examples of order convex subrings will play an important role in the sequel:

Example 5.1 Let $\sigma\in\mathbb{F}$ be an infinitesimal: this means that σ is a positive element satisfying $\sigma<1/n$ for any integer n. An example of an order convex subring of $\mathbb{F}$ is given by the set of elements comparable to σ:

$$\mathcal{O}_\sigma=\{x\in\mathbb{F}:|x|<\sigma^{-k}\text{ for some }k\in\mathbb{N}\};$$

in this case, the maximal ideal can also be characterized as

$$\mathcal{I}_\sigma=\{x\in\mathbb{F}:|x|<\sigma^{k}\text{ for all }k\in\mathbb{N}\}.$$

Example 5.2 Let us assume that $\mathbb{F}$ admits an order compatible valuation v. An example of order convex subring is given by the elements with nonnegative valuation

$$\mathcal{U}=\{x\in\mathbb{F}\mid v(x)\geq 0\},$$

and the maximal ideal can be characterized as

$$\mathcal{M}=\{x\in\mathbb{F}\mid v(x)>0\}.$$

5.2 $\mathcal{O}$–points

Let $\mathcal{O}$ be an order convex subring of $\mathbb{F}$, and let W be a finite-dimensional $\mathbb{F}$–vector space equipped with an $\mathbb{F}$–valued scalar product $(\cdot,\cdot)$. Then we set

$$W(\mathcal{O})=\{v\in W\mid (v,v)\in\mathcal{O}\}\quad\text{and}\quad W(\mathcal{I})=\{v\in W\mid (v,v)\in\mathcal{I}\}.$$

They are $\mathcal{O}$–submodules; if $e_1, \dots, e_m$ is any orthonormal basis of W, then one verifies

$$W(\mathcal{O}) = \sum_{i=1}^{m} \mathcal{O} e_i \quad \text{and} \quad W(\mathcal{I}) = \sum_{i=1}^{m} \mathcal{I} e_i.$$

This implies that the quotient $W_{\mathcal{O}} = W(\mathcal{O})/W(\mathcal{I})$ is an $\mathbb{F}_{\mathcal{O}}$–vector space of dimension $m = \dim(W)$, that the scalar product $(\cdot,\cdot)$ descends to a well-defined scalar product $(\cdot,\cdot)_{\mathcal{O}}$ on $W_{\mathcal{O}}$ and that, if $p_{\mathcal{O}}\colon W(\mathcal{O}) \to W_{\mathcal{O}}$ denotes the quotient map, $\{p_{\mathcal{O}}(e_1), \dots, p_{\mathcal{O}}(e_m)\}$ is again an orthonormal basis of $W_{\mathcal{O}}$. Notice, however, that the map $p_{\mathcal{O}}$ depends on the choice of the scalar product on W.

The subgroup

$$\mathrm{GL}(W)(\mathcal{O}) := \{g \in \mathrm{GL}(W) \mid g(W(\mathcal{O})) = W(\mathcal{O})\}$$

preserves $W(\mathcal{I})$, and we obtain this way a natural homomorphism $\pi_{\mathcal{O}}\colon \mathrm{GL}(W)(\mathcal{O}) \to \mathrm{GL}(W_{\mathcal{O}})$. The choice of an orthonormal basis of W induces an identification of the group $\mathrm{GL}(W)(\mathcal{O})$ with $\mathrm{GL}(m, \mathcal{O})$.

Let $\mathcal{Q}(W)$ be the vector space of $\mathbb{F}$–valued quadratic forms on W. As in Section 2, we associate to $f \in \mathcal{Q}(W)$ the symmetric bilinear form $b_f(\cdot,\cdot)$. We fix a basis $e_1, \dots, e_m$ of W which is orthonormal for $(\cdot,\cdot)$ and let $(A_f)_{ij} = b_f(e_i, e_j)$ be the associated symmetric matrix. We endow $\mathcal{Q}(W)$ with the scalar product $(f, g) = \mathrm{tr}(A_f A_g)$. Our next task is to understand the relationship between $\mathcal{Q}(W_{\mathcal{O}})$ and $\mathcal{Q}(W)_{\mathcal{O}}$.

Lemma 5.3 *For a quadratic form $f \in \mathcal{Q}(W)$, the following are equivalent:*

(1) $f \in \mathcal{Q}(W)(\mathcal{O})$;

(2) $f(W(\mathcal{O})) \subseteq \mathcal{O}$;

(3) $b_f(W(\mathcal{O}), W(\mathcal{O})) \subseteq \mathcal{O}$ *and* $b_f(W(\mathcal{O}), W(\mathcal{I})) \subseteq \mathcal{I}$.

Proof Clearly $\|f\|^2 = \mathrm{tr}(A_f^2) = \sum (A_f)_{ij}^2$ belongs to $\mathcal{O}$ if and only if $(A_f)_{ij}$ belongs to $\mathcal{O}$ for all i, j, which easily implies the desired equivalences. □

Thus, if f belongs to $\mathcal{Q}(W)(\mathcal{O})$, then b_f induces an $\mathbb{F}_{\mathcal{O}}$–valued bilinear symmetric form $\overline{b_f}$ on $W_{\mathcal{O}}$. In turn, $\overline{b_f}$ defines a quadratic form $\overline{f} \in \mathcal{Q}(W_{\mathcal{O}})$. If A_f is the matrix of f with respect to the orthonormal basis $\{e_1, \dots, e_m\}$, then the matrix $A_{\overline{f}}$ representing $\overline{f}$ with respect to the basis $\{p_{\mathcal{O}}(e_1), \dots, p_{\mathcal{O}}(e_m)\}$ is just the reduction modulo $\mathcal{I}$ of the matrix A_f. With this at hand, one verifies easily that the map

$$\overline{p}_{\mathcal{O}}\colon \mathcal{Q}(W)(\mathcal{O}) \to \mathcal{Q}(W_{\mathcal{O}}), \quad f \mapsto \overline{f},$$

induces an isomorphism of $\mathbb{F}_{\mathcal{O}}$–vector spaces

$$\mathcal{Q}(W)_{\mathcal{O}} \to \mathcal{Q}(W_{\mathcal{O}}).$$

We end the discussion concerning quadratic forms with the following remark:

Remark 5.4 Let $f \in \mathcal{Q}(W)$, and let $\{e_1, \dots, e_n\}$ be an orthonormal basis in which f is diagonal, that is, $b_f(e_i, e_j) = \lambda_i \delta_{ij}$. Let

$$m_f = \operatorname{card}\{i \mid \lambda_i > 0\},$$
$$n_f = \operatorname{card}\{i \mid \lambda_i < 0\},$$
$$z_f = \operatorname{card}\{i \mid \lambda_i = 0\}.$$

Then clearly $m_{\bar{f}} \le m_f$, $n_{\bar{f}} \le n_f$ and $z_{\bar{f}} \ge z_f$.

There is also a reduction process for Grassmannians, and it will play an important role for the construction of framings. Thus let $L \in \mathrm{Gr}_l(W)$ be an l–dimensional subspace of W. Then $L(\mathcal{O}) = L \cap W(\mathcal{O})$, and if $e_1, \dots, e_l$ is an orthonormal basis of L, we have $L(\mathcal{O}) = \mathcal{O}e_1 + \dots + \mathcal{O}e_l$. This implies that the image $p_{\mathcal{O}}(L)$ of $L(\mathcal{O})$ in $W_{\mathcal{O}}$ is an $\mathbb{F}_{\mathcal{O}}$–vector subspace of dimension l. In this way, we obtain a map $q_{\mathcal{O}}\colon \mathrm{Gr}_l(W) \to \mathrm{Gr}_l(W_{\mathcal{O}})$ which is equivariant with respect to $\pi_{\mathcal{O}}\colon GL(W)(\mathcal{O}) \to \mathrm{GL}(W_{\mathcal{O}})$.

Remark 5.5 The map $q_{\mathcal{O}}$ does not preserve transversality: if $V = \mathbb{F}^2$ with the standard scalar product, and x is a nonzero element of $\mathcal{I}$, the two distinct lines $\mathbb{F} \cdot (1, 0)$ and $\mathbb{F} \cdot (1, x)$ of $\mathbb{P}V$ have the same image in $\mathbb{P}V_{\mathcal{O}}$.

We apply now the preceding remarks to the following situation. Let V be a $\mathbb{F}$–vector space with a symplectic form $\langle \cdot, \cdot \rangle$, and fix a compatible complex structure J. We will use the associated scalar product $(\cdot, \cdot) := \langle J\cdot, \cdot \rangle$ to define the $\mathcal{O}$ points. If L is a Lagrangian, then JL is orthogonal to L, and if $\{e_1, \dots, e_n\}$ is an orthonormal basis of L, the basis $\mathcal{B} = \{e_1, \dots, e_n, -Je_1, \dots, -Je_n\}$ is orthonormal and symplectic. With this at hand, one shows readily that $J \in \mathrm{Sp}(V)(\mathcal{O}) := \mathrm{Sp}(V) \cap \mathrm{GL}(V)(\mathcal{O})$, and that $\langle \cdot, \cdot \rangle$ induces a symplectic form $\langle \cdot, \cdot \rangle_{\mathcal{O}}$ of $V_{\mathcal{O}}$ compatible with $p_{\mathcal{O}}\colon V(\mathcal{O}) \to V_{\mathcal{O}}$. If in addition, one sets $J_{\mathcal{O}} = \pi_{\mathcal{O}}(J)$, then $J_{\mathcal{O}}$ is a complex structure on $V_{\mathcal{O}}$ compatible with $\langle \cdot, \cdot \rangle_{\mathcal{O}}$ and with associated scalar product $(\cdot, \cdot)_{\mathcal{O}}$. From the above, it follows that, if $L \in \mathrm{Gr}_n(V)$ is a Lagrangian, then $q_{\mathcal{O}}(L) \in \mathrm{Gr}_n(V_{\mathcal{O}})$ is a Lagrangian as well.

Lemma 5.6 *The map*

$$q_{\mathcal{O}}\colon \mathcal{L}(V) \to \mathcal{L}(V_{\mathcal{O}})$$

is surjective.

Proof Let L_0 be a k–dimensional totally isotropic subspace of V, and let $v_0 \in V$ be such that $\langle v, v_0 \rangle \in \mathcal{I}$ for all $v \in L_0$. Let $e_1, \dots, e_k$ be an orthonormal basis of L_0. By completing it to a symplectic basis of V, it is easy to verify that the map

$$V(\mathcal{I}) \to \mathcal{I}^k, \quad w \mapsto (\langle e_1, w \rangle, \dots, \langle e_k, w \rangle),$$

is surjective. Thus we can find $w_0 \in V(\mathcal{I})$ with $\langle e_i, v_0\rangle = \langle e_i, w_0\rangle$ for all $1 \le i \le k$. Then $v_1 = v_0 - w_0$ has the same projection in $V_{\mathcal{O}}$ as v_0 and is orthogonal to L_0 with respect to the symplectic form. The lemma follows then by recurrence on the dimension. □

5.3 Affine charts on Lagrangian Grassmannians and reduction modulo $\mathcal{I}$

Now we turn to a more detailed study of the map $q_{\mathcal{O}}$ and certain transversality properties. Recall from Section 2.1 that given transverse Lagrangians l_1, l_2 in V, we have a map

$$j_{l_1,l_2}\colon \mathcal{Q}(l_1) \to \mathcal{L}(V)^{l_2}$$

which to $f \in \mathcal{Q}(l_1)$ associates the Lagrangian

$$L_f = \{v + T_f v \mid v \in l_1\},$$

where $T_f\colon l_1 \to l_2$ is defined by the equation

$$b_f(v,w) = \langle v, T_f w\rangle = \langle w, T_f v\rangle, \quad v, w \in l_1.$$

If l_1, l_2 are orthogonal for $(\cdot,\cdot)$ and $\{e_1,\dots,e_n\}$ is an orthonormal basis of l_1, then $\{Je_1,\dots,Je_n\}$ is an orthonormal basis for l_2, and the symmetric matrix A_f of f in this basis is given by $(A_f)_{ij} = \langle e_i, T_f(e_j)\rangle = -(Je_i, T_f(e_j))$. Thus it follows from Lemma 5.3 that f belongs to $\mathcal{Q}(l_1)(\mathcal{O})$ if and only if the matrix coefficients of T_f with respect to the basis $\{e_1,\dots,e_n\}$ and $\{Je_1,\dots,Je_n\}$ are in $\mathcal{O}$, which in turn is equivalent to $T_f(l_1(\mathcal{O})) \subseteq l_2(\mathcal{O})$.

Lemma 5.7 *If l_1, l_2 are orthogonal Lagrangians in the symplectic vector space V, then $q_{\mathcal{O}}(l_1)$ and $q_{\mathcal{O}}(l_2)$ are orthogonal, and the diagram*

$$\begin{array}{ccccccc}
\mathcal{Q}(l_1)(\mathcal{O}) & \longrightarrow & \mathcal{Q}(l_1) & \xrightarrow[\sim]{j_{l_1,l_2}} & \mathcal{L}(V)^{l_2} & \longrightarrow & \mathcal{L}(V) \\
\big\downarrow \bar{p}_{\mathcal{O}} & & & & & & \big\downarrow q_{\mathcal{O}} \\
\mathcal{Q}(q_{\mathcal{O}}(l_1)) & & \xrightarrow[\sim]{j_{q_{\mathcal{O}}(l_1),q_{\mathcal{O}}(l_2)}} & & \mathcal{L}(V_{\mathcal{O}})^{q_{\mathcal{O}}(l_2)} & \longrightarrow & \mathcal{L}(V_{\mathcal{O}})
\end{array}$$

commutes. The image under $q_{\mathcal{O}}$ of a Lagrangian that does not belong to $j_{l_1,l_2}(\mathcal{Q}(l_1)(\mathcal{O}))$ is not transverse to $q_{\mathcal{O}}(l_2)$.

Proof Since l_1 and l_2 are orthogonal, we have for $f \in \mathcal{Q}(l_1)$,

$$j_{l_1,l_2}(f)(\mathcal{O}) = \{v + T_f(v) \mid v \in l_1(\mathcal{O}),\ T_f(v) \in l_2(\mathcal{O})\}.$$

First notice that, if f belongs to $\mathcal{Q}(l_1)(\mathcal{O})$, then $T_f(l_1(\mathcal{O}))$ is contained in $l_2(\mathcal{O})$, and thus we get

$$j_{l_1,l_2}(f)(\mathcal{O}) = \{v + T_f(v) \mid v \in l_1(\mathcal{O})\}.$$

Now T_f induces a well-defined map $\overline{T}_f\colon q_{\mathcal{O}}(l_1) \to q_{\mathcal{O}}(l_2)$ with the property that

$$q_{\mathcal{O}}(j_{l_1,l_2}(f)) = \{v + \overline{T}_f(v) \mid v \in q_{\mathcal{O}}(l_1)\}.$$

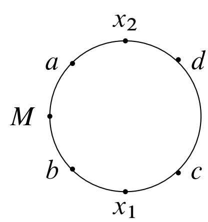
But $b_f(v,w)$ is, by definition, equal to $\langle v, T_f(w)\rangle$, and thus $b_{\overline{f}}(v,w)$ is equal to $\langle v, \overline{T}_f(w)\rangle_{\mathcal{O}}$ for $v,w \in q_{\mathcal{O}}(l_1)$. This implies that $j_{q_{\mathcal{O}}(l_1),q_{\mathcal{O}}(l_2)}(\overline{p}_{\mathcal{O}}(f))$ is equal to $q_{\mathcal{O}}(j_{l_1,l_2}(f))$ and proves the commutativity of the diagram.

If f does not belong to $\mathcal{Q}(l_1)(\mathcal{O})$, we can assume without loss of generality that $T_f(e_1)$ is not in $l_2(\mathcal{O})$. Writing $T_f(e_1) = \sum_{i=1}^n \mu_i J e_i$, let i_0 be such that $|\mu_{i_0}| = \max\{|\mu_i| : 1 \le i \le n\}$. Then μ_{i_0} does not belong to $\mathcal{O}$, and hence $\mu = \mu_{i_0}^{-1}$ belongs to $\mathcal{I}$. This implies that $T_f(\mu e_1)$ belongs to $l_2(\mathcal{O})$ and its e_{i_0} coordinate is equal to 1. Thus $\mu e_1 + T_f(\mu e_1)$ belongs to $j_{l_1,l_2}(f)(\mathcal{O})$, and

$$0 \neq p_{\mathcal{O}}(\mu e_1 + T_f(\mu e_1)) \in q_{\mathcal{O}}(j_{l_1,l_2}(f)) \cap q_{\mathcal{O}}(l_2). \qquad \square$$

Lemma 5.8 *Assume $(a,b,c,d) \in \mathcal{L}(V)^{(4)}$ is a maximal 4–tuple such that $q_{\mathcal{O}}(a)$ is transverse to $q_{\mathcal{O}}(b)$, and $q_{\mathcal{O}}(c)$ is transverse to $q_{\mathcal{O}}(d)$. Then for every $x_1 \in ((b,c))$ and $x_2 \in ((d,a))$, the subspace $q_{\mathcal{O}}(x_1)$ is transverse to $q_{\mathcal{O}}(x_2)$.*

Proof Pick $m \in ((q_{\mathcal{O}}(a), q_{\mathcal{O}}(b)))$ and $M \in \mathcal{L}(V)$ with $q_{\mathcal{O}}(M) = m$ (see Lemma 5.6). As a consequence of Remark 5.4 and the definition of the Kashiwara cocycle, we get that $M \in ((a,b))$. It follows then that (b, x_1, c, d, x_2, a) forms a maximal 6–tuple, and these six Lagrangians are all transverse to M, as illustrated in the following picture:



Thus these points are in the image of $j_{JM,M}\colon \mathcal{Q}(JM) \to \mathcal{L}(V)^M$. Denote by $f_l \in \mathcal{Q}(JM)$ the quadratic form with $j_{JM,M}(f_l) = l \in \mathcal{L}(V)^M$. We have from the maximality property of the 6–tuple that

$$f_b \ll f_{x_1} \ll f_c \ll f_d \ll f_{x_2} \ll f_a;$$

see Lemma 2.10(2). Applying now Lemma 5.7 to $l_1 = JM$ and $l_2 = M$, we deduce, from the fact that $q_{\mathcal{O}}(a)$ and $q_{\mathcal{O}}(b)$ are transverse to $m = q_{\mathcal{O}}(M)$, that f_a and f_b

are in $\mathcal{Q}(JM)(\mathcal{O})$. From the inequalities above, we deduce that f_{x_1} and f_{x_2} are in $\mathcal{Q}(JM)(\mathcal{O})$; it follows then from the commutativity of the diagram in Lemma 5.7 that $q_{\mathcal{O}}(x_1)$ and $q_{\mathcal{O}}(x_2)$ are transverse to $m = q_{\mathcal{O}}(M)$. Also,

$$f_{q_{\mathcal{O}}(x_2)} - f_{q_{\mathcal{O}}(x_1)} \gg f_{q_{\mathcal{O}}(d)} - f_{q_{\mathcal{O}}(c)} \gg 0,$$

where the last inequality follows from the hypothesis that $q_{\mathcal{O}}(d)$ is transverse to $q_{\mathcal{O}}(c)$. Thus $q_{\mathcal{O}}(x_2)$ is transverse to $q_{\mathcal{O}}(x_1)$. □

5.4 Choosing the scale and constructing the maximal framing

Let $\rho\colon \Gamma \to \mathrm{Sp}(V)$ be a representation admitting a maximal framing $\phi\colon S \to \mathcal{L}(V)$. We assume that there is a complex structure J in $\mathbb{X}_V$ and an order convex subring $\mathcal{O}$ of $\mathbb{F}$ such that $\rho(\Gamma) \subset \mathrm{Sp}(V)(\mathcal{O})$. We define then $\rho_{\mathcal{O}}\colon \Gamma \to \mathrm{Sp}(V_{\mathcal{O}})$ as the composition $\rho_{\mathcal{O}} := \pi_{\mathcal{O}} \circ \rho$ and $\phi_{\mathcal{O}}\colon S \to \mathcal{L}(V_{\mathcal{O}})$ as the composition $\phi_{\mathcal{O}} := q_{\mathcal{O}} \circ \phi$. Our goal is to show:

Theorem 5.9 *If ϕ is a maximal S–framing for $\rho\colon \Gamma \to \mathrm{Sp}(V)$, then $\phi_{\mathcal{O}}$ is a maximal S–framing for $\rho_{\mathcal{O}}\colon \Gamma \to \mathrm{Sp}(V_{\mathcal{O}})$.*

Remark 5.10 Since Γ is finitely generated, for any choice of a compatible complex structure J it is possible to find an infinitesimal σ such that $\rho(\Gamma) \subset \mathrm{Sp}(V)(\mathcal{O}_\sigma)$, where $\mathcal{O}_\sigma$ is the order convex subring described in Example 5.1. However, as we will discuss in Section 10, the choice of σ depends on the complex structure J; see Proposition 10.6.

In view of the definition of maximality of triples of Lagrangians and Remark 5.4, in order to prove Theorem 5.9, we have to show that if $x \neq y$ are distinct points in S, then $q_{\mathcal{O}}(\phi(x))$ and $q_{\mathcal{O}}(\phi(y))$ are transverse Lagrangians. As a first step, we show:

Lemma 5.11 *Assume that there exist two distinct points x, y in S such that $q_{\mathcal{O}}(\phi(x))$ and $q_{\mathcal{O}}(\phi(y))$ are not transverse. Then there exists a hyperbolic element $\gamma \in \Gamma$ such that $q_{\mathcal{O}}(\phi(\gamma^+))$ and $q_{\mathcal{O}}(\phi(\gamma^-))$ are not transverse.*

Proof From Lemma 5.8, it follows that we can choose I either $((x, y))$ or $((y, x))$ so that for every t_1, t_2 in I, we have that $q_{\mathcal{O}}(\phi(t_1))$ and $q_{\mathcal{O}}(\phi(t_2))$ are not transverse. Now pick a hyperbolic element $\gamma \in \Gamma$ with $\{\gamma^+, \gamma^-\} \subset I$. □

The strategy of the proof consists in showing that for every hyperbolic element $\gamma \in \Gamma$, the Lagrangians $q_{\mathcal{O}}(\phi(\gamma^-))$ and $q_{\mathcal{O}}(\phi(\gamma^+))$ are transverse. This will be a consequence of the properties of the eigenvalues of $\rho(\gamma)$ using the collar lemma.

We first observe that eigenvalues behave well with respect to reduction modulo $\mathcal{I}$:

Lemma 5.12 *Let $B \in \mathrm{GL}(m, \mathcal{O})$ be a matrix, and denote by $\beta_i \in \mathbb{K}$ the eigenvalues of B. Then:*

(1) $|\beta_i| \in \mathcal{O}$;

(2) *if $\overline{B}$ denotes the image of B in $\mathrm{GL}(m, \mathbb{F}_{\mathcal{O}})$, and $\overline{\beta}_i$ are the images of β_i in $\mathbb{K}_{\mathcal{O}}$, then the eigenvalues of $\overline{B}$ are precisely $\overline{\beta}_i$.*

Proof The first assertion follows from the fact that if β_i is an eigenvalue of B, then there exists a vector $v \in V(\mathcal{O}) \setminus V(\mathcal{I})$ such that $\|Bv\| = |\beta_i|\|v\|$; see Lemma 3.4. The second assertion follows from the fact that the characteristic polynomial of the reduction $\overline{B}$ is the reduction of the characteristic polynomial of B. □

Remark 5.13 Clearly, if g belongs to $\mathrm{GL}(V)(\mathcal{O})$, for each subspace W of V preserved by g, the restriction $g|_W$ belongs to $\mathrm{GL}(W)(\mathcal{O})$, and the restriction commutes with the reduction: $\pi_{\mathcal{O}}(g)|_{q_{\mathcal{O}}(W)} = \pi_{\mathcal{O}}(g|_W)$. However, it is worth pointing out that the Jordan decomposition of a matrix $B \in \mathrm{GL}(m, \mathcal{O})$ is not necessarily defined in $\mathrm{GL}(m, \mathcal{O})$, and in particular, the exponents of the minimal polynomial of a matrix B need not to be related with the exponents of the minimal polynomial of the reduction of B. For example, if ϵ belongs to $\mathcal{I}$, then the reduction of the not diagonalizable matrix $\left(\begin{smallmatrix} 2 & \epsilon \\ 0 & 2 \end{smallmatrix}\right)$ is diagonalizable, and the reduction of the diagonalizable matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1+\epsilon \end{smallmatrix}\right)$ is not diagonalizable.

This last example shows that generalized eigenspaces relative to distinct eigenvalues might not have transverse images in the quotient if the corresponding eigenvalues coincide modulo $\mathcal{I}$. We will now deduce from the collar lemma that, in case of maximal S–framed representations, the intermediate eigenvalues have distinct reductions:

Lemma 5.14 *Let $\rho\colon \Gamma \to \mathrm{Sp}(V)$ be a representation admitting a maximal framing. Assume that $\rho(\Gamma) \subset \mathrm{Sp}(V)(\mathcal{O})$. Then for every hyperbolic element $\gamma \in \Gamma$, we have*

$$|\lambda_n(\gamma)| - 1 \in \mathcal{O} \setminus \mathcal{I},$$

where $|\lambda_1(\gamma)| \geq \cdots \geq |\lambda_n(\gamma)| > 1$ are the eigenvalues of γ of absolute value greater than 1.

Proof Let $\delta \in \Gamma$ be a hyperbolic element with positive intersection number with γ, and let $\lambda_1(\delta)$ be the eigenvalue of $\rho(\delta)$ of largest modulus. If $|\lambda_n(\gamma)| < 2$, then the collar lemma (Theorem 3.3) implies

$$|\lambda_n(\gamma)| - 1 = \frac{|\lambda_n(\gamma)|^2 - 1}{|\lambda_n(\gamma)| + 1} \geq \frac{1}{3|\lambda_1(\delta)|^{2n}}.$$

Now observe that, since $\rho(\delta) \in \mathrm{Sp}(2n, \mathcal{O})$, we have that $|\lambda_1(\delta)|$ belongs to $\mathcal{O}$, from which the claim follows. □

We have now all the necessary ingredients to prove Theorem 5.9:

Proof of Theorem 5.9 Let us assume by contradiction that there exist x, y in S with $q_{\mathcal{O}}(\phi(x))$ nontransverse to $q_{\mathcal{O}}(\phi(y))$. As a consequence of Lemma 5.11, we can find a hyperbolic element γ in Γ such that $q_{\mathcal{O}}(\phi(\gamma^+))$ is nontransverse to $q_{\mathcal{O}}(\phi(\gamma^-))$.

If now $|\lambda_1(\gamma)| \geq \cdots \geq |\lambda_n(\gamma)| > 1$ are the absolute values of the eigenvalues of $\rho(\gamma)|_{\phi(\gamma^+)}$, counted with multiplicity, then it follows from Lemmas 5.14 and 5.12 that the absolute values $|\overline{\lambda_1(\gamma)}| \geq \cdots \geq |\overline{\lambda_n(\gamma)}| > 1$ of the eigenvalues of the restriction of $\rho_{\mathcal{O}}(\gamma)$ to $q_{\mathcal{O}}(\phi(\gamma^+))$ are all strictly larger than 1. Since $|\overline{\lambda_1(\gamma)}|^{-1} \leq \cdots \leq |\overline{\lambda_n(\gamma)}|^{-1} < 1$ are then the absolute values of the eigenvalues of the restriction of $\rho_{\mathcal{O}}(\gamma)$ to $q_{\mathcal{O}}(\phi(\gamma^-))$, this implies that the $\rho_{\mathcal{O}}$–invariant vector space $q_{\mathcal{O}}(\phi(\gamma^+)) \cap q_{\mathcal{O}}(\phi(\gamma^-))$ must be zero since otherwise $\rho_{\mathcal{O}}(\gamma)$ would have at least a nonzero eigenvalue which would be an element in $\mathbb{K}_{\mathcal{O}}$ both of absolute value strictly larger and smaller than 1. Thus $q_{\mathcal{O}}(\phi(\gamma^+)) \cap q_{\mathcal{O}}(\phi(\gamma^-)) = 0$, which is a contradiction. Hence, for every $x \neq y$ in S, we have that $q_{\mathcal{O}}(\phi(x))$ is transverse to $q_{\mathcal{O}}(\phi(y))$. □

6 Fields with valuation and the projection to the building

In this section, $\mathbb{F}$ will denote an ordered field with a compatible valuation $v\colon \mathbb{F} \to \mathbb{R} \cup \{\infty\}$, meaning that we require $v(y) \leq v(x)$ whenever $0 \leq x \leq y$.

Example 6.1 (compare Example 5.1) Let $\mathbb{E}$ be an ordered field, $\sigma \in \mathbb{E}$ an infinitesimal and $\mathcal{O}_\sigma$ the order convex local subring consisting of elements comparable with σ. On $\mathcal{O}_\sigma$, we define the valuation

$$v_\sigma(x) = \sup\{t \in \mathbb{R} : |x| \leq \sigma^t\}.$$

Then v_σ passes to the quotient $\mathbb{E}_\sigma := \mathcal{O}_\sigma / \mathcal{I}_\sigma$ by the maximal ideal $\mathcal{I}_\sigma$ and defines an order compatible valuation.

We introduce on $\mathbb{F}$ the norm $\|x\| := e^{-v(x)}$. This defines an ultrametric norm on $\mathbb{F}$ with valuation ring $\mathcal{U} := \{x \in \mathbb{F} : \|x\| \leq 1\}$ whose maximal ideal is $\mathcal{M} := \{x \in \mathbb{F} : \|x\| < 1\}$. Observe that since the valuation is order compatible, the norm is order compatible as well: if $0 < x < y$, then $\|x\| \leq \|y\|$.

Let $(V, \langle \cdot, \cdot \rangle)$ be a symplectic vector space over $\mathbb{F}$, $J_0 \in \mathbb{X}_V$ a compatible complex structure and $(\cdot, \cdot)_{J_0}$ the corresponding scalar product. We denote by $\mathcal{B}_V$ the affine building associated to $\mathrm{Sp}(V)$; see [20, Section 3.2; 13, Theorem 4.3]. It is well known that the set of vertices $\mathcal{B}_V^0$ of $\mathcal{B}_V$ can be identified with the homogeneous space $\mathrm{Sp}(V)/\mathrm{Sp}(V)(\mathcal{U})$, where, as in Section 5, we define

$$V(\mathcal{U}) = \{v \in V \mid (v,v) \in \mathcal{U}\}$$

and

$$\mathrm{Sp}(V)(\mathcal{U}) = \{g \in \mathrm{Sp}(V) \mid g(V(\mathcal{U})) = V(\mathcal{U})\}.$$

The stabilizer of the complex structure $J_0 \in \mathbb{X}_V$ is

$$\begin{aligned} U(J_0) &= \{g \in \mathrm{Sp}(V) \mid gJ_0g^{-1} = J_0\} \\ &= \{g \in \mathrm{Sp}(V) \mid g \text{ preserves the scalar product } (\cdot,\cdot)_{J_0}\}, \end{aligned}$$

and hence is contained in $\mathrm{Sp}(V)(\mathcal{U})$. As a result, we can define the projection

$$\pi_{\mathcal{B}}\colon \mathbb{X}_V = \mathrm{Sp}(V)/U(J_0) \to \mathcal{B}_V^0 = \mathrm{Sp}(V)/\mathrm{Sp}(V)(\mathcal{U}).$$

Remark 6.2 Parreau [20] gave an explicit description of the building associated to $\mathrm{SL}(2n,\mathbb{F})$ as the space of good norms on $\mathbb{F}^{2n}$ of determinant one. It is possible to verify that, considering the affine building associated to $\mathrm{Sp}(2n,\mathbb{F})$ as a subbuilding of the affine building associated to $\mathrm{SL}(2n,\mathbb{F})$, the map $\pi_{\mathcal{B}}$ corresponds to the map that associates to a point $J \in \mathbb{X}_V$ the corresponding good norm $\eta_J(v) = \|(v,v)_J\|$.

For $\mathbb{F} = \mathbb{R}$, Siegel [26] gave explicit formulas for the Riemannian distance on $\mathcal{X}_{\mathbb{R}}$. We use the cross-ratio R defined in Section 4.1 to define in our context a distance-like function as follows. Observe that, given $X, W \in \mathcal{T}_V$, the cross-ratio $R(X,\sigma(W),W,\sigma(X))$ is always well defined: indeed, the Hermitian form $i\langle\cdot,\sigma(\cdot)\rangle$ is positive definite on X and W and negative definite on $\sigma(W)$ and $\sigma(X)$; in particular, X and $\sigma(W)$ are transverse, and so are W and $\sigma(X)$. Moreover, all the eigenvalues of the cross-ratio $R(X,\sigma(W),W,\sigma(X))$ belong to $\mathbb{F}$ and are between 0 and 1: indeed, since $\mathbb{F}$ is real closed, for each pair $X, W \in \mathcal{X}_{\mathbb{F}}$, we can find $g \in \mathrm{Sp}(V)$ such that $g\cdot X = i\,\mathrm{Id}$ and $g\cdot W = iD$ for a diagonal matrix D with positive entries, and we have

$$gR(X,\sigma(W),W,\sigma(X))g^{-1} = R(i\,\mathrm{Id},-iD,iD,-i\,\mathrm{Id}) = \frac{(\mathrm{Id}-D)^2}{(\mathrm{Id}+D)^2}.$$

We can thus define

$$d(Z,W) = \sqrt{\sum_{i=1}^{n}\left(\ln\left\|\frac{1+\sqrt{r_i}}{1-\sqrt{r_i}}\right\|\right)^2}, \tag{9}$$

where $r_1,\dots,r_n$ are the eigenvalues of $R(X,\sigma(W),W,\sigma(X))$.

In the case we considered above, where $X = i\,\mathrm{Id}$ and $W = iD$, (9) specializes to

$$d(i\,\mathrm{Id}, iD) = \sqrt{\sum_{i=1}^{n} (\ln \|d_i\|)^2},$$

where $d_1, \dots, d_n$ are the entries of D.

The function d is clearly $\mathrm{Sp}(V)$–invariant since the eigenvalues of the cross-ratio are. Denote by $d_{\mathcal{B}}$ the CAT(0) distance on $\mathcal{B}_V$. Using the transitivity of the symplectic group on apartments in $\mathcal{B}_V$ and the invariance of d, one verifies:

Proposition 6.3 *For any $X, Y \in \mathcal{T}_V$, we have*

$$d_{\mathcal{B}}(\pi_{\mathcal{B}}(X), \pi_{\mathcal{B}}(Y)) = d(X, Y).$$

As a result, we get that d is a pseudodistance on $\mathcal{T}_V$, and $\mathcal{B}_V$ is the Hausdorff quotient of $\mathcal{T}_V$ modulo this pseudodistance.

We will denote by $L_{\mathcal{B}}(g)$ the translation length of an element $g \in \mathrm{Sp}(V)$ considered as an isometry of the affine building $\mathcal{B}_V$.

7 On elements with fixed points

We place ourselves in the framework of Section 6 and consider a representation $\rho\colon \Gamma \to \mathrm{Sp}(V)$ admitting a maximal framing (S, ϕ). In this section, we want to analyze how elements of Γ which have zero translation length in the building $\mathcal{B}_V$ interact. As a crucial step in the analysis, we associate to any such $\gamma \in \Gamma$ a pair (b_γ^+, b_γ^-) of points in $\mathcal{B}_V$ which are fixed by $\rho(\gamma)$ and are canonically constructed from the maximal framing ϕ.

Recall from Section 6 that we denote by $\pi_{\mathcal{B}}\colon \mathcal{T}_V \to \mathcal{B}_V$ the $\mathrm{Sp}(V)$–equivariant projection from the Siegel upper half-space to the affine building associated to $\mathrm{Sp}(V)$, and given an element $g \in \mathrm{Sp}(V)$, we denote by $L_{\mathcal{B}}(g)$ the translation length of g on $\mathcal{B}_V$. Moreover, for ease of notation, we will denote by $\mathcal{Y}_\gamma$ the $\mathbb{F}$–tube $\mathcal{Y}_{\phi(\gamma^-),\phi(\gamma^+)}$ and by $\mathbb{Y}_\gamma$ its projection to $\mathcal{B}_V$:

$$\mathbb{Y}_\gamma = \pi_{\mathcal{B}}(\mathcal{Y}_\gamma).$$

It follows from the equivariance of $\pi_{\mathcal{B}}$ that $\mathbb{Y}_\gamma$ is a subbuilding of $\mathcal{B}_V$ associated to a subgroup of $\mathrm{Sp}(V)$ isomorphic to $\mathrm{GL}(n, \mathbb{F})$. Recall from Section 4.3 that given any pair of transverse Lagrangians $a, b \in \mathcal{L}(V)$, we defined an orthogonal projection

$$\mathrm{pr}_{\mathcal{Y}_{a,b}}\colon ((a,b)) \cup ((b,a)) \to \mathcal{Y}_{a,b}.$$

The first goal of the section is to prove:

Proposition 7.1 *Let $\gamma \in \Gamma$ be an element which is not boundary parallel. Assume that $L_{\mathcal{B}}(\rho(\gamma)) = 0$. Then both maps*

$$F_\gamma^+\colon ((\gamma^-, \gamma^+)) \to \mathbb{Y}_\gamma, \quad x \mapsto \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(x)))$$

and

$$F_\gamma^-\colon ((\gamma^+, \gamma^-)) \to \mathbb{Y}_\gamma, \quad x \mapsto \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(x)))$$

are constant.

Denoting by b_γ^+ (resp. b_γ^-) the constant images of the maps F_γ^+ (resp. F_γ^-) in Proposition 7.1 we have:

Corollary 7.2 *The points b_γ^+ and b_γ^- are fixed by $\rho(\gamma)$.*

If $\gamma \in \Gamma$ corresponds to a simple closed geodesic, it is possible to construct examples of representations $\rho\colon \Gamma \to \mathrm{Sp}(V)$ such that the points b_γ^+ and b_γ^- are different. The second main result of the section gives sufficient conditions for the two points to coincide:

Proposition 7.3 *Assume that γ and η in Γ are hyperbolic elements with intersecting axes, and that $L_{\mathcal{B}}(\rho(\gamma)) = L_{\mathcal{B}}(\rho(\eta)) = 0$. Then*

$$b_\gamma^+ = b_\gamma^- = b_\eta^+ = b_\eta^- = \pi_{\mathcal{B}}(\mathcal{Y}_\gamma \cap \mathcal{Y}_\eta).$$

Corollary 7.4 *Assume that $L_{\mathcal{B}}(\rho(\gamma)) = 0$. If the closed geodesic corresponding to γ is not simple, then $b_\gamma^+ = b_\gamma^-$.*

Before proceeding to the proofs of Propositions 7.1 and 7.3, we observe that in certain situations, one can get a uniform lower bound on the translation lengths $L_{\mathcal{B}}(\rho(\gamma))$ for all hyperbolic elements γ crossing a given hyperbolic element η. This is in fact an immediate corollary of the collar lemma:

Corollary 7.5 *Assume that $\eta \in \Gamma$ is a hyperbolic element, and let us denote by $|\lambda_1(\eta)| \geq \cdots \geq |\lambda_n(\eta)| > 1$ the eigenvalues of $\rho(\eta)$ of absolute value larger than 1. If $\delta = \| |\lambda_n(\eta)| - 1 \| < 1$, then for any element γ intersecting η, we have*

$$L_{\mathcal{B}}(\rho(\gamma)) \geq \frac{1}{2n\delta}.$$

In particular, if the closed geodesic represented by η is not simple, $\| |\lambda_n(\eta)| - 1 \| \geq 1$.

Proposition 7.3 also allows us to give sufficient conditions for a representation ρ to have a global fixed point. We say that a generating set X for Γ is connected if the graph (X, E), where E consists of the pairs (s_1, s_2) of elements of X whose axes intersect, is connected.

Corollary 7.6 *Let X be any connected generating set for Γ. If $\rho\colon \Gamma \to \mathrm{Sp}(V)$ is a representation admitting a maximal framing, the following are equivalent:*

(1) *ρ has a global fixed point in $\mathcal{B}_V$;*

(2) *$L_{\mathcal{B}}(\rho(s)) = 0$ for all $s \in X$.*

Remark 7.7 There exist connected generating sets consisting of $2g$ simple closed curves. In particular, Corollary 7.6 refines, in our setting, [22, Corollary 3].

Recall from Section 2.3 that we say that $g \in \mathrm{Sp}(V)$ is Shilov hyperbolic if there exists a g–invariant decomposition $V = L_g^+ \oplus L_g^-$ such that all the eigenvalues of the restriction M_g of g to L_g^+ are in absolute value strictly greater than one. It is however worth remarking that, in general, g does not necessarily have a hyperbolic dynamic on $\mathcal{L}(V)$. It follows from Lemma 5.14 that, as soon as ρ admits a maximal framing, for any hyperbolic element $\gamma \in \Gamma$, its image $\rho(\gamma)$ is Shilov hyperbolic.

Lemma 7.8 *Let $g \in \mathrm{Sp}(V)$ be Shilov hyperbolic, and let $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{F}[i]$ be the set of eigenvalues of M_g. Then*

$$L_{\mathcal{B}}(g) = 2\sqrt{\sum_{i=1}^{n} (\ln \|\lambda_i\|)^2}.$$

Proof Since g is Shilov hyperbolic, it stabilizes the $\mathbb{F}$–tube $\mathcal{Y}_{L_g^+, L_g^-}$, and similarly it stabilizes the projection

$$\mathbb{Y}_{L_g^+, L_g^-} = \pi_{\mathcal{B}}(\mathcal{Y}_{L_g^+, L_g^-}).$$

This latter is a subbuilding of $\mathcal{B}_V$ associated to $\mathrm{GL}(n, \mathbb{F})$. The desired statement then follows from [20]. □

Lemma 7.9 *Let $g \in \mathrm{Sp}(V)$ be Shilov hyperbolic. Then the following are equivalent:*

(1) *$L_{\mathcal{B}}(g) = 0$;*

(2) *$\|\det M_g\| = 1$;*

(3) *$\|\det R(L_g^+, S, gS, L_g^-)\| = 1$ for every S in $((L_g^+, L_g^-))$.*

Proof In view of Lemma 7.8, we have that

$$L_{\mathcal{B}}(g) = 2\sqrt{\sum_{i=1}^{n}(\ln\|\lambda_i\|)^2},$$

while

$$\|\det M_g\| = \prod_{i=1}^{n}\|\lambda_i\| \quad \text{and} \quad \det R(L_g^+, S, gS, L_g^-) = (\det M_g)^{-2}.$$

The equivalence follows easily from the assumption that $|\lambda_i| > 1$ for all i and the order compatibility of the norm. □

Lemma 7.10 *Let us assume that the 5–tuple of Lagrangians $(x_1, x_2, x_3, x_4, x_5)$ is maximal. Then*

$$\det R(x_1, x_2, x_3, x_5) \le \det R(x_1, x_2, x_4, x_5).$$

Proof We may assume $x_1 = 0$ and $x_5 = l_\infty$; then we have $0 \ll x_2 \ll x_3 \ll x_4$. In this case, a computation gives that $R(x_1, x_2, x_3, x_5)$ is conjugate to $y_1 = x_2^{-1/2} x_3 x_2^{-1/2}$, and $R(x_1, x_2, x_4, x_5)$ is conjugate to $y_2 = x_2^{-1/2} x_4 x_2^{-1/2}$. Since each eigenvalue of y_1 is positive and smaller than the corresponding eigenvalue of y_2, one obtains the desired inequality. □

Lemma 7.11 *Assume that (a, x, y, b) in $\mathcal{L}(V)^4$ is maximal. Then*

(1) $\|\det R(a, x, y, b)\| \ge 1$;

(2) $d(\mathrm{pr}_{\mathcal{Y}_{a,b}}(x), \mathrm{pr}_{\mathcal{Y}_{a,b}}(y)) \le \ln\|\det R(a, x, y, b)\| \le \sqrt{n}\, d(\mathrm{pr}_{\mathcal{Y}_{a,b}}(x), \mathrm{pr}_{\mathcal{Y}_{a,b}}(y))$.

Proof Since $\mathrm{Sp}(V)$ is transitive on maximal triples, we can assume that $a = 0, b = l_\infty$ and x corresponds to the matrix $+\mathrm{Id}$. Since the triple (x, y, l_∞) is maximal, y corresponds to a positive-definite matrix Y with all eigenvalues strictly bigger than one. The first statement is immediate since $\det R(a, x, y, b) = \det(Y)$.

It follows from the definition of the orthogonal projection that $\mathrm{pr}_{\mathcal{Y}_{a,b}}(x) = i\,\mathrm{Id}$ and $\mathrm{pr}_{\mathcal{Y}_{a,b}}(y) = iY$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of Y, the explicit formula for the distance d gives

$$d(i\,\mathrm{Id}, iY) = \sqrt{\sum_{i=1}^{n}(\ln\|\lambda_i\|)^2},$$

and we have

$$\ln\|\det R(a, x, y, b)\| = \sum_{i=1}^{n}\ln\|\lambda_i\|.$$

The second assertion in the lemma then follows from Cauchy–Schwartz and the fact that $\ln\|\lambda_i\| \ge 0$ for every i. □

Lemma 7.12 *If $L_{\mathcal{B}}(\rho(\gamma)) = 0$, then for any $x, y \in ((\gamma^-, \gamma^+))$ with $(\gamma^-, x, y, \gamma^+)$ positively oriented, we have*

$$\|\det R(\phi(\gamma^-), \phi(x), \phi(y), \phi(\gamma^+))\| = 1.$$

Proof Since $(\gamma^-, x, y, \gamma^+)$ is positively oriented, and γ^+ is the attractive fixed point of γ, we can pick $n \geq 1$ with $(x, y, \gamma^n x)$ positively oriented. Then by Lemma 7.10, we have

$$\begin{aligned} 1 &\leq \det R(\phi(\gamma^-), \phi(x), \phi(y), \phi(\gamma^+)) \\ &\leq \det R(\phi(\gamma^-), \phi(x), \rho(\gamma)^n \phi(x), \phi(\gamma^+)), \end{aligned}$$

and the latter has norm 1 by Lemma 7.9(3). □

Proof of Proposition 7.1 Let s and t be points in $((\gamma^-, \gamma^+))$ and assume without loss of generality that the quadruple $(\gamma^-, t, s, \gamma^+)$ is positively oriented. Then $(\phi(\gamma^-), \phi(t), \phi(s), \phi(\gamma^+))$ is a maximal quadruple; thus by Lemma 7.11, we have $d(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(t)), \mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(s))) \leq \ln \|\det R(\phi(\gamma^-), \phi(t), \phi(s), \phi(\gamma^+))\|$. The right hand side vanishes by Lemma 7.12, and hence we obtain, using Proposition 6.3, that $\pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(t))) = \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(s)))$. □

Let us now assume that there are two elements γ, η in $\pi_1(\Sigma)$ whose axes intersect. We want to show that if both $\rho(\gamma)$ and $\rho(\eta)$ fix a point in $\mathcal{B}_V$, then they share a fixed point. We begin with a preliminary computation:

Lemma 7.13 *Let γ and η be two hyperbolic elements of Γ with intersecting axes. Assume $L_{\mathcal{B}}(\rho(\gamma)) = L_{\mathcal{B}}(\rho(\eta)) = 0$ and that the quadruple $(\eta^-, \gamma^-, \eta^+, \gamma^+)$ is positively oriented. Then for every $x \in ((\gamma^-, \gamma^+))$, all eigenvalues of the cross-ratio $R(\phi(\eta^-), \phi(\gamma^-), \phi(x), \phi(\gamma^+))$ have the form $1+f$, where $f \in \mathbb{F}^{>0}$ satisfies $\|f\| = 1$.*

Proof Pick $g \in \mathrm{Sp}(V)$ such that $g(\phi(\eta^-), \phi(\gamma^-), \phi(\gamma^+)) = (-\mathrm{Id}, 0, l_\infty)$, and set $p = g(\phi(\eta^+))$; see Figure 2. Now pick $x \in ((\gamma^-, \eta^+))$ and set $q = g(\phi(x))$. Observe that $0 \ll q \ll p$.

By Lemma 7.12, since $L_{\mathcal{B}}(\rho(\gamma)) = 0$, we have

$$\|\det R(\phi(\gamma^-), \phi(x), \phi(\eta^+), \phi(\gamma^+))\| = 1,$$

which implies $\|\det p\| = \|\det q\|$.

Let $\mu_1 \geq \cdots \geq \mu_n > 0$ and $\lambda_1 \geq \cdots \geq \lambda_n > 0$ denote the eigenvalues of q and p, respectively. Since $0 \ll q \ll p$, we deduce that $0 < \mu_i < \lambda_i$ and hence $\|\mu_i\| \leq \|\lambda_i\|$. This implies that $\|\mu_i\| = \|\lambda_i\|$ since we know that their products are equal.

Exploiting that $L_{\mathcal{B}}(\rho(\eta)) = 0$ together with Lemma 7.12 we get

$$\|\det R(\phi(\eta^-), \phi(\gamma^-), \phi(x), \phi(\eta^+))\| = 1,$$

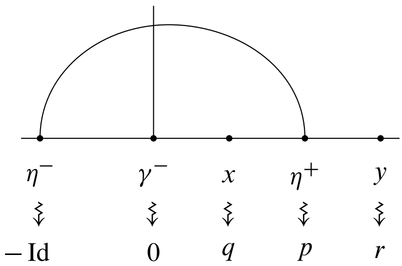


Figure 2: The points needed for the proof of Lemma 7.13

which implies that $\|(\det p)(\det(p-q))^{-1}\det(\mathrm{Id}+q)\| = 1$. From this, we deduce

$$\prod_{i=1}^{n} \|1+\mu_i\| = \|\det(\mathrm{Id}+q)\| = \left\| \frac{\det(p-q)}{\det p} \right\| \le 1,$$

where the last inequality follows from $0 \ll p-q \ll p$. Together with the observation that $1+\mu_i \ge 1$ and the ultrametric inequality, this implies $\|\mu_i\| \le 1$ for all i, and thus $\|\lambda_i\| = \|\mu_i\| \le 1$.

Now let $y \in ((\eta^+, \gamma^+))$ and set $r = g(\phi(y))$. Then $0 \ll p \ll r$. Again by Lemma 7.12 we deduce that

$$\|\det R(\phi(\gamma^-), \phi(\eta^+), \phi(y), \phi(\gamma^+))\| = 1,$$

which implies $\|\det p\| = \|\det r\|$.

Let $\nu_1 \ge \cdots \ge \nu_n > 0$ denote the eigenvalues of r. Since $p \ll r$, we deduce that $0 < \lambda_i < \nu_i$, and hence $\|\nu_i\| \ge \|\lambda_i\|$. This implies, as above, that $\|\nu_i\| = \|\lambda_i\|$. Since $L_{\mathcal{B}}(\rho(\eta)) = 0$, Lemma 7.12 implies that

$$\|\det R(\phi(\eta^+), \phi(y), \phi(\gamma^+), \phi(\eta^-))\| = 1;$$

that is, $\|\det(\mathrm{Id}+r)\| = \|\det(r-p)\|$. Since $0 \ll r-p \ll r$, we obtain $\|\det(r-p)\| \le \|\det r\|$. On the other hand, $0 \ll r \ll \mathrm{Id}+r$, and hence $\|\det(\mathrm{Id}+r)\| = \|\det(r)\|$, or equivalently, $\prod_{i=1}^{n} \|1+(1/\nu_i)\| = 1$. This, together with the information that $\nu_i > 0$ and the ultrametric inequality, implies $\|\nu_i\| \ge 1$, and thus $\|\lambda_i\| = \|\nu_i\| \ge 1$.

To conclude the proof, we observe that $R(\phi(\eta^-), \phi(\gamma^-), \phi(x), \phi(\gamma^+))$ is conjugate to $R(-\mathrm{Id}, 0, q, l_\infty) = \mathrm{Id}+q$ and hence has as all eigenvalues of the form $1+f$ with f positive satisfying $\|f\| = 1$. □

Remark 7.14 Recall from Definition 4.14 that $\mathcal{Y}_\gamma$ and $\mathcal{Y}_\eta$ are orthogonal if and only if $R(\phi(\eta^-), \phi(\gamma^-), \phi(\eta^+), \phi(\gamma^+)) = 2\,\mathrm{Id}$. Lemma 7.13 should be interpreted as a weaker form of orthogonality for the projections $\mathbb{Y}_\gamma$ and $\mathbb{Y}_\eta$.

Lemma 7.15 *Let $(a,c,b,d) \in \mathcal{L}(V)^4$ be a maximal quadruple, and assume that all the eigenvalues of $R(a,c,b,d)$ have the form $1+f$ for some $f \in \mathbb{F}^{>0}$ with $\|f\|=1$. Then the points*

$$\mathrm{pr}_{\mathcal{Y}_{c,d}}(a), \quad \mathrm{pr}_{\mathcal{Y}_{c,d}}(b), \quad \mathrm{pr}_{\mathcal{Y}_{a,b}}(c), \quad \mathrm{pr}_{\mathcal{Y}_{a,b}}(d), \quad \mathcal{Y}_{a,b} \cap \mathcal{Y}_{c,d}$$

have pairwise pseudodistance zero.

Proof Pick $g \in \mathrm{Sp}(V)$ such that $g(a,c,b,d) = (-\mathrm{Id}, 0, D, l_\infty)$ where D is diagonal with strictly positive entries. Then a computation gives $\mathrm{pr}_{\mathcal{Y}_{0,l_\infty}}(-\mathrm{Id}) = i\,\mathrm{Id}$, $\mathrm{pr}_{\mathcal{Y}_{0,l_\infty}}(D) = iD$ and $\mathcal{Y}_{0,l_\infty} \cap \mathcal{Y}_{-\mathrm{Id},D} = i\sqrt{D}$.

Now since $D = \mathrm{diag}(d_1, \dots, d_n)$, the assumption on the eigenvalues implies $\|d_i\| = 1$, and the explicit formula for the distance gives the desired statement. □

Proof of Proposition 7.3 We may assume that $(\eta^-, \gamma^-, \eta^+, \gamma^+)$ is positively oriented. Applying Lemma 7.13 to $x = \eta^+$, we obtain that the pseudodistances of the points $\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(\eta^+))$, $\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(\eta^-))$, $\mathrm{pr}_{\mathcal{Y}_\eta}(\phi(\gamma^+))$, $\mathrm{pr}_{\mathcal{Y}_\eta}(\phi(\gamma^-))$, $\mathcal{Y}_\gamma \cap \mathcal{Y}_\eta$ are all zero. This concludes the proof once one notices that (see Proposition 7.1)

$$\begin{aligned} b_\gamma^+ &= \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(\eta^+))), \\ b_\gamma^- &= \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(\eta^-))), \\ b_\eta^+ &= \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\eta}(\phi(\gamma^-))), \\ b_\eta^- &= \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\eta}(\phi(\gamma^+))). \end{aligned}$$

□

8 Decomposition theorem

Let $\rho\colon \pi_1(\Sigma, x) \to \mathrm{Sp}(V)$ be a representation into a symplectic group over a real closed field $\mathbb{F}$ with valuation, and let $\pi_{\mathcal{B}}\colon \mathcal{T}_V \to \mathcal{B}_V$ denote the projection to the building. Recall from the introduction that if $\Sigma = \bigcup_{v \in \mathcal{V}} \Sigma_v$ is a decomposition of the surface Σ into subsurfaces with geodesic boundary, we consider the associated presentation of Γ as fundamental group of a graph of groups with vertex set $\mathcal{V}$ and vertex groups $\pi_1(\Sigma_v)$. We denote by $\widetilde{\mathcal{V}}$ the vertex set of the associated Bass–Serre tree $\mathcal{T}$. For every $v \in \mathcal{V}$ and $w \in \widetilde{\mathcal{V}}$ lying above v, the stabilizer Γ_w of w in Γ is isomorphic to $\pi_1(\Sigma_v)$. In this section, we prove the result mentioned in the introduction as Theorem 1.8:

Theorem 8.1 *Assume that $\rho\colon \Gamma \to \mathrm{Sp}(V)$ admits a maximal framing. Then there is a decomposition $\Sigma = \bigcup_{v \in \mathcal{V}} \Sigma_v$ of Σ into subsurfaces with geodesic boundary such that*

1. *for every $\gamma \in \Gamma$ whose associated closed geodesic is not contained in any subsurface, $L_{\mathcal{B}}(\rho(\gamma)) > 0$;*

(2) *for every* $v \in \mathcal{V}$, *there is the following dichotomy:*

(PT) *for every* $w \in \widetilde{\mathcal{V}}$ *lying above* v, *and any* $\gamma \in \Gamma_w$ *which is not boundary parallel,* $\rho(\gamma)$ *has positive translation length;*

(FP) *for every* $w \in \widetilde{\mathcal{V}}$ *lying above* v, *there is a point* $b_w \in \mathcal{B}_V$ *which is fixed by* Γ_w.

The proof of the theorem is based on the analysis of the incidence structure of the set

$$\mathcal{L}_\rho = \{\gamma \in \Gamma \mid \gamma \neq e,\ \gamma \text{ hyperbolic},\ L_{\mathcal{B}}(\rho(\gamma)) = 0\}.$$

Let

$$\mathbb{P}\mathcal{L}_\rho = \{\gamma \in \mathcal{L}_\rho \mid \gamma \text{ is primitive}\}/_{\gamma \sim \gamma^{-1}},$$

and denote by $\overline{\gamma} \in \mathbb{P}\mathcal{L}_\rho$ the equivalence class of γ. Let

$$\mathcal{A}_\rho = \{\mathrm{ax}(\gamma) \mid \gamma \in \mathcal{L}_\rho\}$$

denote the set of axes of elements in $\mathcal{L}_\rho$, so there is a bijective correspondence $\mathcal{A}_\rho \cong \mathbb{P}\mathcal{L}_\rho$.

On $\mathbb{P}\mathcal{L}_\rho$ we put a graph structure by requiring that $\overline{\gamma}$ is adjacent to $\overline{\eta}$ if they are distinct and their axes intersect. We denote by $\mathcal{G}_\rho$ this graph and proceed to study its connected components. Let $\mathfrak{C} \subset \mathcal{G}_\rho$ be a connected component with vertex set $V(\mathfrak{C})$. We observe that if the component consists of a single vertex $\overline{\gamma}$, then the closed geodesic associated to γ is simple. Indeed, for each η in Γ, the conjugate $\eta\gamma\eta^{-1}$ belongs to $\mathcal{L}_\rho$ and if $\overline{\eta\gamma\eta^{-1}} \neq \overline{\gamma}$, the corresponding axes do not intersect.

Let us assume from now on that $|V(\mathfrak{C})| \geq 2$, and let

$$\Gamma_{\mathfrak{C}} = \{\gamma \in \Gamma \mid \gamma \text{ stabilizes } \mathfrak{C}\}$$

and

$$\Delta_{\mathfrak{C}} = \bigcup_{\overline{\gamma} \in V(\mathfrak{C})} \{\gamma^-, \gamma^+\}.$$

Then we clearly have that if $\overline{\gamma}$ belongs to $V(\mathfrak{C})$, then γ is an element of $\Gamma_{\mathfrak{C}}$ and $\Delta_{\mathfrak{C}}$ is a subset of the limit set $\Lambda(\Gamma_{\mathfrak{C}}) \subset \partial\mathbb{H}^2$ of $\Gamma_{\mathfrak{C}}$. In particular, since $\Delta_{\mathfrak{C}}$ is $\Gamma_{\mathfrak{C}}$–invariant, we get $\overline{\Delta}_{\mathfrak{C}} = \Lambda(\Gamma_{\mathfrak{C}})$.

Lemma 8.2 *There is a point* $p_{\mathfrak{C}} \in \mathcal{B}_V$ *with* $b_\gamma^\pm = p_{\mathfrak{C}}$ *for all* γ *such that* $\overline{\gamma} \in V(\mathfrak{C})$.

Proof Indeed, if $\overline{\gamma}$ is adjacent to $\overline{\eta}$, we have $b_\gamma^+ = b_\gamma^- = b_\eta^+ = b_\eta^-$; see Lemma 7.15. The lemma follows from the assumption that $\mathfrak{C}$ is connected. □

Lemma 8.3 *For every* $\gamma \in \Gamma_{\mathfrak{C}}$, *we have* $\rho(\gamma) p_{\mathfrak{C}} = p_{\mathfrak{C}}$.

Proof For every $\gamma \in \Gamma_{\mathfrak{C}}$, if η gives a vertex of $V(\mathfrak{C})$, the same holds for $\gamma\eta\gamma^{-1}$. Hence we get

$$b_\eta^\pm = b_{\gamma\eta\gamma^{-1}}^\pm = \rho(\gamma) b_\eta^\pm. \qquad \square$$

Lemma 8.4 *Let g be an oriented geodesic with endpoints g^- and g^+. Assume that $\Delta_{\mathfrak{C}} \cap ((g^-, g^+)) \neq \varnothing$ and $\Delta_{\mathfrak{C}} \cap ((g^+, g^-)) \neq \varnothing$. Then there exists $\overline{\gamma} \in \mathfrak{C}$ with $\operatorname{ax}(\gamma) \cap g \neq \varnothing$.*

Proof Let us choose a class $\overline{\eta} \in \mathfrak{C}$ with $\eta^+ \in ((g^-, g^+))$ and a class $\overline{\tau} \in \mathfrak{C}$ with $\tau^- \in ((g^+, g^-))$. Since $\mathfrak{C}$ is connected, there is a sequence $\overline{\alpha}_1 = \overline{\eta}, \overline{\alpha}_2, \dots, \overline{\alpha}_n = \overline{\tau}$ of classes in $\mathfrak{C}$ such that, for every i, the axis $\operatorname{ax}(\alpha_i)$ intersects $\operatorname{ax}(\alpha_{i+1})$. But then clearly there is an index j such that $\operatorname{ax}(\alpha_j)$ intersects the geodesic g. $\square$

If X is a subset of $\overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial\mathbb{H}^2$, we denote by $\overline{\operatorname{Co}(X)}$ the closed convex hull of X in $\mathbb{H}^2$. To any component $\mathfrak{C}$ we associate the closed convex subset $Y_{\mathfrak{C}}$ of $\mathbb{H}^2$ defined by

$$Y_{\mathfrak{C}} = \overline{\operatorname{Co}(\Lambda(\Gamma_{\mathfrak{C}}))} = \overline{\operatorname{Co}(\Delta_{\mathfrak{C}})}.$$

We say that an element $\gamma \in \Gamma_{\mathfrak{C}}$ is a boundary component if the axis of γ is a boundary component of $Y_{\mathfrak{C}}$.

Proposition 8.5 *For every primitive, hyperbolic element $\gamma \in \Gamma_{\mathfrak{C}}$ which is not a boundary component, we have*

$$\overline{\gamma} \in V(\mathfrak{C}).$$

Proof Since γ stabilizes $\mathfrak{C}$ and is not a boundary component, we have that the intersection $\Delta_{\mathfrak{C}} \cap ((\gamma^-, \gamma^+))$ is not empty, and similarly, $\Delta_{\mathfrak{C}} \cap ((\gamma^+, \gamma^-))$ is not empty. Thus we conclude by Lemma 8.4. $\square$

Our next aim is to show that the image $p(Y_{\mathfrak{C}})$ of $Y_{\mathfrak{C}}$ under the universal covering map $p\colon \mathbb{H}^2 \to \Sigma$ is a compact subsurface of Σ with geodesic boundary.

Proposition 8.6 *Let $\mathfrak{C} \subset \mathcal{G}_\rho$ be a connected component with more than one vertex. For every $\gamma \in \Gamma$, one of the following holds:*

1. $\gamma Y_{\mathfrak{C}} = Y_{\mathfrak{C}}$*;*
2. $\gamma Y_{\mathfrak{C}} \cap Y_{\mathfrak{C}}$ *is a boundary component of* $Y_{\mathfrak{C}}$*;*
3. *the intersection* $\gamma Y_{\mathfrak{C}} \cap Y_{\mathfrak{C}}$ *is empty.*

Proof First we show that if the intersection $\gamma \mathring{Y}_{\mathfrak{C}} \cap Y_{\mathfrak{C}}$ is not empty, then $\gamma\mathfrak{C} = \mathfrak{C}$, and hence $\gamma Y_{\mathfrak{C}} = Y_{\mathfrak{C}}$. Let $x \in \gamma \mathring{Y}_{\mathfrak{C}} \cap Y_{\mathfrak{C}}$, and assume by contradiction that $\gamma\mathfrak{C} \neq \mathfrak{C}$, which implies that $\gamma V(\mathfrak{C}) \cap V(\mathfrak{C}) = \varnothing$.

Claim 1 *The point x does not belong to* $\mathrm{ax}(\eta)$ *for any* $\overline{\eta} \in \mathfrak{C}$.

Proof Assume, instead, that x belongs to $\mathrm{ax}(\eta)$ for some element η with $\overline{\eta} \in \mathfrak{C}$. If the intersection $\Delta_{\gamma\mathfrak{C}} \cap ((\eta^-, \eta^+))$ is empty, then $\Delta_{\gamma\mathfrak{C}}$ is contained in the closed interval $[\![\eta^+, \eta^-]\!]$, and hence $Y_{\gamma\mathfrak{C}}$ is contained in one of the closed halfplanes determined by $\mathrm{ax}(\eta)$. This contradicts the hypothesis that x belongs to the interior of $Y_{\gamma\mathfrak{C}}$. Thus we have that both intersections $\Delta_{\gamma\mathfrak{C}} \cap ((\eta^-, \eta^+))$ and $\Delta_{\gamma\mathfrak{C}} \cap ((\eta^+, \eta^-))$ are not empty. But then, by Lemma 8.4, there is an element $\xi \in \gamma\mathfrak{C}$ whose axis $\mathrm{ax}(\xi)$ intersects $\mathrm{ax}(\eta)$. This implies that either $\overline{\xi} = \overline{\eta}$, or the elements $\overline{\xi}$ and $\overline{\eta}$ are adjacent in the graph $\mathcal{G}_\rho$. Both contradict the fact that $\gamma V(\mathfrak{C}) \cap V(\mathfrak{C}) = \varnothing$, and this proves Claim 1. □

Now we can define $B_{\overline{g}}$, for every $\overline{g} \in \mathfrak{C}$, to be the unique closed interval in S^1 with endpoints $\{g^-, g^+\}$ and such that x does not belong to the convex hull $\overline{\mathrm{Co}(B_g)}$. According to Claim 1, this is well defined.

Claim 2 *For every $\overline{g}$ in $\mathfrak{C}$, the intersection $\Delta_{\gamma\mathfrak{C}} \cap B_{\overline{g}}$ is empty.*

Proof Indeed, assume that the intersection is not empty for some $\overline{g} \in \mathfrak{C}$. Since $\overline{g}$ does not belong to $\gamma\mathfrak{C}$, this implies that the intersection $\Delta_{\gamma\mathfrak{C}} \cap \mathring{B}_g$ is not empty. Since x belongs to $\gamma\mathring{Y}_{\mathfrak{C}}$, we get that the intersection $\Delta_{\gamma\mathfrak{C}} \cap (S^1 \setminus B_g)$ is not empty, and hence, by Lemma 8.4, there is $\overline{\xi} \in \gamma\mathfrak{C}$ whose axis $\mathrm{ax}(\xi)$ intersects $\mathrm{ax}(g)$ nontrivially. This again contradicts the assumption $\gamma V(\mathfrak{C}) \cap V(\mathfrak{C}) = \varnothing$. □

Claim 3 *The union $\bigcup_{\overline{g}\in\mathfrak{C}} B_{\overline{g}}$ is connected.*

Proof Indeed, for any pair of adjacent elements $\overline{\gamma}$ and $\overline{\eta}$ in $\mathfrak{C}$, we have that the intersection $B_{\overline{\gamma}} \cap B_{\overline{\eta}}$ is not empty. Now enumerate $\mathfrak{C}$ by a possibly redundant sequence $\overline{\gamma}_1, \overline{\gamma}_2, \dots$ of consecutive adjacent vertices. Then the union $\bigcup_{i=1}^{\infty} B_{\overline{\gamma}_i}$ is connected. □

Since the union $\bigcup_{\overline{g}\in\mathfrak{C}} B_{\overline{g}}$ is connected, it is an interval of S^1 say with endpoints α_1, α_2, numbered such that

$$((\alpha_1, \alpha_2)) \subset \bigcup_{\overline{g}\in\mathfrak{C}} B_{\overline{g}} \subset [\![\alpha_1, \alpha_2]\!].$$

It follows then from Claim 2 that the intersection $\Delta_{\gamma\mathfrak{C}} \cap ((\alpha_1, \alpha_2))$ is empty; on the other hand, $\Delta_{\mathfrak{C}} \subseteq \bigcup_{\overline{g}\in\mathfrak{C}} B_{\overline{g}} \subset [\![\alpha_1, \alpha_2]\!]$. This implies that $Y_{\mathfrak{C}}$ and $Y_{\gamma\mathfrak{C}}$ lie in different half-planes determined by the geodesic joining α_1 to α_2 and hence the intersection $\gamma\mathring{Y}_{\mathfrak{C}} \cap Y_{\mathfrak{C}}$ is empty. This gives a contradiction.

Assume now that $\gamma Y_{\mathfrak{C}}$ is different from $Y_{\mathfrak{C}}$ and that the intersection $\gamma Y_{\mathfrak{C}} \cap Y_{\mathfrak{C}}$ is not empty. Let x be a point in the intersection $\gamma Y_{\mathfrak{C}} \cap Y_{\mathfrak{C}}$; then x belongs to the boundary of $\gamma Y_{\mathfrak{C}}$ and also to the boundary of $Y_{\mathfrak{C}}$. Let g and g' be the geodesics giving respectively the connected components of $\partial(\gamma Y_{\mathfrak{C}})$ and $\partial(Y_{\mathfrak{C}})$ containing x.

If $g \cap g' = \{x\}$, then the intersection of the interiors $\gamma \mathring{Y}_{\mathfrak{C}} \cap \mathring{Y}_{\mathfrak{C}}$ is not empty which, together with what we proved, implies that the $\gamma \mathring{Y}_{\mathfrak{C}}$ is equal to $\mathring{Y}_{\mathfrak{C}}$ and leads to a contradiction. Thus $g = g' \subseteq \partial(\gamma Y_{\mathfrak{C}}) \cap \partial Y_{\mathfrak{C}}$. Since the intersection $\gamma \mathring{Y}_{\mathfrak{C}} \cap \mathring{Y}_{\mathfrak{C}}$ is empty, we deduce that $\gamma Y_{\mathfrak{C}}$ and $Y_{\mathfrak{C}}$ lie on different sides of g, and hence $(\gamma Y_{\mathfrak{C}}) \cap Y_{\mathfrak{C}} = g$. □

Proposition 8.7 *Let $\mathfrak{C} \subset \mathcal{G}_\rho$ be a component with more than one vertex. Let $\Gamma_{\mathfrak{C}}$ be the stabilizer of $\mathfrak{C}$ in Γ and $Y_{\mathfrak{C}} \subset \mathbb{H}^2$ the closed convex hull of the limit set of $\Gamma_{\mathfrak{C}}$. Then the map*

$$\Gamma_{\mathfrak{C}} \backslash Y_{\mathfrak{C}} \hookrightarrow \Gamma \backslash \mathbb{H}^2$$

induces an embedding with image a compact surface with geodesic boundary.

Proof Let us enumerate the vertices $\{\overline{\gamma}_1, \overline{\gamma}_2, \dots\}$ of $V(\mathfrak{C})$ in such a way that for each i, we have $\overline{\gamma}_i$ adjacent to $\overline{\gamma}_{i+1}$. Let $\widetilde{x}_0$ be the intersection $\mathrm{ax}(\gamma_1) \cap \mathrm{ax}(\gamma_2)$, and define $X_n = \bigcup_{i=1}^n \mathrm{ax}(\gamma_i)$. By construction, X_n is connected. Let furthermore x_0 denote the projection $x_0 = p(\widetilde{x}_0)$.

Let $\Gamma_n < \Gamma$ be the image of the natural map $\pi_1(p(X_n), x_0) \to \pi_1(\Sigma, x_0)$ induced by the inclusion $p(X_n) \hookrightarrow \Sigma$. Then Γ_n is the fundamental group of the surface $\Sigma_n \subseteq \Sigma$ obtained by taking an appropriate tubular neighborhood of $p(X_n) \subseteq \Sigma$ and adding to it all components of the complement which are either simply connected or whose fundamental group is generated by a parabolic element of Γ. Then Σ_n is a subsurface with smooth boundary and of finite topological type. Since $\Gamma_n < \Gamma_{n+1}$, there exists $N \geq 1$ with $\Gamma_n = \Gamma_N$ for all $n \geq N$.

We will finish the proof by showing that $\Gamma_{\mathfrak{C}} = \Gamma_N$. Since $\Gamma_n \widetilde{x}_0 \subset \widetilde{p(X_n)}$, we have $\Gamma_n < \Gamma_{\mathfrak{C}}$. Conversely, let us take $\gamma \in \Gamma_{\mathfrak{C}}$; then $\gamma \widetilde{x}_0 = \mathrm{ax}(\gamma\gamma_1\gamma^{-1}) \cap \mathrm{ax}(\gamma\gamma_2\gamma^{-1})$, and since $\Gamma_{\mathfrak{C}}$ preserves $V(\mathfrak{C})$, we have that $\overline{\gamma\gamma_1\gamma^{-1}}$ and $\overline{\gamma\gamma_2\gamma^{-1}}$ are in $V(\mathfrak{C})$. Thus $\gamma \widetilde{x}_0 \in X_n$ for n large enough, which implies $\gamma \in \Gamma_n$. As a conclusion, we get $\Gamma_N = \Gamma_{\mathfrak{C}}$, which implies that $\Gamma_{\mathfrak{C}} \backslash Y_{\mathfrak{C}}$ in Σ is isotopic to Σ_N. □

Proof of Theorem 8.1 The set of isolated components of $\mathcal{G}_\rho$ is a Γ–invariant subset. Since we know that each isolated component of $\mathcal{G}_\rho$ corresponds to a geodesic of $\mathbb{H}^2$ that projects to a simple closed curve, we have that the projection of all the isolated components is a collection $\mathcal{C}$ of pairwise disjoint simple closed curves which cut the surface Σ in subsurfaces $\{\Sigma_v\}_{v \in \mathcal{V}}$ for some index set $\mathcal{V}$.

Moreover, for any component $\mathfrak{C}$ consisting of more than one element, we have that

$$Y_{\mathfrak{C}} = \overline{\mathrm{Co}\Big(\bigcup_{\overline{\gamma}\in\mathfrak{C}} \mathrm{ax}(\gamma)\Big)}$$

is a subsurface in $\mathbb{H}^2$ which projects to a subsurface of Σ whose boundary consists of elements of $\mathcal{C}$. In particular, there exists $v \in V$ with $p(Y_{\mathfrak{C}}) = \Sigma_v$. □

9 Quasi-isometric embeddings

Let $\rho\colon \pi_1(\Sigma, x) \to \mathrm{Sp}(V)$ be a representation admitting a maximal framing and $\Sigma = \bigcup_{v\in\mathcal{V}} \Sigma_v$ be the corresponding decomposition given by Theorem 8.1. We assume, as usual, that Σ is equipped with a hyperbolic metric of finite area and denote by $p\colon \mathbb{H}^2 \to \Sigma$ the canonical projection, so $\Sigma = \Gamma\backslash\mathbb{H}^2$.

As we have seen in Section 8, the decomposition of the surface Σ comes from a Γ–invariant decomposition

$$\mathbb{H}^2 = \bigcup_{w\in\widetilde{\mathcal{V}}} S_w$$

into subsurfaces with totally geodesic boundary. The Bass–Serre tree $\mathcal{T} = (\widetilde{\mathcal{V}}, E)$ can be identified with the incidence tree of the set $\{S_w \mid w \in \widetilde{\mathcal{V}}\}$. Recall that a pair $\{w_1, w_2\}$ forms an edge if the intersection $S_{w_1} \cap S_{w_2}$ is not empty. In this case, the intersection corresponds to the axis of an element of Γ that acts on the building $\mathcal{B}_V$ with zero translation length and determines an isolated component of the graph $\mathcal{G}_\rho$.

Assume now that for every subsurface Σ_v we are in the second case of the dichotomy in the decomposition theorem. Then for every $w \in \widetilde{\mathcal{V}}$, the stabilizer Γ_w of w in Γ has a canonical fixed point $b_w \in \mathcal{B}_V^0$ which equals $b_\gamma^\pm$ for each $\gamma \in \Gamma_w$.

Theorem 9.1 *The map*

$$\widetilde{\mathcal{V}} \to \mathcal{B}_V^0, \quad w \mapsto b_w,$$

is a Γ–equivariant quasi-isometry.

Let $\eta \in \Gamma$ be an element whose corresponding geodesic is not contained in a subsurface. The axis $\mathrm{ax}(\eta)$ determines a sequence $(w_n)_{n\in\mathbb{Z}}$ of vertices in $\mathcal{T}$, namely the consecutive sequence of surfaces S_{w_n} crossed by $\mathrm{ax}(\eta)$. This gives a geodesic path in $\mathcal{T}$, which is the axis of the isometry of $\mathcal{T}$ induced by η.

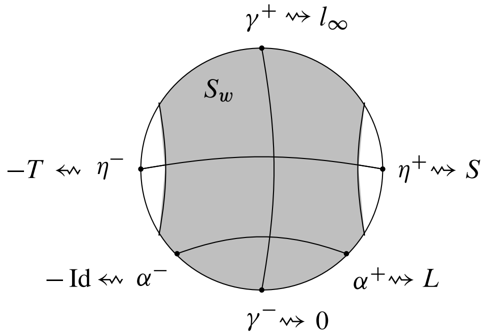


Figure 3: The setting in the proof of Lemma 9.2

Lemma 9.2 *Let us assume that the axis* $\mathrm{ax}(\eta)$ *crosses the surface* S_w. *Let* $\gamma \in \Gamma_w$ *be an element which is not boundary parallel and such that* $\mathrm{ax}(\gamma)$ *intersects* $\mathrm{ax}(\eta)$. *Then*

$$b_w = \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\eta}(\phi(\gamma^+))) = \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\eta}(\phi(\gamma^-))).$$

In particular, b_w *belongs to* $\mathbb{Y}_\eta$.

Proof Without loss of generality, we assume that the 4–tuple $(\eta^-, \gamma^-, \eta^+, \gamma^+)$ is positively oriented. We will show that all of the eigenvalues of the cross-ratio $R(\phi(\eta^-), \phi(\gamma^-), \phi(\eta^+), \phi(\gamma^+))$ have the form $1 + f$ for a positive f satisfying $\|f\| = 1$.

Since γ is not boundary parallel, we can find $\alpha \in \Gamma_w$ such that α^- belongs to $((\eta^-, \gamma^-))$ and α^+ belongs to $((\gamma^-, \eta^+))$; see Figure 3. Since $(\alpha^-, \gamma^-, \alpha^+, \gamma^+)$ is positively oriented, we can pick an element $g \in \mathrm{Sp}(V)$ with $g(\alpha^-, \gamma^-, \gamma^+) = (-\mathrm{Id}, 0, l_\infty)$. For such g, we set $g\phi(\eta^-) = -T$, $g\phi(\eta^+) = S$ and $g\phi(\alpha^+) = L$. With this notation we have $T \gg \mathrm{Id}$ and $S \gg L \gg 0$; moreover, the cross-ratio $R(\phi(\eta^-), \phi(\gamma^-), \phi(\eta^+), \phi(\gamma^+))$ is conjugate to $R(-T, 0, S, l_\infty) = \mathrm{Id} + T^{-1}S$.

First observe that all the eigenvalues of $R(-T, 0, S, l_\infty)$ are smaller than the corresponding eigenvalues of $R(-\mathrm{Id}, 0, S, l_\infty)$. Indeed, the first matrix is conjugate to $\mathrm{Id} + S^{1/2}T^{-1}S^{1/2}$ and the second equals $\mathrm{Id} + S$, moreover all the eigenvalues of T are by assumption greater than 1. Now α and γ cross and have zero translation length since they both belong to Γ_w. Since η^+ belongs to $((\alpha^+, \gamma^+))$, it follows from Lemma 7.13 that all the eigenvalues of $R(\phi(\alpha^-), \phi(\gamma^-), \phi(\eta^+), \phi(\gamma^+))$ have the form $1 + \lambda$ for a positive λ satisfying $\|\lambda\| = 1$. This implies that for each eigenvalue ν_i of $\mathrm{Id} + T^{-1}S$, we have $\|\nu_i - 1\| \le 1$.

On the other hand, all the eigenvalues of $R(-T, 0, S, l_\infty)$ are bigger than the corresponding eigenvalues of $R(-T, 0, L, l_\infty)$: indeed the first matrix is conjugate to $\mathrm{Id} + T^{-1/2} S T^{-1/2}$ and the second is conjugate to $\mathrm{Id} + T^{-1/2} L T^{-1/2}$. This implies that, denoting by μ_i the eigenvalues of $R(-T, 0, L, l_\infty)$ we have that $\|\nu_i - 1\| \geq \|\mu_i - 1\|$. This is enough to conclude: we have by Lemma 4.4 that $R(-T, 0, L, l_\infty) \cong R(L, l_\infty, -T, 0)$, and as a consequence of Lemma 7.13, this latter cross-ratio has all its eigenvalues of the form $1 + f$ for some positive f of norm one.

Now we exploit that b_w is in particular equal to $b_\gamma^\pm$. This latter point is, in view of Proposition 7.1, equal to $\pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(\eta^-)))$. Moreover, we deduce from Lemma 7.15 that

$$\pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\gamma}(\phi(\eta^-))) = \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\eta}(\phi(\gamma^-))) = \pi_{\mathcal{B}}(\mathrm{pr}_{\mathcal{Y}_\eta}(\phi(\gamma^+))),$$

and this concludes. □

Lemma 9.3 *Let $a, b \in \mathcal{L}(V)$ be transverse subspaces, and fix $x_1, \dots, x_k \in ((a, b))$ such that (a, x_i, x_{i+1}, b) is maximal for all i. Then*

$$\sum_{i=1}^{k-1} d(\mathrm{pr}_{\mathcal{Y}_{a,b}}(x_i), \mathrm{pr}_{\mathcal{Y}_{a,b}}(x_{i+1})) \leq \sqrt{n}\, d(\mathrm{pr}_{\mathcal{Y}_{a,b}}(x_1), \mathrm{pr}_{\mathcal{Y}_{a,b}}(x_k)).$$

Proof Since for each pair of symmetric matrices S, T we have $\det R(0, S, T, l_\infty) = \det S^{-1} \det T$, we deduce

$$\det R(a, x_1, x_k, b) = \prod_{j=1}^{k-1} \det R(a, x_j, x_{j+1}, b).$$

Thus we get

$$\ln \|\det R(a, x_1, x_k, b)\| = \sum_{j=1}^{k-1} \ln \|\det R(a, x_j, x_{j+1}, b)\|.$$

From Lemma 7.11, we deduce immediately

$$\sum_{i=1}^{k-1} d(\mathrm{pr}_{\mathcal{Y}_{a,b}}(x_i), \mathrm{pr}_{\mathcal{Y}_{a,b}}(x_{i+1})) \leq \sqrt{n}\, d(\mathrm{pr}_{\mathcal{Y}_{a,b}}(x_1), \mathrm{pr}_{\mathcal{Y}_{a,b}}(x_k)).$$ □

Proof of Theorem 9.1 Let v, w be vertices of $\mathcal{T}$, and pick an element $\eta \in \Gamma$ whose associated axis in $\mathcal{T}$ contains the geodesic path between v and w. Let us name $v_0 = v, v_1, \dots, v_k = w$ the vertices in such path.

We choose, for every i an element $\gamma_i \in S_{v_i}$ whose axis $\mathrm{ax}(\gamma_i)$ intersects the axis $\mathrm{ax}(\eta)$ nontrivially, and with the property that $\gamma_i^+ \in ((\eta^-, \eta^+))$. Then we have that, for every i, the 4–tuple

$$(\phi(\eta^-), \phi(\gamma_i^+), \phi(\gamma_{i+1}^+), \phi(\eta^+))$$

is maximal, and hence by Lemma 9.2 and 9.3, we have

$$\sum_{i=0}^{k-1} d_{\mathcal{B}}(b_{v_i}, b_{v_{i+1}}) \leq \sqrt{n}\, d_{\mathcal{B}}(b_{v_0}, b_{v_k}).$$

Notice that for any pair of adjacent vertices l, r in $\mathcal{T}$, the distance $d_{\mathcal{B}}(b_l, b_r)$ is positive: otherwise it is easy to verify that for each pair of hyperbolic elements $\gamma_l \in \Gamma_l$ and $\gamma_r \in \Gamma_r$, the composition $\gamma_l \gamma_r$ fixes $b_l = b_r$ and corresponds to an element of Γ whose axis crosses the common boundary component of S_l and S_r, contradicting the decomposition of Theorem 8.1.

Now, since the number of Γ–orbits on the set of edges of $\mathcal{T}$ is finite, there are positive constants C_1, C_2 with

$$C_1 \leq d_{\mathcal{B}}(b_l, b_r) \leq C_2$$

for every pair (l, r) of adjacent vertices. Thus we get

$$kC_1 \leq \sqrt{n}\, d_{\mathcal{B}}(b_{v_0}, b_{v_k}),$$

which implies

$$d_{\mathcal{T}}(v_0, v_k) \leq \frac{\sqrt{n}}{C_1} d_{\mathcal{B}}(b_{v_0}, b_{v_k}).$$

The inequality

$$d_{\mathcal{B}}(b_{v_0}, b_{v_k}) \leq C_2 k = C_2 d_{\mathcal{T}}(v_0, v_k)$$

is immediate. □

10 Ultralimits of maximal representations

In this section, we apply the general theory developed so far to the field of hyperreals and the Robinson field in order to deduce the decomposition theorem for ultralimits of maximal representations.

10.1 Hyperreals and Robinson fields

Let $\omega\colon \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ be a nonprincipal ultrafilter on the set of natural numbers. Recall that the ultraproduct $\prod_\omega X_i$ of a sequence $(X_i)_{i\in\mathbb{N}}$, of sets is the quotient of $\prod_{i\in\mathbb{N}} X_i$ by the equivalence relation $(x_i) \sim (y_i)$ if $\omega(\{i \mid x_i = y_i\}) = 1$. We denote by $\lambda_\omega\colon \prod_{i\in\mathbb{N}} X_i \to \prod_\omega X_i$ the quotient map and write X_ω for $\prod_\omega X$. In particular, $\mathbb{R}_\omega$ is the field of hyperreals, and if X_i are vector spaces over $\mathbb{R}$ (resp. $\mathbb{R}$–algebras, groups), then $\prod_\omega X$ is a $\mathbb{R}_\omega$–vector space (resp. an $\mathbb{R}_\omega$–algebra, a group) and λ_ω is a morphism in the appropriate category. For a $\mathbb{R}$–vector space V, the map

$$V \times \mathbb{R}_\omega \to V_\omega, \quad (v, [(l_i)]) \mapsto [(l_i v)],$$

induces an $\mathbb{R}_\omega$–isomorphism $V \otimes_{\mathbb{R}} \mathbb{R}_\omega \to V_\omega$. For V finite-dimensional at least, we deduce from the isomorphism $\mathrm{End}_{\mathbb{R}_\omega}(V \otimes_{\mathbb{R}} \mathbb{R}_\omega) \cong (\mathrm{End}\, V) \otimes_{\mathbb{R}} \mathbb{R}_\omega$ that the map

$$\prod_{i \in \mathbb{N}} \mathrm{End}(V) \to \mathrm{End}(V_\omega), \quad (T_i)_i \mapsto T,$$

where $T([v_i]) = [T_i(v_i)]$ induces an algebra isomorphism $(\mathrm{End}(V))_\omega \cong \mathrm{End}(V_\omega)$ which restricts to a group isomorphism $(\mathrm{GL}(V))_\omega \cong \mathrm{GL}(V_\omega)$. By abuse of notation, we will also denote by $\lambda_\omega\colon \prod_{\mathbb{N}} \mathrm{GL}(V) \to \mathrm{GL}(V_\omega)$ the induced map. Given a symplectic form $\langle \cdot, \cdot \rangle$ on V, let $\langle \cdot, \cdot \rangle_\omega$ denote the symplectic form on V_ω obtained by extending the scalars from $\mathbb{R}$ to $\mathbb{R}_\omega$. Given a sequence of representations $\rho_i\colon \Gamma \to \mathrm{Sp}(V)$, we will denote by ρ_ω the representation of Γ into $\mathrm{Sp}(V_\omega)$ obtained by composing $\prod_{i \in \mathbb{N}} \rho_i$ with λ_ω.

Proposition 10.1 *Assume that $\rho_i\colon \Gamma \to \mathrm{Sp}(V)$ is a sequence of maximal representations. Then $\rho_\omega\colon \Gamma \to \mathrm{Sp}(V_\omega)$ admits a maximal framing.*

The proof uses the following lemma, which is a straightforward verification:

Lemma 10.2 (1) *The map $\prod_{\mathbb{N}} \mathrm{Gr}_k(V) \to \mathrm{Gr}_k(V_\omega)$ defined by $(L_i)_{i \in \mathbb{N}} \mapsto \prod_\omega L_i$ induces a $(\mathrm{GL}(V))_\omega \cong \mathrm{GL}(V_\omega)$–equivariant bijection $(\mathrm{Gr}_k(V))_\omega \cong \mathrm{Gr}_k(V_\omega)$ and restricts to a $(\mathrm{Sp}(V))_\omega \cong \mathrm{Sp}(V_\omega)$–equivariant bijection $\mathcal{L}(V)_\omega \cong \mathcal{L}(V_\omega)$.*

(2) *Let $f_i\colon W_i \to \mathbb{R}$ be quadratic forms with signature $n_i \in \mathbb{Z}$. Assume that the sequence $\dim W_i$ is bounded, and let $f_\omega\colon \prod_\omega W_i \to \mathbb{R}_\omega$ be the quadratic form given by $f_\omega([(v_i)]) = [(f_i(v_i))]$. Then f_ω has signature n where n is defined by $\omega(\{i \mid n_i = n\}) = 1$.*

Proof of Proposition 10.1 Since each ρ_i is maximal, there exists a maximal framing $\phi_i\colon \partial\mathbb{H}^2 \to \mathcal{L}(V)$. Define then $\phi_\omega\colon \partial\mathbb{H}^2 \to \mathcal{L}(V_\omega)$ by composing $\prod \phi_i\colon \partial\mathbb{H}^2 \to \prod_{\mathbb{N}} \mathcal{L}(V)$ with the quotient map $\prod_{\mathbb{N}} \mathcal{L}(V) \to \mathcal{L}(V_\omega)$. The maximality of the so obtained framing follows then from Lemma 10.2(2). □

Let now $\sigma \in \mathbb{R}_\omega$ be an infinitesimal and recall the definition of the local ring

$$\mathcal{O}_\sigma = \{x \in \mathbb{R}_\omega : |x| < \sigma^{-k} \text{ for some } k \in \mathbb{N}\}$$

with associated maximal ideal

$$\mathcal{I}_\sigma = \{x \in \mathbb{R}_\omega : |x| < \sigma^{k} \text{ for all } k \in \mathbb{N}\}.$$

The quotient is the Robinson field $\mathbb{R}_{\omega,\sigma} = \mathcal{O}_\sigma / \mathcal{I}_\sigma$ associated to σ [24; 16].

Remark 10.3 Assuming the continuum hypothesis, a deep result of Erdös, Gillman and Henriksen [8] implies that the field $\mathbb{R}_\omega$ does not depend on the choice of the ultrafilter. And under the same hypothesis, Thornton showed that the normed field $\mathbb{R}_{\omega,\sigma}$ does not depend on the choice of the ultrafilter ω nor on the infinitesimal σ [28, Theorem 2.34].

If instead we assume the negation of the continuum hypothesis, it was shown by Kramer, Shelah, Tent and Thomas [12, Theorem 1.8] that there exists an uncountable set of nonprincipal ultrafilters such that the associated Robinson fields are pairwise nonisomorphic.

If (λ_i) is a divergent sequence of real numbers and we set $\sigma = [(e^{-\lambda_i})] \in \mathbb{R}_\omega$ we have that the field $\mathbb{R}_{\omega,\sigma}$ is the field denoted by $\mathbb{R}_{\omega,\lambda}$ in [22].

Now let ρ_ω be a representation into $\mathrm{Sp}(V_\omega)$ admitting the maximal framing (S, ϕ_ω). Choose a compatible complex structure J_ω and an infinitesimal $\sigma \in \mathbb{R}_\omega$ such that $\rho_\omega(\Gamma) \subseteq \mathrm{Sp}(V_\omega)(\mathcal{O}_\sigma)$, and denote by $V_{\omega,\sigma}$ the vector space $V_\omega(\mathcal{O}_\sigma)/V_\omega(\mathcal{I}_\sigma)$. According to Theorem 5.9, composing ρ_ω with $\pi_\sigma\colon \mathrm{Sp}(V_\omega)(\mathcal{O}_\sigma) \to \mathrm{Sp}(V_{\omega,\sigma})$ we obtain a representation which admits $q_\sigma \circ \phi_\omega\colon S \to \mathcal{L}(V_{\omega,\sigma})$ as maximal framing.

Thus we obtain in particular:

Corollary 10.4 *If $(\rho_i)_{i\in\mathbb{N}}\colon \Gamma \to \mathrm{Sp}(V)$ is a sequence of maximal representations where V is a real symplectic vector space, $\rho_\omega\colon \Gamma \to \mathrm{Sp}(V_\omega)$ the corresponding representation over the field of hyperreals, J_ω a choice of compatible complex structure and σ an infinitesimal such that $\rho_\omega(\Gamma) \subset \mathrm{Sp}(V_\omega)(\mathcal{O}_\sigma)$, then the representation $\rho_{\omega,\sigma}\colon \Gamma \to \mathrm{Sp}(V_{\omega,\sigma})$ admits a maximal framing defined on $\partial\mathbb{H}^2$.*

In the compact case we obtain a converse:

Theorem 10.5 *Assume that the surface $\Gamma\backslash\mathbb{H}^2$ is compact. Then a representation $\rho\colon \Gamma \to \mathrm{Sp}(V_{\omega,\sigma})$ admits a maximal framing if and only if there is a sequence $\rho_i\colon \Gamma \to \mathrm{Sp}(V)$ of maximal representations such that $\rho_{\omega,\sigma} = \rho$.*

Proof Let

$$\mathrm{Rep}_g := \left\{ (A_1, B_1, \dots, A_g, B_g) \in \mathrm{Sp}(V)^{2g} \;\middle|\; \prod_{i=1}^{n} [A_i, B_i] = \mathrm{Id} \right\}$$

be the $\mathbb{R}$–variety of representations of Γ in $\mathrm{Sp}(V)$. Then it follows from [28] that the reduction modulo $\mathcal{I}_\sigma$ induces a surjection $\mathrm{Rep}_g(\mathcal{O}_\sigma) \to \mathrm{Rep}_g(\mathbb{R}_{\omega,\sigma})$. Thus we can lift ρ to a representation $\rho_\omega\colon \Gamma \to \mathrm{Sp}(V_\omega)(\mathcal{O}_\sigma)$ which we represent by a sequence $(\rho_i)_{i\in\mathbb{N}}$ of representations of Γ into $\mathrm{Sp}(V)$.

Let $\phi\colon S \to \mathcal{L}(V_{\omega,\sigma})$ be a maximal framing for ρ. It follows from the collar lemma that for every hyperbolic element $\gamma \in \Gamma$, the image $\rho(\gamma)$ is Shilov hyperbolic. Then $\rho_\omega(\gamma)$ needs also to be Shilov hyperbolic and we have $q_\sigma(L^+_{\rho_\omega(\gamma)}) = L^+_{\rho(\gamma)}$ because of uniqueness of attractive fixed Lagrangians.

Fix a decomposition of $\Sigma = \Gamma\backslash\mathbb{H}^2$ into pairs of pants, let $P \subseteq \Sigma$ denote any such pair of pants and let $\{c_1, c_2, c_3\}$ be standard generators of $\pi_1(P)$; in particular, $c_1c_2c_3 = e$. Let ξ_1, ξ_2, ξ_3 be the attractive fixed points in $\partial\mathbb{H}^2$ of c_1, c_2, c_3. Then (ξ_1, ξ_2, ξ_3) and $(\xi_1, c_1\cdot\xi_3, \xi_2)$ are positively oriented. Thus the images under ϕ of the two triples are maximal, and hence the triples

$$(L^+_{\rho_\omega(c_1)}, L^+_{\rho_\omega(c_2)}, L^+_{\rho_\omega(c_3)}) \quad \text{and} \quad (L^+_{\rho_\omega(c_1)}, \rho_\omega(c_1)L^+_{\rho_\omega(c_3)}, L^+_{\rho_\omega(c_2)})$$

are maximal. It follows that there is a set $E_P \subset \mathbb{N}$ of full ω–measure such that for each i in E_P, $\rho_i(c_1), \rho_i(c_2), \rho_i(c_3)$ are Shilov hyperbolic and both

$$(L^+_{\rho_i(c_1)}, L^+_{\rho_i(c_2)}, L^+_{\rho_i(c_3)}) \quad \text{and} \quad (L^+_{\rho_i(c_1)}, \rho_i(c_1)L^+_{\rho_i(c_3)}, L^+_{\rho_i(c_2)})$$

are maximal. It follows then from [27, Theorem 5] that $\rho_i|_{\pi_1(P)} \to \mathrm{Sp}(V)$ is maximal for each i in E_P. Thus if $P_1, \dots, P_{2g-2}$ is the pair of pants decomposition, we have that for all $i \in \bigcap_{j=1}^{2g-2} E_{P_j}$, the restriction $\rho_i|_{\pi_1(P_j)}$ is maximal. By additivity of the Toledo invariant (see [6, Theorem 1]), we deduce that ρ_i is maximal. Since $\bigcap_{j=1}^{2g-2} E_{P_j}$ is of full ω–measure, this concludes the proof. □

10.2 Asymptotic cones

We finish the paper deducing the statements about ultralimits of maximal representations from the general theory of representations admitting a maximal framing.

Proof of Theorem 1.1 Let $\rho_k\colon \Gamma \to \mathrm{Sp}(V)$ be a sequence of maximal representations, $J_k \in \mathbb{X}_V$ a sequence of basepoints, namely a sequence of compatible complex structures, and $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ an adapted sequence of scales. If the sequence λ is bounded on a set of full ω–measure, then we may assume

$$\sup_{k\in\mathbb{N}} \max_{\gamma\in S} d(\rho_k(\gamma)J_k, J_k) < \infty,$$

and hence, if we conjugate ρ_k by $g_k \in \mathrm{Sp}(V)$ with $g_kJ_k = x$ a fixed basepoint, it follows that the sequence $(\pi_k = g_k\rho_k g_k^{-1})_{k\in\mathbb{N}}$ is relatively compact in the space of representations. In this case, ${}^\omega\mathcal{X}_\lambda$ is just the Siegel space $\mathcal{X}_\mathbb{R}$ with rescaled distance, and ${}^\omega\rho_\lambda$ is an ordinary accumulation point of the sequence $(\pi_k)_{k\in\mathbb{N}}$.

If the sequence λ is unbounded, let $\sigma := (e^{-\lambda_k})_{k\in\mathbb{N}}$, which is an infinitesimal in $\mathbb{R}_\omega$, and let $J_\omega := [(J_k)_{k\in\mathbb{N}}] \in \mathrm{End}(V_\omega)$ which is a compatible complex structure. Then we conclude from the fact that λ is adapted to $(\rho_k, J_k)_{k\in\mathbb{N}}$ that $\rho_\omega(\Gamma) \subset \mathrm{Sp}(V_\omega)(\mathcal{O}_\sigma)$.

Furthermore, it follows from [22] that the action on the Bruhat–Tits building of $\mathrm{Sp}(V_{\omega,\sigma})$ coming from the representation $\rho_{\omega,\sigma}\colon \Gamma \to \mathrm{Sp}(V_{\omega,\sigma})$ coincides with the ultralimit ${}^{\omega}\rho_\lambda\colon \Gamma \to \mathrm{Iso}({}^{\omega}\mathcal{X}_\lambda)$ under the identification of ${}^{\omega}\mathcal{X}_\lambda$ with the Bruhat–Tits building $\mathcal{B}_{V_{\omega,\sigma}}$. Theorem 1.1 follows then from Corollary 10.4 and Theorem 8.1. □

We now characterize the cases which lead to actions without a global fixed point. Recall from the introduction that when S is a finite generating set for Γ, and ρ is a maximal representation we denote by $D_S(\rho)(x)$ the displacement function.

The function $D_S(\rho)$ is convex and, since $\rho(\Gamma)$ is not contained in any proper parabolic subgroup of $\mathrm{Sp}(V)$, we have that for every $C > 0$, the convex set $\{x \mid D_S(\rho)(x) \le C\}$ must be compact; in particular, $D_S(\rho)(x)$ achieves its minimum that we will denote by $\mu_S(\rho) = \min_{x\in\mathcal{X}} D_S(\rho)(x)$.

The function $\rho \mapsto \mu_S(\rho)$ descends then to a proper function

$$\mathrm{Hom}_{\max}(\Gamma, \mathrm{Sp}(V))/\,\mathrm{Sp}(V) \to (0,\infty)$$

on the character variety of maximal representations. Let now $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of maximal representations, $(x_k)_{k\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$ a sequence of basepoints and λ an adapted sequence of scales. Furthermore, let $y_k \in \mathcal{X}$ be such that $\mu_S(\rho_k) = D_S(\rho_k)(y_k)$.

Proposition 10.6 *The representation ${}^{\omega}\rho_\lambda$ on ${}^{\omega}\mathcal{X}_\lambda$ has no global fixed point if and only if*

$$\lim_\omega \frac{\lambda_k}{\mu_S(\rho_k)} < \infty \quad \textit{and} \quad \lim_\omega \frac{d(y_k, x_k)}{\lambda_k} < \infty,$$

in which case ${}^{\omega}\mathcal{X}_\lambda = {}^{\omega}\mathcal{X}_\mu$, the distances on the asymptotic cones are homothetic and the actions ${}^{\omega}\rho_\lambda$ and ${}^{\omega}\rho_\mu$ coincide.

Remark 10.7 We can also deduce the fact that if ${}^{\omega}\rho_\lambda$ has no global fixed point then the limit $\lim_\omega \lambda_k/\mu_S(\rho_k)$ is finite by combining [22, Proposition 4.4] and [21, Corollary 3].

Proof of Proposition 10.6 For the "if" part: changing the sequence on a set of ω–measure zero, we may assume that for some constant $C > 0$, we have $\mu_S(\rho_k)/C \le \lambda_k \le C\mu_S(\rho_k)$ and $d(y_k, x_k) \le C\lambda_k$ for all $k \in \mathbb{N}$. This readily implies that the asymptotic cones ${}^{\omega}\mathcal{X}_\lambda$ and ${}^{\omega}\mathcal{X}_\mu$ are equal, that the induced distances are homothetic with factor $\lim_\omega \lambda_k/\mu_S(\rho_k)$ and that the actions ${}^{\omega}\rho_\lambda$ and ${}^{\omega}\rho_\mu$ coincide. Thus we have to verify that ${}^{\omega}\rho_\mu$ does not have a global fixed point. But this follows immediately from the fact that

$$\max_{\gamma\in S} \frac{d(\rho_k(\gamma)x, x)}{\mu_S(\rho_k)} \ge 1 \quad \text{for all } x \in \mathcal{X}.$$

We next show the "only if" part. Let T be a finite connected generating set, and let us denote by K the maximal length of an element of T with respect to the generating set S. Since ${}^{\omega}\rho_\lambda$ does not have a global fixed point, it follows from Corollary 7.6 that there is $\gamma_0 \in T$ with $L({}^{\omega}\rho_\lambda(\gamma_0)) = \lim_\omega L(\rho_k(\gamma_0))/\lambda_k > 0$. Since

$$L(\rho_k(\gamma_0)) \le d(\rho_k(\gamma_0)y_k, y_k) \le K\mu_S(\rho_k) \le KD_S(\rho_k)(x_k)$$

and $\lim_\omega D_S(\rho_k)(x_k)/\lambda_k < \infty$, we may assume that the sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(\mu_S(\rho_k))_{k\in\mathbb{N}}$ are equivalent, namely that there are positive constants C_1, C_2 such that $C_1\mu_S(\rho_k) \le \lambda_k \le C_2\mu_S(\rho_k)$ for all $k \in \mathbb{N}$.

Pick now two hyperbolic elements γ, η in Γ with intersecting axes. If $\phi_k\colon S^1 \to \mathcal{L}(V)$ denotes the boundary map associated to ρ_k, we have

$$\mathcal{Y}_{\phi_k(\gamma^+),\phi_k(\gamma^-)} \cap \mathcal{Y}_{\phi_k(\eta^+),\phi_k(\eta^-)} = \{z_k\},$$

and the sequence $(z_k)_{k\in\mathbb{N}}$ in ${}^{\omega}\mathcal{X}_\lambda$ represents a point in the intersection $\mathbb{Y}_\gamma^\lambda \cap \mathbb{Y}_\eta^\lambda$; see Section 7. Thus we get $\lim_\omega d(x_k, z_k)/\lambda_k < \infty$. The same applies to ${}^{\omega}\mathcal{X}_\mu$ and hence $\lim_\omega d(y_k, z_k)/\mu_S(\rho_k) < \infty$. Using the triangle inequality and taking into account that the sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(\mu_S(\rho_k))_{k\in\mathbb{N}}$ are equivalent, we deduce

$$\lim_\omega \frac{d(x_k, y_k)}{\lambda_k} < \infty. \qquad \square$$

Proof of Corollary 1.3 The first inequality follows from the collar lemma, while the last follows by contradiction from Proposition 10.6. $\square$

Proof of Corollary 1.4 Applying iteratively Theorem 1.1, it is possible to obtain a canonical decomposition of the surface in subsurfaces with geodesic boundary with the property that all curves strictly contained in a subsurface have the same growth rate. The set $\mathcal{C}$ of curves defining this decomposition is the union of the curves given by Theorem 1.1 and all the curves contained in subsurfaces of type (FP) selected by applying Theorem 1.1 to the restrictions of the representations to those subsurfaces. One can apply Theorem 1.1 at most $3g-3+p$ times corresponding to the case when at each step precisely one curve is added and all the complementary pieces are of type (FP). Hence there are at most $3g-3+p$ distinct growth rates among curves having nontrivial intersection with $\mathcal{C}$. There are three possibilities for the remaining curves: either a curve is contained in a subsurface defined by the decomposition $\mathcal{C}$, or it is one of the curves in $\mathcal{C}$ or it corresponds to a puncture in the surface. The claim follows since there are at most $2g-2+p$ complementary components. $\square$

Appendix

Proposition A.1 *Let $\mathbb{F}$ be a real closed field. Let n be a positive integer and assume that $a_1, \dots, a_n \geq 1$. Then we have*

$$(a_1 a_2 \cdots a_n - 1)^n \geq (a_1^n - 1)(a_2^n - 1) \cdots (a_n^n - 1),$$

with equality if and only if $a_1 = \cdots = a_n$.

For $\mathbb{F} = \mathbb{R}$, this follows easily from the convexity of the function $e^x/(e^x - 1)$; here we reproduce the proof due to Thomas Huber for general real closed fields. We start with a key lemma:

Lemma A.2 *Let n be a positive integer, and let $c, x \geq 1$. Then we have*

$$(cx - 1)^n \geq (c^n x - 1)(x - 1)^{n-1}, \tag{10}$$

with equality if and only if $n = 1$ or $c = 1$.

Proof We use induction. For $n = 1$, the inequality is in fact an equality. By induction,

$$(cx - 1)^{n+1} = (cx - 1)(cx - 1)^n \geq (cx - 1)(c^n x - 1)(x - 1)^{n-1}$$

(observe that all factors are nonnegative), and it suffices to show that

$$(cx - 1)(c^n x - 1) \geq (c^{n+1} x - 1)(x - 1)$$

holds. But the difference of the left and the right hand side factors as

$$x(c - 1)^2(c^{n-1} + \cdots + c + 1)$$

and is clearly nonnegative. □

Now we turn to the proof of the main result and proceed again by induction. For $n = 1$, there is nothing to show; hence let $n \geq 2$. By symmetry, we may assume that $a = a_1 \geq a_i$ for all $i \geq 2$. By the induction hypothesis, the right hand side of the inequality does not decrease when we replace $a_2, \dots, a_n$ by their geometric mean $b = (a_2 \cdots a_n)^{1/(n-1)}$; notice that in a real closed field, positive numbers admit k^{th} roots for any natural number $k \geq 1$. Therefore, it suffices to show the inequality

$$(ab^{n-1} - 1)^n \geq (a^n - 1)(b^n - 1)^{n-1},$$

where $a \geq b \geq 1$. But this is a direct consequence of our lemma: just set $c = a/b \geq 1$ and $x = b^n \geq 1$ in (10). Equality only holds for $c = 1$, that is, for $a = b$. But this implies $a_1 = \cdots = a_n$ by the maximal choice of a_1.

References

[1] **D Alessandrini**, *Tropicalization of group representations*, Algebr. Geom. Topol. 8 (2008) 279–307 MR

[2] **F Bruhat**, **J Tits**, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. 41 (1972) 5–251 MR

[3] **G W Brumfiel**, *The real spectrum compactification of Teichmüller space*, from "Geometry of group representations" (W M Goldman, A R Magid, editors), Contemp. Math. 74, Amer. Math. Soc., Providence, RI (1988) 51–75 MR

[4] **G W Brumfiel**, *The tree of a non-Archimedean hyperbolic plane*, from "Geometry of group representations" (W M Goldman, A R Magid, editors), Contemp. Math. 74, Amer. Math. Soc., Providence, RI (1988) 83–106 MR

[5] **M Burger**, **A Iozzi**, **F Labourie**, **A Wienhard**, *Maximal representations of surface groups: symplectic Anosov structures*, Pure Appl. Math. Q. 1 (2005) 543–590 MR

[6] **M Burger**, **A Iozzi**, **A Wienhard**, *Surface group representations with maximal Toledo invariant*, Ann. of Math. 172 (2010) 517–566 MR

[7] **B Collier**, **Q Li**, *Asymptotics of Higgs bundles in the Hitchin component*, Adv. Math. 307 (2017) 488–558 MR

[8] **P Erdös**, **L Gillman**, **M Henriksen**, *An isomorphism theorem for real-closed fields*, Ann. of Math. 61 (1955) 542–554 MR

[9] **I Kaplansky**, *Linear algebra and geometry: a second course*, revised edition, Dover, Mineola, NY (2003) MR

[10] **L Katzarkov**, **A Noll**, **P Pandit**, **C Simpson**, *Constructing buildings and harmonic maps*, preprint (2015) arXiv

[11] **B Kleiner**, **B Leeb**, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Inst. Hautes Études Sci. Publ. Math. 86 (1997) 115–197 MR

[12] **L Kramer**, **S Shelah**, **K Tent**, **S Thomas**, *Asymptotic cones of finitely presented groups*, Adv. Math. 193 (2005) 142–173 MR

[13] **L Kramer**, **K Tent**, *Asymptotic cones and ultrapowers of Lie groups*, Bull. Symbolic Logic 10 (2004) 175–185 MR

[14] **I Le**, *Higher laminations and affine buildings*, Geom. Topol. 20 (2016) 1673–1735 MR

[15] **G-S Lee**, **T Zhang**, *Collar lemma for Hitchin representations*, Geom. Topol. 21 (2017) 2243–2280 MR

[16] **A H Lightstone**, **A Robinson**, *Nonarchimedean fields and asymptotic expansions*, North-Holland, Amsterdam (1975) MR

[17] **G Lion**, **M Vergne**, *The Weil representation, Maslov index and theta series*, Progress in Mathematics 6, Birkhäuser, Boston (1980) MR

[18] **J Loftin**, *Convex $\mathbb{R}P^2$ structures and cubic differentials under neck separation*, preprint (2015) arXiv

[19] **R Mazzeo**, **J Swoboda**, **H Weiss**, **F Witt**, *Ends of the moduli space of Higgs bundles*, Duke Math. J. 165 (2016) 2227–2271 MR

[20] **A Parreau**, *Immeubles affines: construction par les normes et étude des isométries*, from "Crystallographic groups and their generalizations" (P Igodt, H Abels, Y Félix, F Grunewald, editors), Contemp. Math. 262, Amer. Math. Soc., Providence, RI (2000) 263–302 MR

[21] **A Parreau**, *Sous-groupes elliptiques de groupes linéaires sur un corps valué*, J. Lie Theory 13 (2003) 271–278 MR

[22] **A Parreau**, *Compactification d'espaces de représentations de groupes de type fini*, Math. Z. 272 (2012) 51–86 MR

[23] **A Parreau**, *Invariant subspaces for some surface groups acting on A2–Euclidean buildings*, preprint (2015) arXiv

[24] **A Robinson**, *Function theory on some nonarchimedean fields*, Amer. Math. Monthly 80 (1973) 87–109 MR

[25] **I Satake**, *Algebraic structures of symmetric domains*, Kanô Memorial Lectures 4, Iwanami Shoten, Tokyo (1980) MR

[26] **C L Siegel**, *Symplectic geometry*, Amer. J. Math. 65 (1943) 1–86 MR

[27] **T Strubel**, *Fenchel–Nielsen coordinates for maximal representations*, Geom. Dedicata 176 (2015) 45–86 MR

[28] **B Thornton**, *Asymptotic cones of symmetric spaces*, PhD thesis, The University of Utah (2002) MR Available at `https://search.proquest.com/docview/305520472`

[29] **T Zhang**, *The degeneration of convex $\mathbb{RP}^2$ structures on surfaces*, Proc. Lond. Math. Soc. 111 (2015) 967–1012 MR

[30] **T Zhang**, *Degeneration of Hitchin representations along internal sequences*, Geom. Funct. Anal. 25 (2015) 1588–1645 MR

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