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Thurston norm via Fox calculus

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In 1976 Thurston associated to a 3–manifold N a marked polytope in $H_1(N;\mathbb{R})$, which measures the minimal complexity of surfaces representing homology classes and determines all fibered classes in $H^1(N;\mathbb{R})$. Recently the first and third authors associated to a presentation π with two generators and one relator a marked polytope in $H_1(\pi;\mathbb{R})$ and showed that it determines the Bieri–Neumann–Strebel invariant of π. We show that if the fundamental group of a 3–manifold N admits such a presentation π, then the corresponding marked polytopes in $H_1(N;\mathbb{R}) = H_1(\pi;\mathbb{R})$ agree.

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1 Summary of results

Throughout this paper all 3–manifolds are compact, connected and orientable. Suppose N is a 3–manifold. In 1976 Thurston [49] introduced a seminorm x_N on $H^1(N;\mathbb{R})$, henceforth referred to as the *Thurston norm*, which is a natural measure of the complexity of surfaces dual to integral classes. A class $\phi \in H^1(N;\mathbb{R})$ is *fibered* if ϕ can be represented by a nondegenerate closed 1–form. If ϕ is integral, then ϕ is fibered if and only if it is induced by a surface bundle $N \to S^1$. We refer to Section 2.4 for details.

Thurston [49] showed that the information on the Thurston seminorm and the fibered classes can be encapsulated in terms of a *marked polytope*.

A marked polytope is a polytope in a vector space together with a (possibly empty) set of marked vertices. In order to state Thurston's result precisely we need one more definition. Given a polytope in a vector space V we say that a homomorphism $\phi \in \operatorname{Hom}(V,\mathbb{R})$ *pairs maximally with the vertex* v if $\phi(v) > \phi(w)$ for all other vertices $w \neq v$. In this language, the main result of [49] can be stated as follows:

Theorem 1.1 *Let N be a 3–manifold. There exists a unique symmetric marked polytope $\mathcal{M}_N$ in $H_1(N;\mathbb{R})$ such that for any $\phi \in H^1(N;\mathbb{R}) = \operatorname{Hom}(\pi_1(N),\mathbb{R})$ we have*

$$x_N(\phi) = \max\{\phi(p) - \phi(q) \mid p,q \in \mathcal{M}_N\},$$

and ϕ is fibered if and only if it pairs maximally with a marked vertex of $\mathcal{M}_N$.

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Subsequently, by a $(2,1)$–presentation we mean a group presentation with precisely two generators and one nonempty relator. A $(2,1)$–presentation is *cyclically reduced* if the relator is a cyclically reduced word. Recently, the first and third authors [24] associated to a cyclically reduced $(2,1)$–presentation $\pi = \langle x, y \mid r \rangle$ a marked polytope $\mathcal{M}_\pi$ in $H_1(\pi;\mathbb{R})$.

Now we outline the definition of $\mathcal{M}_\pi$ in the case that $b_1(\pi) = 2$. A different (but equivalent) definition is given in Section 2.6, as well as a definition for cyclically reduced $(2,1)$–presentations π with $b_1(\pi) = 1$.

Identify $H_1(G_\pi;\mathbb{Z})$ with $\mathbb{Z}^2$ such that x corresponds to $(1,0)$ and y corresponds to $(0,1)$. Then the relator r determines a discrete walk on the integer lattice in $H_1(G_\pi;\mathbb{R})$, and the marked polytope $\mathcal{M}_\pi$ is obtained from the convex hull of the trace of this walk as follows:

(1) Start at the origin and walk across $\mathbb{Z}^2$ reading the word r from the left.

(2) Take the convex hull $\mathcal{C}$ of the set of all lattice points reached by the walk.

(3) Mark precisely those vertices of $\mathcal{C}$ which the walk passes through exactly once.

(4) Now consider the unit squares that are completely contained in $\mathcal{C}$ and touch a vertex of $\mathcal{C}$. Mark a midpoint of a square precisely when one (and hence all) vertices of $\mathcal{C}$ incident with the square are marked.

(5) The set of vertices of $\mathcal{M}_\pi$ is the set of midpoints of all of these squares, and a vertex of $\mathcal{M}_\pi$ is marked precisely when it is a marked midpoint of a square.

In Figure 1 we sketch the construction of $\mathcal{M}_\pi$ for the presentation $\pi = \langle x, y \mid r \rangle$, where

$$r = x^2yx^{-1}yx^2yx^{-1}y^{-3}x^{-1}yx^2yx^{-1}yxy^{-1}x^{-2}y^{-1}xy^{-1}x^{-2}y^{-1}xy^3xy^{-1} \cdot x^{-2}y^{-1}xy^{-1}x^{-1}y.$$

This example is due to Dunfield [12] and presents the fundamental group of the exterior of the 2–component link in S^3 shown in Figure 2 (see Section 6.3).

Given two polytopes $\mathcal{P}$ and $\mathcal{Q}$ in a vector space V, we write $\mathcal{P} \doteq \mathcal{Q}$ if the polytopes $\mathcal{P}$ and $\mathcal{Q}$ differ by a translation, ie if there exists $v \in V$ with $\mathcal{P} = v + \mathcal{Q}$. The following is the main theorem of this paper:

Theorem 1.2 *Let N be an irreducible 3–manifold that admits a cyclically reduced $(2,1)$–presentation $\pi = \langle x, y \mid r \rangle$. Then*

$$\mathcal{M}_N \doteq \mathcal{M}_\pi.$$

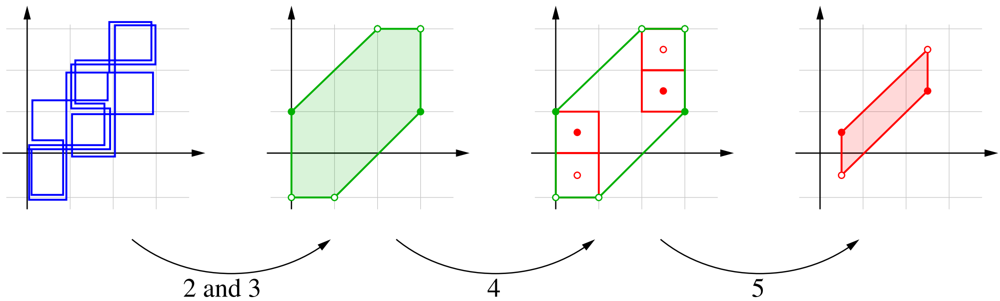

Figure 1: The marked polytope $\mathcal{M}_\pi$ for Dunfield's example, starting from the path determined by the relation r. Marked vertices are filled and unmarked vertices are empty; Labels on arrows correspond to steps in the above algorithm.

This theorem answers in particular a question of Sikorav [48] in the affirmative for 3–manifolds that admit a $(2,1)$–presentation.

The proof of Theorem 1.2 relies on the virtually special theorem of Agol [2], Liu [36], Przytycki and Wise [42; 43] and Wise [56; 57; 58], which we recall in Section 3.1. It also hinges on the following general result, which is of independent interest.

Theorem 1.3 *Let N be an irreducible 3–manifold with empty or toroidal boundary. If N is not a closed graph manifold, then $\pi_1(N)$ is residually a torsion-free and elementary amenable group.*

The proof of Theorem 1.3 uses the virtually special theorem and builds on work of Linnell and Schick [35]. It is proved in Section 3, where we also give several consequences.

We give a brief outline of the proof of Theorem 1.2. The starting point is an alternative definition of the marked polytope $\mathcal{M}_\pi$ using Fox derivatives [17] (see Section 2.6). This definition is less pictorial, but it allows us to relate the polytope $\mathcal{M}_\pi$ to the chain complex of the universal cover of the 2–complex X associated to the presentation π. This makes it possible to study the "size" of $\mathcal{M}_\pi$ using twisted Reidemeister torsions corresponding to finite-dimensional complex representations and corresponding to skew fields of X; see Cochran [8], Friedl [18], Friedl and Vidussi [25], Harvey [27] and Wada [54]. Since X is simple homotopy equivalent to N, these twisted Reidemeister torsions agree with the twisted Reidemeister torsions of N.

In the following we denote by $\mathcal{P}_N$ and $\mathcal{P}_\pi$ the polytopes $\mathcal{M}_N$ and $\mathcal{M}_\pi$ without the markings. Given two polytopes $\mathcal{P}$ and $\mathcal{Q}$ in a vector space V we write $\mathcal{P} \leq \mathcal{Q}$ if there exists $v \in V$ with $v + \mathcal{P} \subset \mathcal{Q}$. The proof of Theorem 1.2 now breaks up into three parts:

(1) We first show that $\mathcal{P}_N \le \mathcal{P}_\pi$. Put differently, we show that $\mathcal{P}_\pi$ is "big enough" to contain $\mathcal{P}_N$. This is achieved with the main theorem of Friedl and Vidussi [26], which states that twisted Reidemeister torsions corresponding to finite-dimensional complex representations detect the Thurston norm of N. This relies on the virtually special theorem. See Section 4.

(2) Next we show the reverse inclusion $\mathcal{P}_\pi \le \mathcal{P}_N$. This means that $\mathcal{P}_\pi$ is "not bigger than necessary". At this stage it is crucial that r is cyclically reduced. Using Theorem 1.3 and the noncommutative Reidemeister torsions of Cochran [8], Friedl [18] and Harvey [27] we show that indeed $\mathcal{P}_\pi \subset \mathcal{P}_N$. See Section 5.

(3) Finally we need to show that the markings of $\mathcal{M}_N$ and $\mathcal{M}_\pi$ agree. This follows immediately from Friedl and Tillmann [24, Theorem 1.1] and Bieri, Neumann and Strebel [4, Theorem E]. See Section 5.3.

The paper is concluded with a conjecture and a question in Section 7.

Convention Throughout this paper, all groups are finitely generated, all vector spaces are finite-dimensional, and all 3–manifolds are compact, connected and orientable.

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2 Polytopes associated to 3–manifolds and groups

2.1 Polytopes

Let V be a real vector space and let $Q = \{Q_1, \dots, Q_k\} \subset V$ be a finite (possibly empty) subset. Denote by

$$\mathcal{P}(Q) = \operatorname{conv}(Q) = \left\{ \sum_{i=1}^{k} t_i Q_i \,\middle|\, \sum_{i=1}^{k} t_i = 1,\ t_i \ge 0 \right\}$$

the *polytope spanned by* Q. A *polytope* in V is a subset of the form $\mathcal{P}(Q)$ for some finite subset Q of V. For any polytope $\mathcal{P}$ there exists a unique smallest subset $\mathcal{V}(\mathcal{P}) \subset \mathcal{P}$ such that $\mathcal{P}$ is the polytope spanned by $\mathcal{V}(\mathcal{P})$. The elements of $\mathcal{V}(\mathcal{P})$ are the *vertices* of $\mathcal{P}$. Note $v \in \mathcal{P}$ is a vertex if and only if there exists a homomorphism $\phi\colon V \to \mathbb{R}$ such that $\phi(v) > \phi(p)$ for every $p \in \mathcal{P}$ with $p \neq v$.

Let V be a real vector space and let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes in V. The *Minkowski sum of* $\mathcal{P}$ *and* $\mathcal{Q}$ is

$$\mathcal{P} + \mathcal{Q} := \{p + q \mid p \in \mathcal{P} \text{ and } q \in \mathcal{Q}\}.$$

It is straightforward to see that $\mathcal{P} + \mathcal{Q}$ is again a polytope. Furthermore, for each vertex u of $\mathcal{P} + \mathcal{Q}$ there exists a unique vertex v of $\mathcal{P}$ and a unique vertex w of $\mathcal{Q}$ such that $u = v + w$. Conversely, for each vertex v of $\mathcal{P}$ there exists a (not necessarily unique) vertex w of $\mathcal{Q}$ such that $v + w$ is a vertex of $\mathcal{P} + \mathcal{Q}$.

If $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are polytopes with $\mathcal{P} + \mathcal{Q} = \mathcal{R}$, then we write $\mathcal{P} = \mathcal{R} - \mathcal{Q}$. We have

$$\mathcal{P} = \{p \in V \mid p + \mathcal{Q} \subset \mathcal{R}\};$$

in particular, $\mathcal{R} - \mathcal{Q}$ is well-defined.

There is a natural scaling operation on polytopes

$$\lambda \cdot \mathcal{P} := \{\lambda p \mid p \in \mathcal{P}\},$$

where $\mathcal{P} \subset V$ is a polytope and $\lambda \in \mathbb{R}^+$. If $k \in \mathbb{N}$, then the Minkowski sum of k copies of $\mathcal{P}$ equals $k\mathcal{P}$.

2.2 Convex sets and seminorms

Let $\mathcal{C}$ be a nonempty convex set in the real vector space V. Given $\phi \in \operatorname{Hom}(V, \mathbb{R})$ we define the *thickness of* $\mathcal{C}$ *in the* ϕ*–direction* by

$$\operatorname{th}_{\mathcal{C}}(\phi) := \max\{\phi(c) - \phi(d) \mid c, d \in \mathcal{C}\}.$$

It is straightforward to see that the function

$$\lambda_{\mathcal{C}}\colon \operatorname{Hom}(V, \mathbb{R}) \to \mathbb{R}_{\geq 0}, \quad \phi \mapsto \operatorname{th}_{\mathcal{C}}(\phi),$$

is a seminorm. Conversely, a seminorm $\lambda\colon \operatorname{Hom}(V, \mathbb{R}) \to \mathbb{R}_{\geq 0}$ defines the convex set

$$\mathcal{C}(\lambda) := \{v \in V \mid \phi(v) \leq 1 \text{ for all } \phi \in \operatorname{Hom}(V, \mathbb{R}) \text{ with } \lambda(\phi) \leq 1\}.$$

Note that $\mathcal{C}(\lambda)$ is *symmetric* since $v \in \mathcal{C}(\lambda)$ implies $-v \in \mathcal{C}(\lambda)$. For any seminorm λ on $\operatorname{Hom}(V, \mathbb{R})$ we have $\lambda_{\mathcal{C}(\lambda)} = \lambda$. On the other hand, if $\mathcal{C}$ is a nonempty convex set of V, then $\mathcal{C}(\lambda_{\mathcal{C}})$ equals the symmetrization of $\mathcal{C}$,

$$\mathcal{C}^{\mathrm{sym}} := \left\{\tfrac{1}{2}(c - d) \mid c, d \in \mathcal{C}\right\}.$$

Finally, given a convex set $\mathcal{C}$ in V the *dual* of $\mathcal{C}$ is

$$\mathcal{C}^* := \{\phi \in \operatorname{Hom}(V, \mathbb{R}) \mid \phi(v) \leq 1 \text{ for all } v \in \mathcal{C}\}.$$

2.3 Marked polytopes

Let V be a real vector space. A *marked polytope* $\mathcal{M}$ in V is a polytope $\mathcal{P}$ and a (possibly empty) subset $\mathcal{V}^+$ of $\mathcal{V}(\mathcal{P})$. The elements of $\mathcal{V}^+$ are the *marked vertices*; the elements of $\mathcal{V}(\mathcal{P}) \setminus \mathcal{V}^+$ are the *unmarked vertices* and $\mathcal{P}$ is the *underlying polytope* of $\mathcal{M}$.

If $\mathcal{M} = (\mathcal{P}, \mathcal{V}^+)$ and $\mathcal{N} = (\mathcal{Q}, \mathcal{W}^+)$ are two marked polytopes, then the *Minkowski sum of* $\mathcal{M}$ *and* $\mathcal{N}$ has underlying polytope the Minkowski sum of the underlying polytopes and set of marked vertices precisely those that are sums of marked vertices:

$$\mathcal{M} + \mathcal{N} = (\mathcal{P} + \mathcal{Q},\ \mathcal{V}(\mathcal{P} + \mathcal{Q}) \cap (\mathcal{V}^+ + \mathcal{W}^+)).$$

The marked polytope $\mathcal{M} = (\mathcal{P}, \mathcal{V}^+)$ is *symmetric* if the underlying polytope $\mathcal{P}$ is symmetric and $\mathcal{V}^+ = -\mathcal{V}^+$.

2.4 The Thurston norm and fibered classes

Let N be a 3–manifold. For each $\phi \in H^1(N;\mathbb{Z})$ there is a properly embedded oriented surface Σ such that $[\Sigma] \in H_2(N, \partial N;\mathbb{Z})$ is the Poincaré dual to ϕ. Letting $\chi_-(\Sigma) = \sum_{i=1}^k \max\{-\chi(\Sigma_i), 0\}$, where $\Sigma_1, \dots, \Sigma_k$ are the connected components of Σ, the *Thurston norm* of $\phi \in H^1(N;\mathbb{Z})$ is

$$x_N(\phi) = \min\{\chi_-(\Sigma) \mid [\Sigma] = \phi\}.$$

The class $\phi \in H^1(N;\mathbb{R})$ is called *fibered* if it can be represented by a nondegenerate closed 1–form. By [50] an integral class $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N), \mathbb{Z})$ is fibered if and only if there exists a fibration $p\colon N \to S^1$ such that $p_* = \phi\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$.

Thurston [49] showed that x_N extends to a seminorm x_N on $H^1(N;\mathbb{R})$ and that the dual $\mathcal{C}(x_N)^*$ to the unit norm ball $\mathcal{C}(x_N)$ of the seminorm x_N is a polytope $\mathcal{P}_N$ with vertices in $\mathrm{Im}\{H_1(N;\mathbb{Z})/\text{torsion} \to H_1(N;\mathbb{R})\}$. Furthermore, Thurston showed that we can turn $\mathcal{P}_N$ into a marked polytope $\mathcal{M}_N$, which has the property that $\phi \in H^1(N;\mathbb{R}) = \mathrm{Hom}(H_1(N;\mathbb{R}), \mathbb{R})$ is fibered if and only if it pairs maximally with a marked vertex.

2.5 The marked polytope for elements of group rings

Let G be a group. Throughout this paper, given $f \in \mathbb{C}[G]$ and $g \in G$ we let f_g denote the g–coefficient of f. Let $\psi\colon G \to H_1(G;\mathbb{Z})/\text{torsion}$ be the canonical map.

We write $V = H_1(G;\mathbb{R})$ and we view $H_1(G;\mathbb{Z})/\text{torsion}$ as a subset of V. With this convention the above map ψ gives rise to a map $\psi\colon G \to V$. Given $f \in \mathbb{C}[G]$ we refer to

$$\mathcal{P}(f) := \mathcal{P}\big(\{\psi(g) \mid g \in G \text{ with } f_g \neq 0\}\big) \subset V$$

as the *polytope of* f. We will now associate to $\mathcal{P}(f)$ a marking. In order to do this we need a few more definitions:

(1) For $v \in V$ we refer to $f^v := \sum_{g \in \psi^{-1}(v)} f_g g$ as the v*–component of* f.

(2) We say that an element $r \in \mathbb{C}[G]$ is a monomial if it is of the form $r = \pm g$ for some $g \in G$.

A vertex v of $\mathcal{P}(f)$ is *marked* precisely when the v–component of f is a monomial. We then refer to the polytope $\mathcal{P}(f)$ together with the set of all marked vertices as the *marked polytope* $\mathcal{M}(f)$ *of* f.

The proof of [24, Lemma 3.2] applies with the above definitions, to give:

Lemma 2.1 *Let G be a group and let $f, g \in \mathbb{C}[G]$. Then the following hold:*

(1) *If for every vertex v of $\mathcal{P}(f)$ the v–component $f^v \in \mathbb{C}[G]$ is not a zero divisor, then $\mathcal{P}(f \cdot g) = \mathcal{P}(f) + \mathcal{P}(g)$.*

(2) *If each vertex of $\mathcal{M}(f)$ is marked, then $\mathcal{M}(f \cdot g) = \mathcal{M}(f) + \mathcal{M}(g)$.*

2.6 The marked polytope for a $(2,1)$–presentation

Let F be the free group with generators x and y. Following [17] we denote by $\partial/\partial x\colon \mathbb{Z}[F] \to \mathbb{Z}[F]$ the *Fox derivative with respect to* x, ie the unique $\mathbb{Z}$–linear map such that

$$\frac{\partial 1}{\partial x} = 0, \quad \frac{\partial x}{\partial x} = 1, \quad \frac{\partial y}{\partial x} = 0 \quad \text{and} \quad \frac{\partial uv}{\partial x} = \frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$$

for all $u, v \in F$. We similarly define the Fox derivative with respect to y, and often write

$$u_x := \frac{\partial u}{\partial x} \quad \text{and} \quad u_y := \frac{\partial u}{\partial y}.$$

In [24] we proved the following proposition:

Proposition 2.2 *Let $\pi = \langle x, y \mid r\rangle$ be a $(2,1)$–presentation with $b_1(\pi) = 2$. Then there exists a marked polytope $\mathcal{M}$, unique up to translation, such that*

$$\mathcal{M} + \mathcal{M}(x-1) \doteq \mathcal{M}(r_y) \quad \textit{and} \quad \mathcal{M} + \mathcal{M}(y-1) \doteq \mathcal{M}(r_x).$$

Denote by $\mathcal{M}_\pi$ the marked polytope of Proposition 2.2. Up to translation it is a well-defined invariant of the presentation, and it is shown in [24] that this definition is equivalent to the one sketched in the introduction.

A $(2,1)$–presentation $\pi = \langle x, y \mid r \rangle$ is *simple* if $b_1(G_\pi) = 1$, x defines a generator of $H_1(\pi;\mathbb{Z})/\text{torsion}$ and y represents the trivial element in $H_1(\pi;\mathbb{Z})/\text{torsion}$. In [24] we showed that given a simple $(2,1)$–presentation $\pi = \langle x, y \mid r \rangle$ there exists a marked polytope $\mathcal{M}_\pi$, unique up to translation, such that

$$\mathcal{M}_\pi + \mathcal{M}(x-1) \doteq \mathcal{M}(r_y).$$

It was shown in [24] that there is a canonical way to associate to any $(2,1)$–presentation $\pi = \langle x, y \mid r \rangle$ with $b_1(G_\pi) = 1$ a simple presentation $\pi' = \langle x', y' \mid r' \rangle$ representing the same group. We then define $\mathcal{M}_\pi := \mathcal{M}_{\pi'}$.

2.7 3–manifold groups which admit $(2,1)$–presentations

Manifolds having fundamental group with a $(2,1)$–presentation are described in Section 6. The only specific result needed to develop our theory is the following, which follows from work of Epstein [16].

Theorem 2.3 *Let N be an irreducible (compact, connected and orientable) 3–manifold such that $\pi := \pi_1(N)$ admits a $(2,1)$–presentation. Then the boundary of N consists of one or two tori.*

Proof Groups that admit a $(2,1)$–presentation have deficiency 1, while the fundamental group of a closed irreducible 3–manifold has deficiency zero [16, Section 3]. Whence N has nonempty boundary, and [16, Lemma 2.2] implies that $\frac{1}{2}\chi(\partial N) = \chi(N) \geq 0$. No boundary component of N is a sphere since we assume N is irreducible and $\pi_1(N) \neq \{1\}$. Since N (and hence each of its boundary components) is orientable, we now have $\chi(\partial N) = 0$ and every boundary component is a torus.

A standard half-lives, half-dies argument shows $b_1(\partial N) \leq 2b_1(N)$. Since $b_1(N) \leq 2$ we deduce that ∂N consists of either one or two tori. □

3 Properties of 3–manifold groups

3.1 The virtually special theorem

As usual, given a property of groups or spaces we say this property is satisfied *virtually* if a finite-index subgroup (not necessarily normal) or a finite-index cover (not necessarily regular) has the property.

In the following, given a 3–manifold N we say that $\phi \in H^1(N;\mathbb{R})$ is *quasifibered* if it is a limit of fibered classes in $H^1(N;\mathbb{R})$. The following theorem is now a variation of the virtually special theorem combined with Agol's virtual fibering theorem [1, Theorem 5.1] (see also [22, Theorem 5.1] for an exposition).

Theorem 3.1 *Let N be an irreducible 3–manifold with empty or toroidal boundary. If N is not a closed graph manifold, then for every $\phi \in H^1(N;\mathbb{R})$ there exists a finite-index cover $p\colon N' \to N$ such that $p^*(\phi)$ is quasifibered.*

The theorem was proved by Agol [2] for all closed hyperbolic 3–manifolds, by Wise [56; 57; 58] for all hyperbolic 3–manifolds with boundary, by Liu [36] and Przytycki and Wise [43] for all graph manifolds with boundary and by Przytycki and Wise [42] for all 3–manifolds with a nontrivial JSJ–decomposition that has at least one hyperbolic JSJ–component. We refer to [3] for precise references.

If we apply the theorem to the zero class we get in particular the following corollary:

Corollary 3.2 *An irreducible 3–manifold with empty or toroidal boundary is virtually fibered unless it is a closed graph manifold.*

3.2 Residual properties of 3–manifold groups

We start with several definitions, most of which are standard. Let $\mathcal{P}$ be a class of groups.

(1) The group π is *residually* $\mathcal{P}$ if for every nontrivial $g \in \pi$, there exists a homomorphism $\alpha\colon \pi \to \Gamma$ to a group in Γ in $\mathcal{P}$ such that $\alpha(g) \neq 1$.

(2) The group π is *fully residually* $\mathcal{P}$ if for every finite subset $\{g_1,\dots,g_n\} \subset \pi\setminus\{1\}$, there exists a epimorphism $\alpha\colon \pi \to G$ to a group in Γ in $\mathcal{P}$ such that $\alpha(g_i) \neq 1$ for all $i = 1,\dots,n$.

(3) The group π has the *$\mathcal{P}$–factorization property* if for every epimorphism $\alpha\colon \pi \to G$ onto a finite group G there exists an epimorphism $\beta\colon \pi \to \Gamma$ to a group Γ in $\mathcal{P}$ such that α factors through β.

We are mostly interested in the following classes of groups.

(1) The class $\mathcal{EA}$ of *elementary amenable groups* is the smallest class of groups that contains all abelian and all finite groups and that is closed under extensions and directed unions.

(2) We denote by $\mathcal{TEA}$ the class of all groups that are torsion-free and elementary amenable. It is clear that $\mathcal{TEA}$ is closed under taking finite direct products.

Using Corollary 3.2 and work of Linnell and Schick [35] we will prove the following theorem:

Theorem 3.3 *Let N be an irreducible 3–manifold with empty or toroidal boundary. If N is not a closed graph manifold, then $\pi_1(N)$ has the $\mathcal{TEA}$–factorization property.*

The question of to what degree this statement holds for closed graph manifolds is discussed in Section 3.4. We postpone the proof of the theorem to Section 3.3, and point out several corollaries.

Theorem 1.3 *Let N be an irreducible 3–manifold with empty or toroidal boundary. If N is not a closed graph manifold, then $\pi_1(N)$ is residually $\mathcal{TEA}$.*

Proof Let $\mathcal{P}$ be any class of groups. If a group π is residually finite and has the $\mathcal{P}$–factorization property, then G is also residually $\mathcal{P}$. The statement of the theorem now follows from Theorem 3.3 and the fact that 3–manifold groups are residually finite [29]. □

Corollary 3.4 *Let π be the fundamental group of an irreducible 3–manifold that has empty or toroidal boundary and is not a closed graph manifold. For every nonzero element $p \in \mathbb{Z}[\pi]$, there exists a homomorphism $\alpha\colon \pi \to \Gamma \in \mathcal{TEA}$ such that $0 \neq \alpha(p) \in \mathbb{Z}[\Gamma]$.*

Proof We write $p = \sum_{i=1}^{k} a_i g_i$, where $a_1, \ldots, a_k \neq 0$ and $g_1, \ldots, g_n \in \pi$ are pairwise distinct. By Theorem 1.3 the group π is residually $\mathcal{TEA}$. Since $\mathcal{TEA}$ is closed under taking finite direct products, π is also fully residually $\mathcal{TEA}$. We can thus find a homomorphism $\alpha\colon \pi \to \Gamma$ to a group $\Gamma \in \mathcal{TEA}$ such that all $\alpha(g_i)$ and all products $\alpha(g_i g_j^{-1})$ with $i \neq j$ are nontrivial. Whence $\alpha(p) \in \mathbb{Z}[\Gamma]$ is nonzero. □

3.3 Proof of Theorem 3.3

The following lemma is probably well-known to the experts.

Lemma 3.5 *Let E be a surface group (ie the fundamental group of a compact orientable surface, possibly with boundary) and let $R \subset E$ be a normal subgroup. Then $E/[R,R]$ is torsion-free.*

Proof Let $g \in E/[R,R]$ be a nontrivial element. We pick a representative for g in E, which by slight abuse of notation we also denote by g. We denote by S the subgroup of E generated by g and R. It suffices to prove the following claim:

Claim *The group $S/[R,R]$ is torsion-free.*

We consider the short exact sequence

$$1 \to [S,S]/[R,R] \to S/[R,R] \to S/[S,S] \to 0.$$

Since R and S are either surface groups or infinitely generated free groups we deduce that $S/[S,S] = H_1(S;\mathbb{Z})$ and $R/[R,R] = H_1(R;\mathbb{Z})$ are torsion-free. The group S/R is generated by one element, which implies that S/R is cyclic, in particular abelian. It follows that $[S,S]\subset R$. We thus see that $[S,S]/[R,R]$ is a subgroup of $R/[R,R]$. So the groups on the left and on the right of the above short exact sequence are torsion-free. It follows that $S/[R,R]$ is torsion-free. $\square$

Proposition 3.6 *If $1\to E\to\pi\to M\to 1$ is an exact sequence with E a surface group and $M\in\mathcal{TEA}$, then π has the $\mathcal{TEA}$–factorization property.*

Proof Let $\alpha\colon \pi\to P$ be a map to a finite group. Let $R=E\cap\operatorname{Ker}\alpha$. By Lemma 3.5 the group $E/[R,R]$ is torsion-free. Furthermore it is elementary amenable by the exact sequence

$$1\to R/[R,R]\to E/[R,R]\to E/R\to 1.$$

Now α factors through $\pi/[R,R]$, and this is in $\mathcal{TEA}$ due to the sequence

$$1\to E/[R,R]\to \pi/[R,R]\to M\to 1. \qquad\square$$

The *profinite completion* of the group π is denoted by $\widehat{\pi}$; see [44, Section 3.2] for a definition and its main properties. Following Serre [47, I.2.6, Exercise 2] we say that a group π is *good* if the natural morphism $H^*(\widehat{\pi};A)\to H^*(\pi;A)$ is an isomorphism for any finite abelian group A with a π–action.

In the proof of the following theorem we will on several occasions use the following standard notation: if Γ is a subgroup of π, then $\Gamma^\pi:=\bigcap_{g\in\pi} g\Gamma g^{-1}$. Note that Γ^π is always a normal subgroup of π, and if Γ is of finite index, then Γ^π is of finite index. We also note that the methods of the proof build heavily on the work of Linnell and Schick [35].

Theorem 3.7 *Let π be a finitely generated torsion-free group that has a finite-dimensional classifying space and which is good. If π admits a finite-index subgroup Γ which has the $\mathcal{TEA}$–factorization property, then π also has the $\mathcal{TEA}$–factorization property.*

Proof Let $\alpha\colon\pi\to G$ be a homomorphism to a finite group. We denote by $K\subset\pi$ the intersection of $\operatorname{Ker}(\alpha)$ and Γ^π. The subgroup K is of finite index in π and is clearly contained in Γ. It follows from Lemma 2.1 of [46] that K also has the $\mathcal{TEA}$–factorization property. We write $Q:=\pi/K$. First suppose that Q is a p–group. It suffices to show there is a subgroup $U\trianglelefteq\pi$ such that the map $\pi\to Q$ factors through π/U and π/U is in $\mathcal{TEA}$.

If no such U exists, then since K has the $\mathcal{TEA}$–factorization property, there is a nontrivial subgroup Q' of Q that splits in the induced sequence of profinite completions

$$1\to\widehat{K}\to\widehat{\pi}\to Q\to 1;$$

see [46, Lemmas 3.4–3.6]. However, putting the following two observations together shows that this is not possible:

(1) The cohomology $H^*(Q', \mathbb{F}_p)$ is nonzero in infinitely many dimensions.

(2) By [47, I.2.6, Exercise 1, page 15] any finite-index subgroup L (such as K or the preimage of Q' under $\pi \to Q$) of π is also good and has a finite-dimensional classifying space. This implies that $H^*(\widehat{L}, \mathbb{F}_p) \cong H^*(L, \mathbb{F}_p)$ is nonzero in only finitely many dimensions.

For the general case, we use a trick from [35]. For each Sylow p–subgroup S of Q, consider the exact sequence $1 \to K \to \pi_S \to S \to 1$, where π_S is the preimage of S. By the above, we get for each S a subgroup U_S such that the quotient π_S/U_S is torsion-free elementary amenable. Let $U = \cap_S U_S$. Since π/U^π is a finite extension of Γ/U^π, elementary amenability follows from [35, Lemma 4.11]. It remains to show that π/U^π is torsion-free.

There is an exact sequence

$$1 \to U_S^\pi/U^\pi \to \pi_S/U^\pi \to \pi_S/U_S^\pi \to 1$$

with U_S^π/U^π and π_S/U_S^π torsion-free [35, Lemma 4.11]. Therefore, π_S/U^π is torsion-free.

Suppose that π/U^π has a nontrivial torsion element γ. By raising γ to some power we get an element γ' that is p–torsion for some prime p. Since K/U^π is torsion-free, γ' would map to some Sylow p–subgroup, in which case $\gamma' \in \pi_S/U^\pi$, which is torsion-free by the above. Therefore, π/U^π is torsion-free. □

Now we are finally in a position to prove Theorem 3.3.

Proof of Theorem 3.3 Let N be an irreducible 3–manifold that has empty or toroidal boundary and that is not a closed graph manifold. According to Corollary 3.2, N has a finite cover M that is fibered. The fundamental group of M is a semidirect product of $\mathbb{Z}$ with a surface group, and hence Lemma 3.5 and Proposition 3.6 imply $\pi_1(M)$ has the $\mathcal{TEA}$–factorization property.

It follows from [47, Exercise 2(b), page 16] that $\pi_1(M)$ is good. By [47, Exercise 1, page 15] the group $\pi_1(N)$ is also good. It is well-known (see eg [3, (A.1), page 44]) that N is aspherical and that in particular $\pi_1(N)$ is torsion-free. Thus we can apply Theorem 3.7 to $\pi_1(N)$ and the finite-index subgroup $\pi_1(M)$, giving the desired result that $\pi_1(N)$ has the $\mathcal{TEA}$–factorization property. □

Remark The same proof also shows that torsion-free virtually cocompact special groups have the $\mathcal{TEA}$–factorization property. Indeed, these groups are virtual retracts of right-angled Artin groups, and therefore contain finite index subgroups that are good and have the $\mathcal{TEA}$–factorization property [46]. The 3–manifold groups that we consider in Theorem 1.3 are not generally known to be virtually cocompact special. However, this observation implies that Theorem 1.3 holds for many other 3–manifold groups, eg for fundamental groups of hyperbolic 3–manifolds with infinite volume. We refer to [3, Theorem 4.3.6] for details and references.

3.4 The case of closed graph manifolds

It is natural to ask for which closed graph manifolds the conclusions of Theorem 3.3 and its corollaries hold. It follows from the work of Liu [36] that the conclusion of the theorem also holds for closed nonpositively curved graph manifolds. The question of which closed graph manifolds are nonpositively curved was treated in detail by Buyalo and Svetlov [7]. In the following we give a short list of examples of graph manifolds that are not nonpositively curved:

(1) spherical 3–manifolds;

(2) Sol– and Nil–manifolds;

(3) Seifert fibered 3–manifolds that are finitely covered by a nontrivial S^1–bundles over a closed surface.

It is clear that the statements do not hold for spherical 3–manifolds with nontrivial fundamental group. The following lemma takes care of the second case:

Lemma 3.8 *The fundamental groups of Sol– and Nil–manifolds are $\mathcal{TEA}$; in particular they have the $\mathcal{TEA}$–factorization property.*

Proof Sol– and Nil–manifolds are finitely covered by torus-bundles over S^1. Hence their fundamental groups are elementary amenable, but the fundamental groups are also torsion-free, so they are $\mathcal{TEA}$. □

Lemma 3.9 *Let N be a Seifert fibered space with infinite fundamental group. Then $\pi_1(N)$ has the $\mathcal{TEA}$–factorization property.*

Proof Since we will not make use of this lemma we only sketch the proof. The manifold N is finitely covered by an S^1–bundle over a surface. By Theorem 3.7 we can thus without loss of generality assume that N is an S^1–bundle over a surface F. Since $\pi_1(N)$ is infinite there exists a short exact sequence

$$1 \to \langle t\rangle \to \pi_1(N) \to \pi_1(F) \to 1,$$

where the subgroup $\langle t\rangle$ is generated by the S^1–fiber. By Proposition 3.6 the group $\pi_1(F)$ has the $\mathcal{TEA}$–factorization property. Let e denote the Euler number of the

S^1–bundle over F and denote by M the total space of the S^1–bundle over the torus with Euler number e. Then there exists a fiber-preserving map from N to M. Since $\pi_1(M)$ is $\mathcal{TEA}$ we have found a homomorphism from $\pi_1(N)$ to a $\mathcal{TEA}$ group which is injective on $\langle t\rangle$. Now it is straightforward to see that $\pi_1(N)$ has the $\mathcal{TEA}$–factorization property. □

The above discussion shows that the fundamental groups of many closed graph manifolds have the $\mathcal{TEA}$–factorization property. Nonetheless we expect that there are many closed graph manifolds whose fundamental groups do not have the $\mathcal{TEA}$–factorization property.

4 Proof of Theorem 1.2, I

The goal of this section is to prove the following proposition.

Proposition 4.1 *Let $\pi = \langle x, y \mid r\rangle$ be a cyclically reduced $(2,1)$–presentation for the fundamental group of an irreducible 3–manifold N. Then*

$$\mathcal{P}_N \leq \mathcal{P}_\pi.$$

The main ingredient in the proof will be the fact that twisted Reidemeister torsions corresponding to finite-dimensional complex representations detect the Thurston norm of 3–manifolds.

4.1 Tensor representations

Let π be a group, let $\alpha\colon \pi \to \mathrm{GL}(k,\mathbb{C})$ be a representation and let $\psi\colon \pi \to H$ be a homomorphism to a free abelian group. We denote by $\mathbb{C}(H)$ the quotient field of the group ring $\mathbb{C}[H]$. The homomorphisms α and ψ give rise to the representation

$$\alpha\otimes\psi\colon \pi \to \mathrm{GL}(k,\mathbb{C}(H)), \quad g \mapsto \alpha(g)\cdot\psi(k),$$

which we refer to as the tensor product of α and ψ. This representation extends to a ring homomorphism $\mathbb{Z}[\pi] \to M(k\times k,\mathbb{C}(H))$, which we also denote by $\alpha\otimes\psi$.

4.2 The definition of the twisted Reidemeister torsion

Let X be a finite CW–complex, $\pi := \pi_1(X)$, and denote by $\widetilde{X}$ the universal cover of X. The action of π via deck transformations on $\widetilde{X}$ equips the chain complex $C_*(\widetilde{X};\mathbb{Z})$ with the structure of a chain complex of $\mathbb{Z}[\pi]$–left modules.

Let $\alpha\colon \pi \to \mathrm{GL}(k,\mathbb{C})$ be a representation. We let $\psi\colon \pi \to H := H_1(X;\mathbb{Z})/\text{torsion}$ be the obvious projection map. Using the representation $\alpha \otimes \psi$ we can now view $\mathbb{C}(H)^k$ as a right $\mathbb{Z}[\pi]$–module, where the action is given by right multiplication on row vectors.

We consider the chain complex

$$C_*(X;\mathbb{C}(H)^k) := \mathbb{C}(H)^k \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X};\mathbb{Z})$$

of $\mathbb{C}(H)$–modules. For each cell in X pick a lift to a cell in $\widetilde{X}$. We denote by $e_1,\dots,e_k$ the standard basis for $\mathbb{C}(H)^k$. The tensor products of the lifts of the cells and the vectors e_i turn $C_*(X;\mathbb{C}(H)^k)$ into a chain complex of based $\mathbb{C}(H)$–vector spaces.

If the chain complex $C_*(X;\mathbb{C}(H)^k)$ is not acyclic, then we define the corresponding twisted Reidemeister torsion $\tau(X,\alpha)$ to be zero. Otherwise we let $\tau(X,\alpha)\in\mathbb{C}(H)\setminus\{0\}$ be the torsion of the based chain complex $C_*(X;\mathbb{C}(H)^k)$. We refer to [52] for the definition of the torsion of a based chain complex. Standard arguments show that $\tau(X,\alpha)\in\mathbb{C}(H)\setminus\{0\}$ is well-defined up to multiplication by an element of the form zh, where $z \in \pm\det(\alpha(\pi))$ and $h \in H$. The indeterminacy arises from the fact that we had to choose lifts and an ordering of the cells.

Suppose N is a 3–manifold and let $\alpha\colon \pi_1(N)\to\mathrm{GL}(k,\mathbb{C})$ be a representation. Choose a CW–structure X for N and define $\tau(N,\alpha) := \tau(X,\alpha)$. It is well-known (see eg [52; 25]) that this definition does not depend on the choice of the CW–structure.

4.3 The polytopes corresponding to twisted Reidemeister torsion

As above, suppose N is a 3–manifold and $\alpha\colon \pi_1(N)\to\mathrm{GL}(k,\mathbb{C})$ a representation. If $\tau(N,\alpha)$ is zero, then we define $\mathcal{T}(N,\alpha)=\varnothing$.

Otherwise we write $\tau(N,\alpha) = p\cdot q^{-1}$ with $p,q\in\mathbb{C}[H]$. If the Minkowski difference $\mathcal{P}(q)-\mathcal{P}(p)$ exists (and by [25, page 53] this is the case if $b_1(N)\geq 2$), then we define

$$\mathcal{T}(N,\alpha) := \frac{1}{k}\cdot(\mathcal{P}(p)-\mathcal{P}(q)),$$

and otherwise define $\mathcal{T}(N,\alpha) := \{0\}$.

Proposition 4.2 *Let $\pi=\langle x,y\mid r\rangle$ be a $(2,1)$–presentation for the fundamental group of an irreducible 3–manifold N. Then for any representation we have*

$$\mathcal{T}(N,\alpha)\subset\mathcal{P}_\pi.$$

In the proof of the proposition we will need one more definition and one more lemma. Let π be a group, $f \in \mathbb{Z}[\pi]$, $\alpha\colon \pi \to \mathrm{GL}(k,\mathbb{C})$ be a representation, and $\psi_\pi\colon \pi \to H := H_1(\pi;\mathbb{Z})/\text{torsion}$ be the canonical epimorphism. Then $\det((\alpha\otimes\psi_\pi)(f)) \in \mathbb{C}[H]$ and we write

$$\mathcal{P}(f,\alpha) := \frac{1}{k}\mathcal{P}\big(\det((\alpha\otimes\psi_\pi)(f))\big) \subset H_1(\pi;\mathbb{R}).$$

Lemma 4.3 *Let π be a group, $f \in \mathbb{Z}[\pi]$ and $\alpha\colon \pi \to \mathrm{GL}(k,\mathbb{C})$ be a representation. Then*

$$\mathcal{P}(f,\alpha) \subset \mathcal{P}(f).$$

Proof We write $f = c_1h_1 + \cdots + c_lh_l$ with $h_1,\dots,h_l \in \pi$ and $c_1,\dots,c_l \neq 0$. We consider

$$S := \{s_1\psi(g_1) + \cdots + s_l\psi(h_l) \mid s_1,\dots,s_l \in \mathbb{C}\}.$$

Put differently, S is the set of all elements in $\mathbb{C}[H]$ with support some subset of $\{\psi(g_1),\dots,\psi(g_l)\}$. For every $p \in S$ we have $\mathcal{P}(p) \subset \mathcal{P}(\psi(g_1),\dots,\psi(g_l)) = \mathcal{P}(f)$. This implies that if $p_1,\dots,p_k$ are elements in S, then

$$\mathcal{P}(p_1\cdots p_k) = \mathcal{P}(p_1) + \cdots + \mathcal{P}(p_k) \subset \mathcal{P}(f) + \cdots + \mathcal{P}(f) = k\mathcal{P}(f).$$

We write $M := (\alpha\otimes\psi)(f) = \sum_{i=1}^{l} c_i\alpha(h_i)\cdot\psi(h_i)$. Each entry of $\det(M)$ lies in S. It follows from the Laplace formula that $\det(M)$ is a sum of products of the form $p_1\cdots p_k$, where each p_i lies in S. By the above we have $\mathcal{P}(p_1\cdots p_k) \subset k\mathcal{P}(f)$. The definitions imply that if $a,b \in \mathbb{C}[\pi]$ are such that $\mathcal{P}(a)$ and $\mathcal{P}(b)$ are contained in a polytope $\mathcal{Q}$, then we have also have $\mathcal{P}(a+b) \subset \mathcal{Q}$. Hence $\mathcal{P}(\det(M)) \subset k\mathcal{P}$. □

Proof of Proposition 4.2 We again denote by $\psi\colon \pi_1(N) \to H_1(N;\mathbb{Z})/\text{torsion}$ the canonical epimorphism. Note that $\psi(x) \neq 0$ or $\psi(y) \neq 0$. Without loss of generality we may assume $\psi(y) \neq 0$.

Theorem 2.3 shows that N has nontrivial toroidal boundary. It thus follows from [32, Theorem A] (see also [25, page 50]) that

$$\tau(N,\alpha) = \det((\alpha\otimes\psi)(r_y))\cdot\det((\alpha\otimes\psi)(y-1))^{-1}.$$

By Lemma 4.3 we have $\mathcal{P}(r_y,\alpha) \subset \mathcal{P}(r_y)$. Since $\psi(y) \neq 0$ we know that $\psi(y)$ and 1 are the two distinct vertices of $\mathcal{P}(y-1)$. Also, we have $\mathcal{P}(y-1,\alpha) = \frac{1}{k}\mathcal{P}\big(\det(\alpha(y)\psi(y) - \mathrm{id}_k)\big)$ and it is straightforward to see that this polytope equals $\mathcal{P}(y-1)$.

Combining these results we obtain

$$\mathcal{T}(N,\alpha) = \mathcal{P}(r_y,\alpha) - \mathcal{P}(y-1,\alpha) = \mathcal{P}(r_y,\alpha) - \mathcal{P}(y-1) \subset \mathcal{P}(r_y) - \mathcal{P}(y-1) = \mathcal{P}_\pi. \quad \square$$

4.4 The proof of Proposition 4.1

Proposition 4.1 is an immediate consequence of Theorem 2.3, Proposition 4.2 and the second statement of the following proposition:

Proposition 4.4 *Let N be a 3–manifold with empty or toroidal boundary and let $\alpha\colon \pi_1(N) \to U(k,\mathbb{C})$ be a unitary representation. Then*

$$\mathcal{T}(N,\alpha) \le \mathcal{P}_N.$$

Furthermore, if N is irreducible, then there exists a unitary representation

$$\alpha\colon \pi_1(N) \to U(k,\mathbb{C})$$

such that

$$\mathcal{T}(N,\alpha) \doteq \mathcal{P}_N.$$

Proof Let N be a 3–manifold with empty or toroidal boundary. We write $\pi = \pi_1(N)$. Let $\alpha\colon \pi \to U(k,\mathbb{C})$ be a unitary representation. If $\tau(N,\alpha) = 0$, then there is nothing to show. So suppose that $\tau(N,\alpha) \neq 0$. In [19, Theorem 1.1; 20, Theorem 3.1] it was shown that for any $\phi \in H^1(N;\mathbb{R}) = \operatorname{Hom}(\pi,\mathbb{R})$ we have

$$\max\{\phi(p) - \phi(q) \mid p,q \in \mathcal{T}(N,\alpha)\} \le x_N(\phi).$$

It follows from the definitions and the discussion in Section 2.2 that $\mathcal{T}(N,\alpha)^{\mathrm{sym}} \le \mathcal{P}_N$. Since α is a unitary representation, it follows from [21, Theorem 1.2] that $\mathcal{T}(N,\alpha)^{\mathrm{sym}} \doteq \mathcal{T}(N,\alpha)$. It thus follows that indeed $\mathcal{T}(N,\alpha) \le \mathcal{P}_N$.

If N is not a closed graph manifold, then, building on Theorem 3.1, it was shown in [26, Corollary 5.10] that there exists a unitary representation $\alpha\colon \pi \to U(k,\mathbb{C})$ such that

$$\max\{\phi(p) - \phi(q) \mid p,q \in \mathcal{T}(N,\alpha)\} = x_N(\phi)$$

for every $\phi \in H^1(N;\mathbb{R})$. The same argument as above then implies that $\mathcal{T}(N,\alpha) \doteq \mathcal{P}_N$. If N is a closed graph manifold, then the same statement holds by [23]. □

5 Proof of Theorem 1.2, II

The goal of this section is to prove the following proposition, and to complete the proof of the main theorem.

Proposition 5.1 *Let $\pi = \langle x, y \mid r\rangle$ be a cyclically reduced $(2,1)$–presentation for the fundamental group of an irreducible 3–manifold N. Then*

$$\mathcal{P}_\pi^{\mathrm{sym}} \le \mathcal{P}_N.$$

In the proof of Proposition 4.1 we used twisted Reidemeister torsions corresponding to finite-dimensional complex representations. In the proof of Proposition 5.1 we use a different but related object, namely noncommutative Reidemeister torsions. In this context they were first studied in [9; 8; 27; 18].

5.1 The Ore localization of group rings and degrees

Let $\Gamma \in \mathcal{TEA}$. It follows from [33, Theorem 1.4] that the group ring $\mathbb{Z}[\Gamma]$ is a domain. Since Γ is amenable it follows from [11, Corollary 6.3] that $\mathbb{Z}[\Gamma]$ satisfies the Ore condition. This means that for any two nonzero elements $x, y \in \mathbb{Z}[\Gamma]$ there exist nonzero elements $p, q \in \mathbb{Z}[\Gamma]$ such that $xp = yq$. By [41, Section 4.4] this implies that $\mathbb{Z}[\Gamma]$ has a classical fraction field, referred to as the Ore localization of $\mathbb{Z}[\Gamma]$, which we denote by $\mathbb{K}(\Gamma)$.

Let $\phi\colon \Gamma \to \mathbb{Z}$ be a homomorphism. For every nonzero $p = \sum_{g\in\Gamma} p_g g \in \mathbb{Z}[\Gamma]$ we define

$$\deg_\phi(p) = \max\{\phi(g) - \phi(h) \mid p_g \neq 0 \text{ and } p_h \neq 0\}.$$

We extend this to all of $\mathbb{Z}[\Gamma]$ by letting $\deg_\phi(0) = -\infty$. Since $\mathbb{Z}[\Gamma]$ has no nontrivial zero divisors it follows that for $p, q \in \mathbb{Z}[\Gamma]$ we have $\deg_\phi(pq) = \deg_\phi(p) + \deg_\phi(q)$. Given $pq^{-1} \in \mathbb{K}(\Gamma)$ we also define

$$\deg_\phi(pq^{-1}) := \deg_\phi(p) - \deg_\phi(q).$$

It is straightforward to see that this is indeed well-defined.

5.2 Noncommutative Reidemeister torsion of presentations

Let X be a finite CW–complex with $G = \pi_1(X)$, and let $\widetilde{X}$ denote the universal cover of X. As in Section 4.2 we view $C_*(\widetilde{X})$ as a chain complex of left $\mathbb{Z}[G]$–modules. Now let $\varphi\colon G \to \Gamma \in \mathcal{TEA}$ be a homomorphism, and consider the chain complex of left $\mathbb{K}(G)$–modules

$$C_*(X; \mathbb{K}(\Gamma)) = \mathbb{K}(\Gamma) \otimes_{\mathbb{Z}[G]} C_*(\widetilde{X}),$$

where G acts on $\mathbb{K}(\Gamma)$ on the right via the homomorphism φ. If $C_*(X; \mathbb{K}(\Gamma))$ is not acyclic, define the corresponding Reidemeister torsion $\tau(X, \varphi)$ to be zero. Otherwise choose an ordering of the cells of X and for each cell in X pick a lift to $\widetilde{X}$. This turns $C_*(X; \mathbb{K}(\Gamma))$ into a chain complex of based $\mathbb{K}(\Gamma)$ left-modules and we define

$$\tau(X, \varphi) \in K_1(\mathbb{K}(\Gamma))$$

to be the Reidemeister torsion of the based chain complex $C_*(X; \mathbb{K}(\Gamma))$. Here $K_1(\mathbb{K}(\Gamma))$ is the abelianization of the direct limit $\lim_{n\to\infty} \mathrm{GL}(n, \mathbb{K}(\Gamma))$ of the general linear groups over $\mathbb{K}(\Gamma)$ (see [37; 45] for details). We write $\mathbb{K}(\Gamma)^\times = \mathbb{K}(\Gamma) \setminus \{0\}$ and denote by $\mathbb{K}(\Gamma)^\times_{\mathrm{ab}}$ the abelianization of the multiplicative group $\mathbb{K}(\Gamma)^\times$. The Dieudonné determinant (see [45]) gives rise to an isomorphism $K_1(\mathbb{K}(\Gamma)) \to \mathbb{K}(\Gamma)^\times_{\mathrm{ab}}$, which we will use to identify these two groups. The invariant $\tau(X,\varphi) \in \mathbb{K}(\Gamma)^\times$ is well-defined up to multiplication by an element of the form $\pm g$, where $g \in \Gamma$. Furthermore, it does not depend on the homeomorphism type of X. We refer to [52; 18; 28] for details.

It follows from $\deg_\phi(p \cdot q) = \deg(p) + \deg(q)$ for $p, q \in \mathbb{K}(\Gamma)^\times$ that $\deg_\phi$ descends to a homomorphism $\deg_\phi\colon \mathbb{K}(\Gamma)^\times_{\mathrm{ab}} \to \mathbb{Z}$. In particular $\deg_\phi(\tau(X,\varphi))$ is defined.

Proof of Proposition 5.1 Let N be an irreducible 3–manifold and suppose $\pi = \langle x, y \mid r \rangle$ is a cyclically reduced $(2,1)$–presentation of its fundamental group. Without loss of generality we may assume that x represents a nonzero element in $H := H_1(N;\mathbb{Z})/\text{torsion}$. We need to show that $\mathcal{P}_\pi \le \mathcal{P}_N$.

We call $\phi \in \mathrm{Hom}(\pi,\mathbb{R})$ *generic* if there are vertices v and w of $\mathcal{P}(r_y)$ such that ϕ pairs maximally with v and ϕ pairs minimally with w.

Claim *For any generic epimorphism $\phi\colon \pi \to \mathbb{Z}$, we have $\mathrm{th}_{\mathcal{P}_\pi}(\phi) \le x_N(\phi)$.*

We denote by v and w the (necessarily unique) vertices of $\mathcal{P}(r_y)$ such that ϕ pairs maximally with v and minimally with w. By Corollary 3.4 and Theorem 2.3 there exists a homomorphism $\alpha\colon \pi_1(N) \to \Gamma \in \mathcal{TEA}$ such that $\alpha(r_y^v \cdot r_y^w) \ne 0$. In particular, $\alpha(r_y^v) \ne 0$ and $\alpha(r_y^w) \ne 0$. Let $\psi\colon \pi \to H$ denote the canonical epimorphism. After possibly replacing α by $\alpha \times \psi$ we can and will assume that ψ factors through α. In particular ϕ factors through α and $\alpha(x)$ is a nontrivial element in Γ.

We denote by X the CW–complex corresponding to the presentation π with one 0–cell, two 1–cells corresponding to the generators x and y and one 2–cell corresponding to the relator r. As in [24] we have $\tau(N,\alpha) = \tau(X,\alpha)$. We then have

$$\begin{aligned}
\mathrm{th}_{\mathcal{P}_\pi}(\phi) &= \mathrm{th}_{\mathcal{P}(r_y)}(\phi) - \mathrm{th}_{\mathcal{P}(x-1)}(\phi)\\
&= (\phi(v) - \phi(w)) - |\phi(x)|\\
&= \deg_\phi(\alpha(r_y)) - \deg_\phi(\alpha(x) - 1)\\
&= \deg_\phi(\alpha(r_y) \cdot \alpha(x-1)^{-1})\\
&= \deg_\phi(\tau(X,\alpha)) = \deg_\phi(\tau(N,\alpha)) \le x_N(\phi).
\end{aligned}$$

Here the first two equalities follows from the definitions and the choice of v and w. The fifth equality is [18, Theorem 2.1] and the last inequality is given by [18, Theorem 1.2] (see also [8; 27; 53]). This concludes the proof of the claim.

It is straightforward to see that the nongeneric elements in $\mathrm{Hom}(\pi, \mathbb{R})$ correspond to a union of proper subspaces of $\mathrm{Hom}(\pi, \mathbb{R})$. By continuity and linearity of seminorms we see that the inequality $\mathrm{th}_{\mathcal{P}_\pi}(\phi) \le x_N(\phi)$ holds in fact for all $\phi \in \mathrm{Hom}(\pi, \mathbb{R})$. It follows from the definitions and the discussion in Section 2.2 that $\mathcal{P}_N^{\mathrm{sym}} \le \mathcal{P}_N$. □

5.3 Proof of the main theorem

For the reader's convenience we recall the statement of Theorem 1.2.

Theorem 1.2 *Let N be an irreducible 3–manifold that admits a cyclically reduced $(2,1)$–presentation $\pi = \langle x, y \mid r \rangle$. Then*

$$\mathcal{M}_N \doteq \mathcal{M}_\pi.$$

Proof It follows from Propositions 4.1 and 5.1 that $\mathcal{P}_N \le \mathcal{P}_\pi$ and $\mathcal{P}_\pi^{\mathrm{sym}} \le \mathcal{P}_N$. By the symmetry of the Thurston norm we also have $\mathcal{P}_N = \mathcal{P}_N^{\mathrm{sym}}$, and this implies $\mathcal{P}_N \doteq \mathcal{P}_\pi$. The fact that the markings agree is an immediate consequence of [24, Theorem 1.1; 4, Theorem E]. □

6 Examples

Currently there is no geometric characterization of those 3–manifolds whose fundamental group may be presented using only two generators and one relator. Waldhausen's question [55] of whether the rank of the fundamental group equals the Heegaard genus gives the conjectural picture that all of these manifolds have tunnel-number one. Li [34] gives examples of 3–manifolds whose rank is strictly smaller than the genus, including closed manifolds, manifolds with boundary, hyperbolic manifolds, and manifolds with nontrivial JSJ decomposition. See also related work of Boileau, Weidmann and Zieschang [6; 5]. However, Waldhausen's question remains open for hyperbolic 3–manifolds of rank 2 and for knot complements in S^3.

6.1 Tunnel-number one manifolds

A *tunnel-number one* 3–manifold is a 3–manifold obtained by attaching a 2–handle to a 3–dimensional 1–handlebody of genus two. The fundamental group has a presentation with two generators from the handlebody and one relator corresponding to the attaching circle of the 2–handle. Theorem 1.2 allows us to compute the unit ball of the Thurston norm with ease, whilst other methods, such as normal surface theory [51; 10] have limited scope (see [13]). Moreover, with Theorem 1.2 one can easily construct examples with prescribed combinatorics or geometry of the unit ball.

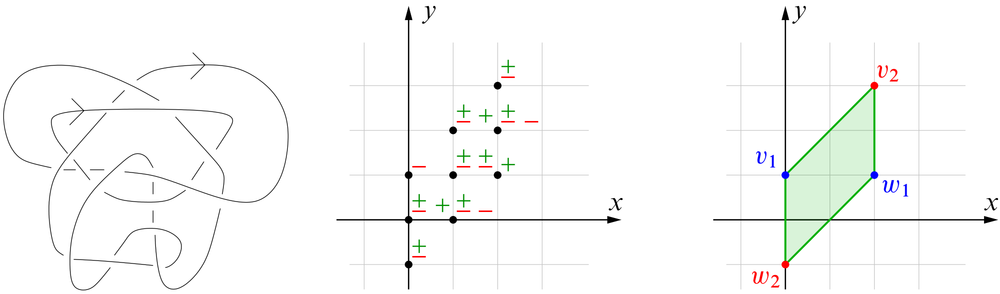

Figure 2: The calculation of $\partial r/\partial x$ (right) for Dunfield's link (left). Center shows the terms appearing in $\partial r/\partial x$ sorted according to their abelianization, with signs as indicated.

Brown's algorithm is an essential ingredient in Dunfield and D Thurston's proof [14] that the probability of a tunnel-number one manifold fibering over the circle is zero. This can be paraphrased as: the probability that the unit ball has a nonempty set of marked vertices is zero. Interesting applications of Theorem 1.2 combined with the methods of [14] would be further predictions about the unit ball of a random tunnel-number one manifold.

6.2 Knots or links in S^3

Norwood [40] showed that if the complement of a given knot in S^3 has fundamental group generated by two elements, then the knot is prime. The complements of *tunnel-number one* knots or links in S^3 are tunnel-number one manifolds. This includes the 2–bridge knots and links, but Johnson [31] showed that there are hyperbolic tunnel-number one knots with arbitrarily high bridge number. There is a complete classification of all tunnel-number one satellite knots by Morimoto and Sakuma [39], and Morimoto [38] also showed that a composite link has tunnel-number one if and only if it is a connected sum of a 2–bridge knot and the Hopf link.

6.3 Dunfield's link

We conclude this section with an explicit calculation for the link L shown in Figure 2, left, which was studied by Dunfield [12]. Write $X_L := S^3 \setminus \nu L$ and write $\pi := \pi_1(X_L)$ for the link group. Then π has the presentation

$$\langle x, y \mid x^2yx^{-1}yx^2yx^{-1}y^{-3}x^{-1}yx^2yx^{-1}yxy^{-1}x^{-2}y^{-1}xy^{-1}x^{-2}y^{-1}xy^3xy^{-1} \cdot x^{-2}y^{-1}xy^{-1}x^{-1}y\rangle,$$

where a meridian for the unknotted component is $y^{-1}x^{-1}yx^2yx^{-1}yx^2yx^{-1}y^{-3}$ and a meridian for the other component is $x^{-1}y^{-1}$. Theorem 1.2 implies that $\mathcal{P}_\pi \doteq \mathcal{P}_N$.

We use the map induced by $x \mapsto (1,0)$ and $y \mapsto (0,1)$ to identify $H_1(X_L;\mathbb{Z}) = H_1(\pi;\mathbb{Z})$ with $\mathbb{Z}^2$.

A straightforward calculation shows that $\mathcal{P}(r_x)$ is the polytope with vertices $v_1=(0,1)$, $v_2=(2,3)$, $w_1=(2,1)$ and $w_2=(0,-1)$ shown in Figure 2, right. Here v_1 and w_1 are opposite vertices of $\mathcal{P}(r_x)$ and v_2 and w_2 are opposite vertices of $\mathcal{P}(r_x)$. Subtracting the underlying polytope of $\mathcal{M}(y-1)$ from $\mathcal{P}(r_x)$ gives $\mathcal{P}_\pi$, and this agrees (up to translation) with Figure 1. The following computation shows that the markings are the same:

$$\begin{aligned}
(r_x)^{v_1} &= x^2yx^{-1}yx^2yx^{-1}y^{-3}x^{-1}yx^2yx^{-1}yxy^{-1}x^{-2}y^{-1}xy^{-1}x^{-2}y^{-1}xy^3 \\
&\qquad \cdot xy^{-1}x^{-2},\\
(r_x)^{w_1} &= x^2yx^{-1}yx^2yx^{-1}y^{-3}x^{-1}yx,\\
(r_x)^{v_2} &= x^2yx^{-1}yx^2yx^{-1}(-1+y^{-3}x^{-1}yx^2yx^{-1}y),\\
(r_x)^{w_2} &= x^2yx^{-1}yx^2yx^{-1}y^{-3}x^{-1}yx^2yx^{-1}yxy^{-1}x^{-2}y^{-1}xy^{-1}x^{-2}y^{-1}\\
&\qquad \cdot (1-xy^3xy^{-1}x^{-2}y^{-1}xy^{-1}x^{-1}).
\end{aligned}$$

7 A conjecture and a question

7.1 A conjecture

We conjecture that Poincaré duality for the 3–manifold can be seen on the level of group presentations as follows:

Conjecture 7.1 *Let $\pi = \langle x, y \mid r\rangle$ be a $(2,1)$–presentation for the fundamental group of a 3–manifold. Then there exists $u \in \left(\frac{1}{2}\mathbb{Z}\right)^2$ such that for any vertex v of $\mathcal{P}(r_x)$ the reflection of v in u, ie the point $w = u-(v-u) = 2u-v$, is also a vertex of $\mathcal{P}(r_x)$. Furthermore we have*

$$(r_x)^v \equiv (-1)^{b_0(\partial N)-1}\,\overline{(r_x)^w}.$$

The twisted Reidemeister torsions of [25] can be computed in terms of Fox derivatives, and the symmetry results for twisted Reidemeister torsions proved in [32; 30; 21] give strong evidence towards this conjecture. Also note that if π is a geometric presentation, ie if it comes the presentation given by a genus-2 handlebody with a 1–handle attached, then r is palindromic, ie reads the same forward and backward (see eg [15, Section 5.2]), and then it is elementary to verify that the conjecture holds.

To give an explicit example, let us return to Dunfield's link. Given the group G and $p, q \in \mathbb{Z}[G]$, write $p \equiv q$ if there exist $g, h \in G$ such that $p = gqh$. Furthermore, denote by $p \mapsto \overline{p}$ the involution of $\mathbb{Z}[G]$ defined by the inversion map $g \mapsto g^{-1}$ for each $g \in G$. We denote by $\pi = \langle x, y \mid r \rangle$ the presentation from Section 6.3. We then note that

$$(r_x)^{v_2} \equiv -1 + y^{-3}x^{-1}yx^2yx^{-1}y (r_x)^{w_2} \equiv 1 - xy^3xy^{-1}x^{-2}y^{-1}xy^{-1}x^{-1}.$$

The relator r is conjugate to

$$yx^2yx^{-1}yx^2yx^{-1}(y^{-3}x^{-1}yx^2yx^{-1}y)xy^{-1}x^{-2}y^{-1}xy^{-1}x^{-2}y^{-1} \cdot (xy^3xy^{-1}x^{-2}y^{-1}xy^{-1}x^{-1}).$$

In particular writing $s = yx^2yx^{-1}yx^2yx^{-1}$ we have the following equality in $\mathbb{Z}[\pi]$:

$$\begin{aligned}(r_x)^{v_2} &\equiv s(r_y)^{v_2}s^{-1} = s(-1 + y^{-3}x^{-1}yx^2yx^{-1}y)s^{-1} \\ &= -1 + (xy^3xy^{-1}x^{-2}y^{-1}xy^{-1}x^{-1})^{-1} \\ &= -\overline{(r_x)^{w_2}}.\end{aligned}$$

7.2 A question

We initially attempted to prove Theorem 1.2 just using twisted Reidemeister torsions corresponding to finite-dimensional representations, noting that Theorem 1.2 follows from the first part of Proposition 4.4 together with an affirmative answer to the following question, which is interesting in its own right.

Question 7.2 *Let N be an aspherical 3–manifold and write $\pi = \pi_1(N)$. Let p be a nonzero element in $\mathbb{Z}[\pi]$. Does there exist a representation $\alpha\colon \pi \to \mathrm{GL}(k, \mathbb{C})$ such that $\det(\alpha(f)) \neq 0$?*

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